Utrecht University

Theoretical Physics

Master Thesis

Holographic nodal spheres

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Abstract

In this thesis, inspired by recent quantum oscillation experiments on ZrSiS [1], we come up with a theoretical model describing nodal spheres holographically. Using the AdS/CFT correspondence, we construct from two bulk Dirac spinors one boundary Dirac spinor and split the Dirac cone in energy space into its two Weyl consituents. The nodal sphere then appears at the Fermi level. Unfortunately, at high energies this result is not in accord with experiment as it does not obey the ARPES sum rule [2]. We try to solve this problem with semiholography, but for the parameters that we considered the nodal sphere then seems to be gapped away.

Op de voorpagina vinden we een houtsnede van *Cirkellimiet* IV (ook wel *Hemel en Hel*) van de Nederlandse kunstenaar M.C. Escher, geïnspireerd op een publicatie van de Canadese professor H.S.M. Coxeter, dan weer gebaseerd op het werk van de Franse wiskundige J.H. Poincaré die in zijn Poincaré schijf deze vorm van hyperbolische meetkunde had gevisualiseerd. Zelf begreep Escher weinig van de wiskunde. Op een poging tot uitleg van Coxeter, reageerde hij: “Drie zijdjes vol met explicaties over wat ik eigenlijk wel gedaan heb. Jammer dat ik er niets, maar dan ook niets, van begrijp.”

Het duale karakter van de zwart-witte engelen vertoont overeenkomst met de twee bulk spinoren die op de rand van een negatief gekromd universum als twee chirale fermionen tot één versmelten, zoals beschreven in dit verslag. Daarnaast deed het mozaïek, als ware het een Adidas Telstar, me denken aan de woorden van Sierd de Vos voorafgaand aan de wedstrijd van Ajax bij het zwart-wit van de Oude Dame Juventus. Dus, naar Frank Boeijen en ter ere aan De Godenzonen: “Denk niet wit. Denk niet zwart. Neehee, denk niet zwart-wit. Dus denk niet zwart-wit! Maar denk in de kleur van je hart.”

Contents

1 Introduction

In 1928, the British theoretical physicist Paul Dirac was the first to write down a relativistic equation describing the behaviour of free electrons. In fact, the Dirac equation gave a correct description not only of electrons but of all relativistic spin-1/2 fermions by successfully combining quantum mechanics and special relativity.

Remarkably enough then, some nonrelativistic systems can be described by the Dirac equation as well. The low-energy excitations in such systems behave as massless Dirac fermions; they have a linear dispersion relation around the point where the conduction and valence band touch, which we call a nodal point. In these pseudorelativistic systems, the velocity of the electrons is however not equal to the speed of light c (of the order 10^8 m/s) but instead to the Fermi velocity v_{F} (of the order 10^6 m/s).

The most famous example of such a system is graphene. Graphene is a (2+1)-dimensional Dirac semimetal, a gapless semiconductor admitting a nodal point (see figure 1). It is described by the Dirac Hamiltonian

$$\mathcal{H}_{\mathrm{Dirac}}(\boldsymbol{k}) = \hbar v_{\mathrm{F}} \left(\sigma^1 k_x + \sigma^2 k_y\right) , \tag{1.1}$$

where σ^i with $i = 1, 2, 3$ denote the Pauli spin matrices and $\hbar$ is Planck's constant. This Hamiltonian enjoys both time-reversal symmetry and spatial inversion symmetry.

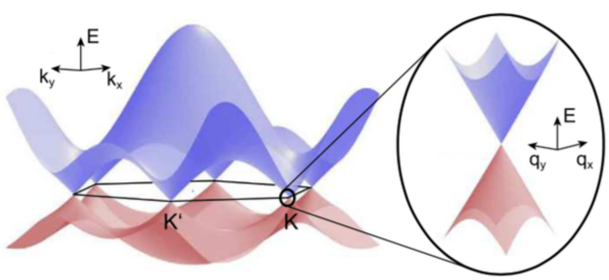

Figure 1: *The electronic band structure of graphene.* In its first Brillouin zone, graphene has the two inequivalent Dirac nodal points indicated by K and K'. Around such a nodal point, we see the material exhibits a pseudorelativistic linear dispersion relation. The red colour indicates the occupied bands whereas the blue colour indicates the empty bands. Adapted from [3].

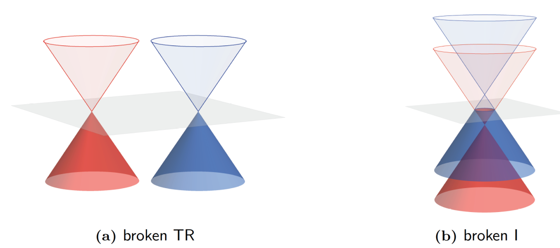

Figure 2: *The separate Weyl cones after breaking symmetry.* By breaking time-reversal (TR) symmetry, the Weyl cones are split in momentum space as seen in (a). By breaking spatial inversion (I) symmetry, the Weyl cones are split in energy space as seen in (b). The red and blue colours correspond with the different chiralities of the cones. The grey plane is the zero energy plane and the dark colour indicates the occupied bands whereas the light colour indicates the empty bands. Taken from [4].

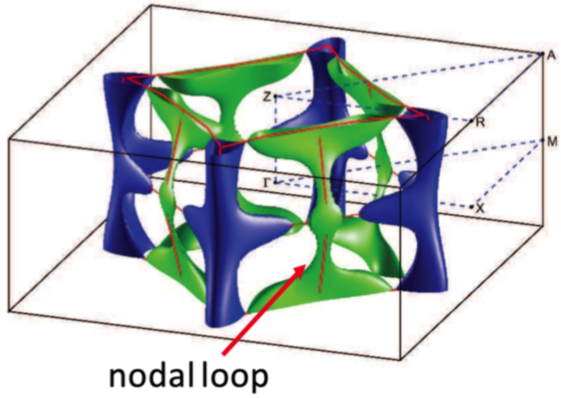

Figure 3: *The three-dimensional Fermi surface of ZrSiS.* Indicated in red, we see that the valence and conduction band touch on a line, dubbed the nodal loop. Adapted from [1].

1.1 Weyl semimetals

Awarded the Nobel Prize in Physics of 2010, graphene caused an explosion of interest in so-called Dirac matter, which is a class of condensed-matter systems that have an effective description by the Dirac equation. Particularly, researchers were looking for other examples of Dirac matter. One such example is given by Weyl semimetals for which either the time-reversal symmetry or the spatial inversion symmetry is broken. Breaking either one of these symmetries, we split up the fourfold degenerate Dirac nodal point into two twofold degenerate Weyl nodal points (see figure 2).

The Hamiltonian needed in order to describe Weyl semimetals is given by a (3+1)-dimensional of the Dirac Hamiltonian

$$\mathcal{H}_{\mathrm{Weyl}}(\boldsymbol{k}) = \hbar v_{\mathrm{F}} \left(\sigma^1 k_x + \sigma^2 k_y + \sigma^3 k_z \right) , \tag{1.2}$$

called the Weyl Hamiltonian. The excitations belonging to a Weyl nodal point are Weyl fermions, which have a definite chirality. In this thesis, we will be concerned with the Weyl nodal points obtained by breaking spatial inversion symmetry.

Recently, in 2017, experimentalists managed to construct a nodal loop for the semimetal ZrSiS (see figure 3) [1]. We aim to construct the nodal sphere, which in a three-dimensional system is easier to construct than a nodal loop because of the higher degree of symmetry of a sphere. We build our model of Weyl nodal points, which we can merge into a sphere, using the holographic duality known as the anti-de Sitter/conformal field theory (AdS/CFT) correspondence.

1.2 The AdS/CFT correspondence

The AdS/CFT correspondence is a gauge/gravity duality that gives the relationship between two seemingly unrelated theories of physics. On the one hand, we have quantum fields in an anti-de Sitter (AdS) spacetime, constituting a theory of quantum gravity (QG) for which we need a string theoretical description. On the other hand, we have in a flat spacetime a conformal field theory (CFT), which is a special type of quantum field theory (QFT). This includes gauge theories similar to Yang-Mills theory, describing elementary particles.

The correspondence, proposed by Juan Maldacena in 1997, was a major breakthrough in the understanding of string theory and QG. By 2015, Maldacena's article was the most highly cited article in high-energy physics with over 10,000 citations.

One part of its success is that it is the most successful realization of the holographic principle, a conjectured property of QG saying that a volume of spacetime can be described by its boundary only. First suggested by Gerard 't Hooft, this principle was advanced by Leonard Susskind who gave it an interpretation within string theory. Steven Gubser, Igor Klebanov, Alexander Polyakov and Edward Witten continued giving important aspects of the AdS/CFT correspondence which culminated in the GKPW rule. This rule can be thought of as the master equation of the correspondence.

Another part of the success of the correspondence comes from providing a way to study strongly coupled QFTs. This use comes from the fact that it represents a strong/weak duality. Strongly interacting fields in the gauge theory correspond to weakly interacting fields in the gravity theory, for which we have the mathematical tools. Hence the correspondence has been used in fields as nuclear and condensed-matter physics to translate hard problems in these fields to string theory where the particular problem might be less hard.

This is also why we will try to describe the nodal sphere holographically. Near any nodal point, the density of states vanishes. Being inversely related to the screening length, a zero density of states means we have a diverging screening length. Since we now have barely any screening effect, the interaction we aimed to describe becomes a bare Coulomb interaction. This interaction is strong and long-ranged. Hence the physics near a nodal point (or loop or sphere) is similar to a CFT, which, according to the AdS/CFT correspondence, can be obtained by studying a weakly coupled gravity dual.

In chapter 2 we take up the AdS/CFT correspondence, explaining general features of a gauge/gravity duality to eventually get a better understanding of the GKPW rule. Its use is then shown through the example of a real massive scalar field in chapter 3, also giving us more insight in certain general aspects of the correspondence. With this simple example, we go on to chapter 4 and explain how to describe spinors holographically. Following, in chapter 5, we will use this knowledge to build a model of the holographic nodal sphere and we will give our results. Finally, in chapter 6, we discuss what we have found and indicate possible further research.

2 AdS/CFT correspondence

In this chapter, we will try to make the AdS/CFT correspondence understandable. By no means we claim to give thorough derivations. This would require a deep understanding of string theory, which is beyond the scope and goal of our thesis. Moreover, even with rigorous derivations most of the ideas would remain conjectural. Instead, we thus aim to make things just a little more intuitive.

We do this, following [5], by starting off with a proposition that is twofold. The first part consitutes the gauge/gravity duality which asserts that some ordinary QFTs are secretly theories of QG. Second, we propose that sometimes the gravity theory has a classical approximation. In those cases, the gravity theory can be used to compute interesting observables of the QFT by using the GKPW rule.

In section 2.1 we explore some general properties any gauge/gravity duality, so also the AdS/CFT correspondence, should have. Then in section 2.2 we look at a precise implementation of this duality in the AdS/CFT correspondence and check that it satisfies the introduced properties of a gauge/gravity duality. From checking the correspondence against the holographic principle, it will become clear how to make the classical approximation of gravity. In section 2.3, this approximation will be applied to the GKPW rule which ultimately gives us a useful equation relating the partition function of the CFT to the partition function of the AdS theory. We give a short discussion on the general strategy we adopt when applying this rule to a model. Sections 2.1 and 2.2 are based on [5–7], which are all excellent references for anyone beginning to learn about the AdS/CFT correspondence. The concluding section 2.3 will follow [6, 8].

2.1 Gauge/gravity duality

To get a better understanding of the gauge/gravity duality we will take three subsequent steps. First we will argue that the QG and the QFT in this duality should not be defined on the same spacetime. Second, following the holographic principle, we will see that the spacetime of the QG must have an extra spatial dimension compared with the spacetime of the QFT. Third, we will explain how this extra spatial dimension can be identified with an energy scale.

To see why the QG and the QFT have different spacetimes, consider quantizing gravity. Inevitably, we come across a massless spin-2 particle known as the graviton. Now, if some ordinary QFTs are indeed secretly theories of QG we must be able to incorporate such a massless spin-2 particle into our QFT. However, this seems to be prohibited by the Weinberg-Witten theorem which says that a QFT with a Poincaré covariant conserved stress-energy tensor $T^{\mu\nu}$ forbids massless particles of spin $j > 1$ which carry momentum (i.e. with $P^\mu = \int d^d x\, T^{0\mu} \neq 0$) [7]. However, if we do want to relate our QFT to QG, we need to find a loophole around this theorem. The solution turns out to be that the graviton and QFT do not need to be defined on the same spacetime.

Next, the holographic principle, which states that the maximum entropy in a spacetime volume is proportional to the surface area of the volume, tells us something about the dimensions of these different spacetimes. Let's first explain the holographic principle. Consider a spacetime volume V with surface area A and entropy S.

Assuming for a moment that the entropy of this spacetime volume is greater than the entropy of a black hole occupying the same volume, we will derive an inconsistency. We know that the entropy of a black hole is proportional to its surface area

$$S_{\mathrm{BH}} = \frac{c^3 A}{4G\hbar} \equiv \frac{A}{4\ell_{\mathrm{P}}^2}\,, \tag{2.1}$$

where G denotes Newton's gravity constant, $\hbar$ the Planck constant, c the speed of light and ℓ_{P} the Planck length. The subscript BH conveniently stands either for black hole or for the physicists Bekenstein and Hawking who came up with this equation. Additionally, we assume that the energy in this volume is less than that of a black hole. Otherwise the energy density would cause gravitational collapse and a black hole would be formed. Now add energy in any form to the volume so that the energy density reaches the value at which a black hole is formed. The configuration we obtain, has less entropy than the one we started with because $S > S_{\mathrm{BH}}$. This violates the second law of thermodynamics. Rather than do that, 't Hooft therefore assumed that a black hole is the configuration with the highest entropy per unit volume. Thus, presumably, the maximum entropy in a spacetime volume is proportional to the surface area of the volume.

We can now look at the consequences of the holographic principle for our two spacetimes. We note that the maximum entropy according to holography, is far smaller than the entropy of a local QFT on the same volume of spacetime, even with some UV cutoff equivalent to a length scale such as the Planck length ℓ_{P}. Such a theory would have a number of degrees of freedom $N_{\mathrm{dof}}^{\mathrm{QFT}} \propto n^{V/l_P^3}$, where n is the number of local degrees of freedom. Hence the maximum entropy of the local QFT is proportional to the volume of the spacetime because $\ln\left(N_{\mathrm{dof}}^{\mathrm{QFT}}\right) \propto V$.

This means that for a QG to be related to a QFT, the QFT would have to be formulated on a spacetime with one less spatial dimension so that the degrees of freedom of the QG and the QFT can at least in principle be equated. Later, in section 2.2, we will come back to this point as we make a check of the conjectured correspondence between a specific choice of the gravity and the gauge theory (spoiler: the AdS/CFT correspondence). We first continue to explain the role of this extra spatial dimension in the gravity theory within the gauge theory.

Thus, finally, the extra degree of freedom in the QG must have some description in the QFT. Wilson taught us that a QFT is best thought of as being sliced up by energy scale, as a family of trajectories of the renormalization group (RG). The most important function associated with RG is the beta function, giving the dependence of a coupling parameter g on the energy scale u:

$$\beta(g(u)) = \frac{dg}{du}(u)\,. \tag{2.2}$$

What is important to note about this equation is that it depends locally on the value of the energy scale u. This means the coupling constants obey the principle of locality. This locality may remind us of GR, where to compute local quantities we only need to know the local geometry of the spacetime. Therefore, it may not be unreasonable to associate the extra spatial dimension we found with the energy scale associated with RG.

Summarizing, we have started by mentioning the Weinberg-Witten theorem. This suggested that if we want to relate a QFT to QG, the graviton should live in a different spacetime than that of the QFT. The holographic principle told us that the gravity theory should have a number of degrees of freedom that grows more slowly than the volume. This implied that the QFT needs to be defined on a spacetime which has one spatial dimension less than the spacetime of the QG. The structure of RG identified the extra dimension of the gravity theory as the energy scale of the gauge theory. Altogether this helps us to interpret the first part of our twofold proposition, the gauge/gravity duality, which stated that some ordinary QFTs are secretly theories of QG.

Continuing, we will make things more concrete by looking at what happens when we set the beta function to zero. This will give us the AdS/CFT correspondence and its master equation known as the GKPW rule. It is within this framework that we will take a look at the second proposition, making use of the classical approximation of gravity to compute observables of the QFT through the GKPW rule.

2.2 AdS/CFT correspondence

We set out to find concrete examples of the gauge/gravity duality. The simplest case of an RG flow is when the beta function is zero, meaning we consider gauge theories in which the couplings do not change with the energy scale. Hence these QFTs, which we take to be d-dimensional, are scale invariant. As a working definition we will call them CFTs. In reality, CFTs have a stronger set of conformal symmetries, but we refer to appendix A for a discussion. In any CFT we thus have that the scale transformation $x^\mu \to \lambda x^\mu$ is a symmetry. If the extra coordinate u in the gravity theory is to be thought of as an energy scale, then dimensional analysis says that u will scale under the scale transformation as $u \to u/\lambda$. The most general $(d+1)$-dimensional metric with this scaling symmetry and Poincaré invariance is of the following form:

$$ds^2 = \frac{u^2}{L^2}\,\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{u^2}\,du^2\,. \tag{2.3}$$

This we recognize as the AdS_{d+1} metric (see appendix B). The parameter L is called the AdS radius and has dimensions of length. For convenience we perform a change of coordinates $z = L^2/u$, in which the metric takes the form

$$ds^2 = \frac{L^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right)\,. \tag{2.4}$$

From this point on, we will always describe our metric in this set of coordinates. In the CFT dual, z will map to the length scale of the system. For z going to the boundary of the AdS spacetime (at $z = 0$) we go to the UV of the CFT. For z going into the interior of the AdS spacetime ($z \to \infty$) we go to the IR of the CFT (see figure 4). We thus have found a correspondence between a d-dimensional CFT and a quantum theory of gravity on AdS_{d+1}.

