# Statistical-physics models for fluctuations and emergent inequality in economic systems

**Jan Mulder**

Supervisors:
Prof. R. A. Duine
Prof. R. van Roij

Special thanks to:
Prof. A. Salomons

**Abstract**

The model of V. M. Yakovenko and J. Barkley Rosser shows how maximization of entropy leads to exponential income distributions [Rev. Mod. Phys. **81**, 1703.]. The simple agent-based model shows how inequality arises solely based on intrinsic statistical fluctuations. Using basic economic principles, we adapt this model multiple times to incorporate the trade between different types of goods. We find steady-state distributions of goods that are unequal, but different from the exponential distribution Yakovenko found. We examine the results and see that unequal income distributions lead to differences in demand and supply curves, and equilibrium prices, as compared to equal income distributions.

A thesis presented for the degree of
Master of Science

Institute for Theoretical Physics
Utrecht University
The Netherlands
June 28th, 2019

# Contents

# 1 Introduction

In the United States income inequality has been rising for years. You will hear or read this very commonly in the news [2][3]. While this is definitely true, it does not mean that the way income is distributed over citizens is changing drastically. Quite the reverse, the income distribution of the lowest earning 95% has remained quite stable, while the top 5% income has been increasing [4]. Graphically, this is shown in Figure 1.1. While economic mobility, the ability to earn more (or less) than your parents, has been decreasing over time [5], it is still possible to move up (or down) in income. Therefore, the distribution cannot be the result of a static microscopic income configuration. This leads to believe that there may be a different underlying mechanism responsible for this stable lower income distribution. What is the nature of this mechanism and what impact has the resulting emergent inequality on quantities like demand and supply?

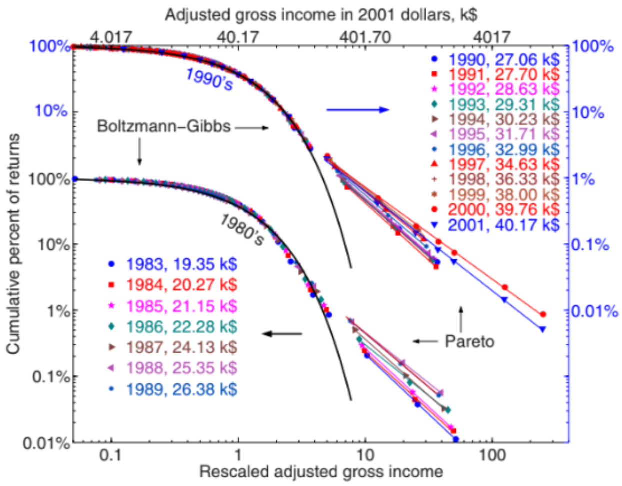



Figure 1.1: Cumulative probability distributions of annual income plotted on log-log scale versus $m/m_0$ (the annual income $m$ divided by the average income $m_0$ in the exponential part of the distribution). The IRS data points are for 1983 to 2001, and the columns of numbers give the values of $m_0$ for the corresponding years [4].

In 2009, physicist Victor Yakovenko attempted to answer these questions and developed a very simple agent-based model that accurately predicts these income distributions solely based on intrinsic statistical fluctuations [1]. It is important that agents in such models are identical so the emergent income inequality can really be pinned on just these statistical fluctuations. Results can be explained mathematically in a similar way as one could do in the field of physics. Specifically, Yakovenko concluded that the *entropy* of the income distribution is maximized by exponential distributions. The field this type of research belongs to is therefore sometimes called *econophysics*.

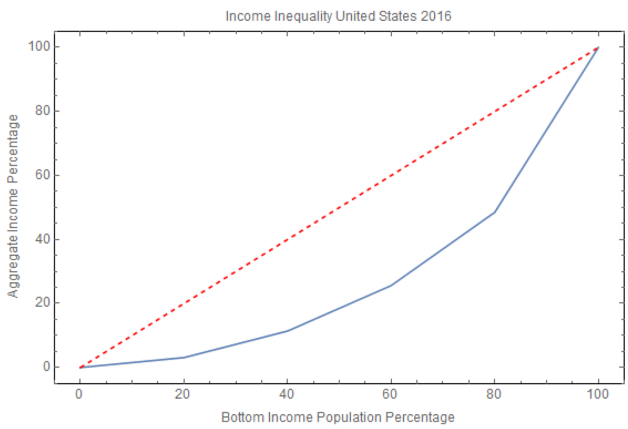



Figure 1.2: (blue) Lorenz curve of income distribution of the United States in 2016 based off research conducted by U.S. Census Bureau [6]. (red, dashed) Perfectly equal distribution.

Under this flag, we intend to reproduce and extend Yakovenko's model to incorporate the trade of different types of goods between agents, by which we try to simulate a real economy slightly better. Moreover, the introduction of two types of goods potentially paves the way for making the connection to cyclic processes in analogy to thermodynamics, although we will only briefly touch upon this aspect as a possible outlook for future research. Along the way, we will try to match resulting distributions with recent data and see if it makes any sense. In Figure 1.2 we plotted recent data on the income distribution of the United States in 2016 in a so-called Lorenz curve. This type of curve plots the aggregate income percentage of the lowest incomes up until certain percentages of the population. Throughout this thesis we will see how well an exponential curve fits this profile, and how unique that actually is. This will also show that recent income distributions (Figure 1.2) are still very similar to income distributions we saw back in the 80s and 90s (Figure 1.1).

The thesis is divided into two parts. In the first part we will reproduce some of Yakovenko's work from his 2009 article [1] and we will see how well it fits the actual money distribution of the United States. We will continue by reproducing a different agent-based model known as the multiplicative exchange model, which we will supplement with a way to solve it numerically. We proceed by adapting the Yakovenko model such that it - arguably - simulates more realistic market-like trades and we find that our adapted model also exhibits emergent inequality, but with different final distributions than the original model.

The second part will revolve around utility theory, which is an economic theory that bases itself off the assumption that there exists a quantity that a person wants to maximize: utility. We managed to develop a more realistic market model with demand and supply curves generated from this utility maximization. This model does not exhibit emergent inequality, but it does show stable unequal distributions. Furthermore, the impact of these unequal distributions compared to a perfectly equal distribution - i.e. the distribution of a system in which all agents own the same amount of goods - on demand, supply, and equilibrium prices is calculated.

Then, finally, an interesting phenomenon we stumbled upon on the way is presented as a possible point of interest for future research. In this side track the behavior of multiple interacting economic systems is compared to the behavior of multiple interacting gases. This leads to results that show economic systems expand and contract adiabatically under circumstances that would normally induce these effects in thermodynamic systems.

# Part I
# Reproduction and Linear Models

## 2 The System of Interacting Agents

Conserved quantities are very important quantities in the field of thermodynamics. However, they do not only show up in physics. For example, in a simple economy one could consider money to be a conserved quantity just as energy is conserved in closed thermodynamic systems. In thermodynamics, the second law states that - in a closed system - *entropy* can never decrease over time. Then, can we say the entropy of the money distribution in an economy is maximized as well?

### 2.1 The Yakovenko Model

#### 2.1.1 Introduction to the Yakovenko Model

A very simple economic model is one described by Yakovenko [1]. This model describes the very simple interaction between a certain number of agents in an economic system. Every agent only has one property: a certain amount of money $m_i > 0$, with $i = 1, 2, ..., N$ labeling the agents, and $N \in \mathbb{N}$ the number of agents in the system. The agents are all given an equal amount of money at the start of the process, $m_i = m_0$ for all agents. The model only consists of one type of event. This event is a transaction of money between two agents; a certain amount of money is transferred from one agent to another. One time step is defined by the following algorithm:

- Agent $i$, and agent $j$ are randomly selected to make a transaction from $i$ to $j$.
- A random amount of money, $\Delta \in [0, \Delta_{max}]$, is chosen, with $\Delta_{max}$ the maximum transaction.
- The system checks if agent $i$ has enough money to afford the payment, i.e. if $m_i \geq \Delta$.
- If the previous condition is satisfied, the money of agent $i$ will be reduced and the money of agent $j$ will be increased by the amount of the transfer:

$$m_i \to m_i - \Delta, \tag{2.1}$$

$$m_j \to m_j + \Delta. \tag{2.2}$$

Doing only one or a few transactions does not lead to interesting behavior. Therefore, a 'sweep' of transactions is introduced. A sweep is defined as $N$ independent transactions, making sure that on average every agent is selected once for receiving, and once for giving an amount of money. After doing more and more sweeps, a system will go to equilibrium, i.e. the probability distribution of the money becomes independent on time.

#### 2.1.2 Solving the Model

A way to calculate this probability distribution in equilibrium is found by looking at the Fokker-Planck equation of this model. This equation describes the change of the distribution over time,

$$\begin{aligned} \frac{dP(m)}{dt} = \int_0^\infty dm' \int_{-\infty}^{\infty} d\Delta \Big[ & f_{[m+\Delta, m'-\Delta] \to [m, m']} P(m+\Delta) P(m'-\Delta) \\ & - f_{[m,m'] \to [m+\Delta, m'-\Delta]} P(m) P(m') \Big] = 0. \end{aligned} \tag{2.3}$$

Here, the $f_{[m+\Delta, m'-\Delta] \to [m, m']}$ is the rate at which two agents with respective money $m + \Delta$ and $m' - \Delta$ trade the amount of $\Delta$ to end up with respective money $m$ and $m'$, and $f_{[m,m'] \to [m+\Delta, m'-\Delta]}$ is the rate at which the opposite event occurs. This equation looks complicated, but is easier to solve once we realize that the two rates are actually equal. If a transaction is possible, i.e. none of the agents involved end with a negative amount of money, then the reversed transaction is also possible. And as every possible transaction has an equal probability to occur, the rates have to be equal. In physics this is what we call *detailed balance* [7].

With this in mind, we can find a solution of the equation in

$$P(m)P(m') = P(m + \Delta)P(m' - \Delta), \tag{2.4}$$

which, for $\Delta \neq 0$, can only be solved by an exponential distribution,

$$P(m) = Ae^{Bm}, \tag{2.5}$$

with constants $A$ and $B$ to be determined out of constraints for normalization and conserved money, which we will look into in a second. We will show that exponential distributions maximize *entropy*, a quantity that is related to the amount of different microscopic configurations with the same distribution of money. The second law of thermodynamics states that - again, in a closed system - entropy can never decrease over time, which means that in thermodynamic equilibrium, entropy is maximized. Entropy $S$ is defined as

$$S = -\langle \log[P(m)] \rangle = -\int_0^\infty dm P(m) \log[P(m)]. \tag{2.6}$$

We calculate the distribution function $P(m)$, at which the entropy is at a maximum. We do this by calculating the functional derivative of the entropy $S$, and then setting it to zero:

$$\frac{\delta S[P(m)]}{\delta P(m)} = 0. \tag{2.7}$$

However, as stated before, the distribution has to be properly normalized, which leads to a first constraint,

$$\int_0^\infty dm P(m) = 1. \tag{2.8}$$

Furthermore, we know the total amount of money, and the number of agents are conserved, and therefore the average amount of money per agent should be constant at all times. So we also set the following constraint,

$$\langle m_i \rangle = \int_0^\infty dm P(m) m = m_0. \tag{2.9}$$

We calculate the functional derivative while taking into account the two Lagrange multipliers $\lambda$, and $\eta$, which belong to the constraints. Their values will later be determined. All together, the next expression is what we will be working out.

$$\frac{\delta}{\delta P(m)} \left\{ -\int_0^\infty dm' P(m') \log P(m') + \lambda \left[ \int_0^\infty dm' P(m') - 1 \right] + \eta \left[ \int_0^\infty dm' P(m') m' - m_0 \right] \right\} = 0. \tag{2.10}$$

Basic calculations will show this leads to a simple expression for $P(m)$.

$$P(m) = e^{-1+\lambda+\eta m}. \tag{2.11}$$

The Lagrange multipliers $\lambda$, and $\eta$ can now be evaluated by filling in the solution for $P(m)$ into the constraints. We start with $\lambda$.

$$\int_0^\infty dm\ e^{-1+\lambda+\eta m} = 1. \tag{2.12}$$

Because this integration goes from 0 to $\infty$, we see already that $\eta < 0$ in order for this integration to converge. So while this is kept in mind, we find

$$e^{-1+\lambda} = -\eta, \text{ with } \eta < 0. \tag{2.13}$$

To calculate $\eta$ we need to fill in our adapted $P(m) = -\eta \exp(\eta m)$ into the second constraint.

$$\int_0^\infty dm\, (-\eta e^{\eta m} m) = m_0. \tag{2.14}$$

We can solve this with partial integration.

$$-\eta \int_0^\infty dm\ e^{\eta m} m = -\eta \int_{m=0}^{m=\infty} d\left(\frac{e^{\eta m}}{\eta}\right) m = \lim_{m\to\infty} \left(-m + \frac{1}{\eta}\right) e^{\eta m} + \lim_{m\to 0} \left(m - \frac{1}{\eta}\right) e^{\eta m} = -\frac{1}{\eta} = m_0. \tag{2.15}$$

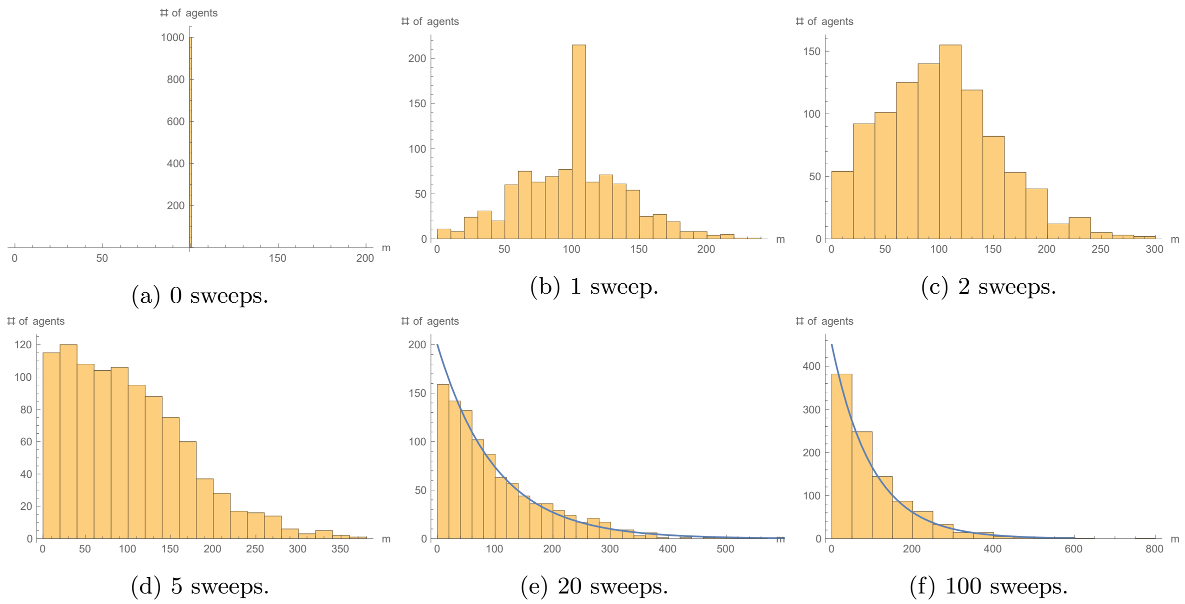
So if we substitute $\eta = -1/m_0$ into our expression for $P(m)$, we get the final result:

$$P(m) = \frac{1}{m_0} e^{-\frac{m}{m_0}}. \tag{2.16}$$

This distribution is called the Boltzmann distribution. It is the result of maximized entropy by the second law of thermodynamics.

### 2.1.3 Results

We now proceed to run the actual simulation and see if our prediction makes sense. We prepare a system with $N = 1000$ agents, all with an amount of money $m_0 = 100$ to start with. Figure 2.1 shows histograms of the money distribution in the system after a specified number of sweeps. It is quite clear that for large numbers of sweeps the distribution tends to the calculated (not normalized) distribution function (plotted in blue). Therefore, it seems that the second law of thermodynamics applies to the Yakovenko model.




(a) 0 sweeps. (b) 1 sweep. (c) 2 sweeps.

(d) 5 sweeps. (e) 20 sweeps. (f) 100 sweeps.

Figure 2.1: Money distribution in the Yakovenko model after certain amounts of sweeps. (blue) Money distribution $P(m) \propto e^{-m/100}$.

Let us finish this section with a first attempt to reconstruct the Lorenz curve shown in Figure 1.2. First, we need to know how to convert distribution functions to Lorenz curves. We introduce the lowest income fraction of the population, $f$, and we want to see how much income they have together. We calculate the highest income, $h_f$, of this fraction by performing the integral,

$$\int_0^{h_f} dm P(m) = f. \tag{2.17}$$

Working this out for the Boltzmann distribution, we get

$$\begin{aligned} \int_0^{h_f} dm \frac{1}{m_0} e^{-m/m_0} &= f \\ 1 - e^{-h_f/m_0} &= f \\ h_f &= -m_0 \log(1-f). \end{aligned} \tag{2.18}$$

Now the income the fraction $f$ accounts for together $M(f)$ is just the expectation value of $m$ in the interval $[0, h_f]$, multiplied by the amount of agents in the fraction, $N$:

$$\begin{aligned} M(f) &= N \int_0^{-m_0 \log(1-f)} dm \, \frac{m}{m_0} e^{-m/m_0} \\ &= N m_0 [f + (1-f)\log(1-f)]. \end{aligned} \tag{2.19}$$

To tune the aggregate income, $M(f)$, to the aggregate income percentage, $m(f)$, we divide over the total amount of money in the system $Nm_0$, and scale to make percentages out of the fractions.

$$m(f) = 100 \left[ \frac{f}{100} + \left(1 - \frac{f}{100}\right) \log\left(1 - \frac{f}{100}\right) \right]. \tag{2.20}$$

In Figure 2.2 this curve is plotted together with the actual data. They align very well, beside the fact that the upper 5% should not be Boltzmann distributed, which should lead to a small correction.

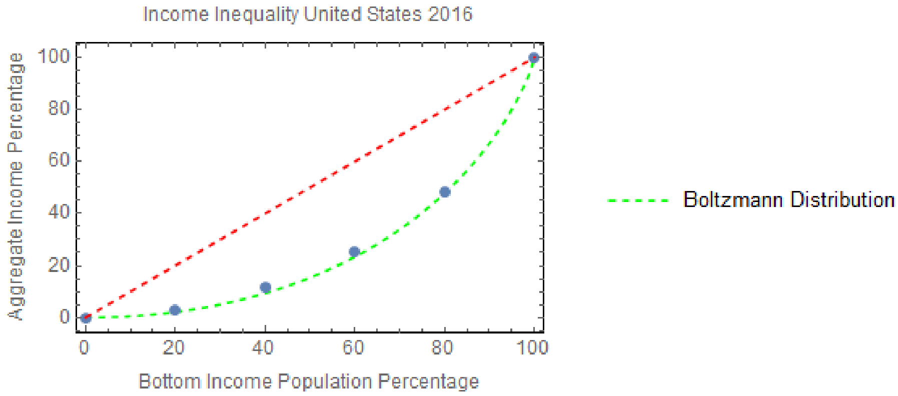



Figure 2.2: (blue) Lorenz curve data points of the United States income in 2016. (green, dashed) Boltzmann distributed income. (red, dashed) Perfectly equal distribution.

## 2.2 Multiplicative Capital Exchange

### 2.2.1 Introduction to MCE

A different model, the multiplicative capital exchange (MCE) model, introduced by Ispolatov, Krapivsky and Redner [9] describes a system in which the amount of money to be transferred depends on the amount of money of the agent that pays the transaction. The agents that will do the transaction are still randomly chosen, but the transaction amount is defined as

$$\Delta m = \gamma m_i, \tag{2.21}$$

with $0 < \gamma < 1$ some fraction of the payer's money, and $m_i$ the amount of money of the payer. Because $\Delta m < m_i$, a transaction can never be denied, it can only become very small for agents with very little money. For 'rich' agents we now see a slight disadvantage, as their outgoing transaction money will on average be larger than their incoming transaction money. For 'poor' agents this goes the other way around. Therefore we would expect that the money will be distributed more evenly than in our previous model.

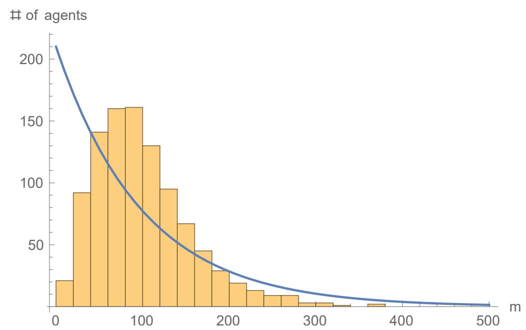



Figure 2.3: Money distribution after 100 sweeps for MCE model, with $N = 1000$, $m_0 = 100$, and $\gamma = 0.25$. (blue) Boltzmann distribution.

In Figure 2.3 we see the money distribution of the MCE model is significantly different than the one we saw with the Yakovenko model. To explain this behavior we will write out a Fokker-Planck equation describing the time evolution of the distribution:

$$\begin{aligned}\frac{\partial P(m)}{\partial t} = \int_0^\infty dm' \int_{-\infty}^{\infty} d\Delta \Big[ &- f_{[m,m'] \to [m-\Delta, m'+\Delta]} P(m) P(m') \\ &+ f_{[m-\Delta, m'+\Delta] \to [m,m']} P(m-\Delta) P(m'+\Delta) \Big].\end{aligned} \tag{2.22}$$

The factor $f_{[m,m'] \to [m-\Delta, m'+\Delta]}$ describes the rate at which agents with the amount of money $m$ do a transaction of $\Delta$ and therefore end up with a different amount of money than $m$. The factor $f_{[m-\Delta, m'+\Delta] \to [m,m']}$ describes the rate at which agents do a transaction of $\Delta$ and arrive at an amount of money $m$. We work these rates out and find

$$\frac{\partial P(m)}{\partial t} = \int_0^\infty dm' \int_{-\infty}^{\infty} d\Delta \Big\{ -\frac{1}{2}\left[\theta(\Delta)\delta(\Delta-\gamma m)+\theta(-\Delta)\delta(\Delta+\gamma m')\right] P(m)P(m') \\ + \frac{1}{2}\left[\theta(\Delta)\delta(\Delta-\gamma(m'+\Delta))+\theta(-\Delta)\delta(\Delta+\gamma(m-\Delta))\right] P(m-\Delta)P(m'+\Delta)\Big\}. \tag{2.23}$$

In this equation we took $t$ in units of the number of sweeps, and therefore, as on average every agent does two transactions per sweep, we multiply the four possible transactions by $1/2$. Furthermore, we use the Dirac delta function $\delta$, and the Heaviside step function $\theta$ to determine the right rates for the different transactions. We find a differential equation for $P(m)$, by integrating the first, second, and fourth term over $\Delta$, and the third term over $m'$:

$$\frac{\partial P(m)}{\partial t} = \frac{1}{2}\Big\{\int_0^\infty dm' \Big[-2P(m)P(m') + \frac{1}{1-\gamma}P\left(\frac{m}{1-\gamma}\right)P\left(m'+\frac{\gamma m}{\gamma-1}\right)\Big] \\ + \int_0^\infty d\Delta \Big[\frac{1}{\gamma}P(m-\Delta)P\left(\frac{\Delta}{\gamma}\right)\Big]\Big\}. \tag{2.24}$$

The first term is simple, as $\int_0^\infty dm' P(m') = 1$. The second term can be simplified in the same way, because $\gamma m/(\gamma-1)$ is strictly negative, while $P(m)=0$, for $m<0$. We get

$$\frac{\partial P(m)}{\partial t} = -P(m) + \frac{1}{2(1-\gamma)}P\left(\frac{m}{1-\gamma}\right) + \frac{1}{2\gamma}\int_0^\infty d\Delta P(m-\Delta)P\left(\frac{\Delta}{\gamma}\right). \tag{2.25}$$

Finally, in the third term we substitute $m' = m-\Delta$, realize again that $P(m')=0$, for $m'<0$, and find

$$\frac{\partial P(m)}{\partial t} = -P(m) + \frac{1}{2(1-\gamma)}P\left(\frac{m}{1-\gamma}\right) + \frac{1}{2\gamma}\int_0^m dm' P(m')P\left(\frac{m-m'}{\gamma}\right). \tag{2.26}$$

At this point, we are interested in finding stationary states, i.e. states for which $\partial P(m)/\partial t = 0$. Let us first take a simple guess of such a state by taking the equilibrium distribution of the Yakovenko model, $P(m) = \exp(-m/m_0)/m_0$.

$$\frac{\partial P(m)}{\partial t} = -\frac{1}{m_0}\exp\left(-\frac{m}{m_0}\right) + \frac{1}{2m_0(1-\gamma)}\exp\left(-\frac{m}{m_0(1-\gamma)}\right) \\ + \frac{1}{2m_0^2\gamma}\int_0^m dm' \exp\left[\frac{-m}{m_0\gamma}+\frac{m'}{m_0}\left(\frac{1}{\gamma}-1\right)\right] = 0. \tag{2.27}$$

Then, we work this out carefully and get

$$-\frac{1}{m_0}\exp\left(-\frac{m}{m_0}\right) + \frac{1}{2m_0(1-\gamma)}\left[\exp\left(-\frac{m}{m_0(1-\gamma)}\right) + \exp\left(-\frac{m}{m_0}\right) - \exp\left(-\frac{m}{m_0\gamma}\right)\right] = 0. \tag{2.28}$$

As we can see this is solved by choosing $\gamma = 1/2$. This means that if the agents randomly exchange half of their money, in equilibrium they are still Boltzmann distributed. In Figure 2.4 the results of such a simulation are shown.

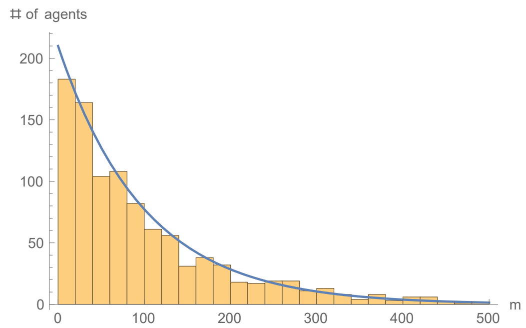



Figure 2.4: Money distribution after 100 sweeps for MCE model, with $N = 1000$, $m_0 = 100$, and $\gamma = 0.5$. (blue) Boltzmann distribution.

### 2.2.2 Analytical Approach

In Figure 2.3 we saw that this Boltzmann distribution did not arise in the case of $\gamma = 0.25$. So what happens for general $0 < \gamma < 1$? We cannot solve Equation 2.26 exactly, but we can look at the behavior of the distribution for large and small $m$. Let us first look at the distribution function at large values of $m$ and assume that it decays exponentially: $P_{\text{large}}(m) = A\exp(-Bm)$, with $B > 0$. We will still look for equilibrium states, and in a very similar way as we previously inserted the Boltzmann distribution we get

$$-Ae^{-Bm} + \frac{A}{2(1-\gamma)}\exp\left(-\frac{Bm}{1-\gamma}\right) + \frac{A^2}{2B(1-\gamma)}\left[e^{-Bm} - \exp\left(-\frac{Bm}{\gamma}\right)\right] = 0. \tag{2.29}$$

Because $0 < \gamma < 1$, and therefore $0 < 1-\gamma < 1$, $\exp[-Bm/(1-\gamma)]$ and $\exp(-Bm/\gamma)$ decay more rapidly than $\exp(-Bm)$. Therefore, for large $m$, we can say

$$Ae^{-Bm} \simeq \frac{A^2}{2B(1-\gamma)}e^{-Bm}, \tag{2.30}$$

which is correct for $A = 2B(1-\gamma)$. So for large values of $m$ we find

$$P_{\text{large}}(m) \simeq 2B(1-\gamma)e^{-Bm}. \tag{2.31}$$

For small values of $m$ we take a similar route. First, we make the ansatz that for very small $m$ the distibution obeys a powerlaw: $P_{\text{small}}(m) = Am^{\lambda}$. Subsituting this into Equation 2.26 gives

$$m^{\lambda} = \frac{m^{\lambda}}{2(1-\gamma)^{\lambda+1}} + \frac{A}{2\gamma^{\lambda+1}}\int_0^m dm'[m'(m-m')]^{\lambda}. \tag{2.32}$$

The integration in the last term looks straightforward, but unfortunately it is not. We try to rewrite it and find an approximate expression for it by shifting to the variable $n = m/2 - m'$:

$$\int_0^m dm'[m'(m-m')]^{\lambda} = \int_{-m/2}^{m/2} dn\left(\frac{m}{2}-n\right)^{\lambda}\left(\frac{m}{2}+n\right)^{\lambda} = \int_{-m/2}^{m/2} dn\left(\frac{m^2}{4}-n^2\right)^{\lambda}. \tag{2.33}$$

Let us now take as an approximation that

$$\int_{-m/2}^{m/2} dn\left(\frac{m^2}{4}-n^2\right)^{\lambda} \sim \int_{-m/2}^{m/2} dn\left(\frac{m^2}{4}\right)^{\lambda} \propto m^{2\lambda+1}, \tag{2.34}$$

because within the integration boundaries $n^2 \leq m^2/4$. If we use this, we see that, for $\lambda > -1$ and $m \ll 1$,

$$m^{\lambda} \simeq \frac{m^{\lambda}}{2(1-\gamma)^{\lambda+1}}, \tag{2.35}$$

which leads to

$$\lambda = -1 - \frac{\log(2)}{\log(1-\gamma)}. \tag{2.36}$$

This expression for $\lambda$ is always larger than $-1$, for $0 < \gamma < 1$. We find $\lambda = 0$ for $\gamma = 1/2$, which is what we found earlier. It predicts that a transition from $P(0) = 0$ to $P(0) > 0$ occurs at $\gamma = 1/2$. An example of a model with $\gamma > 1/2$ is shown in Figure 2.5.

