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Masters Thesis

Thermoelectric transport in strange metals

Author:

J.S. Harms
Theoretical Physics

Supervisor:

Prof. Dr. Ir. H.T.C. Stoof
E. Mauri, MSc.
Institute for Theoretical Physics

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Abstract

Holographic models gained a lot of interest in the condensed-matter community in the last decade. People are interested in these models since they might be able to capture some of the features observed in strongly correlated states of matter. An example of such a highly correlated state of matter is the "strange" metallic phase, which is a metallic phase where the electrons of the system are strongly coupled. In this thesis we try to study these strange metallic phases using a holographic hyperscaling-violating model. Concretely, the thermoelectric transport properties of the system will be determined.

Contents

Chapter 1

Introduction

Condensed-matter theory has continuously been inspired by the creation of new materials. Throughout the years lots of new classes of materials have been discovered. With each new type of material condensed-matter physicists face the challenge to understand these new observed phenomena. This might seem trivial since the basic rules describing electrons and atomic nuclei have long been known. Still, while these basic principles cover the different local descriptions, describing the physics of these new materials is often non-trivial and might be seen just as fundamental as giving the basic rules describing the electrons and the nuclei. Examples of these new physical phenomena include the appearance of fractional charge in the fractional quantum Hall effect and the discovery of what is now known as "conventional" superconductivity.

One of the current phenomena is the appearance of metallic compounds which seem not to be described by the well-established theory for metals. A class of these materials are called "strange" metals, they were first discovered as the metallic phase in high-T_c cuprate superconductors.

Usually metals are described starting from a weakly interacting electron (-like) gas picture (rigorously called a Fermi liquid). This Fermi liquid theory, developed by Landau, for a long time seemed to give the theoretical understanding of all metallic states observed. Yet with the discovery of high-T_c cuprate superconductors amongst others, metallic states were found that are not described by this Fermi liquid theory. Thus the strange thing about these strange metals is that the electron picture is actually the wrong place to start when describing these metallic phases. There are numeral direct and indirect measurements (e.g. Refs. [1–4]) on strange metals showing that these are strongly correlated states of matter that do not have a quasiparticle description.

A striking feature of these cuprate superconductors, is that in the strange metallic regime (so above T_c) the resistivity is two orders lower than that of conventional metals. Additionally the temperature scaling of for instance the resistivity of the material behave very different from the ones observed in conventional metals. This type of behaviour is common in a larger class of strongly correlated electron materials. A famous characteristic of strange metals is the observed linear scaling of the resistivity at all temperatures Fig. 1.1. This is very different from conventional metals, where the resistivity has different power scaling and even saturates at high temperatures. Theoretically these strange metallic phases are very hard to understand

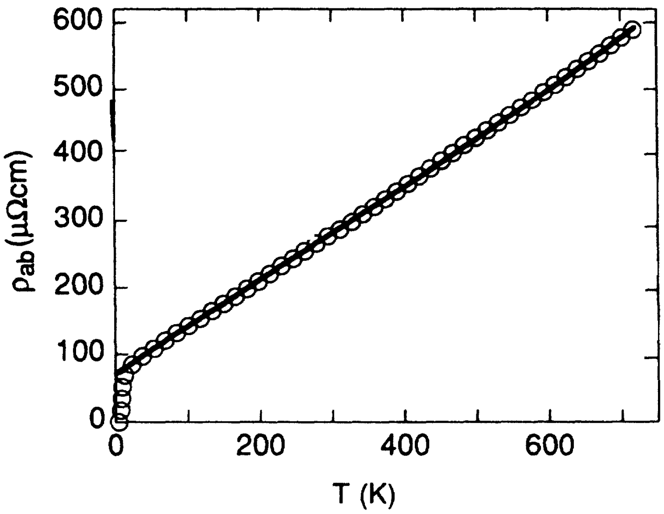

Figure 1.1: Resistivity of the cuprate strange metal $Bi_{2+x}Sr_{2-y}Cu0_{6+\delta}$ with respect to temperature. Source: Ref.[3].

(that's why it's still a current topic). Since these are highly correlated phases (or fluid like phases), they most probably cannot be treated perturbatively starting from a free particle description. So one needs a different framework in which one is able to treat these strongly correlated systems.

This is where the holographic correspondence comes in, since it gives a framework in which one is able to do calculations on these strongly interacting systems. The holographic correspondence was discovered by the string theorist Maldacena at the end of the 20th century in Ref.[5]. According to the correspondence two seemingly unrelated theories (general relativity and quantum field theory) are, under certain conditions, equivalent. The correspondence claims that one may consider a strongly coupled quantum field theory as the hologram of a weakly coupled string theory (in other words gravity). The most important part of the duality is that the difficult to compute quantum field theory is now expressed in terms of an easier to solve gravitational problem. Due to this promising characteristic a lot of attention has gone into describing strongly correlated states of matter using this holographic principle.

In this thesis a special class of gravitational theories is described, namely ones that result in hyperscaling violating gravitational solutions. These theories give rise to strongly coupled field theories in which space and time are not treated on equal footing, giving rise to nontrivial dispersion relations. Additionally these models have anomalous scaling factors in the field theory, so macroscopic quantities like entropy and free energy do not scale with their naive scaling dimension. Furthermore, it turns out that the model studied has an additional anomalous factor in the scaling of the charge of the system.

The outline of the rest of this thesis is as follows:

Chapter 2: Condensed Matter Background
In this chapter we will shortly discuss the well-established Fermi liquid theory for ordinary metals. In specific we consider the resistivity of an ordinary metal. We note that experiments performed on strange metals gives different behaviour than is predicted by Femi liquid theory. In the second part we give a short introduction to strange metals and what makes them different from conventional metals.

Chapter 3: The Holographic correspondence
This chapter will give a short introduction into the essential dictionary of the holographic correspondence. The basic concepts needed later in the thesis will briefly be reviewed. The goal of the chapter is to give the reader some intuition on how one could use holography to model certain strongly correlated field theories.

Chapter 4: Strange Metals as HV Geometries
This chapter discusses the hyperscaling violating model that will be used throughout this thesis. It turns out that this model is more involved than the standard approach (Reissner-Nördstrom) and introduces one additional gauge field and an additional dilaton field on top of the minimal Reissner-Nördstrom description. In the beginning of the chapter the explicit solution giving the model and it thermodynamic properties are discussed. Additionally we will see that these models exhibit additional scaling parameters. At the end of the chapter we discuss general properties of hyperscaling violating gravitational theories and the corresponding quantum critical field theories.

Chapter 5: Quantum Critical Dynamics
In this chapter we introduce the concept necessary to determine the dynamics of the studied hyperscaling violating model. The procedure on how one could obtain the correlation functions in the field theory from the gravitational theory is lined out here. Additionally the identification of the field expansion with field theoretical quantities such as source and vacuum expectation value are being made.

Chapter 6: Thermoelectric Transport of the EMD model
In the final chapter the transport properties of the hyperscaling violating model are discussed. In particular, the frequency dependence of the optical conductivity is determined. The obtained results seem to suggest that momentum conservation is explicitly broken. This cannot be the case since this is a Ward identity of the system, the explanation turns out to be more subtle and it is still work in progress.

Chapter 7: Discussion and Outlook
We will end with a conclusion, discussion and outlook.

Chapter 2

Condensed Matter Background

In order to know why something is strange we first need to know what is normal. In the first part of this section we give a short introduction into the well-established Fermi liquid description of ordinary metals, with a focus on the resistivity of the metals. In the second part we talk about the strange metallic phase and why it can not be described by a Fermi liquid theory. It turns out that the description of the strange metallic phases is quite hard and one thus needs unconventional frameworks to be able to describe them theoretically.

2.1 Ordinary metals and the Fermi-liquid

The following sections will give a brief introduction into the well-established description of ordinary metals, Refs. [6, 7] will be followed quite closely throughout this section. The current understanding of the usual metallic state has been initiated by Drude's work at the beginning of the twentieth century, which described the metal as an gas of weakly interacting electrons. While this was a good model to describe the conductivity of metallic states it had problems explaining the observed heat conductivity. In the classical picture every electron has to contribute $3k_b/2$ to the specific heat of a metal, this is much larger than the specific heat observed in experiments. The solution came when Pauli formulated the Pauli exclusion principle [8], initially for electrons, which was later extended to general fermions

Statement 2.1.1. *No two electrons can occupy the same quantum state.*

In the absence of interactions one finds that the ground state of a gas of free electrons is given by a filled Fermi-sea of occupied states up to a specific momentum k_F (the Fermi momentum), where the higher energy states are unoccupied. Additionally this gives rise to a surface right at the Fermi-momentum (k_F), which is called the Fermi-surface. Low energy excitation of the free electron gas are given by slightly deviating from the ground state around the Fermi-surface, these are called particle-hole excitations. These are thus given by promoting an electron from slightly below the Fermi-energy to above the Fermi-energy.
This already resolved the problem of the observed heat conductivity with respect to Drude's weakly interacting electron gas model. Since in the Fermi-gas picture only a small fraction of the electrons are contributing to the heat capacity. Which is due to the fact that only states near the Fermi surface (so within a energy gap of k_bT) are able to carry heat. The predictions made by the non-interacting free-Fermi-gas picture matched very well with the

temperature dependencies found by measurements done on metals. But it remained unclear how a non-interacting theory of free electrons could so well describe a system where obviously interactions are important, such as the coulomb interaction between the electrons.
The answer to the last question was provided by Landau's Fermi-Liquid theory (1956). The key idea behind Landau's Fermi-Liquid theory rests on the notion of adiabaticity. Consider a low energy eigenstate state of the non-interacting Fermi-gas and suppose the interactions in the system are turned on slowly. Laundau argued that in this case these eigenstates of the non-interacting system would adiabatically transform into the eigenstates of the interacting system. (So labels associated to the low energy eigenstates of the system are robust against perturbations, while the states themselves might not be.) Therefore there is a one-to-one correspondence with the low energy excitations of the Fermi-gas and the interacting Fermi-Liquid, where the particle of the Fermi-Liquid are referred to as quasiparticles. Thus despite the potentially strong interactions between the electrons, the low energy excitations near the Fermi surface still behave like weakly interacting particles and holes, which are called quasi-particles.
The dispersion relation of the quasiparticles resemble those of the free electrons, but with a modified inertial mass $m \to m^*$, which one may see as the effective mas of the quasiparticle. Furthermore it can be shown using Fermi's golden rule that these quasiparticles are long lived near the Fermi surface and thus form a proper bases.

2.1.1 Resistivity of ordinary metals

The resistivity ρ of an ordinary metal may semi-classically be expressed in terms of the mean free path l which is the average distance an electron moves before a collision

$$\rho \propto 1/l. \tag{2.1}$$

It follows that at low temperatures (temperatures lower than the Debeye temperature, $T \ll T_D$) the resistivity of the metallic state is dominated by electron-electron (Umklapp) scattering and scattering of impurities of the metal. Impurities give rise to a constant temperature dependence of the resistivity, where the electron-electron scattering gives rise to a temperature squared dependence of the resistivity

$$\rho \sim A_{imp} + B_{e,e}T^2. \tag{2.2}$$

At larger temperatures (above the Debeye temperature) the electron-phonon interactions start to dominate, this gives rise to a linear scaling of the resistivity with temperature

$$\rho \sim C_{e,ph}T. \tag{2.3}$$

It turns at that at even larger temperatures the resistivity of an ordinary metal start to saturate, this is called the Mott-Ioffe-Regel condition.

Mott-Ioffe-Regel condition

In an ordinary metal the mean free path of electrons conducting the current in the metal is typically much larger than the lattice spacing of the metal. If we heat up the material the mean free path of the electrons should decrease, since it is now more likely to scatter of

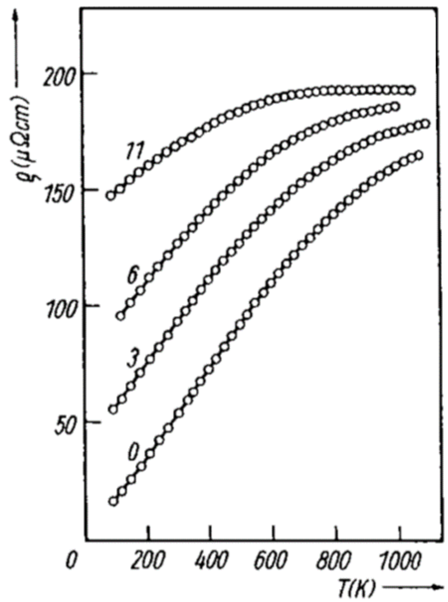

Figure 2.1: Resistivity of $Ti_{1-x}Al_x$ alloys for several percentages x of Aluminium. In this figure we see saturation of the resistivity at high temperatures. Source: Mooij, 1973 and Ref.[7].

phonons in the metal and thus the resistivity of the metal keeps increasing with temperature. The Mott-Ioffe-Regel bound states that the mean free path can't keep decreasing indefinitely, but has to saturate when the mean free path of the electrons becomes comparable to the lattice spacing of the metal. This saturation is due to the fact that a semi-classical picture does not make sense any more when the mean free path l becomes comparable to the lattice spacing d, $l \sim d$. Since at this moment the uncertainty of the wave vector $\mathbf{k}$ becomes of the size of the Brillouin zone, thus quantum mechanically one is not able to localize the electron more than the lattice spacing by the Heisenberg uncertainty principle. Since the mean free path now saturates at large temperatures, the resistivity also has to saturate, again this bound is called the Mott-Ioffe-Regel bound. An experimental example of the resistivity saturation is given in Fig. 2.1. There the resistivity of $Ti_{1-x}Al_x$ for several percentages of doping is plotted against the temperature of the metal, we can clearly see that in these metals the resistivity saturates at high temperatures.

2.2 What is a strange metal?

The strange metallic phase was first discovered as the metallic phase of high-T_c cuprate superconductors, see Fig. 2.2. These are layered materials consisting of superconducting copper oxide layers separated by spacer layers. As already discussed in the introduction, the strange metallic phase is not described by the well-established Fermi-Liquid theory. For instance a famous feature of the strange metallic phase is that the resistivity scales linearly with temperature from T_c up to as high temperatures as one can measure [2], see Fig. 1.1. This violates the quadratic temperature dependence predicted by Fermi liquid theory additionally the strange metallic phase does not satisfy the Mott-Ioffe-Regel bound [4, 7] and the resistivity just keeps

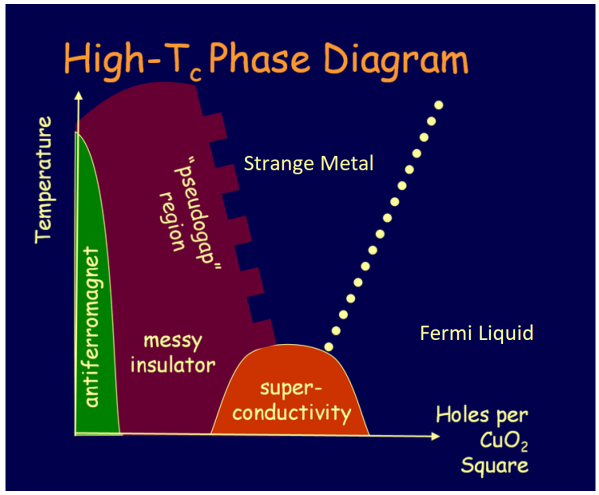

Figure 2.2: The phase diagram of a high-T_c cuprate is shown. On the horizontal axe the doping of the cuprate is given, while on the vertical axes the temperature of the cuprate is given. Source: Ref.[9].

on increasing linearly while raising the temperature of the strange metal. These observations are not consistent with a long lived quasiparticle description of the metallic phase we just mentioned. There are additional measurements for instance on the Hall resistivity, where the temperature dependence is also different from what would be expected in a quasiparticle description. This suggests that a long lived quasiparticle description of strange metals is the wrong place to start. It turns out that the electrons in a strange metallic phase are strongly interacting, due to the chemistry of in this case the copper oxides which amplify the Coulomb interactions between the electron. The result is that the electrons in a strange metal are highly correlated, which means that the conductance in strange metals has more similarities with a fluid-like picture than with the well established gas-like picture described in the previous section.
So we have just seen that the electrons in the strange metallic phase are strongly correlated and there is no quasiparticle description of these strange metallic phases. Understanding these highly correlated fluid-like phases using field theory is quite hard since the electrons are so strongly correlated it is unlike one is able to treat these metallic phases fundamentally starting from a free theory using a perturbative approach. As mentioned in the introduction we need a framework in which we are able to treat these strongly correlated states of matter. It turns out that the holographic correspondence may be of use here. Since using the correspondence we are able to treat our difficult to solve strongly coupled field theory as the hologram of an solvable gravitational theory. This is what will be explored in the next chapters.

Remark. So far we have not discussed the superconducting region of the phase diagram in Fig. 2.2, this is not because we are not interested in the superconducting regime. To understand the superconducting regime of a metal the metallic state from which it forms should must be understood. For instance our current understanding of conventional superconductivity relies on Fermi liquid theory as a starting point. This is some additional motivation for understanding the strange metallic phase better.

Chapter 3

The Holographic correspondence

As mentioned in the previous chapters, in order to treat strange metals properly we need a framework in which we are able to treat a system of strongly coupled particles (electrons in our specific case). The holographic correspondence is such a framework in which we might be able to treat these strongly interacting phases of matter, since it tells us that we may consider strongly interacting quantum field theories (which are near to impossible to describe) as the hologram of a semi-classical gravitational theories, in which calculations are much easier. A downside of using the holographic correspondence to describe strongly correlated states of matter, is that the connection with the underlying microscopic physics is not clear any more. In this chapter we will treat the basic ingredients to use the holographic correspondence as a calculational device. There is already a lot of good literature giving an introduction to the holographic correspondence e.g. Refs.[10–14], for a more extensive introduction into holography we refer to the literature which was just mentioned.

This chapter is organised in the following way, we start with a short motivation on the original correspondence proposed by Maldacena. In Section 3.1.2 we discuss the precise correspondence between gravitational theories and the strongly coupled quantum field theories. In special, we discuss a simple example of how one could use the obtained mapping to determine the correlation functions of strongly coupled field theory using a holographic calculation in gravity. The above tells us how we could determine the response functions of the quantum field theory, but does not tell us what the thermodynamic equilibrium quantities of the quantum field theory such as temperature and density correspond to in the gravitational theory. In short since we want to describe a field theory at nonzero temperature and nonzero density we need to know how to describe those field theories in a holographic context, this is done in Section 3.2. Furthermore in holography it is often not clear which operators of the quantum field theory correspond to which fields in the gravitational theory. Symmetries of the gravitational fields and the corresponding quantum field theory operators are now an important guide. In Section 3.3 we describe how local gauge symmetries of the gravitational fields should correspond global symmetries in operators of the strongly coupled quantum field theories which gives us an useful guide. We end this chapter with a summary of how one could use the holographic correspondence as a computational device, which is usually called the holographic dictionary.

3.1 The basic correspondence

The original correspondence has been proposed by Maldacena in Refs.[5, 15]. It described a duality between a type IIB superstring theory on $AdS_5 \times S^5$ and a $\mathcal{N} = 4$ super-conformal Yang Mills with gauge group $SU(N)$ in $3+1$-dimensions. The general form of the holographic correspondence relates stringy quantum gravity to a class of quantum field theories. Currently stringy quantum gravity is poorly understood and difficult to perform calculations in. However there is a special limit one could consider in the string theory, which reduces the difficult string theory into (well understood) classical general relativity. This limit in the field theory side corresponds to considering the strong coupling 't Hooft matrix large-N limit. This leads to the following statement

Statement 3.1.1. *A d+2 dimensional classical gravitational theory on AdS_{d+2} has a dual description as the large-N limit of a strongly coupled field theory in $d+1$ dimensional flat spacetime.*

So currently we have a duality between the gravitational AdS solution and the super-conformal Yang-Mills theory with four supercharges and gauge group $SU(N)$. It turns out that the above is just an example of the holographic correspondence and many more examples of dualities between gravitational theories and quantum critical field theories exist. At the moment it is only clear that the theories are related and not how they are related, the relation between the gravitational theory and the strongly coupled quantum field theory is explained in Section 3.1.2. But let us first motivate the initial correspondence a bit more.

3.1.1 Motivating the correspondence

It has just been described that gravitational theories living in AdS_{d+2} have a dual strongly coupled field theory description, which is actually a conformal field theory. It namely turns out that the $\mathcal{N} = 4$ super-Yang-Mills theory described above has an extensive symmetry group. Besides being invariant under translations, rotations and Lorentz boost it additionally is invariant under scale transformations. Actually this field theory turns out to be invariant under the full conformal group.

A strong argument in favour of Maldacena's conjecture may be given from basic properties of CFTs and AdS spaces. Namely if two theories are equal it is expected that their symmetries match. It turns out that the isometry group $SO(2, d+1)$ of AdS_{d+2} is exactly matching with the conformal symmetry group $SO(2, d+1)$ of a conformal field theory living in $d+1$ spacetime dimensions. This relation gets even more precise if you move to the boundary of AdS space where the isometries of the spacetime reduce to the conformal transformations [11].

A strange feature of the correspondence stated above is that it conjectures a duality between two theories living in different spacetime dimensions. To get a feeling for this apparent mismatch between the spacetime dimension, one may consider the famous Bekenstein-Hawking area law [16, 17] for the entropy of a black-hole. The statement is that in semi-classical gravity black-holes are thermal objects and have an entropy proportional to the area of the black-hole

$$S_{BH} = \frac{c^3 A}{4G\hbar}.$$

In short the above formula says that the maximum entropy in a spacetime scales with the area of the boundary and not with the volume of spacetime. This entropy is much smaller

then the entropy of a quantum field theory which scales with the volume of the system. One might argue that the entropy of spacetime is related to the entropy of a quantum field theory living in one dimension less.
An interpretation of the additional dimension appears when we deform away form pure AdS and the pure CFT, at that point the radial direction in the gravitational theory is related to the energy scale of the field theory.

3.1.2 The GKPW formula

Soon after the first paper by Maldacena the precise dictionary between properties of the field theory and the dual gravitational theory was worked out by Gubser, Klebanov, Polyakov [18] and independently Witten [19]. Thus it is now called the GKPW rule. In essence the GKPW rule relates the generating functional of the field theory to the partition function of the gravitational theory. This statement will be made a bit more precise later. Let us first take a step back, in a field theory all the correlation functions can be obtained from the generating functional

$$\mathcal{Z}_{QFT}(\{h_i(x)\}) \equiv \left\langle e^{i\sum_i \int dx\, h_i(x)\mathcal{O}_i(x)} \right\rangle_{QFT}, \tag{3.1}$$

where in the above $\mathcal{O}_i(x)$ are the operators of the field theory and $h_i(x)$ are the sources of the generating functional. One can extract all n-point correlation functions by taking functional derivatives of generating functional (3.1) with respect to the sources $h_i(x)$. For a gravitational theory it is a bit more involved to construct the right partition function. In the case where the spacetime has a boundary, observables of the theory can be defined on the boundary of spacetime. One may for instance consider Dirichlet boundary conditions (boundary value is fixed) and thus construct a partition function of the gravitational theory as a function of the boundary values $\{h_i(x)\}$ of the bulk fields $\{\phi_i\}$

$$\mathcal{Z}_{grav.}(\{h_i(x)\}) = \int^{\phi_i \to h_i} (\Pi_i \mathcal{D}\phi_i)\, e^{iS_{grav.}[\{\phi_i\}]}. \tag{3.2}$$

The above is already written down in a very suggestive form. The GKPW rule namely states

$$\mathcal{Z}_{grav.}(\{h_i(x)\}) = \mathcal{Z}_{QFT}(\{h_i(x)\}). \tag{3.3}$$

An essential part of the dictionary is that there is a one-to-one correspondence between operators in the field theory and fields in the gravitational theory. Thus to every operator $\mathcal{O}_i(x)$ there is an associated source $h_i(x)$ which is the boundary value of a gravitational bulk field.
In order to see which operator corresponds to which field one typically looks at the symmetries of the fields and the operators, since there is no general recipe at hand. So for instance a scalar operator in the field theory should correspond to a scalar field in the gravitational bulk.

Statement 3.1.2. *The GKPW rule, identifies the generating functional of the field theory with sources $\{h_i\}$ with the gravitational bulk partition function where the asymptotic boundary values of the fields ϕ_i corresponding to field theory operators $\mathcal{O}_i$ are given by the sources $\{h_i(x)\}$*

$$\mathcal{Z}_{QFT}(\{h_i(x)\}) = \int^{\phi_i \to h_i} (\Pi_i \mathcal{D}\phi_i)\, e^{iS_{grav.}[\{\phi_i\}]}. \tag{3.4}$$

In the large-N limit this reduces to the on-shell action of classical gravity

$$\mathcal{Z}_{QFT}(\{h_i(x)\}) \leftrightarrow e^{iS_{grav.on-shell}[\{\phi_i\}]}. \tag{3.5}$$

Remark. While the dictionary derived by Gubser, Klebanov, Polyakov and Witten [18, 19] was building on the specific examples from Maldecena, it was argued that the AdS/CFT duality might be formulated on general holographic grounds and does not need explicit top-down constructions from string theory. So above the "universal dictionary" is formulated. Since one does not always need the string-like origin, one may engineer the gravitational theory in a "bottom up" approach, to study the field theories of interest. This is what will be done throughout this thesis.

3.1.3 Example: scalar field in AdS spacetime

To make the GKPW formula more explicit let us give an example, next we determine the vacuum expectation value and response function for a scalar field in AdS spacetime by closely following Ref. [10, Chapter 1.5].
To discuss the scalar field in AdS spacetime we first need to discus the the general bulk action that has pure AdS spacetime as a classical solution

$$\mathcal{S} = -\frac{1}{8\pi G}\int d^{d+2}x\,\sqrt{-g}\left(R + \frac{d(d+1)}{L^2}\right). \tag{3.6}$$

The first term in the above action is called the Einstein-Hilbert term with R the Ricci scalar. The second gives a negative cosmological constant (or vacuum energy) characterised by the length scale L. The classical equations of motion for the above action are given by[1]

$$R_{\mu\nu} = -\frac{d+1}{L^2}g_{\mu\nu}. \tag{3.7}$$

A solution to the above equations of motion is the AdS spacetime

$$ds^2 = L^2\left(\frac{-dt^2 + dz^2 + d\vec{x}^2}{z^2}\right). \tag{3.8}$$

In the above $z = 1/r$ is the inverse radial coordinate of the gravitational bulk, so the critical boundary is now located at $z = 0$, while the deep interior of the bulk is located at $z = \infty$. Additionally t and $\vec{x}$ give the time and spatial directional of both the quantum critical field theory and the gravitational bulk. It is a known fact that AdS spacetime is a maximally symmetric spacetime with isometry group $SO(2, d+1)$ [20]. A striking feature of the isometry group is that the isometries of AdS spacetime turn exactly into the conformal transformations (also $SO(2, d+1)$) on the boundary of bulk $z = 0$, this together with the recently discussed GKPW formula suggest that we may consider the quantum critical field theory as living on the boundary of our maximally symmetric spacetime. Let's now discuss this for a scalar field

[1] By taking the trace of the Einstein equations of motion one is able to obtain an expression for the Ricci scalar. This may be used to simplify the Einstein equations of motion by eliminating the Ricci scalar from the equations of motion.

Consider now an additional scalar field ϕ in the bulk[2]

$$\mathcal{S}_{scalar} = -\sqrt{-g}\left(\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2\right), \tag{3.9}$$

where ϕ is thus dual to some scalar operator in the dual quantum critical field theory. Before we can see the effect of the source h of this operator in terms of the scalar field ϕ, we first need to solve the classical bulk equations of motion

$$\nabla^2\phi - m^2\phi^2 = \partial^2\phi - m^2\phi^2 = 0, \tag{3.10}$$

with boundary condition $\phi(x,z) \to h(x)$ on the conformal boundary $z \to 0$. After this we have to evaluate the above solution in the gravitational action (3.9) to obtain the generating functional of the quantum critical field theory via the GKPW rule (3.5). Eq. (3.10) gives rise to a second order differential equation (wave equation) in the AdS_{d+2} background spacetime, the equation of motion may be solved by considering a Fourier transform with respect to only the spacetime coordinates

$$\phi(x,z) = \int \frac{d^dk}{(2\pi)^d}\frac{d\omega}{2\pi}\phi(k,\omega,z)e^{-i\omega t + ik\cdot x}. \tag{3.11}$$

In these coordinates the equation of motion becomes

$$\left(z^2\partial_z^2 - zd\partial_z + z^2\left(\omega^2 - k^2 - \frac{(mL)^2}{z^2}\right)\right)\phi(k,\omega,z) = 0. \tag{3.12}$$

In order to understand the near boundary asymptotics of the scalar field, we consider the above equation near the conformal boundary $z \to 0$, which simplifies to

$$\left(z^2\partial_z^2 - zd\partial_z - (mL)^2\right)\phi(k,\omega,z) = 0. \tag{3.13}$$

By considering an asymptotic series expansion near the conformal boundary $z \to 0$ we see that the solution takes the form

$$\phi(\omega,k,z\to 0) = \phi_0(\omega,k)\left(\frac{z}{L}\right)^{\Delta_-} + \cdots + \phi_1(\omega,k)\left(\frac{z}{L}\right)^{\Delta_+} + \ldots, \tag{3.14}$$

with

$$\Delta_\pm(\Delta_\pm - d - 1) = (mL)^2. \tag{3.15}$$

So there are two integration parameters in expansion (3.14), namely ϕ_0 and ϕ_1. The meaning of ϕ_0 is what was previously called the boundary value h of ϕ and is thus interpreted as the source of the scalar operator $\mathcal{O}$ in conformal field theory. The meaning of ϕ_1 will become clear in a moment.

Remark. Note that to extract the source h from expansion (3.14) we must get rid of the factors in r. These factors have physical implications as we will show now. We have seen that AdS_{d+2} has isometry group $SO(2,d+1)$, one of these isometries gives that AdS spacetime

[2] Note that in principle we should also have included the metric fluctuations of action (3.6). But since this is quite involved and not serving the purpose of this example and consider the scalar field in what is known as the probe limit (no back-reaction of the metric).

is invariant under rescaling $\{t, x, z\} \to \lambda\{t, x, z\}$. Knowing that the scalar field ϕ should also be invariant under this rescaling we find that the source ϕ_0 has to scale as $\phi_0 = h \to \lambda^{-\Delta_-} h$ in order for the scalar field (3.14) to be invariant. By recalling that h couples to the operator $\mathcal{O}$ on the conformal boundary and noticing that the boundary theory should also be scale invariant we deduce that under rescaling

$$\mathcal{O}(x) \to \lambda^{-\Delta_+} \mathcal{O}(\lambda^{-1} x). \tag{3.16}$$

This can easily be seen from the quantum critical boundary action (with $x^\mu \to \lambda \tilde{x}$)

$$\int d^{d+1}x \; h(x)\mathcal{O}(x) = \int \lambda^{d+1} d^{d+1}\tilde{x} \; h(\lambda\tilde{x})\mathcal{O}(\lambda\tilde{x}) \to \int \lambda^{d+1-\Delta_-} d^{d+1}\tilde{x} \; h(\tilde{x})\mathcal{O}(\lambda\tilde{x}), \tag{3.17}$$

using that $\Delta_+ = d + 1 - \Delta_-$ and the fact that the conformal boundary action should be scale invariant. This is an important relation since it relates bulk properties to conformal scaling dimensions of the operators of the quantum field theories. In this specific case the mass of the scalar field determines the conformal scaling dimension of the scalar operator.

