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MASTER'S THESIS

Anisotropic gauge theories at finite charge density

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``I am the wisest man alive, for I know one thing, and that is that I know nothing."

Plato, The Republic.

Abstract

We set up a non-pertrurbative study of anisotropic, charged and scale invariant gauge theories by means of the AdS/CFT correspondence. The gravitational dual theory is realised by the Einstein-Maxwell-Axion-Dilaton, which accounts for the spatial anisotropy and charge density of the gauge theory. In the vacuum the solutions describe RG flows from a conformal theory in the UV to generic scaling solutions in the IR with Lifshitz scaling and hyperscaling violation. We work out a generalization of the holographic c-theorem to the anisotropic charged case given by the null energy condition, constraining the parameters of the 5d metric. At finite temperature, we analyse the blackening factor and temperature in the strongly coupled IR regime. Finally, we study the thermodynamic stability of the black brane concluding the existence of a single stable IR geometry.

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1. Introduction

Strongly interacting systems remain as one of the most poorly understood elements of physics. The reason is that usual techniques, such as perturbation theory, do not apply for such systems. Nevertheless, strongly correlated systems play an important role in physics, from high energy to condensed matter physics. Therefore, the search and improvement of new methods is imperative in order to gain new insights and achieve progress.

One of the most remarkable strongly coupled systems is Quantum Chromodynamics (QCD). Indeed, QCD has the particularity of being asymptotically free due to the negativity of the β -function, and it is therefore strongly coupled in the low energy regime, the infrared (IR). The main motivation of this work is to study the quark-gluon plasma (QGP) created at heavy ion collisions (LHC, RHIC). The QGP is a QCD state of matter composed of quarks and gluons in which particles are deconfined. Contrary to usual QCD where quarks are confined constituting hadrons, in the QGP particles are not bounded together. The QGP constitutes one of the most studied systems from theoretical physics. It is a significant QCD state of matter with unique characteristics to which we have experimental access through heavy ion collisions. And not only that, the QGP has important connections to cosmology and astrophysics, being present instants after the Big Bang and in the core of neutron stars. Besides, experimental data teaches us that the QGP is strongly coupled. Therefore, the QGP constitutes an excellent opportunity to accomplish crucial knowledge in different areas of physics and explore new techniques. Being a strongly correlated system, perturbation theory is not a possible method to use. Furthermore, the QGP behaves like a fluid, for which time dependent phenomena such as the viscosity plays a central role. Consequently, lattice QCD is not a perfect candidate due to its difficulties dealing with the real time phenomena, requiring the spectrum density and carrying statistical uncertainties. Thus, a new approach is needed to face the task.

The AdS/CFT correspondence found by Maldacena [1] offers a new method to study strongly coupled gauge theories. The correspondence gives a particular example of the equivalence between quantum field theory and gravity, the so called gauge/gravity duality, establishing a bridge among the most fundamental theories of nature. The correspondence is a strong/weak duality, meaning that when the gauge theory is strongly coupled the dual gravity theory is weakly curved, and vice versa. Due to this feature, the duality gives a route to study the QGP using Einstein's gravity in the large-N limit, where N is the number of colours of the gauge theory. In this thesis the gauge/gravity duality is the tool we use to study the QGP created at heavy ion collisions; concretely, a deformation of the original AdS_5/CFT_4 correspondence. We are concerned with two important properties of the QGP. The first one, the spatial anisotropy along one axis due to the non-centrality of the collision. The second one, the charge density of the QGP generated from the quarks. The gravitational dual theory is given by the 5d Einstein-Mawell-Axion-Dilaton action [2,3] which mimicks the main QCD/QGP features, namely the renormalization group flow, the anisotorpy and the charge. Using this action the strongly coupled IR regime is explored. In particular, we analyse the IR geometry, null energy conditions and the thermodynamic stability of the system in order to see the effects sourced by the anisotropy [4] as well as the charge density.

The thesis is organized as follows. In chapter 2 the AdS/CFT correspondence is introduced, revealing its most important features as well as the apparatus we make use of. In chapter 3 we present the QGP and its main characteristics, focusing on the anisotropy and culminating with the gravitational action. In chapter 4 we initiate the study of the strongly coupled IR regime. In particular, we use Einstein's equations to find the IR metric and the null energy conditions. In chapter 5 the black hole thermodynamics is displayed. We compute the blackening factor and temperature, to finally analyse the stability of the system.

2. The AdS/CFT correspondence

The discovery of the AdS/CFT correspondence [1] established a new perspective in the understanding of the fundamental theories of physics. For the first time, an explicit example of the holographic principle was found, confirming the idea that a d-dimensional theory can be explained by its degrees of freedom on the (d-1)-dimensional boundary. Furthermore, the discovery of the correspondence made clear the idea that quantum field theories (QFT) and gravity are related in the so called gauge/gravity duality, although in a peculiar and odd manner. Immediatly, the AdS/CFT correspondence was studied by many and its applications started to flourish. One of the most important applications is the study of strongly coupled systems, for which the duality allows us to map the strongly correlated QFT to gravity in a new way of addressing the problem. In this thesis the original AdS_5/CFT_4 will be used to examine the QGP, serving as a tool to explore the strongly coupled regime. In this chapter a first glance of the AdS/CFTcorrespondence is presented together with the necessary apparatus which will be used to model the desired properties of the QGP.

2.1 Anti de-Sitter spacetime

Maximally symmetric spacetimes play an important role in physics due to their particular features, having all the possible Killing vectors and constant curvature everywhere. Anti de-Sitter (AdS) spacetime is the maximally symmetric spacetime which solves Einstein's equation in the vacuum with negative curvature R < 0 [5],

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{2.1}$$

where Λ is the cosmological constant. Contracting with the metric, one obtains the Ricci scalar, $R = \frac{2d\Lambda}{d-2}$. Since R < 0 the cosmological constant is also negative, $\Lambda < 0$. In (d+1) dimensions AdS_{d+1} can be embedded into (d+2)-dimensional Minkowski spacetime with coordinates $(X^0, X^1, ..., X^{d+1})$ with the flat metric $\eta = diag(-, +, +, ..., +, -)$. The metric takes the form

$$ds^{2} = -\left(X^{0}\right)^{2} + \sum_{i=1}^{d} \left(X^{i}\right)^{2} - \left(X^{d+1}\right)^{2} = -L^{2}$$
(2.2)

where L is the AdS radius of curvature. The line element is invariant under the action of the group SO(d, 2) acting on the coordinates in the usual way, $X'^{\mu} = \Lambda^{\mu}_{\nu} X^{\nu}$. A more useful way to represent the line element is by the so called Poincaré patch coordinates, given by

$$X^{0} = \frac{L^{2}}{2r} \left(1 + \frac{r^{2}}{L^{4}} \left(x^{2} - t^{2} + L^{2} \right) \right)$$
(2.3)

$$X^i = \frac{rx^i}{L} \tag{2.4}$$

$$X^{d} = \frac{L^{2}}{2r} \left(1 + \frac{r^{2}}{L^{4}} \left(x^{2} - t^{2} + L^{2} \right) \right)$$
(2.5)

$$X^{d+1} = \frac{rt}{L} \tag{2.6}$$

with r > 0 and $i \in 1, ..., d-1$. Because of the restriction over the r coordinate, this parametrisation covers only half of AdS_{d+1} spacetime. The metric in this coordinates reads ¹

$$ds^{2} = \frac{L^{2}}{r^{2}}dr^{2} + \frac{r^{2}}{L^{2}}\left(\eta_{\mu\nu}dx^{\mu}dx^{\nu}\right) = \frac{L^{2}}{z^{2}}\left(dz^{2} + \eta_{\mu\nu}dx^{\mu}dx^{\nu}\right)$$
(2.7)

where we introduced $z = L^2/r$, in which the metric takes a more compact form. This form of the metric is the one that will be most useful when we apply the AdS/CFTcorrespondence to the QGP. In these coordinates, the invariance under SO(d, 2) is not manifest anymore. Instead, the metric is invariant under the SO(d, 2) subgroups ISO(d-1,1) acting on (t, \vec{x}) which corresponds to Poincaré transformations, and the group SO(1,1) acting on (t, \vec{x}, r) as

$$(t, \vec{x}, r) \to (\lambda t, \lambda \vec{x}, r/\lambda)$$
 (2.8)

which in the context of the AdS/CFT correspondence is identified with the dilatation D of the conformal symmetry group [6]. The line element 2.7 has a nice interpretation. One can understand AdS as a Minkowski spacetime given by the coordinates t and \vec{x} plus an extra dimension, corresponding to the coordinate r. This is represented in Figure 2.1. Furthermore, the conformal boundary of AdS_{d+1} corresponds to $\mathbb{R} \times S^d$ which is precisely the conformal diagram of Minkowski spacetime [6].

¹The cosmological constant in these coordinates is $\Lambda = \frac{d(d-1)}{2L^2}$ which confirms that L is the radius of curvature.



FIGURE 2.1: Taken from [5]. Representation of AdS spacetime in Poincaré coordinates. For a fixed value of r the d-dimensional spacetime corresponds to Minkowski spacetime. The boundary is located at $r \rightarrow \infty$.

Note that the metric 2.7 has a second order pole when $r \to \infty$. It is possible to show that any metric which is asymptotically AdS always has such quadratic divergence for a particular value of r, denoted by r_{\star} . The spacetime slice at $r = r_{\star}$ is the conformal boundary of AdS spacetime²

2.2 Conformal algebra

In particle physics the symmetry group is the Poincaré symmetry ISO(1,3) defined by the Lorentz generators $J_{\mu\nu}$ and the generator P_{μ} of (3 + 1)-dimensional translational symmetry, satisfying the algebra

$$[J_{\mu\nu}, J_{\rho\sigma}] = -(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma})$$

$$[P_{\mu}, J_{\nu\sigma}] = (\eta_{\mu\nu}P_{\rho} - \eta_{\mu\rho}P_{\nu})$$

$$[P_{\mu}, P_{\nu}] = 0.$$
(2.9)

There are also internal symmetries, such as U(1) for electromagnetism or SU(3) for QCD, with the generators satisfying the Lie algebra $[T_r, T_s] = f_{rs}^t T_t$. One can wonder if it is possible to combine the internal symmetries with the Poincaré symmetry. The answer is no, a result due to the Mandula-Coleman theorem [5]. Therefore, if we wish to enlarge the symmetry group we must consider new transformations of spacetime. In particular, the group considered is the conformal group, which is SO(d, 2) in d-dimensions.

The conformal group is the group of transformations which preserve the metric up to an arbitrary factor, $g_{\mu\nu} \to \Omega^2(x)g_{\mu\nu}$. Therefore, the transformation changes the length

²Notice that in the z coordinates the boundary is located at z = 0. When working with the gravity theory we will locate the boundary at the origin.

of an interval, but leaves the angles invariant and thus preserves the causal structure. The conformal group is composed of the Poincaré group, plus the scaling

$$x^{\mu} \to \lambda x^{\mu}$$
 (2.10)

and special tansformations

$$x^{\mu} \to \frac{x^{\mu} + a^{\mu} x^2}{1 + 2x^{\nu} a_{\nu} + a^2 x^2} \tag{2.11}$$

with generators D and K_{μ} respectively. The algebra of the group is defined by the commutation relations

$$[J_{\mu\nu}, K_{\rho}] = i (\eta_{\mu\rho} K_{\nu} - \eta_{\nu\rho} K_{\mu})$$

$$[D, P_{\mu}] = i P_{\mu}$$

$$[D, K_{\mu}] = -i K_{\mu}$$

$$[D, J_{\mu\nu}] = [K_{\mu}, K_{\rho}] = 0$$

$$[K_{\mu}, P_{\nu}] = -2i (\eta_{\mu\nu} D - J_{\mu\nu})$$

(2.12)

together with the Poincaré algebra 2.9. This algebra is isomorphic to the group SO(d, 2), and for d = 2 one obtains the Virasoro algebra [7], familiar from string theory. Recall that in particle physics is possible to have a conformal theory, as long as the spectrum is composed of massless particles. Indeed, there is then no intrinsic scale in the theory.

2.2.1 Fields transformations

Representations of the conformal group are defined in terms of the eigenfunctions of the scaling operator D with eigenvalue $-i\Delta$, where Δ denotes the scaling dimension. Under a transformation 2.10 the field transforms as

$$\phi(x) \to \phi'(x) = \lambda^{\Delta} \phi(x). \tag{2.13}$$

A field which transforms covariantly under an irreducible representation of the conformal algebra has a fixed scaling dimension and it is therefore an eigenstate of the dilatation operator D. From the conformal algebra 2.12 we see that P_{μ} increases the scaling dimension while K_{μ} decreases it. Due to unitarity there is a lower bound on the scaling dimension of the fields, meaning that there are operators of lowest dimension which are annihilated by K_{μ} . This feature allows to define a primary field, which are the fields with lowest scaling dimension, from which it is possible to obtain any other representation by acting P_{μ} on them. The action of the conformal gorup on such a field is given by the relations [8]

$$[P_{\mu}, \phi(x)] = i\partial_{\mu}\phi(x)$$

$$[J_{\mu\nu}, \phi(x)] = [i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) + \Sigma_{\mu\nu}]\phi(x)$$

$$[D, \phi(x)] = -i\Delta\phi(x) - ix^{\mu}\partial_{\mu}\phi(x)$$

$$[K_{\mu}, \phi(x)] = [i(-x^{2}\partial^{\mu} + 2x_{\mu}x^{\rho}\partial_{\rho} + 2x_{\mu}\Delta) - 2x^{\nu}\Sigma_{\mu\nu}]\phi(x)$$
(2.14)

where $\Sigma_{\mu\nu}$ are the matrices of a finite dimensional representation of the Lorentz group. The correlation functions can be computed from the dilatation Ward identity [5]. The 2-point correlation functions of fields of different dimension vanish, while for fields with the same scaling dimension

$$\langle \phi(0)\phi(x)\rangle = \frac{1}{x^{2\Delta}}.$$
(2.15)

3-point functions are of the form

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\rangle = \frac{c_{ijk}}{(x_1 - x_2)^{\Delta_1 + \Delta_2 - \Delta_3} (x_2 - x_3)^{-\Delta_1 + \Delta_2 + \Delta_3} (x_1 - x_3)^{\Delta_1 - \Delta_2 + \Delta_3}}$$
(2.16)

and similar for higher order correlation functions. From these expressions, we see that the most important quantity to know is the scaling dimension of the fields. Important fields which will appear further in this thesis are the energy momentum tensor $T_{\mu\nu}$ and the conserved current J_{μ} , with scaling dimension $\Delta = d$ and $\Delta = d - 1$ respectively. Another important feature of conformal field theories (CFT) is the operator product expansion (OPE), which is used in the radial quantization of the theory [7] reducing it to live on $\mathbb{R} \times S^{d-1}$. The OPE of two operators gives important information about the algebra of the theory, such as the central charge or the weight of the operators. Further, one can define asymptotic states in a CFT using radial quantization. Given an operator $\mathcal{O}(x)$ a state can be defined as $|\phi\rangle = \lim_{x\to 0} \mathcal{O}(x)|0\rangle$ being $x \to 0$ the infinite past.

2.3 Supersymmetry and superconformal algebras

Another way to bypass the Coleman-Mandula theorem is by introducing new generators transforming in the spinor representation. These are fermionic operators (supercharges) denoted by \mathcal{Q}^a , where *a* specifies the number of independent supersymmetries present, i.e., $a = 1, ..., \mathcal{N}$. In order to introduce such operators the Lie algebra is \mathbb{Z}_2 graded, with bosonic generators having grade 0 and fermionic generators 1. Therefore, the product of two bosonic fields is bosonic, of two fermions is bosonic, and of one boson and fermion is fermionic. Given two generators \mathcal{O}_1 and \mathcal{O}_2 with grades g_1, g_2

$$[\mathcal{O}_1, \mathcal{O}_2] = \mathcal{O}_1 \mathcal{O}_2 - (-1)^{g_1 g_2} \mathcal{O}_2, \mathcal{O}_1$$
(2.17)

where the bracket notation means that it can either be a commutator or an anitcommutator. It turns out that it is possible to join together supersymmetry and the conformal algebra as long as $d \leq 6$ [6]. When doing so, new fermionic supercharges S^a are introduced. While the Poincaré supercharges Q are the fermionic superpartners of P_{μ} (they anticommute), S are the fermionic superpartners of K_{μ} . The commutation relations of the superconformal algebra include, in addition to 2.9 and 2.12

$$[D, Q] \simeq -i\frac{Q}{2} \qquad [D, S] \simeq i\frac{S}{2}$$

$$[K, Q] \simeq S \qquad [P, S] \simeq Q$$

$$[Q, Q] \simeq P \qquad [S, S] \simeq K$$

$$[Q, S] \simeq M + D + R$$
(2.18)

where indices and factors have been supressed for simplicity. R represents the nonabelian R-symmetry of the Poincaré supercharges given by

$$\mathcal{Q}^a_\alpha \to \mathcal{Q}^{a'}_\alpha = R^a_b \mathcal{Q}^b_\alpha, \tag{2.19}$$

where R_b^a are the components of an $\mathcal{N} \times \mathcal{N}$ matrix. The superconformal theory which will play an important role in the AdS_5/CFT_4 correspondence is $\mathcal{N} = 4$ Super Yang-Mills (SYM) in d = 4.

2.3.1 $\mathcal{N} = 4$ Super Yang-Mills

 $\mathcal{N} = 4$ Super Yang-Mills (SYM) has 16 Poincaré supercharges and the field content in d = 4 is composed of a gauge field $A_{\mu}(x)$, four Weyl fermions $\lambda^{a}(x)$ and six real scalars $\phi^{i}(x)$. The Lagrangian reads

$$\mathcal{L} = Tr \left[-\frac{1}{2g_{YM}^2} F^2 + \frac{\theta}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - i\bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a - \sum_i D_\mu \phi^i D^\mu \phi^i \right. \\ \left. + g_{YM} \sum_{a,b,i} C_i^{ab} \lambda_a \left[\phi^i, \lambda_b \right] + g_{YM} \sum_{a,b,i} \bar{C}_{iab} \bar{\lambda}^a \left[\phi^i, \bar{\lambda}^b \right] \right.$$

$$\left. + \frac{g_{YM}^2}{2} \sum_{i,j} \left[\phi^i, \phi^j \right]^2 \right]$$

$$\left. + \frac{g_{YM}^2}{2} \sum_{i,j} \left[\phi^i, \phi^j \right]^2 \right]$$

$$\left. + \frac{g_{YM}^2}{2} \sum_{i,j} \left[\phi^i, \phi^j \right]^2 \right]$$

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$$\left. + \frac{g_{YM}^2}{2} \sum_{i,j} \left[\phi^i, \phi^j \right]^2 \right]$$

with $F_{\mu\nu} = \partial_{\mu}A_{\mu} - \partial_{\nu}A_{\mu}$, $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$, $D_{\mu} = \partial_{\mu} + i[A_{\mu}]$ and C_i^{ab} are Clebsch-Gordan coefficients. The lagrangian is invariant under the supersymmetry transformations

$$\delta_{\epsilon}\phi^{i} = \left[\epsilon^{\alpha}_{a}Q^{a}_{\alpha}, \phi^{i}\right] = \epsilon^{\alpha}_{a}C^{iab}\lambda_{\alpha b}$$

$$\delta_{\epsilon}\lambda_{\beta b} = \left[\epsilon^{\alpha}_{a}Q^{a}_{\alpha}, \lambda_{\beta b}\right] = F^{+}_{\mu\nu}\epsilon_{\alpha b}(\sigma^{\mu\nu})^{\alpha}_{\beta} + \left[\phi^{i}, \phi^{j}\right]\epsilon_{\beta a}(C_{ij})^{a}_{b}$$

$$\delta_{\epsilon}\bar{\lambda}^{b}_{\dot{\beta}} = \left[\epsilon^{\alpha}_{a}Q^{a}_{\alpha}, \bar{\lambda}^{b}_{\dot{\beta}}\right] = C^{ab}_{i}\epsilon^{\alpha}_{a}\bar{\sigma}^{\mu}_{\alpha\dot{\beta}}D_{\mu}\phi^{i}$$

$$\delta_{\epsilon}A_{\mu} = \left[\epsilon^{\alpha}_{a}Q^{a}_{\alpha}, A^{\mu}\right] = \epsilon^{\alpha}_{a}(\sigma_{\mu})^{\beta}_{\alpha}\bar{\lambda}^{a}_{\dot{\beta}}.$$

$$(2.21)$$

This field theory has important features. The coupling constant is dimensionless and all fields are massless. Therefore, the action is scale invariant. Further, even when the theory is quantized it remains scale invariant, with a vanishing β -function at all orders. This is the reason for which it is a superconformal field theory. Finally, the action is invariant under the S-duality group $SL(2,\mathbb{Z})$. This implies that the coupling constant changes from g_{YM} to $4\pi/g_{YM}$ and there is a strong-weak duality, so that if $g_{YM} \gg 1$ then $4\pi/g_{YM} \ll 1$ and vice versa. $\mathcal{N} = 4$ SYM plays a central role in the AdS/CFT correspondence since it is the gauge theory present in the CFT side of the duality. Therefore, the equivalence is between $\mathcal{N} = 4$ SYM and a particular gravity theory, which is introduced in the following sections.

2.4 Supergravity

Supergravity is the low-energy effective action for Type II closed superstring theories³, which live in d = 10. The low-energy limit corresponds to the limit $\alpha' \to 0$, for which the massive states disappear from the string spectrum. There are two supergravity actions, denoted as Type IIB and Type IIA. For the purposes of the thesis we will focus on Type IIB supergravity, since it is precisely the gravity theory dual to $\mathcal{N} = 4$ SYM. Type IIB supergravity has 32 supercharges and its action is

$$S_{IIB} = \frac{1}{2\tilde{\kappa}_{10}^2} \left[\int d^{10}x \sqrt{-g} \left(e^{-2\phi} \left(R + 4\partial_M \phi \partial^M \phi - \frac{1}{2} |H_{(3)}|^2 \right) - \frac{1}{2} |F_{(1)}|^2 - \frac{1}{2} |\tilde{F}_{(3)}|^2 - \frac{1}{2} |\tilde{F}_{(5)}|^2 \right) - \frac{1}{2} \int C_{(4)} \wedge H_{(3)} \wedge F_{(3)} \right]$$
(2.22)

where we use the notation $|F_{(p)}|^2 = \frac{1}{p!}g_{M_1N_2}...g_{M_pN_p}\tilde{F}^{M_1...M_p}F^{N_1...N_p}$ and $\tilde{F}_{(p)}$ denotes the complex conjugate of $F_{(p)}$. Moreover, $\tilde{\kappa}_{10}$ is the 10-dimensional gravitational constant, related to the 10-dimensional Newton constant as $\tilde{\kappa}_{10}^2 = 8\pi G_{10}$ and related to the string coupling $g_s = e^{\phi}$ as follows,

$$2\tilde{\kappa}_{10}^2 = (2\pi)^7 \, \alpha'^4 g_s^2, \tag{2.23}$$

where $\alpha' = l_s^2$. The field strength tensors in 2.22 are given by

$$F_{(p)} = dC_{(p-1)}, \quad H_{(3)} = dB_{(2)}, \quad \tilde{F}_{(3)} = F_{(3)} - C_{(0)}H_{(3)}$$

$$\tilde{F}_{(5)} = F_{(5)} - \frac{1}{2}C_{(2)} \wedge H_{(3)} + \frac{1}{2}B_{(2)} \wedge F_{(3)}.$$

(2.24)

In addition, we must impose the self-duality condition $\star \tilde{F}_{(5)} = \tilde{F}_{(5)}$. The field content of the theory is

- The graviton $G_{\mu\nu}$ with 35 bosonic degrees of freedom (d.o.f.)
- The axion and dilaton scalar fields $C + i\phi$, with 2 bosonic d.o.f.
- Two 2-forms $B_{\mu\nu} + iA_{2\mu\nu}$, with 56 d.o.f.
- A 4-form $A_{4\mu\nu\rho\sigma}$ with 35 bosonic d.o.f.
- Two Majorana-Weyl gravitinos $\psi^i_{\mu\alpha}$, i = 1, 2 with 112 fermionic d.o.f.
- Two Majorana-Weyl dilatinos λ_{α}^{i} , i = 1, 2 with 16 fermionic d.o.f.

Different types of string theories are related to each other by the T-duality and the S-duality.

• T-duality: Denotes the equivalence between two superstring theories compactified on different backgrounds. Let's consider Type II superstring theory compactified on a circle, i.e., the coordinate X^9 is periodically identified $X^9 \sim X^9 + 2\pi R$. It

³There are two ways to obtain the supergravity action: from superstring theory and by requiring local supersymmetry as an extension of general relativity, so that the graviton acquires a fermionic partner, the gravitino. We present the first option.

turns out that the spectrum for a compactification with radius R is identical to the spectrum for a compactification with radius α'/R . Therefore, Type IIB superstring theory compactified with radius R is equivalent to Type IIA superstring theory with radius α'/R .

• S-duality: It maps g_s to $1/g_s$. It is a strong-weak coupling duality, since it maps a weak superstring theory to another strongly coupled superstring theory. Indeed, when $g_s \gg 1$ then $1/g_2 \ll 1$ and vice versa.