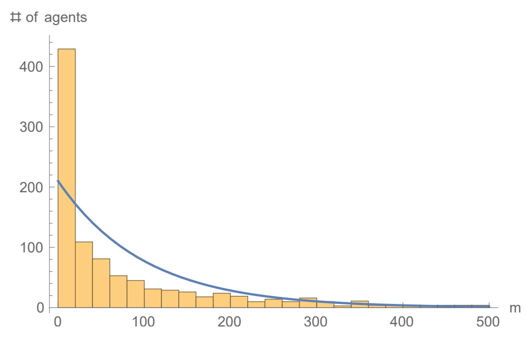



Figure 2.5: Money distribution after 100 sweeps for MCE model, with $N = 1000$, $m_0 = 100$, and $\gamma = 0.75$. (blue) Boltzmann distribution.

### 2.2.3 Numerical Approach

The analytical analysis in the previous section was already covered in the Ispolatov article. This next numerical approach was not.

It would be nice to analyze Equation 2.26 more closely, and see if we can generate the shapes of the observed distributions out of it. In a sense, it is a recurrence relation which we maybe able solve numerically. The problem is the integral. The integral stops us from iteratively setting up our function $P(m)$. We can get rid of it in the following way. Keep in mind we are still looking for steady state solutions, so $\partial P(m)/\partial t = 0$.

The first step is to realize that our function

$$P(m) = 0, \text{ for } m < 0. \tag{2.37}$$

Realizing this, we extend the boundaries of the integral to $-\infty$ and $\infty$. We get

$$-P(m) + \frac{1}{2(1-\gamma)} P\left(\frac{m}{1-\gamma}\right) + \frac{1}{2\gamma}\int_{-\infty}^{\infty} dm' P(m') P\left(\frac{m-m'}{\gamma}\right) = 0. \tag{2.38}$$

Then, we introduce the Fourier transform of our distribution function, $\tilde{P}(n)$, which is defined as

$$\tilde{P}(n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dm P(m) e^{-\mathrm{i}nm}, \tag{2.39}$$

and the other way around:

$$P(m) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dn \tilde{P}(n) e^{\mathrm{i}nm}. \tag{2.40}$$

We then proceed to write Equation 2.38 in terms of its Fourier transform.

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \int dn \left[ -\tilde{P}(n) e^{\mathrm{i}nm} + \frac{1}{2(1-\gamma)} \tilde{P}(n) \exp\left(\mathrm{i}n \frac{m}{1-\gamma}\right)\right] \\ &+ \frac{1}{4\pi\gamma} \int dm' \int dn \int dn' \tilde{P}(n)\tilde{P}(n') \exp\left[\mathrm{i}m'\left(n - \frac{n'}{\gamma}\right) + \mathrm{i}n'\frac{m}{\gamma}\right] = 0. \end{aligned} \tag{2.41}$$

In the third term we integrate $\exp[\mathrm{i}m'(n - n'/\gamma)]$ over $m'$ to get $2\pi\delta\left(n - n'/\gamma\right)$. This delta-function makes the integration over $n'$ straightforward,

$$\int dn \tilde{P}(n) \left[ -\frac{e^{\mathrm{i}nm}}{\sqrt{2\pi}} + \frac{\exp\left(\mathrm{i}n\frac{m}{1-\gamma}\right)}{2(1-\gamma)\sqrt{2\pi}} + \frac{1}{2}\tilde{P}(\gamma n) e^{\mathrm{i}nm}\right] = 0. \tag{2.42}$$

In the second term we make a coordinate shift in integration variables. We go from $n \to n(1-\gamma)$. We obtain an extra factor of $1-\gamma$, which cancels the term in the numerator of the prefactor. Then, all exponentials have the same form, $\exp(\mathrm{i}nm)$. This allows us to write our equation in the following way.

$$\int dn \left[ -\frac{\tilde{P}(n)}{\sqrt{2\pi}} + \frac{\tilde{P}\left[(1-\gamma)\, n\right]}{2\sqrt{2\pi}} + \frac{1}{2}\tilde{P}(n)\tilde{P}(\gamma n)\right] e^{\mathrm{i}nm} = 0. \tag{2.43}$$

Because of the orthogonality of $\exp(\mathrm{i}nm)$, this means that the prefactor of the exponential should be equal to zero! Some rewriting then results in

$$\tilde{P}(n) = \frac{\tilde{P}[(1-\gamma)n]}{2 - \sqrt{2\pi}\tilde{P}(\gamma n)}. \tag{2.44}$$

This recurrence equation looks much friendlier than what we started with, and with the right initial conditions we should be able to calculate it numerically. We get these initial conditions from the Taylor expansion of $\tilde{P}(n)$:

$$\tilde{P}(n) \simeq \tilde{P}(0) + \frac{\partial \tilde{P}(0)}{\partial n} n = \frac{1}{\sqrt{2\pi}} \int dm P(m) - \frac{\mathrm{i}}{\sqrt{2\pi}} \int dm P(m) m = \frac{1}{\sqrt{2\pi}} (1 - \mathrm{i}nm_0). \tag{2.45}$$

So, close to $n = 0$, we know what $\tilde{P}(n)$ looks like.

Before we try to solve this equation numerically, the only solution we had already found, the Boltzmann distribution, can be Fourier transformed to see if it fits the recurrence equation, Equation 2.44, for $\gamma = 1/2$. This Boltzmann distribution had the form

$$P(m) = \begin{cases} \frac{1}{m_0} e^{-\frac{m}{m_0}} & \text{, for } x \geq 0, \\ 0 & \text{, for } x < 0. \end{cases} \tag{2.46}$$

Plugging this into Equation 2.39, we get the following integral for $\tilde{P}(n)$.

$$\tilde{P}(n) = \frac{1}{m_0\sqrt{2\pi}} \int_0^\infty dm \exp\left[-m\left(\frac{1}{m_0} + \mathrm{i}n\right)\right]. \tag{2.47}$$

This integral can easily be performed.

$$\tilde{P}(n) = \frac{1}{m_0\sqrt{2\pi}} \frac{1}{\frac{1}{m_0} + \mathrm{i}n} = \frac{1}{\sqrt{2\pi}(1 + \mathrm{i}m_0 n)}. \tag{2.48}$$

Now, this expression is substituted in the right hand side of Equation 2.44, with $\gamma = 1/2$.

$$\frac{\tilde{P}\left(\frac{n}{2}\right)}{2 - \sqrt{2\pi}\tilde{P}\left(\frac{n}{2}\right)} = \frac{1}{\sqrt{2\pi}(1 + \frac{1}{2}\mathrm{i}m_0 n)} \frac{1}{2 - \frac{\sqrt{2\pi}}{\sqrt{2\pi}(1+\frac{1}{2}\mathrm{i}m_0 n)}} = \frac{1}{\sqrt{2\pi}(2 + \mathrm{i}m_0 n) - \sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}(1 + \mathrm{i}m_0 n)} = \tilde{P}(n). \tag{2.49}$$

So, as it turns out, we confirm that our recurrence equation is correct, at least for $\gamma = 1/2$.

Next, we code an iterative program that will solve Equation 2.44 for arbitrary $\gamma$. Furthermore, it should transform the solution back to our real distribution function according to Equation 2.40. Obviously, we will not be able to find a solution for $\tilde{P}(n)$ for every value of $n \in \mathbb{R}$, so we need to choose a grid size, and a range. For the sake of accuracy we want an as small as possible grid size. However, generally, too small grid sizes lead to either too many datapoints (leading to a large calculation time), or too small ranges (leading to inaccurate solutions). Therefore, let us first check to see what kind of grid sizes and ranges are reasonable for the only known solution given by Equation 2.48, and plotted in Figure 2.6.

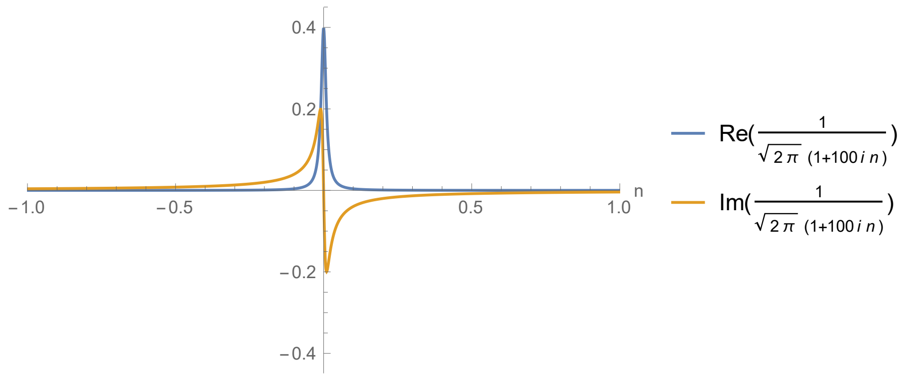



Figure 2.6: (blue) Real and (yellow) imaginary part of the Fourier transform of the Boltzmann distribution for $m_0 = 100$.

As we see, both the real and imaginary part mainly exist locally around $n = 0$, and they damp out to zero for $n \to \pm\infty$. Just to make sure that we will not have miss too much data for Fourier transforms of $P(m)$ for arbitrary $\gamma$ (which in principle could have completely different appearances), for now we will take our range $n \in [-16, 16]$, which we can always adjust if necessary. We choose an number of datapoints that can be handled by any computer, and for which we will have computation times of the order of minutes rather than hours: $M = 8 \times 10^5$. That leaves us with a grid size of $\Delta n = 4 \times 10^{-5}$. This grid size should also capture the sharp peak of the real part of the plotted solution, with width $\sim 0.2$, quite accurately. The entire process we use $m_0 = 100$. For the actual computation we follow the following algorithm.

1. We set the initial conditions of $\tilde{P}(n)$. As we know, for small $n$, $\tilde{P}(n) = (1 - \mathrm{i}nm_0)/\sqrt{2\pi}$. We need to set the value of the first few datapoints near $n = 0$ to the values $\tilde{P}(j\Delta n) = (1 - \mathrm{i}j\Delta n m_0)/\sqrt{2\pi}$. These few datapoints need to span large enough width such that datapoints on the next gridpoints can rely on enough data to calculate their new values, but they need to span small enough width to make sure the approximation still holds. For now, we choose to set initial conditions for $n \in [-5\Delta n, 5\Delta n]$, making sure that $\tilde{P}(6\Delta n)$ can be computed as long as $1/6 \leq \gamma \leq 5/6$.

2. A continuous function of the already obtained values of $\tilde{P}(n)$ is set up, such that, for $n \neq j\Delta n$, $\forall\, j \in [-M, M]$, the function returns an estimation of the value of $\tilde{P}(n)$ based on a linearization of the two nearest datapoints:

$$\tilde{P}_{\text{est}}(n) = \tilde{P}\left[\text{floor}\left(\frac{n}{\Delta n}\right)\Delta n\right] + \left\{\tilde{P}\left[\text{ceiling}\left(\frac{n}{\Delta n}\right)\Delta n\right] - \tilde{P}\left[\text{floor}\left(\frac{n}{\Delta n}\right)\right]\Delta n\right\}\left[n - \text{floor}\left(\frac{n}{\Delta n}\right)\Delta n\right], \tag{2.50}$$

in which the functions floor and ceiling round down and round up their argument to the nearest integer respectively.

3. The next step is the actual computation of the values of $\tilde{P}(j\Delta n)$ and $\tilde{P}(-j\Delta n)$ simultaneously. Starting at $j = 6$ to $j = M/2$ Equation 2.44 is used to calculate the values at every grid point:

$$\begin{aligned}\tilde{P}(j\Delta n) &= \frac{\tilde{P}_{\text{est}}[(1-\gamma)j\Delta n]}{2-\sqrt{2\pi}\tilde{P}_{\text{est}}(\gamma j\Delta n)}\\ \tilde{P}(-j\Delta n) &= \frac{\tilde{P}_{\text{est}}[-(1-\gamma)j\Delta n]}{2-\sqrt{2\pi}\tilde{P}_{\text{est}}(-\gamma j\Delta n)}.\end{aligned} \tag{2.51}$$

4. The inverse Fourier transform is performed according to Equation 2.40. We approximate this integral by doing a Riemann sum that really benefits from a small grid size:

$$P(m) \simeq \frac{1}{\sqrt{2\pi}} \sum_{j=-M/2+1}^{M/2-1} \tilde{P}(j\Delta n) e^{\mathrm{i}j\Delta n m} \Delta n. \tag{2.52}$$

   This can be done for many different $m$ in a certain range depending on the details of the system ($\gamma$, $m_0$, $N$), to get a good idea of the shape of $P(m)$.

5. Both the obtained $\tilde{P}(n)$ and $P(m)$ are then exported to a format that can be read by computer programs that can shape datapoints into graphs.

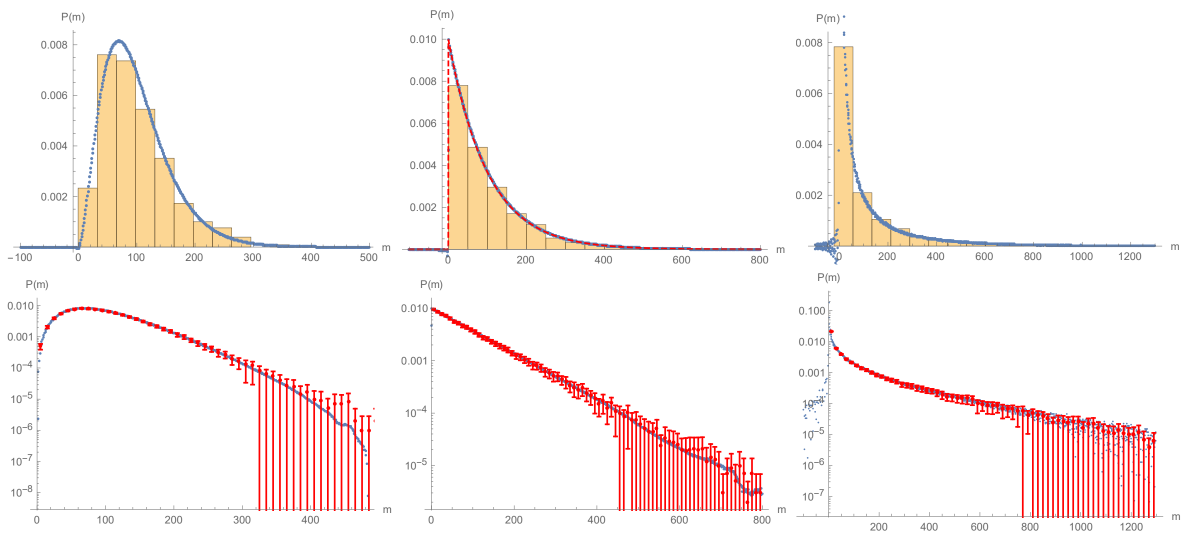
In Figure 2.7 some of the results are plotted. In blue we see the solutions to Equation 2.26. They do seem to align with the histograms quite nicely. For $m < 0$, we nicely see the distribution drop to zero. However, it does not jump as accurately as we had predicted. This, and basically any deviations from the exact solutions are due to the following inaccurate parts of the calculation:

1. The Riemann sum is an approximation of an actual integral, and becoming exact as $\Delta n \to 0$, which is obviously not possible numerically.

2. We had to cut off our solutions of $\tilde{P}(n)$ at some point, because we could not wait infinitely long. However, for the exact solution, we needed all of the values of $\tilde{P}(n)$, with $n \in \mathbb{R}$.

3. The initial conditions needed to be set at a certain width in the beginning from our computation, but this width should theoretically be infinitesimal. This basically comes down to the fact that $\Delta n \neq 0$, again.

4. Lastly, probably the least weighing factor, when computing certain functions like square roots and sines and cosines, a computer always has to make approximations.




(a) $\gamma = 0.25$ (b) $\gamma = 0.5$ (c) $\gamma = 0.75$

Figure 2.7: (blue) Plots of numerically computed distribution functions of the MCE model with $m_0 = 100$, at different values of $\gamma$. (red, dashed) Known shape of the Boltzmann distribution. (histogram) Histograms of the different MCE simulations after 300 sweeps. (red, with error bars) Average and standard deviation of distributions of 100 simulations of mentioned cases.

To finish this section, we would once more like to compare these distributions to actual data from Figure 1.2. This is done in Figure 2.8. As we saw before, the $\gamma = 0.5$, which lead to the same Boltzmann distribution as the Yakovenko model, matches well with the data. The other curves, for $\gamma = 0.25$ and $\gamma = 0.75$, are somewhat more equal and less equal respectively.

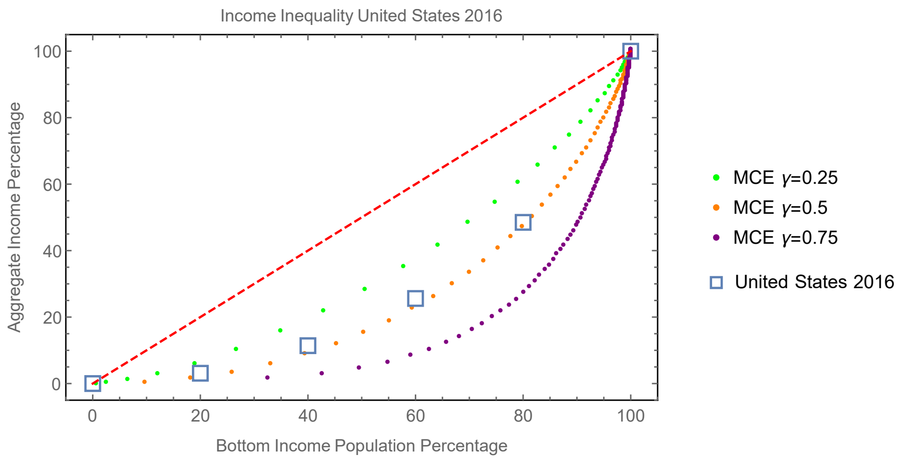



Figure 2.8: Lorenz curves of the distributions for(green) $\gamma = 0.25$, (orange) $\gamma = 0.5$ and (purple) $\gamma = 0.75$ compared to (blue, squares) data United States 2016.

# 3 Linear Demand and Supply Model

The systems we have discussed so far in this thesis are very simple. Agents only have one possession: money. In a slightly less simple model we imagine agents to not only possess money, but also a certain number of products, $V_i \in [0, \infty)$. This could be anything: bananas, coffee machines, or even shares in some company. Let's assume, though, for simplicity there is only one type of product, and we take $V_i$ to be a real number. It is important that these products can be connected to the money somehow. This can be done intuitively by thinking of the price of the product, $p_i$. The price will be determined according to demand and supply and can be different for every individual agent. We discuss this in the next section.

The model is then modified to include trades of products for money or the other way around. How this happens is described in the following steps:

- The variables $i$ (will buy product), and $j$ (will sell product) are assigned a random value from the set $1, 2, ..., N$.
- An amount of products $\Delta V$ is chosen either randomly from $[0, \Delta V_{max}]$, or from demand and supply (see next section). This can either be an integer or a real number, depending on the nature of the product.
- Again, the algorithm checks if agent $i$ has enough money to afford the transaction, i.e. if $m_i \geq p_j \Delta V$. Here $p_j$ is the price at which agent $j$ wants to sell its product.
- The system now also has to check if agent $j$ has at least as many products as it is trying to sell; $V_j \geq \Delta V$.
- If both conditions are satisfied the transaction is done. This means that the money of agent $i$ is reduced and the money of agent $j$ is increased by $p_j \Delta V$. Also the amount of products of agent $i$ is increased and the amount of products of agent $j$ are reduced by $\Delta V$:

$$m_i \to m_i - p_j \Delta V, \tag{3.1}$$

$$m_j \to m_j + p_j \Delta V, \tag{3.2}$$

$$V_i \to V_i + \Delta V, \tag{3.3}$$

$$V_j \to V_j - \Delta V. \tag{3.4}$$

The price $p_j$ is then changed according to the laws of demand and supply. So let us first freshen those up.

## 3.1 Intermezzo: Demand and Supply

Simple economics describes the mechanics of a trade by looking at the the difference between how many products a consumer wants to buy and how many products a supplier can afford to sell at a certain price. The relationship between the number of products that a consumer wants to buy and the price of the product is called the demand curve. The relationship between the number of products that a supplier can afford to sell and the price of the product is called the supply curve. In this section we are going to look at these curves and see if we can use them to determine

- if a consumer wants to buy a product at a certain price,
- how many products a consumer wants to buy at a certain price,
- how a supplier changes its price after he has or has not sold its product.

More detailed descriptions of these curves can be found in *Principles of Microeconomics* by Gregory Mankiw [8].

### 3.1.1 The Demand Curve

Let's start by looking at the demand curve. The demand curve shows the number of products that an agent or a group of agents want to buy given a certain price. This curve is always descending, as nobody would like to buy more products at a higher price.

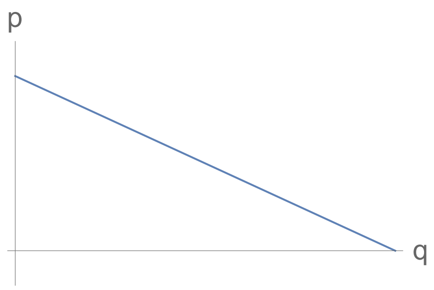



Figure 3.1: A simple example of a linear demand curve. Note that economists plot the price on the vertical axis, whereas this seems unnatural to most physicists.

In Figure 3.1 a simple demand curve is plotted. Note that at prices higher than a certain value no one wants to buy the product anymore. Also, when the price of the product is zero, there is a maximum of products that people want to buy. A demand curve can shift to the left or right depending on any circumstances that influence the consumer. If people have more money to spend, for example, in general the curve will shift to the right, because more products can be bought at the same price. Furthermore, on, for example, a rainy day, the ice cream demand curve will generally be shifted more to the left compared to a sunny day. A linear demand curve can be described by the following equation:

$$q_{\text{dem}}(p) = -\alpha p + C_{\text{dem}}, \text{with } \alpha > 0. \tag{3.5}$$

In this equation $C_{\text{dem}}$ is some variable that shifts the curve to the left or right.

### 3.1.2 The Supply Curve

The supply curve is a price-quantity diagram, that shows the number of products that suppliers are willing to sell at a certain price. At low prices there are few suppliers that can afford to sell their product, and vice versa. The supply curve is therefore always ascending. In Figure 3.2 an example of a supply curve is plotted.

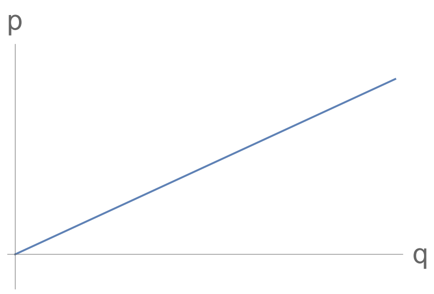



Figure 3.2: A simple example of a linear supply curve.

A supply curve can shift as well as the demand curve. For example, as resources needed to make a product become more expensive, the supply curve will shift to the left, because, at the same price, less suppliers can afford to sell products. A linear supply curve can be described by the following equation:

$$q_{\text{sup}}(p) = \beta p + C_{\text{sup}}, \text{with } \beta > 0. \tag{3.6}$$

Here, $C_{\text{sup}}$ is a variable that shifts the curve to the left or right.

### 3.1.3 Demand and Supply Combined

An equilibrium price and selling quantity for both the consumer and the supplier can be constructed from combining the demand and the supply curve. It is shown in Figure 3.3.

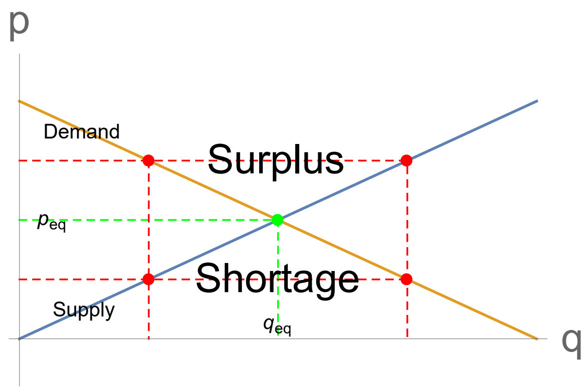



Figure 3.3: Equilibrium price and selling quantity construction from (yellow) demand and (blue) supply curve.

The figure shows that the ideal price and quantity is found at the intersection of the two curves, i.e. by solving

$$q_{\text{sup}}(p_{\text{eq}}) = q_{\text{dem}}(p_{\text{eq}}). \tag{3.7}$$

Furthermore, there is a shortage of a product when the price is lower than the equilibrium price, meaning that at that price people want to buy more of the product than the suppliers can afford to sell. Also, there is a surplus when the price is too high.

Solving Equation 3.7, for the demand and supply curves given in Equation 3.5 and 3.6 respectively, gives:

$$p_{\text{eq}} = \frac{C_{\text{dem}} - C_{\text{sup}}}{\alpha + \beta}, \tag{3.8}$$

$$q_{\text{eq}} = \frac{\alpha C_{\text{sup}} + \beta C_{\text{dem}}}{\alpha + \beta}. \tag{3.9}$$

## 3.2 Demand and Supply in the Yakovenko Model

Earlier we tried to go beyond the Yakovenko model by introducing products and prices. However, we did not specify yet how we should determine and handle the price. Now, it is time to introduce the simple economic background that was previously acquired into the model.

Let us start simple and think of one single system with $N$ agents again. In this new case, however, all agents have three properties:

- the amount of money, $m_i$.
- the amount of products, $V_i$.
- the price, $p_i$, at which the agent would like to sell products.

All agents also have their individual demand and supply curves based on the above properties.

### 3.2.1 Shaping the Demand and Supply Curves

For simplicity, let's first consider a case in which the supply curve of every agent in the system is identical; $\beta = 1$, and $C_{\text{sup}} = 0$. This results in the following curve:

$$q_{i,\text{sup}}(p) = p. \tag{3.10}$$

The equilibrium price is then only changed when the demand curve is shifted. Let us assume $\alpha = 1$, then the only way of changing the curve - and therefore the equilibrium transaction - comes from $C_{\text{dem}}$. A simple way of introducing such a shift is by looking at the amount of money an agent possesses, $m_i$. If an agent has a lot of money, he can afford to buy more products. Therefore, we are looking at an increasing function $C_{\text{dem}}(m_i)$. Let's look at a few possibilities:

- $$C_{\text{dem}}(m_i) = am_i, \text{with } a > 0 \tag{3.11}$$

  is probably the simplest case imaginable. Calculating the equilibrium price and quantity - i.e. inserting the values in Equations 3.8 and 3.9 - gives $p_{i,\text{eq}} = am_i/2$, and $q_{i,\text{eq}} = am_i/2$, making the equilibrium transaction cost $p_{i,\text{eq}}q_{i,\text{eq}} = a^2m_i^2/4$. On the first hand, it seems like there is nothing wrong with this. However, looking more closely we see that there always exist values of $m_i > 0$ for which $p_{i,\text{eq}}q_{i,\text{eq}} = a^2m_i^2/4 > m_i$. This is a problem, because it means that once the system is in equilibrium and an agent has such a large amount of money that this condition is satisfied, it will not be able to pay its equilibrium transaction anymore. Therefore the agent is stuck at the money level it is at.

- $$C_{\text{dem}}(m_i) = a\sqrt{m_i}, \text{with } a > 0 \tag{3.12}$$

  is an easy adaption we can make. This gives $p_{i,\text{eq}} = a\sqrt{m_i}/2$, $q_{i,\text{eq}} = a\sqrt{m_i}/2$, and $p_{i,\text{eq}}q_{i,\text{eq}} = a^2m_i/4$. This suggests that, as long as $a < \sqrt{2}$, the equilibrium transaction will always cost less than the amount of money that the agent has. Next to that, the shape of $C_{\text{dem}}(m_i) = a\sqrt{m_i}$ also seems to fit better, because at larger amounts of money the amount of products an agent wants to buy increases less and less. Which is what we would expect.

- $$C_{\text{dem}}(m_i) = \frac{am_i}{m_i + b}, \text{with } a, b > 0 \tag{3.13}$$

  may even be a more realistic adaption. This function has the property $\lim_{m_i\to\infty} C_{\text{dem}}(m_i) = a = C_{\text{dem,max}}$, which is an asymptote that ensures the demand does not rise further than a certain maximum value even if an agent possesses a very large amount of money. This is what we expect for the demand of certain types of products. For example, a millionaire buys more pairs of shoes than someone who earns minimum wage, but a billionaire does not buy significantly more pairs of shoes than the millionaire. When we look at our equilibrium price and quantity again we find:

  $$p_{i,\text{eq}} = \frac{am_i}{2(m_i + b)}, \tag{3.14}$$

  $$q_{i,\text{eq}} = \frac{am_i}{2(m_i + b)}, \tag{3.15}$$

  $$p_{i,\text{eq}}q_{i,\text{eq}} = \frac{a^2m_i^2}{4(m_i + b)^2}. \tag{3.16}$$

  From Equation 3.16 we can see that for large values of $m_i$, the cost of the equilibrium transaction goes to $a^2/4$. To make sure that $p_{i,\text{eq}}q_{i,\text{eq}}$ never exceeds $m_i$ for $m_i > 0$, we solve

  $$\frac{a^2m^2}{4(m + b)^2} < m, \text{for } m > 0, \tag{3.17}$$

  and find as a result $b > a^2/16$, which is also shown in Figure 3.4.