Let us next focus on the vacuum expectation value and the Green's function of the operator $\mathcal{O}$ corresponding to the scalar field ϕ. From the asymptotic expansion described above and the GKPW formula's Eqs. (3.3) and (3.5) we are able to give a formula for the vacuum expectation value of the operator $\mathcal{O}$

$$\langle \mathcal{O}(x) \rangle = \frac{1}{\mathcal{Z}_{QFT}} \frac{\delta \mathcal{Z}_{QFT}}{\delta h(x)} = \frac{\delta \mathcal{S}_{Grav.}(\phi \to h)}{\delta h(x)}. \tag{3.18}$$

The scalar action (3.9) reduces to a boundary action when evaluated in the classical solution of the field ϕ. Since the volume of the conformal boundary ($z \to 0$) is diverging, we need to take a cutoff at $z = \epsilon$, the boundary action then take the form

$$\begin{aligned} \mathcal{S}_{scalar}[\phi_c] &= -\frac{1}{2} \int_{z=\epsilon} d^{d+1}x \; \sqrt{-\gamma} n^\mu \phi \nabla_\mu \phi \\ &= -\frac{1}{2} \int_{z=\epsilon} d^{d+1}x \left(\frac{z}{L}\right)^d \left(\phi_0(\omega,k)\left(\frac{z}{L}\right)^{\Delta_-} + \cdots + \phi_1(\omega,k)\left(\frac{z}{L}\right)^{\Delta_+} + \dots\right) \\ &\quad \partial_z\left(\phi_0(\omega,k)\left(\frac{z}{L}\right)^{\Delta_-} + \cdots + \phi_1(\omega,k)\left(\frac{z}{L}\right)^{\Delta_+} + \dots\right) \\ &= -\frac{1}{2} \int_{z=\epsilon} d^{d+1}x \; \frac{1}{L}\left(\frac{z}{L}\right)^d \left(\Delta_-\left(\frac{z}{L}\right)^{2\Delta_-} \phi_0(x)^2 + (d+1)\phi_0(x)\phi_1(x) + \dots\right) \end{aligned} \tag{3.19}$$

In the above γ is given by the boundary metric and n^μ is a radially outward pointing vector (i.e. $n^z = \sqrt{g^{zz}}$). By taking the limit $\epsilon \to 0$, we see that the first term in the above action diverges while the second term is finite. The divergent terms are uninteresting to us and should be dealt with more carefully by a renormalization procedure. We will ignore the renormalization and just continue with the finite term here. Additionally note that we are working in the linear response regime, this means $\phi_1(x) = (\delta\phi_1(x)/\delta\phi_0(y))\phi_0(y)$. Using Eqs. (3.18) and (3.19) we thus find

$$\langle \mathcal{O}(x) \rangle \propto \phi_1(x). \tag{3.20}$$

To obtain the correct constant of proportionality we need to consider the renormalised boundary action. We thus see that the subleading integration constant acts as the vacuum expectation value of the scalar operator $\mathcal{O}$, this turns out to be a quite general statement.

Statement 3.1.3. *The field theory source of an operator is given by the leading order integration constant (in this case ϕ_0) in the near boundary expansion of the corresponding bulk field. Additionally the vacuum expectation value of a quantum field theory operator is given by the subleading integration constant of the near boundary expansion (in this case ϕ_1) of the corresponding bulk field.*

Remark. While the above statement turns out to be quite general one should always check if the conformal scaling dimensions of the source and VEV match to the conformal scaling dimensions in the quantum critical field theory. One may already think of several source VEV relations, for instance chemical potential is the source with (charge) density as the VEV. One could also consider is a temperature gradient as source and a heat current as the VEV, or an electrical field as the source and the VEV would be an electrical current.

We could additionally extract the Green's function of the scalar field using a similar procedure. First note that the general n-point connected correlation functions, sourced by the fields $\{\phi_i(x)\}$, are given by

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\rangle = \prod_i^n \frac{\delta}{\delta\phi_n(x_n)} \log \mathcal{Z}_{QFT}\Bigg|_{\phi=0} = \prod_i^n \frac{\delta}{\delta\phi_n(x_n)} \mathcal{S}_{Grav.}\Bigg|_{\phi=0}. \tag{3.21}$$

Thus the Green's function of an scalar operator in the boundary field theory is given by

$$G^R_{\mathcal{O}\mathcal{O}}(x,x') = \langle \mathcal{O}(x)\mathcal{O}(x')\rangle = \frac{\delta^2 \log \mathcal{Z}_{QFT}}{\delta\phi(x)\delta\phi(x')} = \frac{\delta^2 \mathcal{S}_{Grav.}}{\delta\phi(x)\delta\phi(x')}. \tag{3.22}$$

This together with the linear response argument made earlier gives

$$G^R_{\mathcal{O}\mathcal{O}} \propto \frac{\phi_1}{\phi_0}. \tag{3.23}$$

Thus by determining asymptotic near boundary integration constant of the bulk gravitational fields we were able to extract all the correlation functions of the corresponding operators in the strongly coupled field theory.

3.2 Thermodynamics

To describe strongly correlated materials in a realistic setting one needs to be able to describe strongly coupled quantum field theories at nonzero temperatures and nonzero densities. In this section it is discussed what nonzero temperature and nonzero density of the field theory correspond to in the gravitational bulk theory. In other words we consider what should be added to the gravitational theory in order for the corresponding quantum field theory to be at nonzero temperature and nonzero density. We start each part with a short recap of how one could describe nonzero temperature and nonzero density in quantum field theories and then discuss what should be included in the gravitational theory such that the corresponding field theory is at nonzero temperature and nonzero density.

3.2.1 Holography at nonzero temperature

Nonzero temperature in field theories

In field theory temperature and Euclidean time are strongly related. What it boils down to is, studying a field theory at nonzero temperature is equivalent to describing the same field theory in imaginary time (Euclidean time) where the fields need to be periodic or anti-periodic depending on whether we are describing bosons or fermions [21, Chapter. 7]

$$\phi(0,x) = \pm\phi(\beta,x). \tag{3.24}$$

The above periodicity follows from the euclidean time slicing procedure by which the partition function of the field theory is defined. Let's start by noting

$$e^{-\beta\mathcal{H}} = e^{-i\int_0^{-i\beta}\mathcal{H}dt} = e^{-\int_0^{\beta}\mathcal{H}d\tau}, \tag{3.25}$$

so we may think of $e^{-\beta\mathcal{H}}$ as an evolution operator in imaginary time. Using the slicing procedure outlined in Refs.[21, 22] we find that the partition function in terms of the path integral is given by

$$\mathcal{Z} = \int\mathcal{D}\phi\ \langle\phi|\, e^{-\beta\mathcal{H}}\,|\phi\rangle = \int\mathcal{D}\phi\ e^{-\int_0^{\beta}d\tau\mathcal{L}(\tau,\beta)}, \tag{3.26}$$

where $\mathcal{L}$ is the Lagrangian density of the system and the fields ϕ have to satisfy condition Eq. (3.24).

Remark. The periodicity may quickly be seen from the fact that a partition function is given as the trace over the thermal weight factor. Interpreting β as imaginary time, from the slicing procedure we notice that the field ϕ has to return to $\pm$ it's initial value at Euclidean time β.

Nonzero temperature in holography

Next we need to add the concept of temperature in our gravitational theory. As a quick note, introducing temperature adds a scale into the theory and thus breaks the scaling invariance of the system. Since the scaling invariance is recovered at energies well above the characteristic scale of the (in this case temperature) deformation we expect the gravitational theory to be invariant towards it's scale invariant boundary, which correspond to the UV of the field theory. Using that black-holes are thermal objects, we may consider adding a black-hole to our gravitational theory in order to obtain a nonzero temperature in our field theory. It turns out that the temperature and the entropy of the field theory are exactly given by the temperature and entropy of the black-hole, see e.g. Refs. [11, 12]. In what follows next we outline how to obtain the temperature of the field theory from a generic black-hole solution. Consider a general static black-hole metric

$$ds^2 = -g_{tt}(r)dt^2 + \frac{dr^2}{g^{rr}(r)} + g_{xx}(r)d\mathbf{x}^2, \tag{3.27}$$

where g_{tt} and g^{rr} both vanish at the black-hole horizon. Upon performing to Euclidean coordinates by means of a Wick rotation $\tau = it$, we obtain

$$ds^2 = g_{tt}(r)d\tau^2 + \frac{dr^2}{g^{rr}(r)} + g_{xx}(r)d\mathbf{x}^2. \tag{3.28}$$

we make the assumption that the properties of the black-hole are reflected in the geometry near the horizon where g_{tt} and g^{rr} are vanishing. Next we focus on the region near the horizon. Near the horizon r_+ we may expand g_{tt} and g^{rr}, expanding the metric w.r.t r around the black-hole horizon r_+ gives

$$ds_E^2 = g_{tt}'(r_+)(r-r_+)d\tau^2 + \frac{dr^2}{g^{rr\prime}(r_+)(r-r_+)} + g_{xx}(r_+)d\mathbf{x}^2 + \dots \quad . \tag{3.29}$$

Lets consider the following coordinate transformations

$$R = 2\sqrt{r-r_+}/\sqrt{g^{rr\prime}(r_+)}, \tag{3.30}$$

$$\theta = \frac{1}{2}\sqrt{g_{tt}'(r_+)g^{rr\prime}(r_+)}d\tau. \tag{3.31}$$

The near horizon limit may now be written as

$$ds_E^2 = R^2 d\theta^2 + dR^2 + g_{xx}(r_+)d\mathbf{x}^2 + \dots \quad . \tag{3.32}$$

As $R \to 0$ the prefactor in front of $d\theta$ vanishes, this means Euclidean time shrinks to a point. Since the horizon is not a special point, this point is not allowed to be singular. To obtain smoothness at the point $R = 0$ we need to insist the point $R = 0$ may be seen as the origin of a polar coordinate system. We thus see θ needs a period of 2π

$$\theta \sim \theta + 2\pi,$$

this implies τ has a periodicity of

$$\tau \sim \tau + 4\pi/\sqrt{g_{tt}'(r_+)g^{rr\prime}(r_+)}, \tag{3.33}$$

Using our definition of temperature as the periodicity of the fields w.r.t. Euclidean time we find

$$T = \left.\frac{\sqrt{g_{tt}'(r_+)g^{rr\prime}(r_+)}}{4\pi}\right|_{r_+} \tag{3.34}$$

as the temperature of our field theory.

Statement 3.2.1. *Strongly coupled field theories at nonzero temperature are dual to black-hole solutions in the dual gravitational theory. The temperature and the entropy of the black are exactly the temperature and entropy of the dual field theory.*

3.2.2 Holography at nonzero density

Finite density in a field theory

In a quantum field theory particle conservation is described by a global $U(1)$ symmetry. Where the $U(1)$ symmetry is associated with a conserved current J^μ via Noether's theorem, such that the charge of the current is given by $N = \langle Q \rangle \equiv \langle J^0 \rangle$. In statistical physics the grand canonical ensemble describes a thermodynamic system at fixed volume V, temperature T and chemical potential μ. So a system that can exchange particles and heat with a reservoir. The partition function in the grand canonical ensemble is defined as

$$\mathcal{Z} = \mathrm{Tr}\left[e^{-\beta(\mathcal{H}-\mu\mathcal{N})}\right]. \tag{3.35}$$

As derived in Ref.[21, Chapter. 7] in field theory this amounts to modifying the action by the following term

$$\int d\mathbf{x}\, \mu\, \mathcal{N}(\mathbf{x}). \tag{3.36}$$

Finite density in holography

So what does a global $U(1)$ symmetry in the field theory corresponds to in terms of the gravitational theory? In other words what is the dual description of a global $U(1)$ symmetry? As we will see in Section 3.3 the holographic correspondence states

Statement 3.2.2. *Gauge symmetries in the gravitational theory correspond to global symmetries in the dual field theory.*

Thus to describe a global $U(1)$ symmetry in the field theory we therefore need to add a Maxwell field to our spacetime. The minimal gravitational model is thus Einstein-Maxwell theory

$$\mathcal{S} = -\frac{1}{16\pi G}\int d^{d+2}x\sqrt{-g}\left[\left(R + \frac{d(d+1)}{L^2}\right) - \frac{1}{4e^2}F^2\right]. \tag{3.37}$$

Our main purpose is to describe field theories at nonzero temperature and nonzero density. Regarding nonzero temperature, we have seen in the last section that this is achieved by studying a black-hole solution in gravitational theory. Regarding nonzero density we have learned that a gauge field sources a conserved current density in the dual field theory J^μ. If we want nonzero density we need the density (charge) which is given by the time component of the conserved current $\langle\rho\rangle = J^0$ to be nonzero. According to the holographic dictionary the value of the fields near the critical boundary sources dual operators. Remembering that the source of a density ρ is the chemical potential μ, we thus need to impose

$$\lim_{r\to\infty} A_t = \mu, \tag{3.38}$$

to have a finite density in the dual field theory.

3.3 Bulk gauge symmetries are global symmetries of the dual QFT

In Section 3.1.2 it was described how one could obtain the correlation functions of the strongly interacting quantum critical field theory from the corresponding gravitational theory. For this to be useful one must know which fields correspond to which operators. As already discussed in Section 3.1.2 symmetries of the fields and operators are an important guide. A special class of symmetry will be discussed here, namely gauge symmetries. Examples of gauge fields

include Maxwell fields A_μ and metric fields $g_{\mu\nu}$. The claim is that gauge fields couple to conserved currents, thus global symmetries. Let us illustrate this point for both the Maxwell gauge field A_2 and the metric $g_{\mu\nu}$.
Consider a gauge field A_μ that is coupled in the boundary to a current J^μ. the gauge field A_μ transforms to $A_\mu + \nabla_\mu \Lambda$ under a gauge transformation, with Λ a scalar function nonzero on the boundary and ∇_μ is the covariant connection for the metric tensor $\gamma_{\mu\nu}$. Invariance of the boundary action implies

$$\begin{aligned} \int_{\partial M_r} d^{n-1}x\sqrt{-\gamma}\,\{A_\nu J^\nu\} &= \int_{\partial M_r} d^{n-1}x\sqrt{-\gamma}\,\{(A_\nu + \nabla_\nu \Lambda)J^\nu\} \\ = \int_{\partial M_r} d^{n-1}x\sqrt{-\gamma}\,\{A_\nu J^\nu - \Lambda \nabla_\nu J^\nu\}\,. & \end{aligned} \tag{3.39}$$

Invariance of the bulk gauge requires $\nabla_\nu J^\nu = 0$, thus J^μ is a conserved current due to a $U(1)$ global symmetry. One may see that the above procedure is general and validates the statement that gauge fields couple to conserved conserved current of a global symmetry.
Another example of the local global symmetry correspondence is to consider local translations of the metric and the Mawell gauge field which are invariant under diffeomorphisms

$$\begin{aligned} A_\mu &\to A_\mu - \mathcal{L}_\xi A_\mu = A_\mu - \xi^\nu \nabla_\nu A_\mu - (\nabla_\mu \xi^\nu) A_\nu, \\ \gamma_{\mu\nu} &\to \gamma_{\mu\nu} - \mathcal{L}_\xi g_{\mu\nu} = \gamma_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu. \end{aligned} \tag{3.40}$$

For a translation in the ρ-direction one needs to consider $\xi^\nu = \delta^\nu_\rho$. Under such a local translation the boundary metric thus changes as

$$\begin{aligned} \mathcal{S}_r = &\int_{\partial M_r} \sqrt{-\gamma}\left\{\frac{1}{2}T^{ab}\gamma_{ab} + J^\mu A_\mu\right\} \\ \to &\int_{\partial M_r} \sqrt{-\gamma}\Bigg\{\frac{1}{2}T^{ab}\left(\gamma_{ab} - \nabla_a \xi_b - \nabla_b \xi_a\right) \\ &+ J^\mu \left(A_\mu - \xi^\nu \nabla_\nu A_\mu - (\nabla_\mu \xi^\nu) A_\nu\right)\Bigg\} \\ = &\,\mathcal{S}_r + \int_{\partial M_r} \sqrt{-\gamma}\Big\{\xi^\nu \nabla_\mu T^\mu_\nu + \xi^\nu F_{\mu\nu} J^\mu\Big\}. \end{aligned} \tag{3.41}$$

Thus invariance under local translations implies

$$\nabla_\mu T^{\mu\nu} + F^{\mu\nu} J_\mu = 0. \tag{3.42}$$

This is exactly the conservation of energy-momenta for charged matter in a field theory.

Statement 3.3.1. *Bulk gauge fields (e.g. A_μ, $g_{\mu\nu}$) couple to conserved currents of global symmetries in the quantum critical quantum field theory (e.g. $J^\mu, T^{\mu\nu}$).*

In practise matching operators and fields beyond their symmetries is often not possible. But it mostly is not necessary to know the precise correspondence, since the gravitational bulk theory is a self-contained description of a strongly quantum critical field theory. A downside in approaching holography in this way is that the microscopic description of the dual field theory is lost.

3.4 Summary: the dictionary

In the previous sections the necessary tools to use the holographic correspondence as a computational device were discussed. The rules lined out above form a dictionary how to translate from one theory to the other. This goes by the name *holographic dictionary*.

Boundary: field theory	**Bulk: gravitational theory**
field theory partition function	gravitational partition function
scalar operator $\mathcal{O}_\phi$	scalar field ϕ
energy-momentum tensor $T^{\mu\nu}$	metric field $g_{\mu\nu}$
global conserved current J^μ	Maxwell field A_μ
global spacetime isometry	local isometry
global symmetry	local gauge symmetry
source of operator	leading integration constant
VEV of operator	subleading integration constant
nonzero temperature	Hawking temperature of black-hole
entropy	Bekenstein-Hawking entropy of black-hole
nonzero density/ chemical potential	boundary value of Maxwell field A_t

Together with the GKPW formula discussed in Section 3.1.2 this gives us the tools to use holography as a computational device for strongly coupled field theories.

Chapter 4

Strange Metals as Hyperscaling Violating Geometries

In Chapter 1 we mentioned that strange metals are characterised by linear their scaling of the resistivity w.r.t. the temperature for nearly all temperatures. This this scaling is alien to the well established Fermi-Liquid theory used to describe ordinary metals. There are already two things quite different in the strange metallic phase from the Fermi-Liquid phase, in a Fermi-Liquid the resistivity has to scale quadratically with temperature for low temperatures and at very large temperatures the resistivity should saturate at the Mott-Ioffe-Regel boundary. While in the strange metallic phase the resistivity scales linearly with temperature at nearly all temperatures, hence it does not satisfy the Mott-Ioffe-Regel boundary and additionally does not scale quadraticly with temperature for low temperatures. It has already been argued in earlier sections that in strange metals the electrons are strongly correlated and the metallic phase is thus not well-described by quasiparticles any more, but more by something resembling fluid. As already mentioned in the previous chapter the holographic correspondence could be of used to describe strongly correlated states of matter, in special, strange metallic states of matter. Holography namely gives a framework to describe strongly coupled quantum critical field theories, at the expensive of not knowing the underlying microscopic physics any more. In order to describe the strange metallic phase in a holographic context, by the dictionary given in last section, the minimal model has to contain the a Maxwell field and a gravitational field. The Maxwell field is needed to give a finite density to the boundary quantum critical field theory (which is also conducting the current) and one always need a gravitational field, which by the holographic dictionary corresponds to the energy-momentum tensor of the quantum field theory. The minimal action to describe a strange metal is therefore

$$\mathcal{S} = \int_{\mathcal{M}} d^{d+2}x\sqrt{-g}\left(R - d(d+1) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right).$$

In order to describe the field theory at nonzero temperature one must consider the black-hole solution of the above minimal action. The black-hole solution of the above action is well known and is called the Reissner-Nordström black-hole solution. It turns out that this minimal model does not capture all the features of the strange metallic phase yet. For instance it does not reproduce the observed linear resistivity for all temperatures, another problem with the model is that is has nonzero entropy at zero temperature, which makes the solution unstable at low temperatures.

In this chapter we discuss a model which arguably does not have these problems, at the expense of complicating the model, namely we consider an Einstein-Maxwell-dilaton action to describe the gravitational bulk. The action under consideration contains a metric field, which by the holographic dictionary corresponds to the energy-momentum tensor. Additionally the action has two Maxwell fields, where one of them corresponds to a (charge) current in the quantum field theory and the other one by the dictionary should also couple to a conserved current, but the physical interpretation is quite unclear. Lastly the model contains dilaton field, which together with the second gauge field gives rise to a hyperscaling-violating geometry. We will see that observables dual to hyperscaling-violating backgrounds have two additional critical exponents z and θ. These hyperscaling-violating backgrounds are important in describing compressible states of matter which do not have a quasiparticle description.
As we will see, for certain values of these critical exponents resistivity scales linearly with temperature, which as discussed before is an important feature of strange metallic behaviour. Additionally one is able to consider limits of the critical exponents in which the entropy of the corresponding quantum field theory is vanishing at zero temperature.
This chapter is organized in the following way, we start by giving a short general introduction into hyperscaling-violating geometries and how one could interpret the critical exponents z and θ. Additionally the scaling of the entropy and resistivity as a function of temperature is discussed for general hyperscaling violating geometries. In Section 4.2 we discuss a hyperscaling-violating black-hole solution to the studied Einstein-Maxwell-dilaton action, and discuss the thermodynamic equilibrium properties of the corresponding quantum field theory by using the dictionary discussed in last section. Lastly in Section 4.3 we discuss some general anomalous scaling properties of several physical observables that arise in the dual field theory of these Einstein-Maxwell-dilaton theories. Useful references besides the ones cited in the sections are Refs.[10, 11, 22, 23].

4.1 Hyperscaling-violation

Before we start discussing the Einstein-Maxwell-dilaton bulk action, let us first introduce hyperscaling violating geometries in a more general setting. Like the name suggest, we are considering geometries that do not satisfy hyperscaling, we will see shortly what this means. To start hyperscaling-violating metrics give rise to a more general class of scaling metrics than the usual AdS-metric (3.8) and has a more general scaling symmetry $\{t, \vec{x}\} \to \{\lambda^z t, \lambda \vec{x}\}$. The geometry takes the form

$$ds^2 = \left(\frac{r}{R}\right)^{-2\theta/d} \left(-r^{2z}dt^2 + \frac{dr^2}{r^2} + r^2 d\vec{x}^2\right). \tag{4.1}$$

Here R is a constant of integration and should not be mistaken for the Ricci scalar which enters in the gravitational action, z and θ correspond to the dynamical critical exponent and the hyperscaling-violating exponent respectively, of which the meaning will be explained shortly. These metrics are called hyperscaling-violating metrics, since they violate the hyperscaling for nonzero θ, which basically means that macroscopic observables do not scale with their naive scaling dimension any more. Additionally for $z \neq 1$ one finds that the symmetry group only includes the scaling symmetry, spacetime translations and rotations, this is a bit less extensive than the conformal group which follows form AdS spacetime. The metric in Eq. (4.1) does not appear to be scale invariant under Lifshitz rescaling

$$\{t, \mathbf{x}, r\} \to \{\lambda^z t, \lambda x, r/\lambda\}, \tag{4.2}$$

since the metric rescales as $ds^2 \to \lambda^{2\theta/d} ds^2$ near the scale invariant boundary. This would mean that the metric (4.1) does not remain invariant under the Lifshitz scale transform (4.2), but remains conformally invariant. The scaling $\{t, \mathbf{x}, r\} \to \{\lambda^z t, \lambda x, r/\lambda\}$ must be combined with a scaling of parameters in the solution that leave the field theory invariant, so under the above scale transformation we additionally need the integration constant R to also scale $R \to \lambda R$. This transformation of the metric amounts to the fact that operators in the dual field theory acquire an anomalous dimension, since the boundary action obtains factors of R which also scales with λ. Unlike AdS spacetime hyperscaling-violating geometries turn out not to be a solution to pure gravity. Thus we need a slightly more complicated model to describe them, Einstein-Maxwell-dilaton models are one of them. We look at such a model in Section 4.2. Let us now go into a bit more detail on the critical exponents z and θ.

4.1.1 Hyperscaling-violation parameters z and θ and the strange metallic phase

We have seen that hyperscaling-violating metrics have two critical exponents z and θ corresponding to the dynamical critical exponent and the hyperscaling violating exponent respectively. The dynamical exponent or Lifshitz exponent z gives that time scales differently than space. Effectively speaking for large z low-energy excitations are present at a large range of momenta (for finite z, we have $\omega \sim k^z$). As we mentioned before
The hyperscaling violation factor θ gives an anomalous scaling dimension to the free energy (and entropy density) of the critical boundary theory with respect to it's naive engineering dimension. We will see in the next sections that that the free energy density f and the entropy density have the following scaling dimensions

$$[f] = [s] = d - \theta,$$

where we would naive expect this to be the number of spatial dimensions of the quantum field theory. Thus one may think of θ as changing the effective dimension the theory lives in. This scaling property is not present in classical matter and hint's at the fact that strange metals, if properly described by hyperscaling-violating models, are true quantum objects, possibly controlled by long range quantum entanglement.

Example 4.1.1. A conventional example of a theory with a nonzero hyperscaling-violation exponent θ, would be the low temperature thermodynamics of the Fermi-Liquid. To capture the low temperature thermodynamics of a Fermi-Liquid in a hyperscaling-violating geometry one needs

$$z = 1, \quad \theta = d - 1.$$

Where $z = 1$ follows from the linear dispersion relation near the Fermi-surface and θ is given by the dimension of the Fermi-surface which is motivated by the effective dimensionality $d - \theta = 1$ of the chiral fermions near the Fermi-surface [10, Section 4.2].

Having discussed the critical exponent z and θ we now discuss it's implications on for instance the thermodynamic of the corresponding quantum field theory. In addition we like to discuss

what values for the critical exponents z and θ need to be chosen in order to describe the strange metallic phase using a hyperscaling-violating geometry.
In Section 4.2.1 we find that at sufficiently large temperatures the entropy scales as

$$s \propto T^{\frac{(d-\theta)}{z}}. \tag{4.3}$$

Remark. We must note here that this is an approximate expression, where this relation holds at sufficiently large temperatures. Only in the critical limit $z \to \infty$, with $\theta/z = \eta$ fixed, is the above expression exact. Actually the Einstein-Maxwell-dilaton model that we will discuss in the next section still has a nonzero entropy at zero temperature only considering the above limit makes sure that the entropy vanishes at zero temperature.

As outlined in Refs.[24, 25] in the hydrodynamical regime the resistivity of the strange metal has to scale with the viscosity of the the system $\rho \sim \eta$. A famous result from the hydrodynamical region of holography is that the viscosity (η) scales linearly with entropy density (s) [26]

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{4.4}$$

By combining Eqs. (4.3) and (4.4) we notice that in the hydrodynamical regime the resistivity has to scale like

$$\rho \sim T^{\frac{(d-\theta)}{z}}. \tag{4.5}$$

This equation is already quite relevant for describing the strange metallic phase. Since in the strange metallic phase linear scaling of the resistivity w.r.t. temperature[2–4] is observed, that is also what should come out here. This already restricts the possible choices of z and θ a lot and gives $\theta = d - z$.
Let us first discuss an interesting limit, namely the limit $z \to \infty$, with $\theta/z = \eta$ fixed. This limit has the interesting phenomenology of $z = \infty$, thus low-energy excitations are present at all momenta (we can see this from the case for finite z, where we have $\omega \sim k^z$). Additionally, the entropy density vanishes at zero temperature. We immediately notice that the entropy density for this limit behaves like

$$s \propto T^{\eta}, \tag{4.6}$$

in the above limit. Using Eq. (4.5) it is found the resistivity of the strange metallic phase has to scale like

$$\rho \sim T^{\eta}. \tag{4.7}$$

To match the observations done in strange metals e.g. Refs.[2–4] we need linear scaling of the resistivity, we thus need to consider theories with $\eta = 1$.
Lastly let us first briefly comment on the number of spatial dimension needed to describe the strange metallic phase. Since cuprate superconductors are known to behave like two dimensional layered materials, we need to describe a strongly coupled field theory in $2 + 1$ spacetime dimensions in order to describe the strange metallic phase which is observed to be the metallic state in high-T_c cuprates.
Before we start and introduce the EMD dilaton let us first look at the constraints on the critical exponents needed to have a sensible boundary theory.

Remark. Note that Eq. (4.7) is exact and thus extends to zero temperature. Additionally note that we could also have used the method described in Ref.[10, Section 3.4.2] to derive the temperature dependence of the DC conductivity[1].

4.1.2 Null energy conditions

In order to describe a sensible physical theory the bulk theory needs to satisfy certain energy conditions. These energy conditions assure that the energy density can not be negative. The most important one for holographic purposes is the Null-energy condition, the bulk spacetime satisfies the null-energy condition if

$$G^{\mu\nu}k_\mu k_\nu \propto T^{\mu\nu}k_\mu k_\nu \geq 0,$$

where k_μ is an arbitrary future pointing null-vector and $T^{\mu\nu}$ the bulk spacetime energy momentum tensor and $G^{\mu\nu}$ the Einstein tensor. In a holographic context the Null-energy conditions are important since it makes sure the bulk shrinks fast enough as one moves in the bulk [27]. Other reasons why one wants the Null-energy conditions to be satisfied in a holographic context is that it ensures the renormalization procedure makes sense [28]. The above null energy condition for hyperscaling violating spacetimes (4.1) reduces to

$$(d-\theta)\left(d(z-1)-\theta\right) \geq 0,$$
$$(z-1)(d+z-\theta) \geq 0.$$

This means that the exponents z and θ are no longer independent, for instance for positive z we need

$$\theta < d.$$

Furthermore we notice that the limit $z \to \infty$, $\theta \to -\infty$ with $\theta/z = \eta$ fixed does not violate the null energy conditions. Additionally $\theta = d - z$ with $z \geq \frac{2d}{d+1}$ does not violate the above Null energy conditions either. We checked the two cases above explicitly, since these are the cases that give rise to linear scaling of the resistivity w.r.t. temperature according to Eq. (4.5).

4.2 Einstein-Maxwell-dilaton black-brane solution

Having just discussed hyperscaling-violating geometries and some properties of it. We have seen that for certain choices of the critical exponents z and θ these hyperscaling-violating models might be able to capture the physics of the strange metallic phase. In this section we consider a black-hole solution of an Einstein-Maxwell-dilaton model which asymptotically gives rise to a hyperscaling violating geometry. We look at the black-hole solution since by the dictionary discussed in Chapter 3 we need a black hole solution to study the strongly coupled quantum field theory at nonzero temperature. In the next few sections we discuss the hyperscaling-violating black-hole solution of the Einstein-Maxwell-dilaton action under consideration and determine the equilibrium thermodynamics which by the holographic dictionary is given by the bulk background solution.

[1] A calculation in sort following the mentioned reference gave $\sigma_{DC} \sim T^{(d+2\Phi-\theta-2)/z}$, which does not agree with the resistivity mentioned in Eq. (4.5). The meaning of Φ will become clear in Section 4.3.