2.5 D-branes

Dp-branes are extended objects in (p + 1)-dimensions in which open strings can end. The endpoint of a string is charged and thus couples to a gauge field living on the Dp-brane worldvolume. For this reason gauge theory is introduced in string theory using open strings, while closed represent gravity. Let ξ^a denote the coordinates for the worldvolume of the brane. In analogy to the worldsheet area action from the strings, the Dp-brane action is given by the Dirac-Born-Infled (DBI) action

$$S_{DBI} = -\tau_p \int d^{p+1} \xi e^{-\phi} \sqrt{-\det\left(P(g) + 2\pi\alpha' F + P(B)\right)}$$
(2.25)

where P(g), P(B) denote the pullback of the metric and Klab-Ramond fields. F is the field strength living on the Dp-brane. The prefactor in the action is

$$\tau_p = (2\pi)^{-p} \, \alpha'^{-(p+1)/2}. \tag{2.26}$$

Notice that unlike the strings, the Dp-branes are non-perturbative objects since the energy scales as $1/g_s$. Due to the presence of gauge fields on the branes, they can be charged under the forms coming from the string spectrum. But how do we couple the branes to the gauge fields? In (3+1)-dimensional electromagnetism the electric charge within a closed surface is defined as the total electric flux through such a closed surface,

$$Q = \int_{S^2} \star F. \tag{2.27}$$

In this case, the object is identified with a charged particle ⁴. Therefore, a (p + 1)-dimensional object has charge due to the presence of gauge fields coming from field strength forms. The brane couples to the gauge field through a (p + 1)-form gauge potential

$$\mu \int_{W} C_{(p+1)} \tag{2.28}$$

⁴One can define as well a magnetic charge $Q_m = \int_{S^2} F$, which can be extended for Dp-branes as in the case of electric charge. Recall that Stokes theorem together with the Maxwell equation $dF = d^2A = 0$ implies the non-existence of magnetic charges, at least for trivial geometries, i.e., having trivial cohomology groups.

where μ is the charge of the field and $F_{(p+2)} = dC_{(p+1)}$. The total electric charge is then

$$Q = \int_{S^{8-p}} \star F_{(p+2)}.$$
 (2.29)

2.6 Black *p*-branes

Solutions to the supergravity equations of motion are extended objects called black pbranes, living in d = 10. These objects are black holes in p spatial dimensions ⁵. The solution to the Type IIB supergravity equations of motion is

$$ds^{2} = H_{p}(r)^{-1/2} \eta_{\mu\nu} dx^{\mu} dx^{\nu} + H_{p}(r)^{1/2} dy^{i} dy^{j}$$
(2.30a)

$$e^{\phi} = g_s H_p(r)^{(3-p)/4}$$
 (2.30b)

$$C_{(p+1)} = (H_p(r)^{-1} - 1) dx^0 \wedge dx^1 \wedge \dots \wedge dx^p$$
(2.30c)

(2.30c)

$$B_{MN} = 0 \tag{2.30d}$$

where x^{μ} with $\mu = 0, ..., p$ are the coordinates on the worldvolume of the black brane and y^i with i = p + 1, ..., 9 denotes the coordinates transverse to the brane. Moreover, ris defined as $r = \sum_{i=p+1}^{9} y_i^2$. With this ansatz, the equations of motion imply that

$$\Box H_p(r) = 0 \tag{2.31}$$

for $r \neq 0$. Therefore, $H_p(r)$ is a harmonic function and can be written as

$$H_p(r) = 1 + \left(\frac{L_p}{r}\right)^{7-p} \tag{2.32}$$

where L_p is the supergravity length. Notice that when $r \to \infty$ Minkowski spacetime is recovered⁶. To find the constant L_p the charge of the brane must be determined. From the previous section we find the charge

$$Q = \frac{1}{2\kappa_{10}^2} \int_{S^{8-p}} \star F_{(p+2)} = N\mu_p \tag{2.33}$$

where N is the number of branes. Therefore,

$$L_p^{7-p} = (4\pi)^{(5-p)/2} \Gamma\left(\frac{7-p}{2}\right) g_s N \alpha'^{(7-p)/2}.$$
 (2.34)

⁵In order to have a localized object, i.e., a black hole like object it is necessary to have p > 4, with the exception of p = 0 which correspond to usual black holes. This is due to the particular form of the harmonic functions in d < 4.

⁶In Type IIB superstring theory the only possible branes are D1, D3, D5, D7. Therefore $p \leq 7$ which ensures recovering Minkowski in the $r \to \infty$ limit for any case.

In this thesis we will be concerned with D3-branes, so that p = 3 and $L^4 = 4\pi g_s N \alpha'^2$, where we suppressed the subindex p from L_p . L has dimensions of length, and has to be related to the string constant α' . Indeed, the string length is given by $l_s^2 = \alpha'$, and therefore one can relate L to l_s . When $g_s N \ll 1$ the supergravity length will be much smaller than the string length and we do not expect supergravity to be a good approximation. However, since $N \in \mathbb{N}$ the string coupling is of order $g_s \ll 1$, and one can use perturbative string theory. On the other hand, when $g_s N \gg 1$ the supergravity length is much greater than the string length and supergravity is a good approximation. Notice that if N is large enough then we also have $g_s \ll 1$ in this case. This is the large-Nt'Hooft limit, which allows us to ignore quantum fluctuations. The large-N limit is the one we consider in this thesis.

The important feature which the gauge/gravity uses is the fact that Dp-branes, the hypersurfaces where open strings can end, are the same as black *p*-branes, objects which gravitate. Like the charged Reissner-Nordström black hole, it is possible to have charged black *p*-branes, with three different cases depending on the relation between the mass and the charge. When the mass equals the charge of the black hole, the black hole is called extremal. This is the easiest case and the one that we will be concerned about when studying the correspondence. In addition to extremal black branes, there are also near-extremal black hole solutions, for which $Q = N\mu_p$ but for which their mass is no longer proportional to Q. In this case there is a horizon since there is a blackening factor with $f(r_h) = 0$ in the metric solution

$$ds^{2} = H_{p}(r)^{-1/2} \left(-f(r)dt^{2} + dx^{i}dx^{i} \right) + H_{p}(r)^{1/2} \left(\frac{dr^{2}}{f(r)} + r^{2}d\Omega_{8-p}^{2} \right)$$
(2.35a)

$$f(r) = 1 - \frac{r_h^{7-p}}{r^{7-p}}.$$
(2.35b)

2.7 The AdS_5/CFT_4 correspondence

We have now all the elements to present the AdS_5/CFT_4 correspondence, the first concrete example of the gauge/gravity duality which was discovered by Maldacena [1]. Previous to his work it was known that some kind of connection exsits between string theory and QFT. t'Hooft showed that SU(N) Yang-Mills reduces to string theory when taking the $N \to \infty$ limit. Maldacena made specific the idea by telling us which string theory and QFT are equivalent. On one side of the duality, the CFT side, there is $\mathcal{N} = 4$ SYM. On the other side there is string theory in AdS_5 spacetime. Moreover, the correspondence is an example of the concept of holography; the idea that a theory can be explained by the degrees of freedom on its boundary. Indeed, the duality asserts that a d-dimensional field theory can be described by a (d+1)-dimensional string theory (gravity).

In order to motivate the correspondence, let us think about the nature of string theory. String theory has two types of strings, closed and open. But they are actually two faces of the same coin. Two open strings can join to form an open string. But if the endpoints of two open strings can join, the endpoints of a single open string can join as well and produce a closed string. Therefore, if there are open strings there are closed strings⁷. This connection was understood better once Polchinski showed that D-branes, the hypersurfaces on which open strings can end, are the same object as black p-branes, which are solutions to the supergravity (Type IIB for the case in hand) equations of motion and are gravitational objects. Given this specific connection one can think about the Hawking radiation of a p-brane ⁸ in the D-brane picture: two open strings living on the brane join to form a closed string, which detaches from the brane and moves away in the so called bulk as Hawking radiation. Therefore, there should be some relation between the theory living on the D-brane and the gravitational theory in the bulk, which is curved due to the presence of the D-brane.

2.7.1 Why AdS?

In section 2.1 we derived the AdS metric 2.7, with the radial coordinate r having the interpretation of being an extra dimension to Minkowski spacetime, so that slices of constant r correspond to flat spacetime. Consider now a field theory in 4 dimensional Minkowski spacetime $ds^2 = \eta_{\mu\nu} dx^{\mu} dx^{\mu}$. The QFT is subject to the renormalization group (RG) flow due to the dependence of the coupling constant on the energy scale, which we denote by r. At the critical points the β -function vanishes and the theory becomes scale invariant, i.e., $x^{\mu} \to \lambda x^{\mu}$ becomes a symmetry. What happens now if we force r to be included in the Minkowski metric as a coordinate? Dimensional analysis implies that $r \to r/\lambda$ and therefore the metric must be of the form

$$ds^{2} = \frac{L^{2}}{r^{2}}dr^{2} + \frac{r^{2}}{L^{2}}\left(\eta_{\mu\nu}dx^{\mu}dx^{\nu}\right), \qquad (2.36)$$

which is nothing else than the AdS metric 2.7 in 5 dimensions. Therefore, the extra AdS coordinate is the energy scale of the gauge theory, which has now a geometrical interpretation in the AdS/CFT correspondence. Furthermore, a *d*-dimensional CFT field theory is invariant under the conformal group SO(d, 2). As we saw in section 2.1, the isometry group of AdS_{d+1} is SO(d, 2), matching the isometry of the CFT side.

2.7.2 D3-branes and black *p*-branes

In this section we work out the particular example of the AdS_5/CFT_4 correspondence. To do so we work with superstring theory and in particular we make use of the fact that D-branes are the same object as black *p*-branes. Therefore, D-branes can be viewed from two different perspectives: as hypersurfaces where open stirngs are attached (open perspective) or as gravitational objects which solve the supergravity equations (closed perspective). The open string perspective is reliable when $g_s N \ll 1$, so that the coupling of open and closed strings is small and perturbation theory can be used. On the other hand, the closed string perspective is valid when $g_s N \gg 1$, where the length scale *L* is large enough to ensure weak curvature and thus supergravity. If we take the large-*N* limit $N \to \infty$ then $g_s \ll 1$, so that we can also treat perturbatively the strings in the closed perspective.

⁷One can consider string theory with only closed strings. This is not of interest since it does not consider gauge theories.

⁸Specifically, of a near-extremal black brane, since extremal black holes do not emit radiation.

Open string perspective

Let us consider first the perspective in which D-branes are the hypersurfaces on which open strings can end, valid when $g_s N \ll 1$. We study Type IIB superstring theory in flat (9+1)-dimensional Minkowski spacetime where we embed N coincidient D3-branes, which extend along the coordinates (0, 1, 2, 3). We can treat with perturbation theory the strings, which consist of open and closed strings. Concretely, the full action is

$$S = S_{closed} + S_{open} + S_{int}, \tag{2.37}$$

where S_{int} contains the interactions between closed and open strings. S_{closed} corresponds to the 10-dimensional supergravity action 2.22. We can now expand this action using a fluctuation of the metric $g = \eta + \kappa h$, from which we obtain that the bosonic part of S_{closed} reads

$$S_{closed} = -\frac{1}{2} \int d^{10}x \partial_M h \partial^M h + O(\kappa)$$
(2.38)

where κ is given by $2\kappa_{10}^2 = (2\pi)^7 \alpha'^4 g_s^2$. Let us now consider a single D3-brane. The actions S_{open} and S_{int} can be derived from the DBI action 2.25⁹

$$S_{DBI} = -\frac{1}{(2\pi)^3 \alpha'^2 g_s} \int d^4 x e^{-\phi} \sqrt{-\det\left(\gamma + 2\pi\alpha' F\right)}.$$
 (2.39)

Moreover, we can introduce six real scalar fields which can be identified with the tansverse coordinates $x^i = 2\pi \alpha' \phi^i$. Expanding $e^{-\phi}$ and $g = \eta + \kappa$, to leading order in α'

$$S_{open} = -\frac{1}{2\pi g_s} \int d^4x \left(\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^i + O(\alpha') \right)$$
(2.40)

$$S_{int} = -\frac{1}{8\pi g_s} \int d^4 x \kappa \phi F_{\mu\nu} F^{\mu\nu} + \dots$$
 (2.41)

If we now consider a stack of N coincident D3-branes we need to add the potential $V = \frac{1}{2\pi g_s} \sum_{i,j} Tr \left[\phi^i, \phi^j\right]^2$ to S_{open}^{10} . If we now take the low-energy limit $\alpha' \to 0^{11}$, then S_{open} reduces to the bosonic part of $\mathcal{N} = 4$ SYM 2.20, provided that we make the identification

$$2\pi g_s = g_{YM}^2.$$
 (2.42)

 S_{closed} is just the aciton of free supergravity in (9+1)-dimensional Minkowski in the low energy limit, and S_{int} vanishes due to the presence of κ in the action. Therefore, in the low energy limit open and closed strings decouple. To summarize, we obtained that in the low energy limit the dynamics of open strings give rise to $\mathcal{N} = 4$ SYM, while the closed strings are described by supergravity in flat (9+1)-dimensional spacetime.

Closed string perspective

⁹The Kalb-Ramond field is set to zero for simplicity.

¹⁰The scalars and gauge fields are U(N) valued.

¹¹Recall that in this limit $\kappa \to 0$.

We now consider the regime $g_s N \ll 1$. In this limit supergravity is a good approximation and D-branes can be understood as black *p*-branes, solutions of the supergravity Type IIB equations of motion. Since p = 3, from 2.30 we see that

$$ds^{2} = H(r)^{-1/2} \eta_{\mu\nu} dx^{\mu} dx^{\nu} + H(r)^{1/2} \delta_{ij} dx^{i} dx^{j}$$
(2.43a)

$$e^{2\phi} = g_s^2 \tag{2.43b}$$

$$C_{(4)} = (1 - H_{(r)})^{-1} dx^{0} \wedge dx^{1} \wedge dx^{2} \wedge dx^{3}$$
(2.43c)

$$H(r) = 1 + \left(\frac{L}{r}\right)^4 \tag{2.43d}$$

$$L^4 = 4\pi g_s N \alpha'^2. \tag{2.43e}$$

We can consider now the limits $r \ll L$ and $r \gg L$. Recall that L can be understood as the length scale characteristic for the range of the gravitational effects of N D3-branes [9]. If $r \gg L$ then $H(r) \sim 1$ and the metric reduces to 10-dimensional flat spacetime. On the other hand, $r \ll L$ corresponds to $H(r) \sim L^4/r^4$ and the metric reduces to

$$ds^{2} = \frac{r^{2}}{L^{2}} \left(\eta_{\mu\nu} dx^{\mu} dx^{\nu} \right) + \frac{L^{2}}{r^{2}} \left(\delta_{ij} dx^{i} dx^{j} \right) = \frac{L^{2}}{z^{2}} \left(\eta_{\mu\nu} dx^{\mu} dx^{\nu} + dz^{2} \right) + L^{2} ds^{2}_{S^{5}}$$
(2.44)

after introducing the coordinate $z = L^2/r$ as well as spherical coordinates for S^5 given by $\delta_{ij}dx^i dx^j = dr^2 + r^2 ds_{S^5}^2$. This metric corresponds to the $AdS_5 \times S^5$ metric, and the limit is known as near-horizon limit or throat. The two cases are depicted in Figure 2.2.



FIGURE 2.2: Taken from [9]. Representation of the supergravity metric with both limits considered. $r \gg L$ recovers flat spacetime, while the throat region $r \ll L$ reduces the metric to AdS_5 .

This means that we have two kinds of closed stirngs: closed string propagating in flat spacetime and closed strings propagating in the throat. Taking the low energy limit $\alpha' \to 0$ both type of strings decouple. Indeed, using the energy-time relation the energy measured from an observer at infinity is given by

$$E_{\infty} = \sqrt{-g_{00}} E_r, \qquad (2.45)$$

and if we consider the throat limit $r \ll L$ where $H(r) \sim L^4/r^4$ the energy measured from infinity is very small inependently of the value of the energy at a fixed position E_r , since

$$\sqrt{\alpha'}E_{\infty} \sim \frac{r}{L}\sqrt{\alpha'}E_r \to 0.$$
 (2.46)

The observer at infinity therefore sees two different modes: strings propagating in flat spacetime and strings propagating in $AdS_5 \times S^5$ in the throat. To summarize, in the limit low energy limit we find a background which consists of two regions: the throat region and flat spacetime. The dynamics of closed strings in flat spacetime are described by 10-dimensional Type IIB supergravity, while strings in the throat are described by fulctuations about the $AdS_5 \times S^5$ solution of IIB supergravity.

2.7.3 The conjecture

In both pictures we found two decoupled theories:

- Open perspective: $\mathcal{N} = 4$ SYM in flat 4-dimensional spacetime and type IIB supergravity on (9+1)-dimensional spacetime.
- Closed perspective: Type IIB supergravity on $AdS_5 \times S^5$ and type IIB supergravity on (9 + 1)-dimensional spacetime.

Due to the equivalence of open and closed strings both perspectives should be equivalent descriptions of the same object. Type IIB supergravity is present in both perspectives. This leads to conjecture that $\mathcal{N} = 4$ SYM in four dimensions is equivalent to type IIB supergravity on $AdS_5 \times S^5$. This is known as the AdS_5/CFT_4 correspondence, and it was the first prticular example of the gauge/gravity duality. The concept of gauge/gravity duality comes from the fact that the conjecture states the equivalence between a gauge theory withouth gravity, SYM, and a gravity theory, supergravity, in a different number of dimensions. This statement is extremely surprising, since is telling us that gravity is equivalent to another theory without gravity. But one can think a bit about the procedure we followed and make sense of it. We started, in both perspectives, from type IIB supergravity. We then made use of the equivalence between D-branes and black *p*-branes, each one valid in a different regime. Taking then the low energy limit we recovered, on one side, SYM, and in the other, supergravity in AdS_5 . One can think of superstring theory as containing both gravity (closed strings) and gauge theories (open strings), and it can be reasonable to think that under certain limits the theory recovers pure gravity and pure gauge theories, which will be equivalent due to the open-closed srting equivalence. Before moving into the formalism resulting from the duality we need to make more precise the limit of validity of the correspondence.

Wit the presence of N D-branes in string theory is possible to describe the Higgs mechanism, through which strings get mass. This occurs when there is a string stretched

between two branes located at different spacetime points, sparated a certian distance r. If one considers N + 1 D-branes, and separates one brane, the gauge group breaks from $U(N + 1) \rightarrow U(N) \times U(1)$ and the gauge field corresponding to the string stretched between the branes acquiers a mass $m = r/2\pi\alpha'$. Therefore, when taking the low energy limit $\alpha' \rightarrow 0$ all field theory quantities must be fixed, i.e., in particular $u \equiv r/\alpha'$. Therefore, the correct decouping limit to consider is

$$\alpha' \to 0$$
 with $u = \frac{r}{\alpha'}$ kept fixed (2.47)

which is knowns as the Maldacena limit. In this limit we have, from 2.30, $L^4/r^4 \to \infty$, so that we effectively zoom into the throat region since L is a constant value. From the analysis of loop diagrams in field theory we know that perturbation is trustable when

$$g_{YM}^2 N \sim g_s N \sim \frac{L^4}{l_s^4} \ll 1$$
 (2.48)

where we made use of the identification 2.42. On the other hand, the classical gravity description is reliable when the radius of curvature of the spacetime is large compared to the string length,

$$\frac{L^4}{l_s^4} \sim g_s N \sim g_{YM}^2 N \gg 1.$$
 (2.49)

From these relations we see that the criterium of having $g_s N$ large or small translates into having $g_{YM}^2 N$ large or small, i.e., large or small t'Hooft coupling constant $\lambda \equiv g_{YM}^2 N$. Therefore, the question of which perspective must be used translates into the question of whether the gauge theory is strongly or weakly coupled. Further, notice that the regimes 2.48 and 2.49 are incompatible, and the theory in these regimes is very different ¹². This is the reason this correspondence is called duality¹³. Furthermore, given the relations 2.42 and 2.43e together with 2.23,

$$\frac{l_p^8}{L^8} \sim \frac{1}{N^2}, \qquad \frac{l_s^2}{L^2} \sim \frac{1}{\sqrt{\lambda}}.$$
 (2.50)

In the large-N limit and the strongly coupled regime $\lambda \to \infty$

$$\frac{l_p^8}{L^8} \sim \frac{1}{N^2} \ll 1, \qquad \frac{l_s^2}{L^2} \sim \frac{1}{\sqrt{\lambda}} \ll 1.$$
 (2.51)

In this limit the gravity theory reduces to Einstein's gravity. The first condition implies that quantum corrections can be ignored and the spacetime background is fixed. The second condition implies that only the massless low-energy limit modes must be considered, and the extended nature of the strings can be eluded, which means weak curvature. The theory reduces to classical Einstein's gravity, and this is the limit we will

 $^{^{12}}$ This is one of the reasons for which is very difficult to prove, since in each regime only one perspective is valid.

¹³Notice that the condition $g_s N \gg 1$ translates into large-N. Indeed, the S-duality transforms $g_s \to 1/g_s$, and therefore it is necessary to have large-N in order to satisfy the condition.

consider in this thesis. Furthermore, we presented the weak form of the conjecture, for which $N \to \infty, \lambda \to \infty$. The strongest form of the correspondence asserts the duality for any value of N and λ . In this case the only requirement is for the spacetime to be asymptotically AdS as we approach the boundary. This is the most interesting form of the conjecture and the one used in this thesis while considering the classical limit 2.51. With this formulation it is possible to consider all kinds of processes in the five dimensional spacetime, such as the presence of black branes [6].

2.7.4 Holography

The gauge/gravity duality is a particular example of the concept of holography. Such idea was originated from the Bekenstein-Hawking expression for the entropy of a black hole

$$S = \frac{A}{4\pi G} \tag{2.52}$$

where A is the area of the black hole. This expressions shows that the entropy of the black hole, a quantity that is extensive, depends only on the area. Therefore, one expects to be able to explain the system by the degrees of freedom present on the area, i.e., the boundary. This idea is concrete in the correspondence, where a 5-dimensional gravity theory can be described by a 4-dimensional QFT. The 4-dimensional CFT is said to be living "on the boundary" of the AdS theory, where the r coordinate is identified with the extra dimension which is not present in the CFT theory, so that for a slice of constant r of AdS one has the CFT theory in Minkowski spacetime.

2.8 The UV/IR connection

We wish to find a relation between the IR/UV regimes in the field theory and the AdS geometry. We already saw that the extra coordinate r (or z analogously) of AdS 2.7 is identified with the energy scale, and thus the RG flow. From standard calculations we infer that the energy E_{YM} of an object in the gauge theory is related to the proper energy E of this object in the bulk by

$$E_{YM} = \sqrt{-g_{00}}E = \frac{L}{z}E,$$
 (2.53)

given the AdS metric 2.7. We thus see that physical processes in the bulk with identical proper energies but ocurring at different radial positions correspond to different gauge theory processes with energies that scale as $E_{YM} \sim 1/z$. The gauge theory energy is larger if the excitation is nearer the AdS boundary, corresponding to $z \to 0$. This result allows to identify the z-direction with the direction along the RG flow of the gauge theory. In particular, the UV regime corresponds to $z \to 0$, i.e., the boundary, and the IR regime corresponds to $z \to \infty$, i.e., the throat.

2.9 Field-operator map

Given the AdS/CFT correspondence it is natural to expect a relation between operators on the CFT side and fields in the AdS side. In euclidean space, the partition function with sources for the composite operator \mathcal{O} is written as

$$Z_{\mathcal{O}} [\phi_0]_{CFT} = \int D \left[\text{SYM fields} \right] \exp \left(-S_{SYM} + \int d^4 x \phi_0 \mathcal{O} \right)$$
(2.54)

where ϕ_0 is the bulk field evaluated on the boundary, and acts as a source for the operator on the boundary. The natural relation to establish between partition functions, known as the Witten perscription, is

$$Z_{\mathcal{O}}\left[\phi_{0}\right]_{CFT} = Z_{\phi}\left[\phi_{0}\right]_{string}.$$
(2.55)

Thus one can compute correlation functions on the field theory side by using the string action. Therefore, every field on AdS is in a one to one correspondence with an operator in the field theory. It is possible to simplify 2.55 in the limit $\alpha' \to 0$, $g_s \to 0$ and $g_s N \gg 1$, so that there are no string worldsheet corrections or quantum string corrections, and therefore the supergravity approximation is valid. One can then use the saddle point approximation

$$Z_{string} \simeq e^{-S_{SUGRA}} \to Z_{\mathcal{O}} \left[\phi_0\right]_{CFT} = Z_{\phi} \left[\phi_0\right]_{string} = e^{-S_{SUGRA}}.$$
 (2.56)

The path integral is dominated by the minimum action, i.e., the classical on-shell supergravity action. Let us make the statement 2.55 more specific by looking at a Klein-Gordon scalar field in (d + 1) spacetime dimensions. The equation of motion reads

$$z^{d+1}\partial_{z}\left(z^{1-d}\partial_{z}\phi\right) - k^{2}z^{2}\phi - m^{2}L^{2}\phi = 0.$$
(2.57)

Near the boundary $z \to 0$ the second term can be neglected and the equation is solved with a solution of the form

$$\phi \simeq A(x)z^{d-\Delta} + B(x)z^{\Delta}$$
 as $z \to 0$ (2.58)

where $\Delta = d/2 + \nu$, $\nu = \sqrt{m^2 L^2 + d^2/4}$. We therefore find the relation $m^2 L^2 = \Delta(\Delta - d)$ between the mass of the spin two Klein-Gordon field and the dimension of the operator. One can do the same for other fields, and find the relations between scaling dimensions and mass ¹⁴. An interesting result of this solution is that in order to have real exponents in 2.58 the fields must satisfy the Breitenlohner-Freedman bound

$$m^2 L^2 \ge -\frac{d^2}{4}.$$
 (2.59)

It is thus possible to have negative squared-mass, up to a certain limit where the modes grow exponentially in time and the theory is unstable [10]. It turns out that if the bound is satisfied the first term in 2.58 is non-normalisable On the other hand the second term is normalisable and does not affect the leading boundary behaviour.

¹⁴The relation for other fields, such as massless spin two fields, p-forms and others can be found in [6].

Since in 2.58 the behaviour of the field is controlled by A(x), the presence of such nonnormalisable term should correspond to a deformation of the boundary theory of the form

$$S_{CFT} \to S_{CFT} + \int d^d x \phi_0(x) \mathcal{O}(x) \quad \text{with} \quad \phi_0(x) = A(x).$$
 (2.60)

Thus, the non-normalisable term determines the boundary theory Lagrangian. Therefore, the boundary of the field is defined as

$$\phi_0(x) \equiv z^{\Delta - d}\phi. \tag{2.61}$$

On the other hand, the normalisable term is identified with the VEV of the operator, so that $\langle \mathcal{O}(x) \rangle \sim B(x)$. To sum up, the field-operator map teaches us that fields on the gravity side act as sources for operators on the gauge theory. In particular, the non-normalisable terms act as the sources. This can be understood in the following way: on the boundary we have a gauge theory which can have infinities due to the divergences of loop diagrams. Therefore, one identifies this singular behaviour with the non-normalisable terms. This relation is very useful since allows to account for operators on the boundary by introducing the corresponding fields in the bulk which source them. The relation between fields and operators is also known as the holographic dictionary which establishes the relations between fields and operators being like a cooking book, see table 2.1.