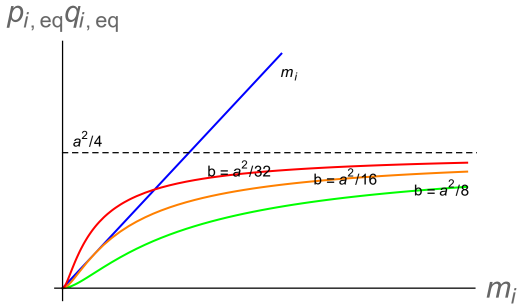



Figure 3.4: Plot of the cost of the equilibrium transaction of agent $i$ as a function of the amount of money of agent $i$, for different values of $b$.

### 3.2.2 Handling the Price

The actual mechanics of the transaction in a simulation have not yet been discussed. These transactions will revolve around the individual property of the agent, the price, $p_i$. Every agent in the system has a personal price at which it wants to sell its product. Accompanied with this price goes a quantity that the agent expects to sell: $q_{i,\mathrm{sup}}(p_i)$. Let's say agent $j$ is randomly selected to sell products to agent $i$. Agent $i$ is presented the price of agent $j$, $p_j$, and determines the quantity he/she wants to buy at that price, $q_{i,\mathrm{dem}}(p_j)$, according to its demand curve. He then proceeds to buy that quantity, if $m_i \geq p_j q_{i,\mathrm{dem}}(p_j)$ and $V_j \geq q_{i,\mathrm{dem}}(p_j)$:

$$m_i \to m_i - p_j q_{i,\mathrm{dem}}(p_j), \tag{3.18}$$

$$m_j \to m_j + p_j q_{i,\mathrm{dem}}(p_j), \tag{3.19}$$

$$V_i \to V_i + q_{i,\mathrm{dem}}(p_j), \tag{3.20}$$

$$V_j \to V_j - q_{i,\mathrm{dem}}(p_j). \tag{3.21}$$

After the transaction agent $j$ has the chance to change his/her price. One of the following three cases can happen.

1. $q_{i,\mathrm{dem}}(p_j) > q_{j,\mathrm{sup}}(p_j)$, i.e. the quantity agent $i$ bought was more than agent $j$ expected to sell. If we look at Figure 3.3 we see that there is a shortage; the consumer bought more than the supplier could afford to sell at the given price. In an attempt to avoid this shortage with future buyers the supplier, agent $j$, increases its price by a fixed amount $\Delta p \in [0, \infty)$.

2. $q_{i,\mathrm{dem}}(p_j) = q_{j,\mathrm{sup}}(p_j)$, i.e. the quantity agent $i$ bought was exactly the quantity agent $j$ expected to sell. This means the consumer and supplier are in equilibrium, and there is no need for the supplier to change the price.

3. $q_{i,\mathrm{dem}}(p_j) < q_{j,\mathrm{sup}}(p_j)$, i.e. the quantity agent $i$ bought was less than agent $j$ expected to sell. This means we are looking at a surplus; the consumer bought less than the supplier expected to sell at the given price. To ensure that future buyers buy more than this buyer, the supplier decreases its price by the fixed amount $\Delta p$.

### 3.2.3 The Demand and Supply Simulation

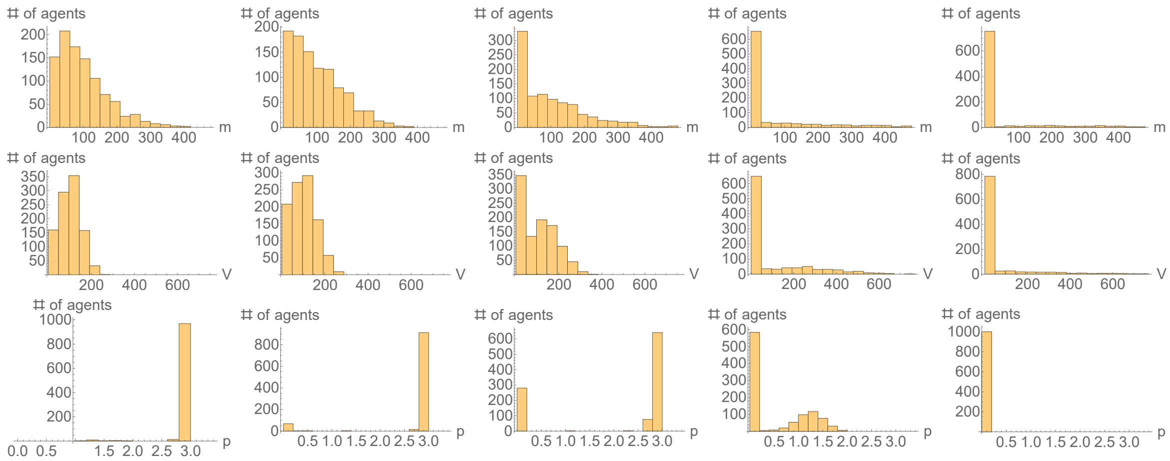
To test our newly defined model, we set up a simulation of a system with $N = 1000$ agents. The agents are all given $m_i = m_0 = 100$ money units, and $V_i = V/N = 100$ products. They all start with selling their products at price $p_i = p_0 = 5$. Let us assume that their demand curves are

$$q_{i,\mathrm{dem}}(p, m_i) = -p + \frac{6m_i}{m_i + 4}, \tag{3.22}$$

and their supply curves are simply

$$q_{i,\mathrm{sup}}(p) = p. \tag{3.23}$$

This means that their demanded quantities of products are maximized at $q_{i,\mathrm{dem}}(p, m_i) = 6$, for $p = 0$, and $m_i \to \infty$. Let us assume that they adjust their price by $\Delta p = 0.01$ every time they do not sell exactly as many products as they had expected. At $t = 0$, we allow the agents to start trading at an average rate of 2 trades per agent per time step (one sweep of the system per time step). We run the simulation for 500000 time steps, and plot some of the results here.




(a) $t = 100000$ (b) $t = 150000$ (c) $t = 250000$ (d) $t = 300000$ (e) $t = 350000$

Figure 3.6: Histograms showing distributions of money, products, and price at different time steps.

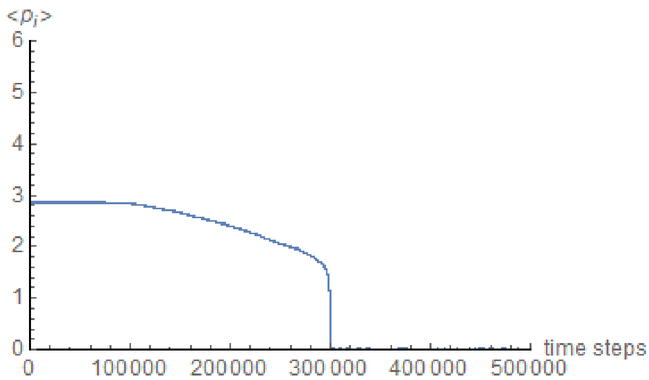



Figure 3.5: Average price $\langle p \rangle$ throughout the simulation.

In Figure 3.5 the evolution of the average price is shown. Initially, the average price drops very quickly from $\langle p \rangle = p_0 = 5$ to some price that seems to be stable for the 100000 first time steps. Then, the average price starts dropping quicker and quicker to finally end up at just above zero. This is a strange, at first unexpected, result that can use some more graphical support to get clarification.

Figure 3.6 shows some snapshots of the distribution functions of $m$, $V$, and $p$. It is clear that the bins at $m = 0$, $V = 0$, and $p = 0$ only increase as time progresses. In the very end most agents have no money and no products, and everyone has adjusted their price to about zero. The exact reason why this happens is not entirely clear yet. However, one can imagine that once an agent - for whatever reason - possesses little money and few products, it is very unlikely that he will climb back out of that position, because he just possesses very little value.

The entropy can be calculated throughout the simulation according to

$$S = - \sum_{m,V,p} P(m, V, p) \log(P(m, V, p)). \tag{3.24}$$

This results in Figure 3.7, showing that this system does not obey the second law of thermodynamics, because the entropy is lowered significantly.

Choosing a different type of demand curve does not change the qualitative behavior of the simulation. In Figure 3.8 we see the time evolution of the average price of the exact same system except for the change to a square root based demand curve:

$$q_{i,\text{dem}}(p, m_i) = -p + \sqrt{m_i}. \tag{3.25}$$

It still drops to zero, but more quickly than the previous case. Adjusting any of the other parameters does not work any better; the system ultimately ends in this 'economic crisis' at all times.

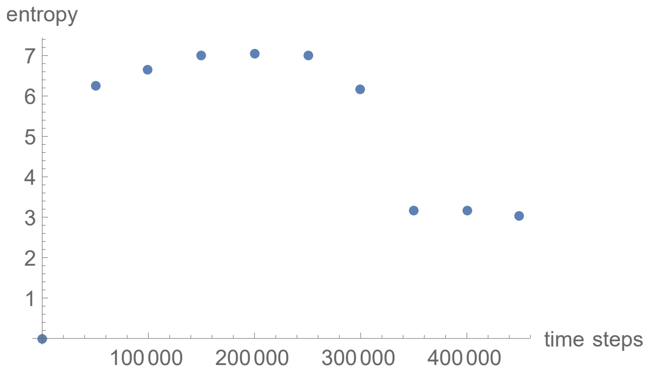



Figure 3.7: Entropy of the Demand and Supply Yakovenko model, with asymptotic demand curve.

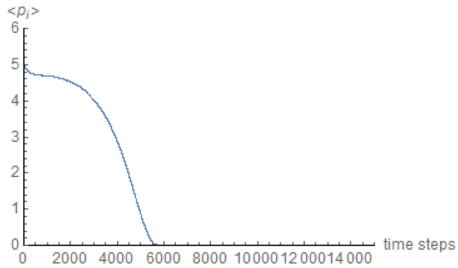
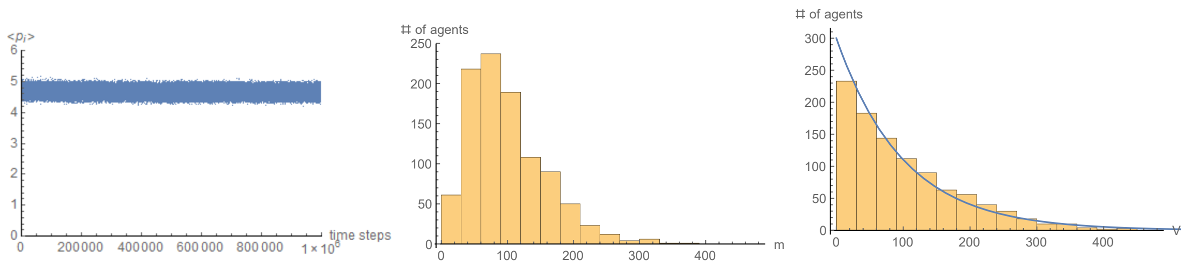



Figure 3.8: Time evolution of the average price, for square root based demand curve.

In the Appendix (Section 7) we attempt to shed some light upon this instability. However, mathematically we bump into a wall quite quickly, at which we will leave the reader only with several comments on the subject.

### 3.2.4 To Solve The Crisis

The collapse of the model, as shown in the subsection above, is something we would like to avoid, not only to circumvent the system going in a very static trivial state, but also to get a better idea of how the economy actually works. Of course, we could try to imitate an actual economy by introducing entities like banks and governments, but that would easily overcomplicate the model, making it hard to analyze it theoretically. Obviously, we would not have written this subsection, if we had not had a different, easy way to stabilize the system. This measure requires us to make a change in the model; a change in the way the price is adjusted. The way the model works as it is, there is no connection between the prices of different agents. In an economy, however, it is generally (by approximation) assumed that there is one single market price for a product [8]. Everyone buys and sells the product at that price. This price can change, but it changes for everyone. This is also exactly the adaption we are going to make in our model; instead of adjusting only the sellers price by $\pm\Delta p$, we now adjust all prices by $\pm\Delta p$. This means that there will no longer be a distribution of the price $p_i$, other than a sharp peak at $p_i = p(t)$, the market price. The impact of one trade on the average transaction will be $N$ times larger than it was before the change, but that is preferred, since we want agents to be able to trade themselves to from rich to poor and the other way around easily. This can only be done by buying products at a high price and selling them at a low price and the other way around. If the difference in the market price is very low, it will probably take a very long time for the system to get to equilibrium distributions.




(a) Time evolution of the average price ($\langle p \rangle = 4.65 \pm 0.10$).

(b) Histogram of money distribution after $10^6$ time steps.

(c) Histogram of product distribution after $10^6$ time steps. (blue) Boltzmann distribution plotted.

Figure 3.9: Results of the square root based demand curve simulation with uniform market price.

Let us now again prepare a system with about the same specifics as the systems we encountered in the previous subsection. We will use the same square root based demand curve, $N = 1000$, $m_0 = 100$, $V_0 = 100$, $p_0 = 5$, $\Delta p = 0.01$, however we run it for $10^6$ time steps, to make sure that we reach an equilibrium.

In Figure 3.9, we see the results of the simulation. Note that it took the entirety of the $10^6$ time steps to reach the product distribution, but the money distribution has been found very quickly. The product distribution is Boltzmann-distributed, whereas the money distribution is not.

## 3.3 The Linear Model

So far, we have mainly discussed the curved demand curves; the asymptotical curve, and the square-root based curve. The simplest form imaginable, however, should also show some interesting behavior. As we discussed earlier, the linear model carries a problem with it. Let us take

$$\begin{aligned} q_{\text{dem}}(p, m) &= -p + am, \\ q_{\text{sup}}(p) &= p. \end{aligned} \tag{3.26}$$

If we apply a mean-field theory, i.e. we ignore fluctuations in money by just solving for average $m_0$, we expect the price to stabilize at

$$q_{\text{dem}}(p, m_0) = q_{\text{sup}}(p) \longrightarrow p = \frac{am_0}{2}. \tag{3.27}$$

The amount of products that is traded over in this mean-field approach is then

$$q = -\frac{am_0}{2} + am_0 = \frac{am_0}{2}. \tag{3.28}$$

This makes the transaction cost an amount of money units of $a^2m_0^2/4$. The quadratic nature of this expression entails that there exist average amounts of money for which the predicted ideal transaction cannot go through, i.e., if

$$m_0 < \frac{a^2m_0^2}{4} \longrightarrow m_0 > \frac{4}{a^2}. \tag{3.29}$$

Furthermore, in this approach, one would expect something interesting to happen when the equilibrium transaction cannot be done due to the lack of products on the seller's side:

$$V_0 < \frac{am_0}{2}. \tag{3.30}$$

In this subsection we will discuss what the model shows in cases where $m_0 < 4/a^2$, and $m_0 > 4/a^2$. Also, we will see if this mean-field approach makes any sense as we know that the distributions we have seen of money and products are very unequal.

### 3.3.1 Product Dependence in the Demand Curve

First, we are going to make the model slightly more interesting, by making an agent's demand depend on the amount of products the agent owns. Simply said, the more products an agent owns, the lower his/her demand. The simplest way to do this is

$$q_{\text{dem}}(p, m, V) = -p + am - bV. \tag{3.31}$$

In the mean-field approach we expect the equilibrium price to be at $(am_0 - bV_0)/2$, the corresponding product quantity to be $(am_0 - bV_0)/2$, and therefore the cost of this transaction is $(am_0 - bV_0)^2/4$. The condition for the transaction to go through on the buyer's side in the mean-field approach is then

$$m_0 > \frac{(am_0 - bV_0)^2}{4}, \tag{3.32}$$

which gives a range of $m_0$:

$$\frac{abV_0 + 2 - 2\sqrt{abV_0 + 1}}{a^2} < m_0 < \frac{abV_0 + 2 + 2\sqrt{abV_0 + 1}}{a^2}. \tag{3.33}$$

The condition for the seller's side is

$$V_0 > \frac{am_0 - bV_0}{2}, \tag{3.34}$$

which sets a condition on $V_0$ or $m_0$:

$$\begin{aligned} V_0 &> \frac{am_0}{2 + b}, \\ m_0 &< \frac{(2 + b)V_0}{a}. \end{aligned} \tag{3.35}$$

If we now set $a = 1$, $b = 1/2$, the initial conditions of $m_0 = 1$, and $V_0 = 1$ should give a fine result in the mean-field approach as shown in Figure 3.10. More specifically, this predicts the system to work fine at $V_0 = 1$, with $0.05 < m_0 < 2.50$.

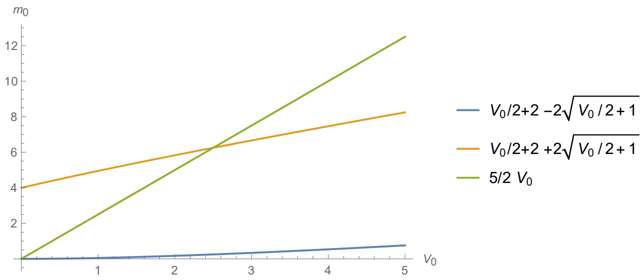



Figure 3.10: $m_0$ should be larger than the blue curve, where it should be smaller than both the yellow and the green curve.

### 3.3.2 Simulation Results

Let us now run our simulation several times with $N = 1000$, and

$$\begin{aligned} q_{\text{dem}}(p, m, V) &= -p + m - \frac{V}{2}, \\ q_{\text{sup}}(p) &= p. \end{aligned} \tag{3.36}$$

We do this simulation for several values of $m_0$, with constant $V_0$, and for several values of $V_0$, with constant $m_0$. The results are plotted in Figure 3.11.

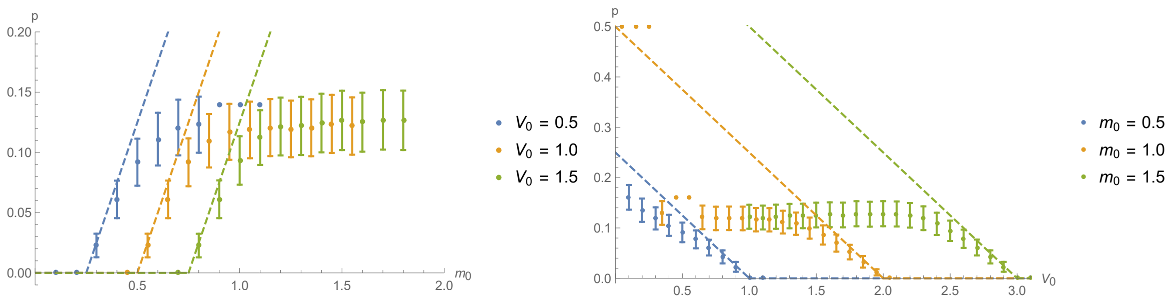
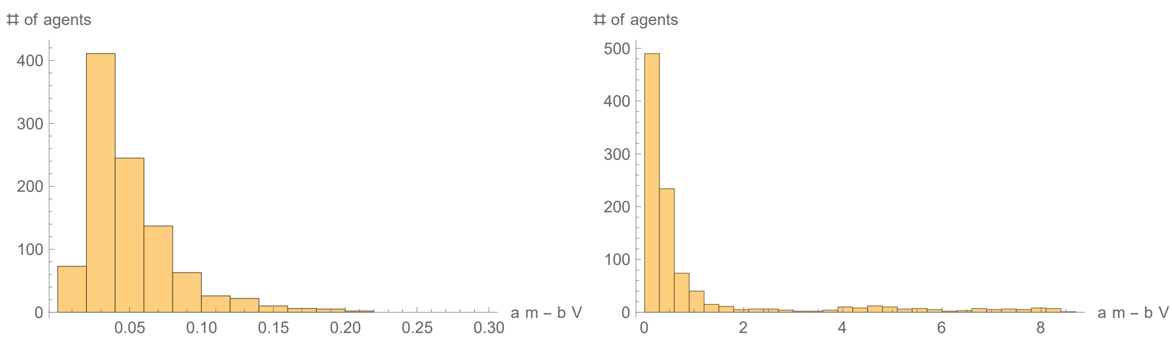



Figure 3.11: Equilibrium price plotted for different combinations of $m_0$ and $V_0$. Simulation is run $10^6$ sweeps. Averages are taken of the set of prices of the last $210^5$ sweeps, and the error bars represent the standard deviation. The dashed line is the expected mean-field equilibrium price.

As we can see, the mean-field approach works decently when the expected price,

$$p = \frac{am_0 - bV_0}{2}, \tag{3.37}$$

is small. Then, the larger it becomes, the more the simulation price deviates from it. Also, the value of $m_0$ from where the simulation price seems to go to an asymptote is unequal to the predicted boundary we saw in Figure 3.10. To analyze this, we take a look at the distributions of the quantity $am - bV = m - V/2$ at a simulation where mean-field theory seems to work decently, and a simulation where it does not work well (Figure 3.12).




(a) $m_0 = 0.55$, $V_0 = 1.00$: mean-field theory works decently.

(b) $m_0 = 1.55$, $V_0 = 1.00$: mean-field theory does not work.

Figure 3.12: Distributions of the quantity $am - bV = m - V/2$.

In the first histogram of Figure 3.12 we see that relatively many agents have a quantity of $am - bV$ that is close to the average amount of $am_0 - bV_0 = 0.05$. In the second histogram we see that there are many agents that have way more or less of the quantity $am - bV$ than the average amount of $am_0 - bV_0 = 1.05$. We may conclude that the better the system is equally distributed, the better mean-field theory works.

Why is there no agent with $am - bV < 0$? Well, let us assume an agent has $am - bV < 0$, his demand is then $q_{\mathrm{dem}} < 0$. So he can only sell products. Say, he sells $\Delta V > 0$ products. His money and products shift:

$$\begin{aligned} V &\longrightarrow V - \Delta V, \\ m &\longrightarrow m + p\Delta V. \end{aligned} \tag{3.38}$$

Therefore, his new value of $am - bV$ becomes

$$a(m + p\Delta V) - b(V - \Delta V) > am - bV, \tag{3.39}$$

because

$$(ap + b)\Delta V > 0. \tag{3.40}$$

This means that when the agent has $am - bV < 0$, he/she can only sell, and by selling, any agent will always increase his/her demand.

### 3.3.3 Median-Field Theory

To explain the behavior shown in Figure 3.11 better, we found that we needed something better than mean-field theory. An interesting point in a distribution is the point where there is an equal amount of agents that have more of the quantity at said point as there are agents that have less of the quantity at said point. The quantity at this point is called the median. Let us introduce $\mu = am - bV$, and its median $\tilde{\mu}$. It is found by solving the following equation.

$$\int_0^{\tilde{\mu}} d\mu P_\mu(\mu) = \int_{\tilde{\mu}}^{\infty} d\mu P_\mu(\mu). \tag{3.41}$$

We will now show why this is an interesting quantity in our model. Let us look at the condition for the market price of our product to increase. This happens when $q_{\mathrm{dem}} > q_{\mathrm{sup}}$, i.e.

$$\mu > 2p. \tag{3.42}$$

The corresponding chance that this happens can be calculated through the distribution function of $\mu$, $P_\mu(\mu)$:

$$P_\uparrow = \int_{2p}^{\infty} d\mu P_\mu(\mu). \tag{3.43}$$

The market price decreases when

$$\mu < 2p, \tag{3.44}$$

with corresponding chance,

$$P_\downarrow = \int_0^{2p} d\mu P_\mu(\mu). \tag{3.45}$$

When the price is in equilibrium $P_\uparrow = P_\downarrow$, or

$$\int_{2p}^{\infty} d\mu P_\mu(\mu) = \int_0^{2p} d\mu P_\mu(\mu). \tag{3.46}$$

This is exactly Equation 3.41, with $p = \tilde{\mu}/2$. In Figure 3.13 the medians of the same simulations as Figure 3.11 are plotted. The results shown in Figure 3.13 show very similar behavior as Figure 3.11 did. For low values of $\mu$, the median is almost equal to the average. For larger values of $\mu$, they start to deviate more and more. To see how well the prediction $p = \tilde{\mu}/2$ works, Figure 3.11 and 3.13 are combined in Figure 3.14. These results are very interesting and show a better prediction of the equilibrium price. The median predictions follow the simulation prices through a bigger set of points. They bend approximately at the same point as the simulation prices, but they do not bend as much, leaving them at $\tilde{\mu}/2 > p_{\mathrm{sim}}$. This can be explained by the transaction success rate. For the simulation that was run in Figure 3.12a, for example, the transaction success rate is about 99%. For Figure 3.12b it is about 67%. In the second simulation there are some agents with a very high demand. This quantity is rarely traded, because most agents do not possess so many products. These agents do count in the median, but they rarely actually trade, and therefore, they rarely influence the price. This makes the median, $\tilde{\mu}$, result in a larger expected equilibrium price, than the measured price.

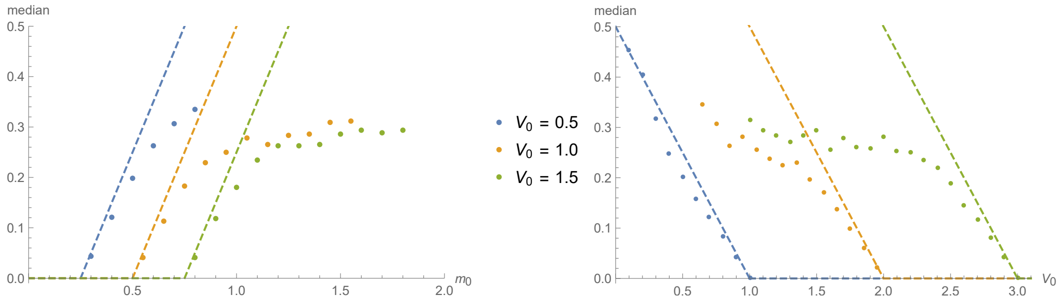



Figure 3.13: Median, $\tilde{\mu}$, plotted for simulations at several combinations of $m_0$ and $V_0$. Dashed lines show the expected average $am_0 - bV_0 = m_0 - V_0/2$.

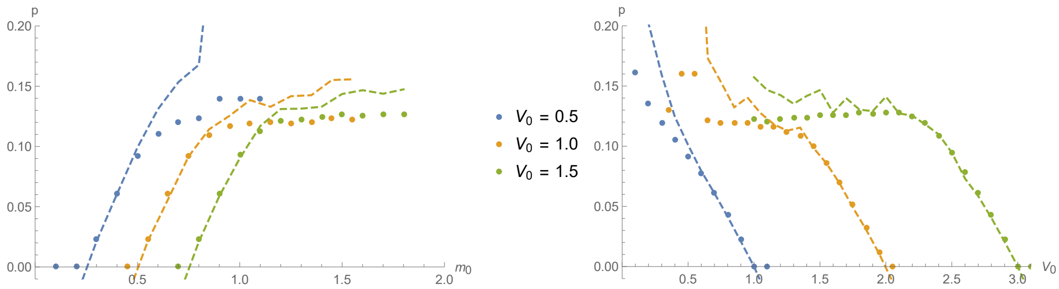



Figure 3.14: Prediction $p = \tilde{\mu}/2$ plotted together with the simulation results at several values of $m_0$ and $V_0$.

## 3.4 Conclusion

The model we discussed in this section shows again how money and products are unequally distributed among identical agents through statistical fluctuations. The model is - in a sense - more relatable to an actual economy than the Yakovenko model. However, many parameters influence the outcome of the simulations, and among those parameters there are hardly any that we can estimate realistic values of. Furthermore, the linear nature of the demand and supply curves and the way they are set up may lack some deeper economic structure. This is why in the next part of this thesis we will dive deeper into economic theories, and we will attempt to make a better model.

But not before we have laid one of the obtained final distributions along the by now so well-known Lorenz curve of the income distribution in the United States we saw in Figure 1.2, to keep our promise. We take the distribution of Figure 3.9b, and convert it to a Lorenz curve in Figure 3.15. The linear demand and supply model results in a more equal distribution than the income distribution of the United States in 2016.

With this we end Part I of this thesis. In the next part we will dive deeper into economic theory and develop models with better economic basis, which - among other benefits - will allow us to interpret results better.

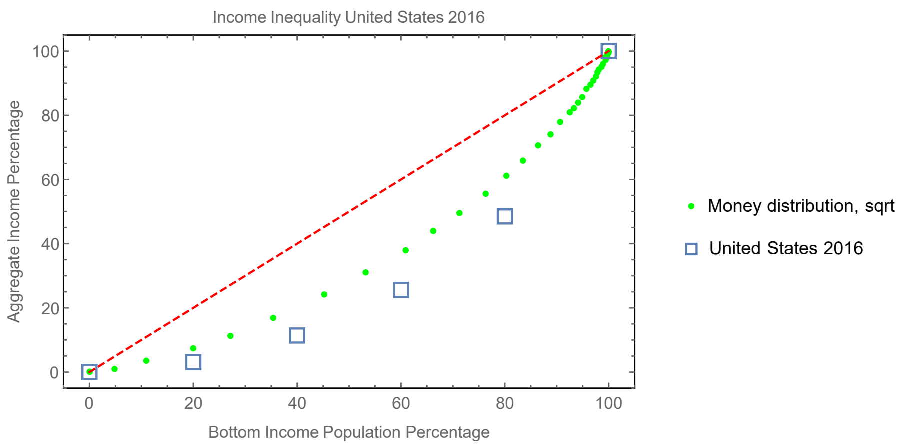



Figure 3.15: (green) Lorenz curve of final money distribution of the linear demand and supply model with square root dependence demand. (blue, squares) Income distribution United States in 2016.

