Mathematical Thinking: Short Circuits Between the Mathematicians' Work and Educational Theories

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Abstract

To better prepare students for society, mathematical thinking has become a key goal of mathematics education, but it is unknown how educational theories on mathematical thinking relate to the thinking and working of mathematicians. The aim of this study is to align the thinking and working of mathematical researchers and theories on mathematical thinking, in order to clarify mathematical thinking as it might be addressed in mathematics education. Ten in-depth semi-structured interviews were conducted with mathematicians from Utrecht University and analyzed based on themes from educational literature on mathematical thinking. As a result, the following aspects of mathematical thinking were identified as most important: posing the right problem before solving it, creativity, and reasoning with examples and visualizations rather than proof language. Furthermore, discussions of educational models on abstraction, problem solving and modelling revealed that problems are rarely solved in a linear way and that mathematicians only generalize when it has a purpose. It is concluded that mathematicians' thinking and working align to some extent with educational theories on mathematical thinking, but not entirely with one single theory. The results of this study give rise to ideas to improve the implementation of mathematical thinking in education: spend more time on thinking processes in the early phase of conceptual development and put more emphasis on mathematical ideas, rather than on formal notation.

Keywords: mathematical thinking, 21st century skills, problem solving, modelling, abstraction, curriculum reform.

Mathematical Thinking: Short Circuits Between the Mathematicians' Work and Educational Theories

What if mathematics education would lead students to 'think like a mathematician,' and to approach mathematical problems in a flexible and creative non-routine way? This possibly more authentic way of doing mathematics might make mathematics more meaningful for students. Firstly, because logical and analytical thinking and qualitative reasoning are crucial abilities for society (Devlin, 2012). Secondly, because thinking out-of-the-box is becoming more important, since companies want to be innovative (Devlin, 2012). That idea of 'thinking like a mathematician' is at the heart of *mathematical thinking*. It is visible in recent educational innovations, for example the American Common Core State Standards (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2012), and in new Dutch curricula (cTWO, 2012). The Dutch innovation will be discussed as a case. We suggest that the problems, and later the solutions, might not be limited to this case.

In 2015, mathematical thinking was placed at the heart of the Dutch mathematics curricula for the two pre-higher education streams, HAVO and VWO (cTWO, 2012). The reform programs put explicit emphasis on mathematical thinking activities. These activities can improve the problem-solving skills of Dutch students. There was reason to: Doorman et al. (2007) noticed that Dutch students are less persistent in solving mathematical tasks that involve many steps, when compared to students from other countries in the PISA test. Mathematical thinking activities are not easy to implement in regular classroom teaching. Therefore, example material was made, and activities were designed (e.g., Bor & Drijvers, 2015). What can withhold teachers in the implementation is that mathematical thinking remains theoretically vague to them: the ideas on mathematical thinking of cTWO (2012) are not concrete enough (Drijvers, 2015) and the theories on mathematical thinking are diverse and lack consensus. The presence of mathematical thinking in national tests has declined already (Drijvers, Kodde-Buitenhuis & Doorman, 2019). As a result of this, teachers (see e.g., Soto y Koelemeijer, 2018) start to wonder if mathematical thinking is really important in the current Dutch curricula.

Currently, implementations of mathematical thinking seem to be solely shaped by ideas from the field of mathematics education. In the meantime, mathematicians are conducting mathematical research in universities, solving mathematical problems, albeit on an entirely different level than students do in math classes. Unfortunately, it is yet unknown how the theories on how students can be taught to approach mathematical problems are related to how mathematical researchers think and how they solve problems:

> Unfortunately, those who are most professionally active rarely teach any undergraduate course related to their scholarly work as mathematicians. Mathematicians seldom teach what they think about—and rarely think deeply about what they teach. (NRC, 1989, p.40)

Professional mathematicians might think very differently about mathematical thinking than teachers, educators and curriculum designers do. For example, Seaman and Szydlik (2007) noticed a difference between the mathematical values and norms of preservice elementary teachers and those of the mathematical community. Sfard (1997) went even further by stating that math educators and mathematicians seem to be talking about entirely different subjects.

We assume that mathematicians think mathematically and that getting inspiration from their way of thinking might clarify the concept of mathematical thinking. Therefore, the research question is: to what extent do thinking processes and work of mathematical researchers align with literature on mathematical thinking in mathematics education?

To answer this question, the first step is to know which theories on mathematical thinking are the most relevant, prominent and interesting to match with mathematicians’ work. We note at this point that we view theory in a broad way: crucial aspects of mathematical thinking are considered theories as well, so theories are not limited to extensive models. The second step is to attempt to match the mathematicians’ activities to the educational theories. The focus will be on two sub-questions: (1) how do mathematicians describe their thinking and work, and (2) what are the mathematicians’ opinions on mathematics education theories on mathematical thinking?

We assume that mathematical thinking and work can be examined together. Firstly, because doing mathematics is mostly a mental activity. Therefore, thinking and doing already coincide because of the nature of the subject. Secondly, because it can be argued that for educational implications, a distinction between doing non-routine mathematics and higher-order thinking skills is not important enough to discuss in depth. A similar assumption is done with respect to other higher order thinking skills, related to mathematics, since some literature is not explicitly on mathematical thinking, but on related aspects such as mathematical proficiency (e.g., Kilpatrick, Swafford & Findell, 2001).

Educational Views on Mathematical Thinking

To get a view of literature on mathematical thinking, we used a database that was used for design research on mathematical thinking (see Bor & Drijvers, 2015). To find more articles, we used snowballing: the most prominent elements were further investigated based on the references used in the articles in the database. The used database is from 2015, so to prevent missing recent publication, we requested some experts in mathematics education to share recent publications. In the examined literature, papers mostly state possible aspects of mathematical thinking, and some zoom in on crucial aspects of mathematical thinking. One clear and usable definition of mathematical thinking was not found.