Holographic dictionary		
Gauge theory	Gravity theory	
Energy-momentum tensor	Metric	
Conserved current	Gauge field	
Fermionic operator	Dirac field	
Conformal dimension	Mass	
Global symmetry	Isometry	

TABLE 2.1: The field-operator map establishes the relation between operators on the gauge theory and fields on the gravity theory which source the operators.

2.10 Charge and chemical potential

In this thesis we wish to consider the charge of the QGP¹⁵ coming from the quarks. To do so, we work with the grand canonical ensemble, being the parameters the temperature and the chemical potential. Within the AdS/CFT the field-operator map teaches us that operators on the boundary are sourced by field in the bulk. On the gauge theory side we have a global four current J^{μ} which is conserved, i.e., $\partial_{\mu}J^{\mu} = 0$. Such current will couple to a bluk field by $\int d^d x \phi_{\mu} J^{\mu}$, being ϕ_{μ} the gravity field. To what field does it couple to? The conformal symmetry present on the gauge theory has a geometrical interpretation in the AdS metric, where the extra dimension is the energy scale. Therefore, the

¹⁵We also wish to consider the spatial anistropy. This will be discussed in the next section.

conformal symmetry in CFT translates into isometries of AdS spacetime, which are nothing else than diffeomorphism (local coordinate transformations). Indeed, recall that both sides are invariant under the group SO(d, 2); in the CFT it refers to global Noether spacetime symmetries and in the AdS refers to isometries. Thus, one concludes that global symmetries in CFT correspond to local transformations on AdS¹⁶. Local transformations translate into gauge transformations, and therefore we are lead to the conclusion that the field sourcing the current on the boundary is a gauge field, i.e., a five-dimensional gauge field $A_{\mu}(z, \vec{x})$. Further, it is also clear that the field source must be a one-form in order to have a well-defined Lagrangian so that it is a scalar quantity. It is also clear that such combination is gauge invariant, as it should be. Therefore, to take into account the charge of the particles one must add the term

$$S' = \int d^4x A^0_{\mu} J^{\mu}, \qquad (2.62)$$

where $A^0_{\mu} \equiv \lim_{z \to 0} A_{\mu}(z, \vec{x})$ is the source field on the boundary. Further, $J^0 \sim Q$, and therefore the simplest way to account for the charge is to consider a gauge field with a non-zero temporal component, so that we actually recover the grand-canonical formalism given by $H - \mu Q$. Indeed,

$$S' = \int d^4x A_t^0 Q \sim \mu Q \tag{2.63}$$

given the identification $A_t^0 \equiv \mu$. Therefore, the boundary value of the gauge field is the chemical potential of the gauge theory. The presence of a gauge field in the bulk has an important meaning: in order to take into account the charge density of the gauge theory we must introduce a charged black hole in the bluk. We can extract another important information for the gauge field. Given a charged black hole, due to spherical symmetry the gauge field can only depen on the radial coordinate, so that the gauge field is $A_{\mu}(z)$. Let us see more specifically the gauge field behaviour on the boundary. The Maxwell equation reads

$$0 = \frac{1}{\sqrt{-g}} \left(\partial_{\mu} \sqrt{-g} F^{\mu\nu} \right) \tag{2.64}$$

so that

$$\frac{5}{z}A_t' + A_t'' = 0, (2.65)$$

where the primes denote derivative respect to the z coordinate. The solution of this equation is easily found to be

$$A(z) = A_t^0 \left(1 + A_t^1 z^2 \right).$$
(2.66)

 A_t^0 is the boundary term (chemical potential), so that at $z \to 0$ we are left with such term. This is precisely the non-normalizable term, which we must identify with

¹⁶This is a very important property of the correspondence, and due to this reason it is said that the correspondence is a local/global duality.

the source term. On the oher hand, the fast falloff of the second term implies that such term is the VEV of the operator, $\langle J^0 \rangle \sim A_t^0 A_t^1$. A comment is in order. In this thesis we use the formalism in which the chemical potential is introduced by deforming the hamiltonian as $H \to H - \mu Q$. This implies the periodic and anti-periodic boundary conditions for bosons and fermions respectively [11]. Another possible formalism is the twisted formalism, in which the charged particles pick up a phase when transported and changes the boundary conditions for the fields. Both frameworks are equivalent so that one choses the most convinient one for the problem.

Within our framework it is necessary to demand $A_t(r_h) = 0$ in order to ensure regularity. The reasoning goes as follows. The Euclidean section of the black hole has the topology of a disc in the (r, τ) directions being τ the periodic variable related to the temperature. Using Stoke's theorem ¹⁷

$$\int_{\partial D} A_0 d\tau = \int_D F_{r0} dr d\tau, \qquad (2.67)$$

where D is a disc with origin at $r = r_h$ and the electric field is in the radial direction due to spherical symmetry. If we now shirnk the disc to zero size the right hand side of the equation vanishes. Thus, the left hand side must vanish as well, which approaches the value $\beta A_0(r_h)$. This implies that $A_0(r_h) = 0^{18}$. Furthermore, due to gauge invariance the meaningful quantity is the difference of the gauge field at different spacetime points. Imagine adding charge ΔQ from the asymptotic ifinity to the black hole. This requires an energy ΔE proportional to the difference of the gauge field between the infinity and the horizon. Therefore, the chemical potential should be given by

$$\mu = A_t^0 - A_t(r_h). \tag{2.68}$$

But given the condition $A_t(r_h) = 0$ the chemical potential reduces to the value of the gauge field on the boundary, i.e., $\mu = A_t^0$.

2.11 Adding temperature

The temperature formalism in $\mathcal{N} = 4$ SYM is introduced in the usual manner by compactifying the temporal direction on the circle and identifying $\beta = 1/T$. This compactification breaks supersymmetry due to the periodic and anti-periodic boundary conditions given to bosons and fermions respectively. The fermions become massive, but the gauge fields and scalars (at classical level) remain massless due to gauge invariance. This means that the theory is pure QCD, with only gauge fields.

The way to introduce temperature in the AdS/CFT is by adding a black hole in the AdS_5 bulk. Black holes emit Hawking radiation ¹⁹. In this way modes on the D3-branes are excited. The AdS/CFT correspondence asserts that the Hawking temperature of the

 $[\]overline{{}^{17}\text{Stoke's theorem states } \int_M d\omega = \int_{\partial M} \omega}$ for a p-form ω . In this case we have a one-form, the gauge field, which satisfies the usual relation F = dA.

¹⁸Within the "twisted" formalism the conditions are $A_0(r_h) = \mu$, $A_t^0 = 0$. In order to satisfy Stoke's theorem the gauge field has a delta type singularity at the horizon, and the topology is the one of a cylinder instead of a disc.

¹⁹Recall that extremal black holes do not emit radiation. Therefore, one wants to introduce a nonextremal black hole in order to have temperature.

black hole is the temperature of the gauge theory, and other themodynamic quantities such as the entropy are also identified. In order to introduce a black hole, the metric of the black D3-branes must have a blackening factor f(r),

$$ds^{2} = H(r)^{-1/2} \left(-f(r)dt^{2} + dx^{2} \right) + H(r)^{1/2} \left(\frac{dr^{2}}{f(r)} + r^{2}d\Omega_{5}^{2} \right)$$
(2.69)

which vanishes at the horizon, $f(r_h) = 0$. The Hawking temperature can be found by compactifying the black hole metric. Consider a general metric

$$ds^{2} = f(r)d\tau^{2} + g(r)^{-1}dr^{2} + r^{2}d\Omega^{2}$$
(2.70)

which is already in euclidean form. We assume that both f(r) and g(r) have a first order zero at $r = r_h$, but the first derivative at such point is not vanishing. Expanding both functions to first order around the horizon as $f(r) \simeq f'(r_h)(r - r_h)$ (same for g(r)) the metric reduces to

$$ds^{2} = f'(r_{h})(r - r_{h})d\tau^{2} + \frac{dr^{2}}{g'(r_{h})(r - r_{h})} + \dots$$

$$\equiv \rho^{2}d\phi^{2} + d\rho^{2} + \dots, \qquad (2.71)$$

with $\rho^2 = \frac{4(r-r_h)}{g'(r_h)}$ and $\phi = \frac{\tau}{2}\sqrt{g'(r_h)f'(r_h)}$, so that (ρ, ϕ) are polar coordinates. Therefore, to ensure regularity on the plane ϕ must have period 2π , impliying

$$\tau = \tau + \frac{4\pi}{\sqrt{f'(r_h)g'(r_h)}}.$$
(2.72)

Due to the compactification τ must be periodic with periodicity 1/T. Therefore,

$$T = \frac{\sqrt{f'(r_h)g'(r_h)}}{4\pi}$$
(2.73)

which is the famous Hawking radiation of a black hole. We have seen that in order to introduce thermodynamics on the CFT theory living on the boundary black holes must be present in the AdS bulk. The easiest black hole to consider is the Shcarwshiwild black hole. But one can also introduce a charged black hole. This black hole will emit also Hawking radiation as long as it is not extremal, and the charge of the black hole will be dual to the charge of the particles on the CFT side. This is the type of black hole that we will deal with in this thesis in order to account for the charge of the QGP.

3. Holographic QCD

We now wish to design a dual gravity theory of QCD using the AdS/CFT correspondence. This task can be done in two ways. The first one is the so called top-down approach. The idea is to consider a setup of D-branes to find an exact dual theory for QCD. This task is extremely difficult, and although much effort has been invested in it, the resulting theories do not have the same spectrum as QCD [12]. Therefore, in the mids 00s the so called bottom-up approach was introduced. The idea is to give up the ambitious goal of finding a precise holographic dual to QCD, and instead construct a UV/IR effective theory to capture the dynamics of the gauge theory. The first bottom-up models were crude [13] and although some behaviour could be reproduced, they led to inconsistencies. In response, the improved holographic QCD models were introduced [14, 15], where a non-constant dilaton profile is considered and reproduces the RG flow of the underlying gauge theory. The idea is to reproduce the basic building blocks of QCD by introducing fields in the bulk which will constrain the gauge theory living on the boundary. To do so, we first need to know the basic features of the QGP and the characteristics we wish to account for. In this chapter we present the QGP and the gravitational theory dual to QCD, the Einstein-Maxwell-Axion-Dilaton 5d action.

3.1 The Quark-Gluon plasma

The QGP is a QCD state in which quarks and gluons are deconfined. This is a very interesting feature which differentiates this state from usual QCD matter, in which quarks and gluons are confined forming hadrons. The QGP can be produced at heavy ion collisions in particle accelerators (RHIC, SPS, LHC) by colliding gold or lead atoms, making it an experimentally accessible system. The QGP created at heavy ion collisions is the one that we will be concerned with, and we will try to reproduce some of its most important features, in particular the anisotropy and electric charge. Besides, it is believed that the QGP can be present in the core of neutron stars, where pressure and temperature is high enough to create the plasma. Further, the QGP was the state of matter present instants after the Big Bang occurred. Therefore, there are many motivations to study it due to its connections to astrophysics and cosmology.

But being such a particular QCD state, the main interest of this thesis lies on studying the QGP to see what we can learn about QCD in the strongly coupled regime. Indeed, the QGP is a strongly coupled QCD state never studied in depth before, which could help to gain substantial understanding of QCD. Even though QCD has been studied for many years and much effort has been invested in trying to achieve progress, its phase diagram remains as an enigma, being only partially understood theoretically and experimentally. From the theoretical point of viev, it is still poorly understood due to the limitations of traditional methods. In particular, the usual perturbative methods and lattice QCD do not apply due to the strong coupling and the real-time phenomena. The AdS/CFT correspondence offers a new technique to study the QGP by using the mapping to low coupling in the gravity side. This is of course not an easy task, which involves a good knowledge of the basics of the plasma in order to be able to formulate a reasonable gravity dual.



FIGURE 3.1: Taken from [9]. QCD phase diagram as a function of the baryon density. As the temperature or the baryon density increases the phase transition occurs, giving place to the deconfined QGP. As indicated in the diagram, the QGP is created at particle accelerators (LHC, RHIC) and it is present in neutron stars. At low temperatures and high density a colour superconductor phase is expected, in which colour and flavour d.o.f. couple to each other. In this thesis we will just be concerned with the QGP created at heavy ion collisions around 175 MeV.

The QGP created at heavy ion collisions is formed at a temperature around 175 MeV, where the phase transition from usual matter to the plasma takes place, see Figure 3.1. When the phase transition occurs¹, quarks and gluons deconfine, giving place to the QGP. Nowadays, it is understood that the QGP is a strongly coupled system due to the data collected at particle accelerators [16]. The data teaches us that the plasma has a very low viscosity [12], even lower than water or helium, making it one of the lowest viscosity fluids known. Low viscosity means that momentum is not well transported along the plasma, and it is not possible to define quasiparticles with long mean free paths, which implies strongly coupled constituents. Therefore, the QGP can be thought as a very low viscosity fluid, for which hydrodynamics is a good description. One of the main features of the QGP created at heavy ion collisions is the spatial anisotropy along a certain axis. When two beams collide they do so in an almond-shaped region, since there is a non-zero impact parameter which makes the collision non-central; see Figure 3.2. Within the context of hydrodynamics, the colliding region can be understood as

¹It is still not known if this phase transition is first or second order.

a droplet which explodes due to the difference of pressure along the different axes. In order to reproduce this behaviour using the gauge/gravity duality, we will introduce an axion field in the bulk, which will make the resulting geometry anisotropic.



FIGURE 3.2: Taken from [9]. Result of the non-centrality of the two beams colliding. The beams collide in a almond-shaped region (yellow) which makes the QGP spatially anisotropic in one direction, the vertical y axis in the figure.

Another important property of the QGP is the electric charge. Quarks have electric charge and therefore the QGP is also charged. But the charge of the QGP cannot be just the sum of the charges of all the quarks, due to screening effects [16]. Furthermore, in the strongly coupled regime perturbation theory cannot be used and renormalization is not a possible technique, so we cannot say much about the exact behaviour of the charge. But as it is thought in condensed matter systems, we can consider the charge of the QGP to be conserved ². Therefore we will consider that there is a global U(1) symmetry, so that the J^{μ} current is conserved and we can make use of the AdS/CFT correspondence. As previously explained in section 2.10 this current will be sourced by the presence of a gauge field in the gravitational bulk theory, corresponding to the chemical potential on the boundary.

3.2 Properties of the QGP

In this section we present some important features of the QGP which will be useful when considering the corresponding bulk theory. There are of course important differences between the QCD and $\mathcal{N} = 4$ SYM, but some of them become relatively unimportant due to the characteristics of the QGP. The first problem dwells in the fact that QCD is a confining theory, while $\mathcal{N} = 4$ SYM being a conformal theory cannot be confining ³. But as we well know, the QGP is a QCD state of matter which is deconfined, and this obstacle disappears. Furthermore, the QGP shows a scale invariant behaviour as the temperature increases (see Figure 3.3), and we will be concerned with this regime of the strongly coupled QGP.

 $^{^{2}}$ Also, when considering such a global symmetry it is possible to develop and effective theory [17].

³The supersymmetry of the theory is broken when temperature is considered due to the different boundary conditions for bosons and fermions.

Energy density



FIGURE 3.3: Taken from [18]. This graph represents the energy density dependence on the temperature for different gauge groups and improved holographic QCD (ihQCD). As the temperature increases, the plasma becomes more and more scale invariant. Further, the graph shows how the method ihQCD gives good results when compared to the theory, and this method is the one in which our gravitational setup is based on [2, 14, 15].

But to make justice there are other aspects which cannot be avoided, such as the fact that for QCD N = 3 instead of $N \to \infty$. Further, even though the QGP is strongly coupled, the limit $\lambda \to \infty$ is of course not a real situation. Taking into account all these corrections still remains as a challenge nowadays, being one of the most current topics in QGP research. The comprenhension of strongly coupled systems is imperative, and the AdS/CFT correspondence has been studied by many authors, hoping that the duality will be able to reproduce the behaviour of the plasma in the strongly coupled regime where traditional techniques do not apply. In the end, what we are doing is what physicists are the best at: modelling a system by just taking into account the most important and simplest features, so that the essence of the phenomena can be captured and hopefully described by the method.

3.3 The domain wall method

The domain wall method consists of constructing a gravitational theory which can be dual to the RG flow of the gauge theory. In order to do so, the dilaton field is responsible for mimicking the RG behaviour. A non trivial potential for the dilaton is introduced which reduces to the AdS cosmological constant at the fixed points and a particular metric is considered, which reproduces important gauge theory properties, such as the c-theorem.

3.3.1 The metric ansatz

The AdS metric 2.7 can be written as

$$ds^{2} = du^{2} + e^{-2u/L} \eta_{\mu\nu} dx^{\mu} dx^{\nu}$$
(3.1)

provided that $r \equiv Le^{u/L}$ ⁴. Recall that L is the radius of curvature, which is positive by definition. In the u coordinates the geometry translates into $u \to -\infty$ for the UV and $u \to \infty$ for the IR. As we approach the UV regime we expect an AdS_5 geometry due to the conformality of QCD at high energies⁵. Let us now consider a more general metric than 3.1. We consider

$$ds^{2} = du^{2} + e^{2A(u)} \left(\eta_{\mu\nu} dx^{\mu} dx^{\nu} \right), \qquad (3.2)$$

known as the domain wall metric. We want this metric to be the AdS metric on the UV boundary. Therefore, this metric recovers 3.1 if we ensure

$$A(u) \to -u/L + A_0 + A_1 e^{4u} + \dots \qquad u \to -\infty.$$
 (3.3)

The radial coordinate r relates to u by $du = e^A dr$. Working out the relation, it is easy to show that

$$A(r) \to -\log r/L + A_c + \frac{A_1 r^4}{L^4} + \dots \qquad r \sim L e^{u/L} \to 0.$$
 (3.4)

We now turn to the IR behaviour. Wilson loops give a test for confinement. In order to have a field theory with a linear potential $V \sim \kappa r$ which increases with the distance r between particles, the Wilson loop expectation value must follow an area law [5],

$$\langle W \rangle \sim e^{-\kappa Area(C)},$$
 (3.5)

where C is the loop which the quark follows⁶. Within the context of the gauge/gravity duality the expectation value of the wilson loop is given by

$$\langle W \rangle \sim e^{-S_{NG,min}},$$
(3.6)

where $S_{NG,min}$ is the on-shell value of the Nambu-Goto action corresponding to a string. For an analysis of the potential energy of a string in AdS spacetime and its computation, we refer to [15]. The question we are concerned with is which IR

⁴We call the z coordinate from 2.1 r. Notice that the change we made is actually the only one that we can make. Indeed, if r was an even power, the horizon and boundary would coincide and would spoil AdS. If the power is any odd power, different from 1, one can always reescale the radial coordinate to get 3.1.

⁵One actually expects to recover AdS with logarithmic corrections. Indeed, as $1/\lambda \sim \log E$ we expect that $e^{-\phi} \sim -\log r$.

⁶Recall that in conformal field theories the only dimensionful quantity is the distance R, and therefore the potential scales as $V \sim 1/R$. Thus, CFT are not confining.

asymptotics give rise to non confining and scale invariant theories. As shown in [15], if we consider a logarithmic behaviour on the IR of the form

$$A(r) \sim -\delta \log r + \dots \quad \delta \ge 1 \tag{3.7}$$

there is no confinement, and the IR corresponds to a scale invariant conformal theory⁷. With this behaviour we are describing the scale invariant QGP detailed in 3.2, and therefore 3.7 is the kind of function we will consider.

3.3.2 The Dilaton as the RG flow

The dilaton is responsable for reproducing the RG flow of the gauge theory. Such identification comes from the AdS/CFT identification 2.42 which relates the string coupling to the gauge coupling. Due to such relation, $e^{\phi} \sim \lambda$, and thus one allows to identify the dilaton to the gauge theory RG flow. In order to reproduce the RG flow using the dilaton a non-trivial potential is introduced, which is reponsable for mimicking the desired behaviour of the AdS metric 3.2. Generalizing, consider the action in (d+1)-dimensions of the dilaton coupled to gravity,

$$S = \int d^{d+1}x \sqrt{-g} \left(\frac{R}{16\pi G} - \frac{1}{2} \partial_{\mu}\phi \partial^{\mu}\phi - V(\phi) \right), \qquad (3.8)$$

which is solved by the metric 3.2 and yields the following dilaton equation of motion and Einstein equation

$$\frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}g^{\mu\nu}\partial_{\nu}\phi\right) - V'(\phi) = 0 \tag{3.9}$$

$$G_{\mu\nu} = 8\pi G \left(\partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} g_{\mu\nu} \partial^{\beta} \phi \partial_{\beta} \phi - g_{\mu\nu} V(\phi) \right) \equiv 8\pi G T_{\mu\nu}.$$
(3.10)

At the stationary points ϕ_i there is a trivial solution to the equations with the dilaton taking a constant value, $\phi(u) = \phi_i$. At this point, the Einstein equation reduces to $G_{\mu\nu} = -8\pi G g_{\mu\nu} V(\phi_i)$. This is identical to the Einstein equation of AdS spacetime $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ if we identify

$$\Lambda_i = 8\pi G V(\phi_i) = -\frac{d(d-1)}{L_i^2}.$$
(3.11)

Therefore, constant scalar fields with AdS_{d+1} geometry of scale L_i are critical solutions which correspond to conformal theories at RG fixed points on the field theory side. In our particular case we have a theory with d = 4, yielding $\Lambda = -12/L^2$. This computation indicates that the first term of the dilaton potential must be the cosmological constant of AdS. Indeed, if we consider the metric 3.2 which solves the action 3.8, we get the Einstein equation

$$12(A'(u))^2 - V(\phi(u))e^{2A} - \frac{1}{2}(\phi'(u)^2) = 0.$$
(3.12)

⁷The constraint $\delta \geq 1$ comes from the NEC.

Upon using $\phi \to 0$ and $A \to -u/L$ when $u \to -\infty$ the potential to first order is

$$V \to -\frac{12}{L^2},\tag{3.13}$$

which recovers AdS at such stationary point. Therefore, the potential has as a first term the cosmological constant. Let us now analyse what happens with the dilaton equation of motion near the boundary given the corresponding asymptotics. The procedure is analog to the one performed in section 2.9. The dilaton equation of motion reduces to

$$\phi'' - 4\phi' - m^2\phi = 0, \tag{3.14}$$

which has the solution $\phi(u) = Ae^{\Delta_+} + Be^{\Delta_-}$. The coefficients are given by

$$\Delta_{+-} = \frac{4 \pm \sqrt{16 + 4m^2}}{2}.\tag{3.15}$$

If we wish to saturate the BF bound, then $m^2 = -4$ with the solution $\phi(u) = e^{2u} (\phi_+ u + \phi_-)$. From the analysis in section 2.9 we see that the first term corresponds to a mass deformation, while the second one is the VEV of the dual operator. The action we will use in this thesis is more complicated than 3.8 since it has more field content, but the idea behind it is the same as the one presented in these previous sections. The dilaton will mimick the RG flow, while other fields (axion, gauge field) will source other features.

3.3.3 Holographic c-theorem

The c-theorem states an important property of the RG flow connecting two conformal field theories. In concrete, the theorem asserts that in two dimensions there is a function which decreases monotonically along the flow from the UV to the IR. Furthermore, at the fixed points such function reduces to the central charge. The extension of such theorem to four dimensions⁸ is the a-theorem [5]. The c-theorem has a natural formulation in holography. One of the Einstein equations coming from the domain wall metric 3.2 reads

$$A''(u) + \frac{8}{3}\phi'^2 = 0. (3.16)$$

This equation implies that A''(u) < 0. Therefore, A'(u) is a monotonically decreasing function of u. Since A'(u) is already negative in the UV due to 3.3, it is forced to be negative in the entire range of u. Thus, A(u) itself should be a monotonically decreasing function of u. In particular, at the fixed points we recover A(u) = -u/L. If the holographic flow is between two field theory fixed points, it should correspond to two AdS points, so $A(u) = -u/L_{UV}$ and $A(u) = -u/L_{IR}$ with $L_{UV} \ge L_{IR}$ due to the condition $A''(u) \le 0$.

⁸In four dimensions there are two central charges of interest: the Weyl anomaly and the anomaly of the VEV of the divergence of the R-symmetry current.
This is the realization of the c-theorem within holography, which has a legitimate dual in the gravity picture thanks to Einstein's equations. The c-theorem can be extracted from the null energy condition (NEC) [19], and the function

$$C(u) = \frac{a_0}{\left(A'\right)^{d-1}} \tag{3.17}$$

is monotonically decreasing along the RG flow towards the IR, recovering the central charge at the stationary points [6]. There are another two possible IR behaviours [15]. The first one corresponds to having a curvature singularity at $r \to \infty$ corresponding to the IR, where the scale factor vanishes and the geometry shrinks to zero size. This is actually the case for 3.7 and the case we will have in hand. The other possibility corresponds to have an spacetime which ends ar a finite value r_0 where there is curvature singularity. All these behaviours come only from the fact that $A''(u) \leq 0$ together with the requirement of recovering AdS at the fixed points.

Recall from equation 2.53 that given the metric 3.2, the energy of the gauge theory is $E = e^{A(r)}$. With the UV behaviour 3.4, we indeed get $E \to 0$ at the UV. For the IR regime it is necessary for the energy to go to zero as $r \to \infty$, which implies that in the IR the function A(r) can be a logarithmic function 3.7, as long as the constant in front is negative. Therefore, $A(r) \to -\infty$ as $r \to \infty$. This is the case for 3.7, so the gauge theory energy is well defined and we get a curvature singularity in the deep IR $r \to \infty$ where the conformal factor $e^{A(r)}$ vanishes.