Unlike AdS spacetimes hyperscaling-violating spacetimes are not pure gravity solutions. Therefore we need a more extensive bulk theory to describe them. In this case we study an Einstein-Maxwell-Dilaton theory with two Maxwell (gauge) fields

$$\mathcal{S} = -\frac{1}{16\pi G}\int d^{d+2}x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 + V(\phi) - \frac{1}{4}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2\right]. \tag{4.8}$$

Here R is the Ricci scalar, ϕ the dilaton field (scalar field), A_i are given by the two Maxwell fields and $V(\phi) = V_0 e^{(2\theta/(d\beta))\phi}$, with $\beta = \sqrt{2d(1+\alpha)(\alpha+z-1)}$. In Appendix A the hyperscaling-violating black-hole solution for this model is derived. Since the derivation of the classical black-hole solution is done in Appendix A, we state the result here. The hyperscaling-violating metric is given by

$$ds^2 = r^{-2\frac{\theta}{d}}\left(-r^{2z}f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\vec{x}^2\right), \tag{4.9}$$

with

$$f(r) = 1 - \frac{m}{r^{z+d-\theta}} + \frac{Q^2}{r^{2(z+d-\theta-1)}}. \tag{4.10}$$

In the above m may be interpreted as the mass of the black-hole and Q as it's charge. We can use the above parameter to set the temperature and chemical potential of the quantum critical field theory respectively. Additionally notice that $f(r) \to 1$ as we approach the critical boundary of the gravity theory ($r \to \infty$), so pure hyperscaling violation is recovered towards the critical boundary which describes the UV of the quantum field theory. The background solutions for the gauge fields and the dilaton field are given by

$$\begin{aligned} F_{1rt} &= \sqrt{2(z-1)(z+d-\theta)}\,e^{\frac{\theta(1-d)/d+d}{\sqrt{2(d-\theta)(z-1-\theta/d)}}\phi_0} r^{d+z-\theta-1}, \\ F_{2rt} &= Q\sqrt{2(d-\theta)(z-\theta+d-2)}\,e^{-\sqrt{\frac{z-1-\theta/d}{2(d-\theta)}}\phi_0} r^{-(z+d-\theta-1)}, \\ e^{\phi} &= e^{\phi_0} r^{\sqrt{2(d-\theta)(z-1-\theta/d)}}. \end{aligned} \tag{4.11}$$

Note that the above solution is not defined for the case $\theta = d$ since we have divergences in our solutions in that case. We see that Eq. (4.10) gives a black-brane solution where the radius of horizon (r_H) is determined by $f(r_h) = 0$

$$r_H^{2(d+z-\theta-1)} - m r_H^{d+z-\theta-2} + Q^2 = 0. \tag{4.12}$$

By the holographic dictionary derived in Chapter 3 we note that the above black-hole solution determines the equilibrium thermodynamics of the system. This is what we will explore in the next few sections.

4.2.1 Thermodynamic properties of the Einstein-Maxwell-dilaton model

The black-hole solution of the above action Eq. (4.8) determines the thermodynamic equilibrium properties of the system. In the next sections these thermodynamic properties for the gravitational black-hole solution Eqs. (4.9) and (4.11) will be determined.

Nonzero density and chemical potential

In Section 3.2.2 it was described how to add a nonzero density into a strongly coupled quantum critical field theory in terms of the gravitational theory. The conclusion of this section was that the chemical potential of the quantum critical field theory corresponds to the source of the density operator, where the holographic dictionary thus gives

$$\lim_{r\to\infty} A_{2,t}(r) = \mu. \tag{4.13}$$

For the our gravitational background solution Eq. (4.11) we thus obtain

$$A_{2,t}(r) = \mu\left(1 - \left(\frac{r}{r_+}\right)^{-(z+d-\theta-2)}\right). \tag{4.14}$$

In terms of the parameters z, θ, ϕ_0 and Q we thus find

$$\mu = Q\sqrt{\frac{2(d-\theta)}{(z+d-\theta-2)}}e^{-\sqrt{\frac{z-1-\theta/d}{2(d-\theta)}}\phi_0} r_+^{-(z+d-\theta-2)}. \tag{4.15}$$

Where z, θ, ϕ_0 and Q are given by the Lifschitz scaling parameter, hyperscaling violating parameter, anomalous parameter and the charge of the potential. The integration constant ϕ_0 will become relevant later since it introduces an additional anomalous scaling for the scaling of the gauge fields.

Nonzero temperature

In Section 3.2.1 we discussed that a nonzero temperature in the strongly coupled field theory corresponds to considering a black-hole(brane) solution in the gravitational theory of the correspondence. We discussed that the temperature of the field theory is identical to the Hawking temperature of the black-hole. Additionally we derived the Hawking temperature by considering the black-hole geometry in the Euclidean time formalism. It was found that the Hawking temperature is given by

$$T = \left.\frac{\sqrt{g'_{\tau\tau}g^{rr\prime}}}{4\pi}\right|_{r_+}, \tag{4.16}$$

with r_+ the outer event horizon. Using that the event horizon is defined by $f(r_+) = 0$, the radial derivates of the above metric components, for the black-hole(brane) solution given in Eqs. (4.9) and (4.10), become

$$g'_{\tau\tau}(r_+) = r^{2z-2\theta/d}f'(r_+), \quad g^{rr\prime}(r+) = r^{2+2\theta/d}f'(r_+).$$

This implies the temperature of the field theory is given by

$$T = \frac{r_+^{1+z}f'(r_+)}{4\pi} = \frac{1}{4\pi}\left(\frac{(z+d-\theta)m}{r_+^{d-\theta}} - \frac{2(z+d-\theta-1)Q^2}{r_+^{z+2(d-\theta-1)}}\right). \tag{4.17}$$

Using that $f(r_+) = 0$ to eliminate m we find

$$T = \frac{(z+d-\theta)}{4\pi} r_+^z \left(1 - \frac{(d+z-\theta-2)Q^2}{(z+d-\theta)} r_+^{-2(z+d-\theta-1)}\right). \tag{4.18}$$

In terms of the chemical potential defined in Eq. (4.15) the temperature is given by

$$T = \frac{(z+d-\theta)}{4\pi} r_+^z \left(1 - \mu^2 \frac{(d+z-\theta-2)^2}{(d-\theta)(z+d-\theta)} e^{\sqrt{\frac{2(z-1-\theta/d)}{(d-\theta)}}\phi_0} r_+^{-2}\right). \tag{4.19}$$

For the entropy it will be useful to determine the horizon radius in terms of the temperature of the black-hole. Starting from Eq. (4.18) we find that for sufficiently large temperatures we may express that outer black-hole horizon r_+ as a series expansion in $1/T$ as

$$\begin{aligned} r_+ &= \left[\frac{4\pi}{(z+d-\theta)}\right]^{1/z} T^{1/z} \left(1 - \mu^2 \frac{(d+z-\theta-2)^2}{(d-\theta)(z+d-\theta)} e^{\sqrt{\frac{2(z-1-\theta/d)}{(d-\theta)}}\phi_0} r_+^{-2}\right)^{-1/z} \\ &\simeq \left[\frac{4\pi}{(z+d-\theta)}\right]^{1/z} T^{1/z} \\ &\left(1 + \mu^2 \left(\frac{4\pi}{(z+d-\theta)}\right)^{-2/z} \frac{(d+z-\theta-2)^2}{z(d-\theta)(z+d-\theta)} e^{\sqrt{\frac{2(z-1-\theta/d)}{(d-\theta)}}\phi_0} T^{-2/z} + h.o.\right). \end{aligned} \tag{4.20}$$

where $h.o.$ stands for higher order terms in inverse temperature.

Entropy

In Section 3.2.1 it was argued that the entropy of our critical field theory is identical to the entropy of the black-hole solution in the gravitational side of of the correspondence. The entropy of a black-hole is given by the Bekenstein-Hawking entropy which states that the black-hole entropy has to scales as the area of the horizon A_H of the black-hole(brane). To make this more concrete, the area of the horizon is defined as

$$A_H = \int_{r=r_+} d^d\mathbf{x} \sqrt{det g_{ij}}, \tag{4.21}$$

where i, j run over all indices except t and r. For the hyperscaling violating background given in Eqs. (4.9) and (4.10) we obtain

$$A_H = r_+^{d-\theta} \mathrm{Vol}_d. \tag{4.22}$$

This implies that the entropy density scales like

$$s \propto r_+^{d-\theta}. \tag{4.23}$$

Using the expansion of the outer black-hole horizon in terms of the temperature Eq. (4.20) an expansion of the entropy density s in terms of temperature T is obtained for sufficiently large temperatures

$$\begin{aligned} s \propto\ & T^{(d-\theta)/z} \left(1 + \mu^2 \left(\frac{4\pi}{(z+d-\theta)}\right)^{-2/z} \right. \\ & \left. \frac{(d-\theta)(d+z-\theta-2)^2}{z(d-\theta)(z+d-\theta)} e^{\sqrt{\frac{2(z-1-\theta/d)}{(d-\theta)}}\phi_0} T^{-2/z} + h.o.\right), \end{aligned} \tag{4.24}$$

where $h.o.$ stands for higher order terms in inverse temperature. At lowest order in $1/T$ we thus see that

$$s \sim T^{\frac{d-\theta}{z}}. \tag{4.25}$$

This result has already been used in Section 4.1.1. There it was used to determine which z and θ might be able to capture strange metallic behaviour.

4.3 Anomalous scaling of transport properties

Having just discussed the thermodynamics of the EMD model used in Section 4.2, in this section we treat this EMD model on a more general footing. Namely we will discuss the quantum critical scaling dimensions of multiple observables in terms of three critical exponents $\{z, \theta, \Phi\}$. It turns out that to properly characterise the scaling of the studied EMD model a a third critical exponent, besides z and θ is needed [29–33, 33, 34]. We have already seen that θ gives rise to an anomalous factor with which the overall metric scales. The additional exponent Φ captures the additional anomalous scaling of the physical bulk Maxwell field[2] A_2. In this section Ref.[33] will mainly be followed.

We have seen in Section 4.1 that the hyperscaling-violating metric is invariant under the following scale transformation

$$\eta: \ \{t, \vec{x}, r\} \rightarrow \{\lambda^z t, \lambda\vec{x}, r/\lambda\}. \tag{4.26}$$

From the above scaling transformation we are able to assign the following scaling dimensions to space and time (and thus also momentum, frequency and temperature)

$$[k] = -[x] = 1, \quad [\omega] = -[t] = [T] = z. \tag{4.27}$$

We have seen in Section 4.2.1 that the entropy has scaling dimension

$$[s] = d - \theta. \tag{4.28}$$

Since we know that the scaling dimension of the free energy is given by $[f] = [s]+[T]$, therefore the free energy density and the energy density acquire the following scaling dimension

$$[f] = [\epsilon] = d - \theta + z. \tag{4.29}$$

Thus we can think of hyperscaling violation as an anomalous dimension in the energy density operator. In Chapter 6 we will see that hyperscaling violating model discussed in Section 4.2 needs an additional critical exponent to be properly characterised. As we have discussed, this additional anomalous scaling happens in the gauge fields, a natural choice to incorporate this into our scaling analyses is to add an exponent Φ that leads to anomalous scaling of the charge density operator. This leads to the observation that now charged critical fluctuations are distinct from critical fluctuations contributing to the entropy

$$[n] = d - \theta + \Phi. \tag{4.30}$$

[2]Note that in principle we should also take the anomalous scaling of the field A_1 into account, but since it is unknown to us what physical observables it corresponds to we omit it in this discussion.

From the conservation of charge and conservation of energy

$$\partial_t n + \nabla j = 0, \quad \partial_t \epsilon + \nabla j^Q = 0, \tag{4.31}$$

we find the critical scaling dimensions of electrical current and the heat current

$$[j] = d - \theta + \Phi + z - 1, \quad [j^Q] = d - \theta + 2z - 1. \tag{4.32}$$

The scaling dimensions of the currents imply the following scaling dimensions of the Maxwell field A_2, the chemical potential and the electrical field

$$[A_{2,i}] = 1 - \Phi, \quad [A_{2,0}] = z - \Phi, \quad [\mu] = z - \Phi, \quad [E] = z + 1 - \Phi. \tag{4.33}$$

So currently we have derived in a somewhat general setting the scaling dimensions of several observables of the strongly coupled quantum field theory[3]. These scaling dimensions will become important in the next chapters since they give an additional way to check whether the identification of the sources and the VEVs of the quantum field theory are correct as described at the end of Section 3.1.2.

4.4 Summary

In this chapter we looked at an EMD model as a model to describe strange metallic phases. These EMD models give rise to hyperscaling-violating geometries, it turns out that these geometries under certain conditions look promising in describing strange metallic phases. In the main part of this chapter we stated the black-hole solution to the studied EMD action and we studied the thermodynamics of the black-hole solution which by the holographic dictionary gives the thermodynamic equilibrium properties of the quantum field theory. In the last part of this section we described the critical scaling dimensions of several observables of the quantum field theory, for instance the electrical current and the heat current. The need for the scaling analyses will become clear in the next chapters, because it will give us a way to check if the identification of the sources and VEVs of the quantum field theory are correct.

[3]The precise value of Φ wil become clear in the next chapters and is most easily obtained from the scaling of the conductivity.

Chapter 5

Quantum Critical Dynamics

In the previous chapter we discussed the equilibrium properties of the black-hole solution to the studied EMD model. In this chapter we set up the necessities to determine the correlation functions (or dynamics) of the studied EMD model. In Section 3.1.3 the correlation functions of a scalar field in AdS spacetime were determined. Determining the correlation functions for the EMD model discussed in the previous chapter will be similar to the example given in Section 3.1.3, only a lot more involved. Indeed we have to determine the correct asymptotic expansion for the fluctuations of the EMD fields. And determine the right sources and VEVs in the expansion by using the holographic dictionary, the second order boundary action and the scaling arguments given at the end of the previous chapter. From the sources and VEVs of the theory, the Green's functions can easily be extracted.
This chapter is organized in the following way, we start with a general discussion of how to extract Green's functions of the field theory by considering dynamic fluctuations on top of a static bulk background. After this general introduction we determine the classical equations of motion for the fluctuations on top of the EMD black-hole solution, which will be done in Section 5.3. Since we want to determine the thermoelectric transport properties in our EMD model it is sufficient to consider only the frequency dependence of fluctuations. For this reason we may consider the zero momentum limit of the equations of motion for the fluctuations, this is carried out in Section 5.4. In Section 5.5 we determine the asymptotic near boundary expansion of the fluctuations, from which by the holographic dictionary discussed in Chapter 3 one should be able to identify the sources and VEVs of the dynamical fluctuations of the quantum field theory. In the remainder of this chapter the unrenormalised boundary action is given as a check for of the identification of the sources and the VEVs. Furthermore critical scaling dimensions of the sources and VEVs of the system are looked at as a check of the identification of the VEVs.

5.1 Holographic response functions

In this the next two sections it is explained how to extract the dynamical correlation function from a general gravitational bulk model. Since the bulk background solution only describes the equilibrium properties of the system, the dynamics has to added in the form of dynamical fluctuations. We will line out this procedure below. Let's start by considering a general set of M fields $\{\Phi_J\}_{J=1,\ldots,M}$ where J denotes the different fields. The corresponding action is given by

$$\mathcal{S}[\{\Phi_J\}] = \int d^{d+2}\mathcal{L}[\{\Phi_J\}].$$

As seen in the previous chapters the classical black-brane solution of this action gives the equilibrium properties of the system. In order to obtain the Green's functions of the system it is needed to expand the action up to second-order in fluctuations of the fields

$$\Phi_J \to \Phi_J + \delta\Phi_J, \tag{5.1}$$

$$\mathcal{S}[\{\Phi_J\}] \to \mathcal{S}[\{\Phi_{c,J}\}] + \delta\Phi_J^\dagger \frac{\delta^2\mathcal{S}[\{\Phi_{c,J}\}]}{\delta\Phi_J\delta\Phi_J}\delta\Phi_J + \ldots \quad . \tag{5.2}$$

The classical solution to the second order action gives what is called the linearised equations of motion. The linearised equations of motion lead to a set of in the case M, coupled ordinary differential equations for the dynamical perturbations (5.1). Since we are truncating the action at second order, we find that the second order action becomes a boundary action when evaluated at the classical solution. This boundary action might in principle contain divergences which need to be regularised by adding the right counter term boundary action to the system. By the holographic dictionary the full boundary action after regularisation assumes the form

$$\delta^2 S_{bdy} = \int_{\partial M} d^{d+1}k \sum_I^M \delta\langle\mathcal{O}_I\rangle\delta\Phi_s^I, \tag{5.3}$$

we thus see that the dynamical fluctuations in the vacuum expectation values $\delta\langle\mathcal{O}_I\rangle$ can be extracted from the boundary action. In the next few sections we cover the holographic computations needed to determine the Green's function of the quantum critical operators in the quantum critical field theory in the case we have coupled operators. Without going into details a general Green's function for multiple coupled operators, in frequency space, is determined by

$$\delta\langle\mathcal{O}_I\rangle(\omega,k) = \sum_J G^R_{\mathcal{O}_I\mathcal{O}_J}(\omega,k)\delta\Phi^s_J(\omega,k). \tag{5.4}$$

In the above $\delta\langle\mathcal{O}_I\rangle(\omega,k)$ is the change in the expectation value of the operator $\mathcal{O}_I$ due to the addition of the dynamical (space and time dependent) sources $\delta\Phi_J$ and $G^R_{\mathcal{O}_I\mathcal{O}_J}(\omega,k)$ is defined as the retarded Green's function. From the holographic dictionary discussed in Chapter 3 it is known that expectation values and sources are given by the near-boundary behaviour of bulk fields $\{\Phi_J\}$ dual to operators $\{\mathcal{O}_J\}$[1]. Therefore

$$G^R_{\mathcal{O}_I\mathcal{O}_J} = \frac{\delta\langle\mathcal{O}_I\rangle}{\delta\Phi^s_J}.$$

In the uncoupled problem using linear-response theory one obtains $G_{\mathcal{O}\mathcal{O}}\delta\Phi_s = \delta\langle\mathcal{O}\rangle$, from which the Green's function is simply extracted by taking the ratio of the expectation value and the corresponding source term.

Notice that in the case of coupled operators a change in a single operator, say $\langle\mathcal{O}_I\rangle$ will be given as a linear combination of changes in all source terms. In the coupled problem it is not enough to know a single solution to the linearised equations of motion to extract the full Green's function, a nice way to proceed from here is lined out in Ref.[35], which we will follow in Section 5.2.

It's important to note that the boundary conditions of the dynamical fields play an important role in the physics that is obtained. Since we are interested in retarded Green's functions we need to impose the correct boundary conditions to obtain them, it turns out that the boundary conditions at the horizon describing retarded Green's functions are the infalling boundary conditions.

5.1.1 Infalling boundary conditions near the horizon

In this section we derive the near-horizon boundary conditions corresponding to retarded Green's functions in the quantum field theory, in doing so Ref.[10, Section 3.3] is closely followed.

Let's start by considering an Euclidean black-hole spacetime solution. In Euclidean spacetime the solution of the dynamical fluctuations near the horizon ($r \to r_+$) assume the form

$$\delta\Phi_{J\pm} \simeq \alpha_J(\omega, k)(r - r_+)^{\pm\beta_J}. \tag{5.5}$$

Where $\alpha_J(\omega, k)$ is a prefactor that does not depend on r. In many cases the exponent assumes the form

$$\beta_J = \omega_n/(4\pi T), \tag{5.6}$$

with ω_n the Matsubara frequency and T the temperature. Since real time retarded Green's functions are obtained from Euclidean Green' functions by analytic continuation in the upper half plane ($\omega_n > 0$), we have to consider $\omega_n > 0$. For $\omega_n > 0$, it is clear that the regular solution of Eq. (5.5) is the one that decays as $r \to r_+$, thus for Eq. (5.6)

$$\delta\Phi_J \simeq \alpha_J(\omega, k)(r - r_+)^{\beta_J}.$$

To obtain the real time equations form the Euclidean equations we have to perform a Wick rotation $i\omega_n \to \omega$. We find that the regular boundary conditions at the horizon in Euclidean time translate to infalling boundary conditions in real time

$$\delta\Phi_{\text{infalling},J} \simeq \alpha_J(\omega, k)(r - r_+)^{-i\beta_J}. \tag{5.7}$$

[1] The near-boundary behaviour has the same form at zero and nonzero temperature since, putting a black-hole in the spacetime changes the geometry in the interior of the gravitational theory (which correspond to the IR physics of the field theory), but leaves the critical boundary of the bulk intact (which corresponds to the UV of the field theory).

In order to see why this behaviour is infalling we have to restore the time dependence (for ease of use consider β_J given by Eq. (5.6))

$$\delta\Phi_{\text{infalling},J} \simeq \alpha_J(\omega,k)e^{-i\omega(t+\frac{1}{4\pi T}\log(r-r_+))},$$

which corresponds to modes moving towards $r = r_+$ as t increases. So in order to study retarded Green's functions we must consider infalling boundary conditions at the black-hole horizon.

5.2 Extract coupled Green's functions

Having established the correct boundary conditions and the near-boundary solution for the dynamical fluctuations, we are able to determine the coupled Green's functions for the quantum critical field theory. The linearised equations of motion which we discussed in the sections above give a set of M at most second-order differential equations. In general we may have that $N \leq M$ differential equations are coupled and therefore need N different initial conditions to describe them. Since we have a system of mostly second-order ordinary differential equations considering infalling boundary conditions restricts one of the two degrees of freedom. The other one is determined by the prefactors α_J in Eq. (5.7). Since the differential equations are linear, we are able to characterise the independent solutions to the linearised equations of motion by considering linearly independent sets of prefactors

$$\{\alpha_J\}_{J=1,\ldots,M}.$$

Example 5.2.1. In order to obtain the coupled Green's function we need to define a set of M linearly independent vectors $\vec{\alpha}^{(m)}(\omega,k)$. An example of such a linearly independent set would be

$$\begin{aligned}
\vec{\alpha}^{(1)} &= \big(1,1,\ldots,1\big)\,,\\
\vec{\alpha}^{(2)} &= \big(1,-1,\ldots,1\big)\,,\\
&\ \ \vdots\\
\vec{\alpha}^{(m)} &= \big(1,1,\ldots,-1,\ldots,1\big)\,,\\
&\ \ \vdots\\
\vec{\alpha}^{(M)} &= \big(1,1,\ldots,-1\big)\,.
\end{aligned}$$

For each of the solutions we are able to find the associated source and expectation value of the fluctuations. We are thus able to construct two matrices where each vector is corresponding to a linearly independent solution

$$\mathbf{A} = \begin{pmatrix} \delta\Phi^1_{s,1} & \delta\Phi^1_{s,2} & \ldots & \delta\Phi^1_{s,M} \\ \delta\Phi^2_{s,1} & \delta\Phi^2_{s,2} & \ldots & \delta\Phi^2_{s,M} \\ \vdots & \vdots & \ddots & \vdots \\ \delta\Phi^M_{s,1} & \delta\Phi^M_{s,2} & \ldots & \delta\Phi^M_{s,M} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \delta\langle\mathcal{O}_1\rangle^1 & \delta\langle\mathcal{O}_2\rangle^1 & \ldots & \delta\langle\mathcal{O}_M\rangle^1 \\ \delta\langle\mathcal{O}_1\rangle^2 & \delta\langle\mathcal{O}_2\rangle^2 & \ldots & \delta\langle\mathcal{O}_M\rangle^2 \\ \vdots & \vdots & \ddots & \vdots \\ \delta\langle\mathcal{O}_1\rangle^M & \delta\langle\mathcal{O}_2\rangle^M & \ldots & \delta\langle\mathcal{O}_M\rangle^M \end{pmatrix}. \tag{5.8}$$

The coupled retarded Green's function is now determined by

$$\mathbf{A}\mathbf{G}^{\mathbf{R}} = \mathbf{B}. \tag{5.9}$$

We have thus seen that if the values of the sources and VEVs of the quantum field theory for different initial conditions are known, that the Green's functions easily can be extracted. The next part of this chapter will be concerned with determining the sources and VEVs for the dynamical fluctuations on top of the studied EMD background.

Remark. Note that we assumed linear response between the vacuum expectation values and the sources of the operators in Eq. (5.9).

5.3 Dynamics of the EMD black-brane solution

In the previous sections we discussed how to obtain the retarded Green's functions in a general setting. In the next sections we will setup the necessities needed to obtain the coupled Green's function for the black-brane solution to the EMD model discussed in Chapter 4. Let's start by considering the EMD action

$$\mathcal{S} = -\frac{1}{8\pi G}\int d^{d+1}x\sqrt{-\gamma}K - \frac{1}{16\pi G}\int d^{d+2}x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 + V(\phi) - \frac{1}{4}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2\right], \tag{5.10}$$

The above action contains a gravitational field $g_{\mu\nu}$, a dilaton field ϕ and two gauge fields A_i. Action (5.10) is equal to the action given in Section 4.2 with the addition of the Gibbons-Hawking-York boundary term. In order to have proper Dirichlet boundary conditions in gravitational theories with a boundary the Gibbons-Hawking-York term should be added to the theory. We thus need this additional boundary term in order to have a well-defined boundary theory.

The classical black-brane solution to the EMD action is given in Eqs. (4.9) to (4.11) and the precise derivation of this background is done in Appendix A. To add dynamics into our holographical setup we need to introduce dynamical fluctuations on top of our classical black-brane solution

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}, \quad \phi \to \phi + \delta\phi, \quad A_{i,\mu} \to A_{i,\mu} + a_{i,\mu}. \tag{5.11}$$

The total action up to second-order in fluctuations is worked out in Appendix B. As described in the first section of this chapter, by considering the classical solution of the obtained action a set of equations is obtained, called the linearised equations of motion. Considering the variational derivative w.r.t. the dilaton fluctatuations $\delta\phi$ of the second order action determined in Appendix B we obtain the linearised equation op motion for the dilaton fluctuations

$$\begin{aligned}&\sqrt{-g}\left(V''(\phi) + \frac{\lambda_i^2\rho_i F_{irt}}{4\sqrt{-g}}\right)\delta\phi + \partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\delta\phi\right) + \frac{1}{2}\sum_{i=1}^{2}\lambda_i\rho_i\partial_r a_{i,t}\\ &+ g^{rr}\partial_r h^\alpha_\alpha + \frac{1}{2}\sqrt{-g}g^{rr}\sum_{i=1}^{2}\lambda_i e^{\lambda_i\phi}F_{i,tr}F_{i,tr}h^{tt} = 0.\end{aligned} \tag{5.12}$$

A similar procedure w.r.t. the gauge field fluctuations $a_{1,\nu}$ and $a_{2,\nu}$ gives the following linearised equations of motion

$$\begin{aligned}&\frac{1}{2}\partial_\mu\left(\sqrt{-g}e^{\lambda_i\phi}F^{\mu\nu}h^\alpha_\alpha\right)-\partial_\alpha\left(\sqrt{-g}e^{\lambda_i\phi}F_\beta{}^\nu h^{\alpha\beta}\right)+\partial_\mu\left(\sqrt{-g}e^{\lambda_i\phi}h^{\nu\beta}F_\beta{}^\mu\right)\\&+\partial_\mu\left(\sqrt{-g}\lambda_i\delta\phi e^{\lambda_i\phi}F^{\mu\nu}\right)+\partial_\mu\left(\sqrt{-g}e^{\lambda_i\phi}f^{\mu\nu}\right)=0.\end{aligned}\tag{5.13}$$

Again, by considering the variational derivative of the second order action determined in Appendix B w.r.t the metric fluctuations $h^{\alpha\beta}$ one obtains the following set of linearised equations of motion

$$\begin{aligned}&\frac{1}{2}h_{\alpha\beta}\left[R-\frac{1}{2}(\partial\phi)^2+V(\phi)-\frac{1}{4}\sum_{i=1}^{2}e^{\lambda_i\phi}F_i^2\right]-\frac{1}{2}g_{\alpha\beta}R_{\mu\nu}h^{\mu\nu}+2R_{\nu\alpha}h^\nu_\beta\\&+\frac{1}{2}g_{\alpha\beta}\nabla_\nu\nabla_\mu h^{\mu\nu}+\frac{1}{2}\nabla_\alpha\nabla_\beta h^\mu_\mu-\frac{1}{2}g_{\alpha\beta}\nabla_\lambda\nabla^\lambda h^\mu_\mu\\&-\frac{1}{2}\nabla_\mu\nabla_\alpha h^\mu_\beta-\frac{1}{2}\nabla_\nu\nabla_\beta h^\nu_\alpha+\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\alpha\beta}\\&+\frac{1}{2}\sum_{i=1}^{2}e^{\lambda_i\phi}\left(F_{i,t\alpha}F_{i,t\beta}\right)h^{tt}\\&+\frac{1}{2}g_{\alpha\beta}\left[\left(-(\partial_\mu\phi)\partial^\mu+V'(\phi)-\frac{1}{4}\sum_{i=1}^{2}\lambda_ie^{\lambda_i\phi}F_i^2\right)\delta\phi-\frac{1}{2}\sum_{i=1}^{2}e^{\lambda_i\phi}F_if_i\right]\\&-(\partial_\alpha\phi)(\partial_\beta\delta\phi)+\frac{1}{2}g^{\mu\nu}\sum_{i=1}^{2}\left(\lambda_ie^{\lambda_i\phi}F_{i,\alpha\mu}F_{i,\beta\nu}\delta\phi+2e^{\lambda_i\phi}F_{i,\alpha\mu}f_{i,\beta\nu}\right)=0.\end{aligned}\tag{5.14}$$

The above set of equations determine the dynamics of the EMD model. We will see in the next sections that by solving these equations near the critical boundary of the bulk, the sources and VEVs of te quantum field theory may be identified. By considering the full solutions to the linearised equations of motion with infalling boundary conditions, the values of the sources and VEVs for different initial values can be obtained and allows to extract the retarded Green's functions of the dual field theory, as described in the previous section.
As we will see in Chapter 6, in order to determine the the response functions needed for thermoelectric transport it is sufficient to consider the equations of motion in the zero momentum limit. This is done in the next section.