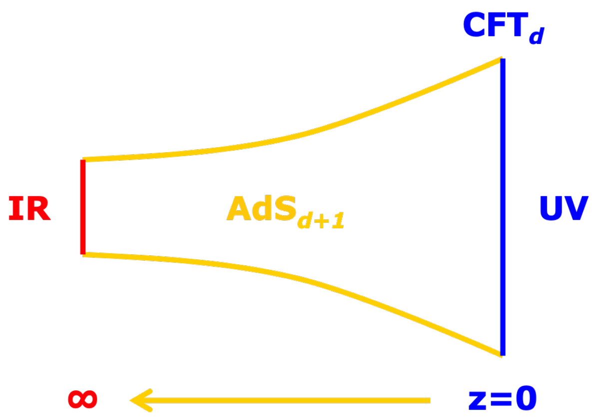

Figure 4: *The idea behind the AdS/CFT correspondence.* We see how the radial coordinate z functions as an (inverse) energy scale. The IR lies in the limit $z \to \infty$, while the CFT of dimension d is found in the UV at $z = 0$. For any fixed value of the radial coordinate we obtain a slice of the AdS$_{d+1}$ spacetime corresponding to a d-dimensional Minkowski spacetime.

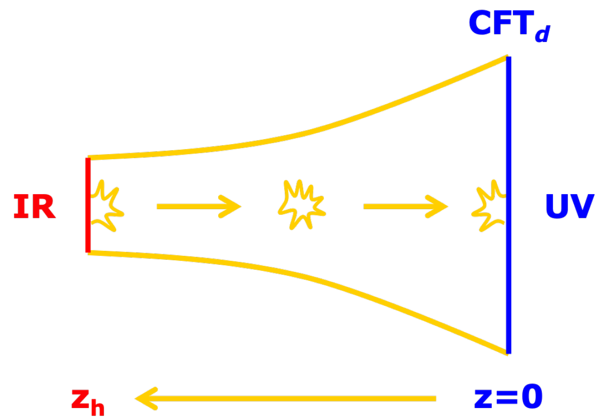

Figure 5: *The open/closed string duality.* On any particular brane one may find open strings corresponding to gauge fields. When such a string closes up on itself, it becomes bosonic and can propagate through the bulk only to open up again on some other brane. From this we see how properties of a black brane at $z = z_h$ can translate to properties of the CFT.

The metric (2.4) is the maximally symmetric solution to the Einstein equations with a negative cosmological constant. The corresponding action, written in SI units and in such a way that all coordinates share the same mass dimension, is given by

$$S_{\text{AdS}} = \int_{\text{AdS}} d^d x \, dz \, \sqrt{-g} \left[\frac{c^3}{16\pi G} (-2\Lambda + R) + \dots \right] . \tag{2.5}$$

Here $\sqrt{-g} \equiv \sqrt{-\det g_{\mu\nu}}$ makes the integral coordinate invariant, c is the speed of light and G is the $(d+1)$-dimensional gravitational constant. In units where $\hbar = c = 1$ the gravitational constant has dimension $[\text{length}]^{d-1} = 1/[\text{mass}]^{d-1}$ and in fact is given by $G = \ell_{\text{P}}^{d-1} = 1/m_{\text{P}}^{d-1}$ with ℓ_{P} the Planck length and m_{P} the Planck mass. The cosmological constant Λ appears in the first term, which is associated with the volume of the spacetime. The second term contains the Ricci scalar R and constitutes the Einstein-Hilbert action, associated with the curvature of the spacetime. By computing the Ricci scalar directly from its definition and comparing it with (B.5), we find that the cosmological constant Λ is related to the AdS radius L as $\Lambda = -d(d-1)/2L^2$. Variation of (2.5) with respect to the metric yields the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - 2\Lambda) = 0 \, . \tag{2.6}$$

Now, we make a subtle but crucial point. Since the spacetime metric is a dynamical variable in a theory of gravity, we only get to specify its boundary behaviour. The AdS metric (2.4) yields a boundary at $z = 0$. Keeping x^μ fixed and moving in the radial direction from a finite value of z to $z = 0$ is actually infinite distance. However, massless particles in an AdS spacetime, such as the graviton, travel along null geodesics and these reach the boundary at $z = 0$ in a finite time. This means that in order to specify the future evolution of the system from some initial data, we also have to specify boundary conditions at $z = 0$. These boundary conditions will play a crucial role in the AdS/CFT correspondence.

Moreover, it is only as $z \to 0$ that we need the metric to take the form (2.4). The metric (2.4) is the unique maximally symmetric solution to (2.6), but there exist other solutions as well. In particular, we have AdS black hole solutions. The most universal deformation away from pure AdS is the AdS Schwarzschild black hole, which is described by the metric

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + d\boldsymbol{x}^2 + \frac{dz^2}{f(z)}\right) \quad \text{with} \quad f(z) = 1 - \left(\frac{z}{z_h}\right)^d . \tag{2.7}$$

Here we have set $c = 1$ and where the position of the black hole horizon is given by z_h. Note that, using the above coordinates, the black holes take the form of black branes. Additionally, we see that $f(z) \to 1$ as $z \to 0$ which means our spacetime is still asymptotically AdS. Also, we see that by putting the black brane horizon z_h infinitely far away, we recover the maximally symmetric AdS solution.

Allowing these black brane solutions is important, because properties of the black brane translate to properties of the CFT through energy rescaling by traversing the bulk AdS spacetime from the IR to the UV. For example, the Hawking temperature of a black hole will give the temperature of the CFT. For further discussion about asymptotically AdS spacetimes we refer to appendix B.2. A useful picture to have in mind is that of the open/closed string duality (see figure 5). Thinking of the $\text{AdS}_(d+1)$ spacetime as being composed of branes of Minkowski spacetimes, strings can attach themselves to these branes. By closing up on themselves, the strings can traverse the larger spacetime to open up again on some other brane. This way properties of the blakc brane get translated to properties of the CFT.

To recap then, starting from the gauge/gravity duality we tried to make matters more concrete by setting the beta function to zero. Including black hole solutions, this led us to a correspondence between CFTs of dimension d and (asymptotically) AdS spacetimes of dimension $d + 1$. As promised earlier, we now come back to the holographic principle for a check of our conjectured correspondence.

Degrees of freedom

For our AdS/CFT correspondence to make any sense, we need to equate the degrees of freedom on both sides. We must find $N_{\text{dof}}^{\text{AdS}} = N_{\text{dof}}^{\text{CFT}}$ where $N_{\text{dof}}^{\text{AdS}}$ is the number of degrees of freedom in the AdS gravity theory and $N_{\text{dof}}^{\text{CFT}}$ the number of degrees of freedom in the CFT. Obviously we have that $N_{\text{dof}}^{\text{CFT}}$ is infinite. From the holographic principle we can conclude that $N_{\text{dof}}^{\text{AdS}}$, being proportional to the area of our spacetime, is infinite as well. To make any real progress, we need to regulate our counting.

Let's regulate the field theory first. There are both UV and IR divergences. We put the CFT on a lattice thereby introducing a short distance cutoff ε, for example the lattice spacing, which gets rid of UV divergences. We also put the CFT in a cubical box of linear size R, getting rid of IR divergences. In d spacetime dimensions the system then has $(R/\varepsilon)^{d-1}$ cells. Let c_{CFT} be the number of degrees of freedom per lattice site, which we will refer to as the central charge. Then the total number of degrees of freedom in the CFT is

$$N_{\text{dof}}^{\text{CFT}} = \left(\frac{R}{\varepsilon}\right)^{d-1} c_{\text{CFT}} \,. \tag{2.8}$$

The central charge is one of the main quantities that characterize a CFT. If the CFT is an SU(N) gauge field theory then the fields are $N \times N$ matrices in the adjoint representation which for large N contain N^2 independent components. Thus, in these cases, we have that $c_{\text{SU}(N)} \propto N^2$.

Next we regulate the area of our AdS_{d+1} spacetime, which is given by

$$A = \int_{\{\mathbb{R}^{d-1},\, z\to 0,\, \text{fixed } t\}} d^{d-1}x \sqrt{-g} \tag{2.9}$$

$$= \int_{\{\mathbb{R}^{d-1},\, z\to 0\}} d^{d-1}x \left(\frac{L}{z}\right)^{d-1} \,. \tag{2.10}$$

As was the case for the CFT, this is infinite for two reasons. First from the integral over x that we have and second from the limit of z that we take. To regulate the integral, we integrate not to $z = 0$ but rather cut it off at $z = \varepsilon$, meaning we perform the integral on a brane close to the CFT. This cutoff of the geometry at $z = \varepsilon$ has the interpretation of a UV cutoff of the CFT. Also, we again place our system in a cubical box of linear size R. This gives us the regulated area of the AdS_{d+1} spacetime

$$A = \int_0^R d^{d-1}x \left.\frac{L^{d-1}}{z^{d-1}}\right|_{z=\varepsilon} = \left(\frac{RL}{\varepsilon}\right)^{d-1} \,. \tag{2.11}$$

After having set $\hbar = c = 1$ and by following equation (2.1), we get the total number of degrees of freedom of the AdS gravity theory

$$N_{\text{dof}}^{\text{AdS}} = \frac{1}{4}\left(\frac{R}{\varepsilon}\right)^{d-1}\left(\frac{L}{\ell_{\text{P}}}\right)^{d-1} \,. \tag{2.12}$$

Looking at both the expression for $N_{\text{dof}}^{\text{AdS}}$ and $N_{\text{dof}}^{\text{CFT}}$, we see that they scale in the same way with the IR cutoff R and the UV cutoff ε. This means that the AdS/CFT correspondence is indeed an implementation of the holographic principle.

We can even learn more from the expressions we have found. For the prefactors to agree we need to relate the AdS radius in Planck units L/ℓ_{P} to the number of degrees of freedom per lattice site N^2 of the SU(N) gauge field theory. Up to numerical prefactors, this implies for SU(N) gauge theories that

$$\left(\frac{L}{\ell_{\mathrm{P}}}\right)^{d-1} \simeq N^2 \,. \tag{2.13}$$

We now have obtained a matching condition between gravity and gauge theories. Note that a theory is (semi)classical when the coefficient multiplying the action is large. The path integral then becomes dominated by a saddle point. Working in units where $\hbar = c = 1$ and hence $G = \ell_{\mathrm{P}}^{d-1}$, we see that we have exactly a factor proportional to N^2 multiplying the action (2.5) of our gravity theory.

In order to make a saddlepoint approximation we thus need to take the so-called large N limit, equivalent to an AdS radius large in Planck units:

$$\left(\frac{L}{\ell_{\mathrm{P}}}\right)^{d-1} \gg 1 \,. \tag{2.14}$$

In conclusion then, we see that a CFT has a classical gravity dual when it has a large number of degrees of freedom per unit volume, or a large number of species N. This touches upon the second part of our twofold proposition, which had to do with a classical approximation of the gravity theory. On the gravity side, taking the large N limit suppresses loops and higher curvature corrections. On the gauge side, the perturbation expansion can be written down in terms of an effecive coupling that is proportional to N, so that the large N limit corresponds to a strong coupling limit. We thus see that the AdS/CFT correspondence is a strong/weak duality. Next, we will show how to use the large N limit in computing observables of the CFT.

2.3 GKPW rule

We are now ready to give the master equation of the AdS/CFT correspondence. This is the Gubser-Klebanov-Polyakov-Witten (GKPW) rule. All previous discussion has been to make the GKPW rule more intuitive, but the true support follows from string theory and we will just state the rule as coming from a divine authority. Its essence is the equality of partition functions on both sides of the correspondence

$$Z_{\mathrm{CFT}}(N) = \int \mathcal{D}\varphi \, e^{iN^2 S_{\mathrm{AdS}}[\varphi]} \,. \tag{2.15}$$

Here φ is meant to denote all fields in the theory and we have made the N^2 dependence explicit by taking this factor out of the action. Let's quickly review both sides.

On the left-hand side of (2.15) we have the full quantum partition function of the field theory. Observables of a QFT are characterized by vacuum correlation functions of local operators in the QFT:

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle_{\mathrm{QFT}} \equiv \frac{\int \mathcal{D}\Phi\, e^{iS_{\mathrm{QFT}}}\, \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)}{\int \mathcal{D}\Phi\, e^{iS_{\mathrm{QFT}}}}\,. \tag{2.16}$$

Here *Phi* is meant to denote all fields in the theory. With the quantum partition function we can compute any observable of the field theory. For clarity, we recall the use of the generating functional in QFT, which is given by

$$Z_{\mathrm{QFT}}[J] = \int \mathcal{D}\Phi\, e^{i[S_{\mathrm{QFT}}+\int J_i(x)\mathcal{O}_i(x)]}\,. \tag{2.17}$$

By taking functional derivatives of the partition function we can compute the vacuum correlation functions of local operators in the QFT as follows

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i(x_i) \right\rangle_{\mathrm{QFT}} = \frac{1}{i^n}\left(\prod_{i=1}^{n} \frac{\delta}{\delta J_i(x_i)}\right) \ln Z_{\mathrm{QFT}}[J]\Bigg|_{J=0}\,. \tag{2.18}$$

On the right-hand side of (2.15) we have a path integral over all fields in the gravity theory. It should be thought of as a partition function of QG in the bulk of an AdS spacetime. Although it might be difficult to get the generating functional for a (strongly interacting) QFT, it certainly is not more difficult than solving QG. However, we have just seen that in the large N limit, the gravity dual becomes classical. In this limit, (2.15) can be used to compute the observables of the CFT from the dynamics of the gravity theory.

Looking at both sides together now, the insight of GKPW thus was how to incorporate the recipe of taking functional derivatives into the right-hand side of (2.15). The source in the CFT should be encoded in a field in AdS. Hence if the CFT can be thought to live at the boundary of the AdS spacetime, then on the gravitational side the source is restricted to this boundary. The sources act therefore as boundary conditions for the classical fields propagating in the bulk gravity theory. In mathematical form this yields the GKPW rule

$$\left\langle \exp\left\{ i\int d^dx\, J(x)\mathcal{O}(x)\right\}\right\rangle_{\mathrm{CFT}} = \int_{\varphi_0(x)\equiv J(x)} \mathcal{D}\varphi\, e^{iN^2 S_{\mathrm{AdS}}[\varphi(x,z)]}\,. \tag{2.19}$$

Here φ_0 is a suitably defined boundary value of the AdS field φ and couples as a source to the boundary conformal field theory operator $\mathcal{O}$. Naively $\varphi_0(x) = \varphi(x, z = 0)$, but, as sometimes the leading order behaviour of $\varphi(x, z \to 0)$ is divergent, in some cases it has to be understood as the coefficient of the leading order term (or even of the subleading term as we will later point out). The point to take home then is that we need the asymptotic behaviour of the field φ to determine the source J.

Of course, we still have to deal with the path integral on the right-hand side of (2.19) which is now a partition function of QG with boundary conditions depending on the source J. At this point we perform the saddle point approximation by taking the large N limit, yielding

$$\left\langle \exp\left\{ i \int d^d x\, J(x)\mathcal{O}(x) \right\} \right\rangle_{\text{CFT}} \approx e^{iS_{\text{AdS}}^{\text{on shell}}} \Big|_{\varphi_0(x)\equiv J(x)} . \tag{2.20}$$

This expresses in a formula the twofold proposition we started with, relating a classical approximation of QG to an ordinary QFT. We conclude this section by taking a closer look at the on-shell action. This will be useful for working out examples as the strategy is always the same.