# Part II

# Models based on Utility Theory

## 4 Utility-based Market Model

In the previous section we discussed a model in which we assumed linear demand and supply curves, that were shifted depending on the possessed money and amount of products of individual agents. Mathematically, this model very quickly became very complicated. Furthermore, the model gave rise to very many unknown parameters and functional dependencies that could be chosen from a generically large set of options. Economists generally have a different approach to these kind of problems: *Utility.* We will try to build the same model, i.e. one with a number of agents having two different quantities they can exchange, but now based off the concept of utility.

### 4.1 Introduction to Utility

In this part we will discuss the concept of utility. Following Stigler's *The Development of Utility Theory* [10], we find that the utility is a quantity that a person wants to maximize. Philosophically, this can be something like pleasure or satisfaction, and this quantity is generally assumed to be impossible to measure. However, in simplified models it is possible to generate a mathematical expression for a utility function that simulates known results.

This utility $U$ is a function of the consumption of one or more goods $x_1, x_2, x_3, ....$ Economists will generally look at a person's income, which is an amount of money that a person can spend, and see what combination of goods this person should buy (with given prices $p_1, p_2, p_3, ...$) and consume to maximize this utility. In order to find this, as it turns out, it is more useful to look at the *marginal utilities* $MU_1, MU_2, MU_3, ...$ rather than the *utility.* In economics, the *marginal* prefix is equal to taking a partial derivative in mathematical terms:

$$MU_i = \frac{\partial U}{\partial x_i}. \tag{4.1}$$

Now, to maximize the utility, Heinrich Gossen was the first to write down what is now often called the fundamental principle of marginal utility theory: "A person maximizes his utility when he distributes his available money among the various goods so that he obtains the same amount of satisfaction from the last unit of money spent on each commodity [10]". This translates into the following equation.

$$\frac{MU_1}{p_1} = \frac{MU_2}{p_2} = \frac{MU_3}{p_3} = ... \tag{4.2}$$

A second law of this marginal utility theory originates from a century before Gossen wrote his principle. This law comes from the physicist and mathematician Daniel Bernoulli, who brought fame to the following paradox: "Peter tosses a coin in the air repeatedly until it falls heads up. If this occurs on the first throw, he pays Paul \$1.00; if this occurs first on the second throw, he pays \$2.00; on the third throw, \$4.00; on the fourth throw, \$8.00; and on the $n$th throw, $\$2.00^{n-1}$. What is the maximum amount Paul should pay for this game?" [10]

The solution would be to look at every individual toss and see - on average - how much money this wins Paul. For the first toss there is a probability of 1/2 that he wins \$1.00, resulting in a \$0.50 average. The second toss gives a probability of 1/4 to win \$2.00, also resulting in a \$0.50 average. This result repeats itself for all tosses, resulting in a $\$0.50 \times \infty = \$\infty$ average prize. This would be a ridiculously large amount of money to pay for such game.

Bernoulli solved this paradox by introducing the *law of diminishing marginal utility.* He basically assumed that the first dollar you win has more value to Paul, than the second one, and the second one has more than the third, and so on. It may be better explained by the following example. If you win the lottery jackpot of ten million dollar, and the next month you win the same amount again, with which prize are you pleasured most? Right, probably the first one. Bernoulli proceeded to solve this by taking the marginal utility to fall off as a function of $x^{-1}$ as an example:

$$\frac{dU}{dx} = \frac{k}{x}. \tag{4.3}$$

This would lead to the following utility function

$$U(x) = k \log \frac{x}{c}, \tag{4.4}$$

in which $c$ is the amount of money necessary for existence; i.e. $U(c) = 0$, and $k$ is just some scaling constant that will drop out in most results, because we will often only be interested in the comparison between two (equally scaled) utilities.

To finish the solution to the paradox, instead of looking at the gain and loss in money, we will now look at the gain and loss in utility. The gained utility $\Delta U^+$ is the summed gain in utility of every individual toss:

$$\begin{aligned}\Delta U^+ &= \sum_{n=1}^{\infty} \frac{1}{2^n} k \left[ \log\left(\frac{x_0 + 2^{n-1}}{c}\right) - \log\left(\frac{x_0}{c}\right)\right] \\ &= \sum_{n=1}^{\infty} \frac{k}{2^n} \log\left(1 + \frac{2^{n-1}}{x_0}\right),\end{aligned} \tag{4.5}$$

where $x_0$ is the extra amount of money Paul has over the money needed for existence. Note that the dependence on $c$ has already dropped. The loss in utility induced by the payment for this game $\Delta U^-$ is calculated similarly,

$$\Delta U^- = k \log\left(1 + \frac{p}{x_0}\right). \tag{4.6}$$

We then proceed by setting $\Delta U^+ = \Delta U^-$ and solving for $p$, which ultimately leads to the following expression,

$$p = x_0 \left[-1 + \prod_{n=1}^{\infty} \left(1 + \frac{2^{n-1}}{x_0}\right)^{2^{-n}}\right]. \tag{4.7}$$

Note that this expression does not depend on $k$. On first glance it seems we are not able to solve this analytically. Numerical methods, however, show that, if Paul has $x_0 = 1000\$$, he should pay $p \approx 5.97\$$ for this game.

In the rest of this section we will use the concept of utility to help us construct the new model.

## 4.2 Demand and Supply from Utility

Let us assume we have good A, and good B. We treat them equivalently, but in principle good B could be thought of as a certain currency, and good A could be a certain good. In our system we have $N$ agents that all have a certain amount of good A $a_i$, and of good B $b_i$ $(i = 1, 2, 3, ..., N)$ in their possession. Now we assume that all these agents have the same utility function, $U$, but their values depend on their individual possessions $a_i$ and $b_i$: $U(a_i, b_i)$. We assume that all trades that are going to take place involve a fixed amount of good A $\Delta a$ and a non-fixed amount of good B represented by the variable $p$ ('p' for price).

Obviously, for the transaction to go through we need the simple condition that the buyer $i$ must have

$$b_i > p, \tag{4.8}$$

and the seller $j$ must have

$$a_j > \Delta a. \tag{4.9}$$

Furthermore, this transaction should only go through if the utilities of both traders are increased or kept equal. So for the buyer $i$ this means that

$$U(a_i + \Delta a, b_i - p) > U(a_i, b_i). \tag{4.10}$$

This condition can be simplified in terms of the marginal utility if we take $\Delta a$ infinitesimally small and we look at the first order Taylor expansion.

$$U(a_i, b_i) + \frac{\partial U(a_i, b_i)}{\partial a_i} \Delta a - \frac{\partial U(a_i, b_i)}{\partial b_i} p > U(a_i, b_i), \tag{4.11}$$

or

$$p < \frac{\partial U(a_i, b_i)/\partial a_i}{\partial U(a_i, b_i)/\partial b_i} \Delta a. \tag{4.12}$$

Similarly, for seller $j$, the transaction is reversed, leading to the following condition.

$$U(a_j - \Delta a, b_j + p) > U(a_j, b_j), \tag{4.13}$$

or again for small $\Delta a$ we find the same expression as we saw for the buyer, but with flipped sign,

$$p > \frac{\partial U(a_j, b_j)/\partial a_j}{\partial U(a_j, b_j)/\partial b_j} \Delta a. \tag{4.14}$$

This means that in principle every agent either wants to buy or sell. The demand of the entire system is determined by just checking all agents if they are able and willing to buy good A at a certain given price.

$$d_A(p) = \Delta a \sum_{i=1}^{N} \theta \left( \frac{\partial U(a_i, b_i)/\partial a_i}{\partial U(a_i, b_i)/\partial b_i} \Delta a - p \right) \theta(b_i - p). \tag{4.15}$$

The global supply of good A is determined similarly:

$$s_A(p) = \Delta a \sum_{i=1}^{N} \theta \left( p - \frac{\partial U(a_i, b_i)/\partial a_i}{\partial U(a_i, b_i)/\partial b_i} \Delta a \right) \theta(a_i - \Delta a). \tag{4.16}$$

If we assume that there is a known distribution of good A and good B of the form $P(a, b)$, with normalization,

$$\int_0^\infty da \int_0^\infty dbP(a, b) = 1, \tag{4.17}$$

we can calculate the expectation values of $d_A(p)$ and $s_A(p)$ by integrating their ability and willingness to trade multiplied by this distribution over $a$ and $b$.

$$\begin{aligned} d_A(p) &= N\Delta a \int_0^\infty da \int_0^\infty dbP(a, b)\theta \left( \frac{\partial U(a, b)/\partial a}{\partial U(a, b)/\partial b} \Delta a - p \right) \theta(b - p), \\ s_A(p) &= N\Delta a \int_0^\infty da \int_0^\infty dbP(a, b)\theta \left( p - \frac{\partial U(a, b)/\partial a}{\partial U(a, b)/\partial b} \Delta a \right) \theta(a - \Delta a). \end{aligned} \tag{4.18}$$

We will use these demand and supply curves to determine the equilibrium price, $p$ in units of good B, that will be paid for $\Delta a$ units of good A. Also, we are about to be in need of a utility function, which is why we will go into a short intermezzo.

## 4.3 Intermezzo: Elasticity of Substitution

The *elasticity of substitution*, $\sigma$, shows how the ratio of the consumption good A compared to good B, $A = a/b$ changes as the ratio of the price of good B compared to good A $P_B = \Delta a/p$ changes.

$$\sigma = \frac{\Delta A/A}{\Delta P_B/P_B} = \frac{dA}{dP_B} \frac{P_B}{A}. \tag{4.19}$$

Generally speaking $\sigma \geq 0$, as we assume that a price increase in good B compared to good A will not lead to an increase in the consumption of good B compared to good A.

A very interesting - and commonly used - utility function is the Constant Elasticity of Substitution (CES) utility function introduced by Solow in 1956 [11]:

$$U(a, b) = \left[ \left( \frac{a}{\alpha} \right)^r + \left( \frac{b}{\beta} \right)^r \right]^{1/r}. \tag{4.20}$$

Let us calculate $\sigma$ for this utility. We start by calculating the marginal utility functions:

$$\begin{aligned} \frac{\partial U}{\partial a} &= \frac{1}{\alpha} \left[ 1 + \left( \frac{\alpha b}{\beta a} \right)^r \right]^{(1-r)/r}, \\ \frac{\partial U}{\partial b} &= \frac{1}{\beta} \left[ 1 + \left( \frac{\beta a}{\alpha b} \right)^r \right]^{(1-r)/r}. \end{aligned} \tag{4.21}$$

Let us now use the fundamental principle of marginal utility theory, Equation 4.2:

$$\frac{1}{p} \frac{\partial U}{\partial a} = \frac{1}{\Delta a} \frac{\partial U}{\partial b}. \tag{4.22}$$

We try to find $A(P_B)$ so that we can calculate the derivative in Equation 4.19. We find

$$A(P_B) = \left( \frac{\alpha}{\beta} \right)^{r/(r-1)} P_B^{1/(1-r)}. \tag{4.23}$$

Therefore, the derivative is easily calculated to be

$$\frac{dA}{dP_B} = \frac{1}{1-r} \left( \frac{\alpha}{\beta} \right)^{r/(r-1)} P_B^{r/(1-r)}. \tag{4.24}$$

So filling this in Equation 4.19 we see that

$$\sigma = \frac{dA}{dP_B}\frac{P_B}{A} = \frac{1}{1-r}\frac{\left(\frac{\alpha}{\beta}\right)^{r/(r-1)} P_B^{r/(1-r)} P_B}{\left(\frac{\alpha}{\beta}\right)^{r/(r-1)} P_B^{1/(1-r)}} = \frac{1}{1-r}. \tag{4.25}$$

We can see that if we do not want $\sigma < 0$, then $r < 1$.

## 4.4 CES Demand and Supply

Let us now use the CES utility to calculate the demand and supply curves. To do this we need to assume a distribution function $P(a, b)$. Let us take the Boltzmann distribution; because we are looking at conserved quantities, i.e. the total amount of good A and B are separately conserved, this is not a very strange guess. Therefore, we take

$$P(a, b) = \frac{1}{\bar{a}} e^{-a/\bar{a}} \frac{1}{\bar{b}} e^{-b/\bar{b}}, \tag{4.26}$$

with $\bar{a}$ and $\bar{b}$ the average amount of good A and good B in the system respectively. These Boltzmann exponentials are easy to integrate, so this may indicate that the solution to Equation 4.18 can be calculated analytically. We arrive at

$$\frac{d_A(p)}{\Delta a} = \frac{N}{\bar{a}\bar{b}} \int_0^\infty da \int_0^\infty db\; e^{-a/\bar{a}} e^{-b/\bar{b}} \theta \left\{ \frac{\Delta a}{\alpha}\left[1 + \left(\frac{\alpha b}{\beta a}\right)^r\right]^{(1-r)/r} - \frac{p}{\beta}\left[1 + \left(\frac{\beta a}{\alpha b}\right)^r\right]^{(1-r)/r} \right\} \theta(b - p). \tag{4.27}$$

We want to reduce the first $\theta$-function to a condition for $a$ or $b$, and we are lucky that is equal to Equation 4.22, the result of which was already shown:

$$\left(\frac{b}{a}\right)_{\text{crit}} = \left(\frac{\beta}{\alpha}\right)^{r/(r-1)} \left(\frac{\Delta a}{p}\right)^{1/(r-1)} = \gamma(p). \tag{4.28}$$

We only need to know if our $b/a$ should be greater or less than this $\gamma(p)$. As it turns out, after careful calculations, for $r < 1$,

$$\theta \left\{ \frac{\Delta a}{\alpha}\left[1 + \left(\frac{\alpha b}{\beta a}\right)^r\right]^{(1-r)/r} - \frac{p}{\beta}\left[1 + \left(\frac{\beta a}{\alpha b}\right)^r\right]^{(1-r)/r} \right\} = \theta\left(b - \gamma(p) a\right). \tag{4.29}$$

We then proceed by merging this $\theta$-function in the integral over $a$:

$$\begin{aligned} \frac{d_A(p)}{\Delta a} &= \frac{N}{\bar{a}\bar{b}} \int_0^\infty db\; e^{-b/\bar{b}} \theta(b - p) \int_0^{b/\gamma(p)} da\; e^{-a/\bar{a}} \\ &= \frac{N}{\bar{b}} \int_p^\infty db\; e^{-b/\bar{b}} \left[1 - \exp\left(-\frac{b}{\gamma(p)\bar{a}}\right)\right] \\ &= N e^{-p/\bar{b}} \left[1 - \frac{\gamma(p)\bar{a}}{\gamma(p)\bar{a} + \bar{b}} \exp\left(-\frac{p}{\gamma(p)\bar{a}}\right)\right]. \end{aligned} \tag{4.30}$$

In a very similar way we find the supply curve.

$$\frac{s_A(p)}{\Delta a} = N e^{-\Delta a/\bar{a}} \left[1 - \frac{\bar{b}}{\gamma(p)\bar{a} + \bar{b}} \exp\left(-\frac{\gamma(p)\Delta a}{\bar{b}}\right)\right]. \tag{4.31}$$

To grasp more of the behavior of these resulting functions, we look at the interesting regions of $p \to 0$, and $p \to \infty$. First, we check what the critical ratio of $b/a$ does in these limit cases. We easily see (still for $r < 1$, and therefore $1/(r-1) < 0$) that $\lim_{p\to 0} \gamma(p) = 0$, and $\lim_{p\to\infty} \gamma(p) = \infty$. Also, we are going to need to see what the quantity of $p/\gamma(p) \propto p^{1-1/(r-1)} = p^{(r-2)/(r-1)}$ in the second exponential in the demand curve expression does: $\lim_{p\to 0} p/\gamma(p) = \infty$, and $\lim_{p\to\infty} p/\gamma(p) = 0$, because $(r-2)/(r-1) < 0$, for $r < 1$. Now we have enough tools to start calculating these limits of the demand and supply curves.

- How much is the demand of good A when its price is zero?

$$\lim_{p\to 0} d_A(p) = \lim_{p\to 0} N \Delta a e^{-p/\bar{b}} \left[1 - \frac{\gamma(p)\bar{a}}{\gamma(p)\bar{a} + \bar{b}} \exp\left(-\frac{p}{\gamma(p)\bar{a}}\right)\right]. \tag{4.32}$$

We see instantly that the first exponential will equal 1 (everyone can pay the price), the fraction drops to zero, as well as the second exponential ($1 - 0 \times 0 = 1$, everyone will improve his/her utility with this transaction), resulting in

$$\lim_{p \to 0} d_A(p) = N\Delta a. \tag{4.33}$$

As you probably would have guessed, if you can get good A without giving anything in return, everyone would agree with this transaction.

- How much is the demand of good A when its price becomes infinitely high?

$$\lim_{p \to \infty} d_A(p) = \lim_{p \to \infty} N\Delta a e^{-p/\bar{b}} \left[1 - \frac{\gamma(p)\bar{a}}{\gamma(p)\bar{a} + \bar{b}} \exp\left(-\frac{p}{\gamma(p)\bar{a}}\right)\right]. \tag{4.34}$$

The first exponential will now drop to zero (nobody can pay the transaction), where the fraction becomes 1, as well as the second exponential ($1 - 1 \times 1 = 0$, nobody improves his/her utility with this transaction, resulting in

$$\lim_{p \to \infty} d_A(p) = 0. \tag{4.35}$$

This result is also very intuitive; nobody wants to buy good A, when he has to pay an infinite price. We expect the demand curve to decay asymptotically to zero.

- How much is the supply of good A when its price is zero?

$$\lim_{p \to 0} s_A(p) = \lim_{p \to 0} N\Delta a e^{-\Delta a/\bar{a}} \left[1 - \frac{\bar{b}}{\gamma(p)\bar{a} + \bar{b}} \exp\left(-\frac{\gamma(p)\Delta a}{\bar{b}}\right)\right]. \tag{4.36}$$

The fraction becomes 1, as well as the second exponential ($1 - 1 \times 1 = 0$, nobody improves his/her utility by selling good A without any return), resulting in

$$\lim_{p \to 0} s_A(p) = 0. \tag{4.37}$$

This is also the result one would have expected; nobody wants to sell their product without any return.

- How much is the supply of good A when its price becomes infinitely high?

$$\lim_{p \to \infty} s_A(p) = \lim_{p \to \infty} N\Delta a e^{-\Delta a/\bar{a}} \left[1 - \frac{\bar{b}}{\gamma(p)\bar{a} + \bar{b}} \exp\left(-\frac{\gamma(p)\Delta a}{\bar{b}}\right)\right]. \tag{4.38}$$

The fraction becomes zero, as well as the second exponential ($1 - 0 \times 0 = 1$, everybody improves his/her utility by selling and getting infinitely much in return), resulting in

$$\lim_{p \to \infty} s_A(p) = N\Delta a e^{-\Delta a/\bar{a}}. \tag{4.39}$$

Here we see a result that may not have been one's first guess. When the price of good A becomes infinitely high, not everybody can sell the amount of $\Delta a$, because not everybody *has* so much of good A to sell. Therefore, the correct guess would have been

$$\lim_{p \to \infty} s_A(p) = N\Delta a \int_{\Delta a}^{\infty} da \frac{1}{\bar{a}} e^{-a/\bar{a}} = N\Delta a e^{-\Delta a/\bar{a}}. \tag{4.40}$$

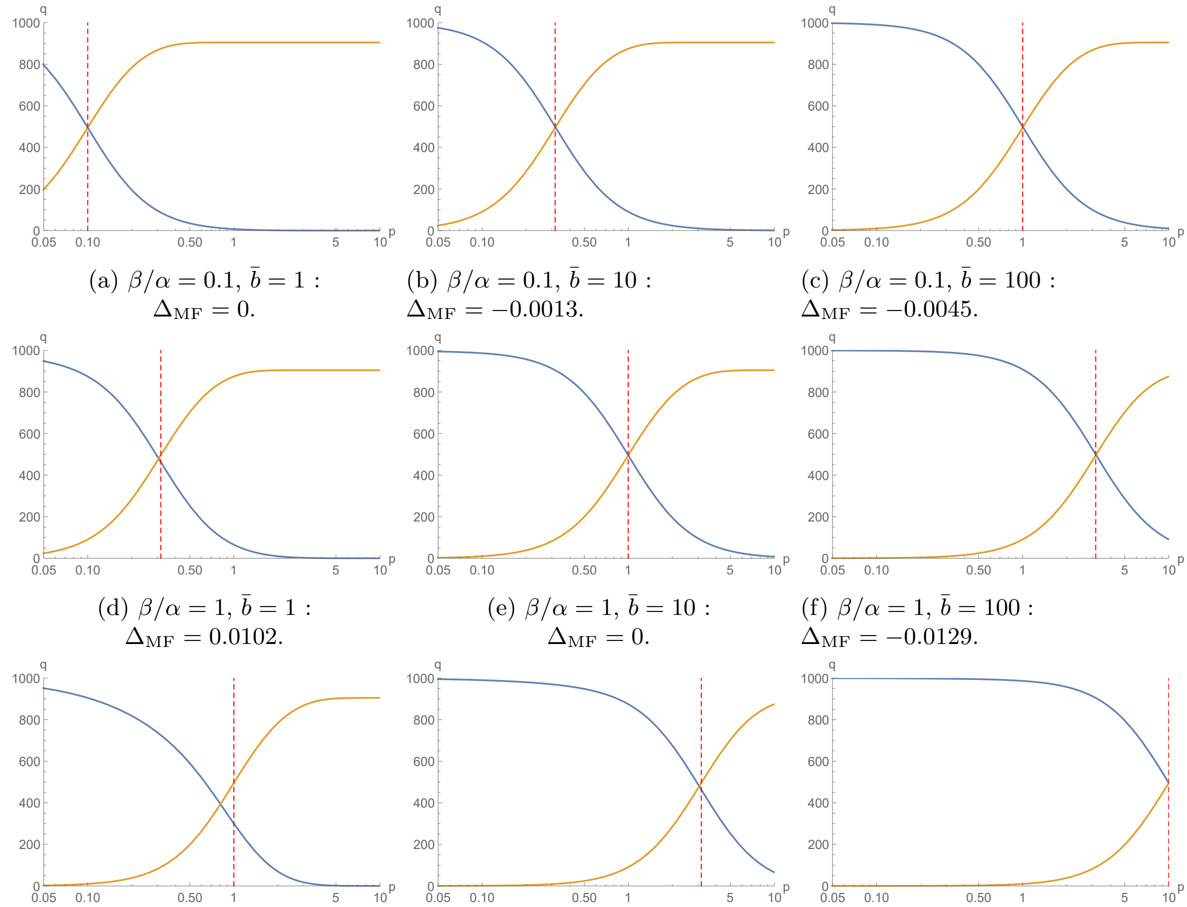
We expect the supply curve to ascend asymptotically to this value.

Another interesting aspect of these demand and supply functions is the price at which they intersect. This is the expected equilibrium price. We have analytic expressions for the functions, but solving

$$d_A(p) = s_A(p) \tag{4.41}$$

analytically is not possible. To get some more information, though, we inspect the case in which $\gamma(p) = \bar{b}/\bar{a}$. Why? Because if an agent's possessions obey $\gamma(p) < b/a$, it means that buying good A improves his utility, and if they obey $\gamma(p) > b/a$, it means that selling good A improves his utility, then, surely, if $\gamma(p) = b/a$, this would mean that this agent's utility is maximized. If we then take a mean-field approach, i.e. we ignore fluctuations and evaluate $a$ and $b$ at their average values, we could say that $\gamma(p) = \bar{b}/\bar{a}$. If we substitute this in Equations 4.30 and 4.31, and fill this in in Equation 4.41, we find

$$e^{-p/\bar{b}} \left(1 - \frac{1}{2} e^{-p/\bar{b}}\right) = e^{-\Delta a/\bar{a}} \left(1 - \frac{1}{2} e^{-\Delta a/\bar{a}}\right). \tag{4.42}$$




(a) $\beta/\alpha = 0.1$, $\bar{b} = 1$ : $\Delta_{\text{MF}} = 0$.

(b) $\beta/\alpha = 0.1$, $\bar{b} = 10$ : $\Delta_{\text{MF}} = -0.0013$.

(c) $\beta/\alpha = 0.1$, $\bar{b} = 100$ : $\Delta_{\text{MF}} = -0.0045$.

(d) $\beta/\alpha = 1$, $\bar{b} = 1$ : $\Delta_{\text{MF}} = 0.0102$.

(e) $\beta/\alpha = 1$, $\bar{b} = 10$ : $\Delta_{\text{MF}} = 0$.

(f) $\beta/\alpha = 1$, $\bar{b} = 100$ : $\Delta_{\text{MF}} = -0.0129$.

(g) $\beta/\alpha = 10$, $\bar{b} = 1$ : $\Delta_{\text{MF}} = 0.1896$.

(h) $\beta/\alpha = 10$, $\bar{b} = 10$ : $\Delta_{\text{MF}} = 0.1012$.

(i) $\beta/\alpha = 10$, $\bar{b} = 100$ : $\Delta_{\text{MF}} = 0$.

Figure 4.1: (blue) Demand curves, and (yellow) supply curves plotted as a function of the price, $p$. Parameters $r = 0.5$, $\Delta a = 1$, and $\bar{a} = 10$ are kept constant. (red, dashed) Line $p = \Delta a \left(\bar{b}/\bar{a}\right)^{1-r} (\beta/\alpha)^r$ is shown to see how much the intersections deviate from the mean-field assumption. $\Delta_{\text{MF}}$ is the absolute difference between this mean-field prediction and the actual intersection.

The substitution of $\gamma(p) = (\beta/\alpha)^{\frac{r}{r-1}} (\Delta a/p)^{\frac{1}{r-1}} = \bar{b}/\bar{a}$ entails in terms of $p$ itself

$$p = \Delta a \left(\frac{\bar{b}}{\bar{a}}\right)^{1-r} \left(\frac{\beta}{\alpha}\right)^r = p_{\text{MF}}. \tag{4.43}$$

The exponent $-p/\bar{b}$, can then be written to look like the exponent $-\Delta a/\bar{a}$:

$$-\frac{p}{\bar{b}} = -\Delta a \frac{\bar{a}^{r-1}}{\bar{b}^r} \left(\frac{\beta}{\alpha}\right)^r = -\frac{\Delta a}{\bar{a}} \left(\frac{\beta \bar{a}}{\alpha \bar{b}}\right)^r. \tag{4.44}$$

From this we can conclude that if

$$\left(\frac{\beta \bar{a}}{\alpha \bar{b}}\right)^r = 1, \text{ or } \frac{\bar{a}}{\alpha} = \frac{\bar{b}}{\beta}, \tag{4.45}$$

then

$$\Delta a \left(\frac{\bar{b}}{\bar{a}}\right)^{1-r} \left(\frac{\beta}{\alpha}\right)^r = \Delta a \frac{\bar{b}}{\bar{a}}, \tag{4.46}$$

and Equation 4.43 is a solution to Equation 4.41.

In Figure 4.1 we see the demand and supply curves plotted, and the compared equilibrium price predicted through mean-field. We can also see all expected features in the limits $p \to 0$, and $p \to \infty$. In general, it seems the mean-field price gives a very good prediction for the actual price. To get a better understanding of this difference we take a closer look at the case in which we saw the biggest difference with mean-field in Figure 4.2. Why do the conditions for this case result in the biggest price difference of the nine possibilities shown?

The answer is actually quite easy to grasp. The expected mean-field price for good A is 1 unit of good B. However, on average agents have only $\bar{b} = 1$ of good B. This means that, in a Boltzmann distributed system,

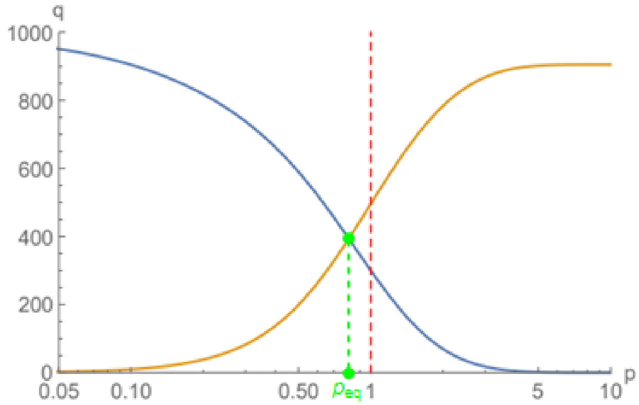



Figure 4.2: (blue) Demand and (yellow) supply curves for $\beta/\alpha = 10$, $\bar{a} = 10$, $\bar{b} = 1$, $r = 0.5$, $\Delta a = 1$, and therefore (red, dashed) $p_{\mathrm{MF}} = 1$. (green, dashed) Actual equilibrium price.

about half of the agents cannot afford to buy good A at a price that is near the mean-field price, and are therefore not a buyer. Furthermore, around $\int_0^1 da \exp(-a/10)/10 \approx 10\%$ of the agents do not have enough of good A to sell $\Delta a$, and are therefore not a seller. So, statistically speaking, about 5% of the agents is neither a buyer nor a seller. This means that these agents do not contribute to the demand or the supply curves around the mean-field price. Furthermore, it means that how ever many goods they *do* have, get taken out of play. Among these goods are - compared to their averages - more of type B than there are of type A, which makes good B worth slightly more and it makes good A worth slightly less. As the equilibrium price is the price of good A, it is then only logical that it is lower than the mean-field price.