We now provide an overview of different mentioned aspects of mathematical thinking in literature. The National Research Council (NRC) stated six mathematical modes of thought, which we consider possible aspects of mathematical thinking: modeling, optimization, symbolism, inference, logical analysis and abstraction (The Mathematical Sciences Education Board and the Board on Mathematical Sciences, National Research Council, 1989). Dreyfus and Eisenberg (1996) mentioned seven facets of mathematical thinking: aesthetics, self-confidence, using analogies, structure, representations, visual reasoning and reverse thinking. Essential in their idea is flexible thinking. The Common Core State Standards initiative (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2012) mentioned eight mathematical practices: make sense of problems and persevere in solving them, reason abstractly and quantitatively, construct viable arguments and critique the reasoning of others, model with mathematics, use appropriate tools strategically, attend to precision, look for and make use of structure, look for and express regularity in repeated reasoning. Kilpatrick, Swafford & Findell (2001) stated five strands of mathematical proficiency that are important to learn to think mathematically:

> Conceptual understanding (comprehension of mathematical concepts, operations, and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently, and appropriately), strategic competence (ability to formulate, represent, and solve mathematical problems), adaptive reasoning (capacity for logical thought, reflection, explanation, and justification) and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.) (p.5)

The Dutch curriculum reform committee cTWO (2012) described mathematical thinking by stating six core aspects: modelling and using algebra (or algebraization), ordering and structuring, thinking analytically and problem solving, manipulating formulas, abstraction, logical reasoning and proving. Two aspects of modelling are visualizing and using schemata and representations. A part of problem solving is using heuristics. Drijvers (2015) argued that three of them are the main aspects: problem-solving, modelling and abstraction. For example, using algebra can be seen as supporting the other mentioned core aspects. Similar reasoning can also be applied to the other aspects of cTWO (2012) that were not mentioned by Drijvers (2015).

Some authors, instead of giving aspects of mathematical thinking, stated one idea that is crucial to mathematical thinking. Creativity is such an idea. Byers (2007) placed central emphasis on creativity in mathematics and mathematical thinking. We decide to use the idea of mathematical creativity that is suitable for education by Schroevers et al. (2019):

> Creating something new and meaningful by breaking away from established mindsets (...), [and also] the cognitive act of combining known concepts in an adequate, but for the pupil new way, thereby increasing or extending the pupil's (correct) understanding of mathematics. (p.324-325)

Devlin (2012) primarily focused his idea of mathematical thinking on solving non-routine problems: "the thinking skills that will allow you to solve novel problems (either practical, real-world problems or ones that arise in math or science) for which you don't know a standard procedure. In some cases, there may not be a standard procedure." (Devlin, 2012, p. 8). Lakoff & Núñez's (2000) idea of mathematical thinking focused on reasoning with metaphors for concepts, often from embodied experiences in real life.

Altogether, one single commonly shared definition of mathematical thinking seems to be absent and there is variety of possible aspects of mathematical thinking. Silver (2003, as cited in Selden & Selden, 2005) commented that "advanced mathematical thinking" tends to be defined as "whatever the author chooses it to mean." (Selden & Selden, 2005, p. 4) Each author's mentioned aspects do not conflict with each other, but combining all aspects will create an idea of mathematical thinking that is possibly too elaborate to be of any use. We wonder which of all these mentioned aspects can be aligned with the thinking processes and the work of mathematicians.

Educational Theories on Problem Solving

Related to the second sub-question, on mathematicians' ideas on didactical models, some of these models will be discussed. Because of the prominence in ideas on mathematical thinking of problem solving, that aspect was examined first. Due to its importance to the Dutch innovation (cTWO, 2012) of mathematical thinking, we also discuss models on modelling and abstraction.

A problem can be two things: a routine task that has to be done, or an interesting non-routine problem that has to be answered (e.g. Schoenfeld, 1992). The first type of problem is for example a list of tasks used as a vehicle for teaching. This type of problem is used to practice and test routines. The second type of problem is a perplexing or difficult problem. The second type of problem solving is the one of interest for mathematical thinking. For example, Tall (1991) mentioned only this type of problem solving: "problem solving is the activity that occurs when the individual (or individuals) concerned is (or are) faced with a problem situation for which the precise nature of the problem and its solution are not initially evident." (Tall, 1991, p.176) This type of problem solving is at the heart of mathematics and this should be very familiar for mathematicians, according to Schoenfeld (1992). It is this second type of problem solving that Devlin (2012) focused on in his idea of mathematical thinking.

Pólya (1962) described four phases of solving a problem: understanding the problem, thinking of an approach, applying the approach and monitoring it to verify if the target was reached. Later, other scholars refined or expanded Pólya's ideas or created new models that describe the problem-solving process. Roth (2012) concluded after analyzing problem-solving models that some are cyclic and some are linear, such as Pólya's (1962) model. Roth (2012) found that cyclic models are more useful, in a study on grade 5. Wilson, Fernandez and Hadaway (1993) stated that many American schoolbooks give linear models for problem solving. This has four problems: problem solving is depicted as linear, as a series of steps; it implies that solving problems is a procedure to be memorized and it leads to an emphasis on answer getting (Wilson, Fernandez & Hadaway, 1993). They propose the alternative model as displayed in Figure 1. Schoenfeld (1992) showed that mathematicians also go through steps of problem solving in a non-linear way. He identified six phases: reading, analyzing, exploring, planning, implementing and verifying. In print, the model can be perceived as linear after the planning phase, see Figure 2. In Schoenfeld's (1992) study, a student spent all time on reading and exploring, and the mathematician went forth and back between phases.