5.4 Dynamics of EMD bulk fields at zero momentum

In order to simplify Eqs. (5.13) and (5.14) let us first exploit the rotational invariance of our system to set the momentum of the gauge and metric fields to be in the x-direction, so that $\mathbf{k}=(\omega,k_x,0)$. Thus the fluctuations on top of the background solution take the form

$$\begin{aligned}A_{i,\mu}&\to A_{i,\mu}+e^{-i\omega t+ik_xx}a_{i,\mu}(r),\\g_{\mu\nu}&\to g_{\mu\nu}+e^{-i\omega t+ik_xx}h_{\mu\nu}(r).\end{aligned}\tag{5.15}$$

By considering a discrete symmetry $y\to-y$, with y a boundary direction orthogonal to x. We see that the fluctuations now decouple by their parity, into a transverse component and longitudinal component. The thermoelectric response functions are in the longitudinal component, which in $d = 2+1$ dimensions is given by

$$\{h_{tt}, h_{tx}, h_{xx}, h_{yy}, a_{i,t}, a_{i,x}, h_{rr}, h_{tr}, h_{xr}, \delta\phi\}. \tag{5.16}$$

For both the metric $h_{\mu\nu}$ and the gauge fields $a_{i,x}$ the corresponding physical observables should be invariant under a gauge transformation in the fields. For the metric this is diffeomorphism invariance, while for the gauge field this is a $U(1)$ gauge symmetry. For our purposes it is convenient to work in a gauge where

$$\begin{aligned} h_{r\mu} &= 0, \\ a_r &= 0. \end{aligned} \tag{5.17}$$

There are still some gauge degrees of freedom left, which lead to pure gauge solutions. Under the above gauge choice the set of longitudinal fluctuations become

$$\{h_{tt}, h_{tx}, h_{xx}, h_{yy}, a_{i,t}, a_{i,x}, \delta\phi\}. \tag{5.18}$$

As will be seen in Chapter 6, only the responses of h_{xt} and $a_{2,x}$ are relevant for describing the thermoelectric transport[2], additionally we will see that it suffices to consider the linearised equations of motion in the zero momentum limit $k_x \to 0$. The relevant linearised equations of motion for thermoelectric transport are now given by the x-component of the linearised gauge field equations (5.13). The other relevant equation is given by the rx-component of the linearised equations Eq. (5.26) w.r.t. to metric perturbation. Those equations were chosen because they form a closed set of differential equations containing just the fields needed in the zero-momentum limit. Note that in principle we could have included the xt-component of the linearised equations of motion of the metric, but this one should be a superposition of the three linearised equations of motion that we have just mentioned. Since the linearised equation of motion w.r.t. rx-component is much simpler than the the one obtained from varying w.r.t. h_{xt}-component we use rx-component.
The x-component of of the linearised equations of motion for the gauge fields $a_{1,x}$ and $a_{2,x}$ (5.13) are given by

$$\partial_\mu\left(\sqrt{-g}e^{\lambda_i\bar{\phi}}f_i^{\mu x}\right) + \partial_\mu\left(\sqrt{-g}e^{\lambda_i\phi}h^{x\beta}F_\beta{}^\mu\right) = 0, \tag{5.19}$$

using gauge choice (5.17) and ansatz (5.15) the x-component of the linearised equations of motion for the gauge fields $a_{1,x}$ and $a_{2,x}$ in the zero momentum limit ($k_x \to 0$) becomes

$$\omega^2\left(\sqrt{-g}g^{tt}g^{xx}e^{\lambda_i\bar{\phi}}a_{i,x}\right) - \partial_r\left(\sqrt{-g}g^{rr}g^{xx}e^{\lambda_i\bar{\phi}}\partial_r a_{i,x}\right) - \partial_r\left(\sqrt{-g}e^{\lambda_i\phi}g^{rr}g^{xx}g^{tt}h_{xt}F_{i,rt}\right) = 0. \tag{5.20}$$

Note that the above equations only depend on $a_{1,x}$, $a_{2,x}$ and h_{xt}.
The rx-component of the linearised equations of motion for the metric fields 5.26 is given by

$$\nabla_\mu\nabla_r h^\mu_x + \nabla_\mu\nabla_x h^\mu_r - \nabla_\lambda\nabla^\lambda h_{rx} - \sum_{i=1}^{2} g^{tt}e^{\lambda_i\phi}F_{irt}f_{xt} = 0, \tag{5.21}$$

[2]Note that the response of the field $a_{1,x}$ is not relevant for thermoelectric transport. Although this field is dual to a conserved current $\delta\langle J_1^x\rangle$, the interpretation of this current is unknown. As we see later $a_{2,x}$ gives rise to fluctuations the electrical current $\delta\langle J_2^x\rangle$ and h_t^x gives rise to fluctuations in the heat current $\delta\langle J^Q\rangle$, so those are the two necessary fields to describe thermoelectric transport.

at zero momentum ($k_x \to 0$), using gauge choice Eq. (5.17), the above equation reduces to

$$\begin{aligned}&\frac{i\omega r^{\theta-2z}h'_{xt}}{f}-\frac{2ih_{xt}\omega r^{\theta-2z-1}}{f}+\frac{i\theta h_{xt}\omega r^{\theta-2z-1}}{f}\\&+\frac{i\omega}{f}\left(e^{\lambda_1\phi_0}r^{2(\theta-z-d)}F_{1,rt}a_{1,x}+e^{\lambda_2\phi_0}r^{-2}F_{2,rt}a_{2,x}\right)=0.\end{aligned} \tag{5.22}$$

Where

$$f(r)=1-\frac{m}{r^{z+d-\theta}}+\frac{Q^2}{r^{2(z+d-\theta-1)}}.$$

Again note that also rx-component of the linearised equation of motion of metric just depends on the fields $a_{1,x},\ a_{2,x}$ and h_{xt} for gauge choice (5.17) in the zero momentum limit. So in the end we have three coupled second order differential equations, which depend on the fields $a_{1,x},\ a_{2,x}$ and h_{xt}, describing thermoelectric transport. Next to the fields h_{xt} and $a_{2,x}$ needed to determine the thermoelectric response of the system we have an additional field $a_{1,x}$ of which the physical meaning is unclear to at the moment of writing.

Supplement: equations of motion for the h_{xt} component

If consistent, this equation should be a superposition of the constraint equation following from the gauge fluctuations of the metric and the spatial fluctuations of the gauge fields.

$$\begin{aligned}&\frac{1}{2}h_{tx}\left[R-\frac{1}{2}(\partial\phi)^2+V(\phi)-\frac{1}{4}\sum_{i=1}^{2}e^{\lambda_i\phi}F_i^2\right]+\left(R^t_t+R^x_x\right)h_{tx}\\&-\frac{1}{2}\nabla_\mu\nabla_\alpha h^\mu_\beta-\frac{1}{2}\nabla_\mu\nabla_\beta h^\mu_\alpha+\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\alpha\beta}\\&+\frac{1}{2}g^{\mu\nu}\sum_{i=1}^{2}\left(e^{\lambda_i\phi}F_{i,t\mu}f_{i,x\nu}+e^{\lambda_i\phi}F_{i,x\mu}f_{i,t\nu}\right)=0.\end{aligned} \tag{5.23}$$

5.4.1 Linearised equations of motion in the EMD background solution at zero momentum

The linearised equations of motion at zero momentum given in the beginning of this section still have background fields in them $F_{i,\mu\nu},\ \phi$ and $g_{\mu\nu}$, for which the expressions are given in Section 4.2. By considering the linearised equations of motion of the gauge fields Eq. (5.20) in the EMD background solution given in Section 4.2. We find that the linearised equation of motion for $a_{1,x}$ becomes

$$\frac{\omega^2\left(r^{\theta-z-d-3}e^{\lambda_1\bar{\phi}_0}a_{1,x}\right)}{f}+\partial_r\left(fr^{\theta+z-d-1}e^{\lambda_1\bar{\phi}_0}\partial_r a_{1,x}\right)+\rho_1\partial_r h^x_t=0. \tag{5.24}$$

Evaluated on the EMD background solution the linearised equation of motion for $a_{2,x}$ gives

$$\begin{aligned}&\frac{\omega^2\left(r^{z-\theta+d-5}e^{\lambda_2\bar{\phi}_0}a_{2,x}\right)}{f}+\partial_r\left(fr^{z-\theta+d-1+\frac{2\theta}{d}}e^{\lambda_2\bar{\phi}}\partial_r a_{2,x}\right)+\rho_2\partial_r h^x_t=\\&\frac{\omega^2\left(r^{z-\theta+d-5}e^{\lambda_2\bar{\phi}_0}a_{2,x}\right)}{f}+\partial_r\left(fr^{3z-\theta+d-3}e^{\lambda_2\bar{\phi}_0}\partial_r a_{2,x}\right)+\rho_2\partial_r h^x_t=0.\end{aligned} \tag{5.25}$$

The linearised equation of motion for the metric component Eq. (5.21) may be simplified by multiplying it with $\sqrt{-g}g^{rr}$. Evaluated on the EMD background solution this modified equation of motion becomes

$$\begin{aligned} \sqrt{-g}g_{xx}g^{tt}g^{rr}\partial_r h_t^x - \sum_{i=1}^{2}\rho_i a_i = 0, \\ \implies r^{-z-\theta+3+d}\partial_r h_t^x + \sum_{i=1}^{2}\rho_i a_i = 0. \end{aligned} \tag{5.26}$$

Using the above three equations of motion we are able to start determining the thermoelectric transport coefficients of the EMD model. To do so we first need to perform the asymptotic boundary expansion in order to identify the sources and VEVs of the system. This will be done in the next section.

Before determining the asymptotic behaviour of the dynamical fluctuations, we need to consider the correct boundary conditions for this system. It is argued in Section 5.1.1 that the correct boundary conditions in the gravitational side of the correspondence (corresponding to retarded Green's functions) are the infalling boundary conditions. In the near-horizon limit of Eqs. (5.24) to (5.26) the infalling boundary conditions of the fields are given by

$$\begin{aligned} a_{1,x} &\propto \alpha_1 (r-r_+)^{-\frac{i\omega}{4\pi T}}, \\ a_{2,x} &\propto \alpha_2 (r-r_+)^{-\frac{i\omega}{4\pi T}}, \\ h_t^x &\propto \alpha_h (r-r_+)^{1-\frac{i\omega}{4\pi T}}. \end{aligned} \tag{5.27}$$

Remark. Note that the constants α_1, α_2 and α_h in the above near-horizon infalling boundary conditions are not linearly independent as we will see in the next section. It turns out that there are only two degrees of freedom in the above infalling near-horizon expansion.

5.5 Asymptotic expansion near critical boundary

In this section we will determine the asymptotic near boundary behaviour of the fields $h_{\mu\nu}$, $a_{i,\mu}$. This behaviour is important since as noted before in Chapter 3 the free integration constants of the expansion may be linked to the sources and VEVs of the dynamical fluctuations of the quantum field theory.

Let's start by considering the ansatz for large r (thus near boundary)

$$a_{1,x}(r) = r^{\alpha_1}\left(\sum_{n=0}^{\infty} a_1^{(n)} r^{-n}\right), \tag{5.28}$$

$$a_{2,x}(r) = r^{\alpha_2}\left(\sum_{n=0}^{\infty} a_2^{(n)} r^{-n}\right), \tag{5.29}$$

$$h_t^x(r) = r^{\gamma}\left(\sum_{n=0}^{\infty} h^{(n)} r^{-n}\right). \tag{5.30}$$

Remark. In principle we should have included additional logarithmic terms in the expansions. For the purpose of this thesis this is not necessary though, since the values of z and θ of interest give rise to even solution in r.

To determine the factors $\alpha_1,\ \alpha_2$ and γ, the leading order behaviour of the linearised equations of motion Eqs. (5.24) to (5.26) is considered with ansatz Eqs. (5.28) to (5.30). This gives rise to the following equations at leading order

$$e^{\lambda_1\phi_0}\partial_r\left(r^{\theta+z-d-1}\partial_r a_{1,x}\right)+\rho_1\partial h_t^x=0, \tag{5.31}$$

$$e^{\lambda_2\bar{\phi}_0}\partial_r\left(r^{3z-\theta+d-3}\partial_r a_{2,x}\right)+\rho_2\partial_r h_t^x=0, \tag{5.32}$$

$$r^{-z-\theta+3+d}\partial_r h_t^x+\sum_{i=1}^{2}\rho_i a_i=0. \tag{5.33}$$

Substituting Eq. (5.33) into Eqs. (5.31) and (5.32) we find

$$a_1^{(0)}e^{\lambda_1\phi_0}\partial_r\left(r^{\theta+z-d-1}\partial_r r^{\alpha_1}\right)-r^{z+\theta-d-3}\left(\rho_1^2a_1^{(0)}r^{\alpha_1}+\rho_1\rho_2a_2^{(0)}r^{\alpha_2}\right)=0, \tag{5.34}$$

$$a_2^{(0)}e^{\lambda_2\bar{\phi}_0}\partial_r\left(r^{3z-\theta+d-3}\partial_r r^{\alpha_2}\right)-r^{z+\theta-d-3}\left(\rho_1\rho_2a_1^{(0)}r^{\alpha_1}+\rho_2^2a_2^{(0)}r^{\alpha_2}\right)=0. \tag{5.35}$$

Next we can make a distinction between two cases.

Case 1. $\alpha_1\neq\alpha_2$, Eq. (5.34) automatically implies $\alpha_1>\alpha_2$. Due to the assumption, Eq. (5.34) assumes the form

$$a_1^{(0)}\left\{e^{\lambda_1\phi_0}\alpha_1(\alpha_1+\theta+z-d-2)-\rho_1^2\right\}r^{\alpha_1+z+\theta-d-3}=0. \tag{5.36}$$

The factors of α_1 that solve the above equation are

$$\alpha_1^{\pm}=\frac{d+2-z-\theta}{2}\pm\sqrt{\frac{(\theta+z-d-2)^2}{4}+e^{-\lambda_1\phi_0}\rho_1^2}. \tag{5.37}$$

Since $\rho_1^2=2(z+d-\theta)(z-1)e^{\lambda_1\phi_0}$ by Eq. (A.29) it follows that the above equation simplifies to

$$\alpha_1^{\pm}=\frac{d+2-z-\theta}{2}\pm\sqrt{\frac{(\theta+z-d-2)^2}{4}+2(z+d-\theta)(z-1)} \tag{5.38}$$

$$=\frac{d+2-z-\theta}{2}\pm\frac{3z+d-\theta-2}{2}. \tag{5.39}$$

In the special case $\theta=d-z$ Eq. (5.38) assumes the form

$$\alpha_1^{\pm}=1\pm(2z-1). \tag{5.40}$$

Remark. Note that the above powers are related to powers in h_t^x by Eq. (5.33) therefore,

$$\begin{aligned}\alpha_h^{\pm}&=\frac{z+\theta-d-2}{2}\pm\sqrt{\frac{(\theta+z-d-2)^2}{4}+2(z+d-\theta)(z-1)},\\&=\frac{z+\theta-d-2}{2}\pm\frac{3z+d-\theta-2}{2}.\end{aligned} \tag{5.41}$$

In the case $\theta = d - z$ this becomes

$$\alpha_h^{\pm} = -1 \pm (2z - 1). \tag{5.42}$$

For Eq. (5.35) we note that the diverging factor Eq. (5.38) is not playing a role in the leading behaviour of Eq. (5.35), therefore the leading order equation for $a_{2,x}(r)$ becomes

$$e^{\lambda_2 \bar{\phi}_0} \partial_r \left(r^{3z-\theta+d-3} \partial_r r^{\alpha_2} \right) = \partial_r \left(r^{z-\theta+d-1} e^{\lambda_2 \bar{\phi}} \partial_r r^{\alpha_2} \right) = 0. \tag{5.43}$$

This implies that

$$\alpha_2 = 0 \text{ or } \alpha_2 = -3z + \theta + d + 4 = \theta - z - d + 2 - \lambda_2 \beta, \tag{5.44}$$

with $\beta = \sqrt{2(d-\theta)(-\theta/d + z - 1)}$.

Case 2. $\alpha_1 = \alpha_2$, it can quickly be seen that this leads to a contradiction and is thus an invalid assumption.

We thus see that the integration constants of the near-boundary expansion are associated to the powers of r determined in **Case 1**, similar to example performed in Section 3.1.3.

Remark. It can be seen from the linearised equations of motion that the prefactors in Eqs. (5.29) and (5.30), related to the subleading powers derived in Eqs. (5.41) and (5.44), are independent of prefactors related with higher powers in r. In short this is due to fact that they have no contribution in the leading part of the equation and this only couples to even further subleading terms in the expansion.

One could next determine all the coefficients in between the leading order and the subleading order determined above. These coefficients should follow from substituting Eqs. (5.28) to (5.30) into the linearised equations of motion to obtain constraint equations for each individual power in r. For the specific case where $\theta = d - z$ a relevant set of constraints is given by[3]

$$\begin{aligned}
\omega^2 e^{\lambda_1 \phi_0} a_{1,x}^{(0)} - 2\rho_1 h^{(-2z)} &= 0, \\
-2h^{(-2z)} + \rho_1 a_{1,x}^{(-2z)} + \rho_2 a_{2,x}^{(0)} &= 0, \\
(2z-2) h^{(0)} + \rho_1 a_{1,x}^{(0)} &= 0, \\
(2z) e^{\lambda_1 \phi_0} a_{1,x}^{(0)} + \rho_1 h^{(0)} &= 0, \\
(2-2z) e^{\lambda_2 \phi_0} a_{2,x}^{(2-2z)} + \rho_2 h^{(0)} &= 0, \\
-2z h^{(2-4z)} + \rho_1 a_{1,x}^{(2-4z)} + \rho_2 a_{2,x}^{(2-2z)} &= 0.
\end{aligned} \tag{5.45}$$

It was found above that there are five free parameter in the near-boundary expansion (given below by the different colours, uncoloured means they depend on both $a_{2,x}^{(s)}$ and $h^{(0)}$)

[3]Note that in determining the constraints we used $f(r) = 1$, while this statement certainly hold for high orders at lower orders in the expansion the terms containing f will be relevant.

$$\begin{aligned} a_{1,x}(r) &= r^{d+z-\theta}\left(\mathbf{a_{1,x}^{(0)}} + a_{1,x}^{(\theta-d-z)} r^{\theta-d-z} + \cdots + \mathbf{a_{1,x}^{(2-d-3z+\theta)}} r^{2-d-3z+\theta} + \dots\right), \\ a_{2,x}(r) &= \mathbf{a_{2,x}^{(s)}} + a_{2,x}^{(2-2z)} r^{2-2z} + \cdots + \mathbf{a_{2,x}^{(v)}} r^{3z-\theta+d-4} + \dots, \\ h_t^x(r) &= r^{2z-2}\left(\mathbf{h^{(0)}} + \mathbf{h^{(2-2z)}} r^{2-2z} + \mathbf{h^{(-2z)}} r^{-2z} + \cdots + \mathbf{h^{(2-d-3z+\theta)}} r^{2-d-3z+\theta} + \dots\right). \end{aligned} \tag{5.46}$$

So in the above boundary expansion we have five free parameters, the holographic dictionary stated in Chapter 3 gives us that the sources and VEVs are probably given by the free integration constants of the theory. In the case of the field $a_{2,x}$ we are quite confident that $a_{2,x}^{(s)}$ is the source for conserved current fluctuations $\delta\langle J_2^x\rangle$ and $a_{2,x}^{(v)}$ is the vacuum expectation value of the conserved current fluctuations $\delta\langle J_2^x\rangle$. For h_t^x it not obvious to determine the right source and VEV from the holographic dictionary since we have three integration constants in the expansion. In first instance one would say that $h^{(0)}$ is the source corresponding to the $\delta\langle T_x^t\rangle$-component of the fluctuations in the energy-momentum tensor, since it is associated to the dominant power. Which integration constant in the expansion takes up the role of the VEV is not clear from the expansion alone. In the next section we take a look at the unrenormalised second order boundary action relevant for thermoelectric transport. Note that the fully renormalized boundary action determines the VEV, but the unrenormalized boundary action might give hints to determine the VEV, just as in Section 3.1.3.

Remark. Note that we obtained four degrees of freedom from the expansion in **Case 1**, the fifth degree of freedom is called $h^{(2-2z)}$ in the expansion above. This is a constant shift in the solution of h_t^x, it is a degree of freedom since only derivatives of h_t^x enter in the linearised equations of motion given in Eqs. (5.24) to (5.26).

Example 5.5.1. In the case $\theta = d - z$, $z = 3$ and $d = 2$ the following near-boundary expansion is found[4]

$$\begin{aligned} a_{1,x}(r) =& -\sqrt{3/2}e^{-\frac{1}{2}\sqrt{\frac{5}{3}}\phi_0} h^{(0)} r^6 - \sqrt{2/3}e^{\frac{1}{2}\sqrt{\frac{5}{3}}\phi_0} r_+^4 \mu a_{2,x}^{(s)} \\ & - (1/\sqrt{6})\left(e^{-\frac{1}{2}\sqrt{\frac{5}{3}}\phi_0} r_+^6 + (2/3)e^{\frac{1}{2}\sqrt{\frac{5}{3}}\phi_0} r_+^4\mu^2 - (1/12)e^{-\frac{1}{2}\sqrt{\frac{5}{3}}\phi_0}\omega^2\right) h^{(0)} \\ & + (1/\sqrt{6})\left(3e^{\frac{1}{2}\sqrt{-\frac{5}{3}}\phi_0} h^{(-10)} - 2e^{\frac{1}{2}\sqrt{\frac{5}{3}}\phi_0} r_+^8 \mu^2 h^{(0)}\right) r^{-4} + \dots, \\ a_{2,x}(r) =& a_{2,x}^{(s)} + r_+^4 \mu h^{(0)} r^{-4} + (1/12)\omega^2 h^{(0)} r^{-6} + a_{2,x}^{(v)} r^{-8} + \dots, \\ h_t^x(r) =& h^{(0)} r^4 + h^{(-4)} - (1/12)\left(12 r_+^6 + 8e^{\sqrt{\frac{5}{3}}\phi_0} r_+^4 \mu^2 + \omega^2\right) h^{(0)} + h^{(-10)} r^{-6} + \dots \quad . \end{aligned} \tag{5.47}$$

[4] The μ^2 terms in the above expansion are coming from the the subleading terms in f.

5.6 The unrenormalised boundary action

The holographic dictionary states that the partition function of the quantum critical field theory and the partition function of classical gravity are equal. Since the second-order perturbed action determined becomes a boundary action when evaluated on the classical solution, as stated in Section 5.1, we see that the correlation functions of the quantum field theory can be extracted form the second order boundary action by the GKPW rule. Thus the holographic correspondence states that the renormalized boundary action has the form

$$\mathcal{S}_{ren} = \int_{\partial M} \sum_I \delta\langle\mathcal{O}_I\rangle \delta\Phi^I.$$

Thus in order to identify the correct vacuum expectation values for the system, we thus need to consider the fully renormalised second order boundary action. Since determining the correct counterterms in the EMD backgrounds is quite involved and one usually goes to the Hamilton-Jacobi formalism to determine the counterterms [36–38], we in first instance look at finite part of the unrenormalized boundary action.
The boundary action relevant for determining thermoelectric transport is given by

$$\begin{aligned}\delta^2\mathcal{S}_{\partial M,\text{optical conductivity}} = \int d^{d+1}x\sqrt{-h}\Bigg\{ &-\frac{1}{2}n_\rho h^{\mu\nu}\nabla_\mu h^\rho_\nu + \frac{3}{4}n_\rho h^{\mu\nu}\nabla^\rho h_{\mu\nu} - n_\rho h^{\rho\nu}\nabla_\mu h^\mu_\nu \\ &+ \frac{1}{2}n_r\sum_i e^{\lambda_i\phi}g^{tt}g^{rr}a_{i,x}h^x_t F_{i,rt} + \frac{1}{2}n_r\sum_i g^{xx}g^{rr}e^{\lambda_i\phi}a_{i,x}f_{i,rx}\Bigg\}.\end{aligned} \tag{5.48}$$

In the EMD background solution, given in Eqs. (4.9) to (4.11), the above boundary action reduces to

$$\begin{aligned}\delta^2\mathcal{S}_{\partial M,\text{optical conductivity}} = \frac{1}{2R^\theta}\int d^{2+1}x\Bigg\{ &h^x_t r^{2+d-z-\theta}\left[r\frac{f'}{f}h^x_t + ((2\theta/d)+2z-4)h^x_t - 3r\partial_r h^x_t\right] \\ &- \sum_{i=1}^{2}\rho_i a_{i,x}h^x_t + e^{\lambda_i\phi}g^{xx}r^{\theta-z-2}a_{i,x}f_{i,rx}\Bigg\}.\end{aligned} \tag{5.49}$$

In terms of the integration parameters of UV expansion (5.46) the above second order boundary action becomes

$$\begin{aligned}\delta^2\mathcal{S}_{bdy} = \frac{1}{R^\theta}\int d^{d+1}x\Bigg\{ &\text{divergent terms} + \kappa_0 h^{(0)}h^{(0)} + \kappa_1 h^{(0)}h^{(2-d-3z+\theta)} + \kappa_2 h^{(2-2z)}h^{(0)} \\ &+ \gamma_1 e^{\lambda_i\phi_0}a^{(s)}_{2,x}a^{(v)}_{2,x} + \text{subleading terms}\Bigg\}.\end{aligned} \tag{5.50}$$

We indeed see that the identified source of current fluctuations $a^{(s)}_{2,x}$ couples to $a^{(v)}_{2,x}$, which may thus be interpreted as the VEV of the current fluctuations as expected. The interpretation of the VEV in the metric expansion is still not fully clear and one probably needs the fully renormalized boundary action to determine the correct expression for the VEV. Since $h^{(2-d-3z+\theta)}$ is the only integration constant that couples in a direct manner to the source $h^{(0)}$ we use this as example of VEV in the next sections.

Remark. The above boundary action is not renormalized, proper renormalization might change the precise form of the boundary action. Additionally in Appendix B we did not take into account second-order fluctuations of the Gibbons-Hawking-York boundary term, this in principle changes the boundary action too. In spite of these difficulties we may still use the above boundary action to check our initial guess for the sources and VEVs of the boundary theory are correct, since they should give finite contributions to the boundary action.

5.7 Critical scaling dimensions of the identified sources and VEVs in position space

As we have seen in Section 4.3 the physical observables of the EMD model have certain critical scaling dimensions. If the identifications of the sources and the VEVs in the previous sections are correct they must obey the scaling dimensions given in Section 4.3. In this section we will check the critical scaling dimensions of the integration constants given in expansion (5.46) and check whether they satisfy the critical scaling dimensions derived in Section 4.3. In the next chapter the critical scaling dimensions of physical sources will be determined, which forms a check of the identification of the sources.

To recap, in Section 4.3 amongst others the general scaling properties of electrical and heat currents were discussed under the scaling transformation (4.26) in the presence of an additional critical scaling exponent Φ, next to z and θ. It turns out that this additional constant is necessary to capture the right scaling behaviour of the studied EMD model. As argued in Section 4.3, it is most natural that this additional exponent Φ captures the anomalous scaling dimension in the charge density operator (4.30)

$$[n] = d - \theta + \Phi.$$

It will be seen in the next chapter (from the conductivity) that in the studied EMD background the value of the critical exponent Φ must be given by

$$\Phi = \frac{\theta}{d} + \frac{\lambda_2 \beta}{2}. \tag{5.51}$$

This additional anomalous scaling factor Φ stems from the fact that parameter $e^{\lambda_2 \phi}$ is not scale invariant but actually scales according to

$$e^{\lambda_2 \phi_0} \to \lambda^{\lambda_2 \beta} e^{\lambda_2 \phi_0},$$

under the scaling transformation[5]

$$\eta : \; \{t, \mathbf{x}, r\} \to \{\lambda^z t, \lambda x, r/\lambda\}. \tag{5.52}$$

[5]In principle we could also rescale in momentum space, which is defined by a Fourier transform. The two scaling dimension can easily be related by a Fourier transformation.

As was discussed in Section 4.1 this scaling must additionally be combined with scaling of the parameter

$$R \to \lambda R,$$

to leave the field theory invariant.
Let us start by determining the critical scaling dimensions of the source and VEV of current fluctuations $\delta\langle J_2^x\rangle$ as determined by the asymptotic expansion of $a_{2,x}$. As we noted before, the holographic dictionary gives that the Maxwell field A_2 must be dual to a conserved current $\langle J_2^x\rangle$ in the boundary field theory. Here we check the scaling dimensions of the source and VEV of the current $\delta\langle J_2^x\rangle$, as determined by the asymptotic expansion of $a_{2,x}$. As mentioned in the beginning of this section, if the identification of the VEV was correct it's critical scaling dimension should be the same as the scaling dimension of the electrical current given in Section 4.3.
Using the scaling properties of the gauge field derived in Section 4.3 it is noted that the x-component of the gauge field A_2 transform like

$$\begin{aligned} a_{2,x}(\eta x) &\to \lambda^{-1+\Phi} a_{2,x}(\eta^{-1}x), \\ a_2^x(\eta x) &\to \lambda^{1-\Phi} a_2^x(\eta^{-1}x). \end{aligned} \tag{5.53}$$

The scaling of the source $a_{2,x}^{(s)}$ in expansion (5.46) is therefore

$$a_{2,x}^{(s)}(x) \to \lambda^{\Phi-1} a_{2,x}^{(s)}(\eta^{-1}x). \tag{5.54}$$

We thus see that the critical scaling dimension of the source $a_{2,x}^{(s)}$ is given by

$$[a_{2,x}^{(s)}] = 1 - \Phi. \tag{5.55}$$

This is precisely the scaling dimension of an electrical field as derived in Section 4.3. Furthermore under scale transformation Eq. (5.52), the VEV $a_{2,x}^{(v)}$ transforms like

$$a_{2,x}^{(v)}(x) \to \lambda^{1+\theta-z-d-\Phi} a_{2,x}^{(v)}(\eta^{-1}x). \tag{5.56}$$

It is thus noted that the VEV $a_{2,x}^{(v)}$ has scaling dimension

$$[a_{2,x}^{(v)}] = d + z + \Phi - \theta - 1, \tag{5.57}$$

This is exactly the scaling dimension of an electrical current as found in Eq. (4.32), which is what we expected from the interpretation as a charge current $\delta\langle J_2^x\rangle$.

Example 5.7.1. The critical scaling in Eq. (5.56) was determined from the known scaling of a gauge field and the scaling of the radial component r (see Section 3.1.3 for an additional example)

$$\begin{aligned} \tilde{a}_{2,x}^{(v)}(\tilde{x})\tilde{r}^{\theta-z-d+2+\frac{2\theta}{d}-\lambda_2\beta} &= \lambda^{-1+\Phi} a_{2,x}^{(v)}(x) r^{\theta-z-d+2+\frac{2\theta}{d}-\lambda_2\beta} \\ &= \lambda^{-1+\Phi} a_{2,x}^{(v)}(x) r^{\theta-z-d+2-2\Phi} \\ &= \lambda^{-1+\Phi+\theta-z-d+2-2\Phi} a_{2,x}^{(v)}(\eta^{-1}\tilde{x})\tilde{r}^{\theta-z-d+2-2\Phi} \\ &= \lambda^{\theta-z-d+1-\Phi} a_{2,x}^{(v)}(\eta^{-1}\tilde{x})\tilde{r}^{\theta-z-d+2-2\Phi}, \\ \implies a_{2,x}^{(v)}(x) &\to \lambda^{1+\theta-z-d-\Phi} a_{2,x}^{(v)}(\eta^{-1}x). \end{aligned} \tag{5.58}$$

Next we determine the critical scaling dimensions of the source and VEV of fluctuations in the tx-component of the energy-momentum tensor $\delta\langle T^t_x\rangle$ as determined by the asymptotic expansion of h^x_t. The source of energy-momentum fluctuations $\delta\langle T^t_x\rangle$ in the metric expansion (5.46) is most probably given by the leading term $h^{(0)}$ in the expansion, as discussed in the previous section. In principle the VEV is everything that couples to the source in the renormalized boundary action. In this case only $h_t^{x(2-d-3z+\theta)}(x)$ is considered since it couples directly to the source in boundary action (5.49). It is fine to consider only the scaling of $h_t^{x(2-d-3z+\theta)}(x)$ since other terms that couple to $h^{(0)}$ in the boundary should have the same critical scaling behaviour in order to have a well defined boundary action.
Under scaling transformation (5.52) the metric should scale like

$$h^x_t(x) \to \lambda^{1-z} h^x_t(\eta^{-1}x), \tag{5.59}$$

Under the above scaling transformation the source $h^{(0)}$ thus has to scale as (see e.g. Example 5.7.1)

$$h_t^{x(0)}(x) \to \lambda^{z-1} h_t^{x(0)}(\eta^{-1}x). \tag{5.60}$$

Thus the critical scaling dimension of the source $h^{(0)}$, of fluctuations in the tx-component of the energy momentum tensor $\delta\langle T^t_x\rangle$, is accordingly given by

$$[h^{(0)}] = 1 - z. \tag{5.61}$$

Like the source of current fluctuations $\delta\langle J^x_2\rangle$, the interpretation of the scaling dimension of $h^{(0)}$ will become clear in the next section. Next under the scale transformation in (Eq. (5.52)) the associated VEV scales like

$$h_t^{x(2-d-3z+\theta)}(x) \to \lambda^{1-2z+\theta-d} h_t^{x(2-d-3z+\theta)}(x). \tag{5.62}$$

We thus see that the critical scaling dimension of the VEV $\delta\langle T^t_x\rangle$ must be given by

$$[h_t^{x(2-d-3z+\theta)}] = d + 2z - \theta - 1. \tag{5.63}$$

This is exactly the critical scaling dimension of the heat current derived in Eq. (4.32), which was to be expected since we need $\delta\langle T^t_x\rangle$ to have the same critical scaling dimension as the heat current[6].
We have thus seen that the identified VEVs in the asymptotic expansion (5.46) exactly scale as was expected by the scaling analysis performed in Section 4.3.