On-shell action

We comment on the standard procedure adopted to obtain the on-shell action. Let us have a diagonal metric depending only on the radial coordinate

$$ds^2 = g_{MN}(z) dx^M dx^N . \tag{2.21}$$

The action we write down as

$$S = \int_{\text{AdS}} d^d x\, dz\, \sqrt{-g}\, \mathcal{L}\left[\varphi, \partial\varphi\right] , \tag{2.22}$$

where φ is meant to denote any field. The variation of the action under a general change $\varphi \to \varphi + \delta\varphi$ reads

$$\delta S = \int_{\text{AdS}} d^d x\, dz\, \sqrt{-g} \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_M \varphi)} \delta(\partial_M \varphi) \right] \tag{2.23}$$

$$= \int_{\text{AdS}} d^d x\, dz \left[\left(\sqrt{-g}\, \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_M \left(\sqrt{-g}\, \frac{\partial \mathcal{L}}{\partial(\partial_M \varphi)} \right) \right) \delta\varphi + \partial_M \left(\sqrt{-g}\, \frac{\partial \mathcal{L}}{\partial(\partial_M \varphi)} \delta\varphi \right) \right] \tag{2.24}$$

$$= \int_{\text{AdS}} d^d x\, dz\, \sqrt{-g} \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \nabla_M \left(\frac{\partial \mathcal{L}}{\partial(\partial_M \varphi)} \right) \right] \delta\varphi + \int_{\text{AdS}} d^d x\, dz\, \sqrt{-g}\, \nabla_M \left[\frac{\partial \mathcal{L}}{\partial(\partial_M \varphi)} \delta\varphi \right] \tag{2.25}$$

$$= \int_{\text{AdS}} d^d x\, dz\, \sqrt{-g} \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \nabla_M \left(\frac{\partial \mathcal{L}}{\partial(\partial_M \varphi)} \right) \right] \delta\varphi + \int_{\partial\text{AdS}} d^d x\, \sqrt{-h} \left[n_M\, \frac{\partial \mathcal{L}}{\partial(\partial_M \varphi)} \right] \delta\varphi\, . \tag{2.26}$$

In the first step we performed integration by parts (and used that partial and functional derivatives commute) and in the second we used the identity $\partial_M\left(\sqrt{-g}\,V^M\right) = \sqrt{-g}\,\nabla_M V^M$. In the final step we used Stokes' theorem $\int_{\mathcal{M}} d^m x\,\sqrt{|g|}\,\nabla_M V^M = \int_{\partial\mathcal{M}} d^{m-1}y\,\sqrt{|h|}\,n_M V^M$ for a general m-dimensional manifold with boundary $\mathcal{M}$, where the boundary is denoted by $\partial\mathcal{M}$. On the boundary we have the determinant of the induced metric, where the induced metric itself is defined as the pullback of the original metric. We also have the unit normal to the boundary n^M which must be chosen outward pointing for a timelike boundary and inward pointing for a spacelike boundary.

Additionally, in the final step we assumed that the fields vanish at infinity of any Minkowski spacetime slice. We are then left only with the boundary of the AdS spacetime in the radial direction. Since $n_M = \pm\sqrt{g_{zz}}\,\delta^z_M$ (see equation (B.20)), we can write the variation of the on-shell action as

$$\delta S^{\text{on shell}} = \int_{z=z_{\text{IR}}} d^d x\,\sqrt{-h g_{zz}}\,\frac{\partial\mathcal{L}}{\partial(\partial_z\varphi)}\,\delta\varphi - \int_{z=z_{\text{UV}}} d^d x\,\sqrt{-h g_{zz}}\,\frac{\partial\mathcal{L}}{\partial(\partial_z\varphi)}\,\delta\varphi\,, \tag{2.27}$$

where we used that the boundary is timelike for the correct sign of the unit normals. The first term, found in the IR, is typically referred to as the horizon term. For a black hole solution we will have $z_{\text{IR}} = z_h$ and the integral is over the event horizon of the black hole. If we consider AdS spacetime, then $z_{\text{IR}} = \infty$ and although no longer designating an actual event horizon, the limit still remains a Killing horizon.

Usually, we will discard the horizon term altogether. Its presence in the on-shell action (2.27) enables us to take appropriate boundary conditions on the horizon. What's more, were we to go to Euclidean signature, the horizon term really vanishes by imposing regularity of the solution in the IR. As path integrals are only well-defined in Euclidean spacetime and we should actually be working in imaginary time from the start only to Wick rotate to real time at the end, it seems legitimate to neglect the IR term.

Anyhow, in light of the GKPW rule we are primarily interested in the UV term. This term corresponds to the CFT and goes under the name of the boundary term. Because it describes UV physics, we will pre-emptively set $z_{\text{UV}} = \varepsilon$ with $\varepsilon \ll 1$. If no UV divergences appear, we can simply take the limit $\varepsilon \to 0$. Otherwise, we will have to go through some renormalization procedure.

The equations (2.23) to (2.26) give the standard AdS/CFT procedure we apply. The bulk term yields the Euler-Lagrange equations in curved spacetime. Taking the action on shell, this term vanishes and only boundary terms remain. The strategy is to solve the equations of motion and substitute the solution in the on-shell action. The leading order term of the solution as $z \to 0$ should be identified with the source of the CFT. Then we can apply the GKPW rule to compute correlation functions.

3 Holographic scalar fields

In this chapter we present an example calculation of two-point functions for conformal operators using the GKPW rule following from the AdS/CFT correspondence. We do so by considering the massive real scalar field. This will exemplify the general discussion of chapter 2 and act as a springboard for the holographic spinor field discussed in chapter 4.

In section 3.1 we introduce the AdS action that we study and from this obtain the equation of motion and on-shell action. Following, we first look at the asymptotic solution of the equation of motion in section 3.2. This will give us some of the more general features encountered in a computation of the holographic Green's function. In particular, it will elucidate a relation between the mass of the bulk field and the scaling dimension of the dual conformal operator, which we will discuss in some detail. Having acquired the general form of our solution, we solve the equation of motion exactly in section 3.3 to obtain the exact results for the holographic Green's function of this theory. For this chapter we follow the line of [9] and specifically rely on [6] for working out the asymptotic solution in section 3.2 and on [8] for the exact solution in section 3.3.

3.1 Klein-Gordon in curved spacetime

We start by giving the action of our AdS spacetime S_{AdS}, which we split into a background part $S_{\text{background}}$ and a matter part S_{scalar}:

$$S_{\text{AdS}} = S_{\text{background}} + S_{\text{scalar}} \,, \tag{3.1}$$

$$S_{\text{background}} = \int_{\text{AdS}} d^d x \, dz \, \sqrt{-g} \left[-2\Lambda + R \right] \,, \tag{3.2}$$

$$S_{\text{scalar}} = \int_{\text{AdS}} d^d x \, dz \, \sqrt{-g} \left[-\frac{1}{2} \left(g^{MN} \, \partial_M \phi \, \partial_N \phi + m^2 \phi^2 \right) \right] \,. \tag{3.3}$$

We adopt the convention of using capital Latin letters as the indices of the full AdS_{d+1} spacetime and reserve the small Greek letters for the indices of any d-dimensional Minkowski spacetime slice that is transverse to the radial direction. By x^M we thus mean (x^μ, z). Considering the maximally symmetric solution, we get from the background action (3.2) our metric

$$ds^2 = \frac{L^2}{z^2} \left(\eta_{\mu\nu} dx^\mu dx^\nu + dz^2 \right) \,. \tag{3.4}$$

From the matter action (3.3) we derive the equation of motion and the on-shell action. The equation of motion is given by

$$\partial_M \left(\sqrt{-g}\, g^{MN}\, \partial_N \phi\right) - \sqrt{-g}\, m^2 \phi = 0 \tag{3.5}$$

and the on-shell action reads

$$S^{\text{on shell}}_{\text{scalar}} = -\frac{1}{2}\int_{\partial\text{AdS}} d^d x\, \sqrt{-h}\, \phi\, n^M\, \partial_M \phi \tag{3.6}$$

$$= \frac{1}{2}\int_{z=\varepsilon} d^d x\, \sqrt{-h g^{zz}}\, \phi\, \partial_z \phi - \frac{1}{2}\int_{z=\infty} d^d x\, \sqrt{-h g^{zz}}\, \phi\, \partial_z \phi\,. \tag{3.7}$$

Our task is to solve the equation of motion (3.5) and substitute the solution in the on-shell scalar action (3.7). Taking advantage of translational invariance $x^\mu \to x^\mu + a^\mu$, we Fourier transform the scalar field $\phi(x, z)$ in the non-radial coordinates

$$\phi(x, z) = \int \frac{d^d k}{(2\pi)^d}\, e^{i k_\mu x^\mu}\, \phi(k, z)\,, \tag{3.8}$$

where $k_\mu = (-\omega, \boldsymbol{k})$. With the metric (3.4), we obtain from the equation of motion the following differential equation which governs the dependence of the Fourier coefficients on the radial coordinate z:

$$\phi'' - \frac{d-1}{z}\, \phi' - \left(\frac{k^2 z^2 + m^2 L^2}{z^2}\right) \phi = 0\,. \tag{3.9}$$

Here the prime denotes differentiation with respect to the radial coordinate and $k^2 = -\omega^2 + \boldsymbol{k}^2$ denotes a contraction of the non-radial coordinates. We generally deal with a differential equation of the form (3.9) using the Frobenius method, which (neglecting eventual logarithmic terms) seeks a power series solution of the form

$$\phi(k, z) = \left(\frac{z}{L}\right)^r \sum_{n=0}^{\infty} f_n(k) \left(\frac{z}{L}\right)^n\,. \tag{3.10}$$

Then our differential equation (3.9) can be seen to yield an exact solution in terms of Bessel or Hankel functions. We come back to this exact solution in 3.3, but first analyze its asymptotic behaviour as it is in the limit $z \to 0$ that we are interested in the solution.

3.2 Asymptotic solution

Looking only at the lowest-order terms, we get from putting the ansatz (3.10) into the differential equation (3.9) for the asymptotic solution $z \to 0$, the following quadratic equation

$$r(r-1) - (d-1)r - m^2L^2 = 0\,. \tag{3.11}$$

The solutions for r are

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2} \tag{3.12}$$

$$\equiv \frac{d}{2} \pm \nu \tag{3.13}$$

and therefore

$$\phi(k, z \to 0) = \left(\phi_0(k)\left(\frac{z}{L}\right)^{\Delta_-} + \dots\right) + \left(\phi_1(k)\left(\frac{z}{L}\right)^{\Delta_+} + \dots\right) \tag{3.14}$$

$$= \phi_0(k)\left(\frac{z}{L}\right)^{\Delta_-} + \phi_1(k)\left(\frac{z}{L}\right)^{\Delta_+} + \dots\,, \tag{3.15}$$

where we call the term with ϕ_0 leading and the term with ϕ_1 subleading. We assumed no terms exist between the leading and subleading term. This then gives us for the on-shell action (3.7) that

$$S_{\text{scalar}}^{\text{on shell}} = \frac{1}{2}\int_{z=\varepsilon} \frac{d^dk}{(2\pi)^d}\left(\frac{L}{z}\right)^{d-1} \phi(k,z)\,\partial_z\phi(-k,z) \tag{3.16}$$

$$= \frac{1}{2L}\int_{z=\varepsilon} \frac{d^dk}{(2\pi)^d}\left[\Delta_-\phi_0(k)\phi_0(-k)\left(\frac{z}{L}\right)^{-2\nu} + \Delta_+\phi_0(k)\phi_1(-k) + \Delta_-\phi_0(-k)\phi_1(k) + \dots\right]. \tag{3.17}$$

Here the dots represent terms that vanish in the limit $z \to 0$. From the first term it follows that we encounter a UV divergence. We deal with this divergence by introducing the renormalized action, which we get by adding a counterterm action to the scalar action

$$S^R_{\text{scalar}} = S_{\text{scalar}} + S_{\text{counter}}\,. \tag{3.18}$$

One can check, by taking it on shell, that the correct form of the counterterm is given by

$$S_{\text{counter}} = -\frac{\Delta_-}{2L}\int_{z=\varepsilon}\frac{d^dk}{(2\pi)^d}\sqrt{-h}\,\phi(k,z)\phi(-k,z)\,. \tag{3.19}$$

By taking the limit $z \to 0$, we can finally apply the GKPW rule to the on-shell renormalized scalar action

$$S^{R,\,\text{on shell}}_{\text{scalar}} = \frac{\nu}{L}\int\frac{d^dk}{(2\pi)^d}\,\phi_0(k)\phi_1(-k) \tag{3.20}$$

from which we then find that

$$\langle\mathcal{O}(k)\rangle = \frac{\nu}{L}\phi_1(k)\,. \tag{3.21}$$

This result, that the one-point function is proportional to the subleading term, is true in general. Moreover, the two-point function is given by the linear response of the one-point function to the source ϕ_0 as

$$\langle\mathcal{O}(k)\rangle = G(k)\phi_0(k)\,, \tag{3.22}$$

where $G(k) \equiv \langle\mathcal{O}(k)\mathcal{O}(-k)\rangle$. Applying the inverse of the source to the right, we find that the Green's function is given by

$$G(k) = \frac{\nu}{L}\frac{\phi_1(k)}{\phi_0(k)}\,. \tag{3.23}$$

Standard and alternative quantization

We now discuss some remarks that were to be made about which term gets identified as the source, but that we skipped over. The identification of the source has to do with the difference between the range of the scaling dimension of the conformal operator $\mathcal{O}$ considered from the AdS perspective and the CFT perspective.

Because we could drop the k^2 term in (3.9), we can straightforwardly reformulate (3.15) in position space to get

$$\phi(x, z \to 0) = \phi_0(x) \left(\frac{z}{L}\right)^{\Delta_-} + \phi_1(x) \left(\frac{z}{L}\right)^{\Delta_+} + \dots \tag{3.24}$$

Using this, it follows from the GKPW rule (2.20) that the term that is added to the Lagrangian of the CFT is given by

$$\int d^d x \, \phi_0(x)\mathcal{O}(x) = \int d^d x \lim_{z\to 0} z^{-\Delta_-} \phi(x,z)\mathcal{O}(x) \,. \tag{3.25}$$

We are now able to determine the scaling dimension Δ of the operator $\mathcal{O}$. Recalling that under a dilation $x' = \lambda x$ we have (see A)

$$\mathcal{O}(x) \to \mathcal{O}'(x') = \lambda^{-\Delta}\mathcal{O}(x) \,, \tag{3.26}$$

we can use the bulk isometry of the AdS spacetime $(x^\mu, z) \to (x'^\mu, z') = (\lambda x^\mu, \lambda z)$ and the fact that the boundary action should be conformally invariant

$$\int d^d x \, \phi_0(x)\mathcal{O}(x) = \int d^d x' \, \phi_0(x')\mathcal{O}(x') \tag{3.27}$$

to obtain

$$\int d^d x \, \phi_0(x)\mathcal{O}(x) = \int d^d x' \, \phi_0(x')\mathcal{O}(x') \tag{3.28}$$

$$= \int d^d x' \lim_{z'\to 0} z'^{-\Delta_-} \phi'(x', z')\mathcal{O}'(x') \tag{3.29}$$

$$= \int d^d (\lambda x) \lim_{z\to 0} (\lambda z)^{-\Delta_-} \phi(x,z)\lambda^{-\Delta}\mathcal{O}(x) \tag{3.30}$$

$$= \lambda^{\Delta_+ - \Delta} \int d^d x \, \phi_0(x)\mathcal{O}(x) \,. \tag{3.31}$$

Here we used that a scalar field is invariant under a change of coordinates. This derivation identifies Δ_+ as the scaling dimension of the boundary operator and, written out explicitly, we thus have

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 L^2}\,. \tag{3.32}$$

We see that the mass of the bulk field determines the scaling dimension of its dual field theory operator and can distinguish the following three cases:

$$\begin{aligned} m^2 > 0 \,\Rightarrow\, \Delta > 0 &\quad \text{irrelevant operator } \mathcal{O} \text{ in the IR} \\ m^2 = 0 \,\Rightarrow\, \Delta = 0 &\quad \text{marginal operator } \mathcal{O} \text{ in the IR} \\ m^2 < 0 \,\Rightarrow\, \Delta < 0 &\quad \text{relevant operator } \mathcal{O} \text{ in the IR}\,. \end{aligned} \tag{3.33}$$

Note that in order to have a relevant operator in the CFT we have to choose a negative mass squared for the bulk field in AdS spacetime. In fact, imposing the scaling dimension to be real, we find the Breitenlohmer-Freedman bound

$$m^2 \geq -\frac{d^2}{4L^2}\,. \tag{3.34}$$

This bound allows slightly negative values for the mass squared which may sound weird as a negative mass squared is usually associated with tachyonic instabilities. However, in contrast to flat spacetimes, negative mass squared solutions are allowed, because AdS spacetimes include a gravitational potential. An easy way to see this is to consider the scalar action (3.3) for a scalar field $\phi = \phi(z)$, redefine the field $\phi \to z^{d/2}\phi$ and make a change of variables $\zeta = \ln z$ as this gives us

$$S_{\text{scalar}} = -\frac{L^{d-1}}{2}\int_{\text{AdS}} d^d x\, dz\, \frac{1}{z^{d+1}}\left[z^2\, \partial_z\phi\, \partial_z\phi + m^2 L^2 \phi^2\right] \tag{3.35}$$

$$= -\frac{L^{d-1}}{2}\int_{\text{AdS}} d^d x\, d\zeta \left[\partial_\zeta\phi\, \partial_\zeta\phi + \left(\frac{d^2}{4} + m^2 L^2\right)\phi^2\right] \tag{3.36}$$

with the metric

$$ds^2 = L^2\left(\eta_{\mu\nu} dx^\mu dx^\nu + d\zeta^2\right)\,. \tag{3.37}$$

Interpreting (3.36) as a scalar field in flat spacetime, we retain stability as long as the Breitenlohmer-Freedman bound is satisfied.

Interesting to note is that the Breitenlohmer-Freedman bound implies for the scaling dimension $\Delta \geq d/2$. However, from the perspective of the CFT we have the unitarity bound, the minimum scaling behaviour of the conformal scaling operator. This gives us the restriction $\Delta > d/2-1$. Describing field theory operators through the AdS/CFT correspondence, we are then left with the question what to do with operators admitting a scaling dimension $d/2-1 < \Delta < d/2$.