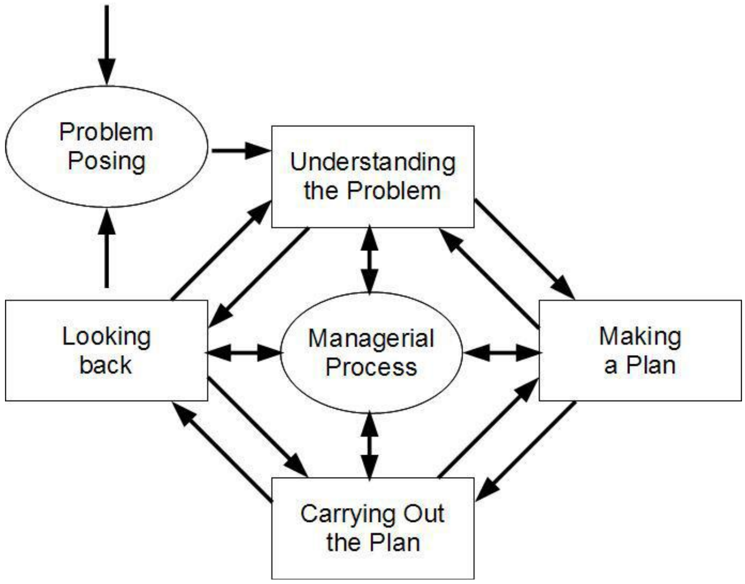

Figure 1: Wilson, Fernandez & Hadaway's (1993) model on problem solving

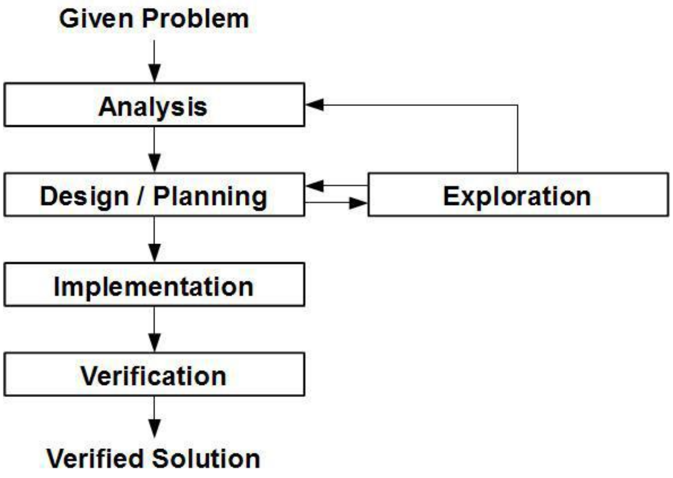

Figure 2: Schoenfeld's (1992) model on problem solving

Mason, Burton and Stacey (1982) emphasized reflecting on experiences as part of problem solving. This is when one tries to generalize a result, searches for essential assumptions and what can be ignored. They also describe "internal enemies" and internal monitors as important to problem solving. An internal enemy should be convinced about the steps and methods in a solving process. The internal monitor will allow you to think strategically.

Mamona-Downs and Downs (2005) wondered what the role is of proof-language in problem solving: "the current problem-solving agenda stresses argumentation at the level of naive mental reasoning, at the same time minimizing the need of the proof language." (Mamona-Downs & Downs, 2005, p.388). But there are indications that proof language is a factor in problem solving and the ability to use the ideas from problem-solving courses in more theoretical and conventional courses afterwards.

Educational Theories on Modelling and Abstraction

A second important aspect of mathematical thinking is *modelling*. Real-world problems can be solved by making a model of the problem in mathematical language. For this process, some theories are present. Spandaw and Zwaneveld (2012) described four phases: a situation will be conceptualized to describe only the relevant aspects. Then, that concept will be translated to mathematics, this is called mathematization. Then, the mathematical problem can be solved. This solution will be interpreted in the situation that the problem started with. In this phase, validation is important, because the mathematical solution can be different or non-applicable with respect to the original problem. One could zoom in on the phase of translating the real-world problem to a mathematical problem, this is called horizontal mathematization, and solving the mathematical problem is then called vertical mathematization (Treffers, 1987).

A third aspect of mathematical thinking is *abstraction*. Tall (2013) described this by conceptualizing either operations on objects or properties of objects. This idea is displayed in Figure 3. The distinction between operations and objects was also made by Sfard (1991). Mathematics can be seen as an iteration of such steps of abstraction. For example, the process of counting can be abstracted – or generalized – to the idea of natural numbers, then, numbers can be abstracted to variables. The distinction between objects and processes is related to the work of Skemp (1976), who noted that there is a major emphasis in classroom mathematics on instrumental understanding, where some properties – or relations in Skemp's work – are viewed as operations by students. We will view generalizing as a form of abstraction.

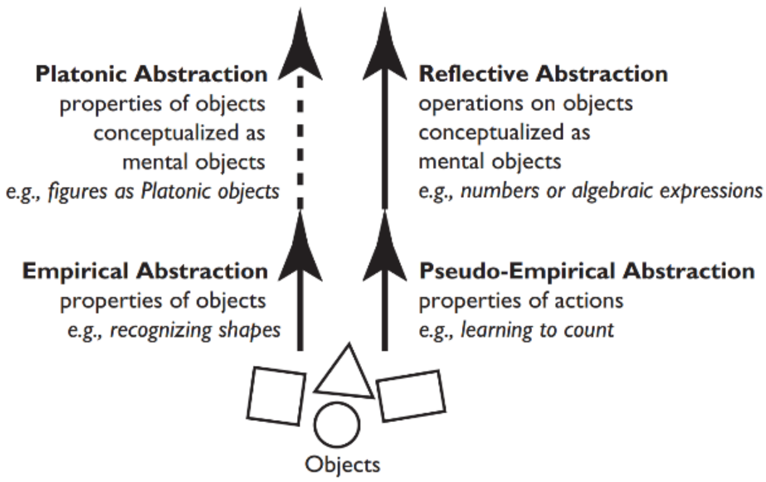

Figure 3: Tall (2013) on abstraction

In relation to viewing objects as conceptualized operations or properties of objects, it can be wondered how mathematicians view concepts. Do mathematicians always reason in symbols, or do they have other ways of reasoning? Burton (2001) described three styles of thinking about mathematics: visual (thinking in pictures), analytic (or thinking symbolically, formalistically); and conceptual (thinking in ideas, classifying).

Methods

To answer the research question, we set out an interview study to try to match the working and thinking of mathematical researchers to ideas from literature on mathematical thinking. Qualitative methods were used, because the way mathematicians think is not easily quantified and an in-depth view of it is needed. Three-staged semi-structured interviews of approximately one hour were conducted with 10 mathematical researchers. The interviews were semi-structured, to allow focusing on unpredicted new ideas from the mathematicians.

Instruments

Two instruments were designed. One is an interview scheme, that will be discussed first. The second instrument consists of a set of possible aspects of mathematical thinking, distilled from literature, to be used in the second stage of the interview.

The interviews consisted of three stages. The scheme can be found in Appendix A. In the first stage of the interview, the mathematicians were asked to describe their mathematical activities, starting with a question on what do they do when they do mathematical research. The focus was on what mathematics in their eyes is, what they do when doing mathematical research, and how they approach problems. This stage's purpose is to provide the mathematicians the room they need to sketch a broad idea on how they work and think. This relates to the first sub-question on how mathematicians describe their thinking and work.. The educational perspective offers a lens in the interpretation of the interviewees' answers.

In the interviews' second stage, a list of possible aspects of mathematical thinking was shown. Relating to the first sub-question, the mathematics education perspective now plays an explicit role, since all discussed aspects are from literature. The aspects form the second research instrument. The mathematicians were asked to divide 10 points over these aspects, based on what they consider the most important aspect of mathematical thinking. We asked the participants to think aloud in this stage. The distribution of points is meant as a primer for a conversation on what they miss on the list, reflecting what is essential and what is less relevant. The 'why' is also important here. This stage is meant to get an idea on what possible aspects are crucial in mathematical thinking and how the theories on mathematical thinking compare to mathematical thinking in the mathematicians' eyes.