Check: invariance of boundary action under scale transformations

If the scaling dimensions of the sources and the VEVs determined above are correct than the second order boundary action should be invariant under the scale transformation (5.52). In this section we check the scaling of the zeroth order part of the second order boundary action. Let is first discuss the part of the boundary action in Eq. (5.50) which only depends on the gauge field $a_{2,x}$. In the boundary action the source and the VEV must couple like

$$\int d^{d+1}x \; a^{(s)}_{2,\mu}(x) a_2^{(v)\mu}(x). \tag{5.64}$$

[6]This can be seen from the definition of the heat current which is defined as $J^{Q,i} \equiv T^{ti} - \mu J^i$. Where T^{ti} is the ti-component of the energy-momentum tensor, μ the chemical potential and J^i is the i-component of the electrical current.

Since we know the scaling of the gauge field in the bulk we are able to extract the conformal scaling behaviour on the field theory side. The scaling of the boundary action is thus given by

$$\begin{aligned}\mathcal{S'}_{bdy} &= \frac{1}{R'^{\theta}}\int d^{d+1}x'\ e^{\lambda_2\phi_0'}a_x'^{(s)}(x')a_2'^{x(v)}(x')\\ &= \frac{1}{\lambda^{\theta}R^{\theta}}\int d^{d+1}(\lambda^{z+d}x)\ \lambda^{\theta-z-d-\lambda_2\beta}\lambda^{\lambda_2\beta}e^{\lambda_2\phi_0'}a_x^{(s)}(x)a_2^{x(v)}(x),\\ &= \int d^{d+1}x\ a_x^{(s)}(x)a_2^{x(v)}(x) = \mathcal{S}_{bdy}.\end{aligned} \tag{5.65}$$

Thus the boundary is scale invariant under the above conformal transformation, which is precisely what was expected.

Next we consider the scaling behaviour of the second order boundary action while considering only the metric field h_t^x. Similar to the previous discussion for the scaling of the identified VEV in the asymptotic expansion of h_t^x, we will only check the scaling of the boundary metric for the term $h^{(0)}h^{(2-d-3z+\theta)}$. This is due to the fact that the other terms in the boundary action should scale in precisely the same way. In the boundary theory the source and VEV of the metric should couple like

$$\int d^{d+1}xh_{xt}^{(s)}(x)h^{(v)xt}(x). \tag{5.66}$$

The scaling of the boundary action is thus given by

$$\begin{aligned}\mathcal{S'}_{bdy} &= \frac{1}{R'^{\theta}}\int d^{d+1}x'h_{xt}'^{(s)}(x')h'^{(v)xt}(x') = \frac{1}{R^{\theta}}\frac{1}{\lambda^{\theta}}\int d^{d+1}(\lambda^{z+d}x)h_{xt}^{(s)}(x)\lambda^{-2z}h^{(v)xt}(x),\\ &= \frac{1}{R^{\theta}}\frac{1}{\lambda^{\theta}}\lambda^{d-z}\int d^{d+1}xh_{xt}^{(s)}(x)h^{(v)xt}(x), = \frac{1}{R^{\theta}}\frac{\lambda^{\theta}}{\lambda^{\theta}}\int d^{d+1}xh_{xt}^{(s)}(x)h^{(v)xt}(x)\\ &= \frac{1}{R^{\theta}}\int d^{d+1}xh_{xt}^{(s)}(x)h^{(v)xt}(x) = \mathcal{S}_{bdy}.\end{aligned} \tag{5.67}$$

Again the boundary action is scale invariant as expected.

So we find that the zeroth order boundary action is invariant under rescaling (5.52) as expected.

In this section we looked at the critical scaling dimensions of the identified sources and VEVs in the asymptotic expansion found in Section 5.5. We found that the identified VEVs scale as was expected from general scaling argument given in Section 4.3. We found that $a_{2,x}^{(v)}$ has the scaling dimension of an electrical current, which strengthens the interpretation of $a_{2,x}^{(v)}$ as the VEV $\delta\langle J_2^x\rangle$. We did not find a closed expression for the VEV $\delta\langle T_x^t\rangle$, since the three terms $\kappa_0 h^{(0)}h^{(0)}$, $\kappa_1 h^{(0)}h^{(2-d-3z+\theta)}$ and $\kappa_2 h^{(2-2z)}h^{(0)}$ enter in the second order boundary action and renormalization might change the relative factor between these three constants of integration $h^{(0)}, h^{(2-d-3z+\theta)}$ and $h^{(2-2z)}$ in the boundary action, which act as the VEV$\delta\langle T_x^t\rangle$. So probably the renormalised boundary action is needed to determine the precise VEV $\delta\langle T_x^t\rangle$. We did find that $h_t^{x(2-d-3z+\theta)}$ has the same critical scaling dimension as the heat current, which is to be expected from the VEV $\delta\langle T_x^t\rangle$. This means that the identification of $h^{(0)}$ as the source probably is correct.

Remark. Note that scale transformation (5.52) also means $R\to\lambda R$ and $e^{\lambda_2\phi_0}\to\lambda^{\lambda_2\beta}e^{\lambda_2\phi_0}$.

5.8 Scaling of Green's function

At the start of this chapter we described that to capture the dynamics of the quantum critical field theory one needs to be able to determine the Green's functions of the field theory. In Section 5.1 it was also explained how to calculate these Green's functions in a holographic context. In this section we take a look at the scaling behaviour of the current-current correlation function $G^{xx}(\omega) = \langle \delta J_2^x \delta J_2^x \rangle = \delta \langle J_2^x \rangle / \delta a_{2,x}$. We treat this Green's function, since this (as will be seen later) determines the conductivity of the material.
One is able to obtain the scaling of the various parts of the Green's function, by considering the scaling of the associated source and VEV. Here as already mentioned the scaling of $G^{xx}(\omega)$ is considered. The scaling behaviour of the different sources and VEVs in position space were given in Section 5.7, this is related to the scaling behaviour in momentum space by a Fourier transformation

$$\tilde{a}(\tilde{k}) = \int d^{d+1}\tilde{x}\ \tilde{a}(\tilde{x}) e^{-i\tilde{k}\tilde{x}} = \int d^{d+1}(\lambda^{d+z}x)\ \lambda^{-\Delta} a(x) e^{-ikx} = \lambda^{d+z-\Delta} a(k). \tag{5.68}$$

The behaviour of the Green's function $G^{xx}(\omega)$ under scale transformation Eq. (5.52) is therefore given by

$$\tilde{G}^{\tilde{x}\tilde{x}}(\tilde{\omega}) \sim \frac{\tilde{a}^{(v)}_{\tilde{x}}(\tilde{\omega})}{\tilde{a}^{(s)}_{\tilde{x}}(\tilde{\omega})} \sim \frac{\lambda^{1+\theta-z-d-\Phi}}{\lambda^{-1+\Phi}} \frac{a^{(v)}_{2,x}(\omega)}{a^{(s)}_{2,x}(\omega)} \sim \lambda^{2+\theta-z-d-2\Phi} G^{xx}(\omega), \tag{5.69}$$

with $\Phi = \frac{\theta}{d} + \frac{\lambda_2 \beta}{2}$. It is thus found that for large frequencies G^{xx} asymptotically scales as

$$G^{xx} \propto \omega^{\frac{d+z-\theta-2+2\Phi}{z}}. \tag{5.70}$$

This scaling will be used to describe the asymptotic behaviour of the optical conductivity in the next section. The scaling dimensions of other components of the Green's function can be obtained using a similar procedure.

5.9 Summary

This was quite a technical chapter. We have seen in the previous chapter the black-hole solution of the considered EMD model gives the thermodynamic equilibrium properties of the model. In order to extract the dynamical properties of the EMD model one has to introduce dynamical fluctuations on top of the equilibrium background solution. We started this chapter with a general discussion on how to introduce dynamical fluctuations into the system and how to extract the correlations functions (Green's functions) from the introduced fluctuations. In Section 5.3 we discussed the equations of motion for these dynamical fluctuations, of which the ones relevant for thermoelectric transport were discussed in Section 5.4. We continued by determining the near boundary asymptotic expansion for the relevant set of equations in Section 5.5. The asymptotic expansion is important since by te holographic dictionary stated in Chapter 3, the near-boundary expansion determines the sources and the VEVs of the field theory in terms integration constants in the asymptotic expansion. In the remaining part of this chapter we gave the unrenormalised boundary action and looked at the scaling behaviour of the identified sources and VEVs in Section 5.5. It turned out that the sources and VEVs had the same scaling behaviour as was derived in Section 4.3. In the end we could properly identify the source and VEV for the electrical current, while to determine the source and VEV for heat transport one probably needs the renormalized boundary action. Lastly we consider the behaviour of the current-current correlation function under scale transformation, from which we could deduce the asymptotic behaviour of the Green's function. This will be useful in the next section where we look at te optical conductivity of the studied EMD model.

Chapter 6

Thermoelectric Transport of the EMD model

As stated previously we are trying to use the Einstein-Maxwell-dilaton described in Chapter 4 as a model for the strongly correlated strange metallic phase. In Chapter 4 we looked at the equilibrium properties of the model, such as the entropy, temperature and the DC resistivity of the metal. Additionally it had been discussed that, in order for the model to have the famous linear resistivity w.r.t. temperature of the strange metallic phase, we needed to restrict the possible choices of the critical exponents z and θ . The previous chapter was spend on setting up the necessities to start determining the dynamical properties of the EMD model. The goal of this section is to use the machinery that was setup in the previous chapter to determine the thermoelectric transport properties of the studied EMD model. In specific we consider the AC conductivity of the EMD model.
This chapter is organized in the following way, we start with a brief general discussion of thermoelectric transport in metals. Next in Section 6.2 we discuss how to incorporate an electrical field and a temperature gradient into the holographic description of the strongly coupled field theory. Finally in Section 6.3 the conductivity of the studied EMD model is discussed, we will see that the imaginary part of the conductivity behaves unexpected and might make this model unsuitable for describing the strange metallic phase.

6.1 Thermoelectric transport

Usually, in in condensed-matter systems, electric and thermal transport couple together. We are interested in response of the electrical current and the heat current to an applied external electric field E and a temperature gradient $-\nabla T$. In general the thermoelectric conductivities are given by the following relation

$$\begin{pmatrix} J^i \\ J^{Q,i} \end{pmatrix} = \begin{pmatrix} \sigma_{ij} & T\alpha_{ij} \\ T\bar{\alpha}_{ij} & T\kappa_{ij} \end{pmatrix} \begin{pmatrix} E_j \\ -\frac{\nabla T}{T} \end{pmatrix} . \tag{6.1}$$

In the above J^Q is the heat current

$$J^{Q,i} \equiv T^{0i} - \mu J^i . \tag{6.2}$$

The coefficients in matrix (6.1) are thus defined as the responses to an external electric field and temperature gradients. One directly sees that σ gives the conductivity of the material since this is given by the response of the electrical current to an electrical field[1]. The resistivity of the material is defined as the the inverse of the conductivity, thus $\rho = \sigma^{-1}$. Response matrix (6.1) gives rise to various thermoelectric transport coefficients used in condensed matter literature. Let us give a few example of transport coefficients one could extract using the response matrix (6.1). The Seebeck coefficient for example is the ratio of the voltage drop to the temperature drop

$$S_{ij} = (\sigma^{-1})_{ik}\alpha_{kj}. \tag{6.3}$$

The thermal conductivity governs the transport of heat in the absence of electrical currents and is therefore given by

$$\bar{\kappa}_{ij} = \kappa_{ij} - T\bar{\alpha}_{ik}(\sigma^{-1})_{kl}\alpha_{lj}. \tag{6.4}$$

Since our setup is isotropic in both spatial dimensions, all the conductivity coefficients are diagonal and it is thus sufficient to consider a single direction to determine them.
With this at hand we can now start to look at determining these transport coefficients in a holographic setting and hopefully start describing the strongly correlated strange metallic phase using the studied EMD model.

Remark. Practically, one would not directly determine matrix (6.1) in holography using the method described in Section 5.2, but rather first determine

$$\begin{pmatrix} J^i \\ T^{0i} \end{pmatrix} = \begin{pmatrix} \sigma_{ij} & T\alpha_{ij} \\ T\bar{\alpha}_{ij} & T\kappa_{ij} \end{pmatrix} \begin{pmatrix} E_j \\ -\frac{\nabla T}{T} \end{pmatrix} \tag{6.5}$$

and compute the transport matrix (6.1) from here.

6.2 Adding an electrical field and temperature gradient in holographic models

Above we defined the responses of the electrical and the heat current w.r.t. to an external electric field and a temperature gradient. To be able to determine these responses for the strongly coupled quantum field theory described by the EMD background, we must be able to include an electric field and and temperature gradient in the holographic calculations. In this section we discuss how to describe an external electric field δE and a temperature gradient $-(\nabla T)/T$ in holographic calculations at zero momentum. We will describe a uniform electric field δE and a uniform temperature gradient $-(\nabla T)/T$, which are frequency dependent.
Introducing an electrical field into the holographic model is done by imposing it through an external gauge field. Recall that the x-component of an electric field is given by $\delta E_x = \delta F_{2,xt} = ik_x\delta A_{2,t} + i\omega\delta A_{2,x}$. We thus see that in the zero momentum limit ($k_x \to 0$) an electrical may be imposed through an external gauge field by

$$\delta A_2 = \frac{1}{i\omega}\delta E_x. \tag{6.6}$$

[1]We must emphasize here that electric or charged means nonzero density in this case, since we did not include Coulomb interactions into the holographic picture.

Introducing a temperature gradient into our system is a bit more involved see e.g. Ref. [10, Section 5.3] which will be followed here. A nice way to introduce a temperature gradient to our system is to Wick transform to Euclidean time and consider the rescaled coordinate $\tilde{t}$

$$t = \frac{\tilde{t}}{T}, \tag{6.7}$$

where $\tilde{t}$ has periodicity $\tilde{t} \sim \tilde{t} + 1$. In the new coordinate the time component of the metric becomes

$$g_{\tilde{t}\tilde{t}} = \frac{1}{T^2} g_{tt}. \tag{6.8}$$

Imposing a temperature gradient $T \to T + x^i \nabla_i T$ we find

$$g_{\tilde{t}\tilde{t}} = \frac{1}{T^2}\left(1 - \frac{2x^i \nabla_i T}{T} + \ldots\right) g_{tt}. \tag{6.9}$$

A constant temperature gradient thus gives

$$\delta g_{\tilde{t}\tilde{t}} = -\frac{2e^{-i\tilde{\omega}\tilde{t}}}{T^2}\frac{x^i \nabla_i T}{T} g_{tt}, \tag{6.10}$$

with $\tilde{\omega} = \omega/T$. Next we perform a coordinate transformation

$$\tilde{t} \to \tilde{t} + \xi^{\tilde{t}}, \quad \xi^{\tilde{t}} = i x^i \frac{\nabla_i T}{T}\frac{e^{-i\tilde{\omega}\tilde{t}}}{\tilde{\omega}}. \tag{6.11}$$

Where the effect of the coordinate transformation is given by

$$\begin{aligned} a_\mu &\to a_\mu - \mathcal{L}_\xi A_\mu = a_\mu - \xi^\nu \partial_\nu A_\mu - (\partial_\mu \xi^\nu) A_\nu \\ &= a_\mu - \xi^\nu \nabla_\nu A_\mu - (\nabla_\mu \xi^\nu) A_\nu, \\ h_{\mu\nu} &\to h_{\mu\nu} - \mathcal{L}_\xi g_{\mu\nu} = h_{\mu\nu} - \xi^\nu \partial_\nu g_{\mu\nu} - (\partial_\mu \xi^\lambda) g_{\lambda\nu} - (\partial_\nu \xi^\lambda) g_{\lambda\mu} \\ &= h_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu. \end{aligned} \tag{6.12}$$

Using Eq. (6.12) it is seen that the change in coordinates (6.11) leads to

$$\delta g_{\tilde{t}\tilde{t}} = 0, \tag{6.13}$$

$$\delta g_{\tilde{t}x} = -(\partial_x \xi^{\tilde{t}}) g_{\tilde{t}\tilde{t}} = i\frac{\nabla_x T}{T}\frac{e^{-i\tilde{\omega}\tilde{t}}}{\tilde{\omega}} g_{\tilde{t}\tilde{t}} = i\frac{\nabla_x T}{T}\frac{e^{-i\tilde{\omega}\tilde{t}}}{\tilde{\omega}T^2} g_{tt}, \tag{6.14}$$

$$\delta A_{i,x} = i\frac{\nabla_x T}{T}\frac{e^{-i\tilde{\omega}\tilde{t}}}{\tilde{\omega}} A_{\tilde{t}} = i\frac{\nabla_x T}{T}\frac{e^{-i\tilde{\omega}\tilde{t}}}{T\tilde{\omega}} A_t. \tag{6.15}$$

Transforming back to our normal time coordinates one obtains

$$\delta g_{tx} = -\frac{\nabla_x T}{T}\frac{e^{-i\omega t}}{i\omega} g_{tt}, \tag{6.16}$$

$$\delta A_{i,x} = -\frac{\nabla_x T}{T}\frac{e^{-i\omega t}}{i\omega} A_t. \tag{6.17}$$

On the boundary of the hyperscaling violating gravitational theory δg_{xt} and δA_x assume the following form[2]

$$\delta g_t^x(r=\infty) = -r^{2z-2}\frac{\nabla_x T}{T}\frac{e^{-i\omega t}}{i\omega}, \tag{6.18}$$

$$\delta A_{2,x}(r=\infty) = -\frac{\nabla_x T}{T}\frac{e^{-i\omega t}}{i\omega}\mu, \tag{6.19}$$

$$\delta A_{1,x}(r=\infty) = -\frac{\nabla_x T}{T}\frac{e^{-i\omega t}}{i\omega}A_{1,t}(t). \tag{6.20}$$

It is thus seen that an electrical field may be sourced through the integration constant $a_{2,x}^{(s)}$ and a temperature gradient is sourced through integration constants $h^{(0)}$ and $a_{2,x}^{(s)}$.

Remark. In principle one could argue that we did not take into account Eq. (6.20). In fact we did since the asymptotic expansion of a_1 contains no free integration constants, this equation just gives that $a_{1,x}^{(0)} \sim h^{(0)}$ sources a temperature gradient. This what we already stated above.

Check of the identification of the sources and VEVs using critical scaling dimensions

If the equations Eqs. (6.18) and (6.19) are correct than the critical scaling dimensions of the integration constants at the orders in the asymptotic expansion suggested by Eqs. (6.18) and (6.19) must have the same scaling dimensions as $\frac{\nabla_x T}{T}\frac{e^{-i\omega t}}{i\omega}$. Let's check that here. From Eq. (4.27) it is seen that $\frac{\nabla_x T}{T}\frac{e^{-i\omega t}}{i\omega}$ has the following scaling units

$$\left[\frac{\nabla_x T}{T}\frac{e^{-i\omega t}}{i\omega}\right] = 1 - z. \tag{6.22}$$

It was seen in Section 5.4 that the source $h^{(0)}$ in metric expansion (5.46) is precisely the coefficient corresponding to the power r^{2z-2}. In Section 5.7 it was seen that the critical scaling dimension of $h^{(0)}$ is given by

$$[h^{(0)}] = 1 - z. \tag{6.23}$$

Which is the same scaling dimension as that of $\frac{\nabla_x T}{T}\frac{e^{-i\omega t}}{i\omega}$ as expected.

For the gauge field A_2 the source $a_{2,x}^{(s)}$ in expansion (5.46) has scaling dimension

$$[a_{2,x}^{(s)}] = 1 - \Phi. \tag{6.24}$$

The scaling dimension of $\frac{\nabla_x T}{T}\frac{e^{-i\omega t}}{i\omega}\mu$ follows from Eq. (4.33)

$$\left[\frac{\nabla_x T}{T}\frac{e^{-i\omega t}}{i\omega}\mu\right] = 1 - \Phi. \tag{6.25}$$

[2] If we perturb the boundary action with Eqs. (6.6) and (6.18) to (6.20) we find

$$\delta\mathcal{S} = \int_{\partial M} d^{d+1}x\left\{T^{\mu\nu}\delta g_{\mu\nu} + \sum_{i=1}^{2} J_i^\mu \delta A_{i,\mu}\right\} = \int_{\partial M} d^{d+1}x\left\{\frac{J^{Q,x}}{i\omega}\frac{-\nabla T}{T} + \frac{J_2^x}{i\omega}\delta E_x\right\}, \tag{6.21}$$

where $J^{Q,x} = T^{tx} - \mu_2 J_2^x - \mu_1 J_1^x$.

This precisely coincides with the scaling dimension of the source $a^{(s)}_{2,x}$.
So it seems that the powers of r in Eqs. (6.18) and (6.19) are correct since they lead to the correct scaling behaviour.

6.3 Optical conductivity

The optical conductivity of a system is the linear response to a small external electric field

$$J^i = \sigma^{ij} E_j, \tag{6.26}$$

where J is the electrical current and E an electric field. To determine the optical conductivity in holography we need to know the response of the current J_x with respect to an external electric field. As we have seen in the previous section an electrical field is imposed through an external gauge field $\delta A_2 = (1/i\omega)\delta E_x$. In linear response the response of the current J^x_2 expectation value to a small perturbation is given by

$$\delta \langle J^x \rangle = G^{xx} \delta a_x. \tag{6.27}$$

Using that $\delta E_x = i\omega \delta A_x$ in the zero momentum limit ($k \to 0$), one finds

$$\delta \langle J^x \rangle = \frac{G^{xx}}{i\omega} \delta E_x. \tag{6.28}$$

It is thus found that the frequency dependent optical conductivity is given by

$$\sigma(\omega) = \frac{G^{xx}}{i\omega}. \tag{6.29}$$

Since the operator $\delta\langle J^x_2\rangle$ does not decouple from for instance the xt-component of the energy-momentum tensor $\delta\langle T^{xt}\rangle$, we need to solve a matrix problem as discussed in Section 5.2, in order to determine the optical conductivity of the studied EMD model. In Section 5.2 it was explained how one could obtain the matrix Green's function (6.5) by determining the sources $\{a^{(s)}_{2,x}, h^{(0)}\}$ and VEV's $\{a^{(v)}_{2,x}, h^{(2-d-3z+\theta)}\}$[3]in the near boundary expansion for infalling boundary conditions for several initial conditions. The Green's function matrix is therefore determined by

$$\begin{pmatrix} a^{(s),1}_{2,x} & h^{(s),1} \\ a^{(s),2}_{2,x} & h^{(s),2} \end{pmatrix} \begin{pmatrix} G^{xx} & G^{xh} \\ G^{hx} & G^{hh} \end{pmatrix} = \begin{pmatrix} a^{(v),1}_{2,x} & h^{(v),1} \\ a^{(v),2}_{2,x} & h^{(v),2} \end{pmatrix}, \tag{6.30}$$

where [1] and [2] on the different sources and VEV's stand for the linearly independent solutions of the linearised equations of motion. We obtained the above Green's function by solving the linearised equations of motion in Eqs. (5.24) to (5.26) numerically for infalling boundary conditions and extracted the integration constants from the asymptotic behaviour for two linearly independent initial conditions.
The real and imaginary part of the optical conductivity for several temperatures in the case[4]$\theta = d - z$, $z = 3$ and $d = 2$ are shown in Figs. 6.1 and 6.2. Where as explained

[3]One could argue that $h^{(2-d-3z+\theta)}$ is not the full vacuum expectation value $\delta\langle T^{xt}\rangle$ and we're thus not determining the matrix Green's function given in Eq. (6.5). That is correct and the reason why we are only focussing on the optical conductivity here. We are determining the linear response of $h^{(2-d-3z+\theta)}$ here though, which gives in important contribution to the xt-component of the energy-momentum tensor $\delta\langle T_{xt}\rangle$.

around Eq. (6.29) the AC optical conductivity was obtained from Green's function (6.30) by $\sigma(\omega) = G^{xx}/i\omega$.
The real part of the conductivity is the part that is most directly measured, it may be interpreted as the inverse resistivity which quantifies how strongly a material resists or conducts electric flow at a given frequency. The imaginary part of the conductivity describes the response of the current by an applied electric field and is thus the reactive part.
Since the conductivity is a retarded Green's function (6.29) and thus analytic in the upper half plane, the real and imaginary part of the conductivity should thus related by the Kramer-Kroning relations[5]

$$\mathrm{Re}[G^R(\omega)] = \frac{1}{\mathcal{P}} \int_{-\infty}^{\infty} \frac{\mathrm{Im}[G^R(\omega')]}{\omega' - \omega}, \tag{6.31}$$

$$\mathrm{Im}[G^R(\omega)] = -\frac{1}{\mathcal{P}} \int_{-\infty}^{\infty} \frac{\mathrm{Re}[G^R(\omega')]}{\omega' - \omega}, \tag{6.32}$$

where $\mathcal{P}$ denotes the Cauchy principle value.
Let us first discuss the expected asymptotic behaviour of the conductivity. Note that, the frequency dependence of the Green's function $G^{xx}(\omega)$ was determined in Section 5.8. Using the expression of the conductivity in term of the Green's function (6.29) one finds that the optical conductivity asymptotically behaves like

$$\sigma(\omega) \sim \omega^{\frac{d-\theta-2+2\Phi}{z}},$$

for large frequencies. This behaviour is indeed asymptotically seen in the numerically determined optical conductivities.
Let's start by discussing the real part of the conductivity in Fig. 6.1, for large temperatures $T \gg \mu$ we see that the presence of the chemical potential is not seen and the conductivities quickly converge to their expected asymptotic scaling. Note that although it may seem that the real part of all the conductivities in Fig. 6.1 are quite featureless, zooming in on the low temperature solution (so where $T \ll \mu$) in Fig. 6.3 shows that the conductivity first decreases before it increases. This is most likely a feature of the chemical potential.
The most surprising figure is the figure for the imaginary part of the conductivity given in Fig. 6.2, one would expect the imaginary part of the conductivity to behave like $1/\omega$ for small values of ω, since this is what is expected in systems where there is momentum conservation. Normally the $1/\omega$ dependence of the imaginary part of the conductivity gives rise to a delta peak at zero frequency in the real part of the conductivity, which is demanded by the Kramers-Kronig relations. Normally this delta-peak in the real part of the conductivity broadens when translational invariance is broken, by for instance a lattice [40], and becomes

[4] These values of z and θ were chosen since they give rise to linear scaling of the resistivity with temperature as discussed in Section 4.1.1. The dimension of $d = 2$ was chosen, since the metallic phase mostly occurs as the metallic phase in cuprate superconductors, which are effectively two dimensional. Additionally we chose $z = 3$ and $\theta = -1$, since this number was already large enough to see non-trivial behaviour due to the critical exponents z and θ, while still small enough to make numerical predictions.

[5] A condition for the above Kramers-Kronig is that the Green's function vanishes for $|\omega| \to \infty$. This is not the case for our Green's function and we need to introduce a modified Green's function δG^R that does satisfy the condition. In Ref. [39] e.g. $\delta G^R(\omega) \equiv G^R(\omega, T) - G^R(\omega, 0)$ was considered. For our purposes something it is probably more useful to subtract the asymptotic behaviour of the Green's function, since we're only interested in using this relation for small frequencies here.

similar to a Drude-peak. This this Drude peak like initial decay of the real part of the conductivity with the frequency $\sim \omega^{-2/3}$ is observed in strange metals [41]. Since the imaginary part of the conductivity in Fig. 6.4 does not have the typical $1/\omega$, it's likely that the real part of the conductivity does not have a delta-peak at zero momentum. As discussed, since a Drude-like peak is observed in strange metals, it is important to understand why there is no delta peak in the studied model.
In the next section we like to give a short discussion of why the delta peak is absent in this particular EMD model, where we note that this is still work in progress so the arguments not complete yet.

Remark. It turned out that numerically determining the values of the sources and VEV's for large values of z and $-\theta$ was quite a challenge. The reason for this is that for large z and large negative values of θ (see Section 4.1.1) the powers of r in the near boundary expansion (5.46) associated to the VEV's in the quantum critical field theory become increasingly large. For instance to determine the VEV's for the case $z = 3,\ \theta = -1$ and $d = 2$, one already needs to obtain correct fit up to eighth order in r w.r.t. the numerically obtained solution for the linearised equations of motion. Furthermore this order keeps increasing in steps of four as one keeps increasing the value of z and accordingly θ. In the end we managed to determine the Green's function for the case where $z = 3,\ \theta = -1$ and $d = 2$, which already turned out to be quite non-trivial. If one likes to go to even larger values of z and $-\theta$, the numerical precision most probably needs to be improved.

6.3.1 A hydrodynamic discussion of the absence of the delta peak

This discussion is still work in progress, which means that the arguments given are not complete yet. We have just discussed the absence of the $1/\omega$ behaviour in Fig. 6.2 and how this likely leads to the absence of an delta peak at zero frequency in the real part of the conductivity. Let us first discuss what actually happens if the gauge field A_1 is switched off. This can easily be done by setting a_1 to zero in the linearised equations of motion, but it is an incorrect procedure since it does not take into account the back reaction of the field a_1 on the metric and the gauge field a_2. Doing so anyway gives a conductivity of which the imaginary part is shown in Fig. 6.4. We thus see the $1/\omega$ behaviour is present, giving rise to a delta peak at zero frequency in the real part of the conductivity. So, we can infer that back-reaction of the gauge field a_1, on the metric and the gauge field a_2, kills the delta peak at zero frequency. In first instance one would think the absence of the delta peak means that momentum conservation is broken. This is not the case since the system under consideration is still translational invariant, from Section 3.3 and Appendix A.2 one finds that due to translational invariance the following relation must hold

$$\partial_\mu T^\mu_\nu + \sum_{i=1}^{2} F_{i,\mu\nu} J^\mu_i = 0. \tag{6.33}$$

Specifically for the hyperscaling violating background Eqs. (4.9) to (4.11) the above equation up to first order in the zero momentum limit ($k \to 0$) becomes

$$\partial_t T^t_x = \delta E_x \rho_2 + i\omega \delta a_{1,x} \rho_1. \tag{6.34}$$

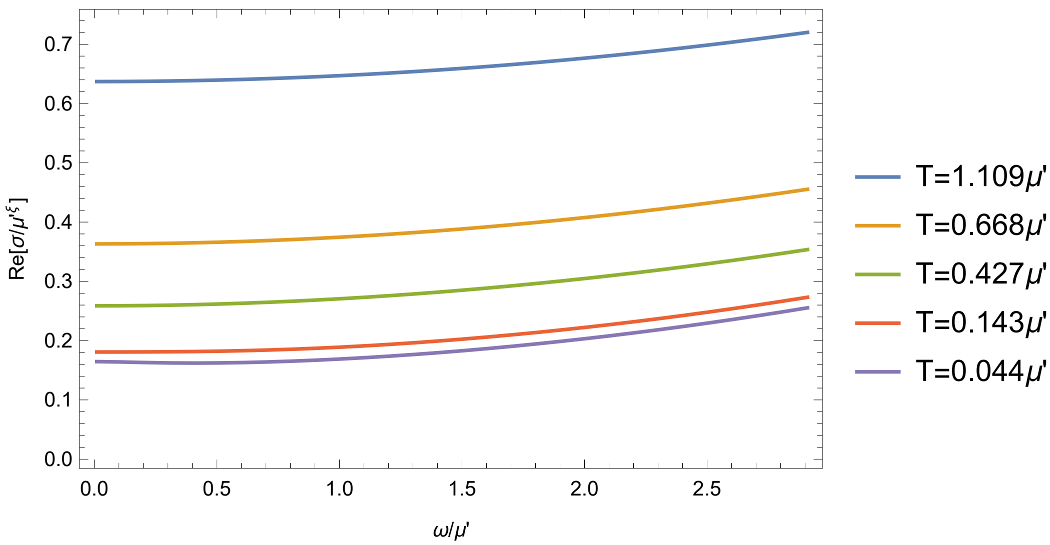

Figure 6.1: The real part of the conductivity for several temperatures is plotted against the frequency of the external electrical field. To make the axes dimensionless σ/μ'^{ξ} and ω/μ' are considered, where $\mu' = \mu^{\frac{z}{z-\Phi}}$ and $\xi = (d - \theta - 2 + 2\Phi)/z$.