To answer this, we have to look at the norm of the scalar field solution. The inner product for two scalar fields $\phi^{(1)}$ and $\phi^{(2)}$ in curved spacetime is given by

$$\langle \phi^{(1)}, \phi^{(2)} \rangle = -i \int_{\{\text{fixed } t\}} d^{d-1}x \, dz \, \sqrt{-g} g^{00} \left(\phi^{(1)} \partial_0 \phi^{(2)} - \phi^{(2)} \partial_0 \phi^{(1)} \right) . \tag{3.38}$$

When we consider a solution of the form $\phi(x,z) \sim z^{\Delta}$, the integrand goes as $z^{2\Delta-d+1}$. Hence the integral is convergent as long as $\Delta > d/2-1$, which we notice to be the same restriction given by the unitarity bound. For the solution (3.24), this means that the term proportional to $z^{\Delta_+} = z^{d/2+\nu}$ is always normalizable whereas the term proportional to $z^{\Delta_-} = z^{d/2-\nu}$ is normalizable only when $0 \leq \nu < 1$ and non-normalizable when $\nu \geq 1$. In case we have a non-normalizable mode, we are bound to choose this as our source. However, in the range $0 \leq \nu < 1$, both modes are normalizable and we are allowed to choose which term corresponds to the source and which to the one-point function.

Naturally, this choice has deep implications on the boundary theory since it corresponds to a choice of operators with different scaling dimensions. In particular, keeping the term with z^{Δ_-} as the source, corresponding to a scaling dimension Δ_+ of $\mathcal{O}$, is called standard quantization. In alternative quantization, the term with z^{Δ_+} acts as the source and the scaling dimension of the operator $\mathcal{O}$ is given by Δ_-, which is bounded below exactly by the unitarity bound.

3.3 Exact solution

To get the explicit expressions for the one-point function (3.21) and two-point function (3.23), we have to determine the coefficients of the expansion (3.15) by solving the differential equation (3.9) exactly. Measuring z in units of L, the solution for spacelike k (that is $k^2 = -\omega^2 + \boldsymbol{k}^2 > 0$) can be written as

$$\phi(k,z) = C_K \, z^{d/2} K_\nu \left(\sqrt{-\omega^2 + \boldsymbol{k}^2} z \right) + C_I \, z^{d/2} I_\nu \left(\sqrt{-\omega^2 + \boldsymbol{k}^2} z \right) , \tag{3.39}$$

where $K_\nu(x)$ and $I_\nu(x)$ are modified Bessel functions of the second kind. In the IR limit $z \to \infty$, their behaviour is given by

$$K_\nu \left(\sqrt{-\omega^2 + \boldsymbol{k}^2} z \right) \sim e^{-\sqrt{-\omega^2+\boldsymbol{k}^2} z} \quad \text{and} \quad I_\nu \left(\sqrt{-\omega^2 + \boldsymbol{k}^2} z \right) \sim e^{\sqrt{-\omega^2+\boldsymbol{k}^2} z} . \tag{3.40}$$

Therefore we have to set $C_I = 0$ to ensure regularity of the solution at the horizon. For timelike k (that is $k^2 = -\omega^2 + \boldsymbol{k}^2 < 0$) the solution can be written as

$$\phi(k,z) = C_{\text{out}}\, z^{d/2} H_\nu^{(1)}\left(\sqrt{\omega^2 - \boldsymbol{k}^2} z\right) + C_{\text{in}}\, z^{d/2} H_\nu^{(2)}\left(\sqrt{\omega^2 - \boldsymbol{k}^2} z\right) , \tag{3.41}$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are Hankel functions of the first and second kind respectively. In the IR limit $z \to \infty$, their behaviour is given by

$$H_\nu^{(1)}\left(\sqrt{\omega^2 - \boldsymbol{k}^2} z\right) \sim e^{-i\sqrt{\omega^2-\boldsymbol{k}^2} z} \quad \text{and} \quad H_\nu^{(2)}\left(\sqrt{\omega^2 - \boldsymbol{k}^2} z\right) \sim e^{i\sqrt{\omega^2-\boldsymbol{k}^2} z} . \tag{3.42}$$

We can now either choose the infalling boundary condition for which $C_{\text{out}} = 0$ or the outgoing boundary condition for which $C_{\text{in}} = 0$. Of course, this choice affects the Green's function that is subsequently found. Since the IR limit considers the horizon term and things tend to fall into black holes, it physically makes more sense to go for the infalling boundary condition. Doing this will yield the retarded Green's function, describing causal processes. Taking the outgoing boundary condition yields the advanced Green's function, describing anti-causal processes.

Upon expanding the Bessel and Hankel functions in the UV limit $z \to 0$ as

$$K_\nu\left(\sqrt{-\omega^2 + \boldsymbol{k}^2} z\right) = \left(\frac{\sqrt{-\omega^2 + \boldsymbol{k}^2} z}{2}\right)^{-\nu} \frac{\Gamma(\nu)}{2} + \left(\frac{\sqrt{-\omega^2 + \boldsymbol{k}^2} z}{2}\right)^{\nu} \frac{\Gamma(-\nu)}{2} + \dots \tag{3.43}$$

$$H_\nu^{(1)}\left(\sqrt{\omega^2 - \boldsymbol{k}^2} z\right) = \left(\frac{\sqrt{\omega^2 - \boldsymbol{k}^2} z}{2}\right)^{-\nu} \left(-\frac{i\Gamma(\nu)}{\pi}\right) + \left(\frac{\sqrt{\omega^2 - \boldsymbol{k}^2} z}{2}\right)^{\nu} \left(-\frac{i\Gamma(-\nu)}{\pi}\, e^{-i\pi\nu}\right) + \dots \tag{3.44}$$

$$H_\nu^{(2)}\left(\sqrt{\omega^2 - \boldsymbol{k}^2} z\right) = \left(\frac{\sqrt{\omega^2 - \boldsymbol{k}^2} z}{2}\right)^{-\nu} \left(\frac{i\Gamma(\nu)}{\pi}\right) + \left(\frac{\sqrt{\omega^2 - \boldsymbol{k}^2} z}{2}\right)^{\nu} \left(\frac{i\Gamma(-\nu)}{\pi}\, e^{i\pi\nu}\right) + \dots , \tag{3.45}$$

we obtain the Green's function

$$G(\omega, \boldsymbol{k}) = \begin{cases} \nu\left(\sqrt{-\omega^2 + \boldsymbol{k}^2}/2\right)^{2\nu} \Gamma(-\nu)/\Gamma(\nu) & k^2 > 0 \\ \nu\left(\sqrt{\omega^2 - \boldsymbol{k}^2}/2\right)^{2\nu} e^{-i\pi\nu}\, \Gamma(-\nu)/\Gamma(\nu) & k^2 < 0 \quad \text{outgoing} \\ \nu\left(\sqrt{\omega^2 - \boldsymbol{k}^2}/2\right)^{2\nu} e^{i\pi\nu}\, \Gamma(-\nu)/\Gamma(\nu) & k^2 < 0 \quad \text{infalling}\,. \end{cases} \tag{3.46}$$

On Fourier transforming the solution for timelike k with infalling boundary conditions back to position space, we find precisely the scaling dependence on the conformal dimension $\Delta = \Delta_+$ expected of a two-point correlation function as in equation (A.5):

$$G(x,y) = i\theta(x_0 - y_0)\,\frac{\nu\Gamma(\Delta)}{\pi^{d/2}|x-y|^{2\Delta}}\,. \tag{3.47}$$

4 Holographic spinor fields

Having seen the example of the scalar field in chapter 3 we turn our attention to spinor fields. Of course, this is a necessary step for describing the holographic nodal sphere in chapter 5. Additionally, this chapter functions as a second example of the AdS/CFT correspondence.

Section 4.1 will review the description of fermions in flat spacetime. After, we generalize the Dirac equation to a curved spacetime in section 4.2 as is needed for a holographic description of spinors in section 4.3. Section 4.1 is a summarized version of a more complete description found in [10, 11], whereas section 4.2 closely follows [12]. Section 4.3 is inspired by [13], but other references on holographic spinors are given by [14, 15].

4.1 Spinors in flat spacetime

In finding a description of fermions, our first goal is to realize a representation of the Lorentz group corresponding to spin 1/2, which we can do using a trick due to Dirac. Say we were given a set of $n \times n$ matrices γ^μ, called gamma or Dirac matrices, satisfying the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_n \,, \tag{4.1}$$

known in general as the Clifford algebra, and in the particular case of $n = 4$ as the Dirac algebra. Here $\eta^{\mu\nu}$ is written in the mostly plus convention. This allows us to write down an n-dimensional representation of the Lorentz algebra

$$\Sigma^{\mu\nu} = \frac{1}{4}\left[\gamma^\mu, \gamma^\nu\right] \,. \tag{4.2}$$

These can be checked to obey the commutation rules of the Lorentz algebra

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = \eta^{\nu\rho}\Sigma^{\mu\sigma} - \eta^{\mu\rho}\Sigma^{\nu\sigma} - \eta^{\nu\sigma}\Sigma^{\mu\rho} + \eta^{\mu\sigma}\Sigma^{\nu\rho} \,. \tag{4.3}$$

For a four-dimensional Minkowski spacetime, these Dirac matrices γ^μ must be at least 4×4. In addition, all 4×4 representations of the Dirac algebra (4.1) are unitarily equivalent meaning we just have to find one explicit realization of the Dirac algebra. The representation we will use, called the Weyl or chiral representation, is given by

$$\gamma^0 = \begin{bmatrix} 0 & -\mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{bmatrix} \quad \text{and} \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix} . \tag{4.4}$$

Here σ^i denote the usual Pauli matrices. A four-component field ψ that transforms under boosts according to Σ^{0i} and under rotations according to Σ^{ij} is called a Dirac spinor. With the transformation law for such a Dirac spinor, we should look for an appropriate field equation. This turns out to be the Dirac equation

$$\left(\partial\!\!\!/ - m\right) \psi = 0 , \tag{4.5}$$

where the slash corresponds to a contraction with the Dirac matrices. Furthermore, to write down a Lagrangian for the Dirac theory, we must figure out how to multiply two Dirac spinors to form a Lorentz scalar. The way to do this, is by multiplying a Dirac spinor ψ with its Dirac adjoint $\bar{\psi} \equiv \psi^\dagger \gamma^0$. This then gives us the Dirac Lagrangian

$$\mathcal{L}_{\text{Dirac}} = i\bar{\psi} \left(\partial\!\!\!/ - m\right) \psi . \tag{4.6}$$

Weyl spinors

By simply working out the generators Σ^{0i} and Σ^{ij} and finding that they have a block diagonal structure, we infer that the Dirac representation of the Lorentz group is reducible. When we consider each block separately, it becomes possible to write down two two-dimensional representations. Writing

$$\psi = \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} , \tag{4.7}$$

where the two-component objects ψ_+ and ψ_- are called Weyl or chiral spinors, the Dirac equation reads

$$\left(\partial\!\!\!/ - m\right) \psi = \begin{bmatrix} -m & -\partial_0 + \sigma^i \partial_i \\ \partial_0 + \sigma^i \partial_i & -m \end{bmatrix} \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = 0 . \tag{4.8}$$

Here we note that the two Weyl spinors ψ_+ and ψ_- are coupled through the mass. If we set $m = 0$, the equations for ψ_+ and ψ_- decouple, yielding the Weyl equations

$$\left(\partial_0 + \sigma^i \partial_i\right) \psi_+ = 0 \quad \text{and} \quad \left(-\partial_0 + \sigma^i \partial_i\right) \psi_- = 0 \,. \tag{4.9}$$

The reason that the generators Σ^{0i} and Σ^{ij} came out in block diagonal form is the specific representation (4.4) that we chose. In fact, this is why it is called the chiral representation. However, choosing a different representation of the Clifford algebra, we would still like to be able to define chiral spinors. We can do this by introducing the fifth gamma matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \,, \tag{4.10}$$

which, together with the other gamma matrices, can be seen to satisfy the Clifford algebra (4.1). With γ^5 we can form the Lorentz invariant projection operators

$$P_R = \frac{1}{2}\left(\mathbb{1}_4 + \gamma^5\right) \quad \text{and} \quad P_L = \frac{1}{2}\left(\mathbb{1}_4 - \gamma^5\right) \,. \tag{4.11}$$

The projection operators give us the four-component right-handed spinor $\psi_R \equiv P_R\psi$ and left-handed spinor $\psi_L \equiv P_L\psi$ in any representation. In the chiral representation, γ^5 takes the form

$$\gamma^5 = \begin{bmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{bmatrix} \tag{4.12}$$

for which we then have that

$$\psi_R = \begin{bmatrix} \psi_+ \\ 0 \end{bmatrix} \quad \text{and} \quad \psi_L = \begin{bmatrix} 0 \\ \psi_- \end{bmatrix} \,. \tag{4.13}$$

4.2 Spinors in curved spacetime

As we are ultimately interested in describing spinors holographically, we first need to know how to describe spinors in curved spacetime. This turns out to be somewhat subtle, because of a different transformation property than usual. In GR we know that under a change of coordinates $x^\mu \to x'^\mu$, a vector field V^μ is transformed by

$$V^\mu \to V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu \,. \tag{4.14}$$

Here the matrix $\partial x'^\mu / \partial x^\nu$ is in general an element of $\mathrm{GL}(n, \mathbb{R})$, the group of invertible real $n \times n$ matrices. The vector V^μ transforms in the fundamental vector representation of this group as is required for the coupling of gravity to matter. Spinors, however, form a representation of $\mathrm{SO}(n)$. Although a $\mathrm{GL}(n, \mathbb{R})$ representation always gives an $\mathrm{SO}(n)$ representation by restriction, the converse is not true; spinors form a representation of $\mathrm{SO}(n)$ that does not arise from a representation of $\mathrm{GL}(n, \mathbb{R})$. Therefore, to couple spinors to gravity, we need to find a framework in which the transformation matrix $\partial x'^\mu / \partial x^\nu$ is replaced by an $\mathrm{SO}(n)$ matrix.

A necessary first step is to introduce at each point x of our spacetime, a basis of orthonormal tangent vectors $e^a_\mu(x)$ with $a \in \{1, \ldots, n\}$. Here μ is an index labelling the components of a vector tangent to the spacetime at the point x and a is just the name of the vector $e^a_\mu(x)$. The vectors e^a_μ form a basis for the tangent space. This basis is called a vielbein or in four dimensions, a vierbein or tetrad. The index a of the vielbein is raised and lowered with the Minkowski metric η_{ab} while the spacetime index μ is raised and lowered with the metric tensor $g_{\mu\nu}$. The vielbeins satisfy $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ and the inverse vielbeins satisfy $e^\mu_a e^a_\nu = \delta^\mu_\nu$ and $e^\mu_a e^b_\mu = \delta^b_a$.

There is a large arbitrariness in the choice of a vielbein. All are equivalent up to a local Lorentz transformation $\tilde{e}^a_\mu(x) = \Lambda^a_b(x) e^b_\mu(x)$ where $\Lambda^a_b(x)$ is an arbitrary spacetime dependent Lorentz transformation. It is the local Lorentz transformation $\Lambda^a_b(x)$ that eventually replaces the $\mathrm{GL}(n, \mathbb{R})$ matrix. We must ensure that our formalism is invariant under local Lorentz transformations so that physical observables are independent of the arbitrary choice of vielbein. As with any local gauge invariance, such a local Lorentz invariance is achieved by introducing a gauge field $\omega^a_{\mu b}(x)$, called the spin connection.

We would like to find a minimal choice of the spin connection with the property that introducing the vielbein and spin connection together does not change the content of GR. Concretely, the notion of taking a covariant derivative should not change depending on whether we write down a vector with its vector indices or its vielbein indices. This is because the V^a contain the same information as V^μ since we can always reconstruct $V^\mu = e^\mu_a V^a$. The covariant derivative of a vector field V^ν is usually defined in GR by saying that

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{4.15}$$

with $\Gamma^\nu_{\mu\lambda}$ denoting the Christoffel connection. In terms of V^a, the natural covariant derivative would be

$$\nabla_\mu V^a = \partial_\mu V^a + \omega^a_{\mu b} V^b \tag{4.16}$$

with $\omega^a_{\mu b}$ denoting the spin connection. If we are to avoid modifying the content of GR, the two notions of taking a covariant derivative of a vector must be equivalent. Since $\nabla_\mu V^a = e^{a\nu} \nabla_\mu V_\nu$, this will be the case if we define the spin connection in such a way that the covariant derivative of the vielbein

$$\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda + \omega^a_{\mu b} e^b_\nu \tag{4.17}$$

will be zero. It follows that

$$\omega^a_{\mu b} = e^a_\nu e^\lambda_b \Gamma^\nu_{\mu\lambda} - e^\lambda_b \partial_\mu e^a_\lambda \,. \tag{4.18}$$

With the spin connection we can describe spinors in curved spacetime. As with any gauge field, the spin connection can be coupled to a field ψ in any required representation of the gauge group. In this case, the gauge group is the Lorentz group and we take ψ to be a field in the spinor representation of the Lorentz group. The covariant derivative of ψ is defined as

$$\nabla_\mu \psi = \partial_\mu \psi + \Omega_\mu \psi \,, \tag{4.19}$$

where Ω_μ is given in the standard way as

$$\Omega_\mu = \frac{1}{2}\omega_{\mu ab}\Sigma^{ab} = \frac{1}{8}\omega_{\mu ab}[\gamma^a, \gamma^b] \,. \tag{4.20}$$

In order to define the Dirac equation, we next need gamma matrices in curved spacetime. This is done straightforwardly by contracting the flat spacetime gamma matrices with the vielbeins:

$$\Gamma^\mu(x) = e^\mu_a(x)\gamma^a \,. \tag{4.21}$$

The curved spacetime gamma matrices obey $\{\Gamma^\mu(x), \Gamma^\nu(x)\} = 2g^{\mu\nu}(x)$. Thus, finally, the Dirac equation in curved spacetime reads

$$\left(\not{\nabla} - m\right)\psi = 0 \,, \tag{4.22}$$

where the slash corresponds to a contraction with the curved spacetime gamma matrices.