The third stage zoomed in on educational models of problem solving, modelling and abstraction. This relates to the second sub-question, on the mathematicians' ideas of educational models. We zoomed in on these models, since these aspects were prominent in de ideas on mathematical thinking, for example in the Dutch innovation (cTWO, 2012). For problem solving, we used two models, the one by Schoenfeld (1992) and the one by Wilson, Fernandez and Hadaway (1993), respectively (see Figure 1). Central was the question whether the mathematicians recognized their own problem-solving behavior in (one of) these models and to what extent they consider problem solving to happen in a linear, cyclic or more chaotic way. For modelling, we zoomed in on Treffers (1987) idea of horizontal and vertical mathematization. Here, we wondered what mathematicians do to translate content from the real world to mathematics and back. For abstraction, we used the model by Tall (2013), see Figure 3 and a (Dutch) variant of the idea of Sfard (1991), see the scheme in Appendix A. We wondered how often and why a step of abstraction is done.

When appropriate, we asked questions that arose from literature. These questions could be asked during any of the three stages. They were indicated because literature did not answer them in a sufficient way, we wanted the mathematicians' opinions on these matters, and we considered possible answers to these questions to be inspiring for math education. The first question was: what is the balance between proof language and intuition? This question came from the work of Mamona-Downs & Downs (2005). The second was: how do mathematicians think about a concept, visual, analytic or conceptual (Burton, 2001)? And lastly, is mathematics essentially cooperative or not (based on the work of Burton, 1998)?

The second instrument is a set of possible aspects of mathematical thinking to be used in stage 2 of the interview. From the earlier discussed lists of possible aspects of mathematical thinking and some separate ideas on mathematical thinking, we created a new, long list of possible aspects. All ideas in italic font were added to the list. We started the list with the ideas phrased by the Dutch curriculum reform committee cTWO (2012): *modelling, ordering and structuring, problem solving, manipulating formulas, abstraction, logical reasoning and proving***.** We added aspects cTWO (2012) mentioned in the examples of their six activities*: using heuristics, visualizing, using schemata and representations.* In their idea of mathematical thinking, the NRC (1989) also stated modeling, symbolism, logical analysis, abstraction, and added *optimization* and *interference***,** that is reasoning from data, premises and graphs, in most cases incomplete and inconsistent sources. Dreyfus and Eisenberg (1996) mentioned the relation of the thinker to mathematics: *aesthetics* and *self-confidence***.** Their other aspects are similar to the ones already added, but *reverse thinking* was not mentioned by the other authors**.** We note that aesthetics of mathematics can be difficult to describe. However, Johnson and Steinberger (2019) showed that people, with and without higher education mathematical knowledge, can recognize aesthetic aspects of mathematical proofs, like they can with art or music, and this seems to be somewhat universal. We decided, although it can be difficult to describe, not to exclude aesthetics. We will view aesthetic aspects of mathematics as reasons to prefer one theorem, proof, method or even a field of mathematics over another. From the mathematical practices of the Common Core State Standards Initiative (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2012) we added, *attend to precision, critique the reasoning of others* and *looking for regularity in repeated reasoning*. We added *strategic thinking* because of Kilpatrick, Swafford & Findell's (2001) focus on strategic competence and adaptive reasoning. Byers (2007) central emphasis on *creativity* was also added to be a part of the list. *Knowing if one is right* was added, based on the idea of an internal enemy or monitor (Mason, Burton and Stacey, 1982). The final list is presented alongside the results in Table 2.

Participants

The participants of the interviews were mathematical researchers. The aim was to get a broad image of the working and thinking of mathematicians of the department of mathematics of Utrecht University, a Dutch research university. We interviewed a total of ten researchers, of which two female, working in theoretic mathematics or applied mathematics, from assistant professors to full professors. The average age was 50 (in a range of 29 to 65 with a standard deviation of 13 years).

But what does the image of one institute's mathematicians show about mathematicians in general? We asked the participants if they did notice a difference between mathematical thinking and doing between Utrecht University and other universities. Five of their responses referred to international differences, that however cannot be correlated with geographic position: "the differences are influenced by emotional distance instead of geographical distance." Some stated that they could better connect to a colleague working in the same field on the other side of the world than their direct neighbor. Three of the statements of the mathematicians varied from "mathematical thinking is universal" to "there is no typical approach to Utrecht; its research style fits an international context and is common for top institutes in the world." One mathematician noted that he saw differences in the approach of mathematics between countries, often correlated to traditions in education. Another mathematician stated that he saw differences between the way in which applied mathematics is conducted. Both of these saw differences in mathematical practice, but did not give a statement on differences in mathematical thinking styles. These responses indicate that the view of mathematical thinking of one institute also gives information on the ideas on mathematical thinking of other mathematicians.

Data and Analysis

The interviews were video-recorded. Video was used to also record drawings and gestures, especially in stage 2 of the interview. We wanted the mathematicians to speak as relaxed and freely as possible. The collected data consists of eight hours and twenty minutes of video data, split over ten participants. Additionally, the interviewer made notes in the interview scheme and wrote down the final point distribution of stage two. The final point distribution for each participant is treated as a separate set of data.

For the analysis of the first stage of the interview, we used a method similar to Bowen's (2006) grounded theory with sensitizing concepts, because the goal was to uncover what mathematical thinking is in the eyes of mathematicians. A coding scheme was designed, based on the earlier described educational theories on mathematical thinking. All words from stage two were used as a starting point for the codebook. We allowed adjustments to the codebook based on the interviews. New codes were allowed as well. The coding served two purposes: being able to find all important information on one specific topic and to make claims about what was mentioned to be important and what was not. Not many adjustments to the codes were necessary, although one code was added: posing the right problem. Some shifts in definitions of the codes were necessary to be able to code each segment in a unique way. To make fragments have one code, prioritizations were added. The final codebook can be found in Appendix B.

The results of the second interview stage is a distribution of points over the list of aspects. We counted how many mathematicians mentioned a specific aspect, because it is possible for one aspect to get a high ranking because one mathematician awarded a lot of points to that aspect. We wondered if there are major differences in the rankings of applied and theoretic mathematicians. Some aspects, for example modelling, could be seen as more related to applied mathematics. Therefore, we split the points over theoretic and applied mathematicians. For each of the three theories of the third stage of the interview, the mathematicians' ideas were synthesized. This was done by going through all fragments that were about the same model, to then look for patterns in there.