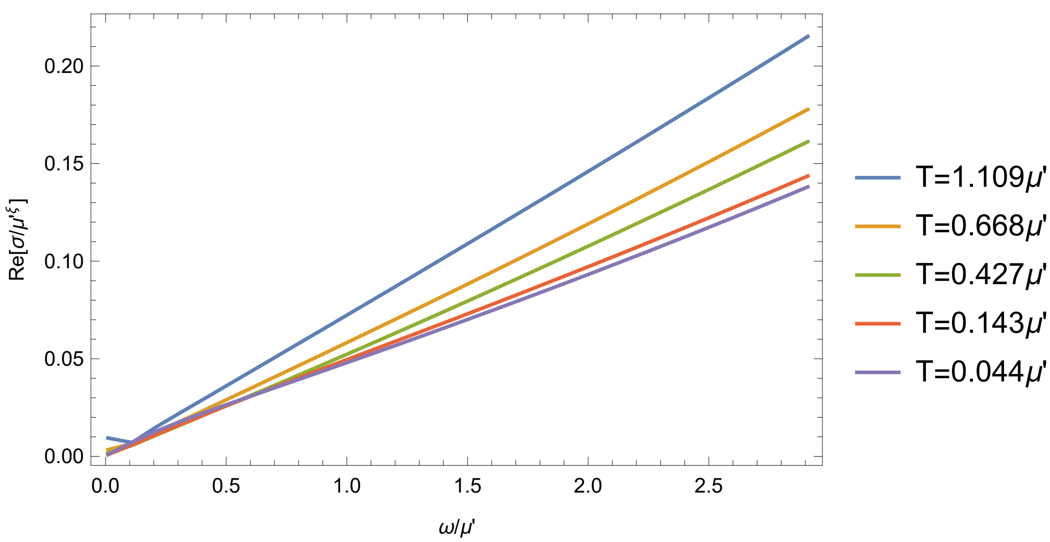

Figure 6.2: The imaginary part of the conductivity for several temperatures is plotted against the frequency of the external electrical field. To make the axes dimensionless σ/μ'^{ξ} and ω/μ' are considered, where $\mu' = \mu^{\frac{z}{z-\Phi}}$ and $\xi = (d - \theta - 2 + 2\Phi)/z$.

Let us first discuss the hydrodynamic transport causing the delta peak in the usual case for a single gauge field. In the case of a single gauge field translational invariance would give

$$\partial_t T^t_x = E_x \rho. \tag{6.35}$$

As described in Ref.[10, Section 5.4.3] there is an additional relation between current and generalised momentum

$$J^x \sim cT^{tx} + \dots, \tag{6.36}$$

with c a constant zeroth order in frequency. It is claimed that Eqs. (6.35) and (6.36) give rise to the presence of the delta peak in the real part of the conductivity, since now

$$J^x \sim cT^{tx} = \frac{cE_x\rho}{i\omega}, \tag{6.37}$$

thus the conductivity seems to have a part that scales like $1/i\omega$

$$\sigma(\omega) \sim \frac{c\rho}{i\omega}.$$

This by the Kramers-Kronig relation gives rise a a delta function at zero frequency in the real part of the conductivity, such that the conductivity is given by

$$\sigma(\omega) \sim c\rho\left[\frac{1}{i\omega} + \pi\delta(\omega)\right] + \dots \quad .$$

In the particular EMD model that is studied throughout this thesis it seems that there is no relation like Eq. (6.36) but rather than a constant term the lowest-order prefactor is first order in frequency

$$J^x_2 \sim \omega T^{tx} + \dots, \tag{6.38}$$

this actually kills the delta peak. All in all still a lot has to be understood in this section, for instance the meaning of the field $a_{1,x}$ in the momentum equation (6.34) is not fully understood. Additionally it is still unclear why the current in Eq. (6.38) is no longer at lowest order in ω scaling with the momentum T^{tx} but as frequency times momentum ωT^{tx}.

6.4 Summary

We started this section with a short introduction into thermoelectric transport in condensed-matter systems. In order to describe thermoelectric transport in holographic calculations we need to know how to treat an electric field and a temperature gradient in a holographic setting, which was done in Section 6.2. We found that an electric field is introduced by imposing through in an external gauge field using Eq. (6.6). Additionally it was found that an temperature gradient could be introduced in the holographic description by perturbing the metric and the gauge fields simultaneously with Eqs. (6.18) and (6.20). Next in Section 6.3 we discussed the optical conductivity of the EMD model that was studied in this thesis. A striking feature was the absence of the $1/\omega$ behaviour in the imaginary part of the conductivity near zero frequency. Normally this behaviour reveals the existence of an zero frequency delta peak in the real part of the conductivity, by the Kramer-Kronig relations. Normally this peak gives rise to a Drude-like peak when momentum conservation is broken and this Drude peak is observed in strange metals [41]. It is thus important to find out why this peak is absent in the studied EMD model. In the last part of this chapter we discussed a possible reason for the absence of the delta peak.

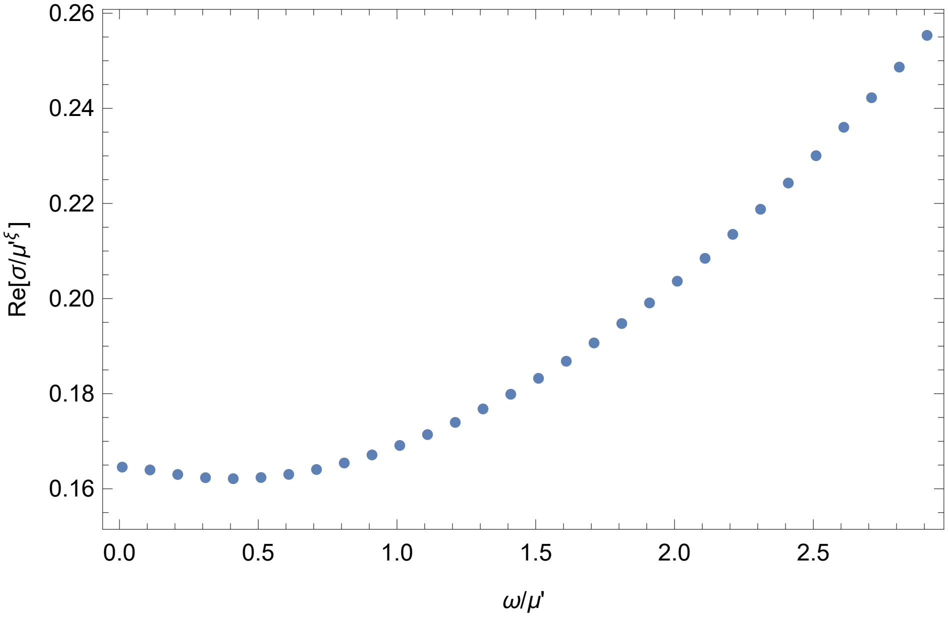

Figure 6.3: The real part of the conductivity for temperature $T = 0.044\mu'$ plotted against the frequency of the external electrical field. To make the axes dimensionless σ/μ'^{ξ} and ω/μ' are considered, where $\mu' = \mu^{\frac{z}{z-\Phi}}$ and $\xi = (d - \theta - 2 + 2\Phi)/z$.

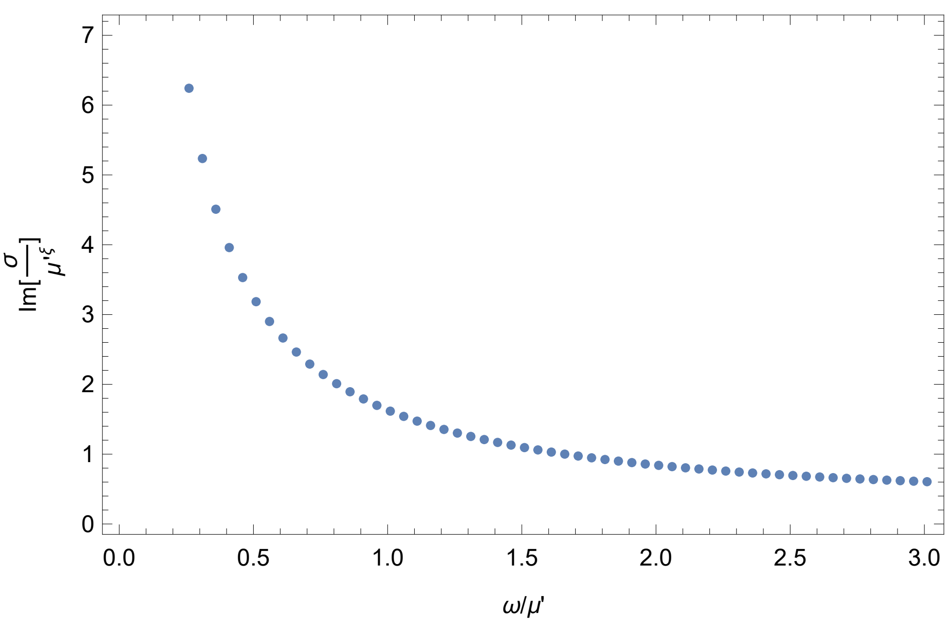

Figure 6.4: The imaginary part of the conductivity where the gauge field A_1 is switched off, for temperature $T = 0.265\mu'$ plotted against the frequency of the external electrical field. It is noted that without the gauge field A_1 the delta peak is recovered. To make the axes dimensionless σ/μ'^{ξ} and ω/μ' are considered, where $\mu' = \mu^{\frac{z}{z-\Phi}}$ and $\xi = (d - \theta - 2 + 2\Phi)/z$.

Chapter 7

Discussion and Outlook

7.1 Conclusion

In this thesis we studied an Einstein-Maxwell-dilaton model, as a description of a strongly interacting field theory using the holographic principle. The hope was that this particular EMD model could be used to model the strange metallic phase, which is a strongly correlated state of matter that for instance occurs as the metallic state of high-T_c superconductors.
In the first chapter of this thesis we gave a short introduction into the physics of ordinary metallic states and briefly discussed Fermi liquid theory. Additionally we introduced what the strange metallic phase is, discussed why it can't be described using Fermi liquid theory and why it might be useful to consider the strange metallic phase from a holographic point of view.
In the succeeding chapter the background to start using the holographic correspondence as a computational device has been treated.
From Chapter 4 onwards the specific EMD model given in [42] was studied. The reason to study such a EMD model is that it gives rise to a hyperscaling violating geometry. Such a hyperscaling-violating geometry is characterised by two additional critical exponents z and θ, these are the Lifshitz critical exponent and the hyperscaling-violating critical exponent respectively, which change the effective scaling dimensions of the theory. Thus we consider such an EMD model since it might be able to capture the physics of the strange metallic phase a bit better than certain minimal models do, due to additional degrees of freedom z and θ.
In Chapter 4 we discussed the equilibrium properties of the EMD model under consideration. Furthermore by considering the DC resistivity we could restrict the number of possible values of the critical exponents z and θ by claiming the resistivity must obey the observed linear scaling w.r.t. temperature in strange metals.
In the succeeding chapters the dynamical properties of the studied EMD model were discussed. Starting with Chapter 5, this chapter starts with a general discussion on how to calculate the dynamical correlation functions in terms of the gravitational bulk fields. In the second part of this chapter we set up the necessities to start determining the dynamical correlation functions of the EMD model under consideration.
In the final chapter thermoelectric transport was discussed. Using the setup of the previous chapter the optical conductivity of the studied EMD model could be determined numerically. Here the optical conductivity for the case $z = 3$, $\theta = -1$ and $d = 2$ has been determined, these

values of z and θ were chosen to satisfy linear scaling with temperature of the DC resistivity observed in strange metals. While the real part of the conductivity seemed to behave as expected. The imaginary part of the conductivity seemed a bit peculiar, since one would expect it to behave like $1/\omega$ for small values of ω. This was actually absent in the behaviour of the imaginary part of the conductivity. This $1/\omega$ scaling is namely crucial since it predicts the presence of a zero frequency delta peak in the real part of the conductivity. This delta peak is as discussed in Chapter 6 becomes a Drude peak when translational invariance is broken. This Drude peak is actually observed in strange metals. It is thus important to find out why this peak is absent in the studied EMD model.

7.2 Discussion and Outlook

To study the dynamics of the system, dynamical fluctuations on top of the hyperscaling-violating background were considered. To treat these dynamical fluctuations we expanded the full action up to second order, where minimizing this action gives us the linearised equations of motion. This perturbative expansion works well when the fluctuations are small compared to the background solution. It might be that the perturbative assumption breaks down in certain regions of the bulk spacetime.

In Section 5.4 we used the unrenormalised boundary action to identify the source and VEV's of the system. Renormalization changes the precise form of the boundary action, this might lead to an altered vacuum expectation value.

Additionally note is that the dynamics in this thesis, have been determined using linear response.

In Chapter 4 we derived an relation for the DC resistivity of the studied EMD model as a function of temperature. We found that the DC resistivity of the EMD model has to scale with temperature like $\rho \sim T^{\frac{d-\theta}{z}}$. Doing so we used that the resistivity in the hydrodynamic regime is proportional to the viscosity of the "fluid". By performing an independent calculation for the DC conductivity following [10, Section 3.4.2] the following DC conductivity was found[1] $\sigma_{DC} \sim T^{(d+2\Phi-\theta-2)/z}$. Since the resistivity and the conductivity are related by $\rho = \sigma^{-1}$, it seems that the two calculations do not go together at first notice. So one could in the future try to understand why these two calculations give different predictions for the conductivity.

An essential future direction is determining the renormalized boundary action of the theory, there are already several papers who are renormalizing similar backgrounds e.g. Refs. [38, 43]. Note that the renormalization of these Lifshitz and hyperscaling-violating backgrounds is mostly done in the Hamilton-Jacobi formalism [37].

Additionally it is interesting to find out why the conductivity of the particular model used in this thesis does not have a delta peak at zero frequency. We already showed that this quite likely has to do with back-reaction of the additional gauge field, since the conductivity without the back-reaction of the additional gauge field does have the zero frequency delta peak.

Once the renormalized boundary action is determined the thermodynamic transport coefficients should also be quickly obtainable, since the VEV $\delta\langle T^{xt}\rangle$ is determined then.

In the future one may like to consider the limit $z \to \infty$ and $\theta \to -\infty$, where $\theta/z = \eta$, since it has interesting physical properties and vanishing ground state entropy. In order to consider

[1] This calculation was performed on short notice and thus not checked for correctness.

this limit, it will be useful to consider coordinate transformation of the form (or something similar)

$$\tilde{r} = r^{\theta/d-1}, \quad \tilde{t} = \left(\frac{\theta - d}{d}\right) t, \quad \tilde{\vec{x}} = \left(\frac{\theta - d}{d}\right) \vec{x}.$$

Using the above coordinate transformation the metric transforms to

$$\left(\frac{\theta - d}{d}\right)^2 d\tilde{s}^2 = \left(-\tilde{r}^{-2(\theta-dz)/(\theta-d)}\tilde{f}(\tilde{r})d\tilde{t}^2 + \frac{\tilde{r}^{(-4\theta+2d)/(\theta-d)}d\tilde{r}^2}{\tilde{f}(\tilde{r})} + \frac{d\tilde{\vec{x}}^2}{\tilde{r}^2}\right).$$

The advantage is that in the above coordinates one might be able to consider the limit $z \to \infty$, with $\theta/z = \eta$ fixed, before doing the numerical calculations. In Refs.[44, 45] something similar has already been done for the zero-temperature case.
Lastly, one could add coulomb interactions to the system by means of a double trace deformation. This has already been for other backgrounds in Refs. [35, 46].

Acknowledgements

I would like to thank Henk and Enea, for their guidance, support and feedback throughout the entire project.

Appendix A

Analytic Solution Einstein-Maxwell-Dilaton Model

A.1 Einstein-Maxwell-Dilaton Model

We will start with the minimal model given in Ref. [42]

$$\mathcal{S} = -\frac{1}{16\pi G}\int d^{d+2}x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 + V(\phi) - \frac{1}{4}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2\right], \tag{A.1}$$

Where we consider the following potential

$$V(\phi) = V_0 e^{\gamma\phi}. \tag{A.2}$$

Equations of motion

To determine the equations of motion we first need

$$\begin{aligned}\det(g+\delta g) &= \exp\left[\mathrm{tr}\log(g+\delta g)\right] \\ &= \det(g)\exp\left[\mathrm{tr}\log(\mathbb{I}+g^{-1}\delta g)\right] \\ &= \det(g)\exp\left[\mathrm{tr}\; g^{-1}\delta g\right] \\ &= \det(g)\left(1+g^{\mu\nu}\delta g_{\mu\nu}\right) \\ &= \det(g)\left(1-g_{\mu\nu}\delta g^{\mu\nu}\right),\end{aligned} \tag{A.3}$$

which implies

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} = -g g_{\mu\nu}\delta g^{\mu\nu}. \tag{A.4}$$

This now gives the following result

$$\delta\sqrt{-g} = -\frac{1}{2}\frac{1}{\sqrt{-g}}\delta g = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{A.5}$$

Also note that $R = R_{\mu\nu}g^{\mu\nu}$. The equations of motion for the bulk part of A.1 are thus given by

$$R_{\mu\nu} - \frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}\sum_{i=1}^{2} e^{\lambda_i\phi}F_{i\;\mu\lambda}F_{i\;\nu}^{\;\;\lambda} = \frac{1}{2}g_{\mu\nu}\left[R - \frac{1}{2}(\partial\phi)^2 + V(\phi) - \frac{1}{4}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2\right],$$
$$\nabla^2\phi = -\frac{\partial V(\phi)}{\partial\phi} + \frac{1}{4}\sum_{i=1}^{2}\lambda_i e^{\lambda_i\phi}F_i^2, \qquad \nabla_\mu\left(\sqrt{-g}e^{\lambda_i\phi}F_i^{\mu\nu}\right) = 0. \tag{A.6}$$

Taking the trace of the first term of Eq. (A.6) we obtain

$$d\left[R - \frac{1}{2}(\partial\phi)^2\right] + (d+2)V(\phi) - (d-2)\frac{1}{4}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2 = 0$$
$$\implies \left[R - \frac{1}{2}(\partial\phi)^2\right] = -\frac{(d+2)}{d}V(\phi) + \frac{(d-2)}{d}\frac{1}{4}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2 \tag{A.7}$$

Plugging Eq. (A.7) back into Eq. (A.6) we obtain

$$R_{\mu\nu} - \frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2}\sum_{i=1}^{2} e^{\lambda_i\phi}F_{i\;\mu\lambda}F_{i\;\nu}^{\;\;\lambda} = \frac{1}{2}g_{\mu\nu}\left[-\frac{2}{d}V(\phi) - \frac{1}{2d}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2\right],$$
$$\nabla^2\phi = -\frac{\partial V(\phi)}{\partial\phi} + \frac{1}{4}\sum_{i=1}^{2}\lambda_i e^{\lambda_i\phi}F_i^2, \qquad \nabla_\mu\left(e^{\lambda_i\phi}F_i^{\mu\nu}\right) = 0. \tag{A.8}$$

Let us now follow [42] by making the following ansatz for the metric, scalar and guage fields

$$ds^2 = r^{2\alpha}\left(-r^{2z}f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\vec{x}^2\right), \quad \phi = \phi(r), \quad F_{irt} \neq 0. \tag{A.9}$$

Solving Gauge fields

Using the Maxwell equations of motion we see

$$\sqrt{-g}\;\nabla_\mu\left(e^{\lambda_i\phi}F_i^{\mu\nu}\right) = \sqrt{-g}\;\partial_\mu\left(e^{\lambda_i\phi}F_i^{\mu\nu}\right) + \sqrt{-g}e^{\lambda_i\phi}\left(\Gamma^\mu_{\mu\lambda}F_i^{\lambda\nu} + \Gamma^\nu_{\mu\lambda}F_i^{\mu\lambda}\right)$$
$$= \partial_\mu\left(\sqrt{-g}e^{\lambda_i\phi}F_i^{\mu\nu}\right) = 0, \tag{A.10}$$

in the third line we used $\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{-g}}\partial_\lambda\sqrt{-g}$. For the ansatz of our metric given in Eq. (A.9) we have

$$\sqrt{-g} = r^{(d-1)+z+(d+2)\alpha}.$$

Plugging this back into Eq. (A.10) we obtain the following expression for F_{irt}

$$\partial_r\left(\sqrt{-g}g^{rr}g^{tt}e^{\lambda_i\phi}F_{irt}\right) = -\partial_r\left(r^{(d-2)\alpha-z+(d+1)}e^{\lambda_i\phi}F_{irt}\right) = 0, \tag{A.11}$$
$$\implies F_{irt} = e^{-\lambda_i\phi}r^{(2-d)\alpha+z-(d+1)}\rho_i. \tag{A.12}$$

Solving the Dilaton field

By combining the tt and rr components of the Einstein equations the following result is obtained

$$R_t^t - R_r^r = -\frac{1}{2} g^{rr} (\partial_r \phi)^2. \tag{A.13}$$

Using the metric ansatz (A.9) we get

$$R_t^t - R_r^r = -d(1+\alpha)(\alpha+z-1)e^{-2\alpha} f(r). \tag{A.14}$$

This implies

$$\begin{aligned} -r^{2-2\alpha}(\partial_r\phi)^2 f(r) &= -2d(1+\alpha)(\alpha+z-1)e^{-2\alpha} f(r), \\ \implies (\partial_r\phi)^2 &= \frac{2d(1+\alpha)(\alpha+z-1)}{r^2}. \end{aligned} \tag{A.15}$$

This implies the solution of ϕ is given by

$$\phi = \phi_0 \pm \log(r)\sqrt{2d(1+\alpha)(\alpha+z-1)}, \tag{A.16}$$

which is equivalent to

$$e^{\phi} = e^{\phi_0} r^{\pm\sqrt{2d(1+\alpha)(\alpha+z-1)}} = e^{\phi_0} r^{\beta}. \tag{A.17}$$

Note that this solution is only well defined for $d(1+\alpha)(\alpha+z-1) \geq 0$. This condition follows from the null energy condition, which implies

$$T_{\mu\nu}\zeta^\mu\zeta^\nu \geq 0,$$

where $\zeta_\mu\zeta^\mu = 0$. Choosing $\zeta^\mu = (\sqrt{-g^{tt}}, \sqrt{g^{rr}}, 0)$, the null energy condition for action A.1 gives

$$T_{\mu\nu}\zeta^\mu\zeta^\nu \propto R_{\mu\nu}\zeta^\mu\zeta^\nu = R_r^r - R_t^t = d(1+\alpha)(\alpha+z-1)e^{-2\alpha} f(r) \geq 0 \tag{A.18}$$

$$\implies d(1+\alpha)(\alpha+z-1) \geq 0. \tag{A.19}$$

Finding the metric

To find the metric for this system we use the ansatz for the metric given in Eq. (A.9). For this metric ansatz the xx components of the Ricci tensor are given by

$$R_x^x = -(\alpha+1) r^{-\alpha(d+2)-z-d+1}\, \partial_r \left(r^{d(\alpha+1)+z} f(r) \right). \tag{A.20}$$

Thus the Einstein equations of motion Eq. (A.8) become

$$\begin{aligned} \partial_r \left(r^{d(\alpha+1)+z} f(r) \right) &= \frac{r^{\alpha(d+2)+z+d-1}}{2(\alpha+1)} \left[\frac{2}{d} V(\phi) + \frac{1}{2d} \sum_{i=1}^{2} e^{\lambda_i\phi} F_i^2 \right] \\ &= \frac{r^{\alpha(d+2)+z+d-1}}{2(\alpha+1)} \left[\frac{2V_0 e^{\gamma\phi}}{d} + \frac{1}{2d} \sum_{i=1}^{2} e^{\lambda_i\phi} \left(2g^{rr} g^{tt} F_{irt}^2 \right) \right] \\ &= \frac{r^{\alpha(d+2)+z+d-1}}{(\alpha+1)} \left[\frac{V_0 e^{\gamma\phi}}{d} - \frac{1}{2d} \sum_{i=1}^{2} e^{-\lambda_i\phi} \rho_i^2 r^{-2d(\alpha+1)} \right] \\ &= \frac{r^{\alpha(d+2)+z+d-1}}{(\alpha+1)} \left[\frac{V_0 e^{\gamma\phi_0} r^{\gamma\beta}}{d} - \frac{1}{2d} \sum_{i=1}^{2} e^{-\lambda_i\phi_0} \rho_i^2 r^{-2d(\alpha+1)-\lambda_i\beta} \right] \end{aligned} \tag{A.21}$$

Note that we inserted Eqs. (A.12) and (A.17) and made use of the ansatz $\phi = \phi(r), F_{irt} \neq 0$. Eq. (A.21) can be integrated out to yield

$$\begin{aligned}\left(r^{d(\alpha+1)+z} f(r)\right) =& - m + \frac{V_0 e^{\gamma\phi_0} r^{\alpha(d+2)+z+d+\gamma\beta}}{d(\alpha+1)(\alpha(d+2)+z+d+\gamma\beta)} \\ &- \frac{1}{2d}\sum_{i=1}^{2} \frac{e^{-\lambda_i\phi_0}\rho_i^2 r^{\alpha(2-d)+z-d-\lambda_i\beta}}{(\alpha+1)(\alpha(2-d)+z-d-\lambda_i\beta)},\end{aligned} \tag{A.22}$$

$$\begin{aligned}\implies f(r) =& - m r^{-d(\alpha+1)-z} + \frac{V_0 e^{\gamma\phi_0} r^{2\alpha+\gamma\beta}}{d(\alpha+1)(\alpha(d+2)+z+d+\gamma\beta)} \\ &- \frac{1}{2d}\sum_{i=1}^{2} \frac{e^{-\lambda_i\phi_0}\rho_i^2 r^{-2d(\alpha+1)+2\alpha-\lambda_i\beta}}{(\alpha+1)(\alpha(2-d)+z-d-\lambda_i\beta)}.\end{aligned} \tag{A.23}$$

Where m is the integration constant, which can later be related to the mass of the black hole. To obtain an asymptotic hyperscaling violating metric (at $r \to \infty$) from Eq. (A.23) we need

$$\gamma = -\frac{2\alpha}{\beta}. \tag{A.24}$$

Thus to find an hyperscaling violating metric we need to a non-trivial potential.

Determining integration constants

From metric ansatz (A.9) the Einstein equation for the dilaton field and Eq. (A.17) we find

$$\begin{aligned}\nabla^2\phi =& \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) = \beta\frac{\partial_r(r^{d(\alpha+1)+z}f(r))}{r^{2\alpha+d(\alpha+1)+z-1}} \\ =& \frac{\beta}{(\alpha+1)}\left[\frac{1}{d}V(\phi) + \frac{1}{4d}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2\right],\end{aligned} \tag{A.25}$$

where the last line was obtained by using the first line of Eq. (A.21). Using the explicit form of the dilaton potential (Eqs. (A.2) and (A.24)) the equation of motion for the scalar field Eq. (A.8) now gives

$$\begin{aligned}&\left(\frac{4\beta}{d(\alpha+1)} - \frac{8\alpha}{\beta}\right) V_0 e^{-2\alpha\phi/\beta} = \sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2\left(\lambda_i - \frac{\beta}{d(\alpha+1)}\right) \implies \\ &\left(\frac{4\beta}{d(\alpha+1)} - \frac{8\alpha}{\beta}\right) V_0 e^{-2\alpha\phi_0/\beta} r^{-2\alpha} = -2\sum_{i=1}^{2} e^{-\lambda_i\phi_0}\rho_i^2 r^{-2d(\alpha+1)-\lambda_i\beta}\left(\lambda_i - \frac{\beta}{d(\alpha+1)}\right)\end{aligned} \tag{A.26}$$

As a solution to the above equation we find

$$\begin{aligned}\lambda_1 =& -\frac{2\alpha(d-1)+2d}{\sqrt{2d(1+\alpha)(\alpha+z-1)}}, \quad \lambda_2 = \sqrt{\frac{2(\alpha+z-1)}{d(\alpha+1)}}, \\ \rho_1^2 =& \frac{\left(\frac{4\beta}{d(\alpha+1)} - \frac{8\alpha}{\beta}\right)}{2\left(\lambda_1 - \frac{\beta}{d(\alpha+1)}\right)} V_0 e^{-2\alpha\phi_0/\beta+\lambda_1\phi_0} = \frac{2V_0(z-1)}{d(\alpha+1)+z-1} e^{-\sqrt{\frac{2d(\alpha+1)}{\alpha+z-1}}\phi_0}\end{aligned} \tag{A.27}$$

ρ_2 remains undetermined and can be identified with the charge of the solution. We fix V_0 by requiring the constant term in $f(r)$ to be one. We thus obtain

$$\begin{aligned} 1 &= \frac{V_0 e^{\gamma\phi_0}}{d(\alpha+1)(\alpha(d+2)+z+d+\gamma\beta)} - \frac{1}{2d}\frac{e^{-\lambda_1\phi_0}\rho_1^2}{(\alpha+1)(\alpha(2-d)+z-d-\lambda_1\beta)}, \\ \implies 1 &= \frac{2V_0 e^{\gamma\phi_0} - e^{-\lambda_1\phi_0}\rho_1^2}{2d(\alpha+1)(d(\alpha+1)+z)} = \frac{V_0 e^{\gamma\phi_0}}{(d\alpha+d+z-1)(d\alpha+d+z)}, \\ \implies V_0 &= (d\alpha+d+z-1)(d\alpha+d+z)e^{-\gamma\phi_0}. \end{aligned} \tag{A.28}$$

And thus

$$\rho_1^2 = 2(z+d-\theta)(z-1)e^{\lambda_1\phi_0} \tag{A.29}$$

Now it remains to check the tt and rr parts of the Einstein equations of motion. We only need to check the tt part explicitly, since the rr-part follows directly from Eq. (A.13) and the tt-part of the Einstein equations.