3.4 Gravitational setup

Quantum many body systems with reduced rotational symmetry have important realizations in nature, such as the QGP produced in heavy ion collisions. Due to the field-operator map of the correspondence, a field in the bluk must be present in order to source an oprtator in the CFT theory. We already saw that in order to introduce charge one should consider a gauge field in the bulk, and now we must consider how to introduce anisotropy. The CP violating term of QCD [17], known as the theta term, is the deformation which breaks isotropy [2,3]. We consider a theta parameter that depends on one spatial direction, namely the z direction. In this manner one can simulate the almond-shape of the QGP due to the non-centrality of the collisions. Furthermore, the dilaton field is dependent on the radial direction and is the responsable to reproduce the RG flow of the gauge theory, as depicted previously. To be more concrete, the field content is:

- The dilaton, ϕ dual to the Yang-Mills operator $Tr[F^2]$ and responsible of the RG flow.
- The five-dimensional metric $g_{\mu\nu}$ dual to $T_{\mu\nu}$ responsible of the geometry.
- The axion field χ dual to the Chern-Simons term $\theta(z)Tr(F \wedge F)$ which sources the anisotropy along the z axis.
- The gauge field A_{μ} dual to the conserved current J^{μ} sourcing the charge density and chemical potential on the boundary.

It is worth mentioning that there are no fermionic gauge invariant operators in pure YM, and thus we do not expect the dual string theory to containt spacetime fermions.

Furthermore, if we consider the case in which $N_f = 0$ the tachyon is not present in the spectrum because there is no such gauge invariant operator in the gauge theory [14].

3.4.1 The gravitational action

The gravitational theory dual to the anisotorpic charged plasma is defined by the Einstein-Maxwell-Axion-Dilaton action in 5d. The action is given by [2-4]

$$S = \frac{1}{2\kappa^2} \int d^5 x \sqrt{-g} \left(R - \frac{1}{2} (\partial \phi)^2 + V(\phi) - \frac{1}{2} Z(\phi) (\partial \chi)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} B(\phi) \right)$$
(3.18)

with $\kappa \sim 1/N^{29}$. The axion potential is $Z(\phi) = e^{2\gamma\phi}$ with $\gamma > 0$ and the gauge field potential is $B(\phi) = e^{2k\phi}$ with k > 0. These potentials come from the Type IIB superstring action 2.22. We consider now the form of the dilaton potential in order to reproduce the desired behaviour. Unlike in the ihQCD method, where the potential is matched with the beta function, here we present a different approach for the UV behaviour, easier and more studied in the literature. We consider that the dilaton behaves as $\phi \to 0$ in the UV, while $\phi \to -\infty$ in the IR, as demanded by 2.42. In the papers [21,22], the potential

$$V(\phi) = -\frac{12}{L^2} + \frac{m^2}{2}\phi^2 + \mathcal{O}(3) \quad \text{as} \quad \phi \to 0$$
(3.19)

with a maximum at $\phi = 0$ is considered in order to reproduce the equation of state of QCD. Notice that this potential satisfies the condition 3.13 at the maximum. On the other hand, for large values of the dilaton, one considers the potential $V = V_0 e^{\sigma\phi}$, where V_0 is the AdS cosmological constant 3.13. This potential reproduces the speed of sound of QCD and it is the simplest choice at the same time. It turns out that the simplest potential which interpolates between 3.19 for small values of the dilaton and the exponential form for large values is given by

$$V(\phi) = -\frac{12}{L^2}\cosh(\sigma\phi) + b\phi^2, \qquad (3.20)$$

where $\sigma \geq \sqrt{2/3}$ to have a confining theory [15]. This potential is the one that we will use in this thesis, and in particular it reduces to $V \sim 6e^{\sigma\phi}$ in the IR, i.e., when $r \to \infty$. What about the anisotropy? A linear axion ansatz automatically satisfies the equations of motion of the action 3.18 and breaks isotropy in the z direction while preserving translational invariance,

$$ds^{2} = e^{2A(r)} \left(-f(r)dt^{2} + d\vec{x}_{\perp}^{2} + e^{2h(r)}dz^{2} + \frac{dr^{2}}{f(r)} \right), \qquad (3.21)$$

with $\chi = az$. The constant *a* is the so called anisotropy factor, since for a = 0 the axion field is null and the metric which solves the equations would preserve isotropy. We now proceed to study the behaviour of the metric 3.21. The spacetime must be

⁹The action is written in the Einstein frame. The Einstein frame $g_{\mu\nu}$ is related to the string frame $g_{S\mu\nu}$ by $g_{\mu\nu} = e^{-4\phi/3}g_{S\mu\nu}$ in five dimensions. For a further explanation we refer the reader to [20].

asymptotically AdS on the boundary $r \to 0$ where the theory is conformal. In order for that to happen, the spacetime should diverge on the boundary. Therefore, the metric functions and dilaton behave as

$$A(r) \rightarrow -\log r$$

$$f(r) \rightarrow 1$$

$$h(r) \rightarrow 0$$

$$\phi(r) \rightarrow 0$$

(3.22)

whenever $r \rightarrow 0^{-10}$. The potentials are exponential functions of the dilaton and therefore it is possible to do a Taylor expansion of the potentials around zero in the UV regime,

$$B(\phi) \to B_0 + B_1 \phi + \dots \qquad u \to -\infty \tag{3.23}$$

$$Z(\phi) \to Z_0 + Z_1 \phi + \dots \qquad u \to -\infty, \tag{3.24}$$

according to the Taylor expansion of the exponential. The domain-wall coordinates are defined by $du = e^A dr$ in which the metric reads

$$ds^{2} = \frac{du^{2}}{f(u)} + e^{2A(u)} \left(-f(u)dt^{2} + d\vec{x}_{\perp}^{2} + e^{2h(u)}dz^{2} \right).$$
(3.25)

¹⁰Notice the similarity with the behaviour of the domain wall metric 3.2.

4. The IR geometry

We initiate the study of the strongly coupled anisotropic charged plasma defined with the holographic gravitational action 3.18. In particular, we consider the scale invariant QGP behaviour depicted in 3.2 given by the conformal factor 3.7. The strongly correlated regime of QCD corresponds to the low energy IR, being the limit $r \to \infty$ of the radial coordinate within the AdS/CFT correspondence. Einstein's equations are the gravity equations from which it is possible extract the IR geometry, i.e., the IR metric, as well as NEC dual to the c-theorem 3.3.3. The work presented is an extension of the paper [4], where the authors studied the QGP considering anisotropy, but not the charge. This new element is introduced here.

4.1 Equations of motion

The gravitational theory dual to the QGP is given by the 5d Einstein-Maxell-Axion-Dilaton action 3.18. The dilaton equation of motion in domain wall coordinates 3.25 and radial coordinates 3.21 is given by

$$(4A'(u) + h'(u))f(u)\phi'(u) + f'(u)\phi'(u) + f(u)\phi''(u) + V'(\phi(u)) - \frac{Z'(\phi(u))a^2}{2e^{2h(u)}} + \frac{Q^2}{2e^{6A(u)+2h(u)}B^2} = 0$$
(4.1)

$$(3A'(r) + h'(r))f(r)\phi'(r) + f'(r)\phi'(r) + f(r)\phi''(r) + e^{2A}V'(\phi(r)) - \frac{Z'(\phi(r))a^2}{2e^{2h(r)}} + \frac{Q^2}{2e^{4A(r)+2h(r)}B^2} = 0.$$
(4.2)

The derivatives are taken respect to u and r respectively. Throughout the thesis we will not make a distinction in notation for the derivatives since we always make explicit which coordinate is used. The reason that we are using both coordinates is due to the fact that the domain wall coordinates u are more useful when solving differential equations and studying the asymptotic behaviour of the fields. On the other hand, the

radial coordinate r has the advantatge of being directly related to the energy of the gauge theory 2.53. The equation of motion for the gauge field is ¹

$$\partial_{\mu}(\sqrt{-g}B(\phi)F^{\mu\nu}) = 0. \tag{4.3}$$

As mentioned in section 2.10 the temporal component of the gauge field resembles the chemical potential at the AdS boundary. Therefore, we consider the simplest possible gauge field, consisting of $A_{\mu} = (A_t(u), 0, 0, 0, 0)^2$. Using the usual definition of the strength tensor, i.e., F = dA, the field strength reads $F_{u0} = A'_t(u)$, $F^{0u} = \frac{A'_t(u)}{e^{2A(u)}}$. For the radial coordinate F has the components $F_{r0} = A'_t(r)$, $F^{0r} = \frac{A'_t(u)}{e^{4A(u)}}$. Thus, the equation of motion 4.3 reads

$$\partial_u(e^{2A(u)+h(u)}B(\phi(u))A'_t(u)) = 0$$
(4.4)

$$\partial_r(e^{A(r)+h(r)}B(\phi(r))A'_t(r)) = 0 \tag{4.5}$$

with the solution

$$A_t(u) = \mu_b - Q \int^u \frac{du'}{e^{2A(u) + h(u)}B(\phi(u))}$$
(4.6)

$$A_t(r) = \mu_b - Q \int_0^r \frac{dr'}{e^{A(r) + h(r)} B(\phi(r))}.$$
(4.7)

The gauge strength square is given by

$$F^{2} = -\frac{2Q^{2}}{e^{6A(u)+2h(u)}B(\phi)^{2}},$$
(4.8)

which is the same in both coordinates, as it should be since it is a physical invariant quantity. The constant Q corresponds to the charge of the black hole. To see this, we compute the Komar integral for the charge of a black hole [23,24],

$$Q_{BH} = \int_{\partial \Sigma} d^3x \sqrt{\gamma^{(3)}} n_\mu \sigma_\nu F^{\mu\nu} B(\phi).$$
(4.9)

The integral is calculated on a constant time slice at spatial infinity, i.e., at $u \to -\infty$. Given the metric 3.25 $\sqrt{\gamma^{(3)}} = e^{3A(u)+h(u)}$, $n_0 = -e^{A(u)}\sqrt{f(u)}$ and $\sigma_u = 1/\sqrt{f(u)^3}$. Thus, the integral gives

$$Q_{BH} = Q \int_{\partial \Sigma} d^3 x = Q V_3, \qquad (4.10)$$

¹The equation of motion of the axion does not need to be considered. Indeed, the axion term has order 1/N, and therefore it can be neglected in the large-N limit [14].

²In r coordinates the gauge field takes the same form, $A_{\mu} = (A_t(r), 0, 0, 0, 0)$.

³The normalization is $n^{\mu}n_{\mu} = -1$ and $\sigma^{\mu}\sigma_{\mu} = 1$ since the hypersurface is spacelike and therefore the normal vector must be pointing outwards, while after applying Stokes the hypersurface is timelike.

showing that indeed Q is the charge of the black brane, up to a constant. Notice from the solution to the Maxwell equation that we do not know if Q is positive or negative. In the previous sections 3.4 we introduced the metric ansatz and the corresponding asymptotic behaviour of the metric functions and dilaton on the UV boundary. Given such behaviour we can now look at the asymptotic value of the gauge field on the boundary. To do so, we just plug in the behaviour of A(u), h(u) and the dilaton 3.22 into the expression 4.6⁴, which yields

$$A_t(u) = \mu_b - \frac{Q}{2B_0}e^{2u} + \dots \qquad u \to -\infty.$$
 (4.11)

We are now able to find the chemical potential, given by 2.68. Indeed,

$$\mu = A_t^0 - A_t(u_h) = \mu_b - A_t(u_h) = \mu_b \tag{4.12}$$

where in the first step we used 4.11 and in the last one we imposed $A_0(u_h) = 0$ to ensure regularity. We see that the chemical potential is then the constant term of the gauge field, as it should be due to the corresponding boundary condition $A_t^0 \equiv \mu$. Furthermore, given the AdS/CFT correspondence which identifies the source with the boundary value and the VEV of the operator with the fast falloff, we infer that the VEV of the current is $\langle J^0 \rangle \sim \frac{Q}{2B_0}$.

4.2 Einstein's equations

The AdS/CFT correspondence gives a route to study gauge theories. Indeed, the gauge theory is dual to gravity living in one more dimension. Since the correspondence is a strong/weak coupling duality, we can use this fact to study a strongly coupled gauge theory in the large-N limit by means of Einstein's gravity in the bulk. In order to find Einstein's equations we must find the Einstein tensor and the energy momentum tensor of the gravitational action 3.18. The Einstein tensor, in domain wall coordinates, reads

$$G_{tt} = -\frac{1}{2}e^{2A(u)}f(u)\left(f'(u)\left(3A'(u) + h'(u)\right) + 2f(u)\left(3A''(u) + 4A'(u)h'(u) + 6A'(u)^2 + h''(u) + h'(u)^2\right)\right)$$
(4.13a)

$$G_{xx} = G_{yy} = \frac{1}{2} e^{2A(u)} \left(7A'(u)f'(u) + 2f(u) \left(3A''(u) + 4A'(u)h'(u) \right) \right)$$
(4.13b)

$$+ 6A'(u)^{2} + h''(u) + h'(u)^{2} + f''(u) + 2f'(u)h'(u))$$

$$G_{zz} = \frac{1}{2} e^{2(A(u)+h(u))} \left(7A'(u)f'(u) + 6f(u)\left(A''(u) + 2A'(u)^2\right) + f''(u)\right)$$
(4.13c)

$$G_{uu} = \frac{f'(u)\left(3A'(u) + h'(u)\right)}{2f(u)} + 3A'(u)\left(2A'(u) + h'(u)\right).$$
(4.13d)

⁴The asymptotics 3.22 are in the radial coordinate, in the domain wall coordinate $A(u) \rightarrow -u$ on the boundary, while the other functions behave as in 3.22.

The energy momentum tensor can be derived from the action by varying respect to the metric,

$$T_{\mu\nu} = \frac{1}{2} \partial_{\mu} \phi \partial_{\nu} \phi + \frac{Z}{2} \partial_{\mu} \chi \partial_{\nu} \chi + \frac{1}{2} F_{\beta\nu} F_{\alpha\mu} g^{\alpha\beta} B + \frac{g_{\mu\nu}}{2} \left(-\frac{1}{2} (\partial\phi)^2 + V(\phi) - \frac{Z(\phi)}{2} (\partial\chi)^2 - \frac{1}{4} B(\phi) F_{\mu\nu} F^{\mu\nu} \right).$$

$$(4.14)$$

The non zero components are

$$T_{tt} = \frac{1}{4} (\phi'(u))^2 f(u)^2 e^{2A(u)} - \frac{V(\phi)}{2} f(u) e^{2A(u)} + \frac{Z(\phi)f(u)a^2}{4e^{2h(u)}} + \frac{f(u)Q^2}{4B(\phi)e^{4A(u)+2h(u)}}$$
(4.15a)

$$T_{xx} = T_{yy} = -\frac{1}{4} (\phi'(u))^2 f(u) e^{2A(u)} + \frac{V(\phi)}{2} e^{2A(u)} - \frac{Z(\phi)a^2}{4e^{2h(u)}} + \frac{Q^2}{4B(\phi)e^{4A(u)+2h(u)}}$$
(4.15b)

$$T_{zz} = -\frac{1}{4} (\phi'(u))^2 f(u) e^{2(A(u)+h(u))} + \frac{V(\phi)}{2} e^{2(A(u)+h(u))} + \frac{Z(\phi)a^2}{4} + \frac{Q^2}{4B(u) + AA(u)}$$
(4.15c)

$$T_{uu} = \frac{1}{4} (\phi'(u))^2 + \frac{V(\phi)}{2f(u)} - \frac{Z(\phi)a^2}{4f(u)e^{2(h(u)+A(u))}} - \frac{Q^2}{4B(\phi)f(u)e^{6A(u)+2h(u)}}$$
(4.15d)

Therefore, the Einstein equations reduce to the following four equations

$$6A''(u) + 2A'(u)h'(u) + 2h''(u) + 2h'(u)^2 + \phi'(u)^2 = 0$$
(4.16)

$$a^{2}e^{-2A(u)-2h(u)}Z(\phi(u)) + 8f(u)A'(u)h'(u) + 2f'(u)h'(u) + 2f(u)h''(u) + 2f(u)h'(u)^{2} = 0$$
(4.17)

$$4A'(u)f'(u) - \frac{Q^2 e^{-6A(u) - 2h(u)}}{B(\phi(u))} + f''(u) + f'(u)h'(u) = 0$$
(4.18)

$$12f(r)(A'(u))^{2} + 6A'(u)h'(u)f(u) + 3f'(u)A'(u) + f'(u)h'(u) - V(\phi(u)) - \frac{1}{2}(\phi'(u)^{2})f(u) + \frac{e^{-2h(u)-2A(u)}}{2}Z(\Phi(u))a^{2} + \frac{Q^{2}}{2B(\phi(u))}e^{-6A(u)-2h(u)} = 0.$$
(4.19)

If we do the same in the radial coordinates, the equations reduce to

$$6A''(r) - 6A'(r)^2 + 2h''(r) + 2h'(r)^2 + \phi'(r)^2 = 0$$
(4.20)

$$\frac{1}{2}a^2e^{-2h(r)}Z(\phi(r)) + 3f(r)A'(r)h'(r) + f'(r)h'(r) + f(r)h''(r) + f(r)h'(r)^2 = 0 \quad (4.21)$$

$$3A'(r)f'(r) - \frac{Q^2 e^{-4A(r) - 2h(r)}}{B(\phi(r))} + f''(r) + f'(r)h'(r) = 0$$
(4.22)

$$12f(r)(A'(r))^{2} + 6A'(r)h'(r)f(r) + 3f'(r)A'(r) + f'(r)h'(r) - V(\phi(r))e^{2A} - \frac{1}{2}(\phi'(r)^{2})f(r) + \frac{e^{-2h(r)}}{2}Z(\Phi(r))a^{2} + \frac{3Q^{2}}{2B(\phi(r))}e^{-4A(r)-2h(r)} = 0.$$

$$(4.23)$$

4.3 The ansatz functions

Once we have obtained the gravity equations we are in position to start the study of the bulk theory. We will solve Einsteins's equations in order to see the form of the metric 3.21 in the IR. We focus on the scale invariant QGP 3.2, for which the conformal factor A(r) behaves as 3.7. In analogy, we take the following ansatz for the metric and dilaton functions⁵

$$A(r) = \Delta_1 \log(ar) + \log L \tag{4.24}$$

$$h(r) = \Delta_2 \log(ar) + \log c_1 \tag{4.25}$$

$$\phi(r) = \Delta_3 \log(ar) + \log c_3 \tag{4.26}$$

with L, c_1, c_3 constants and $\Delta_1, \Delta_2, \Delta_3$ real parameters; $\Delta_i \in \mathbb{R}, i = 1, 2, 3$. Given the identification between the dilaton and the gauge theory coupling 2.42, in the IR regime $r \to \infty$ the dilaton must grow monotonically in order to have a strongly coupled theory. Also, the conformal factor represents the energy scale by $E = e^{A(r)}$ and therefore we demand $A(r) \to -\infty$ so that the relation represents a suitable energy scale for the gauge theory. Therefore, the ansatz functions are constrained by the following conditions

$$\Delta_1 < 0$$
 and $\Delta_3 > 0$ in the IR regime $r \to \infty$. (4.27)

The objective when solving Einstein's equations is to find the parameters $\Delta_1, \Delta_2, \Delta_3$ in terms of the parameters γ, σ, k of the potentials of the axion, dilaton and gauge field respectively 3.18. Therefore, by restricting the parameters of the ansatz functions with conditions such as 4.27 the alloed parameter space of the potentials will be constrained.

4.4 Thermal gas

When we study the gauge theory by means of gravity we are interested in the Hawking-Page transition, which is the phase transition from a global AdS spacetime to a AdS-Schwarzscheid metric with a black hole, i.e., with a blackening factor f(r) which defines

⁵Taking these functions directly implies a > 0.

an horizon by $f(r_h) = 0$. Within the context of the AdS/CFT correspondence this phase transition is dual to the confinement-deconfinement phase transition of the gauge theory. Black holes are dual to a deconfined phase, since the string tension vanishes at the horizon [12], and the Polyakov loop has non-vanishing expectation value. On the other hand, the thermal gas background is confining. The thermal gas (or ground state) is defined by the non-presence of a black hole, and is obtained by taking a black hole solution and letting the horizon area approach zero. This is in accordance with the requirement that the IR singularity 3.3.3 should be of the "good" type, i.e. it should be possible to clock the singularity with a horizon of infinitesimal area [25] in order to avoid a naked singularity. In the Euclidean coordinates this geometry is same as the zero temperature background with Euclidean time circle compactified. In order to study the gauge theory, we first focus on the thermal gas, in which we will compute the IR metric, and later on we will study the thermodynamics. To start, notice that equation 4.22 can be written as

$$\left(e^{3A(r)+h(r)}f'(r)\right)' + QA'_t(r) = 0.$$
(4.28)

The solution to this differential equation is the blackening factor

$$f(r) = A - C \int_0^r dr' e^{-3A(r) - h(r)} - Q \int_0^r dr' A_t(r) e^{-3A(r) - h(r)}$$
(4.29)

with A and C constants that must be determined. To do so, we use the conditions $f(r) \to 1$ when $r \to 0$ which ensures to recover AdS on the boundary, as well as $f(r_h) = 0$, which defines the horizon of the black brane. Thus,

$$C = \frac{1 - Q \int_{-\infty}^{r_h} dr' A_t(r) e^{-3A(r) - h(r)}}{\int_{-\infty}^{r_h} dr' e^{-3A(r) - h(r)}}$$
(4.30a)

$$A = 1. \tag{4.30b}$$

The function 4.29 defines the black brane, and therefore allows to introduce temperature as explained in section 2.11. To obtain the ground state we compute the entropy and send it to zero in accordance with the good IR singularity. Taking 4.29, the Hawking temperature 2.73 reads

$$T = \frac{|f'(r_h)|}{4\pi} = \frac{1}{4\pi} e^{-3A(r_h) - h(r_h)} |C + QA_t(r_h)| = \frac{|C|}{4\pi} e^{-3A(r_h) - h(r_h)}$$
(4.31)

where in the last step we made use of the regularity condition $A_t(r_h) = 0$. The entropy is computed from the expression 2.52⁶,

$$S = \frac{A}{4G} \tag{4.32}$$

⁶The constant G is the five-dimensional Newton constant.

where the area of the black hole is given by $A = \int d^3x \sqrt{g_3}|_{r_h} = e^{3A(r_h) + h(r_h)} \int d^3x = e^{3A(r_h) + h(r_h)}V$. Therefore, the density entropy $s \equiv S/V$ is given by

$$s = \frac{e^{3A(r_h) + h(r_h)}}{4G}.$$
(4.33)

This entropy corresponds to the entropy of the strongly coupled QGP. We have been able to compute the entropy of the QGP with a much more simpler calculation than the corresponding one within field theory. This is just an example of the power behind the gauge/gravity duality and its applications. With all these elements, we can write the temperature 4.31 as a function of the entropy,

$$C = 16\pi GTs. \tag{4.34}$$

The ground state corresponds to s = 0. Thus, C = 0 and f(r) 4.29 and the gauge field 4.7 reduce to

$$f^{TG}(r) = 1 - Q_0 \int_0^r dr' A_t^{TG} e^{-3A(r) - h(r)}, \qquad (4.35)$$

$$A_t^{TG}(r) = \mu_{b0} - Q_0 \int_0^r \frac{dr'}{e^{A(r) + h(r)} B(\phi(r))}.$$
(4.36)

which defines the thermal gas⁷. The charge and chemical potential of the ground state are denoted by Q_0 and μ_{b0} respectively and they can be different to the black hole charge and chemical potential. Notice that if we had considered a system without charge, the ground state would take the simpler form f(r) = 1. This is the result used in [4]. Further, using the ansatz functions 4.24-4.26 the entropy 4.33 reads

$$s = \frac{L^3 c_1}{4G} a^{3\Delta_1 + \Delta_2} r_h^{3\Delta_1 + \Delta_2}.$$
 (4.37)

From this result we can extract important information concerning the parameters Δ_1 and Δ_2 . The IR regime corresponds to $r \to \infty$, which means that $r_h \to \infty$ since the radial coordinate is defined within $0 < r < r_h$. For the thermal gas the entropy must be zero, so we must demand

$$3\Delta_1 + \Delta_2 < 0 \tag{4.38}$$

in the deep IR. This already imposes restrictions on the ansatz functions, and will be used to restrict the solutions to Einstein's equations. Before computing the gauge field and f(r) in the thermal gas, we must emphasize that the ansatz functions 4.24-4.26 are only valid in the IR limit. Therefore, in order to compute the integrals of 4.29 and 4.7 we must consider the ansatz to be true from a certain cutoff r_0 until the horizon of the black hole r_h , see figure 4.1. Therefore, the results we obtain will be valid within the range $r_0 \ll r < r_h$.

⁷In the ground state the Hawking temperature 4.31 is zero.



FIGURE 4.1: The radial coordinate is defined within the range $0 < r < r_h$. Since the ansatz functions 4.24-4.26 are only valid for $r \gg 1$ and $r_h \rightarrow \infty$ its range of validity is taken to be from a certain r_0 to r_h , colored in brown.