For data analysis, NVivo, software for qualitative data analysis, was used. Inter-rater reliability was calculated based on a sample of a total of one hour of video from stage one from three interviews, of a total of three hours for stage one. Coding had a different function for the first stage than the other two, so interrater reliability was most important for this stage, furthermore because this stage of the interview was the most open. We found the inter-rater reliability to be substantial ($\kappa = 0.700$) (Landis & Koch, 1977).

NVivo was used for stage two and stage three as well, primarily to retrieve everything said on one code. For the second stage, emphasis was on which aspects of mathematical thinking were seen as the most important ones in the eyes of the mathematicians. This data was used to support the conclusions of the first stage.

Results

We first address the themes that turned out to be important for mathematical thinking, because they were mentioned in stage one. Then, the second stage's results will be discussed. Although not always discussed in stage three, all ideas stated on problem solving, modelling and abstraction in any stage, will be discussed after that.

Important Themes from the first Stage of the Interview

Table 1 displays the frequency of each code, and how many mathematicians considered it related to mathematical thinking. Appendix B gives each code's used definition. All other codes were assigned less than 3 times. Because of their prominence in the interviews, the themes creativity, being cooperative, posing the right problem, conceptual thinking and the relation between a mathematical idea and a proof will be discussed. There is a connection between use of examples and conceptual thinking, because use of examples was often discussed as a way of reasoning. Therefore, the use of examples will be discussed as a part of conceptual thinking. Mentioned examples on how you can know when one is right were: use examples, write down the proof, have it peer reviewed or use another form of cooperation. Because of the connections with the other themes, knowing if one is right will not be discussed further.

Table 1
Code frequency of stage 1

Code	# References	# Unique interviews
Being creative	14	9
Being cooperative	11	10
Posing the right problem	10	7
Conceptual thinking	9	9
Discussing idea versus proof	9	9
Using examples	6	5
Knowing if one is right	5	5

Note. For stage 1, all other codes were used in a maximum of 3 instances in a maximum of 3 files.

First being *creative* will be discussed. Aspects related to creativity, or creativity itself, were mentioned frequently. At least eight mathematicians gave a typical example of mathematical practice, where they looked for regularities between methods or ideas in different fields of mathematics, to try to apply that idea or technique to a different field. The word *associating* was used often to describe this. Combining fields, adapting and transferring mathematical practice to other fields were seen by us as creative practice and are important parts of mathematical thinking in the participants' eyes.

Besides this associative creativity, more examples of creative practice were mentioned. One of those is looking for underlying structures. This is related to abstracting or generalizing. One mathematician described the explicit creation of new algorithms as typical for his mathematical practice.

A second important theme was *being cooperative*. We asked the participants if mathematics is something that is primarily done alone or together. Most mathematicians considered peers to be important. Some mathematicians worked alone primarily, but worked with peers as well. Another mathematician mentioned that "mathematics is in essence cooperative, since it only becomes mathematics when one convinces another person. Therefore, communication or cooperation should be an explicit part of mathematical thinking." The others' ideas were in between these two views, all accepting the idea that mathematics cannot be done primarily in solitude.

Reasons for cooperating varied. Many mathematicians mentioned that one could use the capacities of others, therefore making more progress. Others mentioned that the role of peers is verification of your ideas. Some of the applied mathematicians stated that (vertical) modelling steps can be done best in cooperation with an expert from the application. Then you know best which assumptions can be done. One mathematician stated that "cooperation is essential to abolish the fundamental reclusion of the mathematician. It is good to know that, after times of working on your research in loneliness, there are others on the world that care about your topic of research."

A third important theme, that emerged in seven interviews, was the necessity to pose the right problem. The participants illustrated that in their work, they often have to ask or modify the problems themselves, unlike a textbook that has preprinted questions. One went further, by saying that "a mathematical problem that starts with *solve* is not mathematical thinking. A part of mathematical thinking is knowing what to solve."

We will discuss when a problem is the right problem: when is a formulation of a problem good? Often, the participants mentioned two out of the three following criteria. The first reason for a problem to be good is that the problem should have a good balance between being do-able and being non-trivial. This means that a problem for which mathematicians have no clue on how to start it is not a good problem, but a problem that is solved by simply applying a theory from the same field in a direct way is also not good. The second reason is that the solution or method will provide new mathematical insights. This frequently happens when the problem is not isolated, but has connections with other mathematical problems. One mathematician expanded this by saying that it is essential for a problem to have relevance for applications. A third criterion is that the formulation of a problem should be simple, for example when it has a short formulation, or it is a natural variation on a known problem.

In the discussions of posing the right problem, three mathematicians mentioned aesthetic arguments to know when a problem is the right one. This indicates a connection between aesthetics of mathematics and posing the right problem. We note that there is a link between the three reasons for a problem to be good and aesthetics: the balance between applicability and generality is a matter of preference and experience. Connections to other fields were mentioned as an aesthetic aspect. When a formulation is short and nice is related to aesthetics as well.

As a side note, the theme of aesthetics was mentioned explicitly without reference to right problems, only two times, for example: "the solution should be nice." Another mathematician found aesthetic arguments important in solutions and proofs or in the formulations of problems. At least three mathematicians linked aesthetics to ordering knowledge: "when difficult things come together in an easy and nice way." Others to problem solving: they explained that when a difficult problem has a relatively simple or short solution, then that is the preferable solution. Another mathematician found this type of aesthetics too subjective to be useful. We conclude that aesthetics could have a link to some other aspects, especially the aspect posing the right problem.

Mathematical reasoning was discussed, for example visually and in formulas. Both styles were recognized by the mathematicians. One aspect that was not prompted by literature was brought up as well: using strong examples. When thinking about abstract concepts, the mathematicians often thought about something at a lower level of abstraction that should behave the same as the abstract theory, to reason about the concept. Although we write *using*, part of this is finding or creating a strong example.

The geometric or visual reasoning style was mentioned frequently. One mathematician mentioned that images help him to grasp mathematical content, for example, a system of equations as geometric conditions in the plane. Another stated that he could "only reason about something when it is concrete enough. For example, when a generic example is visualized. But it can also be a description in words, not necessarily using formulas." One mathematician visualized matching problems in graph theory in a dynamic way; he visualized the nodes as people pointing to other people. Only one mathematician recognized the type of reasoning that primarily uses symbols as a style, but he also used the other styles. Some mathematicians thought that only using symbols to reason does not provide enough insight: "It is only mathematics when the symbols come to life." One mathematician explained he primarily reasoned in a visual way, specifically when working with sets. This is because that was showed to him this way in his education. He therefore suggests that the primary way of reasoning can be dictated by education.