Hyperscaling violating solution

To summarise action (A.1) gives rise to a hyperscaling violating charged black brane solution given by

$$\begin{aligned} ds^2 &= r^{-2\frac{\theta}{d}}\left(-r^{2z}f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\vec{x}^2\right), \\ F_{1rt} &= \sqrt{2(z-1)(z+d-\theta)}\, e^{\frac{\theta(1-d)/d+d}{\sqrt{2(d-\theta)(z-1-\theta/d)}}\phi_0} r^{d+z-\theta-1}, \\ F_{2rt} &= Q\sqrt{2(d-\theta)(z-\theta+d-2)}\, e^{-\sqrt{\frac{z-1-\theta/d}{2(d-\theta)}}\phi_0} r^{-(z+d-\theta-1)}, \\ e^{\phi} &= e^{\phi_0} r^{\sqrt{2(d-\theta)(z-1-\theta/d)}}, \end{aligned} \tag{A.30}$$

with

$$f(r) = 1 - \frac{m}{r^{z+d-\theta}} + \frac{Q^2}{r^{2(z+d-\theta-1)}}. \tag{A.31}$$

We used

$$\begin{aligned} Q^2 &= -\frac{1}{2d}\frac{e^{-\lambda_2\phi_0}\rho_2^2}{(\alpha+1)(\alpha(2-d)+z-d-\lambda_2\beta)}, \\ &= \frac{1}{2}\frac{e^{-\sqrt{\frac{2(\alpha+z-1)}{d(\alpha+1)}}\phi_0}\rho_2^2}{(\alpha d+d)(\alpha d+z+d-2)}. \end{aligned}$$

Where we defined $\alpha = -\theta/d$, with θ the hyperscaling exponent. Note that the above solution is not defined for the case $\theta = d$ since we have divergences in our solutions in that case. We see that Eq. (4.10) gives a black brane solution where the radius of horizon (r_H) is determined by $f(r_h) = 0$

$$r_H^{2(d+z-\theta-1)} - m r_H^{d+z-\theta-2} + Q^2 = 0. \tag{A.32}$$

A.2 Conserved currents in the QFT from bulk gauge degees of freedom

First recall that Stokes theorem gives

$$\int_M d^n x \sqrt{|g|} \nabla_\mu V^\nu = \int_{\partial M} d^{n-1} y \sqrt{|\gamma|}\, n_\mu V^\nu, \tag{A.33}$$

Where γ_{ij} is the induced metric on the boundary of the manifold M and n_μ is a vector orthonormal to the boundary ∂M which is given by $n^\mu = (\sqrt{g^{rr}}, 0)$ for metric ansatz (A.9). We are now able to integrate by parts such that the first variation of the action describing the bulk will be zero for classical solutions and we will only remain with boundary terms

$$\delta \mathcal{S}_r = \int_{\partial M_r} \sqrt{-\gamma} \left\{ \frac{1}{2} T^{ab} \delta\gamma_{ab} + \sum_{i=1}^{2} J_i^\mu \delta A_{i,\mu} + \mathcal{O}_\phi \delta\phi \right\} \Bigg|_r . \tag{A.34}$$

A.2.1 Conserved currents due to $U(1)$ gauge symmetry

For the gauge fields the boundary term for the first variation of the action w.r.t. the gauge fields is given by

$$\begin{aligned} &\int_M d^n x \sqrt{-g} \sum_{i=1}^{2} \nabla_\mu \left(4\delta A_{i,\nu} F_i^{\mu\nu} e^{\lambda_i \phi} \right) \\ &= \int_{\partial M_r} d^{n-1} x \sum_{i=1}^{2} \sqrt{-\gamma}\, n_\mu \left(4 F_i^{\mu\nu} e^{\lambda_i \phi} \right) \delta A_{i,\nu}, \\ &= \int_{\partial M_r} d^{n-1} x \sum_{i=1}^{2} \delta A_{i,\nu} J_i^\nu, \end{aligned} \tag{A.35}$$

where $J_i^\nu = \sqrt{-\gamma}\, n_\mu \left(4 F_i^{\mu\nu} e^{\lambda_i \phi} \right)$. This boundary action should be invariant under $U(1)$ gauge symmetry, which implies that gauge field couple to conserved currents. Let me elaborate a bit on this, the gauge field A_μ transforms to $A_\mu + \partial_\mu \Lambda$ under a gauge transformation. Invariance of the boundary action now implies

$$\begin{aligned} &\int_{\partial M_r} d^{n-1} x \sum_{i=1}^{2} \delta A_{i,\nu} J_i^\nu \rightarrow \int_{\partial M_r} d^{n-1} x \sum_{i=1}^{2} (\delta A_{i,\nu} + \partial_\nu \Lambda_i) J_i^\nu \\ &= \int_{\partial M_r} d^{n-1} x \sum_{i=1}^{2} \delta A_{i,\nu} J_i^\nu - \Lambda_i \partial_\nu J_i^\nu \end{aligned} \tag{A.36}$$

Invariance of the bulk gauge requires $\partial_\nu J_i^\nu = 0$.

A.2.2 Conserved currents due to translational invariance

The action should be diffeomorphism invariant

$$\begin{aligned} a_\mu &\to a_\mu - \mathcal{L}_\xi A_\mu = a_\mu - \xi^\nu \partial_\nu A_\mu - (\partial_\mu \xi^\nu) A_\nu \\ &= a_\mu - \xi^\nu \nabla_\nu A_\mu - (\nabla_\mu \xi^\nu) A_\nu, \\ h_{\mu\nu} &\to h_{\mu\nu} - \mathcal{L}_\xi g_{\mu\nu} = h_{\mu\nu} - \xi^\nu \partial_\nu g_{\mu\nu} - (\partial_\mu \xi^\lambda) g_{\lambda\nu} - (\partial_\nu \xi^\lambda) g_{\lambda\mu} \\ &= h_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu. \end{aligned} \tag{A.37}$$

Specifically under a translation in the ρ-direction ξ^ν is given by $\xi^\nu = \delta^\nu_\rho$ the boundary action thus changes like

$$\begin{aligned} \delta \mathcal{S}_r &= \int_{\partial M_r} \left\{ \frac{1}{2} T^{ab} \delta\gamma_{ab} + \sum_{i=1}^{2} J^\mu_i \delta A_{i,\mu} \right\} \\ &\to \int_{\partial M_r} \left\{ \frac{1}{2} T^{ab} (\delta\gamma_{ab} - \xi^\nu \partial_\nu \gamma_{ab} - (\partial_a \xi^\lambda)\gamma_{\lambda b} - (\partial_b \xi^\lambda)\gamma_{\lambda a}) \right. \\ &\quad \left. + \sum_{i=1}^{2} J^\mu_i (\delta A_{i,\mu} - \xi^\nu \partial_\nu A_\mu - (\partial_\mu \xi^\nu) A_\nu) \right\} \\ &\overset{P.I.}{\to} \delta\mathcal{S}_r + \int_{\partial M_r} \left\{ \xi^\nu \partial_\mu T^\mu_\nu + \xi^\nu \sum_{i=1}^{2} F_{i,\mu\nu} J^\mu_i \right\}. \end{aligned} \tag{A.38}$$

With $T^{ab} = 2\sqrt{-\gamma}\left(K^{ab} - K\gamma^{ab}\right)$, the unnormalized conjugate momentum to γ_{ab}. Note that K^{ab} is given by the extrinsic curvature [47, 48] which is defined by $\nabla_{(a} n_{b)}$, with n the radially outward pointing vector of unit length. Thus invariance under gauge transformations

$$\partial_\mu T^\mu_\nu + \sum_{i=1}^{2} F_{i,\mu\nu} J^\mu_i = 0. \tag{A.39}$$

In the zero momentum limit the boundary of our hyperscaling violating bulk space-time the above above expression gives

$$\partial_0 T^0_x + \sum_{i=1}^{2} F_{i,0x} J^0_i = 0. \tag{A.40}$$

Up to first order this gives

$$\partial_0 T^0_x + \delta E_x J^0_2 - i\omega \delta a^0_{1,x} J^0_1 = 0. \tag{A.41}$$

Appendix B

Fluctuations in the Einstein-Maxwell-Dilaton Action

In this appendix we determine the second order fluctuations of action (B.1)

$$\mathcal{S} = -\frac{1}{16\pi G}\int d^{d+2}x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 + V(\phi) - \frac{1}{4}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2\right] - \frac{1}{8\pi G}\int d^{d+1}x\sqrt{-\gamma}K, \tag{B.1}$$

where $K = \gamma^{\mu\nu}\nabla_\nu n_\nu$. We will consider fluctuations of the type

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + h_{\mu\nu}, \\ \phi &= \bar{\phi} + \delta\phi, \\ A_\mu &= \bar{A}_\mu + a_\mu. \end{aligned} \tag{B.2}$$

B.1 Fluctations of dilaton field and gauge fields

Second order fluctuations of the dilaton and gauge fields

$$\begin{aligned} \delta^2\mathcal{S} = -\frac{1}{16\pi G}\int d^{d+2}x\sqrt{-g}\bigg[&-\frac{1}{2}(\partial\delta\phi)^2 + \frac{1}{2}V''(\phi)\delta\phi^2 \\ &- \frac{1}{4}\sum_{i=1}^{2}\left(e^{\lambda_i\phi}(f_i)^2 + 2\lambda_i e^{\lambda_i\phi}\delta\phi f_{i,\mu\nu}F_i^{\mu\nu} + \frac{1}{2}\lambda_i^2\delta\phi^2 e^{\lambda_i\phi}F_i^2\right)\bigg], \end{aligned} \tag{B.3}$$

We can split this into a second order bulk action and a boundary action

$$\begin{aligned} \delta^2\mathcal{S}_{bulk} = \frac{1}{2}\sqrt{-g}\bigg[&\delta\phi\left(\Box + V''(\bar{\phi}) - \frac{1}{4}\sum_{i=1}^{2}(\lambda_i^2 e^{\lambda_i\bar{\phi}}\bar{F}_i^2)\right)\delta\phi \\ &- \frac{1}{2}\bigg\{\sum_{i=1}^{2} a_{i,\mu}\nabla_\nu\left(e^{\lambda_i\bar{\phi}}f_i^{\mu\nu}\right) - a_{i,\nu}\nabla_\mu\left(e^{\lambda_i\bar{\phi}}f_i^{\mu\nu}\right) \\ &+ \sum_{i=1}^{2}\delta\phi\left(\lambda_i e^{\lambda_i\bar{\phi}}\bar{F}_i^{\mu\nu}\nabla_\mu\right)a_{i,\nu} - a_{i,\nu}\left(\lambda_i e^{\lambda_i\bar{\phi}}\bar{F}_i^{\mu\nu}\nabla_\mu\right)\delta\phi\bigg\}\bigg]. \end{aligned} \tag{B.4}$$

B.2 Metric fluctuations

In this section we're interested in fluctuations with respect to the metric. We're starting by just considering the fluctuations of the Einstein-Hilbert action. The fluctuation of the volume factor $\sqrt{-g}$ is given by

$$\begin{aligned}\sqrt{-g-\delta g} \simeq \sqrt{-g}\exp\left\{\frac{1}{2}g^{\mu\nu}h_{\mu\nu} - \frac{1}{4}\delta g^{\mu\nu}h_{\mu\nu}\right\} \simeq \\ \sqrt{-g}\left(1+\frac{1}{2}g^{\mu\nu}h_{\mu\nu} + \frac{1}{4}\left(\frac{1}{2}(g^{\mu\nu}h_{\mu\nu})^2 - \delta g^{\mu\nu}h_{\mu\nu}\right)\right).\end{aligned} \tag{B.5}$$

And where the fluctuation of the inverse metric is

$$\begin{aligned}g_{\mu\nu} &= \bar{g}_{\mu\nu} + h_{\mu\nu} \\ g^{\mu\nu} &= \bar{g}_{\mu\nu} - g^{\mu\alpha}g^{\nu\beta}h_{\alpha\beta} + g_{\alpha\beta}\delta g^{\mu\beta}\delta g^{\alpha\nu},\end{aligned} \tag{B.6}$$

where $\delta g^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}h_{\alpha\beta}$ and $\delta^2 g^{\mu\nu} = g_{\alpha\beta}\delta g^{\mu\beta}\delta g^{\alpha\nu}$. Next

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \delta R_{\mu\nu} + \delta^2 R_{\mu\nu}, \tag{B.7}$$

with

$$R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\rho}_{\nu\mu} - \partial_\nu \Gamma^{\rho}_{\lambda\mu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}{}_{\nu\mu} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu}. \tag{B.8}$$

Up to second order the variation of the Ricci tensor is

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \bar{\nabla}_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \bar{\nabla}_\nu(\delta\Gamma^{\rho}_{\lambda\mu}), \tag{B.9}$$

$$\delta^2 R^{\rho}{}_{\mu\lambda\nu} = \bar{\nabla}_\lambda(\delta^2\Gamma^{\rho}_{\nu\mu}) - \bar{\nabla}_\nu(\delta^2\Gamma^{\rho}_{\lambda\mu}) + \delta\Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \delta\Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu}, \tag{B.10}$$

where we used

$$\bar{\nabla}_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) = \partial_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}{}_{\nu\mu} - \Gamma^{\sigma}_{\lambda\nu}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\lambda\mu}\delta\Gamma^{\rho}_{\nu\sigma}. \tag{B.11}$$

Using Eq. (B.9) the expansion of the Riemann tensor can be written as

$$\begin{aligned}R^{\rho}{}_{\mu\lambda\nu} =& \bar{R}^{\rho}{}_{\mu\lambda\nu} + \bar{\nabla}_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \bar{\nabla}_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) + \bar{\nabla}_\lambda(\delta^2\Gamma^{\rho}_{\nu\mu}) - \bar{\nabla}_\nu(\delta^2\Gamma^{\rho}_{\lambda\mu})\delta\Gamma \\ &+ \delta\Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \delta\Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu} \\ =& \bar{R}^{\rho}{}_{\mu\lambda\nu} + \nabla_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \nabla_\nu(\delta\Gamma^{\rho}_{\lambda\mu}) + \bar{\nabla}_\lambda(\delta^2\Gamma^{\rho}_{\nu\mu}) - \bar{\nabla}_\nu(\delta^2\Gamma^{\rho}_{\lambda\mu})\delta\Gamma \\ &- \delta\Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\sigma}_{\nu\mu} + \delta\Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\lambda\mu}.\end{aligned} \tag{B.12}$$

Where ∇ is the covariant connection for the metric tensor $g_{\mu\nu} + h_{\mu\nu}$. And the first order variation of the Christoffel symbol is given by

$$\bar{\nabla}_\sigma(h_{\mu\nu}) = \delta(\nabla_\sigma g_{\mu\nu}) + \delta\Gamma^{\lambda}_{\sigma\mu}\bar{g}_{\lambda\nu} + \delta\Gamma^{\lambda}_{\sigma\nu}\bar{g}_{\mu\lambda} \tag{B.13}$$

$$\implies \delta\Gamma^{\lambda}_{\sigma\mu}\bar{g}_{\lambda\nu} + \delta\Gamma^{\lambda}_{\sigma\nu}\bar{g}_{\mu\lambda} = \bar{\nabla}_\sigma(h_{\mu\nu}) \tag{B.14}$$

$$\implies 2\delta\Gamma^{\lambda}_{\mu\nu}\bar{g}_{\sigma\lambda} = \bar{\nabla}_\mu h_{\nu\sigma} + \bar{\nabla}_\nu h_{\sigma\mu} - \bar{\nabla}_\sigma h_{\mu\nu} \tag{B.15}$$

$$\implies \delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\bar{g}^{\sigma\rho}\left(\bar{\nabla}_\mu h_{\nu\sigma} + \bar{\nabla}_\nu h_{\sigma\mu} - \bar{\nabla}_\sigma h_{\mu\nu}\right). \tag{B.16}$$

In the last line of Eq. (B.12) we see terms which give a boundary contribution, while the last term in Eq. (B.12) gives rise to a term in the linearised Einstein equations of motion of the bulk. The second variation of the Einstein-Hilbert term is thus given by

$$\begin{aligned}\delta^2\left(\sqrt{-g}R_{\mu\nu}g^{\mu\nu}\right) &= \left(\delta^2\sqrt{-g}\right)\bar{R} + \left(\delta\sqrt{-g}\right)\bar{R}_{\mu\nu}\delta g^{\mu\nu} + \left(\delta\sqrt{-g}\right)\delta R_{\mu\nu}g^{\mu\nu} \\ &\quad + \sqrt{-\bar{g}}\left(\delta^2 R_{\mu\nu}\right)\bar{g}^{\mu\nu} + \sqrt{-\bar{g}}\delta R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{-\bar{g}}\bar{R}_{\mu\nu}\delta^2 g^{\mu\nu}.\end{aligned} \tag{B.17}$$

So these terms should just give rise to a boundary action and will not influence the linearised equations of motion. So the bulk action obtained from the Ricci scalar must be given by

$$\begin{aligned}\delta^2\mathcal{S} &= \int d^{d+2}x\sqrt{-g}\Bigg[\frac{1}{4}\left(\frac{1}{2}(\delta g^\mu_\mu)^2 + \delta g^{\mu\nu}h_{\mu\nu}\right)R - \frac{1}{2}g_{\alpha\beta}\delta g^{\alpha\beta}R_{\mu\nu}\delta g^{\mu\nu} \\ &\quad - \left(\delta\Gamma^\lambda_{\lambda\sigma}\delta\Gamma^\sigma_{\nu\mu} - \delta\Gamma^\lambda_{\nu\sigma}\delta\Gamma^\sigma_{\lambda\mu}\right)g^{\mu\nu} + R_{\mu\nu}(\delta g^\nu{}_\lambda\delta g^{\lambda\mu})\Bigg] \\ &= \int d^{d+2}x\sqrt{-g}\Bigg[\frac{1}{4}\left(\frac{1}{2}(\delta g^\mu_\mu)^2 - g_{\mu\alpha}g_{\nu\beta}\delta g^{\mu\nu}\delta g^{\alpha\beta}\right)R - \frac{1}{2}g_{\alpha\beta}\delta g^{\alpha\beta}R_{\mu\nu}\delta g^{\mu\nu} \\ &\quad - \frac{1}{2}(\nabla_\nu\delta g^\alpha_\alpha)(\nabla_\mu\delta g^{\mu\nu}) + \frac{1}{4}(\nabla_\nu\delta g^\alpha_\alpha)(\nabla^\nu\delta g^\beta_\beta) \\ &\quad + \frac{1}{2}(\nabla_\sigma\delta g^{\lambda\mu})(\nabla_\lambda\delta g^\sigma_\mu) - \frac{1}{4}(\nabla^\lambda\delta g^\mu_\sigma)(\nabla_\lambda\delta g^\sigma_\mu) + R_{\mu\nu}(\delta g^\nu{}_\lambda\delta g^{\lambda\mu})\Bigg].\end{aligned} \tag{B.18}$$

The above result can be checked against Ref.[49], where we note that there is a sign difference between the answer obtained above and the one obtained in Ref.[49] and the same sign as obtained in Ref.[50, Eq. (22.12)]. The reason for this is a different definition of the Ricci tensor, where the difference in the indices of contraction precisely explain the sign difference. Throughout this thesis we followed the conventions used in Ref.[20]. The boundary term corresponding to the above action is given by

$$\begin{aligned}&\int d^{d+2}x\sqrt{-g}\Bigg\{\frac{1}{2}\nabla_\rho\left(h^\alpha_\alpha\delta\Gamma^\rho_{\mu\nu}g^{\mu\nu}\right) - \frac{1}{2}\nabla^\mu\left(h^\alpha_\alpha\delta\Gamma^\rho_{\rho\mu}\right) - \nabla_\rho\left(h^{\mu\nu}\delta\Gamma^\rho_{\mu\nu}\right) \\ &- \nabla_\nu\left(h^{\mu\nu}\delta\Gamma^\rho_{\rho\mu}\right) + \nabla_\lambda\left(\delta^2\Gamma^\lambda_{\mu\nu}g^{\mu\nu}\right) - \nabla^\mu\left(\delta^2\Gamma^\lambda_{\lambda\mu}\right)\Bigg\} \\ =&\int d^{d+1}x\sqrt{-\gamma}\Bigg\{\frac{1}{4}n_\rho\left(h^\gamma_\gamma\left(2\nabla_\mu h^{\rho\mu} - \nabla^\rho h^\alpha_\alpha\right)\right) - \frac{1}{4}n^\mu\left(h^\gamma_\gamma\nabla^\mu h^\alpha_\alpha\right) - \frac{1}{2}n_\rho\left(h^{\mu\nu}\left(\nabla_\mu h^\rho_\nu + \nabla_\nu h^\rho_\mu - \nabla^\rho h_{\mu\nu}\right)\right) \\ &- \frac{1}{2}n_\nu\left(h^{\mu\nu}\nabla_\mu h^\alpha_\alpha\right) - \frac{1}{2}n_\lambda\left(h^{\lambda\sigma}\left(2\nabla_\mu h^\mu_\sigma - \nabla_\sigma h^\alpha_\alpha\right)\right) + \frac{1}{2}n^\mu\left(h^{\lambda\sigma}\nabla_\mu h_{\lambda\sigma}\right)\Bigg\}\end{aligned} \tag{B.19}$$

By performing partial integration on Eq. (B.18) we obtain

$$\begin{aligned}\delta^2\mathcal{S} = \int d^{d+2}x\sqrt{-g}\Bigg\{&\delta g^{\alpha\beta}\left[\frac{1}{4}\left(\frac{1}{2}(g_{\alpha\beta}g_{\mu\nu}) - g_{\alpha\mu}g_{\beta\nu}\right)R - \frac{1}{2}g_{\alpha\beta}R_{\mu\nu} + g_{\beta\mu}R_{\nu\alpha}\right]\delta g^{\mu\nu}\\ &+ \frac{1}{4}h^{\alpha}_{\alpha}\nabla_\nu\nabla_\mu h^{\mu\nu} + \frac{1}{4}h^{\alpha\beta}\nabla_\alpha\nabla_\beta h^{\mu}_{\mu} - \frac{1}{4}h^{\alpha}_{\alpha}\nabla_\lambda\nabla^\lambda h^{\mu}_{\mu}\\ &- \frac{1}{4}h^{\alpha\beta}\nabla_\mu\nabla_\alpha h^{\mu}_{\beta} - \frac{1}{4}h^{\beta}_{\mu}\nabla_\nu\nabla_\beta h^{\mu\nu} + \frac{1}{4}h^{\alpha\beta}\nabla_\lambda\nabla^\lambda h_{\alpha\beta}\Bigg\}\end{aligned} \tag{B.20}$$

Where the total second order boundary action is given by

$$\begin{aligned}\delta^2\mathcal{S}_{\partial M} = \int d^{d+1}x\sqrt{-h}\Bigg\{&\frac{1}{4}n_\rho\left(h^{\gamma}_{\gamma}\left(2\nabla_\mu h^{\rho\mu} - \nabla^\rho h^{\alpha}_{\alpha}\right)\right) - \frac{1}{4}n_\mu\left(h^{\gamma}_{\gamma}\nabla^\mu h^{\alpha}_{\alpha}\right)\\ &- \frac{1}{2}n_\rho\left(h^{\mu\nu}\left(\nabla_\mu h^{\rho}_{\nu} + \nabla_\nu h^{\rho}_{\mu} - \nabla^\rho h_{\mu\nu}\right)\right)\\ &- \frac{1}{2}n_\nu\left(h^{\mu\nu}\nabla_\mu h^{\alpha}_{\alpha}\right) - \frac{1}{2}n_\lambda\left(h^{\lambda\sigma}\left(2\nabla_\mu h^{\mu}_{\sigma} - \nabla_\sigma h^{\alpha}_{\alpha}\right)\right) + \frac{1}{2}n^\mu\left(h^{\lambda\sigma}\nabla_\mu h_{\lambda\sigma}\right)\\ &- \frac{1}{4}n_\nu\left(h^{\alpha}_{\alpha}\nabla_\mu h^{\mu\nu}\right) - \frac{1}{2}n_\mu\left(h^{\mu\nu}\nabla_\nu h^{\alpha}_{\alpha}\right) + \frac{1}{4}n_\nu\left(h^{\alpha}_{\alpha}\nabla^\nu h^{\beta}_{\beta}\right)\\ &+ \frac{1}{4}n_\sigma\left(h^{\lambda\mu}\nabla_\lambda h^{\sigma}_{\mu}\right) + \frac{1}{4}n_\lambda\left(h^{\sigma}_{\mu}\nabla_\sigma h^{\lambda\mu}\right) - \frac{1}{4}n^\lambda\left(h^{\mu\sigma}\nabla_\lambda h_{\mu\sigma}\right)\Bigg\}.\end{aligned} \tag{B.21}$$

In the above h is the boundary metric and $n_\mu = (\sqrt{g_{rr}}, 0)$ is the normal to the boundary. Explicitly working out the second order terms of

$$\begin{aligned}\delta\left(\sqrt{-g}\left[\bar{\nabla}_\lambda(\delta\Gamma^{\rho}_{\nu\mu}) - \bar{\nabla}_\nu(\delta\Gamma^{\rho}_{\lambda\mu})\right]g^{\mu\nu}\right) = &\frac{1}{2}\delta g^{\alpha}_{\alpha}\left(\nabla_\lambda\nabla_\mu\delta g^{\mu\nu} - \nabla_\lambda\nabla^\lambda\delta g^{\beta}_{\beta}\right)\\ &- \delta g^{\mu\nu}\left(\nabla_\lambda\nabla_\nu\delta g^{\lambda}_{\mu} - \frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\mu\nu}\right)\\ &+ \frac{1}{2}\delta g^{\mu\nu}\nabla_\nu\nabla_\mu\delta g^{\lambda}_{\lambda}.\end{aligned} \tag{B.22}$$

Recall Stokes theorem Eq. (C.1)

$$\int_M d^n x\sqrt{|g|}\nabla_\mu V^\nu = \int_{\partial M} d^{n-1}y\sqrt{|\gamma|}\,n_\mu V^\nu. \tag{B.23}$$

The second order fluctuation of the Christoffel symbols is given by

$$\begin{aligned}0 = \delta^2\left(\nabla_\sigma g_{\mu\nu}\right) &= \left((\delta^2\nabla_\sigma)g_{\mu\nu} + (\delta\nabla_\sigma)\delta g_{\mu\nu}\right)\\ &= \delta^2\Gamma^{\rho}_{\mu\sigma}g_{\rho\nu} + \delta^2\Gamma^{\rho}_{\nu\sigma}g_{\rho\mu} + \delta\Gamma^{\rho}_{\mu\sigma}h_{\rho\nu} + \delta\Gamma^{\rho}_{\nu\sigma}h_{\rho\mu},\\ \implies \delta^2\Gamma^{\rho}_{\mu\nu} &= -\frac{1}{2}h^{\rho\sigma}\left(\nabla_\mu h_{\nu\sigma} + \nabla_\nu h_{\sigma\mu} - \nabla_\sigma h_{\mu\nu}\right).\end{aligned} \tag{B.24}$$

B.3 Interaction between matter fields and the metric

The remaining second order fluctuations are given by

$$
\begin{aligned}
\delta^2\mathcal{S} = \int d^{d+2}x\sqrt{-g}\Bigg\{ &\frac{1}{4}\left(\frac{1}{2}(\delta g^{\mu}_{\mu})^2 - \delta g^{\mu\nu}h_{\mu\nu}\right)\left[-\frac{1}{2}(\partial\phi)^2 + V(\phi) - \frac{1}{4}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2\right] \\
&- \frac{1}{2}g_{\sigma\gamma}\delta g^{\sigma\gamma}\Bigg(\frac{1}{2}\delta g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \left(\delta g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\mu}\delta g^{\beta\nu}\right)\frac{1}{4}\sum_{i=1}^{2} e^{\lambda_i\phi}F_{i,\alpha\beta}F_{i,\mu\nu} \\
&- (\partial\phi)(\partial\delta\phi) + V'(\phi)\delta\phi - \frac{1}{4}\sum_{i=1}^{2}\left(\lambda_i\delta\phi e^{\lambda_i\phi}F_i^2 + 2e^{\lambda_i\phi}F_i f_i\right)\Bigg) \\
&+ \Bigg(\delta g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\delta\phi) - \frac{1}{4}\left(\delta g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\mu}\delta g^{\beta\nu}\right)\sum_{i=1}^{2}\big(\lambda_i e^{\lambda_i\phi}F_{i,\alpha\beta}F_{i,\mu\nu}\delta\phi \\
&+ e^{\lambda_i\phi}f_{i,\alpha\beta}F_{i,\mu\nu} + e^{\lambda_i\phi}F_{i,\alpha\beta}f_{i,\mu\nu}\big)\Bigg) \\
&+ \frac{1}{4}\delta g^{\alpha\mu}\delta g^{\beta\nu}\sum_{i=1}^{2} e^{\lambda_i\phi}F_{i,\alpha\beta}F_{i,\mu\nu}\Bigg\}.
\end{aligned}
\tag{B.25}
$$

By simplifying we obtain

$$
\begin{aligned}
\delta^2\mathcal{S} = \int d^{d+2}x\sqrt{-g}\Bigg\{ &\delta g^{\alpha\beta}\Bigg[\frac{1}{4}\left(\frac{1}{2}(g_{\alpha\beta}g_{\mu\nu}) - g_{\alpha\mu}g_{\beta\nu}\right)\left[-\frac{1}{2}(\partial\phi)^2 + V(\phi) - \frac{1}{4}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2\right] \\
&- \frac{1}{4}g_{\alpha\beta}(\partial_\mu\phi)(\partial_\nu\phi) + \frac{1}{4}g_{\alpha\beta}g^{\gamma\sigma}\sum_{i=1}^{2} e^{\lambda_i\phi}\left(F_{i,\mu\gamma}F_{i,\nu\sigma}\right) + \frac{1}{4}\sum_{i=1}^{2} e^{\lambda_i\phi}F_{i,\alpha\mu}F_{i,\beta\nu}\Bigg]\delta g^{\mu\nu} \\
&- \left[\delta\phi\frac{1}{2}\left(\partial_\mu\phi\nabla^\mu\right) + \frac{1}{2}\sum_{i=1}^{2} a_{i,\mu}e^{\lambda_i\phi}F_i^{\mu\nu}\nabla_\nu\right]g_{\alpha\beta}\delta g^{\alpha\beta} \\
&- \left(\delta g^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\delta\phi) + \frac{1}{2}\delta g^{\alpha\beta}g^{\mu\nu}\sum_{i=1}^{2}\left(\lambda_i e^{\lambda_i\phi}F_{i,\alpha\mu}F_{i,\beta\nu}\delta\phi + 2e^{\lambda_i\phi}F_{i,\alpha\mu}f_{i,\beta\nu}\right)\right)\Bigg\}.
\end{aligned}
\tag{B.26}
$$

B.4 Fixing the gauge

We choose the same gauge as was chosen in [51]. We thus chose the gauge in which

$$
\begin{aligned}
h_{r\mu} &= 0, \\
a_r &= 0.
\end{aligned}
\tag{B.27}
$$

There is are still some gauge degrees of freedom left, and we will construct pure gauge solutions as a solution to this.

B.5 Linearised equations of motion

Using Eqs. (B.4), (B.20) and (B.26), we determine the linearised equations of motion for our system.