In the other cases, the group of agents that are neither a buyer nor a seller, is significantly smaller, making the mean-field price a better approximation to the actual price. Furthermore, in the cases where $\bar{a}/\alpha = \bar{b}/\alpha$, an equivalent amount of good A and good B is taken out of play, leading to the exact same equilibrium price, as predicted.

## 4.5 Dynamics

Now imagine again a system of $N$ agents. These agents can all possess both an amount of good A as well as good B. We would like to see what happens when we allow these agents to trade based on maximizing their utility. The trades occur at the *market*, which is a place that all agents go to every time step. At this market, a price for the amount of goods $\Delta a$ is established through demand and supply, such that $d_A(p) = s_A(p)$. This means that there are an equal amount of sellers as there are buyers at this price. Next, all agents that want to sell good A at price $p$ throw an amount of $\Delta a$ of good A on a pile, and all agents that want to buy good A at price $p$ throw an amount of $p$ of good B on a second pile. Then, the buyers grab $\Delta a$ of good A from the first pile and sellers grab $p$ of good B from the second pile. One could also say that all sellers find a buyer to trade $\Delta a$ for $p$, but this way it is more clear that it does not matter who sells to whom, because all trades are identical.

To start, we could choose to assign them all an amount of $\bar{a}$ of good A and an amount of $\bar{b}$ of good B. However, with initial conditions that make them all identical, i.e. $P(a, b) = \delta(a - \bar{a})\delta(b - \bar{b})$, we will see that they have identical needs. For the demand and supply curves this leads to

$$\begin{aligned} d_A(p) &= N\Delta a \int_0^\infty da \int_0^\infty db \delta(a - \bar{a})\delta(b - \bar{b})\theta[b - \gamma(p)a]\theta(b - p) \\ &= N\Delta a \theta[\bar{b} - \gamma(p)\bar{a}]\theta(\bar{b} - p), \end{aligned} \tag{4.47}$$

$$\begin{aligned} s_A(p) &= N\Delta a \int_0^\infty da \int_0^\infty db \delta(a - \bar{a})\delta(b - \bar{b})\theta[\gamma(p)a - b]\theta(a - \Delta a) \\ &= N\Delta a \theta[\gamma(p)\bar{a} - \bar{b}]\theta(\bar{a} - \Delta a). \end{aligned} \tag{4.48}$$

A graphic example of the implied demand and supply curves is shown in Figure 4.3. The equilibrium price is easily established at $\gamma(p) = \bar{b}/\bar{a}$, which is at $p = p_{\mathrm{MF}}$, as was shown in Equation 4.43. However, it is not clear which agents are going to sell and which agents are going to buy good A. In fact, it does not matter to them if they sell or buy at price $p_{\mathrm{MF}}$. With an eye on the way we would like to simulate the dynamics of this system, we need to say that this is highly impractical. Because of issues regarding precision we obviously cannot evaluate the demand and supply curve at all values of $p$, which means that we would need to choose the values of $p$ at which we *will* evaluate the curves to contain the value of *exactly* $p = p_{\rm MF}$. A computer obviously only has a finite precision, which complicates this matter.

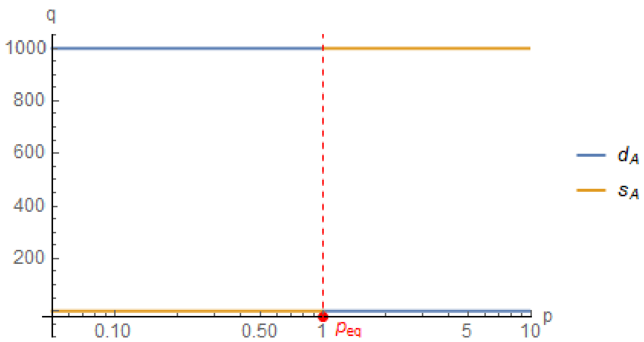



Figure 4.3: (blue) Demand and (yellow) supply curves for uniformly distributed goods. For simplicity assuming that all agents have enough goods to both pay for all prices $p$, and to supply the static $\Delta a$.

To circumvent this problem, we could impose inequality on this distribution, by initially distributing both good A and B exponentially and uncorrelated. We then get demand and supply curves in the form we discussed in the previous subsection and were shown in Figure 4.1. This should work fine, beside the fact that statistically speaking there will be agents that have less of good A than $\Delta a$, *and* less of good B than roughly $p_{\rm MF}$. These agents will never be able to trade, if the market price never drops much below $p_{\rm MF}$. One could argue if this ultimately is a problem in the dynamics, though. The initial distribution would then be for example

$$P(a,b) = \frac{1}{\bar{a}\bar{b}} e^{-a/\bar{a}} e^{-b/\bar{b}}. \tag{4.49}$$

Other ways to impose inequality include giving all agents a random amount of good A on the interval $[0, 2\bar{a}]$, and a random amount of good B on the interval $[0, 2\bar{b}]$. Statistically, this leaves the average amount of goods $\langle a \rangle \neq \bar{a}$, and $\langle b \rangle \neq \bar{b}$, but for large $N$ this should not be a very significant difference, and it would not influence the dynamics very much. The problem with excluding certain agents because they do not have enough possessions to trade still remains, though. The initial distribution would be

$$P(a,b) = \frac{1}{4\bar{a}\bar{b}} \theta(a)\theta(2\bar{a} - a)\theta(b)\theta(2\bar{b} - b). \tag{4.50}$$

So a similar way to approach this is, for every agent, generating a random variable $\lambda \in [0,1]$, then assigning the agent $2\lambda\bar{a}$ of good A and $2(1-\lambda)\bar{b}$ of good B. This way, agents with very little A, have a lot of B, and the other way around, making sure that every agent has enough possessions to at least either sell or buy good A. The initial distribution would be

$$P(a,b) = \frac{1}{2\bar{a}} \theta(a)\theta(2\bar{a} - a)\delta\left[b - 2\bar{b}\left(1 - \frac{a}{2\bar{a}}\right)\right]. \tag{4.51}$$

Let us now run the simulation for these three initial distributions. We will start with the very simplest case imaginable; $\bar{a} = \bar{b} = 10$, $\alpha = \beta = 1$, $r = 0.5$, and $\Delta a = 1$. This means that the average of A is equal to average of B (good A and good B are equally common). Furthermore, agents get equal utility from good A and good B (no preference). Therefore, we know the mean-field price, $p_{\rm MF} = 1$, and the stable ratio the agents want their goods to be in is one-to-one.

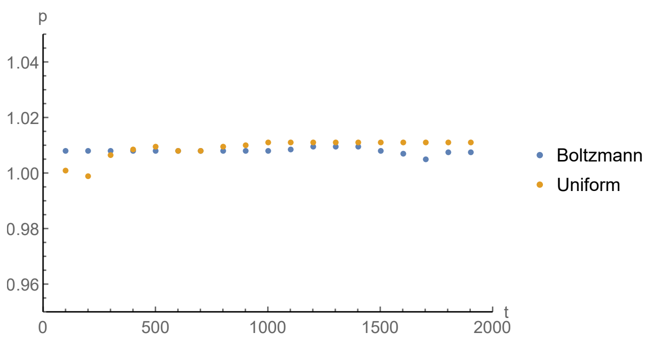
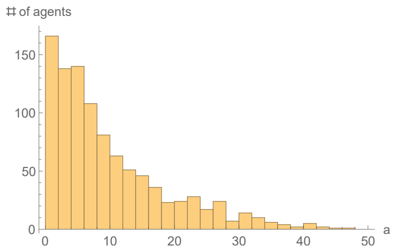
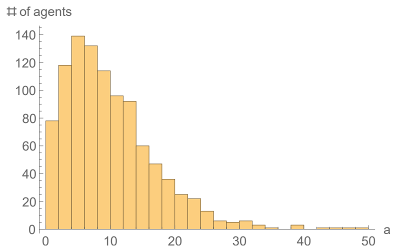


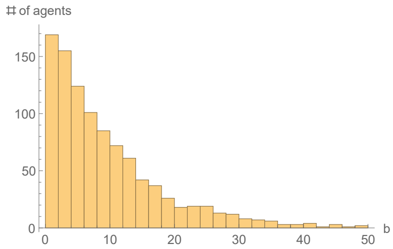

Figure 4.4: Evolution of the established market price for two differently initialized systems.



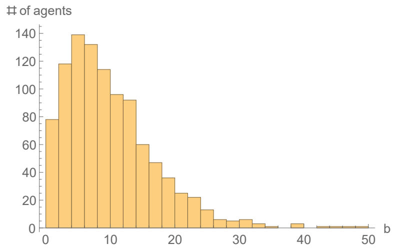

(a) Distribution of A at $t = 0$.



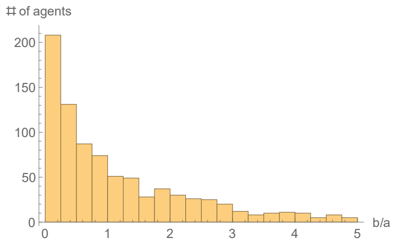

(b) Distribution of A at $t = 2000$.



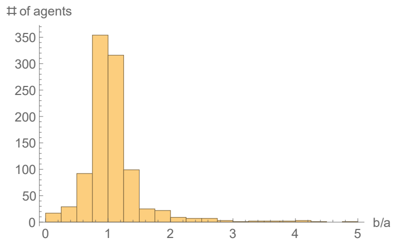

(c) Distribution of B at $t = 0$.



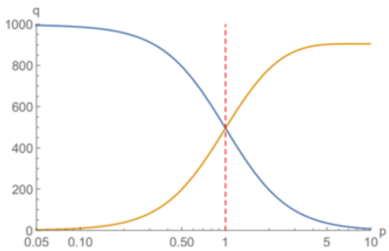

(d) Distribution of B at $t = 2000$.



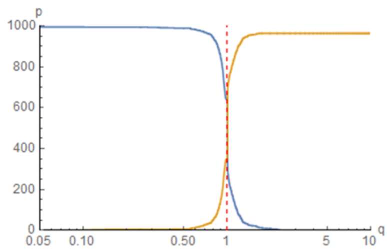

(e) Distribution of $b/a$ at $t = 0$.




(f) Distribution of $b/a$ at $t = 2000$.




(g) (blue) Demand and (yellow) supply curves at $t = 0$.



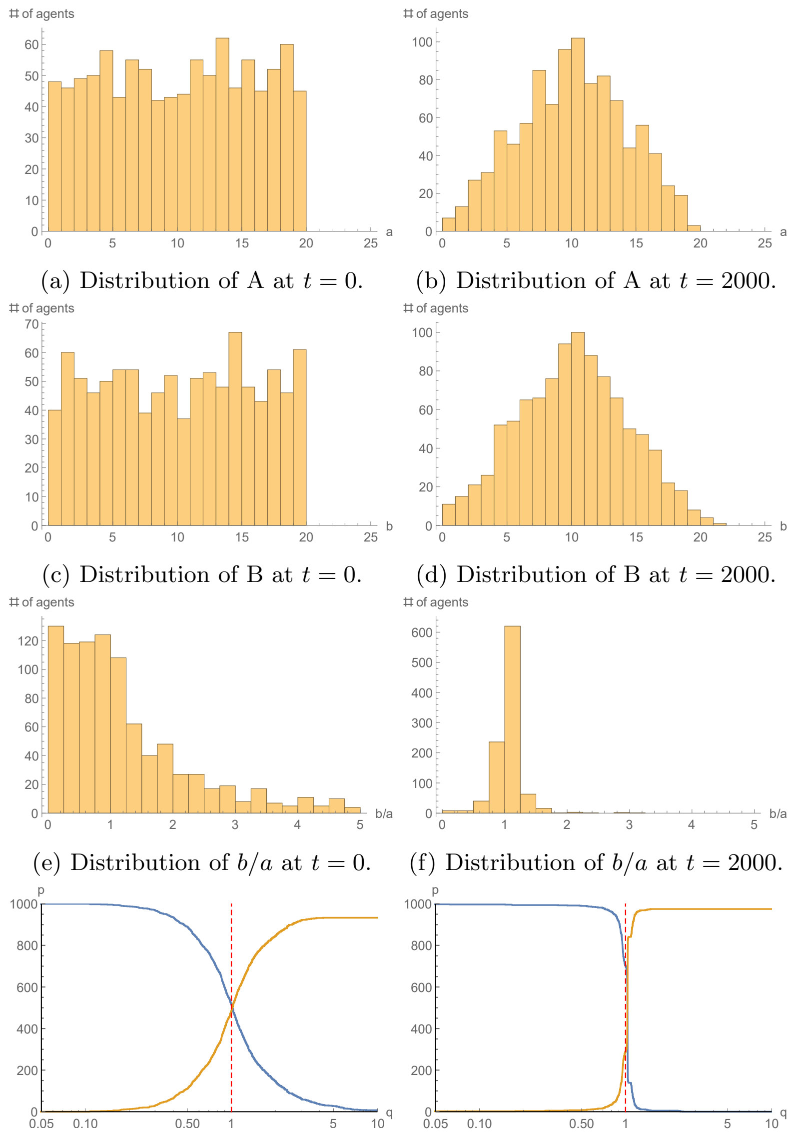

(h) (blue) Demand and (yellow) supply curves at $t = 2000$.

Figure 4.5: Evolution of system due to market trades with initial Boltzmann distribution.

The results are shown in Figure 4.5 and 4.6. It is clear to see that the final distributions of good A and B of the different initialized systems are very different, but the agents definitely try to reach this ratio of $b/a = 1$. In this very symmetric case, we can quite easily calculate the reason for this difference in distribution. First, we look at the price evolution throughout the simulation time (Figure 4.4). x As we can see, the established market price is kept quite constant throughout the time, and it is also quite close to the mean-field price $p_{\text{MF}} = 1$. So let us approximate this simulation to have only included trades of $\Delta a = 1$ for $p = 1$. This would mean that all agents keep their total amount of goods, $c = a + b$, constant. We know the agents strive to have that one-to-one ratio of the two goods, so that would mean they can only divide their initial total amount of goods, $c_{\text{init}}$, to get $a_{\text{final}} = b_{\text{final}} = c_{\text{init}}/2 = c/2$. Now, let $P_C(c)$ be the distribution of the total amount of goods, which we know will not change throughout the simulation time. To calculate this distribution from the known $P(a, b)$ we need to calculate

$$P_C(c) = \int_0^c da P(a, c - a), \tag{4.52}$$

i.e. the added up probabilities of the ways one could construct $c = a + b$. Let us calculate this for the different




(a) Distribution of A at $t = 0$. (b) Distribution of A at $t = 2000$.

(c) Distribution of B at $t = 0$. (d) Distribution of B at $t = 2000$.

(e) Distribution of $b/a$ at $t = 0$. (f) Distribution of $b/a$ at $t = 2000$.

(g) (blue) Demand and (yellow) supply curves at $t = 0$. (h) (blue) Demand and (yellow) supply curves at $t = 2000$.

Figure 4.6: Evolution of system due to market trades with initial uniform distribution.

distributions we used. We start with the Boltzmann distribution.

$$
\begin{aligned}
P_C(c) &= \frac{1}{\bar{a}\bar{b}} \int_0^c da\ e^{-a/\bar{a}} e^{-(c-a)/\bar{b}} \\
&= \frac{e^{-c/\bar{b}}}{\bar{a}\bar{b}} \int_0^c da\ \exp\left[a\left(\frac{1}{\bar{b}} - \frac{1}{\bar{a}}\right)\right] \\
&= \frac{e^{-c/\bar{b}}}{\bar{a}\bar{b}} \frac{1}{\frac{1}{\bar{b}} - \frac{1}{\bar{a}}} \left\{\exp\left[c\left(\frac{1}{\bar{b}} - \frac{1}{\bar{a}}\right)\right] - 1\right\} \\
&= \frac{e^{-c/\bar{a}} - e^{-c/\bar{b}}}{\bar{a} - \bar{b}}.
\end{aligned}
\tag{4.53}
$$

We know that for the final distribution of good A, $a = c/2$, so

$$
P_A(a) \propto \frac{e^{-2a/\bar{a}} - e^{-2a/\bar{b}}}{\bar{a} - \bar{b}}, \tag{4.54}
$$

which we need to normalize properly. So let us calculate

$$
\int_0^\infty da\ P_A(a) = \frac{1}{\bar{a} - \bar{b}} \left(\frac{\bar{a}}{2} - \frac{\bar{b}}{2}\right) = \frac{1}{2}, \tag{4.55}
$$

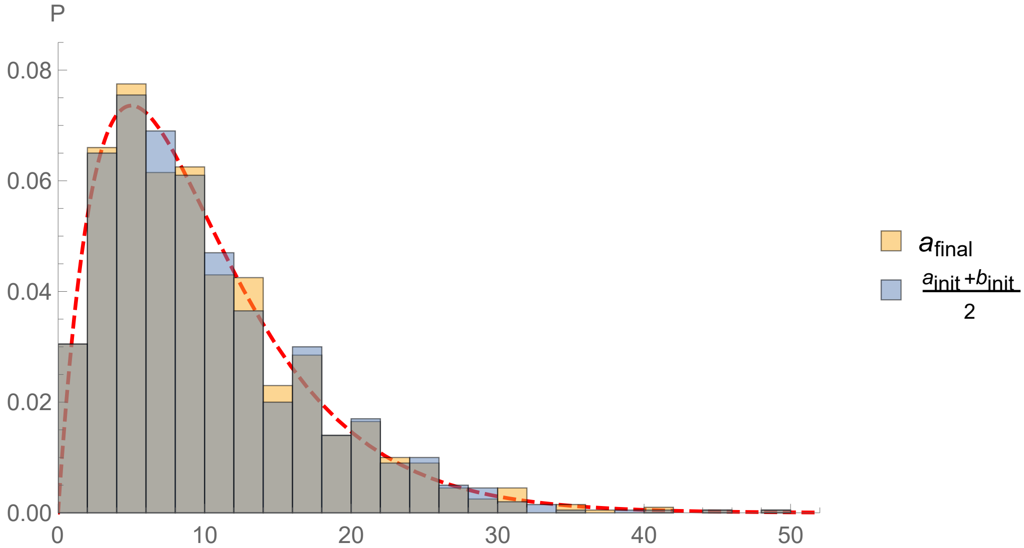



Figure 4.7: Comparison between (yellow) the final distribution of good A after 2000 time steps, with initial Boltzmann distribution, and (blue) the initial distribution of the average amount of goods per agent. (red, dashed) The prediction based off the assumption that trades were exclusively one-for-one and the ideal ratio was one-to-one, Equation 4.57.

which makes the properly normalized final distribution of A

$$P_A(a) = \frac{2e^{-2a/\bar{a}} - 2e^{-2a/\bar{b}}}{\bar{a} - \bar{b}}. \tag{4.56}$$

As it turns out, we see that it is not quite clear what happens to this expression as $\bar{b} \to \bar{a}$, because both numerator and the denominator approach zero at this limit. To calculate this limit case we use L'Hospital's rule.

$$\lim_{\bar{b}\to\bar{a}} P_A(a) = \lim_{\bar{b}\to\bar{a}} \frac{-4ae^{-2a/\bar{b}}}{-\bar{b}^2} = \frac{4a}{\bar{a}^2} e^{-2a/\bar{a}}. \tag{4.57}$$

The accuracy of this approximation is illustrated in Figure 4.7.

The same calculation can be done for the system that started out with uniform distribution. We start from Equation 4.52 and fill in Equation 4.50, which gets us

$$P_C(c) = \frac{1}{4\bar{a}\bar{b}} \int_0^c da\theta(a)\theta(2\bar{a} - a)\theta(c - a)\theta(2\bar{b} - c + a). \tag{4.58}$$

We immediately see that because of the integration boundaries $\theta(a)$ and $\theta(c-a)$ are always 1, because $a \in [0, c]$. This leaves us only with an upper ($2\bar{a}$) and a lower boundary ($c - 2\bar{b}$) for $a$ in the integrand. There are several parts of the integration to be separated:

- The upper integrand boundary is *larger* than the upper integration boundary, and the lower integrand boundary is *smaller* than the lower integration boundary;

$$\theta(2\bar{a} - c)\theta(2\bar{b} - c) \int_0^c da\ 1 = c\ \theta(2\bar{a} - c)\theta(2\bar{b} - c). \tag{4.59}$$

- The upper integrand boundary is *larger* than the upper integration boundary, and the lower integrand boundary is *larger* than the lower integration boundary;

$$\theta(2\bar{a} - c)\theta(c - 2\bar{b}) \int_{c-2\bar{b}}^c da\ 1 = 2\bar{b}\ \theta(2\bar{a} - c)\theta(c - 2\bar{b}). \tag{4.60}$$

- The upper integrand boundary is *smaller* than the upper integration boundary, and the lower integrand boundary is *smaller* than the lower integration boundary;

$$\theta(c - 2\bar{a})\theta(2\bar{b} - c) \int_0^{2\bar{a}} da\ 1 = 2\bar{a}\ \theta(c - 2\bar{a})\theta(2\bar{b} - c). \tag{4.61}$$

- The upper integrand boundary is *smaller* than the upper integration boundary, and the lower integrand boundary is *larger* than the lower integration boundary, *and* the upper integrand boundary is *larger* than the lower integrand boundary;

$$\theta(c-2\bar{a})\theta(c-2\bar{b})\theta(2\bar{a}+2\bar{b}-c)\int_{c-2\bar{b}}^{2\bar{a}} da\ 1 = \theta(c-2\bar{a})\theta(c-2\bar{b})\theta(2\bar{a}+2\bar{b}-c)(2\bar{a}+2\bar{b}-c). \tag{4.62}$$

- Any other case; 0.

This can be written slightly more compact by using a matrix notation.

$$P_C(c) = \frac{1}{4\bar{a}\bar{b}}\begin{bmatrix}\theta(2\bar{b}-c) & \theta(c-2\bar{b})\end{bmatrix}\begin{bmatrix} c & 2\bar{a} \\ 2\bar{b} & \theta(2\bar{a}+2\bar{b}-c)(2\bar{a}+2\bar{b}-c)\end{bmatrix}\begin{bmatrix}\theta(2\bar{a}-c) \\ \theta(c-2\bar{a})\end{bmatrix}. \tag{4.63}$$

To find the corresponding final distribution of good A we substitute $2a$ for $c$, and by doing so we effectively half the surface under the curve in terms of $a$, as we saw with the initially Boltzmann distributed system. Therefore, we find the properly normalized distribution function for good A,

$$P_A(a) = \frac{1}{\bar{a}\bar{b}}\begin{bmatrix}\theta(\bar{b}-a) & \theta(a-\bar{b})\end{bmatrix}\begin{bmatrix} a & \bar{a} \\ \bar{b} & \theta(\bar{a}+\bar{b}-a)(\bar{a}+\bar{b}-a)\end{bmatrix}\begin{bmatrix}\theta(\bar{a}-a) \\ \theta(a-\bar{a})\end{bmatrix}. \tag{4.64}$$

In the limit where $\bar{b} = \bar{a}$, this reduces to

$$\lim_{\bar{b}\to\bar{a}} P_A(a) = \frac{1}{\bar{a}^2}\left[a\ \theta(\bar{a}-a) + (2\bar{a}-a)\theta(2\bar{a}-a)\theta(a-\bar{a})\right], \tag{4.65}$$

which should look like a symmetric roof, with its top at $a = \bar{a}$. To compare the accuracy, some results are plotted in Figure 4.8.

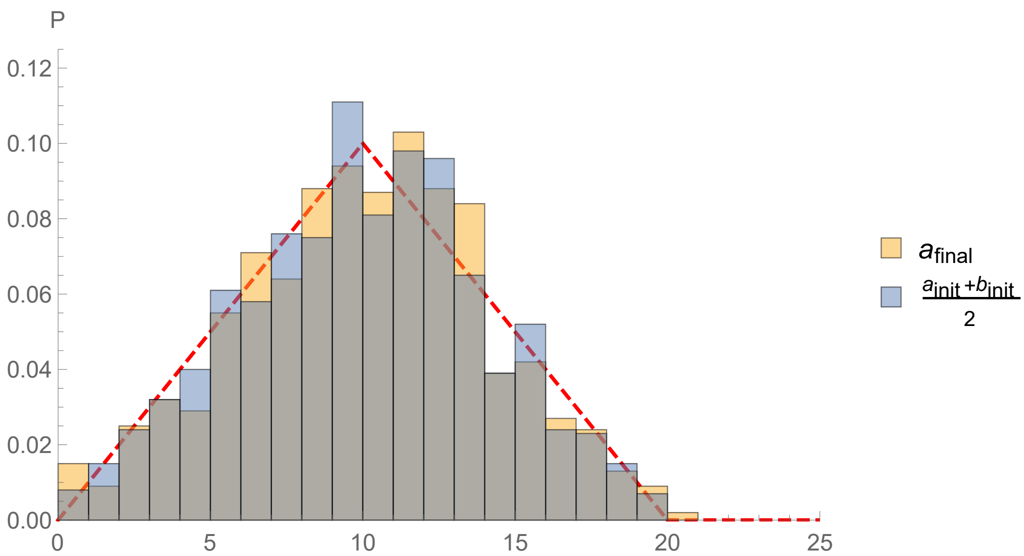



Figure 4.8: Comparison between (yellow) the final distribution of good A after 2000 time steps, with initial uniform distribution, and (blue) the initial distribution of the average amount of goods per agent. (red, dashed) The prediction based off the assumption that trades were exclusively one-for-one and the ideal ratio was one-to-one, Equation 4.65.

The calculation of the final distribution of good A for the last type of initial distribution is straightforward after our last one. Let us have a quick look at it.

$$\begin{aligned} P_C(c) &= \frac{1}{2\bar{a}}\int_0^c da\ \theta(a)\theta(2\bar{a}-a)\delta\left[c-a-2\bar{b}\left(1-\frac{a}{2\bar{a}}\right)\right] \\ &= \frac{1}{2\bar{a}}\int_0^c da\ \theta(2\bar{a}-a)\delta\left[\left(\frac{\bar{b}}{\bar{a}}-1\right)a+c-2\bar{b}\right] \\ &= \frac{1}{2\bar{a}|\frac{\bar{b}}{\bar{a}}-1|}\int_0^c da\ \theta(2\bar{a}-a)\delta\left(a-\frac{2\bar{b}-c}{\frac{\bar{b}}{\bar{a}}-1}\right). \end{aligned} \tag{4.66}$$

Now we need to make sure that the condition for $a$ in the delta function is inside the interval $[0, c]$, in any other case the outcome is zero. Also, we can absorb the $\bar{a}$ in the absolute sign, because it is strictly positive.

$$P_C(c) = \frac{1}{2|\bar{b}-\bar{a}|}\theta\left(2\bar{a} - \frac{2\bar{b}-c}{\frac{\bar{b}}{\bar{a}}-1}\right)\theta\left(\frac{2\bar{b}-c}{\frac{\bar{b}}{\bar{a}}-1}\right)\theta\left(c - \frac{2\bar{b}-c}{\frac{\bar{b}}{\bar{a}}-1}\right). \tag{4.67}$$

Then, we separate two cases; one in which $\bar{b} > \bar{a}$, and therefore $\bar{b}/\bar{a} - 1 > 0$, and one in which $\bar{b} < \bar{a}$, and therefore $\bar{b}/\bar{a} - 1 < 0$. Then we can divide the inequalities inside the Heaviside-step functions and make sure we get the right signs. We get

$$\begin{aligned} P_C(c) = \frac{1}{2|\bar{b}-\bar{a}|}\Big[&\theta(\bar{b}-\bar{a})\theta(c-2\bar{a})\theta(2\bar{b}-c)\theta(c-2\bar{a}) \\ &+\theta(\bar{a}-\bar{b})\theta(2\bar{a}-c)\theta(c-2\bar{b})\theta(2\bar{a}-c)\Big], \end{aligned} \tag{4.68}$$

which shows that the first and the third conditions effectively are the same. Therefore, one of them can be omitted.

$$\begin{aligned} P_C(c) = \frac{1}{2|\bar{b}-\bar{a}|}\Big[&\theta(\bar{b}-\bar{a})\theta(2\bar{b}-c)\theta(c-2\bar{a}) \\ &+\theta(\bar{a}-\bar{b})\theta(c-2\bar{b})\theta(2\bar{a}-c)\Big]. \end{aligned} \tag{4.69}$$

Then, the first condition of the two terms is already implied by the other two conditions, so we can leave those out to get the final expression,

$$P_C(c) = \frac{1}{2|\bar{b}-\bar{a}|}\left[\theta(2\bar{b}-c)\theta(c-2\bar{a}) + \theta(c-2\bar{b})\theta(2\bar{a}-c)\right], \tag{4.70}$$

which is basically just a uniform distribution on the interval $[2\bar{a}, 2\bar{b}]$ for $\bar{b} > \bar{a}$, and on the interval $[2\bar{b}, 2\bar{a}]$ for $\bar{b} < \bar{a}$. The corresponding distribution for A is

$$P_A(a) = \frac{1}{|\bar{b}-\bar{a}|}\left[\theta(\bar{b}-a)\theta(a-\bar{a}) + \theta(a-\bar{b})\theta(\bar{a}-a)\right]. \tag{4.71}$$

Obviously, as $\bar{b} \to \bar{a}$, this distribution approaches a delta-function.

$$\lim_{\bar{b}\to\bar{a}} P_A(a) = \delta\left(a - \bar{a}\right). \tag{4.72}$$

This is what we would expect as well, because every agent gets $c_{\text{init}} = 2\lambda\bar{a} + 2(1-\lambda)\bar{a} = 2\bar{a}$ goods, which when distributed to get the one-to-one ratio should mean it goes to an amount of $\bar{a}$ of each good.