4.3 Spinors in holography

With the Dirac equation in curved spacetime (4.22), we are ready to consider spinors in the context of the AdS/CFT correspondence. First of all, we set the spatial dimension of our gravity theory $d = 4$ as this not only simplifies our discussion but also corresponds to a CFT of dimension $3+1$, resembling many interesting condensed matter systems. Then the action of the AdS spacetime is given by

$$S_{\mathrm{AdS}} = S_{\mathrm{background}} + S_{\mathrm{Weyl}} \tag{4.23}$$

$$S_{\mathrm{background}} = \int_{\mathrm{AdS}} d^4x\, dz\, \sqrt{-g}\, [12 + R] \tag{4.24}$$

$$S_{\mathrm{Weyl}} = i g_f \int_{\mathrm{AdS}} d^4x\, dz\, \sqrt{-g}\, \left[\bar{\psi}\left(\nabla\!\!\!/ - m\right)\psi\right] + S_\partial\,. \tag{4.25}$$

From the background action we get our metric

$$ds^2 = \frac{L^2}{z^2}\left(\eta_{\mu\nu}\, dx^\mu\, dx^\nu + dz^2\right)\,. \tag{4.26}$$

Although the integrand of the Weyl action is the Dirac Lagrangian, we call the action a Weyl action (and not a Dirac action) as it will describe a Weyl fermion (and not a Dirac fermion) in the CFT. The slash corresponds with a contraction under $\Gamma^M(x) = e^M_{\underline{N}}(x)\Gamma^{\underline{N}}$, where we note that we will underline indices referring to flat spacetime. Because d is even, we have that $\Gamma^{\underline{\mu}} = \gamma^{\underline{\mu}}$ and $\Gamma^{\underline{z}} = \gamma^{\underline{5}} = i\gamma^{\underline{0}}\gamma^{\underline{1}}\gamma^{\underline{2}}\gamma^{\underline{3}}$. We work in the Weyl representation and, for convenience, we again give the Dirac matrices in this representation:

$$\Gamma^{\underline{0}} = \begin{bmatrix} 0 & -\mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{bmatrix}, \quad \Gamma^{\underline{i}} = \begin{bmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{bmatrix} \quad \text{and} \quad \Gamma^{\underline{z}} = \begin{bmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{bmatrix} \quad \text{with} \tag{4.27}$$

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \text{and} \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{the Pauli matrices.} \tag{4.28}$$

Following section 4.2, the covariant derivative is given by

$$\nabla_M = \partial_M + \Omega_M = \partial_M + \frac{1}{8}\omega_{M\underline{AB}}\left[\Gamma^{\underline{A}}, \Gamma^{\underline{B}}\right]\,. \tag{4.29}$$

Finally, we need S_∂ in (4.25) for a well-defined variational principle. To see why, consider the variation of the action as in section 2.3 which, neglecting bulk terms, yields

$$\delta S_{\text{Weyl}} = -ig_f \int_{\partial \text{AdS}} d^4x \sqrt{-h}\, n_M \left[\bar{\psi}\, \Gamma^M \delta\psi\right] + \delta S_\partial \tag{4.30}$$

$$= ig_f \int_{z=\varepsilon} d^4x \sqrt{-h g_{zz}} \left[\bar{\psi}\, \Gamma^z \delta\psi\right] - ig_f \int_{z=\infty} d^4x \sqrt{-h g_{zz}} \left[\bar{\psi}\, \Gamma^z \delta\psi\right] + \delta S_\partial \,. \tag{4.31}$$

The problem we have now, is that $\delta\psi(x, z = 0)$ can not be set to zero. The underlying reason is that the Dirac equation in the bulk imposes a relation between the two chiral components found on the boundary. Therefore, we must choose between a source $\psi_R(x)$ for which $\delta\psi_R(x, z = 0) = 0$ and a source $\psi_L(x)$ for which $\delta\psi_L(x, z = 0) = 0$. This choice is arbitrary and relates to changing the sign of the mass as will become evident later. Writing out equation (4.31) further, discarding the horizon term, gives us

$$\delta S_{\text{Weyl}} = ig_f \int_{z=\varepsilon} d^4x \sqrt{-h} \left[\bar{\psi}\, \Gamma^{\underline{z}} \delta\psi\right] + \delta S_\partial \tag{4.32}$$

$$= ig_f \int_{z=\varepsilon} d^4x \sqrt{-h} \left[\psi_-^\dagger \delta\psi_+ + \psi_+^\dagger \delta\psi_-\right] + \delta S_\partial \,. \tag{4.33}$$

Imposing boundary conditions $\delta\psi_R(x, z = 0) = 0$, i.e., $\delta\psi_+(x, z = 0) = 0$, such that the source corresponds with the boundary value of the chiral spinor ψ_+, we find for the boundary term in the action

$$S_\partial = -ig_f \int_{z=\varepsilon} d^4x \sqrt{-h} \left[\psi_+^\dagger \psi_-\right] \tag{4.34}$$

$$= ig_f \int_{z=\varepsilon} d^4x \sqrt{-h} \left[\bar{\psi}_R \psi_L\right] \,. \tag{4.35}$$

Here we mean $\bar{\psi}_R = \psi_R^\dagger \Gamma^{\underline{0}}$. Imposing boundary conditions $\delta\psi_L(x, z = 0) = 0$, i.e., $\delta\psi_-(x, z = 0) = 0$, such that the source corresponds with the boundary value of the chiral spinor ψ_-, we find for the boundary term in the action

$$S_\partial = -ig_f \int_{z=\varepsilon} d^4x \sqrt{-h} \left[\psi_-^\dagger \psi_+\right] \tag{4.36}$$

$$= -ig_f \int_{z=\varepsilon} d^4x \sqrt{-h} \left[\bar{\psi}_L \psi_R\right] \,. \tag{4.37}$$

Here we again mean $\bar{\psi}_L = \psi_L^\dagger \Gamma^{\underline{0}}$. As said before, the choice depends on the sign of the mass. For now, let us impose $\delta\psi_L(x, z = 0) = 0$ corresponding to a positive mass sign, such that the Weyl action reads

$$S_{\text{Weyl}} = ig_f \int_{\text{AdS}} d^4x\, dz\, \sqrt{-g}\, \left[\bar{\psi}\left(\not{\nabla} - m\right)\psi\right] - ig_f \int_{z=\varepsilon} d^4x\, \sqrt{-h}\, \left[\bar{\psi}_L \psi_R\right] . \tag{4.38}$$

Having defined the action of our system, we can proceed by giving the equation of motion

$$\left(\not{\nabla} - m\right)\psi = 0\,, \tag{4.39}$$

which with the metric (4.26) and using the identities (B.24) and (B.25) becomes

$$\left(\not{\partial} - \frac{2}{L}\,\Gamma^{\underline{z}} - m\right)\psi = 0\,. \tag{4.40}$$

Here the slash denotes a contraction with the flat spacetime gamma matrices of the non-radial coordinates. Taking advantage of translational invariance $x^\mu \to x^\mu + a^\mu$ we Fourier transform $\psi(x, z)$ in the non-radial coordinates

$$\psi(x, z) = \int \frac{d^dk}{(2\pi)^d}\, e^{ik_\mu x^\mu}\, \psi(k, z)\,, \tag{4.41}$$

where $k_\mu = (-\omega, \boldsymbol{k})$. From this, we obtain the differential equation governing the dependence of the Fourier coefficients $\psi(k, z)$ on the radial coordinate z:

$$\Gamma^{\underline{z}}\,\psi' + \left(\frac{i\not{k}z - 2\Gamma^{\underline{z}} - mL}{z}\right)\psi = 0\,. \tag{4.42}$$

The prime denotes differentiation with respect to the radial coordinate. We try solutions of the form

$$\psi(k, z) = \left(\frac{z}{L}\right)^r \sum_{n=0}^{\infty} f_n(k)\left(\frac{z}{L}\right)^n . \tag{4.43}$$

Looking only at the lowest-order terms, we find that for the asymptotic $(z \to 0)$ solution we must have

$$(r-2)\,\Gamma^{\underline{z}}\, f_0 - mLf_0 = 0\,. \tag{4.44}$$

The solutions for r are

$$\Delta_\pm = 2 \pm mL\,, \tag{4.45}$$

where for Δ_+ we need $P_L f_0 = 0$ and for Δ_- we need $P_R f_0 = 0$. Therefore

$$\psi(k, z \to 0) = \left(\psi_L(k)\left(\frac{z}{L}\right)^{\Delta_-} + \ldots\right) + \left(\psi_R(k)\left(\frac{z}{L}\right)^{\Delta_+} + \ldots\right) \tag{4.46}$$

$$= \psi_L(k)\left(\frac{z}{L}\right)^{\Delta_-} + \psi_R(k)\left(\frac{z}{L}\right)^{\Delta_+} + \ldots\,, \tag{4.47}$$

where ψ_L is the leading term and ψ_R the subleading term, as long as the mass is positive. We assumed no terms exist between the leading and subleading term. We now see that ψ_L indeed corresponds to the source for positive masses, meaning we chose the right boundary term (4.37) in the action (4.38). For negative masses, ψ_R would be the leading term and we would have to choose the other boundary action (4.35). However, as we have seen in the example of the holographic scalar field, within a certain mass range we have the possibility of going into alternative quantization. In alternative quantization, the source would be ψ_R for positive masses and ψ_L for negative masses. This then also implies that we need to choose the other boundary action in the Weyl action. That there are these different options to choose from, is something we will make use of when holographically describing a Dirac fermion in the CFT in chapter 5. As for now, with our choice of positive mass and standard quantization, the on-shell action is given by

$$S_{\text{Weyl}}^{\text{on shell}} = -ig_f \int_{z=\varepsilon} \frac{d^4k}{(2\pi)^4}\left(\frac{L}{z}\right)^4 \bar{\psi}_L(k,z)\psi_R(k,z) \tag{4.48}$$

$$= -ig_f \int \frac{d^4k}{(2\pi)^4}\,\bar{\psi}_L(k)\psi_R(k)\,. \tag{4.49}$$

Applying the GKPW rule with the adjoint of the source, we then find

$$\langle \mathcal{O}(k)\rangle = -ig_f\psi_R(-k)\,. \tag{4.50}$$

The retarded Green's function with infalling boundary conditions can be obtained from this as it is worked out to give [2]

$$G(\omega, \boldsymbol{k}) = \frac{1}{g_m e^{-i\pi(m+1/2)} p^{2m+1}} \left(\omega \mathbb{1}_2 + \boldsymbol{\sigma} \cdot \boldsymbol{k}\right) , \tag{4.51}$$

where

$$g_m = g_f 2^{-2m} \frac{\Gamma(1/2 - m)}{\Gamma(1/2 + m)} \varepsilon^{-2m+2} \quad \text{and} \quad p = \sqrt{\omega^2 - \boldsymbol{k}^2} \, . \tag{4.52}$$

5 Holographic nodal spheres

In the previous chapter we have seen how to obtain the holographic Green's function for a spinor field. In this chapter we will use this knowledge to obtain the Green's function for a theory corresponding to holographic nodal surfaces.

In section 5.1 we will make our model more physical by putting the system at a nonzero temperature and chemical potential through the introduction of a Reissner-Nordström black hole in the background action. In section 5.2 we will add a second spinor to the matter action in order to describe a Dirac spinor, not just a chiral spinor, on the boundary. This will give us a first result, but will come with a flaw as its full behaviour can not be compared to experiment. We propose a solution in section 5.3 by introducing an action defined only on the UV brane of the AdS theory. The whole chapter is based on previous work from [16].

5.1 Thermodynamic variables

If we want to put our system at a nonzero temperature and chemical potential, then for a positive mass and in standard quantization we have to consider the following action (see appendix B.2)

$$S_{\text{AdS}} = S_{\text{background}} + S_{\text{Weyl}} \tag{5.1}$$

$$S_{\text{background}} = \int_{\text{AdS}} d^4x\, dz\, \sqrt{-g}\left[12 + R - \frac{1}{4}F^2\right] \tag{5.2}$$

$$S_{\text{Weyl}} = ig_f \int_{\text{AdS}} d^4x\, dz\, \sqrt{-g}\left[\bar{\psi}\left(\not{D} - m\right)\psi\right] - ig_f \int_{z=\varepsilon} d^4x\, \sqrt{-h}\left[\bar{\psi}_L \psi_R\right] , \tag{5.3}$$

where in the background action we have the vacuum permeability μ_0 and we have introduced the gauge potential A^M with $F_{MN} = \partial_M A_N - \partial_N A_M$. This allows us to put charge on the black hole. In the Weyl action we then have $D_M = \nabla_M - iqA_M$ with ∇_M the spinor covariant derivative and q the fermion bulk charge, so that the chemical potential of the spinor is $\mu = qA_0(0)$. It follows, from the background action, that the metric we consider is

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + d\boldsymbol{x}^2 + \frac{dz^2}{f(z)}\right) \tag{5.4}$$

with

$$f(z) = 1 - \left(1 + \frac{2z_h^2}{3L^2}\left(A_0^{(0)}\right)^2\right)\left(\frac{z}{z_h}\right)^d + \frac{2z_h^2}{3L^2}\left(A_0^{(0)}\right)^2\left(\frac{z}{z_h}\right)^{2(d-1)} . \tag{5.5}$$

In the previous chapter, the problem of finding the holographic Green's function was very similar. The only, computational, difference is the new and more complicated metric given by equations (5.4) and (5.5). In fact, this prohibits us from finding an analytical solution, so we have to turn to numerics.

Because of this, we will adapt our strategy and aim to find an equation for the holographic Green's function itself. This equation we then solve numerically, which has the benefit of immediately giving us our object of interest. Of course, the following discussion then also holds for an AdS spacetime where $f(z) = 1$.

We start by noting that in writing out the Dirac equation in chiral components, we can relate the chiral components as

$$\psi_+(k, z) = -i\xi(k, z)\psi_-(k, z) . \tag{5.6}$$

As a consequence, the Weyl action (5.3) in momentum space taken on shell can be written as

$$S_{\text{Weyl}}^{\text{on shell}} = -g_f \int_{z=\varepsilon} \frac{d^4k}{(2\pi)^4}\sqrt{-h}\,\psi_-^\dagger(k, z)\xi(k, z)\psi_-(k, z) . \tag{5.7}$$

From the above action it is clear that ξ is proportional to the holographic Green's function for the chiral boundary operator that is sourced by the boundary value of the chiral spinor ψ_-. Using infalling boundary conditions, the holographic retarded Green's function follows from

$$G(k) = \lim_{z\to 0} z^{-2m}\xi(k, z) . \tag{5.8}$$

We thus need to find a differential equation for ξ such that we can directly get the Green's function by numerically solving for ξ. Before we do so, we first rescale the spinor to reduce the spinor covariant derivative to a partial derivative and give the subsequent equations of motion for the chiral components.

To get rid of the spinor connection, note that because the only non-vanishing components of the spin connection are $\omega_{0\underline{0}\underline{z}} = -\omega_{0\underline{z}\underline{0}}$ and $\omega_{i\underline{i}\underline{z}} = -\omega_{i\underline{z}\underline{i}}$ the spinor covariant derivative can be written as

$$\nabla\!\!\!/\,\psi = \partial\!\!\!/\,\psi + F(z)\Gamma^{\underline{z}}\,\psi , \tag{5.9}$$

where F is a function depending on the radial coordinate only. Defining $p(z) \equiv \exp\left(-\int^z dz'\, F(z')\right)$ we get that

$$\nabla\!\!\!/ (p\psi) = p\, \partial\!\!\!/ \psi\,. \tag{5.10}$$

Hence, the rescaling $\psi \to p\psi$ gets rid of the spin connection terms in the Dirac equation, which now becomes

$$\left(\partial\!\!\!/ - iqA\!\!\!/ - m\right)\psi = 0\,. \tag{5.11}$$

Moreover, this rescaling does not affect the matrix ξ. In momentum space this yields

$$\left(i\Gamma^\mu k_\mu + \Gamma^z \partial_z - iq\Gamma^0 A_0 - m\right)\psi = 0\,, \tag{5.12}$$

where $k_\mu = (-\omega, \boldsymbol{k})$. Writing ψ in its chiral components, we find that

$$\left(e^z_{\underline{z}}\partial_z - m\right)\psi_+ + i\left(e^0_{\underline{0}}\left(\omega + qA_0\right) + e^i_{\underline{i}}\boldsymbol{k}\cdot\boldsymbol{\sigma}\right)\psi_- = 0\,, \tag{5.13}$$

$$\left(e^z_{\underline{z}}\partial_z + m\right)\psi_- + i\left(e^0_{\underline{0}}\left(\omega + qA_0\right) - e^i_{\underline{i}}\boldsymbol{k}\cdot\boldsymbol{\sigma}\right)\psi_+ = 0\,, \tag{5.14}$$

where $\boldsymbol{\sigma}$ is the vector of Pauli matrices. At this point, we are ready to derive the differential equation for ξ. Taking the derivative of $\psi_+ = -i\xi\psi_-$ gives us

$$ie^z_{\underline{z}}\partial_z\psi_+ - \left(e^z_{\underline{z}}\partial_z\xi\right)\psi_- - \xi\left(e^z_{\underline{z}}\partial_z\psi_-\right) = 0\,. \tag{5.15}$$

First replacing derivatives of fields with the equations of motion (5.13) and (5.14), then using the substitution $\psi_+ = -i\xi\psi_-$ to eliminate ψ_+, we get the differential equation for ξ

$$\left(e^z_{\underline{z}}\partial_z - 2m\right)\xi - \xi\left(e^0_{\underline{0}}\left(\omega + qA_0\right) - e^i_{\underline{i}}\boldsymbol{k}\cdot\boldsymbol{\sigma}\right)\xi - \left(e^0_{\underline{0}}\left(\omega + qA_0\right) + e^i_{\underline{i}}\boldsymbol{k}\cdot\boldsymbol{\sigma}\right) = 0\,. \tag{5.16}$$

We can reduce the amount of equations we need to solve by exploiting rotational symmetry to set $k_\mu = (-\omega, 0, 0, k_3)$. By symmetry, we can write down the matrix structure of ξ as

$$\xi = \xi_3 \sigma^3 + \xi_c \mathbb{1}_2 \tag{5.17}$$

$$= \begin{bmatrix} \xi_c + \xi_3 & 0 \\ 0 & \xi_c - \xi_3 \end{bmatrix} . \tag{5.18}$$

This shows that there are only two degrees of freedom for which we have to solve. It is more insightful to write the equation (5.16) in terms of $\xi_\pm \equiv \xi_c \pm \xi_3$, which after some work yields

$$\left(e^z_{\underline{z}}\partial_z - 2m\right)\xi_\pm - \left(e^0_{\underline{0}}\left(\omega + qA_0\right) \mp e^3_{\underline{3}}k_3\right)\xi_\pm^2 - \left(e^0_{\underline{0}}\left(\omega + qA_0\right) \pm e^3_{\underline{3}}k_3\right) = 0\,. \tag{5.19}$$

These equations can be solved numerically to obtain the matrix ξ. As they are first-order ordinary differential equations, we need to impose one initial condition for each component. Since only $e^0_{\underline{0}}$ diverges at the horizon, we demand that in both equations the coefficient of this factor vanishes at the horizon. The equation (5.19) then gives that $\xi_+(k, z_h) = \pm i$ and $\xi_-(k, z_h) = \pm i$. Infalling boundary conditions require that we choose $\xi_+(k, z_h) = \xi_-(k, z_h) = i$ [4].