The last important theme that will be discussed is the relationship between intuitions or ideas and formal proofs in mathematical language. None of the participants considered the exact proof a more important part of mathematical thinking than the idea. While all found intuitive or visual reasoning an important part of mathematical thinking, ideas on its relationship with the exact proof varied. They agreed that the proof comes after the idea. It should be noted at this point that this does not at all mean that they considered routine, precise language or the craft of writing down a proof, not to be part of mathematical thinking. These are also crucial for doing mathematics.

The mathematicians gave some concrete ideas on the relation between proof and idea. Some stated that writing down your proof is an important way to conform to the rules of the field. Some expanded this, by adding that sometimes, writing down the proof leads to notice small overseen mistakes: “a proof is like a diary of ideas, like a musician who presses ‘record’ after a creative jam, sometimes leading to the insight that there was a minor mistake.” One mathematician noted that it is important to find precise language for intuition, to write down your ideas. Unique in his answer was the idea that the other way around needs emphasis as well: “one should be able to interpret a proof and give meaning to its symbols.” He noted that a precise account of an intuitive idea is actually a proof: one can write down a proof without mathematical symbols. “These are actually two sides of the same coin.”

Results from the Second Interview Stage

The points were divided as shown in Table 2. One mathematician wanted six aspects to have one point each. The others all used the full ten points. Almost all mathematicians gave two points to one to three categories, and none used more than two for a category. This suggests the recognizability of many different aspects.

Table 2
Point distribution stage 2

Category	Theoretic	Applied	Total points	Unique interviewees
Creativity	7	7	14	9
Visualising	6	4	10	8
Problem solving	5	3	8	7
Abstracting	4	4	8	8
Aesthetics	6	2	8	8
Proving	4	3	7	7
Using schemata and representations	4	2	6	5
Using heuristics	3	2	5	5
Regularities in repeated reasoning	3	2	5	3
Modelling	1	3	4	3
Logical reasoning	1	3	4	3
Manipulating formulas	3	0	3	2
Reverse thinking	3	0	3	2
Attend to precision	1	2	3	3
Ordening	1	1	2	2
Optimization	0	2	2	2
Interference	2	0	2	1
Self-confidence	2	0	2	2
Critique the reasoning of others	0	0	0	0
Strategic thinking	0	0	0	0
Internal enemy or monitor	0	0	0	0

Note. The right column *unique interviewees* displays the amount of mathematicians that awarded points to that category.

Notable are the similarities in the ordering of the aspects between theoretic and applied mathematicians. Some interesting remarks are that creativity and visualizing got two points from many mathematicians, and one or two points from respectively all but one and all but two mathematicians. Up until and including heuristics all categories got points from at least five participants, and regularity in repeated reasoning and further from only three.

Some aspects were often discarded, because they were considered part of another aspect. Reverse thinking for example, was often discarded because it could be an example of a heuristic. Logical reasoning, manipulating formulas and proving were often found to be closely related and then only one was awarded points. Another reason for which aspects were often discarded was that some skills are also important in fields other than mathematical thinking, for example attend to precision.

Results from the Third Interview Stage

In stage 3, models that are about respectively problem solving, modelling and abstraction were discussed. We will discuss the mathematicians' ideas on these models.

The models on *problem solving* were discussed with all participants. Two models were discussed, one of Schoenfeld (1992) and one of Wilson, Fernandez and Hadaway (1993). The main question is to what extent problem solving goes in a linear way. That idea was rejected by all mathematicians, even Schoenfeld's model was too linear. Most mathematicians recognized a more chaotic model for problem solving. They illustrated this by saying that they often have to go back to another stage, or find out that a chosen approach will not work. While one could initially think that a specific approach would work, that could turn out to be untrue during the execution of such a strategy.

Despite the recognizability of the non-linear model, its usefulness was questioned: firstly, because the model would not guide you how to approach a problem. The second reason is a crucial aspect in the model: central in the model of Wilson, Fernandez and Hadaway (1993) is *managerial processes*. The mathematicians supported the idea that one should not pointlessly go from one phase to the next, but the mathematicians could not clarify what managerial process is and how it works. Therefore, it is understandable why schoolbooks would prefer to present a linear model for problem solving. Some mathematicians stated that ideally, they would try the linear approach, while knowing that sometimes you have to go back to an earlier phase.

Two further critical comments were made. Both models missed the idea that the problems that the interviewed mathematicians worked on, often have to be divided in partial problems. These partial problems can sometimes be solved in a linear way. Furthermore, one mathematician questioned the emphasis on problem solving. All models displayed a focus on finding the solution of a problem. The mathematician stated that problem solving should not be focused on finding solutions of problems, but on the process, the ideas. “Problem solving is a party, and finding the solution to a problem is the exit of the party.”

Ideas from the teaching of mathematics on *modelling* were discussed with six mathematicians, with emphasis on the model of Treffers (1987). This model splits horizontal mathematization and vertical mathematization: modelling the world to mathematics and back, respectively modelling inside mathematics. The mathematicians recognized these two types of modelling. Some of them noted that the model works differently for them than presented in Appendix A: when solving a problem from real life, one should not forget that real life problem. This means that steps of horizontal mathematization are carried out more often than at the beginning and at the end of the procedure, and that a step of vertical mathematization will only be done when it has a meaning for the application. For example, generalizing equations to all real numbers, when the application will only have positive parameters, is not necessary.

The answers on the question whether translating the world to mathematicians is part of mathematical thinking or mathematics were divided. Even some of the applied mathematicians considered this to not be a part of mathematics. At the same time, some working in theoretic mathematics did consider horizontal mathematization part of mathematics.

One participant doubted the difference between horizontal and vertical mathematization. He noted that our perception of reality, displayed in numbers, is already a mathematical model. An applied mathematician doubted the difference between vertical mathematization and abstracting: "A step in modelling is an abstraction."

Models on abstraction were discussed with all ten mathematicians. They recognized that abstraction is an important part of mathematics, but did not have many remarks about the different types of abstractions Tall (2013) mentioned.

The model inspired by Sfard (1991), where mathematical concepts are formed by steps of reification, was found interesting by the mathematicians. "All concepts are rooted in experience," one applied mathematician stated. The mathematicians noted however that they rarely think this way about mathematical knowledge. The role of this 'history' of a concept was discussed with some of the participants. Another mathematician explained that mathematicians are good in – sometimes temporarily – forgetting the history and abstraction steps made to arrive at a specific concept or level of abstraction, only thinking about the current concept. Then, it is a skill to know when the roots of a concepts are useful to use and when not.