For more general choices of $\alpha$ and $\beta$ we try to find the distributions in a similar fashion. To do this, we still need to assume that the market price $p$ is relatively constant. We can for example assume that $p \simeq p_{\text{MF}}$, which according to Figure 4.1, is not that bad of an assumption. Let us again assume that $P^{\text{init}}(a,b) = P_A^{\text{init}}(a)P_B^{\text{init}}(b)$ or $P^{\text{i}}(a,b) = P_A^{\text{i}}(a)P_B^{\text{i}}(b)$. In that sense we are looking for $P_A^{\text{final}}(a) = P_A^{\text{f}}(a)$ and $P_B^{\text{final}}(b) = P_B^{\text{f}}(b)$. Note that because of the constant price, we only see trades of $\Delta a \longleftrightarrow p$. Agents tend to the desired ratio $\gamma(p)$ and to do so they do $n$ net trades of gaining $\Delta a$ and losing $p$. We allow $n$ to be negative as well, meaning that the agent does $-n$ net trades of losing $\Delta a$ and gaining $p$. In mathematical terms,

$$\begin{aligned} a^{\text{f}} &= a^{\text{i}} + n\Delta a, \\ b^{\text{f}} &= b^{\text{i}} - np. \end{aligned} \tag{4.73}$$

We assumed that after the trades the ratio of $b/a = \gamma(p)$, therefore,

$$\begin{aligned} \frac{b^{\text{f}}}{a^{\text{f}}} &= \frac{b^{\text{i}} - np}{a^{\text{i}} + n\Delta a} = \gamma(p), \\ n(a^{\text{i}}, b^{\text{i}}) &= \frac{b^{\text{i}} - \gamma(p)a^{\text{i}}}{p + \gamma(p)\Delta a}. \end{aligned} \tag{4.74}$$

If an agent starts with an amount of A of $a^{\text{i}}$, how many of goods B $b^{\text{i}}$ would he/she need to end up both in a ratio $b^{\text{f}}/a^{\text{f}} = \gamma(p)$, and at some given final amount of good A, $a^{\text{f}}$? We can calculate this by filling Equation 4.74 in in Equation 4.73.

$$\begin{aligned} a^{\text{f}} &= a^{\text{i}} + \frac{b^{\text{i}} - \gamma(p)a^{\text{i}}}{p + \gamma(p)\Delta a}\Delta a, \\ b^{\text{i}} &= \frac{p}{\Delta a}(a^{\text{f}} - a^{\text{i}}) + a^{\text{f}}\gamma(p). \end{aligned} \tag{4.75}$$

We calculate the final probability distribution of good A by adding up the probabilities of the different possibilities of ending up with a certain amount of $a^{\mathrm{f}}$.

$$P_A^{\mathrm{f}}(a^{\mathrm{f}}) \propto \int_0^\infty da^{\mathrm{i}} P^{\mathrm{i}}\left[a^{\mathrm{i}}, \frac{p}{\Delta a}(a^{\mathrm{f}} - a^{\mathrm{i}}) + a^{\mathrm{f}}\gamma(p)\right]. \tag{4.76}$$

In a very similar way we can find the distribution of good B.

$$P_B^{\mathrm{f}}(b^{\mathrm{f}}) \propto \int_0^\infty db^{\mathrm{i}} P^{\mathrm{i}}\left[\frac{\Delta a}{p}(b^{\mathrm{f}} - b^{\mathrm{i}}) + \frac{b^{\mathrm{f}}}{\gamma(p)}, b^{\mathrm{i}}\right]. \tag{4.77}$$

The proportionality signs will be translated into constants through normalization afterwards. Let us look at the first example of initial distributions, $P^{\mathrm{i}}(a,b) = 1/(\bar{a}\bar{b})\exp(-a/\bar{a})\exp(-b/\bar{b})\theta(a)\theta(b)$; Boltzmann distributions. Note that the requirements that $a > 0$, and $b > 0$ are added. They will be useful in the calculation. We fill this into Equation 4.76:

$$\begin{aligned} P_A^{\mathrm{f}}(a^{\mathrm{f}}) &\propto \frac{1}{\bar{a}\bar{b}} \int_0^\infty da^{\mathrm{i}} e^{-a^{\mathrm{i}}/\bar{a}} \exp\left[-\frac{\frac{p}{\Delta a}(a^{\mathrm{f}} - a^{\mathrm{i}}) + a^{\mathrm{f}}\gamma(p)}{\bar{b}}\right] \theta\left[\frac{p}{\Delta a}(a^{\mathrm{f}} - a^{\mathrm{i}}) + a^{\mathrm{f}}\gamma(p)\right] \\ &= \frac{1}{\bar{a}\bar{b}} \int_0^\infty da^{\mathrm{i}} \exp\left[\left(\frac{p}{\bar{b}\Delta a} - \frac{1}{\bar{a}}\right) a^{\mathrm{i}}\right] \exp\left\{-\frac{\left[\frac{p}{\Delta a} + \gamma(p)\right] a^{\mathrm{f}}}{\bar{b}}\right\} \theta\left[\frac{p}{\Delta a}(a^{\mathrm{f}} - a^{\mathrm{i}}) + a^{\mathrm{f}}\gamma(p)\right]. \end{aligned} \tag{4.78}$$

As we have seen quite often now, we need to rewrite the condition in terms of $a^{\mathrm{i}}$, the integration variable, so we can adjust the boundaries.

$$\begin{aligned} \frac{p}{\Delta a}(a^{\mathrm{f}} - a^{\mathrm{i}}) + a^{\mathrm{f}}\gamma(p) &> 0 \\ a^{\mathrm{i}} &< a^{\mathrm{f}}\left[1 + \frac{\Delta a}{p}\gamma(p)\right]. \end{aligned} \tag{4.79}$$

We can substitute this expression in the upper boundary of the integration:

$$\begin{aligned} P_A^{\mathrm{f}}(a^{\mathrm{f}}) &\propto \frac{1}{\bar{a}\bar{b}} \exp\left\{-\frac{\left[\frac{p}{\Delta a} + \gamma(p)\right] a^{\mathrm{f}}}{\bar{b}}\right\} \int_0^{\left[1+\frac{\Delta a}{p}\gamma(p)\right]a^{\mathrm{f}}} da^{\mathrm{i}} \exp\left[\left(\frac{p}{\bar{b}\Delta a} - \frac{1}{\bar{a}}\right) a^{\mathrm{i}}\right] \\ &= \frac{1}{\bar{a}\bar{b}} \frac{1}{\frac{p}{\bar{b}\Delta a} - \frac{1}{\bar{a}}} \exp\left\{-\frac{\left[\frac{p}{\Delta a} + \gamma(p)\right] a^{\mathrm{f}}}{\bar{b}}\right\} \left(\exp\left\{\left[\frac{p}{\bar{b}\Delta a} - \frac{1}{\bar{a}}\right]\left[1 + \frac{\Delta a}{p}\gamma(p)\right] a^{\mathrm{f}}\right\} - 1\right) \\ &= \frac{1}{\frac{p}{\Delta a}\bar{a} - \bar{b}} \left(\exp\left\{-\frac{\left[1 + \frac{\Delta a}{p}\gamma(p)\right] a^{\mathrm{f}}}{\bar{a}}\right\} - \exp\left\{-\frac{\left[\frac{p}{\Delta a} + \gamma(p)\right] a^{\mathrm{f}}}{\bar{b}}\right\}\right). \end{aligned} \tag{4.80}$$

We proceed to normalize this properly, for that purpose we calculate

$$\begin{aligned} \int_0^\infty da P_A(a) &= \frac{1}{\frac{p}{\Delta a}\bar{a} - \bar{b}} \left(\frac{\bar{a}}{1 + \frac{\Delta a}{p}\gamma(p)} - \frac{\bar{b}}{\frac{p}{\Delta a} + \gamma(p)}\right) \\ &= \frac{1}{\frac{p}{\Delta a}\bar{a} - \bar{b}} \left(\frac{\frac{p}{\Delta a}\bar{a}}{\frac{p}{\Delta a} + \gamma(p)} - \frac{\bar{b}}{\frac{p}{\Delta a} + \gamma(p)}\right) \\ &= \frac{1}{\frac{p}{\Delta a} + \gamma(p)}, \end{aligned} \tag{4.81}$$

and divide the proportional expression by this factor:

$$P_A^{\mathrm{f}}(a) = \frac{p + \gamma(p)\Delta a}{p\bar{a} - \Delta a\bar{b}} \left(\exp\left\{-\frac{\left[1 + \frac{\Delta a}{p}\gamma(p)\right] a}{\bar{a}}\right\} - \exp\left\{-\frac{\left[\frac{p}{\Delta a} + \gamma(p)\right] a}{\bar{b}}\right\}\right). \tag{4.82}$$

To check if this aligns properly with Equation 4.56, we look at the limit of $\gamma(p) \to 1$, $\Delta a \to 1$, and $p \to 1$. It is easily observed that this gives the same expression.

We prepare an example of a system with $\bar{a} = 10$, $\bar{b} = 20$, $\alpha = 2$, $\beta = 1$, and $r = 0.5$. Therefore, $p_{\mathrm{MF}} = 1$, and $\gamma(p_{\mathrm{MF}}) = 2$. Furthermore, $\bar{a}/\alpha = 5 \neq \bar{b}/\beta = 20$, and therefore the mean-field price is not the equilibrium price. However, according to Figure 4.1 it is still a good approximation. Figure 4.9 shows both the actual distribution and the predicted distribution for this setup.

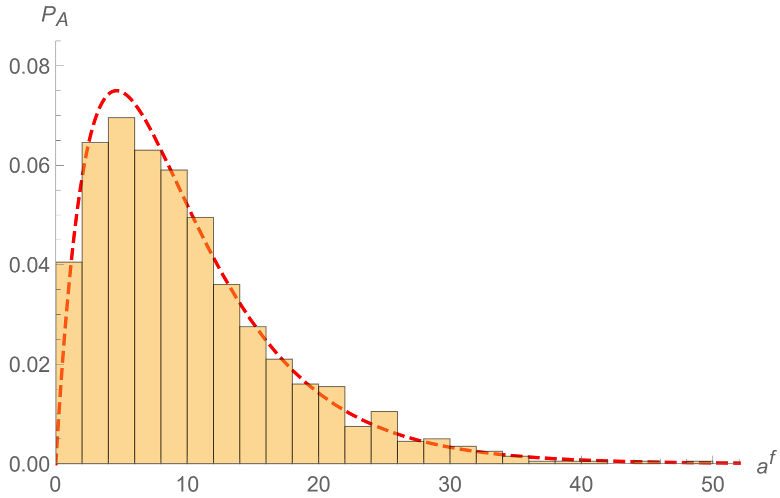



Figure 4.9: Distribution of the initially Boltzmann-distributed simulation with $\bar{a} = 10$, $\bar{b} = 20$, $\alpha = 2$, $\beta = 1$, and $r = 0.5$ compared with predicted distribution: $P_A^{\mathrm{f}}(a^{\mathrm{f}}) = \frac{3}{10}\left(e^{-3a^{\mathrm{f}}/20} - e^{-3a^{\mathrm{f}}/10}\right)$.

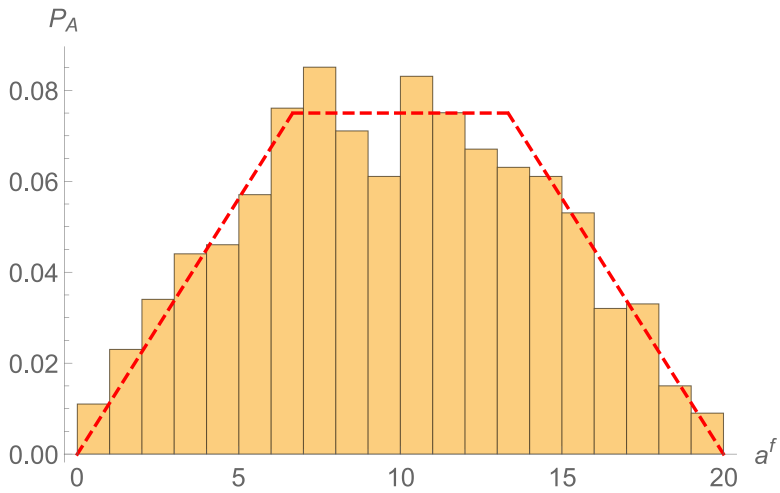



Figure 4.10: Distribution of the initially uniformly distributed simulation with $\bar{a} = 10$, $\bar{b} = 20$, $\alpha = 2$, $\beta = 1$, and $r = 0.5$ compared with predicted distribution.

Again, in a very similar way we can find a prediction for the initially uniformly distributed system. We will not show the derivation explicitly here, but it should be straightforward from the previous calculations.

$$P_A^{\mathrm{f}}(a) = \frac{\frac{p}{\Delta a} + \gamma(p)}{4\bar{a}\bar{b}} \begin{bmatrix} \theta(\frac{2\bar{a}}{1+\frac{\Delta a}{p}\gamma(p)} - a) & \theta(a - \frac{2\bar{a}}{1+\frac{\Delta a}{p}\gamma(p)}) \end{bmatrix} \begin{bmatrix} \left(1 + \frac{\Delta a}{p}\gamma(p)\right) a & 2\frac{\Delta a}{p}\bar{b} \\ 2\bar{a} & \theta(\frac{2(p\bar{a}+\Delta a\gamma(p))}{p+\Delta a\gamma(p)} - a)(2\bar{a} + 2\frac{\Delta a}{p}\bar{b} - \left(1 + \frac{\Delta a}{p}\gamma(p)\right) a) \end{bmatrix} \begin{bmatrix} \theta(\frac{2\Delta a\bar{b}}{p+\Delta a\gamma(p)} - a) \\ \theta(a - \frac{2\Delta a\bar{b}}{p+\Delta a\gamma(p)}) \end{bmatrix}. \tag{4.83}$$

For the same parameters as the previous check, the result is shown in Figure 4.10.

## 4.6 Conclusion

Utility provides a nice mathematical framework to implement demand and supply in the discussed model. Compared to the linear demand and supply model, this has less unknown factors, and the parameters we use are easier to interpret. Simulations of the model do not really show emergent inequality; rather, they show stable inequality. Final distributions are heavily dependent on initial distributions and the former can generally be calculated from the latter. final distributions tend to be more equal than the initial distributions, because agents want to possess their goods in a certain ratio. An agent can only end with very little or very much of a good if he was initially given very little or very much of the *other* good respectively. These odds are slimmer, because the initial distributions are generally not correlated. Because of the desire to possess the two goods in the constant ratio, the final distributions of the two goods are heavily correlated.

In the next section, we will take a step back from dynamics and look at the demand of two competing goods in an unequally distributed environment. Up until now, we have seen that the Boltzmann distribution fitted reality best, so we will impose such distributions on our system. Speaking of which, how well did the distributions in this section do? Figure 4.11 shows us that these distributions are more equal than the actual income distribution of the United States in 2016.

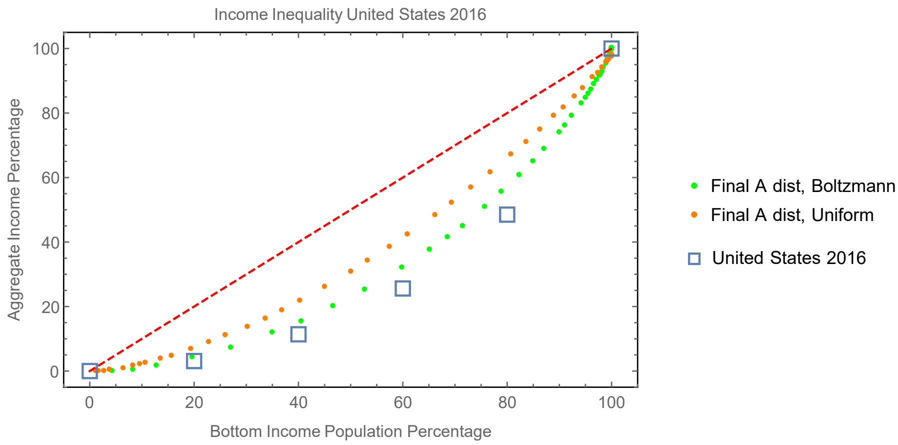



Figure 4.11: Final distributions of initially Boltzmann-distributed (Figure 4.5) and initially uniformly distributed (Figure 4.6) systems compared to actual income distribution of the United States in 2016 (from Figure 1.2).

# 5 Competing Goods

## 5.1 Marshallian Demand

Up until now we have looked at systems in which agents have two types of possessions: a certain good, and a certain currency to pay for the other good. However, economists often look at the utility of *consumption*, and as you cannot *consume* your money, these utility functions do not depend on the amount of money a person possesses. In general there is a difference between gaining utility from a good you *possess* and gaining utility from a good you *consume*. You can say that often this utility through possession is actually the same as consuming a part of this possession; e.g. by sitting on the couch you slowly wear it down. However, we would like to claim that this is not always the case, but more on that later.

For now let us focus on this utility through consumption. We will follow chapter 3 of the book of Mas-Colell [12]. Assume we have a certain budget, $m$, to spend on $L$ different goods. These goods all have their own prices, stored in the vector $p \in \mathrm{R}_+^L$. We buy $x_i$ goods of the $i^{\text{th}}$ type, which are all elements of the consumption vector $x \in \mathrm{R}_+^L$. The utility function $U(x), \mathrm{R}_+^L \to \mathrm{R}_+$ is in effect. The problem we face: How do we spend our budget to get the most utility from it? Or

$$\max_{x \geq 0}[U(x)], \text{ s.t. } p \cdot x \leq m. \tag{5.1}$$

Obviously this consumption vector $x(p, m)$ will depend on the price vector $p$ and the total budget. The resulting $x(p, m)$ is called the Marshallian (or Walrasian) *demand* function, because it is basically a set of $L$ demand curves.

We start by looking at an example in which $L = 2$, and we use the familiar CES utility function,

$$U(a, b) = \left[\left(\frac{a}{\alpha}\right)^r + \left(\frac{b}{\beta}\right)^r\right]^{1/r}, \tag{5.2}$$

where $a$, and $b$ are respectively the amount of goods of type A and type B we consume. Its partial derivatives are

$$\begin{aligned} \frac{\partial U}{\partial a} &= \frac{1}{\alpha}\left[1 + \left(\frac{\alpha b}{\beta a}\right)^r\right]^{(1-r)/r}, \\ \frac{\partial U}{\partial b} &= \frac{1}{\beta}\left[1 + \left(\frac{\beta a}{\alpha b}\right)^r\right]^{(1-r)/r}. \end{aligned} \tag{5.3}$$

These derivatives are strictly positive, meaning that the utility function never descends or becomes flat. Because of this it is straightforward to show that it is always best to spend the entirety of our budget.

We agree to spend $m_A = p_A a$, on goods of type A, where $p_A$ is the price of good A. We spend $m_B = m - m_A = p_B b$ on goods of type B, with $p_B$ the price of good B. The utility function in terms of these quantities becomes

$$U\left(\frac{m_A}{p_A}, \frac{m-m_A}{p_B}\right) = \left[\left(\frac{m_A}{\alpha p_A}\right)^r + \left(\frac{m-m_A}{\beta p_B}\right)\right]^{1/r}. \tag{5.4}$$

To maximize this we set the derivative of $U$ to $m_A$ to zero.

$$\begin{aligned}\frac{\partial U}{\partial m_A} &= \frac{1}{r}\left[\frac{r}{\alpha p_A}\left(\frac{m_A}{\alpha p_A}\right)^{r-1} - \frac{r}{\beta p_B}\left(\frac{m-m_A}{\beta p_B}\right)^{r-1}\right]\left[\left(\frac{m_A}{\alpha p_A}\right)^r + \left(\frac{m-m_A}{\beta p_B}\right)\right]^{(1-r)/r} \\ &= \left[\left(\frac{1}{\alpha p_A}\right)^r m_A^{r-1} - \left(\frac{1}{\beta p_B}\right)^r (m-m_A)^{r-1}\right]\left[\left(\frac{m_A}{\alpha p_A}\right)^r + \left(\frac{m-m_A}{\beta p_B}\right)\right]^{(1-r)/r} \\ &= 0.\end{aligned} \tag{5.5}$$

Because the second part of the product cannot become zero for $m > 0$, this means that the first part has to be zero at a maximum or minimum.

$$\begin{aligned} m_A^{r-1} - \left(\frac{\alpha p_A}{\beta p_B}\right)^r (m-m_A)^{r-1} &= 0, \\ m_A &= \left(\frac{\alpha p_A}{\beta p_B}\right)^{r/(r-1)} (m-m_A), \\ m_A(p_A, p_B, m) &= \frac{m}{\left(\frac{\beta p_B}{\alpha p_A}\right)^{r/(r-1)} + 1}. \end{aligned} \tag{5.6}$$

We have to inspect whether this is a maximum ($\partial^2 U/\partial m_A^2 < 0$) or a minimum ($\partial^2 U/\partial m_A^2 > 0$).

$$\begin{aligned}\frac{\partial^2 U}{\partial m_A^2} &= \left[\left(\frac{1}{\alpha p_A}\right)^r (r-1) m_A^{r-2} + \left(\frac{1}{\beta p_B}\right)^r (r-1)(m-m_A)^{r-2}\right]\left[\left(\frac{m_A}{\alpha p_A}\right)^r + \left(\frac{m-m_A}{\beta p_B}\right)\right]^{1/r-1} \\ &+ r\left[\left(\frac{1}{\alpha p_A}\right)^r m_A^{r-1} - \left(\frac{1}{\beta p_B}\right)^r (m-m_A)^{r-1}\right]^2 \left(\frac{1}{r} - 1\right)\left[\left(\frac{m_A}{\alpha p_A}\right)^r + \left(\frac{m-m_A}{\beta p_B}\right)^r\right]^{1/r-2} \\ &= (r-1)\Bigg\{\left[\left(\frac{1}{\alpha p_A}\right)^r m_A^{r-2} + \left(\frac{1}{\beta p_B}\right)^r (m-m_A)^{r-2}\right]\left[\left(\frac{m_A}{\alpha p_A}\right)^r + \left(\frac{m-m_A}{\beta p_B}\right)\right]^{1/r-1} \\ &+ \left[\left(\frac{1}{\alpha p_A}\right)^r m_A^{r-1} - \left(\frac{1}{\beta p_B}\right)^r (m-m_A)^{r-1}\right]^2 \left[\left(\frac{m_A}{\alpha p_A}\right)^r + \left(\frac{m-m_A}{\beta p_B}\right)^r\right]^{1/r-2}\Bigg\}.\end{aligned} \tag{5.7}$$

A close look reveals that for $0 < m_A < m$ everything in the $\{\}$'s has to be positive. Therefore, for $r < 1$ and $m_A$ anywhere in the interval $[0, m]$ (which it should be), we know that $\partial^2 U/\partial m_A^2 < 0$. This makes sure that we are dealing with a maximum here, which makes the Marshallian demand function

$$a(p_A, p_B, m) = \frac{m_A(p_A, p_B, m)}{p_A} = \frac{1}{p_A}\frac{m}{\left(\frac{\beta p_B}{\alpha p_A}\right)^{r/(r-1)} + 1}. \tag{5.8}$$

In the same way, and also found by switching $\alpha$'s for $\beta$'s and $A$'s for $B$'s, we find the Marshallian demand function for good B.

$$b(p_A, p_B, m) = \frac{1}{p_B}\frac{m}{\left(\frac{\alpha p_A}{\beta p_B}\right)^{r/(r-1)} + 1}. \tag{5.9}$$

These are the demand functions if we look at the consumption of good A and good B. For some more clarity the distribution of the budget is plotted in Figure 5.1. An example of two goods would be Apples and Bread; assume we only consume apples and bread (how ever stupid this may sound). Let us say an apple costs $0.50\$ = p_A$, and a loaf of bread costs $1.00\$ = p_B$. Also, we assume that, ideally, we eat one apple and a third of a loaf of bread every day, and in that ratio they are equally important. This means that one loaf of bread is worth three times the worth of one apple to us. Therefore, $\alpha/\beta = 3$, while $p_B/p_A = 2 < \alpha/\beta$. This means that we will spend more money on bread, than we will spend on apples, because we get more utility per money unit from bread. Even though we would ideally eat them at a ratio of three apples to one loaf of bread, we will eat slightly more bread and less apples.

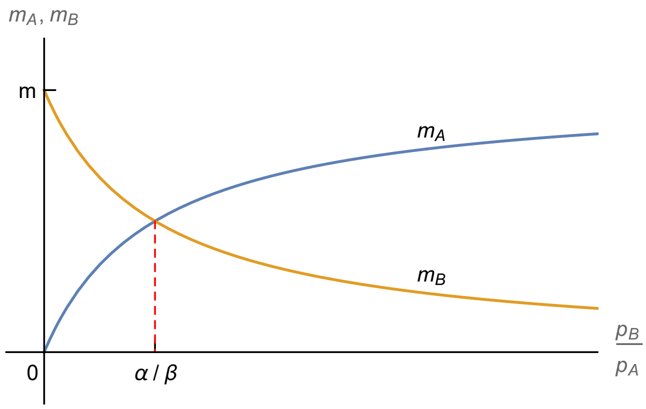



Figure 5.1: Display of the distribution of the budget as a function of the price ratio $p_B/p_A$, with $r = 0.5$.

## 5.2 Utility of Possession

Some goods you consume, and some goods you just collect. Think about an art collector who collects both Rembrandts and Van Goghs. He tries to maximize his utility when he has an amount of money to spend, however he may base that off of how many he already has. For example, if he has ten Rembrandts and only two Van Goghs, he may be inclined to buy more Van Goghs, at least, if he does not have a huge preference for Rembrandts.

Let us say we again have two goods of type A and B. We have a budget of $m = m_A + m_B$, which we spend on an amount of goods of type A, $\Delta a = m_A/p_A$, and of type B, $\Delta b = (m - m_A)/p_B$. However, now we assume we already have in our possession an amount of goods of type A, $a$, and of type B, $b$. We can then proceed to do the same type of calculation of the Marshallian demand as we did in the previous section. We start by maximizing the utility.

$$U\left(a + \frac{m_A}{p_A}, b + \frac{m - m_A}{p_B}\right) = \left[\left(\frac{a}{\alpha} + \frac{m_A}{\alpha p_A}\right)^r + \left(\frac{b}{\beta} + \frac{m - m_A}{\beta p_B}\right)^r\right]^{1/r}. \tag{5.10}$$

We calculate the derivative of $U$ to $m_A$ again to find

$$\begin{aligned}
\frac{\partial U}{\partial m_A} = &\left[\frac{1}{\alpha p_A}\left(\frac{p_A a + m_A}{\alpha p_A}\right)^{r-1} - \frac{1}{\beta p_B}\left(\frac{p_B b + m - m_A}{\beta p_B}\right)^{r-1}\right] \\
&\times \left[\left(\frac{a}{\alpha} + \frac{m_A}{\alpha p_A}\right)^r + \left(\frac{b}{\beta} + \frac{m - m_A}{\beta p_B}\right)^r\right]^{1/r-1} = 0, \\
\frac{\beta p_B}{\alpha p_A}\left(\frac{p_A a + m_A}{\alpha p_A}\right)^{r-1} &= \left(\frac{p_B b + m - m_A}{\beta p_B}\right)^{r-1}, \\
\left(\frac{\beta p_B}{\alpha p_A}\right)^{r/(r-1)}(p_A a + m_A) &= p_B b + m - m_A, \\
m_A(a, b, p_A, p_B, m) &= \frac{p_B + m - p_A a\psi}{1 + \psi},
\end{aligned} \tag{5.11}$$

with

$$\psi = \left(\frac{\beta p_B}{\alpha p_A}\right)^{r/(r-1)}. \tag{5.12}$$

There is still one problem left with this expression for $m_A$: it can be less than zero and it can exceed $m$. We cannot allow ourselves to spend more than $m$ on one good. This is why we still have to cut off this function to make sure $0 \leq m_A \leq m$:

$$m_A(a, b, p_A, p_B, m) = \min\left[\max\left(\frac{p_B + m - p_A a\psi}{1 + \psi}, 0\right), m\right]. \tag{5.13}$$

Similarly, we find

$$m_B(a, b, p_A, p_B, m) = \min\left[\max\left(\frac{p_A + m - p_B b\psi^{-1}}{1 + \psi^{-1}}, 0\right), m\right]. \tag{5.14}$$

These functions are plotted for several values of $a$ and $b$ in Figure 5.2. The true meaning of the results require some thinking. Also, it is important to note that the sole $p_B/p_A$ dependence we saw in Equation 5.6, has gone, though the $\beta/\alpha$ dependence has remained.