5.2 Holographic Green's function

In section 5.1 we have seen how to describe a Weyl fermion on the boundary at nonzero temperature and chemical potential. However, in order to describe a Dirac fermion, we will somehow need to introduce both chiralities on the boundary. This is done by defining two bulk spinors $\psi^{(1)}$ and $\psi^{(2)}$ with which we would like to derive an action similar to (5.7), but this time with four-component spinors.

Using the Dirichlet boundary conditions $\delta\psi_L^{(1)} = 0$ and $\delta\psi_R^{(2)} = 0$, we can derive such an action that contains the two chiral fermions $\psi_-^{(1)}$ and $\psi_+^{(2)}$ as sources. For $m > 0$ and in standard quantization we need to consider the following action

$$S_{\text{AdS}} = S_{\text{background}} + S_{\text{Dirac}} \tag{5.20}$$

$$S_{\text{background}} = \int_{\text{AdS}} d^4x\, dz\, \sqrt{-g}\left[12 + R - \frac{1}{4}F^2\right] \tag{5.21}$$

$$\begin{aligned} S_{\text{Dirac}} = ig_f \int_{\text{AdS}} d^4x\, dz\, \sqrt{-g}\left[\bar{\psi}^{(1)}\left(\not{D}^{(1)} - m\right)\psi^{(1)} + \bar{\psi}^{(2)}\left(\not{D}^{(2)} + m\right)\psi^{(2)}\right] \\ + ig_f \int_{z=\varepsilon} d^4x\, \sqrt{-h}\left[-\bar{\psi}_L^{(1)}\psi_R^{(1)} + \bar{\psi}_R^{(2)}\psi_L^{(2)}\right] , \end{aligned} \tag{5.22}$$

where in the Dirac action we have the derivative operators $D^{(1)}_M = \nabla_M - iq_1 A_M$ and $D^{(2)}_M = \nabla_M - iq_2 A_M$ corresponding to the two different fermion bulk charges q_1 and q_2. These different charges result in the two different chemical potentials $\mu_1 = q_1 A_0(0)$ and $\mu_2 = q_2 A^{(0)}_0$. Changing the charge parameters in the theory will then effectively allow us to shift the two Weyl cones in energy.

Proceeding as before, we get our metric from the background action

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + d\boldsymbol{x}^2 + \frac{dz^2}{f(z)}\right), \tag{5.23}$$

with

$$f(z) = 1 - \left(1 + \frac{2z_h^2}{3L^2}\left(A_0^{(0)}\right)^2\right)\left(\frac{z}{z_h}\right)^d + \frac{2z_h^2}{3L^2}\left(A_0^{(0)}\right)^2\left(\frac{z}{z_h}\right)^{2(d-1)}. \tag{5.24}$$

To obtain the on-shell Dirac action and the equations of motion, we now perform the generalized substitution

$$\eta(k,z) = -i\Xi(k,z)\chi(k,z) \quad \text{where} \quad \eta = \begin{bmatrix}\psi_+^{(1)}\\ -\psi_-^{(2)}\end{bmatrix},\ \chi = \begin{bmatrix}\psi_+^{(2)}\\ \psi_-^{(1)}\end{bmatrix} \quad \text{and} \quad \Xi = \begin{bmatrix}0 & \xi^{(1)}\\ \xi^{(2)} & 0\end{bmatrix}. \tag{5.25}$$

We note that with our choice of Dirichlet boundary conditions, the Dirac spinor χ contains the sources. This way we get for the on-shell Dirac action

$$S^{\text{on shell}}_{\text{Dirac}} = -g_f \int_{z=\varepsilon} \frac{d^4k}{(2\pi)^4}\sqrt{-h}\,\bar{\chi}\,\Xi\,\chi \tag{5.26}$$

so that, using infalling boundary conditions, the holographic Green's function is given by

$$G(k) = \lim_{z\to 0} z^{-2m}\,\Gamma^{\underline{0}}\,\Xi(k,z)\,. \tag{5.27}$$

Analogously to equations (5.18) and (5.19), the equations of motion for the matrix Ξ are given by

$$\left(e^z_{\underline{z}}\partial_z - 2m\right)\xi^{(1)}_\pm - \left(e^0_{\underline{0}}\left(\omega + q_1 A_0\right) \mp e^3_{\underline{3}}k_3\right)\left(\xi^{(1)}_\pm\right)^2 - \left(e^0_{\underline{0}}\left(\omega + q_1 A_0\right) \pm e^3_{\underline{3}}k_3\right) = 0 \tag{5.28}$$

$$\left(e^z_{\underline{z}}\partial_z - 2m\right)\xi^{(2)}_\pm + \left(e^0_{\underline{0}}\left(\omega + q_2 A_0\right) \mp e^3_{\underline{3}}k_3\right)\left(\xi^{(2)}_\pm\right)^2 + \left(e^0_{\underline{0}}\left(\omega + q_2 A_0\right) \pm e^3_{\underline{3}}k_3\right) = 0\,, \tag{5.29}$$

where thus $\xi_{\pm}^{(1)} = \xi_c^{(1)} \pm \xi_3^{(1)}$ and $\xi_{\pm}^{(2)} = \xi_c^{(2)} \pm \xi_3^{(2)}$. Infalling boundary conditions require that $\xi_{\pm}^{(1)}(k, z_h) = i$ and $\xi_{\pm}^{(2)}(k, z_h) = -i$.

The holographic spectral function

We are now able to obtain the holographic Green's function (5.27) of our theory by numerically solving equations (5.28) and (5.29). Given a Green's function $G(\omega, \boldsymbol{k})$, we can compute the spectral function defined as

$$\rho(\omega, \boldsymbol{k}) = \frac{1}{\pi} \operatorname{Im} \operatorname{Tr} G(\omega, \boldsymbol{k}) \,. \tag{5.30}$$

The spectral function can be considered as a generalized density of states, giving the probability that a particle of momentum $\boldsymbol{k}$ carries an energy ω.

Experimentally, the spectral function is an important quantity. Performing spectroscopy on a material, for example in angular resolved photoemission spectroscopy (ARPES) experiments or scanning tunneling spectroscopy (STS) experiments, the quantity you measure can be related to the spectral function.

With the holographic Green's function (5.27) we can get the holographic spectral function of our theory. In figure 6 we see the result of the holographic spectral function in standard quantization for $q_1 = 1$ and $q_2 = -1$ at $T/\mu = 0.01$. The two Weyl cones, somewhat vaguely present because of all the filled states, are shifted in energy space according to the charges q_1 and q_2. In figure 7 we have plotted the spectral function as a function of momentum at the Fermi level. There clearly are two peaks, corresponding to two nodal points. By rotational symmetry in momentum space, these points will constitute a nodal sphere. Figures 8 and 9 concern the alternative quantization, all other parameters being equal. They give the same result as the standard quantization, only now the two Weyl cones are more clearly visible since the states not on the cone are left unoccupied.

We also give the holographic spectral function in standard (figure 10) and alternative (figure 11) quantization for $q_1 = 3$ and $q_2 = -2$ at $T/\mu = 0.01$. Around the origin, we see some nontrivial behaviour in the existence of multiple levels of Fermi surfaces. These appearing bands have been found before by lowering temperature or, equivalently, increasing the chemical potential in [4]. They are presumed to be the result of fermionic bound states.

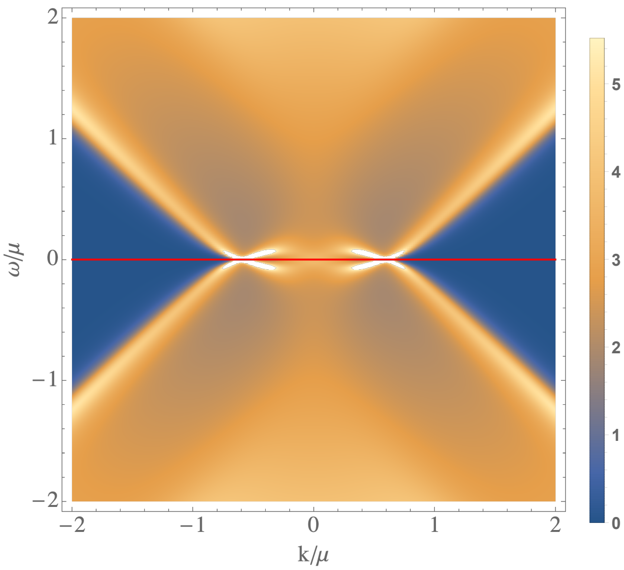

Figure 6: *The holographic spectral function in standard quantization for equal but opposite charges.* The parameters set for this result are $q_1 = 1$, $q_2 = -1$ and $T/\mu = 0.01$. The horizontal red line corresponds to the Fermi level.

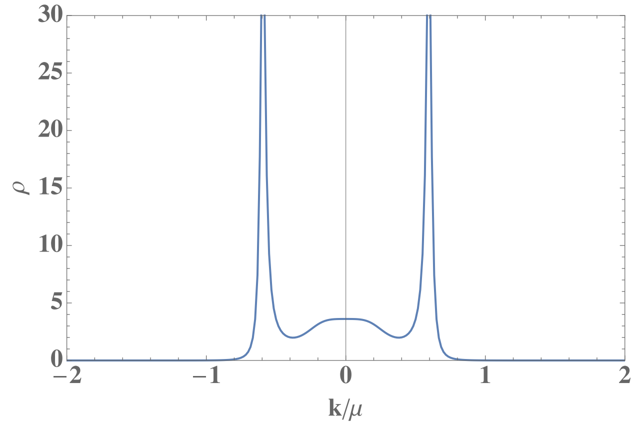

Figure 7: *The holographic spectral function in standard quantization for equal but opposite charges at the Fermi level.* The parameters correspond to those of figure 6. The nodal points are clearly visible as peaks in this function.

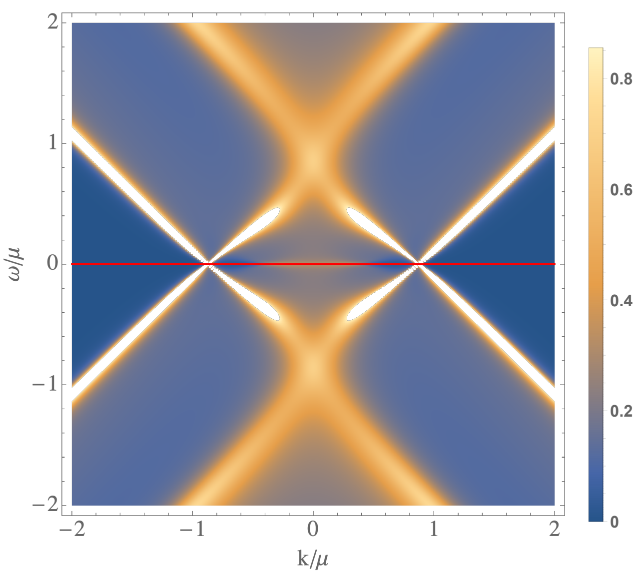

Figure 8: *The holographic spectral function in alternative quantization for equal but opposite charges.* The parameters set for this result are $q_1 = 1$, $q_2 = -1$ and $T/\mu = 0.01$. The horizontal red line corresponds to the Fermi level.

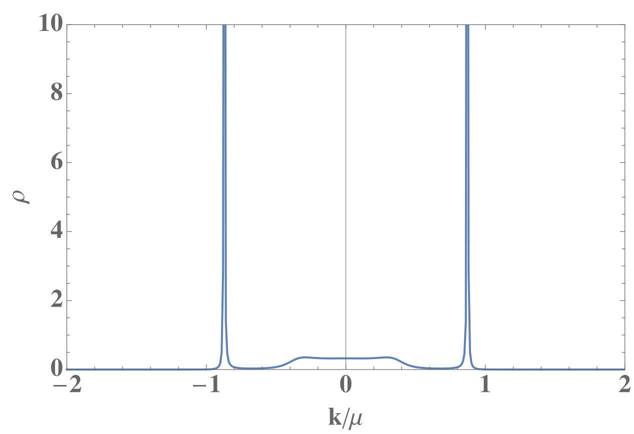

Figure 9: *The holographic spectral function in alternative quantization for equal but opposite charges at the Fermi level.* The parameters correspond to those of figure 8. The nodal points are clearly visible as peaks in this function.

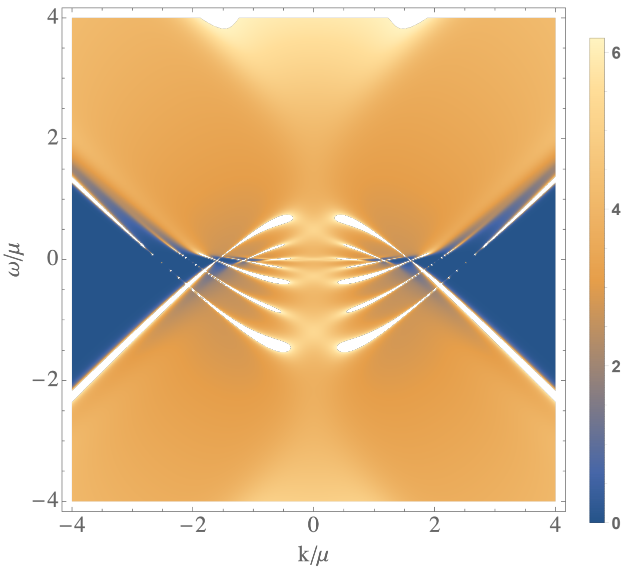

Figure 10: *The holographic spectral function in standard quantization for unequal charges.* The parameters set for this result are $q_1 = 3$, $q_2 = -2$ and $T/\mu = 0.01$. Around the origin we see multiple bands appear.

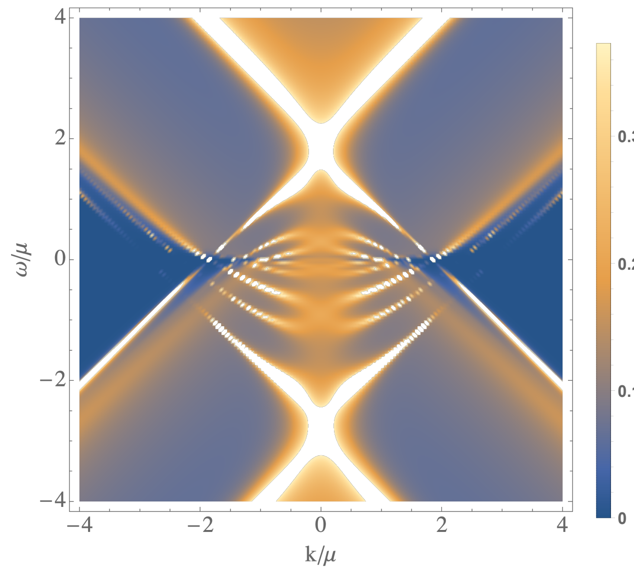

Figure 11: *The holographic spectral function in alternative quantization for unequal charges.* The parameters set for this result are $q_1 = 3$, $q_2 = -2$ and $T/\mu = 0.01$. Around the origin we see multiple bands appear.