Linearised equations of motion $\delta\phi$ component

The linearised equations for the $\delta\phi$ component is given by

$$\begin{aligned}
&\left(\Box + V''(\phi) - \frac{1}{4}(\lambda_i^2 e^{\lambda_i \phi}\bar{F}_i^2)\right)\delta\phi - \frac{1}{2}\sum_{i=1}^{2}\left(\lambda_i e^{\lambda_i\phi}\bar{F}_i^{\mu\nu}\nabla_\mu\right)a_{i,\nu} \\
&+ \frac{1}{2}\left((\Box\phi) + V'(\phi) - \frac{1}{4}\sum_{i=1}^{2}\lambda_i e^{\lambda_i\phi}F_i^2 + \partial_\mu\phi\nabla^\mu\right)g^{\alpha\beta}h_{\alpha\beta} \\
&- \left(\nabla_\beta\left(\partial_\alpha\phi h^{\alpha\beta}\right) - \frac{1}{2}g^{\mu\nu}\sum_{i=1}^{2}\lambda_i e^{\lambda_i\phi}F_{i,\alpha\mu}F_{i,\beta\nu}h^{\alpha\beta}\right) = 0, \qquad \text{(B.28)} \\
\implies &\sqrt{-g}\left(\Box + V''(\phi) + \frac{\lambda_i^2\rho_i F_{irt}}{4\sqrt{-g}}\right)\delta\phi + \frac{1}{2}\sum_{i=1}^{2}\lambda_i\rho_i\left(\delta_t^\nu\nabla_r - \delta_r^\nu\nabla_t\right)a_{i,\nu} \\
&+ g^{rr}\partial_r h_\alpha^\alpha - \left(\nabla_\beta\left(\partial_\alpha\phi h^{\alpha\beta}\right) - \frac{1}{2}g^{\mu\nu}\sum_{i=1}^{2}\lambda_i e^{\lambda_i\phi}F_{i,\alpha\mu}F_{i,\beta\nu}h^{\alpha\beta}\right) = 0.
\end{aligned}$$

In the above we used $\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right)$ and $\left(\sqrt{-g}g^{tt}g^{rr}e^{\lambda_i\phi}F_{irt}\right) = -\rho_i$. In the chosen gauge the above equation becomes

$$\begin{aligned}
&\sqrt{-g}\left(V''(\phi) + \frac{\lambda_i^2\rho_i F_{irt}}{4\sqrt{-g}}\right)\delta\phi + \partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\delta\phi\right) + \frac{1}{2}\sum_{i=1}^{2}\lambda_i\rho_i\partial_r a_{i,t} \\
&+ g^{rr}\partial_r h_\alpha^\alpha + \frac{1}{2}\sqrt{-g}g^{rr}\sum_{i=1}^{2}\lambda_i e^{\lambda_i\phi}F_{i,tr}F_{i,tr}h^{tt} = 0.
\end{aligned} \qquad \text{(B.29)}$$

Linearised equations gauge field components

The linearised equations of motion for the gauge fields give:

$$\begin{aligned}
&\nabla_\mu\left(e^{\lambda_i\bar{\phi}}f_i^{\mu\nu}\right) + \frac{1}{2}\left(\lambda_i e^{\lambda_i\bar{\phi}}\bar{F}_i^{\mu\nu}\nabla_\mu\right)\delta\phi + \left(\frac{1}{2}e^{\lambda_i\phi}F_i^{\mu\nu}\nabla_\mu\right)g^{\alpha\beta}h_{\alpha\beta} \\
&+ \sum_{i=1}^{2}\left\{-\nabla_\beta\left(e^{\lambda_i\phi}F_i^{\alpha\nu}h_\alpha^\beta\right) + e^{\lambda_i\phi}F^{\alpha\beta}\nabla_\beta h_\alpha^\nu\right\} = 0, \\
\implies &\nabla_\mu\left(\sqrt{-g}e^{\lambda_i\bar{\phi}}f_i^{\mu\nu}\right) - \frac{1}{2}\rho_i\left(\delta_t^\nu\nabla_r - \delta_r^\nu\nabla_t\right)\left(\lambda_i\delta\phi + h_\alpha^\alpha\right) \\
&+ \sum_{i=1}^{2}\sqrt{-g}\left\{-\nabla_\beta\left(e^{\lambda_i\phi}F_i^{\alpha\nu}h_\alpha^\beta\right) + e^{\lambda_i\phi}F^{\alpha\beta}\nabla_\beta h_\alpha^\nu\right\} = 0.
\end{aligned} \qquad \text{(B.30)}$$

Without covariant derivatives this becomes

$$\begin{aligned}
&\frac{1}{2}\partial_\mu\left(\sqrt{-g}e^{\lambda_i\phi}F^{\mu\nu}h_\alpha^\alpha\right) - \partial_\alpha\left(\sqrt{-g}e^{\lambda_i\phi}F_\beta{}^\nu h^{\alpha\beta}\right) + \partial_\mu\left(\sqrt{-g}e^{\lambda_i\phi}h^{\nu\beta}F_\beta{}^\mu\right) \\
&\qquad + \partial_\mu\left(\sqrt{-g}\lambda_i\delta\phi e^{\lambda_i\phi}F^{\mu\nu}\right) + \partial_\mu\left(\sqrt{-g}e^{\lambda_i\phi}f^{\mu\nu}\right) = 0.
\end{aligned} \qquad \text{(B.31)}$$

Linearised equations of motion metric components

The linearised equations of motion for the metric components are given by

$$
\begin{aligned}
&\frac{1}{4}\left((g_{\alpha\beta}h^{\mu}_{\mu}) - 2h_{\alpha\beta}\right)\left[R - \frac{1}{2}(\partial\phi)^2 + V(\phi) - \frac{1}{4}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2\right] - \frac{1}{2}g_{\alpha\beta}R_{\mu\nu}h^{\mu\nu} - \frac{1}{2}R_{\alpha\beta}h^{\mu}_{\mu} + 2R_{\nu\alpha}h^{\nu}_{\beta} \\
&+ \frac{1}{2}g_{\alpha\beta}\nabla_\nu\nabla_\mu h^{\mu\nu} + \frac{1}{2}\nabla_\alpha\nabla_\beta h^{\mu}_{\mu} - \frac{1}{2}g_{\alpha\beta}\nabla_\lambda\nabla^\lambda h^{\mu}_{\mu} \\
&- \frac{1}{2}\nabla_\mu\nabla_\alpha h^{\mu}_{\beta} - \frac{1}{2}g_{\alpha\mu}\nabla_\nu\nabla_\beta h^{\mu\nu} + \frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\alpha\beta} \\
&+ \frac{1}{4}g_{\alpha\beta}(\partial_\mu\phi)(\partial_\nu\phi)h^{\mu\nu} + \frac{1}{4}(\partial_\alpha\phi)(\partial_\beta\phi)h^{\mu}_{\mu} + \frac{1}{4}g_{\alpha\beta}g^{\gamma\sigma}\sum_{i=1}^{2} e^{\lambda_i\phi}\left(F_{i,\mu\gamma}F_{i,\nu\sigma}\right)h^{\mu\nu} \\
&+ \frac{1}{4}g^{\gamma\sigma}\sum_{i=1}^{2} e^{\lambda_i\phi}\left(F_{i,\alpha\gamma}F_{i,\beta\sigma}\right)h^{\mu}_{\mu} + \frac{1}{2}\sum_{i=1}^{2} e^{\lambda_i\phi}F_{i,\alpha\mu}F_{i,\beta\nu}h^{\mu\nu} \\
&+ \frac{1}{2}g_{\alpha\beta}\left[\left(-(\partial_\mu\phi)\partial^\mu + V'(\phi) - \frac{1}{4}\sum_{i=1}^{2}\lambda_i e^{\lambda_i\phi}F_i^2\right)\delta\phi - \frac{1}{2}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i f_i\right] \\
&- \left((\partial_\alpha\phi)(\partial_\beta\delta\phi) - \frac{1}{2}g^{\mu\nu}\sum_{i=1}^{2}\left(\lambda_i e^{\lambda_i\phi}F_{i,\alpha\mu}F_{i,\beta\nu}\delta\phi + 2e^{\lambda_i\phi}F_{i,\alpha\mu}f_{i,\beta\nu}\right)\right) = 0
\end{aligned}
\tag{B.32}
$$

With gauge choice (B.27) the above equation becomes

$$
\begin{aligned}
&\frac{1}{2}h_{\alpha\beta}\left[R - \frac{1}{2}(\partial\phi)^2 + V(\phi) - \frac{1}{4}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i^2\right] - \frac{1}{2}g_{\alpha\beta}R_{\mu\nu}h^{\mu\nu} + 2R_{\nu\alpha}h^{\nu}_{\beta} \\
&+ \frac{1}{2}g_{\alpha\beta}\nabla_\nu\nabla_\mu h^{\mu\nu} + \frac{1}{2}\nabla_\alpha\nabla_\beta h^{\mu}_{\mu} - \frac{1}{2}g_{\alpha\beta}\nabla_\lambda\nabla^\lambda h^{\mu}_{\mu} \\
&- \frac{1}{2}\nabla_\mu\nabla_\alpha h^{\mu}_{\beta} - \frac{1}{2}\nabla_\nu\nabla_\beta h^{\nu}_{\alpha} + \frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\alpha\beta} \\
&+ \frac{1}{2}\sum_{i=1}^{2} e^{\lambda_i\phi}\left(F_{i,t\alpha}F_{i,t\beta}\right)h^{tt} \\
&+ \frac{1}{2}g_{\alpha\beta}\left[\left(-(\partial_\mu\phi)\partial^\mu + V'(\phi) - \frac{1}{4}\sum_{i=1}^{2}\lambda_i e^{\lambda_i\phi}F_i^2\right)\delta\phi - \frac{1}{2}\sum_{i=1}^{2} e^{\lambda_i\phi}F_i f_i\right] \\
&- (\partial_\alpha\phi)(\partial_\beta\delta\phi) + \frac{1}{2}g^{\mu\nu}\sum_{i=1}^{2}\left(\lambda_i e^{\lambda_i\phi}F_{i,\alpha\mu}F_{i,\beta\nu}\delta\phi + 2e^{\lambda_i\phi}F_{i,\alpha\mu}f_{i,\beta\nu}\right) = 0.
\end{aligned}
\tag{B.33}
$$

B.5.1 In summary; the equations of motion

So in summary we obtain the following equations of motion

Linearised equation of motion dilaton field

$$\begin{aligned} &\sqrt{-g}\left(V''(\phi)+\frac{\lambda_i^2\rho_i F_{irt}}{4\sqrt{-g}}\right)\delta\phi+\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\delta\phi\right)+\frac{1}{2}\sum_{i=1}^{2}\lambda_i\rho_i\partial_r a_{i,t}\\ &+g^{rr}\partial_r h^\alpha_\alpha+\frac{1}{2}\sqrt{-g}g^{rr}\sum_{i=1}^{2}\lambda_i e^{\lambda_i\phi}F_{i,tr}F_{i,tr}h^{tt}=0, \end{aligned} \tag{B.34}$$

Linearised equations of motion gauge fields

$$\begin{aligned} &\frac{1}{2}\partial_\mu\left(\sqrt{-g}e^{\lambda_i\phi}F^{\mu\nu}h^\alpha_\alpha\right)-\partial_\alpha\left(\sqrt{-g}e^{\lambda_i\phi}F_\beta{}^\nu h^{\alpha\beta}\right)+\partial_\mu\left(\sqrt{-g}e^{\lambda_i\phi}h^{\nu\beta}F_\beta{}^\mu\right)\\ &+\partial_\mu\left(\sqrt{-g}\lambda_i\delta\phi e^{\lambda_i\phi}F^{\mu\nu}\right)+\partial_\mu\left(\sqrt{-g}e^{\lambda_i\phi}f^{\mu\nu}\right)=0, \end{aligned} \tag{B.35}$$

Linearised equations of motion metric perturbation

$$\begin{aligned} &\frac{1}{2}h_{\alpha\beta}\left[R-\frac{1}{2}(\partial\phi)^2+V(\phi)-\frac{1}{4}\sum_{i=1}^{2}e^{\lambda_i\phi}F_i^2\right]-\frac{1}{2}g_{\alpha\beta}R_{\mu\nu}h^{\mu\nu}+2R_{\nu\alpha}h^\nu_\beta\\ &+\frac{1}{2}g_{\alpha\beta}\nabla_\nu\nabla_\mu h^{\mu\nu}+\frac{1}{2}\nabla_\alpha\nabla_\beta h^\mu_\mu-\frac{1}{2}g_{\alpha\beta}\nabla_\lambda\nabla^\lambda h^\mu_\mu\\ &-\frac{1}{2}\nabla_\mu\nabla_\alpha h^\mu_\beta-\frac{1}{2}\nabla_\nu\nabla_\beta h^\nu_\alpha+\frac{1}{2}\nabla_\lambda\nabla^\lambda h_{\alpha\beta}\\ &+\frac{1}{2}\sum_{i=1}^{2}e^{\lambda_i\phi}\left(F_{i,t\alpha}F_{i,t\beta}\right)h^{tt}\\ &+\frac{1}{2}g_{\alpha\beta}\left[\left(-(\partial_\mu\phi)\partial^\mu+V'(\phi)-\frac{1}{4}\sum_{i=1}^{2}\lambda_i e^{\lambda_i\phi}F_i^2\right)\delta\phi-\frac{1}{2}\sum_{i=1}^{2}e^{\lambda_i\phi}F_i f_i\right]\\ &-(\partial_\alpha\phi)(\partial_\beta\delta\phi)+\frac{1}{2}g^{\mu\nu}\sum_{i=1}^{2}\left(\lambda_i e^{\lambda_i\phi}F_{i,\alpha\mu}F_{i,\beta\nu}\delta\phi+2e^{\lambda_i\phi}F_{i,\alpha\mu}f_{i,\beta\nu}\right)=0. \end{aligned} \tag{B.36}$$

Appendix C

Some First Order Boundary Terms

These are some notes on the first order boundary action.

C.1 First order boundary action

The boundary terms obtained from partial integration are not important for the classical field equations in the bulk, but they are important to make the system finite on the boundary of the space. First recall that Stokes theorem gives

$$\int_M d^n x \sqrt{|g|} \nabla_\mu V^\nu = \int_{\partial M} d^{n-1} y \sqrt{|\gamma|}\, n_\mu V^\nu, \tag{C.1}$$

Where γ_{ij} is the induced metric on the boundary of the manifold M and n_μ is a vector orthonormal to the boundary ∂M which is given by $n^\mu = (\sqrt{g^{rr}}, 0)$ for metric ansatz (A.9). We are now able to integrate by parts such that the first variation of the action describing the bulk will be zero for classical solutions and we will only remain with boundary terms

$$\delta \mathcal{S}_r = \int_{\partial M_r} \left\{ \frac{1}{2} T^{ab} \delta\gamma_{ab} + \sum_{i=1}^{2} J_i^\mu \delta A_{i,\mu} + \mathcal{O}_\phi \delta\phi \right\} \Bigg|_r . \tag{C.2}$$

We will give the boundary terms below

metric For the metric tensor we need to add the Gibbons-Hawking-York boundary-term and add a counter-term in the boundary. The counter-term is the same one as one would obtain from the energy momentum tensor and may thus be inspired by [47]

dilaton field Using stokes theorem we obtain the following boundary term for the dilaton field The boundary action of the first variation w.r.t. the dilaton field gives

$$\int_M d^n x \sqrt{-g} \nabla_\mu \left(\phi \nabla^\mu \phi\right) = \int_{\partial M} d^{n-1} x \sqrt{-\gamma}\, n_\mu \left(\phi \nabla^\mu \phi\right). \tag{C.3}$$

The first variation of the action w.r.t. the scalar field is given by

$$\begin{aligned} \delta_\phi S_c &= \int_{\partial M_r} d^{n-1} x \sqrt{-\gamma}\, \left(2 n_\mu \nabla^\mu \phi\right) \delta\phi, \\ \implies \mathcal{O}_\phi &= \sqrt{-\gamma}\, \left(2 n_\mu \nabla^\mu \phi\right). \end{aligned} \tag{C.4}$$

Substituting the solution from Eq. (4.11) we obtain

$$\begin{aligned}
\delta_\phi S_c &= \int_{\partial M_r} d^{n-1}x\sqrt{-\gamma}\,(2n_\mu \nabla^\mu \phi)\,\delta\phi, \\
&= \int_{\partial M_r} d^{n-1}x\,\sqrt{-\gamma}\sqrt{g^{rr}}\,(2\partial_r \phi)\,\delta\phi, \\
&= \int_{\partial M_r} d^{n-1}x\,\sqrt{-g}g^{rr}\,(2\partial_r \phi)\,\delta\phi \\
&= \int_{\partial M_r} d^{n-1}x\,\sqrt{8d(1+\alpha)(\alpha+z-1)}f(r)r^{(d-1)+z+(d+2)\alpha+2-2\alpha-1}\delta\phi, \\
&\overset{r\to\infty}{\to} \int_{\partial M_r} d^{n-1}x\,\sqrt{8d(1+\alpha)(\alpha+z-1)}\left(r^{d+z-\theta}\right)\delta\phi.
\end{aligned} \tag{C.5}$$

In the above equation we used

$$\sqrt{-\gamma} = \sqrt{f(r)}r^{d+z+(d+1)\alpha} = \sqrt{f(r)}r^{d+z-(d+1)\theta/d} \tag{C.6}$$

and noted the fact that the leading term in $f(r)$ is 1.

gauge fields For the gauge fields the boundary term for the first variation of the action w.r.t. the gauge fields is given by

$$\begin{aligned}
&\int_M d^n x\sqrt{-g}\sum_{i=1}^{2}\nabla_\mu\left(4\delta A_{i,\nu}F_i^{\mu\nu}e^{\lambda_i\phi}\right) \\
&= \int_{\partial M_r} d^{n-1}x\sum_{i=1}^{2}\sqrt{-\gamma}\,n_\mu\left(4F_i^{\mu\nu}e^{\lambda_i\phi}\right)\delta A_{i,\nu}, \\
&= \int_{\partial M_r} d^{n-1}x\sum_{i=1}^{2}\sqrt{-\gamma}\,\sqrt{g^{rr}}g^{tt}\left(4F_{i,rt}e^{\lambda_i\phi}\right)\delta A_{i,t}, \\
&= -\int_{\partial M_r} d^{n-1}x\sum_{i=1}^{2} r^{d-z+\alpha(d-2)+1}\left(4r^{(2-d)\alpha+z-(d+1)}\rho_i\right)\delta A_{i,t}, \\
&= -\int_{\partial M_r} d^{n-1}x\sum_{i=1}^{2} 4\rho_i\,\delta A_{i,t}.
\end{aligned} \tag{C.7}$$

Divergence on shell action

$$\begin{aligned}
\mathcal{S} &= -\frac{1}{16\pi G}\int d^{d+2}x\sqrt{-g}\left[-\frac{1}{2}(\partial\phi)^2+V(\phi)\right] \\
&\propto -\int d^{d+2}x\sqrt{-g}\left[-\frac{g^{rr}}{2}(\partial_r\phi)^2+V_0e^{\gamma\phi}\right] \\
&= -\int d^{d+2}xr^{(d-1)+z+(d+2)\alpha}\left[-\frac{f(r)r^{2(1-\alpha)}}{2}\left(\frac{\beta}{r}\right)^2+V_0e^{\gamma\phi_0}r^{\gamma\beta}\right] \\
&= -\int d^{d+2}xr^{(d-1)+z+d\alpha}\left[-\frac{f(r)\beta^2}{2}+V_0e^{\gamma\phi_0}\right] \\
&= -\int d^{d+2}xr^{(d-1)+z-\theta}\left[-\frac{f(r)\beta^2}{2}+V_0e^{\gamma\phi_0}\right] \\
&= -\int_{\partial\mathcal{M}_r} d^{d+1}x\; r^{d+z-\theta}\left[-\frac{f(r)\beta^2}{2}+V_0e^{\gamma\phi_0}\right]
\end{aligned} \tag{C.8}$$

Bibliography

[1] P. Giraldo-Gallo, J. A. Galvis, Z. Stegen, K. A. Modic, F. F. Balakirev, J. B. Betts, X. Lian, C. Moir, S. C. Riggs, J. Wu, A. T. Bollinger, X. He, I. Bozovic, B. J. Ramshaw, R. D. McDonald, G. S. Boebinger, and A. Shekhter. Scale-invariant magnetoresistance in a cuprate superconductor. 361(6401):479–481. ISSN 0036-8075, 1095-9203. doi: 10.1126/science.aan3178. URL http://arxiv.org/abs/1705.05806.

[2] R. A. Cooper, Y. Wang, B. Vignolle, O. J. Lipscombe, S. M. Hayden, Y. Tanabe, T. Adachi, Y. Koike, M. Nohara, H. Takagi, Cyril Proust, and N. E. Hussey. Anomalous criticality in the electrical resistivity of la2–xSrxCuO4. 323(5914):603–607. ISSN 0036-8075, 1095-9203. doi: 10.1126/science.1165015. URL https://science.sciencemag.org/content/323/5914/603.

[3] S. Martin, A. T. Fiory, R. M. Fleming, L. F. Schneemeyer, and J. V. Waszczak. Normal-state transport properties of $\mathrm{bi}_{2+x}\mathrm{sr}_{2-y}\mathrm{cuo}_{6+\delta}$ crystals. 41(1):846–849. doi: 10.1103/PhysRevB.41.846. URL https://link.aps.org/doi/10.1103/PhysRevB.41.846.

[4] N. E. Hussey, K. Takenaka, and H. Takagi. Universality of the mott-ioffe-regel limit in metals. 84(27):2847–2864. ISSN 1478-6435, 1478-6443. doi: 10.1080/14786430410001716944. URL http://arxiv.org/abs/cond-mat/0404263.

[5] Juan M. Maldacena. The large n limit of superconformal field theories and supergravity. 38(4):1113–1133, . ISSN 00207748. doi: 10.1023/A:1026654312961. URL http://arxiv.org/abs/hep-th/9711200.

[6] A. J. Schofield. Non-fermi liquids. 40(2):95–115. ISSN 0010-7514, 1366-5812. doi: 10.1080/001075199181602. URL http://www.tandfonline.com/doi/abs/10.1080/001075199181602.

[7] O. Gunnarsson, M. Calandra, and J. E. Han. Saturation of electrical resistivity. 75 (4):1085–1099. ISSN 0034-6861, 1539-0756. doi: 10.1103/RevModPhys.75.1085. URL http://arxiv.org/abs/cond-mat/0305412.

[8] W. Pauli. Über gasentartung und paramagnetismus. 41(2):81–102. ISSN 0939-7922. doi: 10.1007/BF01391920. URL https://doi.org/10.1007/BF01391920.

[9] Matteo Baggioli and Mikhail Goykhman. Phases of holographic superconductors with broken translational symmetry. page 34.

[10] Sean A. Hartnoll, Andrew Lucas, and Subir Sachdev. Holographic quantum matter. . URL http://arxiv.org/abs/1612.07324.

[11] Jan Zaanen, Yan Liu, Ya-Wen Sun, and Koenraad Schalm. Holographic duality in condensed matter physics by jan zaanen. URL /core/books/holographic-duality-in-condensed-matter-physics/ED1E01756C18AE0352269202C31FE2B3.

[12] Martin Ammon and Johanna Erdmenger. Gauge/gravity duality by martin ammon. URL /core/books/gaugegravity-duality/16CCA2E431B24AEF1B51B0F9C5BE755E.

[13] Christopher P Herzog. C. n. yang institute for theoretical physics, department of physics and astronomy stony brook university, stony brook, NY 11794. page 58.

[14] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz. Large n field theories, string theory and gravity. 323(3):183–386. ISSN 03701573. doi: 10.1016/S0370-1573(99)00083-6. URL http://arxiv.org/abs/hep-th/9905111.

[15] Juan Maldacena. The gauge/gravity duality. . URL http://arxiv.org/abs/1106.6073.

[16] Jacob D. Bekenstein. Black holes and entropy. 7(8):2333–2346. doi: 10.1103/PhysRevD.7.2333. URL https://link.aps.org/doi/10.1103/PhysRevD.7.2333.

[17] S. W. Hawking. Black hole explosions? 248(5443):30. ISSN 1476-4687. doi: 10.1038/248030a0. URL https://www.nature.com/articles/248030a0.

[18] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov. Gauge theory correlators from non-critical string theory. 428(1):105–114. ISSN 03702693. doi: 10.1016/S0370-2693(98)00377-3. URL http://arxiv.org/abs/hep-th/9802109.

[19] Edward Witten. Anti de sitter space and holography. URL http://arxiv.org/abs/hep-th/9802150.

[20] Sean M. Carroll. Lecture notes on general relativity. URL http://arxiv.org/abs/gr-qc/9712019.

[21] Henk T. C. Stoof, Koos B. Gubbels, and Dennis B. M. Dickerscheid. *Ultracold quantum fields*. Theoretical and mathematical physics. Springer. ISBN 978-1-4020-8762-2 978-1-4020-8763-9. OCLC: ocn310153808.

[22] Andrea Amoretti. *Condensed Matter Applications of AdS/CFT*. Springer Theses. Springer International Publishing. ISBN 978-3-319-61874-6 978-3-319-61875-3. doi: 10.1007/978-3-319-61875-3. URL http://link.springer.com/10.1007/978-3-319-61875-3.

[23] Nabil Iqbal. Holography and strongly correlated systems. URL http://dspace.mit.edu/handle/1721.1/68873.

[24] Andrew Lucas, Subir Sachdev, and Koenraad Schalm. Scale-invariant hyperscaling-violating holographic theories and the resistivity of strange metals with random-field disorder. 89(6). ISSN 1550-7998, 1550-2368. doi: 10.1103/PhysRevD.89.066018. URL http://arxiv.org/abs/1401.7993.

[25] Sean A. Hartnoll, Raghu Mahajan, Matthias Punk, and Subir Sachdev. Transport near the ising-nematic quantum critical point of metals in two dimensions. 89(15):155130, . ISSN 1098-0121, 1550-235X. doi: 10.1103/PhysRevB.89.155130. URL http://arxiv.org/abs/1401.7012.

[26] Nabil Iqbal and Hong Liu. Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm. 79(2):025023. ISSN 1550-7998, 1550-2368. doi: 10.1103/PhysRevD.79.025023. URL http://arxiv.org/abs/0809.3808.

[27] Idse Heemskerk and Joseph Polchinski. Holographic and wilsonian renormalization groups. 2011(6):31. ISSN 1029-8479. doi: 10.1007/JHEP06(2011)031. URL http://arxiv.org/abs/1010.1264.

[28] D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner. Renormalization group flows from holography–supersymmetry and a c-theorem. URL http://arxiv.org/abs/hep-th/9904017.

[29] B. Goutéraux. Charge transport in holography with momentum dissipation. 2014(4):181, . ISSN 1029-8479. doi: 10.1007/JHEP04(2014)181. URL http://arxiv.org/abs/1401.5436.

[30] B. Goutéraux and E. Kiritsis. Quantum critical lines in holographic phases with (un)broken symmetry. 2013(4):53. ISSN 1029-8479. doi: 10.1007/JHEP04(2013)053. URL http://arxiv.org/abs/1212.2625.

[31] B. Goutéraux. Universal scaling properties of extremal cohesive holographic phases. 2014 (1):80, . ISSN 1029-8479. doi: 10.1007/JHEP01(2014)080. URL http://arxiv.org/abs/1308.2084.

[32] Jakob Gath, Jelle Hartong, Ricardo Monteiro, and Niels A. Obers. Holographic models for theories with hyperscaling violation. 2013(4). ISSN 1029-8479. doi: 10.1007/JHEP04(2013)159. URL http://arxiv.org/abs/1212.3263.

[33] Sean A. Hartnoll and Andreas Karch. Scaling theory of the cuprate strange metals. 91 (15):155126. doi: 10.1103/PhysRevB.91.155126. URL https://link.aps.org/doi/10.1103/PhysRevB.91.155126.

[34] Andreas Karch. Conductivities for hyperscaling violating geometries. 2014(6):140. ISSN 1029-8479. doi: 10.1007/JHEP06(2014)140. URL http://arxiv.org/abs/1405.2926.

[35] E. Mauri. Fluctuations in the holographic superconductor. URL http://dspace.library.uu.nl/handle/1874/351598.

[36] Wissam Chemissany and Ioannis Papadimitriou. Lifshitz holography: the whole shebang. 2015(1):52. ISSN 1029-8479. doi: 10.1007/JHEP01(2015)052. URL https://doi.org/10.1007/JHEP01(2015)052.

[37] J. de Boer, E. Verlinde, and H. Verlinde. On the holographic renormalization group. 2000(8):003–003. ISSN 1029-8479. doi: 10.1088/1126-6708/2000/08/003. URL http://arxiv.org/abs/hep-th/9912012.

[38] Brandon W. Langley and Philip W. Phillips. Quantum critical diffusion and thermodynamics in lifshitz holography. URL http://arxiv.org/abs/1812.08164.

[39] Daniel R. Gulotta, Christopher P. Herzog, and Matthias Kaminski. Sum rules from an extra dimension. 2011(1):148. ISSN 1029-8479. doi: 10.1007/JHEP01(2011)148. URL http://arxiv.org/abs/1010.4806.

[40] Gary T. Horowitz, Jorge E. Santos, and David Tong. Optical conductivity with holographic lattices. 2012(7):168. ISSN 1029-8479. doi: 10.1007/JHEP07(2012)168. URL https://doi.org/10.1007/JHEP07(2012)168.

[41] D. van der Marel, H. J. A. Molegraaf, J. Zaanen, Z. Nussinov, F. Carbone, A. Damascelli, H. Eisaki, M. Greven, P. H. Kes, and M. Li. Powerlaw optical conductivity with a constant phase angle in high tc superconductors. 425(6955):271–274. ISSN 0028-0836, 1476-4687. doi: 10.1038/nature01978. URL http://arxiv.org/abs/cond-mat/0309172.

[42] Mohsen Alishahiha, Eoin Ó Colgáin, and Hossein Yavartanoo. Charged black branes with hyperscaling violating factor. 2012(11). ISSN 1029-8479. doi: 10.1007/JHEP11(2012)137. URL http://arxiv.org/abs/1209.3946.

[43] Ioannis Papadimitriou. Hyperscaling violating lifshitz holography. URL http://arxiv.org/abs/1411.0312.

[44] Sean A. Hartnoll and Edgar Shaghoulian. Spectral weight in holographic scaling geometries. 2012(7):78. ISSN 1029-8479. doi: 10.1007/JHEP07(2012)078. URL https://doi.org/10.1007/JHEP07(2012)078.

[45] Liza Huijse, Subir Sachdev, and Brian Swingle. Hidden fermi surfaces in compressible states of gauge-gravity duality. 85(3):035121. ISSN 1098-0121, 1550-235X. doi: 10.1103/PhysRevB.85.035121. URL http://arxiv.org/abs/1112.0573.

[46] Alexander Krikun, Aurelio Romero-Bermúdez, Koenraad Schalm, and Jan Zaanen. The anomalous attenuation of plasmons in strange metals and holography. URL http://arxiv.org/abs/1812.03968.

[47] Vijay Balasubramanian and Per Kraus. A stress tensor for anti-de sitter gravity. 208 (2):413–428. ISSN 1432-0916. doi: 10.1007/s002200050764. URL https://doi.org/10.1007/s002200050764.

[48] J. David Brown and James W. York. Quasilocal energy and conserved charges derived from the gravitational action. 47(4):1407–1419. ISSN 0556-2821. doi: 10.1103/PhysRevD.47.1407. URL https://link.aps.org/doi/10.1103/PhysRevD.47.1407.

[49] G. 't Hooft and M. J. G. Veltman. One loop divergencies in the theory of gravitation. URL http://dspace.library.uu.nl/handle/1874/4705.

[50] Matthias Blau. Lecture notes on general relativity. page 963.

[51] Enea Mauri and Henk Stoof. Screening of coulomb interactions in holography. URL http://arxiv.org/abs/1811.11795.