This implies that the gauge field and blackening factor must be computed splitting the integration range,

$$A_t(r) = \mu_b - Q \int_0^{r_0} \frac{dr'}{e^{A(r') + h(r')} B(\phi(r'))} - Q \int_{r_0}^r \frac{dr'}{e^{A(r') + h(r')} B(\phi(r'))}$$
(4.39)

$$f(r) = 1 - C \int_0^{r_o} dr' e^{-3A(r) - h(r)} - Q \int_0^{r_0} dr' A_t(r) e^{-3A(r) - h(r)} - C \int_{r_0}^r dr' e^{-3A(r) - h(r)} - Q \int_{r_0}^r dr' A_t(r) e^{-3A(r) - h(r)}.$$
(4.40)

The behaviour from $0 < r < r_0$ it is unknown but the result of the integrals within this range are independent of r, and we will therefore take them as a constant which depends on the cutoff r_0 . Computing the gauge field 4.36 and $f^{TG}(r)$ 4.35 yields

$$A_t^{TG}(r) = \mu_{b0} - \kappa Q_0 \left(ar\right)^{-\Delta_1 - \Delta_2 - 2k\Delta_3 + 1} + Q_0 \Gamma(r_0)$$
(4.41)

$$f^{TG}(r) = 1 - mQ_0(\mu_{b0} + Q_0\Gamma(r_0))(ar)^{-3\Delta_1 - \Delta_2 + 1} + Q_0\Omega(r_0) + \lambda Q_0^2(ar)^{-4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2}$$
(4.42)

with $\lambda \equiv \frac{1}{a^2 L^4 c_1^2(c_3)^{2k}(1-\Delta_1-\Delta_2-2k\Delta_3)(2-4\Delta_1-2\Delta_2-2k\Delta_3)}$, $\kappa \equiv \frac{1}{Lc_1(c_3)^{2k}a(1-2k\Delta_3-\Delta_1-\Delta_2)}$ and $m \equiv \frac{1}{aL^3c_1(1-3\Delta_1-\Delta_2)}$. $\Gamma(r_0)$ and $\Omega(r_0)$ denote the integrals within the range of $0 < r < r_0$ for which the IR ansatz is not valid. This gauge field and f(r) are the correct expressions as long as the exponent of the gauge field is not zero⁸. If it is zero, we must compute 4.36 again, obtaining

$$A_t^{TG}(r) = \mu_{b0} - \frac{Q_0 \log r}{aL^3 c_1(c_3)^{2k}} + Q_0 \Gamma(r_0)$$
(4.43)

$$f^{TG}(r) = 1 + Q_0 \Omega(r_0) - mQ_0 (\mu_{b0} + Q_0 \Gamma(r_0)) (ar)^{-3\Delta_1 - \Delta_2 + 1} + \xi Q_0^2 (ar)^{-3\Delta_1 - \Delta_2 + 1} (-1 + (-3\Delta_1 - \Delta_2 + 1) \log r)$$
(4.44)

with $\xi \equiv \frac{1}{L^4 a^2 c_1^2 (c_3)^{2k} (-3\Delta_1 - \Delta_2 + 1)^2}$.

⁸Notice that given 4.38 the exponent $-3\Delta_1 - \Delta_2 + 1$ of 4.42 cannot be zero.

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4.5 The c-theorem/NEC

In the previous chapter the holographic c-theorem was introduced, which is given by the NEC and thus from Einstein's equations [19]. Imposing the bulk null energy condition in 5-dimensional AdS, one obtains the conditions (in domain wall coordinates) [26]

$$G_0^0 - G_u^u = T_0^0 - T_u^u \le 0 \tag{4.45a}$$

$$G_0^0 - G_x^x = T_0^0 - T_x^x \le 0 \tag{4.45b}$$

$$G_0^0 - G_z^z = T_0^0 - T_z^z \le 0 \tag{4.45c}$$

which translate into the following restrictions

$$f(u)\left(3A''(u) + A'(u)h'(u) + h''(u) + h'(u)^2\right) = -\frac{f(u)}{2}(\phi')^2 \le 0$$
(4.46a)

$$\frac{1}{2}\left(-4A'(u)f'(u) - f''(u) - f'(u)h'(u)\right) = -\frac{Q^2}{2Be^{6A+2h}} \le 0$$
(4.46b)

$$-2A'(u)\left(f'(u) - 2f(u)h'(u)\right) - \frac{f''(u)}{2} + \frac{1}{2}f'(u)h'(u) + f(u)h''(u) + f(u)h''(u)h''(u) + f(u)h''(u) + f(u)h''(u) + f(u)h''(u) + f(u)h''($$

Notice that equation 4.46a is the Einstein equation 4.16 and it is independent of the presence of a gauge field and f(u). The second condition 4.46b is the Einstein equation 4.18. The last condition is a combination of the Einstein equations 4.17 and 4.18. These equations are a realization of the a-theorem in quantum field theories, where a certian functions decreases along the RG flow from the UV to the IR. This implies that we must be able to write these conditions as total derivatives. The first equation can be written as

$$\frac{d}{du}\left(\left(A'(u) + \frac{h'(u)}{3}\right)e^{\frac{h(u)}{3}}\right) \le 0.$$
(4.47)

This condition holds without the presence of charge, and it is nothing else than the ctheorem 3.16 with anisotropy. The fact that it does not depend on the charge is because the corresponding Einstein equation does not depen on the charge of the system; but the most interesting feature is that it holds for any function f(u), both for the thermal gas and the black hole solution. In radial coordinates 4.47 takes the form

$$A''(r) - A'(r)^2 + \frac{h''(r)}{3} + \frac{h'(r)^2}{9} \le 0.$$
(4.48)

But notice that such condition does not correspond to the first Einstein equation 4.20 due to the factor of the $h'(r)^2$ term. This is because if we take the condition $A''(r) - A'(r)^2 + \frac{h''(r)}{3} + \frac{h'(r)^2}{3} \leq 0$ (which is true) from the Einstein equation, it cannot be written as a total derivative, and thus it cannot correspond to the a-theorem. But this one is more restrictive than 4.48, and it implies it. Therefore, we will make use

of the more restrictive one in order to constrain the parameters of the ansatz functions 4.24-4.26. The second condition 4.46b can be written as well as a total derivative,

$$\frac{d}{du} \left(f'(u) e^{4A(u) + h(u)} \right) \ge 0, \tag{4.49}$$

which in the radial coordinates reads

$$3A'(r)f'(r) + f''(r) + f'(r)h'(r) \ge 0$$
(4.50)

and is the Einstein equation 4.22. The last condition 4.46c has a more complicated expression. It can be written as

$$\frac{d}{du}\left(\left(f(u)h'(u) - \frac{f'(u)}{2}\right)e^{4A(u) + h(u)}\right) \le 0$$
(4.51)

which translates into the following condition in the radial coordinates

$$\frac{f'(r)h'(r)}{2} + 3A'(r)h'(r)f(r) + f(r)h''(r) - \frac{f''(r)}{2} - \frac{3}{2}A'(r)f'(r) + f(r)h'(r)^2 \le 0.$$
(4.52)

This condition comes from substracting 4.21 and 4.22. Further, from equation 4.21 one could derive the condition

$$3f(r)A'(r)h'(r) + f'(r)h'(r) + f(r)h''(r) + f(r)h'(r)^2 \le 0,$$
(4.53)

which does not come directly from the NEC, but which combined with 4.50 would give us 4.52. These conditions will allow us to impose contrains on the IR metric, and in particular we will use the conditions in their radial coordinate form.

4.6 The IR metric

The Einstein's equations 4.20-4.23 are the gravity equations that we must solve. Using the ansatz functions 4.24-4.26 the Einstein equations read⁹

$$\frac{a^2 c_3^{2\gamma}(ar)^{2\gamma\Delta_3 - 2\Delta_2}}{2c_1^2} + \frac{\Delta_2 f'(r)}{r} + \frac{\Delta_2 f(r)(3\Delta_1 + \Delta_2 - 1)}{r^2} = 0$$
(4.54)

$$-6\Delta_1^2 - 6\Delta_1 + 2\Delta_2^2 - 2\Delta_2 + \Delta_3^2 = 0$$
(4.55)

⁹Recall from section 3.4.1 that the dilaton potential in the IR is $V = 6e^{\sigma\phi}$.

$$\frac{1}{2} \left(\frac{a^2 c_3^{2\gamma} (ar)^{2\gamma \Delta_3 - 2\Delta_2} + \frac{3Q_0^2 c_3^{-2k} (ar)^{-2(2\Delta_1 + \Delta_2 + \Delta_3 k)}}{L^4}}{c_1^2} - 12L^2 c_3^{\sigma} (ar)^{2\Delta_1 + \Delta_3 \sigma} + \frac{f(r) \left(24\Delta_1^2 + 12\Delta_1 \Delta_2 - \Delta_3^2\right)}{r^2} \right) + \frac{(3\Delta_1 + \Delta_2)f'(r)}{r} = 0$$

$$(4.56)$$

where the equations are left as a function of f(r) and the equation 4.22 is not considered because is precisely the one from which we found f(r), 4.28. Notice that the second equation 4.55 does not depend on f(r) and it is already in terms of only the parameters of the ansatz functions; this equation exactly corresponds to equation 4.20. We are concerned with the thermal gas, which is given by 4.42

$$f^{TG}(r) = 1 - mQ_0(\mu_{b0} + Q_0\Gamma(r_0)) (ar)^{-3\Delta_1 - \Delta_2 + 1} + Q_0\Omega(r_0) + \lambda Q_0^2 (ar)^{-4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2}.$$
(4.57)

In order to be able to solve the system of equations 4.54-4.56 we need to consider the leading terms in the IR. In the IR regime where $r \to \infty$ we can consider which term dominates depending on the relation between the exponents of each term. Therefore, we need to compare the coefficients, $0, -3\Delta_1 - \Delta_2 + 1$ and $-4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2$ and consider the leading terms as $r \to \infty$. The possible cases are:

- Case 1: The first and easiest case corresponds to consider f(r) = 1. For this to happen one might consider the relations $-3\Delta_1 \Delta_2 + 1 < 0$ and $-4\Delta_1 2\Delta_2 2k\Delta_3 + 2 < 0$. But this is actually incorrect since it violates the condition 4.38 and therefore it is not a possible situation. The correct statement for having f(r) = 1 is to take $Q_0 = 0$. This is the case studied in [4].
- Case 2: Whenever $-3\Delta_1 \Delta_2 + 1 > 0$ and $-3\Delta_1 \Delta_2 + 1 > -4\Delta_1 2\Delta_2 2k\Delta_3 + 2$ the ground state is given by

$$f(r) = -mQ_0(\mu_{b0} + Q_0\Gamma(r_0)) (ar)^{-3\Delta_1 - \Delta_2 + 1}$$
(4.58)

• Case 3: Whenever $-4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2 > 0$ and $-4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2 > -3\Delta_1 - \Delta_2 + 1$ the thermal gas corresponds to

$$f(r) = \lambda Q_0^2 (ar)^{-4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2}.$$
(4.59)

• Case 4: $-3\Delta_1 - \Delta_2 + 1 = -4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2 > 0$. This implies $\Delta_1 + \Delta_2 + 2k\Delta_3 - 1 = 0$ and therefore 4.57 is singular, since the gauge field 4.41 is singular. This precisely corresponds to 4.43-4.44, and therefore in the IR

$$f^{TG}(r) = \xi Q_0^2(ar)^{-3\Delta_1 - \Delta_2 + 1} (-3\Delta_1 - \Delta_2 + 1) \log r$$
(4.60)

since $r^{-3\Delta_1-\Delta_2+1}\log r$ grows faster than $r^{-3\Delta_1-\Delta_2+1}$ when $r \to \infty$ and $-3\Delta_1 - \Delta_2 + 1 > 0$, which is the case due to 4.38.

- Case 5: $-3\Delta_1 \Delta_2 + 1 = 0$ and $-4\Delta_1 2\Delta_2 2k\Delta_3 + 2 = 0$, so that f(r) is singular. To see what happens in such cases one needs to go back to the expressions 4.36-4.35 and compute them again by setting such conditions. But his case is actually in contradiction with the condition 4.38, and thus it is ruled out as a possible IR solution.
- Case 6: One of the exponents is zero and greater than the other one. Thus, there are two possibilities:

$$- -3\Delta_1 - \Delta_2 + 1 = 0 > -4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2$$
$$- -4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2 = 0 > -3\Delta_1 - \Delta_2 + 1.$$

Both of these options are in contradiction with 4.38, and they are discarded as possible IR behaviours.

We now study each case in order to find the corresponding IR metric parameters of the ansatz functions, by solving the Einstein equations $4.54-4.56^{10}$.

Case 1: Chargeless case, $Q_0 = 0$

For $Q_0 = 0$ we have f(r) = 1. The Einstein equations 4.54-4.56 reduce to

$$\frac{a^2 c_3^{2\gamma}(ar)^{2\gamma\Delta_3 - 2\Delta_2}}{2c_1^2} + \frac{\Delta_2(3\Delta_1 + \Delta_2 - 1)}{r^2} = 0$$
(4.61)

$$-6\Delta_1^2 - 6\Delta_1 + 2\Delta_2^2 - 2\Delta_2 + \Delta_3^2 = 0$$
(4.62)

$$\frac{a^2 c_3^{2\gamma} (ar)^{2\gamma\Delta_3 - 2\Delta_2}}{c_1^2} - \frac{12L^2 r^2 c_3^{\sigma}(ar)^{2\Delta_1 + \Delta_3 \sigma} - 24\Delta_1^2 - 12\Delta_1 \Delta_2 + \Delta_3^2}{r^2} = 0.$$
(4.63)

And so we reduce the problem to solve these algebraic equations. From equation 4.62 it is possible to find Δ_3 in function of the other constants. From equation 4.61 we see that $2\gamma\Delta_3 - 2\Delta_2 = -2$. From the last equation we get $2\Delta_1 + \Delta_3\sigma = -2$. Therefore, we need to solve the following system of equations

$$-6\Delta_1^2 - 6\Delta_1 + 2\Delta_2^2 - 2\Delta_2 + \Delta_3^2 = 0$$
(4.64a)

$$2\gamma\Delta_3 - 2\Delta_2 = -2 \tag{4.64b}$$

$$2\Delta_1 + \Delta_3 \sigma = -2 \tag{4.64c}$$

¹⁰Recall that equation 4.55 is independent of f(r). Therefore, it will be the same for every case.

which has the following solution

$$\Delta_1 = \frac{2\gamma(\sigma - 2\gamma) - 2}{4\gamma^2 - 3\sigma^2 + 2}$$
(4.65a)

$$\Delta_2 = \frac{6\gamma\sigma - 3\sigma^2 + 2}{4\gamma^2 - 3\sigma^2 + 2}$$
(4.65b)

$$\Delta_3 = \frac{6\sigma - 4\gamma}{4\gamma^2 - 3\sigma^2 + 2}.$$
 (4.65c)

The metric 3.21 can be written in the IR as [4]

$$ds^{2} = L^{2}(ar)^{\frac{2\theta}{3\alpha}} \left(\frac{-dt^{2} + d\vec{x}_{\perp}^{2} + dr^{2}}{a^{2}r^{2}} + \frac{c_{1}dz^{2}}{(ar)^{\frac{2}{\alpha}}} \right)$$
(4.66)

with $\alpha = \frac{4\gamma^2 - 3\sigma^2 + 2}{2\gamma(2\gamma - 3\sigma)}$ and $\theta = \frac{3\sigma}{2\gamma}$. For $\theta = 0$ the metric exhibits a Lifshitz scaling under the transformation

$$t \to \lambda t, \quad \vec{x}_{\perp} \to \lambda \vec{x}_{\perp}, \quad r \to \lambda r, \quad z \to \lambda^{1/\alpha} z$$

$$(4.67)$$

which maps solutions to solutions. Lifshitz scalings are typical from condensed matter systems. Such scaling indicates that the system is non-relativistic, since for relativistic systems every coordinate scales in the same manner. Therefore, $1/\alpha$ is called the dynamical exponent. For $\theta \neq 0$ the metric has hyperscaling violation under 4.67, $ds \rightarrow \lambda^{\theta/3\alpha} ds$, which indicates a non trivial change in the volume. This kind of behaviour implies a dependence on the dynamical exponent for the entropy and other themodynamic quantities [5].

Case 2: $-3\Delta_1 - \Delta_2 + 1 > 0$ and $-3\Delta_1 - \Delta_2 + 1 > -4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2$

The ground state is given by 4.58 and the Einstein equations 4.54-4.56 reduce to

$$\frac{a^2 c_3^{2\gamma} (ar)^{2\gamma\Delta_3 - 2\Delta_2}}{2c_1^2} = 0 \tag{4.68}$$

$$-6\Delta_1^2 - 6\Delta_1 + 2\Delta_2^2 - 2\Delta_2 + \Delta_3^2 = 0 \tag{4.69}$$

$$\frac{(ar)^{-2\Delta_2} \left(a^2 c_3^{2\gamma} (ar)^{2\gamma\Delta_3} + \frac{3Q_0^2 c_3^{-2k} (ar)^{-2(2\Delta_1 + \Delta_3 k)}}{L^4}\right)}{2c_1^2} - 6L^2 c_3^\sigma (ar)^{2\Delta_1 + \Delta_3 \sigma} - \frac{aQ_0 m \left(6\Delta_1 (\Delta_1 + 1) - 2(\Delta_2 - 1)\Delta_2 - \Delta_3^2\right) (ar)^{-3\Delta_1 - \Delta_2} (\mu_{\rm b0} + \Gamma(r_0)Q_0)}{2r} = 0.$$

$$(4.70)$$

The first equation is telling us that $a = 0^{-11}$ which is a contradiction since we are considering anisotropy with an axion field $\chi = az$. In order to see if this case truly implies zero anisotropy we must look at the subleading terms of the equations, which corresponds to consider the terms with order 0 and $-4\Delta_1 - 2\Delta_2 - 2k\Delta_3 - 2$. Equation 4.69 does not change because it is independent of f(r) and always takes the same form. Equation 4.68 takes the form

$$\frac{a^2 c_3^{2\gamma} (ar)^{2\gamma\Delta_3 - 2\Delta_2}}{2c_1^2} - a^2 \lambda \Delta_2 Q_0^2 (\Delta_1 + \Delta_2 + 2\Delta_3 k - 1) (ar)^{-2(2\Delta_1 + \Delta_2 + \Delta_3 k)} + \frac{\Delta_2 Q_0 \Omega(r_0) (3\Delta_1 + \Delta_2 - 1)}{r^2} + \frac{\Delta_2 (3\Delta_1 + \Delta_2 - 1)}{r^2} = 0$$

$$(4.71)$$

and thus we must look for the relation of the exponents $-4\Delta_1 - 2\Delta_2 - 2\Delta_3 k$ and -2. If $-4\Delta_1 - 2\Delta_2 - 2\Delta_3 k + 2 > 0$ we impose $2\gamma\Delta_3 - 2\Delta_2 = -4\Delta_1 - 2\Delta_2 - 2\Delta_3 k$; otherwise we have $2\gamma\Delta_3 - 2\Delta_2 = -2$. The third equation 4.70 already contains subleading terms, from which we obtain $2\Delta_1 + \Delta_3\sigma = -3\Delta_1 - \Delta_2 - 1$ as a leading order. For $-4\Delta_1 - 2\Delta_2 - 2\Delta_3 k + 2 > 0$ the system reads

$$-6\Delta_1^2 - 6\Delta_1 + 2\Delta_2^2 - 2\Delta_2 + \Delta_3^2 = 0 \tag{4.72a}$$

$$2\gamma\Delta_3 - 2\Delta_2 = -4\Delta_1 - 2\Delta_2 - 2\Delta_3 k \tag{4.72b}$$

$$2\Delta_1 + \Delta_3 \sigma = -3\Delta_1 - \Delta_2 - 1. \tag{4.72c}$$

This system has the following solutions

$$\Delta_{1} = -\frac{2(\gamma+k)}{6\gamma + \sqrt{4\sigma(\gamma+k) - 8(\gamma+k)^{2} + \sigma^{2} - 4} + 6k - 3\sigma}$$
(4.73a)
$$\Delta_{2} = \frac{1}{(8\gamma^{2} + 8k^{2})^{2}}$$

$$\Delta_{2} = \frac{1}{-20\sigma(\gamma+k) + 22(\gamma+k)^{2} + 4\sigma^{2} + 2} \left(8\gamma^{2} + 8k^{2} + k\left(16\gamma - 5\sqrt{4\sigma(\gamma+k) - 8(\gamma+k)^{2} + \sigma^{2} - 4} - 7\sigma\right) + 2\sigma\left(\sqrt{4\sigma(\gamma+k) - 8(\gamma+k)^{2} + \sigma^{2} - 4} + \sigma\right) - \gamma\left(5\sqrt{4\sigma(\gamma+k) - 8(\gamma+k)^{2} + \sigma^{2} - 4} + 7\sigma\right) - 2\right)$$

$$\Delta_{3} = \frac{4}{6\gamma + \sqrt{4\sigma(\gamma+k) - 8(\gamma+k)^{2} + \sigma^{2} - 4} + 6k - 3\sigma}$$
(4.73c)

and

¹¹Notice that it is not possible to consider $c_3 = 0$ since it is inside a logarithm.

$$\Delta_{1} = \frac{2(\gamma + k)}{-6\gamma + \sqrt{4\sigma(\gamma + k) - 8(\gamma + k)^{2} + \sigma^{2} - 4} - 6k + 3\sigma}$$
(4.74a)
$$\Delta_{2} = \frac{1}{-20\sigma(\gamma + k) + 22(\gamma + k)^{2} + 4\sigma^{2} + 2} \left(8\gamma^{2} + 8k^{2} + k\left(16\gamma + 5\sqrt{4\sigma(\gamma + k) - 8(\gamma + k)^{2} + \sigma^{2} - 4} - 7\sigma\right)\right)$$
(4.74b)

$$+2\sigma \left(\sigma - \sqrt{4\sigma(\gamma+k) - 8(\gamma+k)^2 + \sigma^2 - 4}\right)$$

$$+\gamma \left(5\sqrt{4\sigma(\gamma+k) - 8(\gamma+k)^2 + \sigma^2 - 4} - 7\sigma\right) - 2\right)$$

$$\Delta_3 = -\frac{4}{-6\gamma + \sqrt{4\sigma(\gamma+k) - 8(\gamma+k)^2 + \sigma^2 - 4} - 6k + 3\sigma}.$$
(4.74b)
$$(4.74b) = (4.74c)$$

The system corresponding to having $-4\Delta_1 - 2\Delta_2 - 2\Delta_3 k + 2 < 0$ is

$$-6\Delta_1^2 - 6\Delta_1 + 2\Delta_2^2 - 2\Delta_2 + \Delta_3^2 = 0 \tag{4.75a}$$

$$2\gamma\Delta_3 - 2\Delta_2 = -2 \tag{4.75b}$$

$$2\Delta_1 + \Delta_3 \sigma = -3\Delta_1 - \Delta_2 - 1, \qquad (4.75c)$$

which yields the solutions

$$\Delta_{1} = \frac{1}{44\gamma^{2} - 12\gamma\sigma - 6\sigma^{2} + 25} \left(\gamma \left(\sqrt{-32\gamma^{2} + 24\gamma\sigma + 9\sigma^{2} - 36} + 11\sigma \right) + \sigma \left(\sqrt{-32\gamma^{2} + 24\gamma\sigma + 9\sigma^{2} - 36} + 3\sigma \right) - 12\gamma^{2} - 10 \right)$$
(4.76a)

$$\Delta_2 = \frac{-5\gamma \left(\sqrt{-32\gamma^2 + 24\gamma\sigma + 9\sigma^2 - 36} + 3\sigma\right) + 16\gamma^2 - 6\sigma^2 + 25}{44\gamma^2 - 12\gamma\sigma - 6\sigma^2 + 25}$$
(4.76b)

$$\Delta_3 = -\frac{36}{-5\sqrt{-32\gamma^2 + 24\gamma\sigma + 9\sigma^2 - 36} + 28\gamma + 3\sigma}$$
(4.76c)

and

$$\Delta_{1} = \frac{1}{44\gamma^{2} - 12\gamma\sigma - 6\sigma^{2} + 25} \left(\gamma \left(\sqrt{-32\gamma^{2} + 24\gamma\sigma + 9\sigma^{2} - 36} - 11\sigma \right) + \sigma \left(\sqrt{-32\gamma^{2} + 24\gamma\sigma + 9\sigma^{2} - 36} - 3\sigma \right) + 12\gamma^{2} + 10 \right)$$

$$(4.77a)$$

$$\Delta_2 = \frac{5\gamma \left(\sqrt{-32\gamma^2 + 24\gamma\sigma + 9\sigma^2 - 36} - 3\sigma\right) + 16\gamma^2 - 6\sigma^2 + 25}{44\gamma^2 - 12\gamma\sigma - 6\sigma^2 + 25}$$
(4.77b)

$$\Delta_3 = -\frac{36}{5\sqrt{-32\gamma^2 + 24\gamma\sigma + 9\sigma^2 - 36} + 28\gamma + 3\sigma}.$$
(4.77c)

We wish to write the metric in a similar fashion to the IR metric 4.66 where $Q_0 = 0$. The important difference with the case in hand is that the ground state does not correspond to f(r) = 1 due to the gauge field, and therefore the temporal and radial components of the metric cannot be joined together with the spatial isotorpic directions in a similar way. Given the metric 3.21 together with the ansatz functions 4.24-4.26 and the ground state 4.58

$$ds^{2} = L^{2}(ar)^{2\Delta_{1}} \left(-A(ar)^{-3\Delta_{1}-\Delta_{2}+1}dt^{2} + d\vec{x}_{\perp}^{2} + c_{1}^{2}(ar)^{2\Delta_{2}}dz^{2} + A^{-1}(ar)^{3\Delta_{1}+\Delta_{2}-1}dr^{2} \right),$$

$$(4.78)$$

with $A \equiv -Q_0 m (\mu_{b0} + \Gamma(r_0)Q_0)$. Thus, we indeed see that if we wish to join the isotropic directions with the temporal and radial component the metric will not take the same form as 4.66. But in anology we can propose the form

$$ds^{2} = L^{2}(ar)^{2\theta/3\alpha} \left(\frac{-A(ar)^{p}dt^{2} + d\vec{x}_{\perp}^{2} + A^{-1}(ar)^{-p}dr^{2}}{(ar)^{2}} + \frac{dz^{2}}{(ar)^{2/\alpha}} \right).$$
(4.79)

Therefore, to find the parameters of this metric we need to solve the system 12

$$\frac{2\theta}{3z} - 2 = 2\Delta_1 \tag{4.80a}$$

$$\frac{2\theta}{3z} - \frac{2}{\alpha} = 2\Delta_1 + 2\Delta_2 \tag{4.80b}$$

$$\frac{2\theta}{3z} + p - 2 = -\Delta_1 - \Delta_2 + 1 \tag{4.80c}$$

¹²The last equation can be also $\frac{2\theta}{3z} - p - 2 = 5\Delta_1 + \Delta_2 - 1$ due to the inverse role of the blackening factor in the temporal and radial component. The result of the system is the same for both cases.

which yields

$$\theta = -\frac{3(\Delta_1 + 1)}{\Delta_2 - 1} \tag{4.81a}$$

$$\alpha = \frac{1}{1 - \Delta_2} \tag{4.81b}$$

$$p = -3\Delta_1 - \Delta_2 + 1. \tag{4.81c}$$

We can now try to see if this geometry corresponds to a Lifshitz geometry when $\theta = 0$, as in the case of 4.66. We perform the scaling transformations $t \to \lambda^b t$, $\vec{x}_{\perp} \to \lambda^d \vec{x}_{\perp}$, $r \to \lambda^a r$, and $z \to \lambda^c z$ and impose invariance under such coordinate change. The system we encounter is

$$ap - 2a + 2b = 0 \tag{4.82a}$$

$$2d - 2a = 0$$
 (4.82b)

$$2a - pa - 2a = 0 (4.82c)$$

$$2c - \frac{2a}{\alpha} = 0. \tag{4.82d}$$

But this system yields a trivial result, a = b = c = d = 0. Even when one considers the case $\theta \neq 0$ the result is trivial. But what happens if we decide to rescale a parameter as well as the coordinates? Let us consider the possibility of rescaling the charge and the chemical potential. Given 4.79, it turns out that the metric for $\theta = 0$ is invariant under the scaling

$$t \to \lambda t, \quad \vec{x}_{\perp} \to \lambda \vec{x}_{\perp}, \quad r \to \lambda r, \quad z \to \lambda^{1/\alpha} z, \quad Q_0 \to \lambda^{-p/2} Q_0, \quad \mu_{b0} \to \lambda^{-p/2} \mu_{b0}.$$
(4.83)

Therefore, the metric exhibits a Lifshitz scaling upon scaling the charge and chemical potential¹³. Besides, the metric suffers a hyperscaling violation for $\theta \neq 0$ given by $ds \rightarrow \lambda^{\theta/3\alpha} ds$, signaling the dependence of thermodynamic quantities on the dynamical exponents.