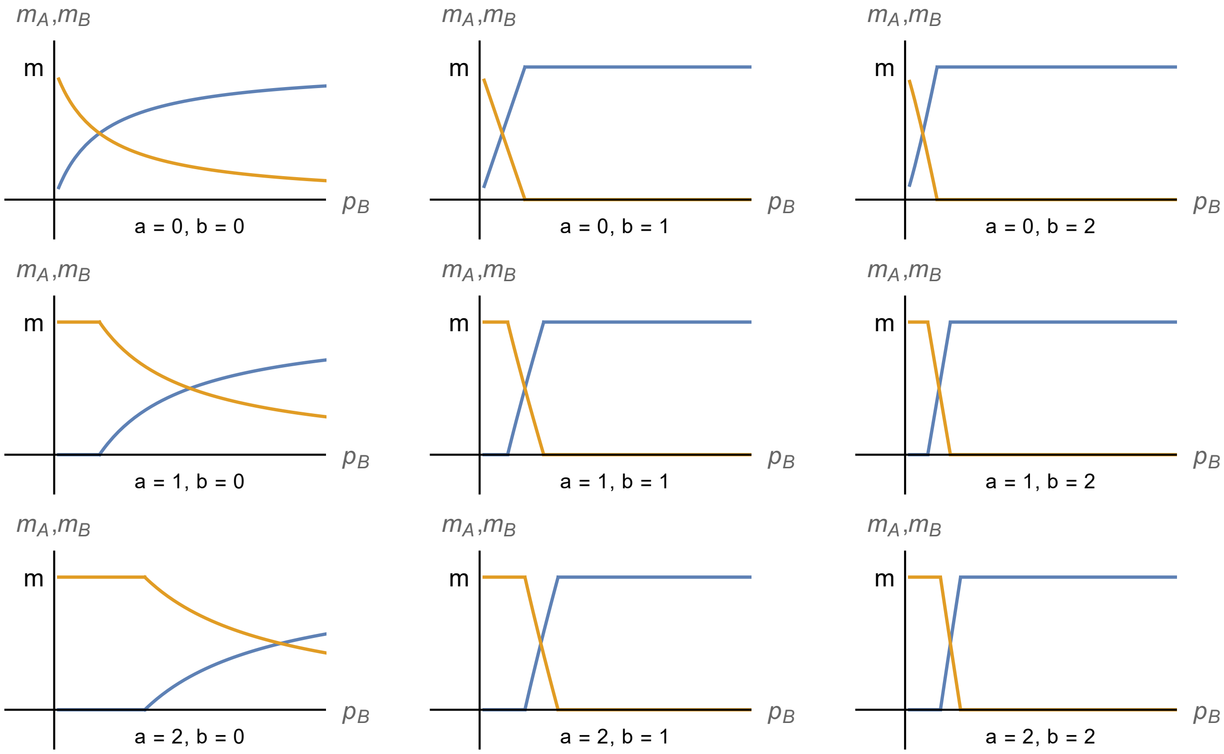



Figure 5.2: Several examples of the Marshallian demand functions of (blue) good A and (yellow) good B as a function of the price of good B, $p_B$, while $p_A = 1$ is kept constant. For $a = 0$, $b = 0$, we recover the functions we saw in the previous section. In the first column we see the most fluent graphs, because it takes a while for $p_B$ to become large enough to compete with the total lack of goods of type B, that really urges the agent to buy more type B goods. The last column shows that at a nearly zero price of good B, it is obviously better to buy good B, but it is quickly overturned to a preference for good A, especially when the agent lacks this good type.

The utility function divides the desire of the agents into two points:

- If prices are some what equal, agents desire to possess the two goods in a certain ratio, because every extra good of the same type involves a larger penalty.
- If an agent can get a good price for either one of the goods, though, he allows this ratio to shift to favor this good.

When an agent possesses few goods, the first point dominates. Purchases of goods have very big impacts on the ratio $b/a$ and at low values of $a$ or $b$ the difference in penalty is still very big. To clarify this, think about the law of diminishing marginal utility: the utility as a function of $a$ or $b$ has to be a decreasingly ascending curve. Therefore, in this limit, differences in prices lead to small differences in demand.

On the other hand, when an agent possesses more goods, the second point comes into play more. The agent's possessions are in a region of the utility where there is less extra penalty for possessing more of a type of a good. If prices are equal, he still prefers this fixed ratio. However, when this agent gets a good price, it is easier for to overcome this extra penalty. The demand, therefore, gets more sensitive to this price.

This ratio can simply be calculated by dividing $c = a + b$ goods over type A and type B and seeing what the best possible distribution is for the utility. We have the following utility function.

$$U(a, c-a) = \left[\left(\frac{a}{\alpha}\right)^r + \left(\frac{c-a}{\beta}\right)^r\right]^{1/r}. \tag{5.15}$$

We derive this utility function to $a$ and set it to zero,

$$\frac{\partial U}{\partial a} = \frac{1}{r}\left[\frac{r}{\alpha}\left(\frac{a}{\alpha}\right)^{r-1} - \frac{r}{\beta}\left(\frac{c-a}{\beta}\right)^{r-1}\right]\left[\left(\frac{a}{\alpha}\right)^r + \left(\frac{c-a}{\beta}\right)^r\right]^{1/r-1} = 0. \tag{5.16}$$

We then work this out to get the following ratio.

$$\begin{aligned}\frac{\beta}{\alpha}\left(\frac{a}{\alpha}\right)^{r-1} &= \left(\frac{c-a}{\beta}\right)^{r-1}, \\ \left(\frac{\beta}{\alpha}\right)^{r/(r-1)} a &= c-a, \\ a &= \frac{c}{\left(\frac{\beta}{\alpha}\right)^{r/(r-1)}+1},\end{aligned} \tag{5.17}$$

or,

$$\frac{b}{a} = \frac{c-a}{a} = \left(\frac{\beta}{\alpha}\right)^{r/(r-1)}. \tag{5.18}$$

This is the ratio that the possessions of agents will tend to, if the prices for the goods are equal. However, as the price of good A becomes larger or smaller than the price of good B, this ratio shifts accordingly. We can estimate this adapted ratio by dividing Equation 5.9 over Equation 5.8, which results in

$$\frac{b}{a} = \frac{p_A}{p_B}\frac{\psi+1}{\psi^{-1}+1} = \frac{p_A}{p_B}\frac{\psi(\psi+1)}{\psi+1} = \frac{p_A}{p_B}\psi. \tag{5.19}$$

This is what we also saw when we were using the fundamental principle of marginal utility theory back at Equation 4.23.

With the findings of Equation 5.13 and 5.14, we can now calculate a global demand of good A or good B, when we assume that we know the distribution of good A and good B in the system. We are interested in this weighted demand,

$$\langle m_A \rangle = N\int_0^\infty da \int_0^\infty db P(a,b) \min\left[\max\left(\frac{p_B b + m - p_A a\psi}{\psi+1}, 0\right), m\right]. \tag{5.20}$$

As we have seen throughout this thesis, the Boltzmann distribution very well reflects reality. Furthermore, it makes sure we can perform this integral, because there is only linear dependence on $a$ and $b$ in the demand function.

$$P(a,b) = \frac{1}{\bar{a}}e^{-a/\bar{a}}\frac{1}{\bar{b}}e^{-b/\bar{b}}. \tag{5.21}$$

Then, we will divide the integral into two parts. We go from zero to $\mu$, from $\mu$ to $\eta$, and from $\eta$ to $\infty$, which are denoted in Figure 5.3.

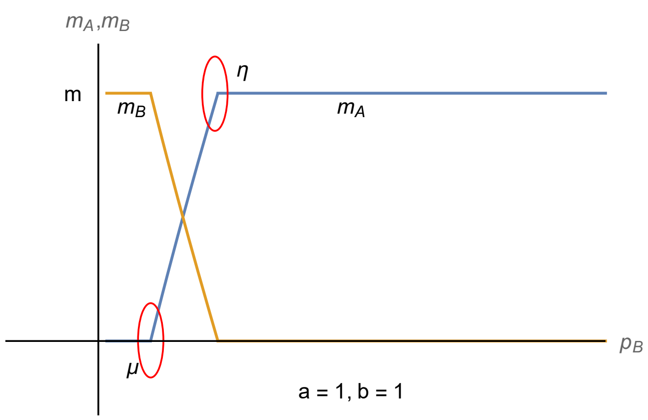



Figure 5.3: Plot of $m_A$ and $m_B$ as a function of $p_B$, with $p_A = 1$, $r = 0.5$, and $\beta/\alpha = 1$. Indication of integration boundaries $\mu$ and $\eta$ are shown.

To find this lower boundary we solve

$$\begin{aligned} m_A(a,b) &> 0, \\ p_B b + m &> p_A a\psi, \\ b &> \frac{p_A}{p_B}a\psi - \frac{m}{p_B} = \mu(a). \end{aligned} \tag{5.22}$$

And to find the upper boundary we solve

$$\begin{aligned} m_A(a,b) &< m, \\ p_B b + m - p_A a\psi &< m(1+\psi), \\ b &< \frac{\psi}{p_B}(m + p_A a) = \eta(a). \end{aligned} \tag{5.23}$$

Now, if $\mu(a) > 0$, the first part of the integration does not contribute to the weighted demand of good A. If $\mu(a) < 0$, we need to make sure that we do not start integrating at $b = \mu(a)$, but at $b = 0$. Incorporating this into the integral looks as follows.

$$\langle m_A \rangle = N \int_0^\infty da \int_{\max[\mu(a),0]}^{\eta(a)} dbP(a,b)m_A(a,b) + N \int_0^\infty da \int_{\eta(a)}^\infty dbP(a,b)m. \tag{5.24}$$

The first integral can then be separated into two integrals depending on whether $\mu(a)$ is larger than zero.

$$\begin{aligned} \langle m_A \rangle = N \int_0^\infty da \Bigg[ \theta[\mu(a)] \int_{\mu(a)}^{\eta(a)} dbP(a,b)m_A(a,b) \\ + \theta[-\mu(a)] \int_0^{\eta(a)} dbP(a,b)m_A(a,b) + \int_{\eta(a)}^\infty P(a,b)m \Bigg]. \end{aligned} \tag{5.25}$$

Let us proceed by filling in our Boltzmann distribution.

$$\begin{aligned} = \frac{N}{\bar{a}\bar{b}} \int_0^\infty dae^{-a/\bar{a}} \Bigg[ \frac{\theta[\mu(a)]}{\psi+1} \int_{\mu(a)}^{\eta(a)} dbe^{-b/\bar{b}}(p_B b + m - p_A a\psi) \\ + \frac{\theta[-\mu(a)]}{\psi+1} \int_0^{\eta(a)} dbe^{-b/\bar{b}}(p_B b + m - p_A a\psi) + m \int_{\eta(a)}^\infty dbe^{-b/\bar{b}} \Bigg] \end{aligned} \tag{5.26}$$

All of the integrals are of one of the following forms.

$$\begin{aligned} \int dx\ e^{-x/\bar{x}} &= -\bar{x}e^{-x/\bar{x}}, \\ \int dx\ xe^{-x/\bar{x}} &= -\bar{x}e^{-x/\bar{x}}(\bar{x} + x). \end{aligned} \tag{5.27}$$

Now all integrals over $b$ can be performed.

$$\begin{aligned} = \frac{N}{\bar{a}\bar{b}} \int_0^\infty dae^{-a/\bar{a}} \Bigg[ \frac{\theta[\mu(a)]}{\psi+1} \left( p_B \left[ -\bar{b}e^{-b/\bar{b}}(\bar{b}+b) \right]_{\mu(a)}^{\eta(a)} + (m - p_A a\psi) \left[ -\bar{b}e^{-b/\bar{b}} \right]_{\mu(a)}^{\eta(a)} \right) \\ + \frac{\theta[-\mu(a)]}{\psi+1} \left( p_B \left[ -\bar{b}e^{-b/\bar{b}}(\bar{b}+b) \right]_0^{\eta(a)} + (m - p_A a\psi) \left[ -\bar{b}e^{-b/\bar{b}} \right]_0^{\eta(a)} \right) \\ + m \left[ -\bar{b}e^{-b/\bar{b}} \right]_{\mu(a)}^{\eta(a)} \Bigg]. \end{aligned} \tag{5.28}$$

Working this out gives us

$$\begin{aligned} = \frac{N}{\bar{a}} \int_0^\infty da\ e^{-a/\bar{a}} \Bigg( \frac{1}{\psi+1} \Bigg\{ \theta[\mu(a)]e^{-\mu(a)/\bar{b}} \left(p_B + p_B\mu(a) + m - p_A a\psi\right) \\ + \theta[-\mu(a)](p_B\bar{b} + m - p_A a\psi) \\ - e^{-\eta(a)/\bar{b}}(p_B\bar{b} + p_B\eta(a) + m - p_A a\psi) \Bigg\} \\ + me^{-\eta(a)/\bar{b}} \Bigg). \end{aligned} \tag{5.29}$$

The restriction $\theta[\mu(a)]$ entails $\mu(a) > 0$, which sets the condition

$$a > \frac{m}{p_A\psi}. \tag{5.30}$$

The restriction $\theta[-\mu(a)]$ sets the condition

$$a < \frac{m}{p_A\psi}. \tag{5.31}$$

We then split the equation into four final integrals which we will perform separately

$$\begin{aligned}\langle m_A\rangle = &\underbrace{\frac{N}{\bar a}\frac{1}{\psi+1}\int_{m/(p_A\psi)}^{\infty} da\ e^{-\mu(a)/\bar b - a/\bar a}[p_B\bar b + m + p_B\mu(a) - p_A a\psi]}_{1}\\ &\underbrace{+\frac{N}{\bar a}\frac{1}{\psi+1}\int_0^{m/(p_A\psi)} da\ e^{-a/\bar a}(p_B\bar b + m - p_A a\psi)}_{2}\\ &\underbrace{-\frac{N}{\bar a}\frac{1}{\psi+1}\int_0^{\infty} da\ e^{-\eta(a)/\bar b - a/\bar a}[p_B\bar b + m + p_B\eta(a) - p_A a\psi]}_{3}\\ &\underbrace{+\frac{Nm}{\bar a}\int_0^{\infty} da\ e^{-\eta(a)/\bar b - a/\bar a}}_{4}.\end{aligned} \tag{5.32}$$

In the following enumeration we will explain the process that leads us to a final expression.

1. Looking at Equation 5.22 for $\mu(a)$, we are lucky to recognize that $m + p_B\mu(a) - p_A a\psi = 0$. This leaves us with just the exponential integral with the factor $p_B\bar b$.

$$\begin{aligned}&= \frac{N}{\bar a}\frac{1}{\psi+1}e^{m/(p_B\bar b)}p_B\bar b\left[-\frac{1}{\frac{p_A\psi}{p_B\bar b}+\frac{1}{\bar a}}\exp\left\{-\left(\frac{p_A\psi}{p_B\bar b}+\frac{1}{\bar a}\right)a\right\}\right]_{m/(p_A\psi)}^{\infty}\\ &= \frac{Np_B\bar b}{(\psi+1)\left(1+\frac{p_A\bar a}{p_B\bar b}\psi\right)}\exp\left(-\frac{m}{p_A\bar a\psi}\right).\end{aligned} \tag{5.33}$$

2. We have less luck in this term. We integrate the full expression.

$$\begin{aligned}&= \frac{N}{\bar a}\frac{1}{\psi+1}\left((p_B\bar b + m)\left[-\bar a e^{-a/\bar a}\right]_0^{m/(p_A\psi)} - p_A\psi\left[-\bar a e^{-a/\bar a}(\bar a + a)\right]_0^{m/(p_A\psi)}\right)\\ &= \frac{N}{\psi+1}\left[p_B\bar b + m - p_A\bar a\psi + \exp\left(-\frac{m}{p_A\bar a\psi}\right)(p_A\bar a\psi - p_B\bar b)\right].\end{aligned} \tag{5.34}$$

3. In this term we recognize that $p_B\eta(a) - p_A a\psi = m\psi$, which rids us from the linear $a$ dependence.

$$\begin{aligned}&= -\frac{N}{\bar a}\frac{1}{\psi+1}\exp\left(-\frac{m\psi}{p_B\bar b}\right)[p_B\bar b + (1+\psi)m]\left[-\frac{1}{\frac{p_A\psi}{p_B\bar b}+\frac{1}{\bar a}}\exp\left\{-\left(\frac{p_A\psi}{p_B\bar b}+\frac{1}{\bar a}\right)a\right\}\right]_0^{\infty}\\ &= -\frac{N}{1+\frac{p_A\bar a}{p_B\bar b}\psi}\left(\frac{p_B\bar b}{\psi+1}+m\right)\exp\left(-\frac{m\psi}{p_B\bar b}\right).\end{aligned} \tag{5.35}$$

4. The fourth and final term is straightforward.

$$\begin{aligned}&= \frac{Nm}{\bar a}\exp\left(-\frac{m\psi}{p_B\bar b}\right)\left[-\frac{1}{\frac{p_A\psi}{p_B\bar b}+\frac{1}{\bar a}}\exp\left\{-\left(\frac{p_A\psi}{p_B\bar b}+\frac{1}{\bar a}\right)a\right\}\right]_0^{\infty}\\ &= \frac{Nm}{1+\frac{p_A\bar a}{p_B\bar b}\psi}\exp\left(-\frac{m\psi}{p_B\bar b}\right).\end{aligned} \tag{5.36}$$

Now, finally we can add up these equations and simplify it a bit, we get

$$\frac{\langle m_A\rangle}{N} = \frac{p_B\bar b + m - p_A\bar a\psi}{\psi+1} + \frac{1}{(\psi+1)\left(1+\frac{p_A\bar a}{p_B\bar b}\psi\right)}\left[\frac{p_A^2\bar a^2}{p_B\bar b}\psi^2\exp\left(-\frac{m}{p_A\bar a\psi}\right) - p_B\bar b\exp\left(-\frac{m\psi}{p_B\bar b}\right)\right]. \tag{5.37}$$

In Figure 5.4 we can see the result of this demand curve weighted according to a Boltzmann distribution compared to a result involving a perfectly equal distribution. The results show a remarkable difference other than there intersection at $p_B = p_A$. This can be explained mathematically, but the symmetric argument is more beautiful. Because $\alpha = \beta$ and $\bar a = \bar b$, and the Boltzmann distribution is symmetric, the entire setup is perfectly symmetric in the type of good. Therefore, if $p_A = p_B$, we can only expect the global demand to have no preference for good A nor good B. This results in a perfect division of the budget, spending $m/2$ at both good A and good B.

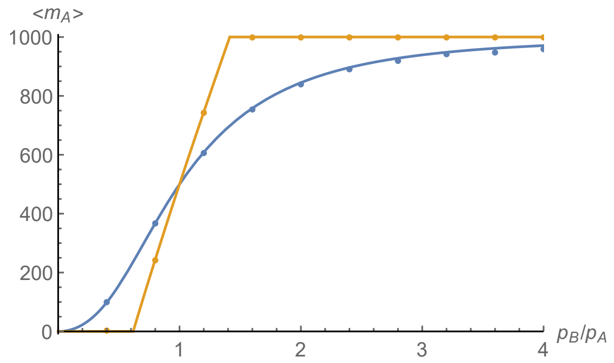



Figure 5.4: (blue) weighted demand of good A competing against good B as a function of the price of good B $p_B$, while $p_A = 1$ is kept constant, $\alpha = \beta = 1$, $\bar{a} = \bar{b} = 1$, $N = 1000$, $m = 1$. The demand is weighted over a Boltzmann distribution. (yellow) Demand curve for perfectly equal distribution (Figure 5.2 middle plot). The plotted points are the confirming result of a numerical approach of the same calculation. Note that, as $p_A = 1$, the demand of good A $d_A = \langle m_A \rangle / p_A = \langle m_A \rangle$ is the same curve.

# 6 In Conclusion

## 6.1 Conclusion and Discussion

We started this thesis by showing the impressive work of Yakovenko. He claims that by entropy maximization lower income distributions are exponential, and we saw how well this matches with actual data. These unequal distributions arise solely based on statistical fluctuations and the way these fluctuations maximize the entropy, which is closely related to the amount of possibilities that income can be distributed among these economic agents. Based on this, our thesis has served two purposes. Firstly, we have looked at adaptations of the Yakovenko model to see how well or how badly their stable distributions resemble actual data. By looking at these other distributions, we saw how uniquely these Boltzmann distributions fit the data. Then, secondly, we examined the impact of inequality (of our newly found distributions, but especially of the Boltzmann distributions as well) on demand and supply.

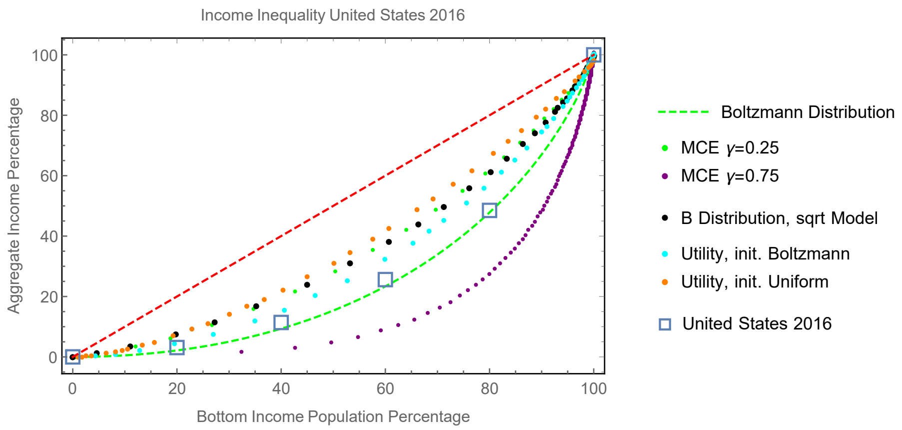



Figure 6.1: Overview of all income distributions we found throughout this thesis.

Most dynamics actually resulted in more equal income distributions than the Boltzmann distribution, and we saw that these curves did not match the actual income distribution of the United States. Basically, this means that the complexity of a real economy cannot be captured by a model with just one type of trade. It seems that the great amounts of different trades and transactions make the real economy of the United States so complex that the particular details of trades even out and it *can* be described by the simple Yakovenko model. An overview of what we found is shown in Figure 6.1. This figure really shows how well this Boltzmann distribution does and it is remarkable that such a complex system like an economy 'knows' about entropy. Furthermore, in all cases we saw emergent, or at least stable inequality.

We managed to generate demand and supply curves based on income distributions and utility theory. Because of this dependence on the income distribution, one can easily see what impact inequality has on these curves. Unequal distributions generally tend to make these curves smoother than the rugged equality-based curves. Moreover, opposed to an equal distribution, unequal distributions can inhabit agents that are 'poor'. In some cases these agents struggle to participate to the dynamics of the model, therefore affecting the total amounts of goods that are active, which can lead to differences in equilibrium prices.

Throughout this thesis we have seen several models with many parameters, initial values, and other to-be-made choices that influence the outcome of said model. Obviously, one can never explore all possibilities of combinations of these inputs. Especially, because there are often simulation times involved. We have intended to show interesting results, while still capturing the general behavior of the model.

We recognize that in doing the research we have grown and that - in that sense - some earlier results seem slightly naive. However, the thesis may well show how a physicist slowly learns about economics and embraces its well-known basic principles.

## 6.2 Outlook

To conclude, allow us to look at an interesting side track we came across when we worked on the model discussed in Section 4. In this section we will use this same model, with the same parameters, but the macroscopic dynamics mostly interest us.

Let us imagine two systems with both their own independent parameters, $\bar{a}$, $\bar{b}$, $\alpha$, $\beta$. Every time step we let the agents trade within their own systems in the exact same way as we did in the previous section, but now for two systems simultaneously. Furthermore, every time step we allow a fraction $s$ of the agents to go to an 'international' market, where an equilibrium price is established for the group of agents, completely disregarding what system they belong to.

What will happen? Let us call the local equilibrium prices $p_1$ and $p_2$ and we will see what happens on the international market,

- if $p_1 > p_2$: The group of agents that want to buy at any given price will - on average - consist of more agents from system 1 than agents from system 2. For selling, it obviously goes the opposite way around. This means that - on average - after every international trade, system 1 will decrease its ratio $b/a$, and system 2 will increase its ratio $b/a$. As roughly $p \sim (b/a)^{1-r}$, and $1 - r > 0$, this results in a decreasing price in system 1 and an increasing price in system 2.

- if $p_1 < p_2$: Exactly the opposite of the previous statement will hold. Therefore, this leads to an increasing price in system 1 and a decreasing price in system 2.

- if $p_1 = p_2$: The systems are in equilibrium. Agents of both systems are equally likely to be buyer or seller at any given price. Therefore, the resulting price difference should - on average - be non-existent.

As physicists this should remind us of a thermodynamic situation called *mechanical equilibrium*, a state at which the pressure of two interacting gasses or liquids cancel each other out, which makes sure that the fluids do not expand or contract.

This interesting phenomenon makes us believe that maybe the two interacting systems behave as thermodynamic fluids. We impose the following analogies.

| Economics | Thermodynamics |
|---|---|
| $N$ (agents) | $N$ (particles) |
| $N\bar{a}$ (products) | $V$ ($m^3$) |
| $\bar{b}$ (\$) | $k_BT$ ($J$) |
| $p/\Delta a$ (\$/product) | $p$ ($N/m^2$) |

To check this theory, we are going to look at a gas that undergoes an *adiabatic expansion*, which is an expansion during which no heat flows in or out ($dQ = 0$). An example of a setup is given in Figure 6.2. As system 2 slowly cools down, its pressure, $p_2$, will slowly decrease, causing system 1 to expand, i.e. $V_1$ grows. Also, the system stays in mechanical equilibrium, $p_1 = p_2$, therefore $p_1$ decreases, while system 1 also cools down.