Unfortunately, the results we have thus far shown come with a problem. The spectral function of an elementary Dirac fermion, which is a measurable quantity containing information about the spectrum of fermion dynamics, obeys the sum rule

$$\int_{-\infty}^{\infty} d\omega\, \rho(\omega, \boldsymbol{k}) = 4\,, \tag{5.31}$$

where the right-hand side equals the number of degrees of freedom (which for a Dirac fermion equals four). The holographic spectral functions from figures 6 to 11 do not adhere to equation (5.31), whereas we would like them to be in point of view of experiment.

This problem is inherent to the holographic procedure. It boils down to the fact that classical computations in gravity provide correlation functions of operators in a strongly coupled CFT. This strong coupling reduces the full Green's function of the theory, given by the Dyson equation

$$G = \frac{1}{G_0^{-1} + \Sigma}\,, \tag{5.32}$$

to just the self-energy Σ. Thus, by taking the limit where gravity becomes classical, necessary to do any holographic computation at all, the free kinetic term corresponding to the bare propagator G_0 is suppressed. In order to have a spectral function obeying the sum rule, we will have to recover this bare propagator. A suggestion on how to do this, is made by what is called semiholography.

5.3 Semiholographic Green's function

We would like to come up with a semiholographic Green's function satisfying the sum rule

$$\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega \, \mathrm{Im}\, \mathrm{Tr}\, G_{\mathrm{semi}}(\omega, \boldsymbol{k}) = 4 \,. \tag{5.33}$$

In order to recover the kinetic part of the Green's function, we make the source χ from (5.25) dynamical by adding to the AdS action (5.20) the free action defined on the UV brane

$$S_{\mathrm{UV}} = iZ \int_{z=\varepsilon} d^4x \, \sqrt{-h} \left[\bar{\chi} \left(\partial\!\!\!/ - \tilde{M} \right) \chi \right] \,. \tag{5.34}$$

Here Z is a proportionality factor and $\tilde{M}$ is the mass of the source spinor. Introducing this action is allowed, in the sense that the variational principle remains unchanged, precisely because we chose the Dirichlet boundary condition corresponding to $\delta\chi = 0$. Its role is to provide the right UV dynamics while being an irrelevant perturbation to the CFT. With S_{UV}, the total effective boundary action becomes

$$S_{\mathrm{eff}} = -\varepsilon Z \int_{z=\varepsilon} \frac{d^4k}{(2\pi)^4} \sqrt{-h} \left[\bar{\chi} \left(\Gamma^{\underline{\mu}} k_\mu + \frac{i\tilde{M}}{\varepsilon} + \frac{g_f}{\varepsilon Z}\, \Xi \right) \chi \right] , \tag{5.35}$$

where we used that $e^{\mu}_{\underline{\nu}}(\varepsilon) \approx \varepsilon \delta^{\mu}_{\underline{\nu}}$ (see equations (B.21) to (B.23)). By noting that the kinetic term becomes canonically normalized upon rescaling the fields $\chi \to \chi/\sqrt{\varepsilon Z \sqrt{-h}}$, we can take the limit $\varepsilon \to 0$ while letting

$$\begin{aligned} g_f \to 0 \quad &\text{such that} \quad g \equiv g_f \varepsilon^{2m-1}/Z = \text{constant} \\ \tilde{M} \to 0 \quad &\text{such that} \quad M \equiv \tilde{M}/\varepsilon = \text{constant} \,. \end{aligned} \tag{5.36}$$

The effective action then reads

$$S_{\mathrm{eff}} = \int \frac{d^4k}{(2\pi)^4} \, \chi^\dagger \, G^{-1}_{\mathrm{semi}} \, \chi \,. \tag{5.37}$$

Here the inverse of the semiholographic Green's function is given by

$$G^{-1}_{\text{semi}}(k) = \begin{bmatrix} \sigma^\mu k_\mu & iM\mathbb{1}_2 \\ -iM\mathbb{1}_2 & -\bar{\sigma}^\mu k_\mu \end{bmatrix} - \Sigma\,, \tag{5.38}$$

where $\sigma^\mu = (\mathbb{1}_2, \sigma^i)$ and $\bar{\sigma}^\mu = (-\mathbb{1}_2, \sigma^i)$ and where we have defined the self-energy

$$\Sigma(k) \equiv g \lim_{z\to 0} z^{-2m}\, \Gamma^{\underline{0}}\, \Xi(k,z)\,. \tag{5.39}$$

By using once again rotational symmetry to set $k_\mu = (-\omega, 0, 0, k_3)$ we are able to numerically compute the semiholographic Green's function (5.38).

The semiholographic spectral function

The semiholographic Green's function indeed satisfies the sum rule such that

$$\int_{-\infty}^{\infty} d\omega\, \rho_{\text{semi}}(\omega, \boldsymbol{k}) = 4 \quad \text{with} \quad \rho_{\text{semi}} = \frac{1}{\pi}\, \text{Im}\, \text{Tr}\, G_{\text{semi}}(\omega, \boldsymbol{k})\,. \tag{5.40}$$

The reason for this is the following. By adding the UV action (5.34) to our theory, we have essentially coupled an elementary Dirac fermion living in a (3+1)-dimensional Minkowski spacetime to a CFT (described by a dual gravity theory). By elementary we mean that the fermionic creation and annihilation operators satisfy the canonical equal-time anticommutation relations

$$\left\{\hat{\xi}(t, \boldsymbol{x}), \hat{\xi}^\dagger(t, \boldsymbol{x}')\right\} = \delta^3(\boldsymbol{x}, \boldsymbol{x}')\,. \tag{5.41}$$

As a consequence, the resulting retarded Green's function of the elementary fermionic operators defined by

$$G(x - x') = -i\theta(t - t')\left\langle \left\{\hat{\xi}(x), \hat{\xi}^\dagger(x')\right\}\right\rangle \tag{5.42}$$

satisfies the zeroth-order frequency sum rule [4]

$$\int_{-\infty}^{\infty} d\omega\, \text{Im}\, G(\omega^+, \boldsymbol{k}) = -\pi\,. \tag{5.43}$$

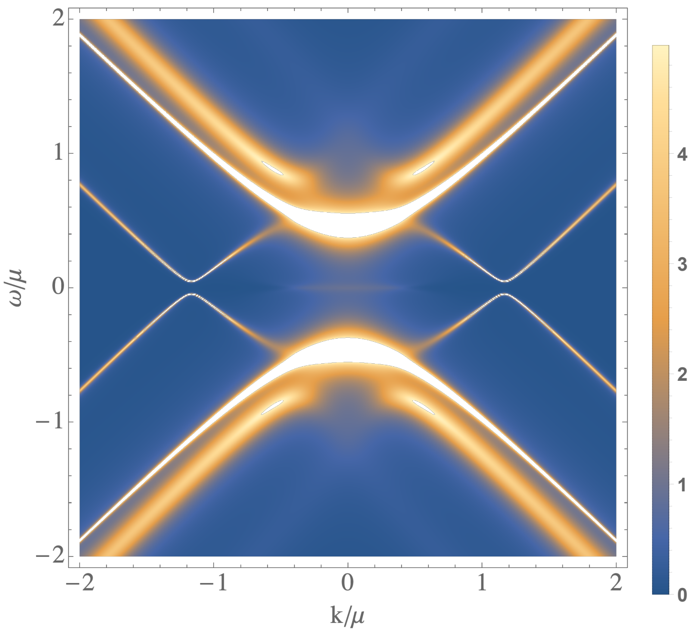

Figure 12: *The semiholographic spectral function in alternative quantization for equal but opposite charges.* The parameters set for this result are $q_1 = 1$, $q_2 = -1$, $T/\mu = 0.01$, $g = 1$ and $M = 1$. The nodal points appear to have been gapped away in this case.

This means that the semiholographic Green's function (5.38) is experimentally accessible and that experimental results obtained by spectroscopy can in principle be compared with the semiholographic spectral function (5.40).

In figure 12 we see the semiholographic spectral function in alternative quantization for $q_1 = 1$ and $q_2 = -1$ at $T/\mu = 0.01$. Also we have set $g = 1$ and $M = 1$. In addition to the two Weyl cones, there now exists some gapped Fermi surface corresponding to the dynamics of the source fermion. The fermion interacts with the CFT nontrivially and the nodal points seem to be gapped away for this choice of parameters.

6 Conclusion

In chapter 1 we started out with the goal of describing nodal spheres holographically. First we developed a necessary understanding of the AdS/CFT correspondence in chapter 2, after which we saw the correspondence in action in chapter 3 for scalar fields. In order to describe nodal spheres, we needed a holographic description of spinors as was given in chapter 4. Finally, in chapter 5, we were ready to give our model and obtained some results. We now discuss our results in section 6.1 and give suggestions on possible further research in section 6.2.

6.1 Discussion

Perhaps most notably, we have the appearance of the multiple Fermi surfaces around the origin when the Weyl cones are split further apart. As mentioned before, they have been seen already in [4]. They might be interpreted as multiple-particle bound states corresponding to molecule formation, which are structured in the same way, but this would require further study.

What might also be interesting to study, is the spin structure of the semiholographic spectral function. For the holographic spectral function it is clear that the Fermi surfaces for which $\omega = +v_{\rm F}k$ should have opposite spin states as should the Fermi surfaces for which $\omega = -v_{\rm F}k$. What happens to this spin structure in the semiholographic picture is yet unclear.

Finally, we have refrained from studying the dependence of the model on its parameters. We might wonder what happens to the semiholographic spectral function when changing the coupling constant g or the mass

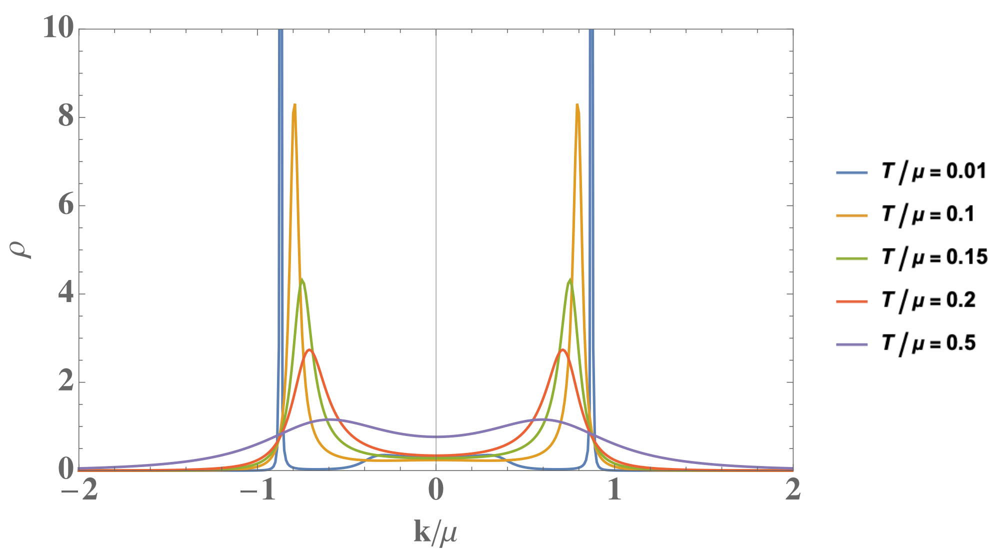

Figure 13: *The temperature dependence of the holographic spectral function in alternative quantization for equal but opposite charges at the Fermi level.* The parameters set for this result are $q_1 = 1$ and $q_2 = -1$ with varying T/μ.

M. Also, we could look more closely into the role of temperature for the holographic spectral function. From figure 13 it becomes clear that by increasing the temperature, the peaks broaden out. Finding precise relations between their width and the temperature is left for further research.

6.2 Outlook

Besides studying the results our model yields, there is also room to work on the model itself. To start, it might be interesting to introduce a magnetic field by considering a nonzero spatial part of the gauge potential. This would allow for quantum oscillations, which are used experimentally to determine the Fermi surface of a metal. In extension, it would be useful to introduce a more physical background action and consider, for example, Lifshitz backgrounds.

Finally, we refer back to the introduction in which we said that a nodal loop had been experimentally realized. To theoretically describe nodal loops, we somehow need to break the symmetry of our nodal sphere. This might be done through the introduction of axion fields in the background action, which has been suggested as a way to cause spontaneous symmetry breaking in other models. Also, it might be possible to break the symmetry by letting the mass of the source fermion be dependent on the momentum.

A Conformal field theories

A conformal field theory is a theory that is invariant under conformal transformations. A conformal transformation is a change of coordinates such that the metric changes according to

$$g_{\mu\nu}(x) \to \Omega(x)^2 g_{\mu\nu}(x)\,. \tag{A.1}$$

Notice that the set of conformal symmetries contains scale transforations, corresponding to $\Omega(x)$ being constant. Taking $\Omega(x) = 1$ we obtain the isometries of the field theory. For Minkowski spacetimes $g_{\mu\nu}(x) = \eta_{\mu\nu}$ and the isometries are given by the usual Poincaré group. Additionally, the field theory has the property of being scale invariant.

Although conformal invariance is a stronger requirement, the most commonly studied QFTs that present scale invariance are also invariant under the full conformal group. The full set of conformal transformations consists of

$$\begin{aligned} \text{translations} \quad & x'^{\mu} = x^{\mu} + a^{\mu} \\ \text{rigid rotations} \quad & x'^{\mu} = M^{\mu}_{\nu} x^{\nu} \\ \text{dilations} \quad & x'^{\mu} = C x^{\mu} \\ \text{special conformal transformations} \quad & x'^{\mu} = \left(x^{\mu} - b^{\mu} x^2\right) / \left(1 - 2 b_{\mu} x^{\mu} - b^2 x^2\right)\,. \end{aligned} \tag{A.2}$$

From this we can deduce the number of generators in d dimensions. There are d generators of translation and $d(d-1)/2$ generators of rotation. For dilations, the number of generators is obviously equal to one. For the special conformal transformations the number of generators is equal to d, making a total number of $(d+1)(d+2)/2$ generators (cf. the number of Killing vectors of an AdS spacetime in appendix B).

All these symmetries impose significant restrictions on the form of the correlation functions of the field theory. In particular, an operator $\mathcal{O}(x)$ in a scale invariant theory transforms under a dilation $x' = \lambda x$ as

$$\mathcal{O}(x) \to \mathcal{O}'(x') = \lambda^{-\Delta} \mathcal{O}(x)\,, \tag{A.3}$$

where Δ is called the scaling or conformal dimension. Using the invariance under dilations, the two-point function of two scalar conformal operators $\mathcal{O}_1$ and $\mathcal{O}_2$ with scaling dimensions Δ_1 and Δ_2 respectively, transforms as

$$\langle \mathcal{O}_1(\lambda x) \mathcal{O}_2(\lambda x) \rangle = \lambda^{-\Delta_1} \lambda^{-\Delta_2} \langle \mathcal{O}_1(x) \mathcal{O}_2(x) \rangle\,. \tag{A.4}$$

Moreover, due to Poincaré invariance, the two-point function $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle$ can only depend on $(x_1 - x_2)^2$ from which we then obtain

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{C}{|x_1 - x_2|^{\Delta_1+\Delta_2}}\,. \tag{A.5}$$

The full conformal invariance imposes additional constraints. In particular, by performing an inversion $x'^{\mu} = x^{\mu}/x^2$ (which is implicitly present in a special conformal transformation as a special conformal transformation is equivalent to an inversion followed by a translation followed by an inversion, but where we note that the inversion itself is not necessarily a symmetry of the theory), we see that the correlation function is zero unless both fields have the same scaling dimension.

B Anti-de Sitter spacetimes

In this appendix we will discuss anti-de Sitter spacetimes. Naturally, understanding these spacetimes is useful in understanding the AdS/CFT correspondence. In section B.1 we will introduce the maximally symmetric AdS spacetime and next in section B.2, we will consider spacetimes that are asymptotically AdS. Throughout we make use of [9]. For section B.1 we also looked at [6] and in the discussion of the chemical potential in section B.2 we closely followed [17].

B.1 AdS spacetimes

We give a brief introduction to the geometry of AdS spacetimes. We classify a spacetime through its symmetries, which are the coordinate transformations that leave the metric invariant. Hence each spacetime symmetry has its own Killing vector. Since under a general change of coordinates $\delta x^\mu = \xi^\mu(x)$ the metric transforms as $\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ we get that a Killing vector K^μ is a vector such that

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0\,. \tag{B.1}$$

The simplest spacetimes are those with the most symmetries and thus the most Killing vectors. These spacetimes are called maximally symmetric. Because a d-dimensional symmetric metric has $d(d+1)/2$ independent components, maximally symmetric spacetimes are those yielding $d(d+1)/2$ Killing vectors. Together the Killing vectors form a group. It turns out there are only three possibilities. These are $\mathrm{SO}(d,1)$ corresponding to a d-dimensional de Sitter spacetime, $\mathrm{SO}(d-1,1)$ corresponding to a d-dimensional Minkowski spacetime and $\mathrm{SO}(d-2,2)$ corresponding to a d-dimensional anti-de Sitter spacetime.