Case 3:
$$-4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2 > 0$$
 and $-4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2 > -3\Delta_1 - \Delta_2 + 1$
The ground state is given by 4.50. The resulting Einstein equations 4.54.4.56 and

The ground state is given by 4.59. The resulting Einstein equations 4.54-4.56 are

$$c_3^{2\gamma}(ar)^{-2\Delta_2+2\Delta_3\gamma} - 2c_1^2\Delta_2\lambda Q_0^2(ar)^{-2(2\Delta_1+\Delta_2+\Delta_3k)}(\Delta_1+\Delta_2+2\Delta_3k-1) = 0 \quad (4.84)$$

$$-6\Delta_1^2 - 6\Delta_1 + 2\Delta_2^2 - 2\Delta_2 + \Delta_3^2 = 0$$
(4.85)

 $^{^{13}}$ This is in analogy with the AdS black brane for which one needs to rescale not only the coordinates but also the horizon in order to have scale invariance.

$$\left(-a^2 c_1^2 \lambda L^4 Q_0^2 c_3^{2k} (ar)^{-2(2\Delta_1 + \Delta_2 + \Delta_3 k)} \left(4(\Delta_2 - 1)\Delta_2 + \Delta_3^2 + 4\Delta_1 (2\Delta_2 + 3\Delta_3 k - 3) + 4\Delta_2 \Delta_3 k \right) + a^2 L^4 c_3^{2(\gamma+k)} (ar)^{-2\Delta_2 + 2\Delta_3 \gamma} \right)$$

$$\left(4.86 \right)$$

$$\left(-12c_1^2 L^6 c_3^{2k+\sigma} (ar)^{2\Delta_1 + \Delta_3 \sigma} + 3Q_0^2 \right) = 0.$$

From the first equation we get $2\gamma\Delta_3 = -4\Delta_1 - 2\Delta_3 k$. From the second one is possible to get Δ_3 in function of the rest. The last equation has three free exponents which must be taken into account, $2\Delta_1 + \Delta_3\sigma$, $-2\Delta_2 + 2\Delta_3\gamma$ and $-2\Delta_2 - 2\Delta_3 k - 4\Delta_1$. If we use the equality $2\gamma\Delta_3 = -4\Delta_1 - 2\Delta_3 k$ from the first equation and plug it into the second free exponent we get $-2\Delta_2 + 2\Delta_3\gamma = -2\Delta_2 - 2\Delta_3 k - 4\Delta_1$. Therefore, two exponents are the same and one of them is free, from which we conclude that $2\Delta_1 + \Delta_3\sigma = -2\Delta_2 + 2\Delta_3\gamma$. Therefore, we are left with the system of equations

$$2\gamma\Delta_3 = -4\Delta_1 - 2\Delta_3 k \tag{4.87a}$$

$$-6\Delta_1^2 - 6\Delta_1 + 2\Delta_2^2 - 2\Delta_2 + \Delta_3^2 = 0$$
(4.87b)

$$2\Delta_1 + \Delta_3 \sigma = -2\Delta_2 + 2\Delta_3 \gamma. \tag{4.87c}$$

The solution to such system is the following¹⁴

$$\Delta_1 = -\frac{(\gamma+k)(2k+\sigma)}{-6\gamma^2 + 6\gamma\sigma + 2k^2 + 2k\sigma - \sigma^2 - 2}$$
(4.88a)

$$\Delta_2 = \frac{(2k+\sigma)(3\gamma+k-\sigma)}{-6\gamma^2+6\gamma\sigma+2k^2+2k\sigma-\sigma^2-2}$$
(4.88b)

$$\Delta_3 = -\frac{2(2k+\sigma)}{6\gamma^2 - 2k^2 - 2\sigma(3\gamma+k) + \sigma^2 + 2}.$$
(4.88c)

Another solution is for all the coefficients to be zero, which of course is not a relevant physical situation since corresponds to take the ansatz functions 4.24-4.26 to be constant. As in the previous cases we wish to write the metric in the form 4.66. Following previous arguments, we propose the form

$$ds^{2} = L^{2}(ar)^{2\theta/3\alpha} \left(\frac{-A(ar)^{p}dt^{2} + d\vec{x}_{\perp}^{2} + A^{-1}(ar)^{-p}dr^{2}}{(ar)^{2}} + \frac{c_{1}dz^{2}}{(ar)^{2/\alpha}} \right)$$
(4.89)

¹⁴One could consider the system taking the last equation to be $2\Delta_1 + \Delta_3 \sigma = -2\Delta_2 - 2\Delta_3 k - 4\Delta_1$. The solution is the same, as it should be since both exponents are the same.

with $A \equiv \lambda Q_0^2$ and

$$\theta = -\frac{3(\Delta_1 + 1)}{\Delta_2 - 1} \tag{4.90a}$$

$$\alpha = \frac{1}{1 - \Delta_2} \tag{4.90b}$$

$$p = -2(2\Delta_1 + \Delta_2 + \Delta_3 k - 1).$$
(4.90c)

As in the previous case the metric exhibits a scaling upon scaling the charge. The Lifshitz scaling corresponds to

$$t \to \lambda t, \quad \vec{x}_{\perp} \to \lambda \vec{x}_{\perp}, \quad r \to \lambda r, \quad z \to \lambda^{1/\alpha} z, \quad Q_0 \to \lambda^{-p/2},$$
 (4.91)

with a hyperscaling violation $ds \to \lambda^{\theta/3\alpha} ds$ for $\theta \neq 0$.

Case 4: $-3\Delta_1 - \Delta_2 + 1 = -4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2 > 0$

The ground state is given by 4.60. The Einstein's equations 4.54-4.56 read

$$(ar)^{-2\Delta_2} \left(\frac{ac_3^{2\gamma}(ar)^{2\gamma\Delta_3}}{c_1^2} - 2a\Delta_2\xi Q_0^2(3\Delta_1 + \Delta_2 - 1)(ar)^{-3\Delta_1 + \Delta_2 - 1} \right) = 0$$
(4.92)

$$-6\Delta_1^2 - 6\Delta_1 + 2\Delta_2^2 - 2\Delta_2 + \Delta_3^2 = 0$$
(4.93)

$$\frac{r(ar)^{-2\Delta_2} \left(a^2 c_3^{2\gamma}(ar)^{2\gamma\Delta_3} + \frac{3Q_0^2 c_3^{-2k}(ar)^{-2(2\Delta_1+\Delta_3k)}}{L^4}\right)}{c_1^2} - 6L^2 c_3^{\sigma} r(ar)^{2\Delta_1+\Delta_3\sigma}$$

$$- a\xi Q_0^2 (3\Delta_1 + \Delta_2 - 1)(ar)^{-3\Delta_1-\Delta_2} \left(2(3\Delta_1 + \Delta_2) + \log(r)\left(6\Delta_1(\Delta_1 + 1)\right)\right)$$

$$-2(\Delta_2 - 1)\Delta_2 - \Delta_3^2\right) = 0.$$

$$(4.94)$$

From the first equation we obtain $2\gamma\Delta_3 - 2\Delta_2 = -3\Delta_1 - \Delta_2 - 1$. In the third equation the logarithmic term vanishes because the factor in front is precisely the second equation. Therefore, we obtain the system

$$2\gamma\Delta_3 = -1 - 3\Delta_1 + \Delta_2 \tag{4.95a}$$

$$-6\Delta_1^2 - 6\Delta_1 + 2\Delta_2^2 - 2\Delta_2 + \Delta_3^2 = 0$$
(4.95b)

$$-3\Delta_1 - \Delta_2 - 1 = 2\Delta_1 + \Delta_3 \sigma \tag{4.95c}$$

which has the solutions

$$\Delta_1 = \frac{2\gamma \left(\sqrt{-8\gamma^2 + 6\gamma \sigma - 3} + 5\sigma\right) + \sigma \sqrt{-8\gamma^2 + 6\gamma \sigma - 3} - 12\gamma^2 - 4}{44\gamma^2 - 36\gamma \sigma + 3\sigma^2 + 16}$$
(4.96a)

$$\Delta_{2} = \frac{1}{44\gamma^{2} - 36\gamma\sigma + 3\sigma^{2} + 16} \left(2\gamma \left(5\sqrt{-8\gamma^{2} + 6\gamma\sigma - 3} + 9\sigma \right) + 3\sigma \left(\sqrt{-8\gamma^{2} + 6\gamma\sigma - 3} + \sigma \right) + 16\gamma^{2} + 4 \right)$$
(4.96b)

$$\Delta_3 = \frac{6}{4\sqrt{-8\gamma^2 + 6\gamma\sigma - 3} + 2\gamma - 3\sigma} \tag{4.96c}$$

 and

$$\Delta_1 = -\frac{2\gamma \left(\sqrt{-8\gamma^2 + 6\gamma \sigma - 3} - 5\sigma\right) + \sigma \sqrt{-8\gamma^2 + 6\gamma \sigma - 3} + 12\gamma^2 + 4}{44\gamma^2 - 36\gamma \sigma + 3\sigma^2 + 16}$$
(4.97a)

$$\Delta_{2} = \frac{1}{44\gamma^{2} - 36\gamma\sigma + 3\sigma^{2} + 16} \left(2\gamma \left(5\sqrt{-8\gamma^{2} + 6\gamma\sigma - 3} - 9\sigma \right) + 3\sigma \left(-\sqrt{-8\gamma^{2} + 6\gamma\sigma - 3} + \sigma \right) + 16\gamma^{2} + 4 \right)$$
(4.97b)

$$\Delta_3 = \frac{6}{-4\sqrt{-8\gamma^2 + 6\gamma\sigma - 3} + 2\gamma - 3\sigma}.$$
(4.97c)

The metric 3.21 in the IR reads

$$ds^{2} = L^{2}(ar)^{2\theta/3\alpha} \left(\frac{-A(ar)^{p}\log rdt^{2} + d\vec{x}_{\perp}^{2} + A^{-1}(ar)^{-p}(\log r)^{-1}dr^{2}}{(ar)^{2}} + \frac{c_{1}dz^{2}}{(ar)^{2/\alpha}} \right)$$
(4.98)

with $A \equiv \xi Q_0^2(-3\Delta_1 - \Delta_2 + 1)$ and $p \equiv -3\Delta_1 - \Delta_2 + 1$. Due to the logarithm the metric does not exhibit a Lifshitz scaling. This expected since the logarithmic behaviour indicates the presence of an anomaly, concretely the Weyl anomaly [27].

4.7 Restrictions from the NEC

Once we have the IR metric for each case we can restrict the parameter space of the solutions. The solutions must satisfy the conditions 4.27 and 4.38, but they also must satisfy the NEC conditions found in section 4.5. Let us now see how these conditions translate when considering the different IR ground states found previously. The first condition 4.48 implies

$$A''(r) - A'(r)^2 + \frac{h''(r)}{3} + \frac{h'(r)^2}{3} \le 0 \to -\Delta_1 - \Delta_1^2 - \frac{\Delta_2}{3} + \frac{\Delta_2^2}{3} \le 0.$$
(4.99)

This condition is independent of f(r) and therefore it is valid both in the thermal gas and the black hole solution, taking such form for every IR solution. The other conditions are more difficult to analyse due to the presence of the blackening factor. In this section we simply present the conditions obtained for each case; the corresponding calculations can be found in Appendix B. As a summary, we obtain the following restrictions ¹⁵

• Conditions which are true for every case:

$$a > 0$$
 (4.100a)

- $\Delta_1 < 0 \tag{4.100b}$
- $\Delta_3 > 0 \tag{4.100c}$

$$3\Delta_1 + \Delta_2 < 0 \tag{4.100d}$$

$$-\Delta_1 - \Delta_1^2 - \frac{\Delta_2}{3} + \frac{\Delta_2^2}{3} \le 0.$$
 (4.100e)

• Case 1 [4]:

$$3\Delta_1\Delta_2 + \Delta_2^2 - \Delta_2 \le 0$$

• Case 2 4.58:

$$Q_0(\Gamma(r_0)Q_0 + \mu_{b0}) \le 0 \tag{4.101a}$$

$$\Delta_2(1+Q_0\Gamma(r_0)) \ge 0 \iff -4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2 < 0$$
 (4.101b)

$$\frac{\Delta_2}{(2\Delta_1 + \Delta_2 + \Delta_3 k - 1)} \ge 0 \iff -4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2 > 0.$$
(4.101c)

• Case 3 4.59:

$$\frac{\Delta_2}{(2\Delta_1 + \Delta_2 + \Delta_3 k - 1)} \ge 0 \tag{4.102a}$$

$$\frac{6\Delta 1}{\Delta 1 + \Delta 2 + 2\Delta 3k - 1} + \frac{\Delta 2}{2\Delta 1 + \Delta 2 + \Delta 3k - 1} + 1 \ge 0.$$
(4.102b)

• Case 4 4.60:

$$\Delta_2 \le 0. \tag{4.103}$$

Therefore, the holographic c-theorem realised by the NEC restricts the parameter space of the IR ansatz functions 4.24-4.26 and therefore the IR metric. Recall that due to the presence of the gauge field the conditions 4.48, 4.50, 4.52 and 4.53 take a different form in the thermal gas respect to the case without charge. Indeed, for such case the ground state is f(r) = 1 and the conditions simplify considerably, being the most significant difference that condition 4.50 is not present.

 $^{^{15}\}mathrm{We}$ also present the restrictions coming from the case without charge.

5. Thermodynamics

One of the most remarkable aspects of the AdS/CFT correspondence lies in its power to compute thermodynamic quantities. Thermal field theory is a rather complex field, which requires difficult computations. Thanks to the gauge/gravity duality, the thermodynamics of the field theory can be computed from the gravity side, where it is easier and more familiar to standard thermodynamic computations. Further, it is possible to compute macroscopic quantities such as specific heats without making reference to the underlying microscopic theory. The important aspect is that due to the AdS/CFT correspondence we know the underlying microscopic gauge theory which gives rise to the macroscopic properties of the system. In this chapter we consider the presence of a black hole in the bulk theory. In particular, a charged black hole due to the presence of a gauge field. Therefore, the blackening factor and Hawking temperature turn on and the system is not anymore in the thermal gas state. We study the thermodynamic quantities of the black brane in the IR limit as well as the stability of the solutions, which provides further constraints on the IR geometry.

5.1 Gauge field

Due to the presence of a black brane in the bluk the gauge field 4.7 must satisfy $A(r_h) = 0$. This condition allows us to find the charge in terms of the horizon and chemical potential,

$$Q = \frac{\mu_b}{\int_0^{r_h} \frac{dr'}{e^{A(r) + h(r)}B(\phi(r))}}.$$
(5.1)

Using the ansatz functions 4.24-4.26 the charge and gauge field in terms of the charge and chemical potential read

$$Q = \frac{\mu_b}{\Gamma(r_0) + \kappa(ar_h)^{\delta}}$$
(5.2)

$$A_t(r) = \mu_b \left[1 - \frac{\Gamma(r_0) + \kappa(ar)^{\delta}}{\Gamma(r_0) + \kappa(ar_h)^{\delta}} \right]$$
(5.3)

with $\delta \equiv -\Delta_1 - \Delta_2 - 2k\Delta_3 + 1$, $\kappa \equiv \frac{1}{Lc_1(c_3)^{2k}a(1-2k\Delta_3-\Delta_1-\Delta_2)}$. Notice that δ is (of course) the exponent of the gauge field 4.41. The reason to denote it as δ is because we

will need to explicitly account for its relation to zero in the calculations, and it is a more compact way of seeing its relation to other exponents. Indeed, in the IR limit $r_h \to \infty$ we must distinguish between positive and negative values of δ , as well as $\delta = 0$.

Case 2 $\delta < 0$: Corresponds to case 2 4.58

$$Q \to \frac{\mu_b}{\Gamma(r_0)} \tag{5.4a}$$

$$A_t(r) = -\frac{\kappa\mu_b}{\Gamma(r_0)} \left((ar)^{\delta} - (ar_h)^{\delta}) \right)$$
(5.4b)

Case 3 $\delta > 0$: Corresponds to case 3 4.59

$$Q \to 0$$
 (5.5a)

$$A_t(r) = \mu_b \left(1 - \left(\frac{r}{r_h}\right)^\delta \right) \tag{5.5b}$$

Case 4 $\delta = 0$: Corresponds to case 4 4.60. Due to the singular behaviour the integrals must be performed again, yielding

$$Q = \frac{\mu_b}{\Gamma(r_0) + \theta \log r_h} \to 0 \tag{5.6a}$$

$$A_t(r) = \mu_b \left(1 - \frac{\Gamma(r_0) + \theta \log r}{\Gamma(r_0) + \theta \log r_h} \right) \to \mu_b \left(1 - \frac{\log r}{\log r_h} \right)$$
(5.6b)

with $\theta \equiv 1/Lc_1(c_3)^{2k}$.

These expressions are valid only in the IR limit $r < r_h \to \infty$ and thus they satisfy $A_t(r_h) = 0$, but not necesserally $A_t(0) = \mu_b$.

5.2 Blackening factor and temperature

In section 4.4 we derived the blackening factor 4.29. In order to compute the blackening factor and temperature in the IR limit we distinguish between different values of the exponent δ of the gauge field 5.3, as done previously. The full computations of the blackening factor and temperature can be found in Appendix C; here we present the blackening factor and temperature for each case.

Case 1, Q = 0

We present the blackening factor and temperature without the presence of charge [4],

$$f(r) = 1 - \left(\frac{r}{r_h}\right)^{\delta_2} \tag{5.7}$$

$$4\pi T = \frac{\delta_2}{r_h} \tag{5.8}$$

with $\delta_2 \equiv -3\Delta_1 - \Delta_2 + 1^1$. We will see how this blackening factor appears in the case with charge together with new contributions coming from the presence of the gauge field.

¹Recall that this power is precisely the same of the one present in the second term of the thermal gas 4.57. The relation of this exponent to δ is important and we therefore use this more compact notation.

For the temperature we will see how in the IR limit the contribution $1/r_h$ is subleading respect to leading terms containing charge contributions.

Case 2 ($\delta < 0$) 4.58

The only possibility corresponds to $\delta < \delta_2$ since $\delta_2 > 0$ due to 4.38. The blackening factor reads

$$f(r) = 1 - \frac{\mu_b \Omega}{\Gamma(r_0)} - \left(\frac{r}{r_h}\right)^{\delta_2} \left[1 - \frac{\mu_b \Omega(r_0)}{\Gamma(r_0)} + \frac{\mu_b^2 (ar_h)^{\delta + \delta_2}}{\Gamma(r_0)^2 \omega} \left(\frac{\kappa - 2}{\delta_2} + \frac{\kappa}{\delta_2 + \delta}\right)\right] + \frac{\mu_b^2 (ar)^{\delta_2} (ar_h)^{\delta}}{\Gamma(r_0)^2 \omega} \left(\frac{\kappa - 2}{\delta_2} + \frac{\kappa}{\delta_2 + \delta}\right).$$
(5.9)

This blackening factor contains the term 5.7 of the Q = 0 case with new terms coming from the presence of charge/chemical potential. If $\mu_b = 0$ it reduces to the blackening factor 5.7, as expected. The temperature is given by

$$4\pi\omega T = -Q\mu_b(ar_h)^{\delta_2 - 1} + Q^2\Gamma(r_0)(ar_h)^{\delta_2 - 1}.$$
(5.10)

Case 3 ($\delta > 0$) 4.59

• If $\delta > \delta_2$:

$$f(r) = 1 - \left(\frac{r}{r_h}\right)^{\delta_2} \tag{5.11}$$

which is the same as 5.7. This does not mean that this case corresponds to not having charge. Indeed, to leading order it is effectively the same backening factor, but there are subleading terms containing charge contributions.

• If $\delta < \delta_2$:

$$f(r) = 1 - \left(\frac{r}{r_h}\right)^{\delta_2} + \frac{\mu_b^2(ar)^{\delta + \delta_2}}{\kappa\omega(ar_h)^{\delta}} \left[\frac{(ar)^{-\delta} - (ar_h)^{-\delta}}{\delta_2 + \delta} - \frac{(ar)^{-\delta} - (ar_h)^{-\delta}}{\delta_2}\right]$$
(5.12)

which has 5.7 plus contributions coming from the chemical potential/charge. Again, for $\mu_b = 0$ the blackening factor reduces to 5.7 as one expects when setting the chemical potential to zero.

The temperature reads

$$4\pi T = \frac{Q^2 \kappa \delta_2(ar_h)^{\delta_2 + \delta_{-1}}}{a\omega(\delta + \delta_2)}.$$
(5.13)

Case 4 ($\delta = 0$) 4.60

For the singular case we obtain the following blackening factor and temperature

$$f(r) = 1 - \left(\frac{r}{r_h}\right)^{\delta_2} \left[1 - \frac{\mu_b^2 (ar_h)^{\delta_2}}{\theta \omega \delta_2 \log r_h} + \frac{\mu_b^2 \Gamma (ar_h)^{\delta_2}}{2\delta_2 \omega \theta (\theta + \Gamma) \log r_h} + \frac{\mu_b^2 \delta_2 (ar_h)^{\delta_2}}{2a\delta_2^2 \omega (\theta + \Gamma)}\right] - \frac{\mu_b^2 (ar_h)^{\delta_2}}{\theta \omega \delta_2 \log r_h} + \frac{\mu_b^2 \Gamma (ar_h)^{\delta_2}}{2\delta_2 \omega \theta (\theta + \Gamma) \log r_h} + \frac{\mu_b^2 (ar)^{\delta_2} \log r}{2a\omega \delta_2 (\Gamma + \theta) \log r_h}$$
(5.14)

with temperature

$$4\pi T = \frac{Q^2 \theta(ar_h)^{\delta_2 - 1} \log r_h}{a\delta_2}.$$
 (5.15)

The blackening factor contains again the 5.7 term, which has no logarithmic behaviour. Such behaviour comes from the gauge field being singular, which is not present in the Q = 0 case. Also, setting $\mu_b = 0$ recovers 5.7. Notice that due to the singular behaviour the temperature contains such characteristic.

5.3 Stability

The stability condition of a thermodynamic state corresponds for such a state to have maximum entropy/minimum energy. In terms of the grand canonical potential G, stability translates into its concavity respect to the temperature and chemical potential,

$$\left(\frac{\partial^2 G}{\partial T^2}\right)_{\mu} \le 0 \quad \text{and} \quad \left(\frac{\partial^2 G}{\partial \mu^2}\right)_T \le 0.$$
 (5.16)

Furthermore, it is also necessary to ask for the positivity of the determinant of the hessian matrix [28,29]

$$H = \begin{pmatrix} -\frac{\partial^2 G}{\partial T^2} & -\frac{\partial^2 G}{\partial \mu \partial T} \\ -\frac{\partial^2 G}{\partial T \partial \mu} & -\frac{\partial^2 G}{\partial^2 \mu} \end{pmatrix} = \begin{pmatrix} \frac{\partial s}{\partial T} & \frac{\partial s}{\partial \mu} \\ \frac{\partial Q}{\partial T} & \frac{\partial Q}{\partial \mu} \end{pmatrix}$$
(5.17)

$$\det H = \left(\frac{\partial^2 G}{\partial T^2}\right)_{\mu} \left(\frac{\partial^2 G}{\partial \mu^2}\right)_T - \left(\frac{\partial^2 G}{\partial \mu \partial T}\right)^2 \ge 0.$$
(5.18)

where all the derivatives with respect to T and μ are taken with the other one constant². One can prove that these conditions are equivalent to the following restrictions [30] ³

$$C_Q = T\left(\frac{\partial s}{\partial T}\right)_Q \ge 0, \quad C_\mu = T\left(\frac{\partial s}{\partial T}\right)_\mu \ge 0 \quad \text{and} \quad \chi = \left(\frac{\partial Q}{\partial \mu}\right)_T \ge 0$$
 (5.19)

²See Appendix **D** for a review on thermodynamics.