The mathematicians rejected the idea that one should try to keep abstracting. "Abstracction should not be a goal per se," one stated. Abstractions should only be made when they make sense and offer new insights. Some mathematicians stated that they stop trying to generalize when they do not understand the structure sufficiently. Abstractions should give some new insight, a powerful new theory or useful ideas.

Conclusion and Discussion

Our research question is to what extent do thinking processes and work of mathematical researchers align with literature on mathematical thinking in mathematics education? The sub-questions are (1) how do mathematicians describe their thinking and work, and (2) what are the mathematicians' opinions on mathematics education theories on mathematical thinking? The first sub-question relates to the first two stages of the interview. The second sub-question relates to the third stage of the interview. To answer the main question, we first look at the sub-questions.

With respect to the first sub-question, on the mathematicians' work and thinking processes, mathematicians mentioned creativity and posing the right question as important to mathematical thinking. They discussed the role of strong examples and visual reasoning styles, cooperation and the relation between a proof and the intuition behind it. When prompted by the possible aspects from literature, they mentioned similar aspects and added aesthetics and abstraction.

The second sub-question is about the mathematicians' opinions on the didactical models. For the models of stage three of the interview, the practical applicability was doubted, or information was missing in the mathematicians' eyes. Problem-solving models that are linear, such as Schoenfeld's (1992), turned out to be somewhat useful, because they describe an ideal situation. However, this theoretic ideal situation is seldom how solving an actual problem goes for our participants. The models described in the literature lack an idea on dividing a problem in partial problems. Treffers' (1987) distinction between two types of mathematization was questionable, and it was even wondered if mathematization and abstraction are different. Tall's (2013) and Sfard's (1991) models on abstraction were found interesting, but the practical applicability of these models was considered doubtful. The participants noted that abstractions should serve a purpose and not be a goal on itself.

What does this imply in relation to the main research question? The mathematicians' views aligned with *parts* of the total examined literature, but not with one source alone. Aspects that were mentioned in literature and indeed turned out to be important for mathematical thinking were creativity, visualizing, cooperation, the intuition behind the proof, aesthetics of mathematics and abstraction. Other aspects were recognized by some participants. Ordering, optimizing, interference, self-confidence, critique the reasoning of others, strategic thinking and internal enemy or monitor were recognized by few to none. As noticeable in Table 2, some aspects from literature are largely recognized, some were rarely recognized. Posing a good problem was of major importance, and not found in literature. The importance of strong examples turned out to be bigger than thought. The didactical models were seen as somewhat useful, but have limitations.

Before discussing these conclusions, it is important to note some limitations that could inspire possible further research. Firstly, it should be noted that this study is done with participants from one mathematical institute. The participants considered Utrecht University's mathematics department's work and thinking styles to be similar to those of other mathematics research groups, but to make claims about mathematicians in general, one should increase the number of participants and think about generalizability in a broader way. For example, in the answers of the mathematicians on the questions about universality of mathematical thinking, they compared it mostly to Western mathematical institutes.

Some limitations have to do with the use of literature. One limitation is that we cannot claim completeness of the body of literature that was examined. More systematic research is needed to lay claim on the relation between *the* literature and thinking of mathematicians. Another important limitation is that some of the conclusions were possibly prompted by some questions in the interview form. This is specifically true for some of the first stage's results: cooperation, idea versus proof and conceptual thinking were in the interview scheme (Appendix A) as topics to be discussed. Although they were motivated by literature and we and the mathematicians did find these very important, these results might not have become as important as they were now, if they would not have been in the interview format.

More research could be done in the relation of doing and thinking. We wonder what the relation is between what the mathematicians said they do and how they actually think. For this research, we assumed that those are similar. But, for example, we wonder how cooperation is related to thinking. It did turn out to be an important aspect of authentic mathematical practice, but it is questionable if this is (mathematical) thinking. Further research and discussion could clarify this.

Mathematical thinking is also done at other places than mathematical research institutes. Other scientists, for example physicist or computer scientist, also use mathematical thinking. In companies, for example in finance, mathematical reasoning also plays a role. The mathematical thinking in other places of society is something not taken into consideration. Future research can be conducted on mathematical thinking outside of mathematical research institutes.

At this point, we note that we barely discussed the uniqueness of mathematical thinking and its relation to other higher order thinking skills. It can be wondered how mathematical thinking is related to 21st century skills, scientific literacy, statistical literacy, computational thinking. Questions such as what is the difference between these higher order thinking skills, what is unique to mathematical thinking and which valuable assets of mathematical thinking should be integrated in other subjects, can be asked. Opportunities lie in connecting the knowledge from these current goals for education.

Let us now discuss what we, reflecting on these findings, consider typical mathematical thinking. We believe it all starts with having a goal: a specific problem, either from the real world or mathematics itself. One should know why this problem is interesting and allow adjustment of the problem. Solving the problem often comes from creatively using ideas from other fields of mathematics or methods from similar problems. The process of problem solving can be supported by making visualizations or using typical examples of the structure that is examined. In some steps, cooperation can speed up the process significantly. The idea has to be written down in proof language, sometimes to get new ideas on why the idea of the proof was wrong, or to see possible new or broader theories or problems. While this optimal scenario seems linear, the real process from creating the problem to solving it, will rarely be linear and one often has to go back to another phase.

Another point of discussion is what to do with the didactical models. They were not discarded, or deemed wrong. Often, aspects of them were recognized, or highlighted interesting patterns in what the mathematicians had always done. Some were called incomplete, some were called impractical. Based on this, fortunately, we can state that the discussed didactical models are not *wrong* or in contrast with what actual mathematicians do. The mathematicians' comments can inspire improvements of the models, or their use in class. Primary examples of this are to stress the non-linearity of real problem solving, not forgetting the application when modeling and to view abstractions as a way, not a goal.

To conclude the discussion, we discuss the connection between creativity and routine for mathematical thinking. It seems that the latter did not receive any points in stage two of the interview: aspects that can be seen as connected to algebraic skills such as manipulating formulas, attend to precision and logical reasoning were not mentioned frequently. But when one puts these points together, it is possible to say that algebraic skills is somewhat important. Furthermore, proving was given points by a majority of the participants in stage two. In stage one, the participants discussed that the exact part of a proof, where one uses proof language, comes after the creative idea of the proof, sometimes as just a verification to conform to the rules of the field, but sometimes to shine a light on why the idea might not have been sufficiently precise. Many mathematicians discussed that they used examples and visualizations instead of precise reasoning. We conclude that for mathematical thinking, creativity and flexibility are important, but routine skills – primarily algebraic skills - are also needed. However, mathematical thinking does not only focus on being good and flexible with symbol language, but also with visualizations and examples.