The simplicity of maximally symmetric spacetimes is reflected through their curvature. Since these spacetimes look the same at every point, the curvature can not depend on derivatives. This implies we can write the Riemann tensor as

$$R_{\mu\nu\rho\sigma} = C\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)\,, \tag{B.2}$$

where C is a constant. Contracting with the metric twice, we obtain the Ricci scalar

$$R = Cd\left(d-1\right)\,. \tag{B.3}$$

The three possible maximally symmetric spacetimes then correspond to the Ricci scalar being positive, zero or negative respectively. To fix the constant C, we contract the vacuum Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(R - 2\Lambda \right) = 0 \,, \tag{B.4}$$

where Λ is the cosmological constant. This gives us for $d > 2$ that

$$R = \frac{2d}{d-2} \, \Lambda \tag{B.5}$$

and thus the Riemann tensor for maximally symmetric spacetimes is given by

$$R_{\mu\nu\rho\sigma} = \frac{2\Lambda}{(d-1)(d-2)} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right) \,. \tag{B.6}$$

Equivalently then, a maximally symmetric spacetime is determined completely by the cosmological constant. Note that in Euclidean signature the solution with positive cosmological constant corresponds to a d-dimensional sphere, the solution with zero cosmological constant to $\mathbb{R}^d$ and the solution with negative cosmological constant to a hyperboloid.

Since a hyperboloid in d-dimensional Euclidean space is given by the solution to

$$-X_{-1}^2 + X_0^2 + \ldots + X_{d-1}^2 = -L^2 \,, \tag{B.7}$$

an anti-de Sitter spacetime of dimension d is defined as the solution to

$$-X_{-1}^2 - X_0^2 + \ldots + X_{d-1}^2 = -L^2 \,. \tag{B.8}$$

Here the positive constant L is called the AdS radius. A solution of (B.8) is given by

$$\begin{aligned}
X_{-1} &= L\cosh\rho\cos\tau \\
X_0 &= L\cosh\rho\sin\tau \\
X_1 &= L\sinh\rho\cos\theta_1 \\
X_2 &= L\sinh\rho\sin\theta_1\cos\theta_2 \\
&\vdots \\
X_{d-2} &= L\sinh\rho\sin\theta_1\cdots\sin\theta_{d-3}\cos\theta_{d-2} \\
X_{d-1} &= L\sinh\rho\sin\theta_1\cdots\sin\theta_{d-2}\,.
\end{aligned} \tag{B.9}$$

This gives the metric

$$ds^2 = L^2\left(-\cosh^2\rho\,d\tau^2 + d\rho^2 + \sinh^2\rho\,d\Omega_{d-2}^2\right)\,, \tag{B.10}$$

where $d\Omega_{d-2}^2$ is the metric on the $(d-2)$-dimensional sphere. This metric with $\rho \in \mathbb{R}_{>0}$ and $\tau \in [0, 2\pi]$ covers the Minkowski hyperboloid exactly once, but gives closed timelike curves because of the periodic time coordinate. However, this is not a property of the spacetime itself but just of our choice of the solution. The AdS spacetime is then defined as the universal cover of this where we take $\tau \in \mathbb{R}$ and there are no closed timelike curves. Since these coordinates cover all of the spacetime, they are called global.

By making the coordinate change $\rho = \operatorname{arcsinh}\tan\tau$ with $\tau \in [0, \pi/2]$, we can better understand the topology of AdS spacetime. The metric now reads

$$ds^2 = \frac{L^2}{\cos^2\theta}\left(d\theta^2 - d\tau^2 + \sin^2\theta\,d\Omega_{d-2}^2\right)\,. \tag{B.11}$$

This is topologically equivalent to a cylinder with a radial direction given by θ and a longitudinal direction given by τ and where each point on this cylinder is a $(d-2)$-dimensional sphere. The metric is multiplied by a conformal factor dependent on θ. Note that at infinity, corresponding to $\theta = \pi/2$, the AdS spacetime has a timelike boundary.

There is another set of coordinates useful to describe AdS spacetime. Given a vector x^μ of dimension $d-1$ and a coordinate $r \in \mathbb{R}_{>0} \cup \infty$ we can write a solution to (B.8) as

$$
\begin{aligned}
X_{-1} &= \frac{1}{2r}\left(1 + r^2\left(L^2 + x_0^2 - \sum_i x_i^2\right)\right) \\
X_0 &= Lrx_0 \\
X_1 &= Lrx_1 \\
&\vdots \\
X_{d-2} &= Lrx_{d-2} \\
X_{d-1} &= \frac{1}{2r}\left(1 - r^2\left(L^2 + x_0^2 - \sum_i x_i^2\right)\right) .
\end{aligned}
\tag{B.12}
$$

This gives the metric

$$
ds^2 = L^2\left(r^2\left(\eta_{\mu\nu}dx^\mu dx^\nu\right) + \frac{dr^2}{r^2}\right) , \tag{B.13}
$$

which is called the Poincaré patch. We note that it covers only half of the hyperboloid. Taking spacetime slices of constant r, we simply get the Minkowski spacetime metric multiplied by a warp factor r^2. By moving in the radial direction, the warp factor effectively rescales all lengths in a Minkowski slice by a factor of r.

Moving from the boundary, which is the surface at $r = \infty$, to the interior of the AdS spacetime, the length scale of the Minkowski spacetime slices becomes increasingly larger. The point at $r = 0$ is a horizon, since it corresponds to a surface where the Killing vector ∂_t has a norm equal to zero.

B.2 Asymptotically AdS spacetimes

In section 2.2 we have seen that there is a correspondence between CFTs of dimension d and asymptotically AdS spacetimes of dimension $d+1$. We argued that allowing black brane solutions is important because properties of the CFT can be described through a black brane and gave the example of an AdS Schwarzschild black brane being dual to a finite temperature. In terms of static and isotropic backgrounds there is not much more to be done with pure Einstein gravity. In order to describe more features of the dual field theory, we must add structure to the bulk theory.

In subsection B.2.1 we derive an expression for the Hawking temperature of a black hole and in subsection B.2.2 we add some structure to the bulk theory to describe CFTs with a chemical potential. We first focus on a general asymptotically AdS spacetime and give as a reference some identities that can be derived from the metric, which is given by

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)c^2dt^2 + d\boldsymbol{x}^2 + \frac{dz^2}{f(z)}\right) . \tag{B.14}$$

Here we must have $f(z) \to 1$ as $z \to 0$. Note that choosing $f(z) = 1$ simply gives us the metric for AdS_{d+1}, such that the following identities all hold for pure AdS spacetimes as well. The non-zero Christoffel symbols are given by

$$\Gamma^0_{0z} = \Gamma^0_{z0} = -\frac{1}{z} + \frac{f'(z)}{2f(z)} \tag{B.15}$$

$$\Gamma^i_{iz} = \Gamma^i_{zi} = -\frac{1}{z} \tag{B.16}$$

$$\Gamma^z_{00} = \frac{f(z)\left(-2f(z) + zf'(z)\right)}{2z} \tag{B.17}$$

$$\Gamma^z_{ii} = \frac{f(z)}{z} \tag{B.18}$$

$$\Gamma^z_{zz} = -\frac{1}{z} - \frac{f'(z)}{2f(z)} . \tag{B.19}$$

The unit normal in the radial direction, useful in determining the on-shell action as in section 2.3, is given by

$$n_M = \pm\sqrt{g_{zz}}\,\delta^z_M = \pm\frac{L}{z\sqrt{f(z)}}\,\delta^z_M , \tag{B.20}$$

where the sign determines whether it is the outward or inward pointing normal. We choose our vielbeins, introduced in section 4.2, to be

$$e^0_{\underline{0}} = \sqrt{-g^{00}} = \frac{z}{L\sqrt{f(z)}} \tag{B.21}$$

$$e^i_{\underline{i}} = \sqrt{g^{ii}} = \frac{z}{L} \tag{B.22}$$

$$e^z_{\underline{z}} = \sqrt{g^{zz}} = \frac{z\sqrt{f(z)}}{L} . \tag{B.23}$$

Finally the non-zero spin connections, introduced in section 4.2 as well, then read

$$\omega_{0\underline{0z}} = -\omega_{0\underline{z0}} = -e^{\underline{0}}_0\, e^z_{\underline{z}}\,\Gamma^0_{0z} = \frac{f(z)}{z} - \frac{f'(z)}{2} \tag{B.24}$$

$$\omega_{i\underline{iz}} = -\omega_{i\underline{zi}} = e^{\underline{i}}_i\, e^z_{\underline{z}}\,\Gamma^i_{iz} = -\frac{\sqrt{f(z)}}{z} . \tag{B.25}$$

For completeness we also give the relevant metrics in SI units:

$$\begin{aligned} &\text{AdS} \quad f(z) = 1 \\ &\text{Schwarzschild} \quad f(z) = 1 - \left(\frac{z}{z_h}\right)^d \\ &\text{Reissner-Nordström} \quad f(z) = 1 - \left(1 + \frac{8\pi G}{\mu_0 c^6}\frac{d-2}{d-1}\frac{z_h^2}{L^2}\left(A_0^{(0)}\right)^2\right)\left(\frac{z}{z_h}\right)^d + \frac{8\pi G}{\mu_0 c^6}\frac{d-2}{d-1}\frac{z_h^2}{L^2}\left(A_0^{(0)}\right)^2\left(\frac{z}{z_h}\right)^{2(d-1)} . \end{aligned}$$

B.2.1 Hawking temperature

We have seen that an AdS black hole metric introduces a black brane at $z = z_h$. Since we identified the radial direction with the energy scale, this black brane horizon is consistent with our intuition of the effect of a nonzero temperature in the boundary theory. Namely, turning on a temperature in a field theory sets an energy scale that breaks conformal invariance, modifying the IR physics. The effects of excitations with an energy lower than the scale set by the temperature are just modified by thermal excitations and only higher energy effects, corresponding in the gravitational dual to scale $z < z_h$, can be observed. On the other hand, for energies much higher than this scale, the theory is not sensitive to the effects of the finite temperature and we recover the CFT, that corresponds to the spacetime being asymptotically AdS for $z \to 0$.

There are several ways for computing the Hawking temperature of a black hole. Hawking first derived it by quantizing matter fields in a black hole background. Nonetheless, there is a simpler derivation that does not require field quantization. In a general background geometry the Hawking temperature can be computed by performing a Wick rotation to Euclidean gravity and requiring the black hole solution to be smooth.

The relation between a temperature and Euclidean time is known in QFT. To describe a system at a finite temperature T, we analytically continue to Euclidean signature ($t \to -i\tau$) and let τ be periodic:

$$\tau \sim \tau + \hbar\beta \,. \tag{B.26}$$

Conversely, if the Euclidean continuation of a QFT is periodic in the time direction then we can conclude that the QFT is at a finite temperature. A way to think about this analogy is the following. The partition function for a statistical system in thermal equilibrium in the grand canonical ensemble and in the Schrödinger picture at a constant time $t = 0$ reads

$$Z = \text{Tr}\left[e^{-\beta H}\right] = \sum_\psi \langle\psi(0)|\, e^{-\beta H}\, |\psi(0)\rangle \ , \tag{B.27}$$

where H is the Hamiltonian of the system and we are summing over a complete set of states. On the other hand, we know that time evolution of a state is given by $|\psi(t)\rangle = e^{-itH}|\psi(0)\rangle$. This allows us to think of the Boltzmann factor $e^{-\beta H}$ as a time evolution operator in imaginary time so that we can write

$$Z = \mathrm{Tr}\left[e^{-\beta H}\right] = \sum_{\psi} \langle\psi(0)|\psi(-i\beta)\rangle \; . \tag{B.28}$$

The left hand side now represents the vacuum amplitude as we are evolving the state from $\tau = 0$ to $\tau = \beta$. Requiring the final state to be the same as the initial state forces τ to be periodic. We can use this relation between temperature and Euclidean time to get an expression for the Hawking temperature.

We analytically continue the metric (EQ) with $t \to -i\tau$ to obtain the Euclidean metric

$$ds_E^2 = \frac{L^2}{z^2}\left(f(z)c^2 d\tau^2 + d\boldsymbol{x}^2 + \frac{dz^2}{f(z)}\right) \; . \tag{B.29}$$

We want to study the periodicity near the horizon. However, at the horizon we have $f(z_h) = 0$ meaning that the metric blows up there. Hence we Taylor expand $f(z)$ near the horizon giving us the near-horizon Euclidean metric

$$ds_E^2 \approx \frac{L^2}{z_h^2}\left(f'(z_h)\,(z - z_h)\,c^2 d\tau^2 + d\boldsymbol{x}^2 + \frac{dz^2}{f'(z_h)\,(z - z_h)}\right) \; . \tag{B.30}$$

If we now perform a coordinate change with

$$R = \frac{2L}{z_h}\sqrt{\frac{-\,(z - z_h)}{f'(z_h)}} \quad \text{and} \quad \theta = \frac{c}{2}\,f'(z_h)\,\tau \equiv K\tau \,, \tag{B.31}$$

the near-horizon Euclidean metric can be written as

$$ds_E^2 \approx -R^2 d\theta^2 + \frac{L^2}{z_h^2} d\boldsymbol{x}^2 - dR^2 \,. \tag{B.32}$$

We see that the time and the radial component are simply Euclidean polar coordinates in $\mathbb{R}^2$. This metric presents a conical singularity unless θ is periodic with period 2π, which implies that the Euclidean time must have the periodicity

$$\tau \sim \tau - \frac{2\pi}{K}\,. \tag{B.33}$$

From earlier consideration we had $\tau \sim \tau + \hbar\beta$ so that we get the expression for the Hawking temperature

$$T = \frac{1}{k_B\beta} = -\frac{\hbar K}{2\pi k_B} = -\frac{\hbar c}{4\pi k_B} f'(z_h)\,. \tag{B.34}$$

B.2.2 Chemical potential

In a condensed matter system we generally have a U(1) symmetry, for example the electromagnetic U(1) symmetry, present. In this subsection we will consider the gravitational dual of theories with a global U(1) symmetry. In processes where virtual photons are not important, the electromagnetic symmetry can be treated as a global symmetry.

By considering symmetries, one can deduce that there is a duality between global internal symmetries of the boundary field theory and local gauge symmetries of the bulk gravity theory. In order to give a description of such a theory, we need to consider the Einstein-Maxwell action as the background part of our AdS action:

$$S_{\text{background}} = \int_{\text{AdS}} d^d x\, dz\, \sqrt{-g} \left[\frac{c^3}{16\pi G}\left(-2\Lambda + R\right) - \frac{1}{4\mu_0 c} F^2\right]. \tag{B.35}$$

Here μ_0 is the vacuum permeability and $F_{MN} = \partial_M A_N - \partial_N A_M$ is the electromagnetic field strength.

Preserving rotational symmetry, we are now able to introduce two new background scales in thermal equilibrium. One is the chemical potential $\mu \propto A_0^{(0)}$ and the other, only preserving rotational symmetry in $(2+1)$-dimensions, is a background magnetic field $B = F_{12}^{(0)}$. Following the GKPW rule, we have to consider the boundary value of the bulk Maxwell potential

$$A_M(z \to 0) = A_M^{(0)} + \dots \tag{B.36}$$

In general, we need to look for solutions to Einstein-Maxwell theory of the form (B.14) with a gauge potential reading

$$A = A_0(z)\, dt + B(z) x\, dy\,. \tag{B.37}$$

However, we are not interested in the second term which will break the isotropy of the field theory unless our bulk theory only has three spatial dimensions. Considering no magnetic field and looking for solutions to the Einstein equation of motion as well as the Maxwell equation of motion in arbitrary dimension, we find the AdS Reissner-Nordström black hole for which in (B.14) we have

$$f(z) = 1 - \left(1 + \frac{8\pi G}{\mu_0 c^6} \frac{d-2}{d-1} \frac{z_h^2}{L^2} \left(A_0^{(0)}\right)^2\right) \left(\frac{z}{z_h}\right)^d + \frac{8\pi G}{\mu_0 c^6} \frac{d-2}{d-1} \frac{z_h^2}{L^2} \left(A_0^{(0)}\right)^2 \left(\frac{z}{z_h}\right)^{2(d-1)} . \tag{B.38}$$

The scalar potential is

$$A_0 = \frac{\mu}{q} \left[1 - \left(\frac{z}{z_h}\right)^{d-2}\right] . \tag{B.39}$$

C Units

Following [16], we discuss how we obtain dimensionless units. We consider the general metric

$$S = \int d^4x\, dz\, \sqrt{-g} \left[\frac{c^3}{16\pi G} \left(-2\Lambda + R\right) - \frac{1}{4\mu_0 c} F^2 - \frac{1}{2} \left((\partial\phi)^2 + \frac{m_\phi^2 c^2}{\hbar^2} \phi^2 \right) + i g_f \bar{\psi} \left(\not{D} - \frac{m_\psi c}{\hbar} \right) \psi \right]$$

where G is Newton's constant and μ_0 is the vacuum permeability, both in four spatial dimensions. The AdS radius, given by $L^2 = -6/\Lambda$, allows us to scale all lengths by L implying that $\Lambda = -6$. We have put the Boltzmann constant k_B equal to one. This means that all energies, for example $k_B T$ or $m_\phi c^2$, are in units of $\hbar c/L$. The dimensionless fields are then obtained by

$$\tilde{A}_0 = \sqrt{\frac{16\pi G}{\mu_0 c^6}} A_0 \tag{C.1}$$

$$\tilde{\phi} = \sqrt{\frac{16\pi G}{c^3}} \phi \tag{C.2}$$

$$\tilde{\psi} = \frac{\sqrt{\hbar}}{L^2} \psi\,, \tag{C.3}$$

where g_f is taken to be dimensionless. Finally, we have the dimensionless charge that is implicit in the covariant derivative

$$\tilde{q} = \frac{\sqrt{\mu_0 c^6}}{16\pi G} \frac{L}{\hbar c} q\,. \tag{C.4}$$

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