³The stability conditions 5.16-5.18 imply that in order to have a stable system all three response functions must be positive. If one is negative, the system is unstable.

where C_Q and C_{μ} are the specific heat at constant charge and chemical potential, and χ is the isothermal permittivity⁴. The specific heat conditions determine the thermal stability of the black hole. Since the entropy of a black hole is proportional to its area, these conditions indicate whether a thermal fluctuation results in a increase or decrease in size of the black hole. Because black holes radiate at higher temepratures when they are smaller, stability follows from $C \geq 0$. The permittivity has the following interpretation. It is negative if the black hole is unstable to electrical fluctuations, which happens when the potential decreases as a result of placing more charge on it. In an attempt to make it harder to move the system from equilibrium, the potential should increase. In order to prove 5.19 we make use of the relation [28]

$$\left(\frac{\partial^2 F}{\partial T^2}\right)_Q = \frac{\left(\frac{\partial^2 G}{\partial \mu^2}\right)_T \left(\frac{\partial^2 G}{\partial T^2}\right)_\mu - \left(\frac{\partial^2 G}{\partial \mu \partial T}\right)^2}{\left(\frac{\partial^2 G}{\partial \mu^2}\right)_T} = -\frac{C_Q}{T},\tag{5.20}$$

where F is the Helmholtz canonical potential. Thus, we find

$$C_Q = -T\left(\left(\frac{\partial^2 G}{\partial T^2}\right)_{\mu} - \left(\frac{\partial^2 G}{\partial \mu \partial T}\right)^2 \left(\frac{\partial^2 G}{\partial \mu^2}\right)_T^{-1}\right),\tag{5.21}$$

which allows to write the determinant 5.18 as

$$\det H = \frac{\chi C_Q}{T}.$$
(5.22)

Demanding detH > 0 implies the positivity of C_Q and χ . The positivity of C_{μ} comes from the concavity of the grand potential respect to the temperature at constant chemical potential, 5.16. Therefore, we obtain the three conditions on the response functions 5.19.

5.4 Stability of the solutions

Given the stability conditions 5.19 we examine the stability of each case. The explicit calculations can be found in appendix C; in this section we present the results and conditions.

Case 1

In the Q = 0 case we do not have the conditions 5.19. Instead, we are interested in the specific heat at constant volume [4]. This leads to the restriction

$$c_v = \frac{d\log s}{d\log T} \ge 0 \to 3\Delta_1 + \Delta_2 \le 0, \tag{5.23}$$

which is nothing else than 4.38.

⁴Do not confuse the isothermal permitivity with the axion field also denoted by χ . If we do not say otherwise, within this chapter χ always refers to the isothermal permitivity.

Case 2 ($\delta < 0$) 4.58, 5.9, 5.10

- $C_Q \ge 0 \rightarrow -4\Delta_1 2\Delta_2 k\Delta_3 + 1 < 0$
- $C_{\mu} \ge 0 \rightarrow -4\Delta_1 2\Delta_2 k\Delta_3 + 1 < 0$
- $\chi \ge 0 \to \frac{1}{\Gamma(r_0)} \ge 0.$

This case is stable as long as these conditions are true.

Case 3 ($\delta > 0$) 4.59, 5.11, 5.12, 5.13

- $C_Q \ge 0 \rightarrow -4\Delta_1 2\Delta_2 k\Delta_3 + 1 < 0$
- $C_{\mu} \ge 0 \rightarrow -2\Delta_1 2k\Delta_3 1 < 0$
- $\chi \ge 0 \rightarrow \frac{-4\Delta_1 2\Delta_2 k\Delta_3 + 1}{-2\Delta_1 2k\Delta_3 1} \ge 0.$

This case is not a possible IR behaviour because combining the conditions from C_Q and C_{μ} we reach a contradiction with 4.38.

Case 4 ($\delta = 0$) 4.60, 5.14, 5.15

- $C_Q = \epsilon < 0$ with ϵ an infinitesimal number close to zero.
- $C_{\mu} = -1$
- $\chi \ge 0 \to -\theta \log a + \frac{\theta}{\delta_2 1} \log 4\pi T a \delta_2 \frac{2\theta}{\delta_2 1} \log \mu_b + \frac{2\theta}{\delta_2 1} \ge 0.$

This case unstable. We conclude that the only stable solution in the IR is the one corresponding to case 2 4.58, 4.79, 5.9, 5.10, as long as we demand the corresponding stability restriction

Case 2 is stable
$$\iff -4\Delta_1 - 2\Delta_2 - k\Delta_3 + 1 < 0.$$
 (5.24)

5.5 The stable solution

The stable solution is case 2 with 4.58, metric 4.79, blackening factor 5.9 and temperature 5.10. For this case we found two solutions, depending on the relation between $-4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2$ and 0. Given all the constrains, i.e, 4.38, 4.27, NEC 4.100b-4.100e, 4.101b-4.101c and the stability condition 5.24 it is possible to restrict the parameter space of the potentials given by γ, σ, k . Let us consider the first solution 4.73. The conditions, in terms of the parameters of the potentials, read

$$4\sigma(\gamma + k) - 8(\gamma + k)^2 + \sigma^2 - 4 \ge 0$$
(5.25)

$$11k + \gamma - 2\sigma - 2\sqrt{4\sigma(\gamma + k) - 8(\gamma + k)^2 + \sigma^2 - 4} > 0$$
(5.26)

$$-4k + \sigma - 4\gamma + \sqrt{4\sigma(\gamma + k) - 8(\gamma + k)^2 + \sigma^2 - 4} \ge 0$$
 (5.27)

where the first condition comes from imposing $\Delta_i \in \mathbb{R}$ and the other two from the NEC, stability and other conditions. Given these restrictions it is possible to find the

values of σ, γ and k which satisfy the allowed region. We present a 3d plot as well as a 2d plot by considering slices of constant k.



FIGURE 5.1: 3d plot of the allowed parameter space 5.25-5.27 of the first solution 4.73.



FIGURE 5.2: 2d plot of the allowed parameter space 5.25-5.27 of the first solution 4.73, with slices of constant k. This graph represents the upper tip of the triangle within the corresponding 3d plot 5.1.

Consider the second solution 4.74. The conditions obtained are

$$4\sigma(\gamma + k) - 8(\gamma + k)^2 + \sigma^2 - 4 \ge 0$$
(5.28)

$$3\gamma - \sqrt{4\sigma(\gamma+k) - 8(\gamma+k)^2 + \sigma^2 - 4} + 5k - 3\sigma > 0$$
 (5.29)

$$\frac{4\gamma + \sqrt{4\sigma(\gamma+k) - 8(\gamma+k)^2 + \sigma^2 - 4} + 4k - \sigma}{2\left(-3\gamma + \sqrt{4\sigma(\gamma+k) - 8(\gamma+k)^2 + \sigma^2 - 4} - k + \sigma\right)} \ge 0$$
(5.30)



FIGURE 5.3: 3d plot of the allowed parameter space 5.28-5.30 of the second solution 4.74.



FIGURE 5.4: 2d plot of the allowed parameter space 5.28-5.30 of the second solution 4.74, with slices of constant k.

These two solutions corresponded to the case in which $-4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2 > 0$ 4.73-4.74. For the case $-4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2 < 0$ we also had two solutions, 4.76-4.77. For the first one 4.76 the conditions take the following form

$$-32\gamma^2 + 24\gamma\sigma + 9\sigma^2 - 36 \ge 0 \tag{5.31}$$

$$-33k - 3\gamma + 15\sigma - 3\sqrt{-32\gamma^2 + 24\gamma\sigma + 9\sigma^2 - 36} < 0$$
(5.32)

$$\frac{\left(\sqrt{-32\gamma^{2}+24\gamma\sigma+9\sigma^{2}-36}\right)(3\gamma-7\sigma)+96\gamma^{2}-59\gamma\sigma-15\sigma^{2}+45}{44\gamma^{2}-12\gamma\sigma-6\sigma^{2}+25} - \frac{8\left(\sqrt{-32\gamma^{2}+24\gamma\sigma+9\sigma^{2}-36}-11\gamma-9k+3\sigma\right)}{5\sqrt{-32\gamma^{2}+24\gamma\sigma+9\sigma^{2}-36}-28\gamma-3\sigma} > 0$$
(5.33)

$$4\gamma \left(35\sqrt{-32\gamma^{2}+24\gamma\sigma+9(\sigma^{2}-4)}-96\sigma\right)+8\gamma^{2}+450 + 3\sigma \left(5\sqrt{-32\gamma^{2}+24\gamma\sigma+9(\sigma^{2}-4)}-39\sigma\right) \le 0$$
(5.34)



FIGURE 5.5: 3d plot of the allowed parameter space 5.31-5.34 of the third solution 4.76.



FIGURE 5.6: 2d plot of the allowed parameter space 5.31-5.34 of the third solution 4.76, with slices of constant k. It is not shown in the plot, but there is an overlap of regions for different values of k. In particular, the regions with higher values of k overlap with the ones of lower k, so that when increasing k the allowed region increases, as it can be seen in the 3d plot 5.5.

For the last solution 4.77 it turns out that in order to satisfy the conditions the parameters γ, σ, k must be complex, which is in contradiction with the restriction $\Delta_i \in \mathbb{R}, i = 1, 2, 3$. Thus, such solution is not a possible IR behaviour. Therefore, from the stable solution three of the solutions are possible 4.73, 4.74, 4.76, while one is discarded 4.77. In conclusion, case 2 4.58, 5.9, 5.10 is a stable solution if the condition 5.24 is satisfied. The IR geometry is given by the metric 4.79 which exhibits a Lifshitz scaling upon scaling the charge/chemical potential, with three possible solutions. These solutions can be constrained using the NEC together with 4.27, 4.38 and therefore the potentials of the gravitational action 3.18 are restricted in the IR regime.

6. Conclusions and Outlook

In this thesis we have studied the IR regime of the anisotropic charged QGP. The Einstein-Maxwell-Axion-Dilaton action has been the holographic theory exploited, from which Einstein's equations and Maxwell's equations gave us the gravity expressions to study the bulk theory.

Our work was based on [4], where the spatial anisotropy of the gauge theory was considered. In this thesis we extended such investigation by introducing a gauge field in the bulk, in order to account for the charge density of the QGP. First, we found that the IR metric has three possible forms in accordance with the requirement of the good type IR singularity. As in the case without charge we identified that the metric exhibits a Lifshitz scaling, upon scaling the charge/chemical potential, and hyperscaling violation, signaling the dependence of the thermodynamic quantities on the dynamical exponents. Crucially, one of the cases correspondend to a singular behaviour of the gauge field, for which the IR metric does not show Lifshitz scaling and indicates an anomaly. We also accounted for the realisation of the c-theorem in holography given by the NEC, providing constraints to the allowed parameter space of the solutions.

At finite temperature we computed the blackening factor and Hawking radiation obtaining new contributions originated from the charge density. Finally, we investigated the thermodynamic stability of the IR solutions based on the positivity of the response functions, leading to the conclusion that only one IR geometry is stable, corresponding to 4.58, 5.9 and 5.10. For such a case, the IR metric parameters were constrained using the conditions coming from the good singularity behaviour 4.38, the AdS/CFT correspondence 4.27, the NEC 4.7 and the stability restriction 5.24. Following this route we have been able to constrain the potentials of the gravitational action 3.18.

There is, of course, much work that would be interesting to complete and we have not been able to perform in this thesis. In particular, the next possible step would be to calculate the free energy of the system in order to see the confined-deconfined plasma transition, and in particular to examine how the transition temperature changes with the presence of charge. For the anisotropic case [4] the transition temperature decreases due to the anisotorpy, and one expects some dependence from the presence of charge. The calculation of transport and diffusion quantities, such as the viscosity and butterfly velocities, would also be relevant to see their dependence on the dynamical exponents as well as their saturation values, to examine how they change due to the presence of a gauge field.
Appendices

A. Quantum Chromodynamics

QCD is a non-abelian gauge theory invariant under SU(3). The fermions of the theory are the quarks transforming in the fundamental representation, while the gluons are the bosonic gauge fields carrying the colour charge which transform in the adjoint representation. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + i \overline{q}_i \gamma^\mu D_\mu q_i - m \overline{q}_i q_i \tag{A.1}$$

where q_i are the spinor fields describing the quarks with i = u, d, c, s, t, b, i.e., the six flavours. The covariant derivative and field strength read

$$D_{\mu} = \partial_{\mu} - igA^a_{\mu}\sigma_a \tag{A.2}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a_{bc} A^b_\mu A^c_\mu \tag{A.3}$$

where A^a_{μ} is the gluon gauge field, g the coupling constant, σ_a the Gell-Mann matrices and f^a_{bc} the structure constants satisfying $[\sigma_b, \sigma_c] = f^a_{bc}\sigma_a$. QCD presents divergences in both the IR and UV regimes. Within renormalization, the regularization technique gives the β -function

$$\frac{\beta(g)}{g} = -\frac{33 - 2N_f}{3} \frac{g^2}{16\pi^2} - \frac{306 - 38N_f}{3} \left(\frac{g^2}{16\pi^2}\right) + \mathcal{O}(6) \tag{A.4}$$

with $N_f = 6$. The β -function contains the information about the RG flow of the gauge theory. In particular, it is a measure of the rate of change of the coupling constant respect to the energy scale. For QCD, $N_f = 6$ and thus the β -function is negative, indicating an interesting property of QCD: asymptotic freedom. Contrary to QED, the coupling constant of QCD decreases towards the UV regime, and in the far UV the theory becomes asymptotically free. On the other hand, in the low energy IR regime the coupling constant increases and QCD becomes strongly coupled. Another improtant property of QCD is confinement. Confinement refers to the behaviour of the potential interaction between two quarks; it increases linearly with distance, so that the quarks are glued together forming hadrons and they are therefore confined, meaning that is not possible to find them as free particles. Contrary to usual QCD matter, the QGP is deconfined, being one of the reasons for which is an interesting QCD state.

B. c-theorem/NEC

In section 4.7 we presented the constrains coming from the NEC for each case, i.e., 4.58, 4.58 and 4.60. In this Appendix we show the calculations involved in obtaining such conditions. In order to study condition 4.50 we consider the blackening factor 4.29, so that

$$f'(r) = -e^{-3A(r) - h(r)}(QA_t(r) + C)$$
(B.1a)

$$f''(r) = e^{-3A(r) - h(r)} \left(Q(A_t(r)(3A'(r) + h'(r)) - A'_t(r)) + C(3A'(r) + h'(r)) \right).$$
(B.1b)

Plugging in these functions into the condition 4.50 we obtain

$$3A'(r)f'(r) + f''(r) + f'(r)h'(r) \ge 0 \to QA'_t(r) \le 0.$$
(B.2)

Given the gauge field 4.7 together with the ansatz functions 4.24-4.26,

$$A'_t(r) = \frac{Q}{Lc_1(c_3)^{2k}} (ar)^{-\Delta_1 - \Delta_2 - 2k\Delta_3},$$
(B.3)

which yields

$$\frac{Q^2 c_3^{-2k} (ar)^{-\Delta_1 - \Delta_2 - 2\Delta_3 k}}{c_1 L} \ge 0 \to a \ge 0.$$
(B.4)

But this condition is not giving new information, since we already knew from the ansatz functions that the anisotropy factor has to be positive. To analyse condition 4.52 we use the ground state 4.42. By doing so, we obtain

$$Ar^{-2} + Br^{-3\Delta_1 - \Delta_2 - 1} + Cr^{-4\Delta_1 - 2\Delta_2 - 2k\Delta_3} \le 0$$
(B.5)

with

$$A = 2\Delta_2(1 + Q_0\Omega(r_0))(3\Delta_1 + \Delta_2 - 1)$$
(B.6a)

$$B = \frac{-6\Delta_1 Q_0}{c_1 L^3} \left(\mu_{b0} + Q_0 \Gamma(r_0)\right)$$
(B.6b)

$$C = -\left(\frac{6\Delta 1}{\Delta 1 + \Delta 2 + 2\Delta 3k - 1} + \frac{\Delta 2}{2\Delta 1 + \Delta 2 + \Delta 3k - 1} + 1\right)$$
(B.6c)

Therefore, we have to consider the relation between the different exponents and see which term leads the order as $r \to \infty$, i.e., in the IR. Looking case by case

- Case 2: Implies that $B \leq 0 \rightarrow Q_0(\Gamma(r_0)Q_0 + \mu_{b0}) \leq 0$
- Case 3: Implies $C \leq 0$ so

$$\frac{6\Delta 1}{\Delta 1 + \Delta 2 + 2\Delta 3k - 1} + \frac{\Delta 2}{2\Delta 1 + \Delta 2 + \Delta 3k - 1} + 1 \ge 0.$$
(B.7)

For case 4 4.60 the ground state is different to the other cases due to the divergence of the gauge field, and it must be analysed on its own. If one does so for condition 4.52 with the corresponding ground state 4.60

$$\xi Q_0^2 (3\Delta_1 + \Delta_2 - 1)(-9\Delta_1 - 3\Delta_2 + 6\Delta_1 (3\Delta_1 + \Delta_2 - 1)\log(r) + 1) \le 0$$
 (B.8)

which implies $\Delta_1 \leq 0$ as $r \to \infty$. This does not provide new information since we already knew it from 4.27. The extra condition 4.53 is also taken into account. Given the thermal gas 4.57

$$-\frac{\Delta_2 Q_0^2 c_3^{-2k} (ar)^{-2(2\Delta_1 + \Delta_2 + \Delta_3 k)}}{2c_1^2 L^4 (2\Delta_1 + \Delta_2 + \Delta_3 k - 1)} + \frac{\Delta_2 Q_0 \Omega(r_0) (3\Delta_1 + \Delta_2 - 1)}{r^2} + \frac{\Delta_2 (3\Delta_1 + \Delta_2 - 1)}{r^2} \le 0$$
(B.9)

which translates into the following restrictions:

• If $-2(2\Delta_1 + \Delta_2 + \Delta_3 k) > -2$

$$\frac{\Delta_2}{(2\Delta_1 + \Delta_2 + \Delta_3 k - 1)} \ge 0 \tag{B.10}$$

which happens for case 3 4.59 and for one of the particular possibilities of case 2, 4.73 and 4.74.

• If $-2(2\Delta_1 + \Delta_2 + \Delta_3 k) < -2$

$$\Delta_2(1+Q_0\Gamma(r_0)) \ge 0 \tag{B.11}$$

which happens for one of the solutions of case 2, 4.76 and 4.77.

Again for case 4 4.60 we need to use the corresponding thermal gas 4.60. Then condition 4.53 takes the form

$$-\Delta_2(3\Delta_1 + \Delta_2 - 1) \le 0 \tag{B.12}$$

which implies $\Delta_2 \leq 0$ due to 4.38 and constrains directly 4.25. Therefore, we obtain the following restrictions for every case:

• Case 2 4.58:

$$Q_0(\Gamma Q_0 + \mu_{b0}) \leq 0$$

$$\Delta_2(1 + Q_0 \Gamma(r_0)) \geq 0 \iff -4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2 < 0$$

$$\frac{\Delta_2}{(2\Delta_1 + \Delta_2 + \Delta_3 k - 1)} \geq 0 \iff -4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 2 > 0.$$

• Case 3 4.59:

$$\frac{\Delta_2}{(2\Delta_1 + \Delta_2 + \Delta_3 k - 1)} \ge 0$$
$$\frac{6\Delta 1}{\Delta 1 + \Delta 2 + 2\Delta 3k - 1} + \frac{\Delta 2}{2\Delta 1 + \Delta 2 + \Delta 3k - 1} + 1 \ge 0.$$

• Case 4 4.60:

$$\Delta_2 \leq 0.$$

C. Blackening factor, temperature and stability

In chapter 5 we study the thermodynamics of the 5-dimensional black brane present in the bulk. In such chapter the results were presented, omiting the full computation of the blackening factor and temperature as well as the computation of the response functions 5.19 controlling the thermodynamic stability. In this appendix the corresponding computations are presented.

C.1 Blackening factor

We present the computation of the full blackening factor, from which we obtained the different cases 5.11, 5.12, 5.9, 5.14 of section 5.2. The blackening factor is given by 4.4,

$$f(r) = A - C \int_0^r dr' e^{-3A(r) - h(r)} - Q \int_0^r dr' A_t(r) e^{-3A(r) - h(r)}$$
(C.1)

$$C = \frac{1 - Q \int_{-\infty}^{r_h} dr' A_t(r) e^{-3A(r) - h(r)}}{\int_{-\infty}^{r_h} dr' e^{-3A(r) - h(r)}}$$
(C.2a)

$$A = 1. \tag{C.2b}$$

The gauge field 4.7 reads

$$Q = \frac{\mu_b}{\Gamma(r_0) + \kappa(ar_h)^{\delta}} \tag{C.3}$$

$$A_t(r) = \mu_b \left[1 - \frac{\Gamma(r_0) + \kappa(ar)^{\delta}}{\Gamma(r_0) + \kappa(ar_h)^{\delta}} \right]$$
(C.4)

with $\delta \equiv -\Delta_1 - \Delta_2 - 2k\Delta_3 + 1$.

Case $\delta \neq 0$

Using the IR ansatz 4.24-4.26 and the gauge field 5.3 the blackening factor reads¹

$$\begin{split} f(r) &= 1 - C \left(\Lambda(r_0) + \frac{(ar)^{\delta_2}}{\omega a \delta_2} \right) - \frac{Q \mu_b(ar)^{\delta_2}}{a \omega \delta_2} + \frac{Q^2 \kappa(ar)^{\delta_2 + \delta}}{\omega(\delta_2 + \delta)} + \frac{Q^2 \Gamma(r_0)(ar)^{\delta_2}}{a \omega \delta_2} - Q \Omega(r_0) \\ &= 1 - C \left(\Lambda(r_0) + \frac{(ar)^{\delta_2}}{\omega a \delta_2} \right) - \frac{\mu_b \Omega(r_0)}{\Gamma(r_0) + \kappa(ar_h)^{\delta}} - \frac{\mu_b^2 (ar)^{\delta_2}}{a \omega \delta_2 (\Gamma(r_0) + \kappa(ar_h)^{\delta})} \\ &+ \frac{\mu_b^2 \kappa(ar)^{\delta_2 + \delta}}{\omega(\delta_2 + \delta) (\Gamma(r_0) + \kappa(ar_h)^{\delta})^2} + \frac{\mu_b^2 \Gamma(r_0)(ar)^{\delta_2}}{a \omega \delta_2 (\Gamma(r_0) + \kappa(ar_h)^{\delta})^2} \end{split}$$
(C.5)

where in the second equality we introduced the charge given by 5.2 and we defined $\omega \equiv L^3 c_1, \, \delta_2 \equiv -3\Delta_1 - \Delta_2 + 1$. The constant C is

$$C = \frac{1 - \frac{Q\mu_{b}(ar)^{\delta_{2}}}{a\omega\delta_{2}} + \frac{Q^{2}\kappa(ar)^{\delta_{2}+\delta}}{\omega(\delta_{2}+\delta)} + \frac{Q^{2}\Gamma(r_{0})(ar)^{\delta_{2}}}{a\omega\delta_{2}} - Q\Omega(r_{0})}{\Lambda(r_{0}) + \frac{(ar_{h})^{\delta_{2}}}{a\omega\delta_{2}}} = \frac{1 - \frac{\Omega(r_{0})\mu_{b}}{\Gamma(r_{0}) + \kappa(ar_{h})^{\delta}} - \frac{\mu_{b}^{2}(ar_{h})^{\delta_{2}}}{\omega\delta_{2}(\Gamma(r_{0}) + \kappa(ar_{h})^{\delta})} + \frac{\mu_{b}^{2}\Gamma(r_{0})(ar_{h})^{\delta_{2}}}{\omega\delta_{2}(\Gamma(r_{0}) + \kappa(ar_{h})^{\delta})^{2}} + \frac{\mu_{b}^{2}\kappa(ar_{h})^{\delta_{2}+\delta}}{\omega(\delta_{2}+\delta)(\Gamma(r_{0}) + \kappa(ar_{h})^{\delta})^{2}}}{\Lambda(r_{0}) + \frac{(ar_{h})^{\delta_{2}}}{a\omega\delta_{2}}}.$$
(C.6)

Introducing C into the blackeing factor C.5

$$f(r) = 1 - \frac{\Lambda(r_0) + \frac{(ar)^{\delta_2}}{a\omega\delta_2}}{\Lambda(r_0) + \frac{(ar_h)^{\delta_2}}{a\omega\delta_2}} \left[1 - \frac{\Omega(r_0)\mu_b}{\Gamma(r_0) + \kappa(ar_h)^{\delta}} - \frac{\mu_b^2(ar_h)^{\delta_2}}{\omega\delta_2(\Gamma(r_0) + \kappa(ar_h)^{\delta})} + \frac{\mu_b^2\Gamma(r_0)(ar_h)^{\delta_2}}{\omega\delta_2(\Gamma(r_0) + \kappa(ar_h)^{\delta})^2} \right] \\ + \frac{\mu_b^2\kappa(ar_h)^{\delta_2+\delta}}{\omega(\delta_2+\delta)(\Gamma(r_0) + \kappa(ar_h)^{\delta})^2} \left] - \frac{\Omega(r_0)\mu_b}{\Gamma(r_0) + \kappa(ar_h)^{\delta}} - \frac{\mu_b^2(ar)^{\delta_2}}{\omega\delta_2(\Gamma(r_0) + \kappa(ar_h)^{\delta})} \right] \\ + \frac{\mu_b^2\Gamma(r_0)(ar)^{\delta_2}}{\omega\delta_2(\Gamma(r_0) + \kappa(ar_h)^{\delta})^2} + \frac{\mu_b^2\kappa(ar)^{\delta_2+\delta}}{\omega(\delta_2+\delta)(\Gamma(r_0) + \kappa(ar_h)^{\delta})^2}.$$
(C.7)

In the limit $r_h \to \infty$, we must distinguish between δ positive or negative and its relation to δ_2 . Recall that $\delta_2 > 0$ due to 4.38.

Case 2 ($\delta < 0$) 4.58

 $^{{}^{1}\}Lambda(r_{0})$ denotes the integral of the C term for which the IR ansatz is not satisfied. Since C = 0 in the thermal gas this term did not appear in the previous chapter.