The curve that is made in the $p, V$-plane is distinctive for adiabatic processes and it can be calculated quite easily if we take these gases to be ideal.

$$\begin{aligned} dQ &= dU + dW = 0 \\ 3Nk_BdT &+ pdV = 0. \end{aligned} \tag{6.1}$$

Now, we make use of the ideal gas law,

$$\frac{Nk_BT}{V} = p, \tag{6.2}$$

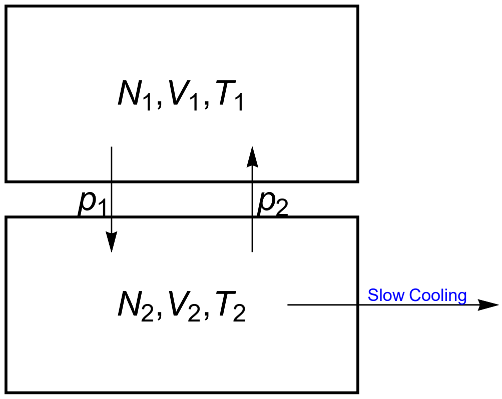



Figure 6.2: Setup in which system 1 will undergo adiabatic expansion.

and write the infinitesimal $dT$ in terms of $dp$ and $dV$.

$$\begin{aligned} 3pdV + 3Vdp + pdV &= 0 \\ \frac{4}{V}dV &= -\frac{3}{p}dp \\ 4\log\frac{V}{V_0} &= -3\log\frac{p}{p_0} \\ \left(\frac{V}{V_0}\right)^4 &= \left(\frac{p}{p_0}\right)^{-3} \\ p(V) = p_0 &\left(\frac{V}{V_0}\right)^{-4/3}. \end{aligned} \tag{6.3}$$

Here $p_0$ and $V_0$ are the pressure and the volume respectively at a moment in time in which they are known, like for example $t = 0$. The conserved quantity in this system during the process is therefore

$$pV^{4/3} = p_0 V_0^{4/3}. \tag{6.4}$$

In a similar way we can attempt to do this for our economic setup. We do want to stress that we do not believe the microscopic mechanics of the thermodynamic and the economic setup are equal. However, if we want to predict the global behavior of the economic setup, we could at least try the same approach we would take in the thermodynamic case. To start, we need a new 'ideal gas law', an equation that links the quantities $N\bar{a}$, $\bar{b}$, and $p$ to one another. One that fits this description is Equation 4.28, the critical $b/a$ ratio,

$$\frac{\bar{b}}{\bar{a}} = \gamma(p) = \left(\frac{\beta}{\alpha}\right)^{r/(r-1)} \left(\frac{\Delta a}{p}\right)^{1/(r-1)} \rightarrow \epsilon p^{1/(1-r)}. \tag{6.5}$$

In an attempt to make the calculation as orderly as possible, we take $\epsilon = (\beta/\alpha)^{r/(r-1)}$, and we translate $p/\Delta a \rightarrow p$. We see immediately, that with $\bar{b} \longleftrightarrow k_B T$ and $\bar{a} \longleftrightarrow V/N$, and, $\epsilon = 1$ and $r = 0$, this exactly recovers the ideal gas law. So for $r = 0$ the calculation is as straightforward as the previous calculation.

$$\begin{aligned} dQ = dU + dW &= 0 \\ Nd\bar{b} + Npd\bar{a} &= 0. \end{aligned} \tag{6.6}$$

We only have a factor of 1 instead of 3 in the $dU$-term, because the economic setup has only one degree of freedom, where the thermodynamic setup has three.

$$\begin{aligned} N\frac{d\bar{b}}{d\bar{a}}d\bar{a} + N\frac{d\bar{b}}{dp}dp + Npd\bar{a} &= 0 \\ N\epsilon pd\bar{a} + N\epsilon\bar{a}dp + Npd\bar{a} &= 0 \\ \frac{1+\epsilon}{\bar{a}}d\bar{a} &= -\frac{\epsilon}{p}dp \\ (1+\epsilon)\log\frac{\bar{a}}{\bar{a}_0} &= -\epsilon\log\frac{p}{p_0} \\ p(\bar{a}) = p_0 &\left(\frac{\bar{a}}{\bar{a}_0}\right)^{-(1+\epsilon)/\epsilon}. \end{aligned} \tag{6.7}$$

The conserved quantity is

$$p\bar{a}^{(1+\epsilon)/\epsilon} = p_0\bar{a}_0^{(1+\epsilon)/\epsilon}. \tag{6.8}$$

Then, obviously, there is the case in which $r \neq 0$, in which we have to work harder to get the adiabatic curve, because the $p$-integration will be significantly more difficult. We start from the same equation,

$$\begin{aligned} N\frac{d\bar{b}}{d\bar{a}}d\bar{a} + N\frac{d\bar{b}}{dp}dp + Npd\bar{a} &= 0 \\ N\epsilon p^{1/(1-r)}d\bar{a} + \frac{\epsilon}{1-r}N\bar{a}p^{r/(1-r)}dp + Npd\bar{a} &= 0 \\ \left(\epsilon p^{1/(1-r)} + p\right) d\bar{a} &= \frac{\epsilon}{r-1}\bar{a}p^{r/(1-r)}dp \\ \frac{1}{\bar{a}}d\bar{a} &= \frac{\epsilon}{r-1}\frac{p^{r/(1-r)}}{\epsilon p^{1/(1-r)} + p}dp \\ \frac{1}{\bar{a}}d\bar{a} &= \frac{1}{r-1}\frac{1}{p + \frac{1}{\epsilon}p^{(1-2r)/(1-r)}}dp. \end{aligned} \tag{6.9}$$

While, the left-hand-side of the equation is still a very easy integration, the right-hand-side has become quite a handful. So let us take a closer look there.

$$\int dp\frac{1}{r-1}\frac{1}{p+\frac{1}{\epsilon}p^{(1-2r)/(1-r)}} = \frac{1}{r-1}\int dp\frac{1}{p}\frac{1}{1+\frac{1}{\epsilon}p^{r/(r-1)}}. \tag{6.10}$$

The integral is not straightforward so we choose a new integration variable $u$, that simplifies this expression. Let us choose

$$u = 1 + \frac{1}{\epsilon}p^{r/(r-1)}, \tag{6.11}$$

and therefore

$$p = [\epsilon(u-1)]^{(r-1)/r}, \tag{6.12}$$

and

$$\frac{dp}{du} = \epsilon^{(r-1)/r}\frac{r-1}{r}(u-1)^{-1/r}. \tag{6.13}$$

Let us substitute these equations in Equation 6.10.

$$\begin{aligned} &\frac{1}{r-1}\int dp\frac{1}{p}\frac{1}{1+\frac{1}{\epsilon}p^{r/(r-1)}} \\ =&\frac{1}{r-1}\int du\frac{dp}{du}[\epsilon(u-1)]^{(1-r)/r}u^{-1} \\ =&\frac{1}{r-1}\int du\ \epsilon^{(r-1)/r}\frac{r-1}{r}(u-1)^{-1/r}\epsilon^{(1-r)/r}(u-1)^{(1-r)/r}u^{-1} \\ =&\frac{1}{r}\int du(u-1)^{-1/r+(1-r)/r}u^{-1} \\ =&\frac{1}{r}\int du\frac{1}{u(u-1)}. \end{aligned} \tag{6.14}$$

This seems like a big improvement in the level of difficulty of the integration. Especially once we realize that

$$\frac{1}{r}\int du\frac{1}{u(u-1)} = \frac{1}{r}\int du\left(\frac{1}{u-1} - \frac{1}{u}\right) = \frac{1}{r}\log\frac{u-1}{u}. \tag{6.15}$$

We then proceed to substitute the $u$s back to the original functions of $p$, to get the final indefinite integral

$$\int dp\frac{1}{r-1}\frac{1}{p+\frac{1}{\epsilon}p^{(1-2r)/(1-r)}} = \frac{1}{r}\log\frac{\frac{1}{\epsilon}p^{r/(r-1)}}{1+\frac{1}{\epsilon}p^{r/(r-1)}} = -\frac{1}{r}\log\left(\epsilon p^{r/(1-r)} + 1\right). \tag{6.16}$$

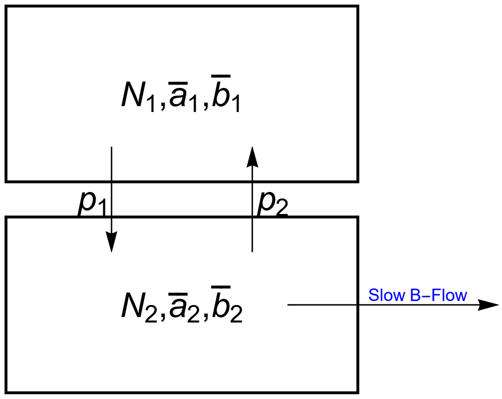



Figure 6.3: Setup in which system 1 will undergo an economic adiabatic expansion.

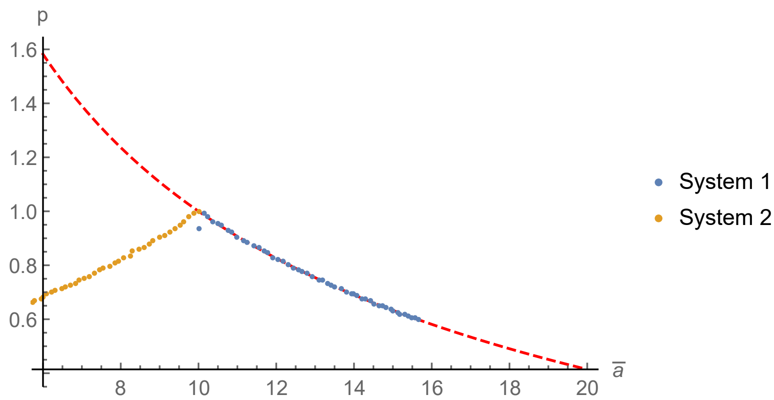



Figure 6.4: The prices, $p$, of both system 1 and 2 plotted against their average amounts of goods $\bar{a}$ for initial system setup with $N_1 = N_2 = 1000$, $\bar{a}_1 = \bar{a}_2 = 10$, $\bar{b}_1 = \bar{b}_2 = 10$, $s = 0.1$, tax $= 0.01$, $\alpha = \beta = 1$, and $r = 0.5$. The simulation is run for 5000 time steps. The predicted curve, Equation 6.17, is shown (red, dashed).

Finally, we go back to the original equation that relates $p$ to $\bar{a}$. We left it at

$$\begin{aligned}
\frac{1}{\bar{a}} d\bar{a} &= \frac{1}{r-1} \frac{1}{p + \frac{1}{\epsilon} p^{(1-2r)/(1-r)}} dp \\
\int_{\bar{a}_0}^{\bar{a}} d\bar{a}' \frac{1}{\bar{a}'} &= \int_{p_0}^{p} dp' \frac{1}{r-1} \frac{1}{p' + \frac{1}{\epsilon} p'^{(1-2r)/(1-r)}} \\
\log \frac{\bar{a}}{\bar{a}_0} &= -\frac{1}{r} \log \frac{\epsilon p^{r/(1-r)} + 1}{\epsilon p_0^{r/(1-r)} + 1} \\
\left(\frac{\bar{a}_0}{\bar{a}}\right)^r &= \frac{\epsilon p^{r/(1-r)} + 1}{\epsilon p_0^{r/(1-r)} + 1} \\
p(\bar{a}) &= \left\{ \frac{1}{\epsilon} \left[ \left(\frac{\bar{a}_0}{\bar{a}}\right)^r \left(\epsilon p_0^{r/(1-r)} + 1\right) - 1 \right] \right\}^{(1-r)/r}.
\end{aligned} \tag{6.17}$$

The conserved quantity in this case is

$$\left(\epsilon p^{r/(1-r)} + 1\right) \bar{a}^r = \left(\epsilon p_0^{r/(1-r)} + 1\right) \bar{a}_0^r. \tag{6.18}$$

To check if the economic setup really obeys this adiabatic curve, we setup a simulation as shown in Figure 6.3. In system 2 the 'government' has put a tax on all local transactions, and as this government does not return this to the system, good B slowly gets drained.

The systems have identical parameters before the simulation starts. Let us take $N_1 = N_2 = 1000$, $\bar{a}_1 = \bar{a}_2 = 10$, and $\bar{b}_1 = \bar{b}_2 = 10$. We Boltzmannize the systems in the familiar Yakovenko procedure before the start of the simulation. The government in system 2 takes 1 percent tax from every local transaction. Every time step 10 percent of the agents go to trade at the international market. The utility functions in both systems are equal, with $\beta = \alpha = 1$, therefore $\epsilon = 1$, and $r = 0.5$. The simulation is run for 5000 time steps. The curve both systems make in the $p, \bar{a}$-plane is shown in Figure 6.4, and it seems as though Equation 6.17 makes a very good prediction.

On microscopic level our economic model does not remotely resemble the microscopic dynamics of a physical system. So why does it act like one on a macroscopic scale? While the two types of dynamics do not resemble each other on microscopic scale, they *do* resemble each other on macroscopic scale. It seems that all that is needed for this adiabatic expansion, is two quantities ($V$ and $T$, or $\bar{a}$ and $\bar{b}$) that are linked by an equation of state (ideal gas law, or $\bar{b}/\bar{a} = \gamma(p)$) involving a third quantity (pressure, or price). The first two quantities are exchanged while the equation of state is kept satisfied. This is what makes these systems so similar; not their microscopic behavior. Furthermore, both the thermodynamic and the economic setup have a good (but different) reason to keep their respective pressures and prices at the same level, which is also a macroscopic similarity.

We have obviously not explored the full potential of these similarities, and are curious what it may bring. Adiabatic expansions and contractions are essential parts of the so-called *Carnot cycle*. Together with *isothermal* expansions and contractions they provide a way to turn heat into work in the most efficient way. An interesting question is if this would also be possible in economics. And what economic concepts would be converted into one another? These questions we will leave to future explorers to answer.

# 7 Appendix

The simulations that were run in Section 3.2.3, involving the system in which the price was distributed, all showed instability. In this appendix we will attempt to show why this is the case. However, calculations will get sufficiently difficult that it is only a few interesting comments we will make.

For some conditions the average price dropped to zero almost straight away, for some it took longer, but eventually the would all end in a 'crisis', where almost every agent did not have any money or products anymore. In order to analyze this we set up a Fokker-Planck equation just like we did in Section 2.2. It gets somewhat more complicated, though. This is the equation we want to write out in terms of the rates:

$$\begin{aligned}\frac{\partial P(m,V,p)}{\partial t} = \int_0^\infty dm' \int_0^\infty dV' \int_0^\infty dp' \int_{-\infty}^\infty d\Delta_m \int_{-\infty}^\infty d\Delta_p \int_{-\infty}^\infty d\Delta_{p'} \int_0^\infty dp_{\text{sell}} \Big[ \\ - f_{[m,V,p;m',V',p'] \to [m+\Delta_m, V-\frac{\Delta_m}{p_{sell}}, p+\Delta_p; m'-\Delta_m, V+\frac{\Delta_m}{p_{sell}}, p'+\Delta_{p'}]} \\ P(m,V,p)P(m',V',p') \\ + f_{[m+\Delta_m, V-\frac{\Delta_m}{p_{sell}}, p+\Delta_p; m'-\Delta_m, V+\frac{\Delta_m}{p_{sell}}, p'+\Delta_{p'}] \to [m,V,p;m',V',p']} \\ P\left(m+\Delta_m, V-\frac{\Delta_m}{p_{\text{sell}}}, p+\Delta_p\right) P\left(m'-\Delta_m, V+\frac{\Delta_m}{p_{sell}}, p'+\Delta_{p'}\right)\Big].\end{aligned} \tag{7.1}$$

In writing this out, there is a distinction between transaction amount, $\Delta_m$, being positive or negative. The sign of $\Delta_m$ determines to which side the transaction goes, and therefore which price is to be used, and which side should update their price.

$$\begin{aligned}\frac{\partial P(m,V,p)}{\partial t} = \int_0^\infty dm' \int_0^\infty dV' \int_0^\infty dp' \int_{-\infty}^\infty d\Delta_m \int_{-\infty}^\infty d\Delta_p \int_{-\infty}^\infty d\Delta_{p'} \int_0^\infty dp_{\text{sell}} \Big[ \\ -\frac{1}{2}\Big(\theta(\Delta_m)\theta(m'-\Delta_m)\theta(V-\frac{\Delta_m}{p_{\text{sell}}})\delta(\Delta_m - p_{\text{sell}} q_{\text{dem}}(p_{\text{sell}}, m')) \\ \delta\Big(\Delta_p - \Delta p\big(2\theta(q_{\text{dem}}(p_{\text{sell}}, m') - q_{\text{sup}}(p_{\text{sell}})) - 1\big)\Big)\delta(\Delta_{p'})\delta(p_{\text{sell}} - p) \\ +\theta(-\Delta_m)\theta(m+\Delta_m)\theta(V'+\frac{\Delta_m}{p_{\text{sell}}})\delta(\Delta_m + p_{\text{sell}} q_{\text{dem}}(p_{\text{sell}}, m)) \\ \delta(\Delta_p)\delta\Big(\Delta_{p'} - \Delta p\big(2\theta(q_{\text{dem}}(p_{\text{sell}}, m) - q_{\text{sup}}(p_{\text{sell}})) - 1\big)\Big)\delta(p_{\text{sell}} - p)\Big) \\ P(m,V,p)P(m',V',p') \\ +\frac{1}{2}\Big(\theta(\Delta_m)\theta(m'-\Delta_m)\theta(V-\frac{\Delta_m}{p_{\text{sell}}})\delta(\Delta_m - p_{\text{sell}} q_{\text{dem}}(p_{\text{sell}}, m+\Delta_m)) \\ \delta(\Delta_p)\delta\Big(\Delta_{p'} - \Delta p\big(2\theta(q_{\text{dem}}(p_{\text{sell}}, m+\Delta_m) - q_{\text{sup}}(p_{\text{sell}})) - 1\big)\Big)\delta(p_{\text{sell}} - p') \\ +\theta(-\Delta_m)\theta(m+\Delta_m)\theta(V'+\frac{\Delta_m}{p_{\text{sell}}})\delta(\Delta_m + p_{\text{sell}} q_{\text{dem}}(p_{\text{sell}}, m'-\Delta_m)) \\ \delta\Big(\Delta_p - \Delta p\big(2\theta(q_{\text{dem}}(p_{\text{sell}}, m'-\Delta_m) - q_{\text{sup}}(p_{\text{sell}})) - 1\big)\Big)\delta(\Delta_{p'})\delta(p_{\text{sell}} - p)\Big) \\ P\left(m+\Delta_m, V-\frac{\Delta_m}{p_{\text{sell}}}, p+\Delta_p\right) P\left(m'-\Delta_m, V'+\frac{\Delta_m}{p_{\text{sell}}}, p'+\Delta_{p'}\right)\Big].\end{aligned} \tag{7.2}$$

If we look at the new model, in which the price is the same for everyone, we get a much simpler Fokker-Planck equation. Let us start with the rates again.

$$\begin{aligned}\frac{\partial P(m,V)}{\partial t} = \int_0^\infty dm' \int_0^\infty dV' \int_{-\infty}^\infty d\Delta \Big[ \\ - f_{[m,V;m',V'] \to [m+\Delta, V-\frac{\Delta}{p}; m'-\Delta, V'+\frac{\Delta}{p}]} P(m,V)P(m',V') \\ + f_{[m+\Delta, V-\frac{\Delta}{p}; m'-\Delta, V'+\frac{\Delta}{p}] \to [m,V;m',V']} P\left(m+\Delta, V-\frac{\Delta}{p}\right) P\left(m'-\Delta, V'+\frac{\Delta}{p}\right)\Big].\end{aligned} \tag{7.3}$$

Here, we have assumed that the price has found a constant value, i.e. $p(t) = p$. It is interesting to see what happens if we allow the price to fluctuate, but for now we want to see if we can find the equilibrium distribution function $P(m,V)$, and we assume the price to be constant in equilibrium. For now we accept that we will not find an equilibrium price this way, at best we find $P(m,V)$ in a steady state in terms of $p$. We write out the rates and get

$$
\begin{aligned}
\frac{\partial P(m,V)}{\partial t} = & \int_0^\infty dm' \int_0^\infty dV' \int_{-\infty}^\infty d\Delta \Bigg[ \\
& -\frac{1}{2}\Big(\theta(\Delta)\theta(m'-\Delta)\theta(V-\frac{\Delta}{p})\delta(\Delta - pq_{\text{dem}}(p,m')) \\
& \quad +\theta(-\Delta)\theta(m+\Delta)\theta(V'+\frac{\Delta}{p})\delta(\Delta + pq_{\text{dem}}(p,m))\Big) \\
& \quad P(m,V)P(m',V') \\
& +\frac{1}{2}\Big(\theta(\Delta)\theta(m'-\Delta)\theta(V-\frac{\Delta}{p})\delta(\Delta - pq_{\text{dem}}(p,m+\Delta)) \\
& \quad +\theta(-\Delta)\theta(m+\Delta)\theta(V'+\frac{\Delta}{p})\delta(\Delta + pq_{\text{dem}}(p,m'-\Delta))\Big) \\
& \quad P\left(m+\Delta, V-\frac{\Delta}{p}\right) P\left(m'-\Delta, V'+\frac{\Delta}{p}\right)\Bigg].
\end{aligned} \tag{7.4}
$$

The system obeys detailed balance if a transaction is equally likely to occur in both ways. If this is the case, the condition, $f_{[m,V;m',V']\to[m+\Delta,V-\frac{\Delta}{p};m'-\Delta,V'+\frac{\Delta}{p}]} = f_{[m+\Delta,V-\frac{\Delta}{p};m'-\Delta,V'+\frac{\Delta}{p}]\to[m,V;m',V']}$, should be satisfied. If it does not obey detailed balance it can still be stable, though. We can see that we only have detailed balance if $m' = m+\Delta$. This means there is no detailed balance. However, we can explain the specific condition for it in the following way.

Let us take a look at a transaction where agent $i$ buys products from agent $j$. Before the transaction we have

$$
\begin{aligned}
m_i &= m_{i,0}, \\
m_j &= m_{j,0}.
\end{aligned} \tag{7.5}
$$

Now, after the transaction agent $i$ has paid a certain fixed amount of money to agent $j$:

$$
\begin{aligned}
m_i &\to m_{i,0} - pq_{\text{dem}}(p,m_{i,0}), \\
m_j &\to m_{j,0} + pq_{\text{dem}}(p,m_{i,0}).
\end{aligned} \tag{7.6}
$$

To see if there would be detailed balance, we have to look if it is equally likely to go back to the initial state. The odds of the simulation randomly choosing agent $j$ to buy products back from agent $i$ are equal, so let us have a look what happens. We assume the price has not changed after the transaction.

$$
\begin{aligned}
m_i &\to m_{i,0} - pq_{\text{dem}}(p,m_{i,0}) + pq_{\text{dem}}(p,m_{j,0} + pq_{\text{dem}}(p,m_{i,0})), \\
m_j &\to m_{j,0} + pq_{\text{dem}}(p,m_{i,0}) - pq_{\text{dem}}(p,m_{j,0} + pq_{\text{dem}}(p,m_{i,0})).
\end{aligned} \tag{7.7}
$$

If there had been detailed balance, $m_i$ and $m_j$ at the end of the two transactions should equal their initial values, $m_{i,0}$ and $m_{j,0}$. This is true for

$$pq_{\text{dem}}(p,m_{i,0}) = pq_{\text{dem}}(p,m_{j,0} + pq_{\text{dem}}(p,m_{i,0})), \tag{7.8}$$

which is only leads to

$$m_{i,0} = m_{j,0} + pq_{\text{dem}}(p,m_{i,0}) = m_{j,0} + \Delta. \tag{7.9}$$

This means that in order for the pair of agents to make the exact same transaction back and forth their first transaction has to make the two end up with exactly eachother's initial amount of money. This makes sure that the second transaction has the exact equal demand and therefore is the exact same amount.

This has not even taken into account the change in the price that would happen after the first transaction, which would make the condition harder to calculate.

We now proceed to attempt to solve the simplified model with fixed price, i.e. Equation 7.4, in the simplest form of the demand curve: $q_{\text{dem}}(p,m) = -\alpha p + am$, for $m > \frac{\alpha p}{a}$, and $q_{\text{dem}}(p,m) = 0$, otherwise. We start by working out the delta-functions. We introduce the following four variables corresponding to the roots of the functions inside the delta-functions, to help us write out the calculation as compact as possible.

$$
\begin{aligned}
\Delta_1(m') &= pq_{\text{dem}}(p,m'), \\
\Delta_2(m) &= -pq_{\text{dem}}(p,m), \\
\Delta_3 = pq_{\text{dem}}(p,m+\Delta_3) &\to \Delta_3(m) = -\frac{\Delta_2(m)}{1-ap}, \\
\Delta_4 = -pq_{\text{dem}}(p,m'-\Delta_4) &\to \Delta_4(m') = -\frac{\Delta_1(m')}{1-ap}.
\end{aligned} \tag{7.10}
$$

The integral over $\Delta$ in Equation 7.4 can then be performed to obtain

$$\begin{aligned}\frac{\partial P(m,V)}{\partial t} = &\int_0^\infty dm' \int_0^\infty dV' \Bigg[ \\ &-\frac{1}{2}\Big(\theta(\Delta_1(m'))\theta(m'-\Delta_1(m'))\theta(V-\frac{\Delta_1(m')}{p}) \\ &\quad+\theta(-\Delta_2(m))\theta(m+\Delta_2(m))\theta(V'+\frac{\Delta_2(m)}{p})\Big)P(m,V)P(m',V') \\ &+\frac{1}{2|1-ap|}\theta(\Delta_3(m))\theta(m'-\Delta_3(m))\theta(V-\frac{\Delta_3(m)}{p}) \\ &\quad P(m+\Delta_3(m),V-\frac{\Delta_3(m)}{p})P(m'-\Delta_3(m),V'+\frac{\Delta_3(m)}{p}) \\ &+\frac{1}{2|1-ap|}\theta(-\Delta_4(m'))\theta(m+\Delta_4(m'))\theta(V'+\frac{\Delta_4(m')}{p}) \\ &\quad P(m+\Delta_4(m'),V-\frac{\Delta_4(m')}{p})P(m'-\Delta_4(m'),V'+\frac{\Delta_4(m')}{p})\Bigg].\end{aligned} \tag{7.11}$$

We make the assumption that $0 < ap < 1$ (we also get to choose these constants), and therefore $1 - ap > 0$. From this we can see that the first theta-function in every term, i.e. $\theta(\Delta_1(m'))$, $\theta(-\Delta_2(m))$, $\theta(\Delta_3(m))$, and $\theta(-\Delta_4(m'))$, are automatically satisfied, because $\Delta_1(m')$ and $\Delta_3(m)$ can only be positive, and $\Delta_2(m)$ and $\Delta_4(m')$ can only be negative. Without this assumption thee third and fourth term would vanish instantly, forcing the other two terms to equal zero too (because of the conservation of money and products). This would make it impossible to trade. We now proceed to look at all theta-functions independently to see what conditions they lay on the variables $m$, $V$, $m'$, and $V'$. Let us first look at the first two terms.

1. - $\theta(m' - \Delta_1(m'))$ gives the condition $m' > \Delta_1(m') = \max(0, -\alpha p^2 + apm')$. We know that $m' > 0$, $\alpha > 0$, and $0 < ap < 1$. From this we can see that this condition will always hold.
   - $\theta(V - \frac{\Delta_1(m')}{p})$ gives the condition $V > \frac{\Delta_1(m')}{p} = \max(0, -\alpha p + am')$. We prefer conditions for the primed variables, because we can write them in the boundaries of the integral. This gives the following condition for $m'$:

$$m' < \frac{V}{a} + \frac{\alpha p}{a}. \tag{7.12}$$

2. - $\theta(m + \Delta_2(m))$ gives the condition $m > -\Delta_2(m) = \max(0, -\alpha p^2 + apm)$. In the same way as the first term of the previous term, we can see that this always holds.
   - $\theta(V' + \frac{\Delta_2(m)}{p})$ gives the condition $V' > -\frac{\Delta_2(m)}{p} = \max(0, -\alpha p + am)$. We know that $V' > 0$, so we can simply just say

$$V' > -\frac{\Delta_2(m)}{p} \geq -\alpha p + am. \tag{7.13}$$

For the third and fourth term we can actually incorporate the theta-functions into the distribution functions by making use of the same trick as we earlier used in the MCE model. That way, we do not have to change the integration boundaries. We assume that

$$P(m,V) = P_m(m)P_V(V), \tag{7.14}$$

and

$$\begin{aligned} P_m(m) &= 0, \quad \text{for } m < 0, \\ P_V(V) &= 0, \quad \text{for } V < 0. \end{aligned} \tag{7.15}$$

We will now remove as many theta-functions as possible, by rewriting the conditions into the integration boundaries and incorporating the theta-functions of the third and fourth term.

$$\begin{aligned}\frac{\partial P(m,V)}{\partial t} = &-\frac{1}{2}\int_0^{\frac{V+\alpha p}{a}} dm' \int_0^\infty dV' P(m,V)P_m(m')P_V(V') \\ &-\frac{1}{2}\int_0^\infty dm' \int_{-\frac{\Delta_2(m)}{p}}^\infty dV' P(m,V)P_m(m')P_V(V') \\ &+\frac{1}{2-2ap}\int_0^\infty dm' \int_0^\infty dV' P(m+\Delta_3(m),V-\frac{\Delta_3(m)}{p})P_m(m'-\Delta_3(m))P_V(V'+\frac{\Delta_3(m)}{p}) \\ &+\frac{1}{2-2ap}\int_0^\infty dm' \int_0^\infty dV' P(m+\Delta_4(m'),V-\frac{\Delta_4(m')}{p})P_m(m'-\Delta_4(m'))P_V(V'+\frac{\Delta_4(m')}{p}).\end{aligned} \tag{7.16}$$

After we have done this step, we can see that in every term we can perform one integral: the $V'$ integral in the first and the fourth term, and the $m'$ integral in the second and the third term. We know these, because we know

$$\int_0^\infty dm P_m(m) = \int_0^\infty dV P_V(V) = 1. \tag{7.17}$$

We find

$$\begin{aligned}\frac{\partial P(m,V)}{\partial t} &= -\frac{1}{2}P(m,V)\Big[\int_0^{\frac{V+\alpha p}{a}} dm' P_m(m') + \int_{-\frac{\Delta_2(m)}{p}}^\infty dV' P_V(V')\Big] \\ &+\frac{1}{2-2ap}\Big[\int_0^\infty dV' P(m+\Delta_3(m), V-\frac{\Delta_3(m)}{p})P_V(V'+\frac{\Delta_3(m)}{p}) \\ &\quad+\int_0^\infty dm' P(m+\Delta_4(m'), V-\frac{\Delta_4(m')}{p})P_m(m'-\Delta_4(m'))\Big].\end{aligned} \tag{7.18}$$

Now, let us write all integrals in the same form; with boundaries from zero to infinity. That means we have to do a coordinate transformation in the first two terms, and we have to rewrite the first term such that we can adjust the lower boundary. We get

$$\begin{aligned}\frac{\partial P(m,V)}{\partial t} &= -\frac{1}{2}P(m,V)\Big[1-\int_0^\infty dm' P_m(m'+\frac{V+\alpha p}{a}) + \int_0^\infty dV' P_V(V'-\frac{\Delta_2(m)}{p})\Big] \\ &+\frac{1}{2-2ap}\Big[\int_0^\infty dV' P(m+\Delta_3(m), V-\frac{\Delta_3(m)}{p})P_V(V'+\frac{\Delta_3(m)}{p}) \\ &\quad+\int_0^\infty dm' P(m+\Delta_4(m'), V-\frac{\Delta_4(m')}{p})P_m(m'-\Delta_4(m'))\Big].\end{aligned} \tag{7.19}$$

This is how far we seem to be able to go mathematically.

# References


[1] V. M. Yakovenko, and J. Barkley Rosser, Jr. *Statistical mechanics of money, wealth, and income*, Rev. Mod. Phys. **81**, 1703. December 2009.

[2] T. Piketty. *Capital in the Twenty-First Century*, Harvard University Press. 2013.

[3] L. DePillis. *America's wealth gap is bigger than ever*, CNN Money. November 3, 2017.

[4] A. C. Silva, and V. M. Yakovenko. *Temporal evolution of the thermal and superthermal income classes in the USA during 1983 - 2001*, Europhys. Lett. **69**, 304310. 2005.

[5] R. Chetty, et al. *The fading American dream: Trends in absolute income mobility*, SCIENCE. **356**, 6336, 398-405, April 2017.

[6] Report on income, earnings, income inequality and poverty conducted by U.S. Census Bureau. *https://www.census.gov/content/census/en/library/publications/2018/demo/p60-263.html*

[7] L. Boltzmann. *Lectures on gas theory*, U. of California Press. 1964.

[8] N. Gregory Mankiw. *Principles of Microeconomics* (5th edition), Cengage Learning. 2008.

[9] S. Ispolatov, P. L. Krapivsky, and S. Redner. *Wealth distributions in asset exchange models*, Eur. Phys. J. B **2**, 267276. 1998.

[10] G. J. Stigler. *The Development of Utility Theory. I and II*, Journal of Political Economy. **58**, 307-327. 1950.

[11] R. M. Solow. *A contribution to the theory of economic growth*, The Quarterly Journal of Economics. **70** (1), 6594. 1956.

[12] A. Mas-Colell, M. Whinston, and J. Green. *Microeconomic Theory*, Oxford: Oxford University Press. 1995.