• The only possibility corresponds to $\delta < \delta_2$. Using the Taylor expansion $1/(1+x) \sim 1-x$ we obtain

$$f(r) = 1 - \frac{\mu_b \Omega(r_0)}{\Gamma(r_0)} - \left(\frac{r}{r_h}\right)^{\delta_2} \left[1 - \frac{\mu_b \Omega(r_0)}{\Gamma(r_0)} + \frac{\mu_b^2 (ar_h)^{\delta + \delta_2}}{\Gamma(r_0)^2 \omega} \left(\frac{\kappa - 2}{\delta_2} + \frac{\kappa}{\delta_2 + \delta}\right)\right] + \frac{\mu_b^2 (ar)^{\delta_2} (ar_h)^{\delta}}{\Gamma(r_0)^2 \omega} \left(\frac{\kappa - 2}{\delta_2} + \frac{\kappa}{\delta_2 + \delta}\right).$$
(C.8)

Case 3 ($\delta > 0$) 4.59

• If $\delta > \delta_2$ then, in the IR limit we obtain

$$f(r) = 1 - \left(\frac{r}{r_h}\right)^{\delta_2}.$$
 (C.9)

• If $\delta < \delta_2$ the leading terms in the IR are the $\delta + \delta_2$ terms and thus

$$f(r) = 1 - \left(\frac{r}{r_h}\right)^{\delta_2} + \frac{\mu_b^2(ar)^{\delta + \delta_2}}{\kappa\omega(ar_h)^{\delta}} \left[\frac{(ar)^{-\delta} - (ar_h)^{-\delta}}{\delta_2 + \delta} - \frac{(ar)^{-\delta} - (ar_h)^{-\delta}}{\delta_2}\right].$$
(C.10)

Case 4 ($\delta = 0$) 4.60

This case corresponds to the singular gauge field behaviour 5.6b, 4.60. The constant C reads

$$C = \frac{1 - Q\Omega(r_0) - \frac{Q\mu_b(ar_h)^{\delta_2}}{\omega\delta_2} + \frac{Q^2(ar_h)^{\delta_2}}{\omega\delta_2} + \frac{\theta Q^2(ar_h)^{\delta_2}(-1+\delta_2\log r_h)}{a\omega\delta_2^2}}{\Lambda(r_0) + \frac{(ar_h)^{\delta_2}}{\omega}} \\ \frac{1 - \frac{\Omega(r_0)\mu_b}{\Gamma(r_0) + \theta\log r_h} - \frac{\mu_b^2(ar_h)^{\delta_2}}{(\Gamma(r_0) + \theta\log r_h)\omega\delta_2} + \frac{\mu_b^2(ar_h)^{\delta_2}}{(\Gamma(r_0) + \theta\log r_h)^2\omega\delta_2} + \frac{\theta \mu_b^2(ar_h)^{\delta_2}(-1+\delta_2\log r_h)}{(\Gamma(r_0) + \theta\log r_h)^2a\omega\delta_2^2}}{\Lambda(r_0) + \frac{(ar_h)^{\delta_2}}{\omega}},$$
(C.11)

and the blackening factor

$$\begin{split} f(r) =& 1 - \frac{\Lambda + \frac{(ar)^{\delta_2}}{a\omega\delta_2}}{\Lambda + \frac{(ar_h)^{\delta_2}}{a\omega\delta_2}} \left[1 - \frac{\Omega\mu_b}{\Gamma + \theta\log r_h} - \frac{\mu_b^2(ar_h)^{\delta_2}}{(\Gamma + \theta\log r_h)\omega\delta_2} + \frac{\mu_b^2(ar_h)^{\delta_2}}{(\Gamma + \theta\log r_h)^2\omega\delta_2} \right] \\ &+ \frac{\theta\mu_b^2(ar_h)^{\delta_2}(-1 + \delta_2\log r_h)}{(\Gamma + \theta\log r_h)^2a\omega\delta_2^2} - \frac{\Omega\mu_b}{\Gamma + \theta\log r_h} - \frac{\mu_b^2(ar)^{\delta_2}}{(\Gamma + \theta\log r_h)\omega\delta_2} \\ &+ \frac{\mu_b^2(ar)^{\delta_2}}{(\Gamma + \theta\log r_h)^2\omega\delta_2} + \frac{\theta\mu_b^2(ar)^{\delta_2}(-1 + \delta_2\log r)}{(\Gamma + \theta\log r_h)^2a\omega\delta_2^2}. \end{split}$$
(C.12)

In the IR limit,

$$f(r) = 1 - \left(\frac{r}{r_h}\right)^{\delta_2} \left[1 - \frac{\mu_b^2 (ar_h)^{\delta_2}}{\theta \omega \delta_2 \log r_h} + \frac{\mu_b^2 \Gamma (ar_h)^{\delta_2}}{2\delta_2 \omega \theta (\theta + \Gamma) \log r_h} + \frac{\mu_b^2 \delta_2 (ar_h)^{\delta_2}}{2a\delta_2^2 \omega (\theta + \Gamma)}\right] - \frac{\mu_b^2 (ar_h)^{\delta_2}}{\theta \omega \delta_2 \log r_h} + \frac{\mu_b^2 \Gamma (ar_h)^{\delta_2}}{2\delta_2 \omega \theta (\theta + \Gamma) \log r_h} + \frac{\mu_b^2 (ar)^{\delta_2} \log r}{2a\omega \delta_2 (\Gamma + \theta) \log r_h}.$$
(C.13)

Notice that for every case, setting Q = 0 (or $\mu_b = 0$) recovers, as expected, the case without charge [4]

$$f(r) = 1 - \left(\frac{r}{r_h}\right)^{\delta_2}.$$
 (C.14)

C.2 Temperature

The temperature is given by 4.31, and it is therefore given in terms of the constant C. Again, in the IR limit we distinguish different cases depending on the value of the exponent δ of the gauge field 5.3.

Case
$$\delta \neq 0$$

$$4\pi T = \frac{\delta_2 (1 - Q\Omega(r_0))}{r_h} - \frac{Q\mu_b (ar_h)^{\delta_2 - 1}}{\omega} + \frac{Q^2 \Gamma(r_0) (ar_h)^{\delta_2 - 1}}{\omega} + \frac{Q^2 \kappa \delta_2 (ar_h)^{\delta_2 + \delta_1 - 1}}{a\omega(\delta + \delta_2)}$$
$$= \delta_2 r_h^{-1} - \frac{\delta_2 \Omega(r_0) \mu_b}{(\Gamma(r_0) + \kappa(ar_h)^{\delta}) r_h} - \frac{\mu_b^2 (ar_h)^{\delta_2 - 1}}{\omega(\Gamma(r_0) + \kappa(ar_h)^{\delta})} + \frac{\mu_b^2 \Gamma(r_0) (ar_h)^{\delta_2 - 1}}{a\omega(\Gamma(r_0) + \kappa(ar_h)^{\delta})^2}$$
$$+ \frac{\mu_b^2 \kappa \delta_2 (ar_h)^{\delta_2 + \delta_1 - 1}}{a\omega(\delta_2 + \delta)(\Gamma(r_0) + \kappa(ar_h)^{\delta})^2}$$
(C.15)

Case 2 ($\delta < 0$) C.8

In the IR the temperature reduces to

$$4\pi\omega T = -Q\mu_b (ar_h)^{\delta_2 - 1} + Q^2 \Gamma(r_0) (ar_h)^{\delta_2 - 1}.$$
 (C.16)

Case 3 ($\delta > 0$) C.9-C.10

In the IR regime the temperature reduces to

$$4\pi T = \frac{Q^2 \kappa \delta_2 (ar_h)^{\delta_2 + \delta_{-1}}}{a\omega(\delta + \delta_2)}.$$
 (C.17)

$$\begin{aligned} 4\pi T &= \frac{\delta_2}{r_h} - \frac{Q\Omega(r_0)\delta_2}{r_h} - \frac{Q\mu_b(ar_h)^{\delta_2 - 1}}{\omega\delta_2} + \frac{Q^2(ar_h)^{\delta_2 - 1}}{\omega\delta_2} + \frac{Q^2\theta(ar_h)^{\delta_2 - 1}(-1 + \delta_2\log r_h)}{a\delta_2^2} \\ &= \frac{\delta_2}{r_h} - \frac{\mu_b}{\Gamma(r_0) + \theta\log r_h} \frac{\Omega(r_0)\delta_2}{r_h} - \frac{\mu_b}{\Gamma + \theta\log r_h} \frac{\mu_b(ar_h)^{\delta_2 - 1}}{\omega\delta_2} \\ &+ \left(\frac{\mu_b}{\Gamma(r_0) + \theta\log r_h}\right)^2 \frac{(ar_h)^{\delta_2 - 1}}{\omega\delta_2} + \left(\frac{\mu_b}{\Gamma(r_0) + \theta\log r_h}\right)^2 \frac{\theta(ar_h)^{\delta_2 - 1}(-1 + \delta_2\log r_h)}{a\delta_2^2}. \end{aligned}$$
(C.18)

The temperature in the IR reduces to

$$4\pi T = \frac{Q^2 \theta(ar_h)^{\delta_2 - 1} \log r_h}{a\delta_2}.$$
(C.19)

Recall that setting Q = 0 (or $\mu_b = 0$) for C.15 and C.18 recovers, as expected, the temperature for C.14 [4],

$$T = \frac{\delta_2}{r_h}.\tag{C.20}$$

C.3 Stability

Case $\delta = 0$ C.12

In chapter 5 the results coming from the stability conditions were shown. In this section the computation of such response functions is shown.

Case 2 $\delta < 0$

In order to compute C_Q we need to find s = s(T, Q). Using the corresponding temperature C.16 and charge expressions 5.2, we can extract the $r_h = r_h(T, Q)$,

$$r_h = \frac{1}{a} \left(\frac{4\pi T}{Q^2 \kappa}\right)^{\frac{1}{\delta_2 + \delta - 1}}.$$
 (C.21)

Therefore, the entropy 4.37 reads

$$s = \frac{L^3 c_1}{4G} \left(\frac{4\pi T}{Q^2 \kappa}\right)^{\frac{3\Delta_1 + \Delta_2}{\delta_2 + \delta - 1}}.$$
(C.22)

We can now compute the specific heat at constant charge. In particular, we consider the dimensionless quantity

$$c_q \equiv \frac{T}{s}C_Q = \left(\frac{d\log s}{d\log T}\right)_Q \ge 0 \to \delta_2 + \delta - 1 = -4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 1 < 0 \quad (C.23)$$

where we made use of the condition 4.38. In order to compute the specific heat at constant chemical potential we need $r_h = r_h(\mu_b, T)$ to get $s = s(\mu_b, T)$. Using the temperature C.16 and charge 5.2

$$4\pi T\omega = \frac{\mu_b^2 (ar_h)^{\delta_2 - 1}}{\Gamma + \kappa (ar_h)^{\delta}} \left(\frac{\Gamma}{\Gamma + \kappa (ar_h)^{\delta}} - 1\right).$$
(C.24)

Since $\delta < 0$, in the IR limit

$$r_h = \frac{1}{a} \left(\frac{4\pi T \omega \Gamma^2}{\mu_b^2 \kappa} \right)^{\frac{1}{\delta_2 + \delta - 1}} \tag{C.25}$$

and the entropy reads

$$s = \frac{L^4 c_1}{4G} \left(\frac{4\pi T \omega \Gamma^2}{\mu_b^2 \kappa} \right)^{\frac{3\Delta_1 + \Delta_2}{\delta_2 + \delta - 1}}.$$
 (C.26)

As in the case of C_Q , we compute the dimensionless quantity

$$c_{\mu} \equiv \frac{d\log s}{d\log T} \ge 0 \to \delta_2 + \delta - 1 = -4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 1 < 0 \tag{C.27}$$

In order to find the isothermal permitivity we find the charge in terms of the temperature and chemical potential,

$$Q = \frac{\mu_b}{\Gamma} - \frac{\kappa}{\Gamma^2} \left(4\pi T \omega \Gamma^2\right)^{\frac{1}{\delta_2 + \delta - 1}} \mu_b^{\frac{\delta_2 + \delta - 3}{\delta_2 + \delta - 1}} \tag{C.28}$$

and thus, to leading order,

$$\chi = 1/\Gamma(r_0) \ge 0. \tag{C.29}$$

Therefore, this case is stable as long as such conditions are satisfied; in particular it must satisfy $-4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 1 < 0$, having $\Gamma(r_0)$ positive.

Case 3 $\delta > 0$

Using C.17 and 5.2 we find the horizon as a function of the charge and temperature,

$$r_h = \frac{1}{a} \left(\frac{4\pi T a \omega (\delta_2 + \delta)}{Q^2 \kappa \delta_2} \right)^{\frac{1}{\delta_2 + \delta - 1}}, \tag{C.30}$$

and therefore

$$s = \frac{L^3 c_1}{4G} \left(\frac{4\pi T a \omega (\delta_2 + \delta)}{Q^2 \kappa \delta_2} \right)^{\frac{3\Delta_1 + \Delta_2}{\delta_2 + \delta - 1}} \rightarrow c_q \ge 0 \rightarrow -4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 1 < 0 \quad (C.31)$$

Following the same procedure as before, we find the horizon as a function of the temperature and chemical potential. Since $\delta > 0$,

$$r_h = \frac{1}{a} \left(\frac{4\pi T a \omega \kappa (\delta_2 + \delta)}{\delta_2 \mu_b^2} \right)^{\frac{1}{\delta_2 - \delta - 1}} \tag{C.32}$$

$$s = \frac{L^3 c_1}{4G} \left(\frac{4\pi T a \omega \kappa (\delta_2 + \delta)}{\delta_2 \mu_b^2} \right)^{\frac{3\Delta_1 + \Delta_2}{\delta_2 - \delta - 1}} \to c_\mu \ge 0 \to -2\Delta_1 - 1 + 2k\Delta_3 < 0.$$
(C.33)

This condition, together with the condition from c_q C.31 is in contradiction with 4.38. For the isothermal permitivity we find the charge

$$Q = \frac{\mu_b}{\Gamma(r_0) + \kappa \left(\frac{4\pi T a \omega \kappa(\delta_2 + \delta)}{\delta_2 \mu_b^2}\right)^{\frac{\delta}{\delta_2 - \delta - 1}}}.$$
 (C.34)

To leading order

$$\chi = \frac{-4\Delta_1 - 2\Delta_2 - 2k\Delta_3 + 1}{-2\Delta_1 - 1 + 2k\Delta_3} \ge 0,$$
 (C.35)

which is automatically satisfied from the previous two conditions C.31, C.33. This case is not a possible IR behaviour because it violates the good singularity condition 4.38.

Case 4 $\delta = 0$

We first compute c_{μ} . Using the expressions C.19 and 5.6b, we obtain $r_h = r_h(\mu_b, T)$

$$r_{h} = \frac{1}{a} \left(\frac{4\pi T a \delta_{2}}{\mu_{b}^{2}} \right)^{\frac{1}{\delta_{2} - 1}} \tag{C.36}$$

which implies

$$s = \frac{L^3 c_1}{4G} \left(\frac{4\pi T a \delta_2}{\mu_b^2}\right)^{-1} \to c_\mu = -1,$$
(C.37)

giving an unstable system. Computing the isothermal permitivity we obtain

$$-\theta \log a + \frac{\theta}{\delta_2 - 1} \log 4\pi T a \delta_2 - \frac{2\theta}{\delta_2 - 1} \log \mu_b + \frac{2\theta}{\delta_2 - 1} \ge 0.$$
 (C.38)

To compute c_q we use the chain rule

$$\left(\frac{\partial s}{\partial T}\right)_Q = \left(\frac{\partial s}{\partial r_h}\right) \frac{1}{\left(\frac{\partial T}{\partial r}\right)} \tag{C.39}$$

$$\left(\frac{\partial s}{\partial r_h}\right) = (-\delta_2 + 1)r_h^{-\delta_2} \tag{C.40}$$

$$\left(\frac{\partial T}{\partial r}\right) = Q^2 r_h^{\delta_2 - 2} \left((\delta_2 - 1) \log r_h + 1 \right) \tag{C.41}$$

and thus

$$\left(\frac{\partial s}{\partial T}\right)_{Q} = \frac{-(\delta_{2} - 1)r_{h}^{-2(\delta_{2} - 1)}}{Q^{2}((\delta_{2} - 1)\log r_{h} + 1)}.$$
(C.42)

Since every term is positive, in the limit $r_h \to \infty$ the derivative goes to zero from the left, i.e., $C_Q \sim \epsilon < 0$ where ϵ is a number infinitessimally close to zero from the left. Therefore, this case is thermodynamically unstable.

D. Statistical physics and thermodynamics

D.1 Canonical ensemble

In the canonical ensemble the system under study is in contact with a heat bath at temperature T, being a closed system, i.e., it does not interchange particles. The variables which define the system are T, Q, V. The potential associated to this ensmeble is the Helmholtz potential F. The partition function is given by

$$Z = \sum_{r} e^{-\beta E_r} \tag{D.1}$$

so that the potential is found as

$$F = -k_b T \ln Z. \tag{D.2}$$

Recall that the Helmholtz potential is F = U - TS, so that F = F(T, V, Q). This is related to the euclidean action by $S_E - S_E^0 = -\beta F$ within the context of the AdS/CFTcorrespondence. The first law of thermodynamics takes the form

$$dE = Tds + \mu dQ,\tag{D.3}$$

which in terms of F reads

$$dF = -sdT + \mu dQ. \tag{D.4}$$

Thus, we find the relations

$$s = -\left(\frac{\partial F}{\partial T}\right)_Q, \quad \mu = \left(\frac{\partial F}{\partial Q}\right)_T.$$
 (D.5)

D.2 Grand canonical ensemble

Within the grand canonical ensemble the system considered has energy E variable, but also the charge Q of the system is allowed to change. The system is in equilibrium with a heat bath at temperature T and in chemical equilibrium with a reservoir of particles with a chemical potential μ . Therefore, the states are described by T,V and μ . The partition function is given by

$$Q = \sum_{Q_s=0}^{\infty} Z(N_s, T, V) z^{Q_s}$$
(D.6)

with $z = e^{\beta\mu}$ and $Z(Q_s, T, V)$ the canonical partition function D.1 of a subsystem. The grand potential is

$$G = -k_b T \ln \mathcal{Q},\tag{D.7}$$

which is also defined as $G = U - TS - \mu Q$. The Gibbs free energy can be defined from the potential F as the Legendre transformation $G = F - \mu Q$, so that $G = G(T, V, \mu)$. Within the context of the AdS/CFT correspondence the potential is found for a particular solution of the euclidean continuation of the Lorentzian action by $S_E - S_E^0 = -\beta G$. The first law is expressed as

$$dE = \mu dQ + TdS \tag{D.8}$$

which takes the following form in terms of G

$$dG = -sdT - Qd\mu. \tag{D.9}$$

Given these expressions one can find the relations

$$s = -\left(\frac{\partial G}{\partial T}\right)_{\mu}, \qquad Q = -\left(\frac{\partial G}{\partial \mu}\right)_{T}.$$
 (D.10)

The fundamental difference between the canonical and gran canonical ensemble is the presence of the particle reservoir in the grand canonical formalism. This means that the chemical potential is constant in the grand canonical ensemble, and thus one works with the intensive variables T and μ . On the other hand, in the canonical ensemble one works with the intensive variable T and the extensive variable Q.

D.3 Response functions

The reponse functions are the themodynamic quantites which establish the dependence of thermodynamic functions respect to external changes of a certain variable. In particular, the specific heat is defined as

$$C_X = \left(\frac{\partial E}{\partial T}\right)_X.$$
 (D.11)

This means that the specific heat at constant variable X is the variation of the energy respect to the temperature while keeping X constant. The specific heat at constant charge and chemical potential read

$$C_Q = \left(\frac{\partial E}{\partial T}\right)_Q = T \left(\frac{\partial S}{\partial T}\right)_Q = -T^2 \left(\frac{\partial^2 F}{\partial T^2}\right)_Q \tag{D.12}$$

$$C_{\mu} = \left(\frac{\partial E}{\partial T}\right)_{\mu} = T \left(\frac{\partial S}{\partial T}\right)_{\mu} = -T^2 \left(\frac{\partial^2 G}{\partial T^2}\right)_{\mu} \tag{D.13}$$

where in the last step we used the relations D.5, D.10. There are other response functions, such as the susceptibilities. In this thesis we are interested in the isothermal permitivity

$$\chi = \left(\frac{\partial Q}{\partial \mu}\right)_T.\tag{D.14}$$

D.4 Fluctuations

In the previous section we computed the response functions which signal the stability of the system 5.19. We can go further in trying to describte the underlying field theory and compute the fulctuations of the system. Within the canonical ensemble, the charge is constant (being the analog of the volume for hydrostatic systems), and the variance of the system is given by [28]

$$\langle \delta E^2 \rangle = T C_Q. \tag{D.15}$$

For the grand canonical ensemble, the charge and energy are free to vary, and the fluctuations are given by

$$\langle \delta E^2 \rangle = TC_Q + T\mu \left(\frac{\partial E}{\partial \mu}\right)_T$$
 (D.16)

$$\langle \delta Q^2 \rangle = T\chi \tag{D.17}$$

$$\langle \delta E \delta Q \rangle = T^2 \alpha_\mu + T \mu \chi, \qquad (D.18)$$

where $\alpha_{\mu} = \left(\frac{\partial Q}{\partial T}\right)_{\mu}$ is the thermal dilatation coefficient.

Bibliography

- Juan Maldacena. The large-n limit of superconformal field theories and supergravity. International journal of theoretical physics, 38(4):1113-1133, 1999.
- [2] David Mateos and Diego Trancanelli. Thermodynamics and instabilities of a strongly coupled anisotropic plasma. Journal of High Energy Physics, 2011(7):54, 2011.
- [3] Tatsuo Azeyanagi, Wei Li, and Tadashi Takayanagi. On string theory duals of lifshitz-like fixed points. Journal of High Energy Physics, 2009(06):084, 2009.
- [4] Dimitrios Giataganas, Umut Gürsoy, and Juan F Pedraza. Strongly coupled anisotropic gauge theories and holography. *Physical review letters*, 121(12):121601, 2018.
- [5] Martin Ammon and Johanna Erdmenger. Gauge/gravity duality: Foundations and applications. Cambridge University Press, 2015.
- [6] Ofer Aharony, Steven S Gubser, Juan Maldacena, Hirosi Ooguri, and Yaron Oz. Large n field theories, string theory and gravity. *Physics Reports*, 323(3-4):183–386, 2000.
- [7] Ralph Blumenhagen and Erik Plauschinn. Introduction to conformal field theory: with applications to string theory, volume 779. Springer Science & Business Media, 2009.
- [8] G Mack and Abdus Salam. Finite-component field representations of the conformal group. In Selected Papers Of Abdus Salam: (With Commentary), pages 255-283. World Scientific, 1994.
- [9] Jorge Casalderrey-Solana, Hong Liu, and David Mateos. *Gauge/string duality, hot QCD and heavy ion collisions.* Cambridge University Press, 2014.
- [10] Peter Breitenlohner and Daniel Z Freedman. Positive energy in anti-de sitter backgrounds and gauged extended supergravity. *Physics Letters B*, 115(3):197–201, 1982.
- [11] Karl Landsteiner, Eugenio Megías, and Francisco Pena-Benitez. Anomalous transport from kubo formulae. In Strongly Interacting Matter in Magnetic Fields, pages 433-468. Springer, 2013.

- [12] Umut Gursoy. Improved holographic qcd and the quark-gluon plasma. arXiv preprint arXiv:1612.00899, 2016.
- [13] Joseph Polchinski and Matthew J Strassler. Hard scattering and gauge/string duality. *Physical Review Letters*, 88(3):031601, 2002.
- [14] U Gürsoy and E Kiritsis. Exploring improved holographic theories for qcd: Part i. Journal of High Energy Physics, 2008(02):032, 2008.
- [15] U Gürsoy, E Kiritsis, and F Nitti. Exploring improved holographic theories for qcd: Part ii. Journal of High Energy Physics, 2008(02):019, 2008.
- [16] Sourav Sarkar, Helmut Satz, and Bikash Sinha. The physics of the quark-gluon plasma: introductory lectures, volume 785. Springer, 2009.
- [17] Matthew D Schwartz. Quantum field theory and the standard model. Cambridge University Press, 2014.
- [18] Marco Panero. Thermodynamics of the qcd plasma and the large-n limit. Physical review letters, 103(23):232001, 2009.
- [19] Daniel Z Freedman, Steven Scott Gubser, Krzysztof Pilch, and Nicholas P Warner. Renormalization group flows from holography-supersymmetry and a c-theorem. arXiv preprint hep-th/9904017, 1999.
- [20] Ralph Blumenhagen, Dieter Lüst, and Stefan Theisen. Basic concepts of string theory. Springer Science & Business Media, 2012.
- [21] Steven S Gubser and Abhinav Nellore. Mimicking the qcd equation of state with a dual black hole. *Physical Review D*, 78(8):086007, 2008.
- [22] Steven S Gubser, Abhinav Nellore, Silviu S Pufu, and Fabio D Rocha. Thermodynamics and bulk viscosity of approximate black hole duals to finite temperature quantum chromodynamics. *Physical review letters*, 101(13):131601, 2008.
- [23] Sean M Carroll and Jennie Traschen. Spacetime and geometry: An introduction to general relativity. *Physics Today*, 58(1):52, 2005.
- [24] Christos Charmousis, Blaise Gouteraux, Bom Soo Kim, Elias Kiritsis, and Rene Meyer. Effective holographic theories for low-temperature condensed matter systems. Journal of High Energy Physics, 2010(11):151, 2010.
- [25] Steven S Gubser. Curvature singularities: The good, the bad, and the naked. arXiv preprint hep-th/0002160, 2000.
- [26] Peter Koroteev and Maxim Libanov. On existence of self-tuning solutions in static braneworlds without singularities. *Journal of High Energy Physics*, 2008(02):104, 2008.
- [27] Mark Henningson and Kostas Skenderis. The holographic weyl anomaly. Journal of High Energy Physics, 1998(07):023, 1998.
- [28] Herbert B Callen. Thermodynamics and an introduction to thermostatistics, 1998.
- [29] Antón F Faedo, David Mateos, Christiana Pantelidou, and Javier Tarrío. Towards a holographic quark matter crystal. *Journal of High Energy Physics*, 2017(10):139, 2017.

[30] Andrew Chamblin, Roberto Emparan, Clifford V Johnson, and Robert C Myers. Holography, thermodynamics, and fluctuations of charged ads black holes. *Physical Review D*, 60(10):104026, 1999.