Before mentioning ideas on how the findings of this study can inspire mathematics teaching practice, we want to note that these ideas possibly need design research to be properly implemented. Furthermore, a discussion on what is feasible and desirable in education is required to translate this research's ideas to educational practice: not everything a mathematician does is necessary to be expected of students.

A first group of opportunities for the teaching of mathematics has to do with aesthetics and posing the right problem. One opportunity related to *posing the right problem* for classroom mathematics is besides solving given problems, thinking about the formulations of problems. An example is thinking about what can be changed in conditions of problems or theorems, or even thinking about problem statements yourself. Furthermore, some theorems could be formulated by students themselves, based on patterns. Lastly, the aesthetical aspect of mathematical thinking can also be emphasized by challenging students to think about how parts of mathematics are related to each other and how different representations of a mathematical concept are related. We note that the current Dutch implementation seems to focus on mathematical thinking in solving assignments, mostly at the end of a chapter after a concept has been discussed. Based on this study and the mentioned ideas, we note that mathematical thinking can also be done in the introductory phase of a concept. These ideas are also related to creativity, because the mathematical idea is the creative part of mathematics.

Another opportunity focusses on the findings on the importance of mathematical ideas in proofs. An idea is to focus on ideas or heuristics, and not always on proving something precisely. Textbooks sometimes give precise proofs for theorems. Because discussing this precise proofs can be time-consuming, teachers probably skip these from time to time. As an alternative, a sketch of the proof, the crucial intuition or step in the proof, can be mentioned in less time. Something similar is possible for assignments in textbooks, where currently there seems to be a focus on precise proof, calculation, but not on sketches or ideas. More emphasis could be put on the mathematical idea in assignments. Activities that focus on mathematical thinking are possibly even better when they are done cooperatively, because working together can make difficult activities less frustrating when students get stuck on them and ideas can be discussed together.

What to do with the mostly linear models in textbooks that tell you how to solve problems? They describe an ideal situation, but do not teach students that sometimes the solving process is less linear. This remark, or examples on when and why you have to go back to earlier phase in steps on solving a specific model, should find its place to textbooks or the classroom. Activities that support this idea, are assignments that do not start at the first step of the model of solving that type of assignment: something has to be done before step one is possible, or one assignment starts at step two. In this way, the results of this study may inform teaching practice, even if this requires further research and implementation.

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Appendix A: Short version of the interview scheme

[Introduction: context is the Dutch curriculum reform, aim is to make mathematical thinking more clear. Primary interest is on the participant's mathematical thinking]

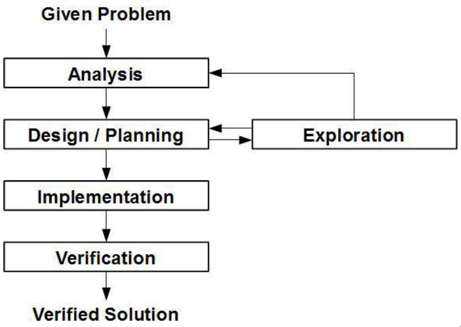
1.
You are mathematical researcher. What do you do when doing mathematical research, when doing 'real mathematics'?
What is the core of mathematical thinking processes needed for that? What are crucial processes and thinking steps?
What is crucial in getting unstuck on a problem?
What is mathematical thinking? What is a mathematical thinking activity?

Ask follow up questions on (also possible in **2** or **3)**

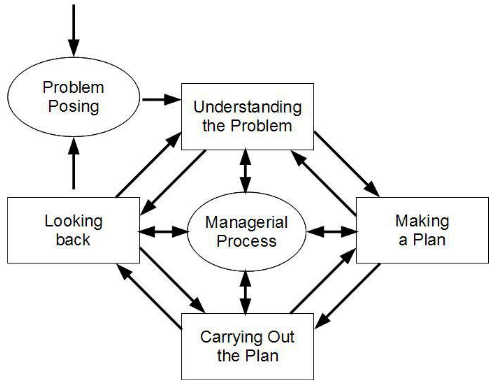
What is the role of proof language versus intuition?
How do you reason about a concept?
Do you do mathematics together or alone?
How can you verify your steps?

2.
Rank the following words, based on if they describe *your* mathematical thinking?
Note: each time, the second word is the actual Dutch word used.
Modelling - Modelleren
ordering and structuring - Ordenen
problem solving - Probleemoplossen
manipulating formulas – Formules manipuleren
abstraction - Abstraheren
logical reasoning – Logisch redeneren
proving - Bewijzen
using heuristics – Heuristieken gebruiken
visualizing - Visualiseren
using schemata and representations – Schema's en representaties gebruiken
optimization - Optimaliseren
interference (incomplete sources) – Interfereren (werken met incomplete bronnen)
Aesthetics – Schoonheid van wiskunde
self confidence - zelfvertrouwen
Reverse thinking – terugredeneren / omkeerbaarheid van denken
attend to precision – precies werken
critique the reasoning of others – het werk van anderen bekritiseren
looking for regularity in repeated reasoning – overeenkomsten in redeneringen of methoden zoeken
strategic thinking – strategisch denken
creativity – creativiteit
Internal enemy/monitor – Weten wanneer je gelijk hebt

3.1 Problem Solving



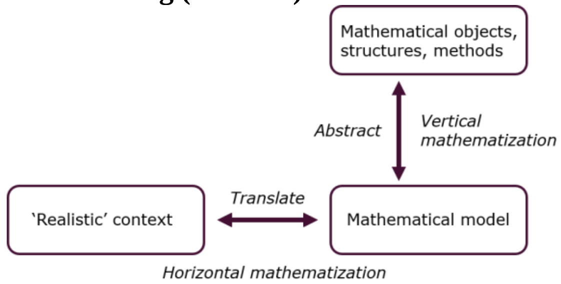
Schoenfeld, 1992



Wilson et al, 1993

What do you do when you get a problem?
[Show the models]
Do you recognize this? How does it work?
Which of the two models is the best / most useful?
In Wilson's model, what are managerial processes?

3.2 Modelling (Treffers)



Do you recognize this?
How does this translation from world to mathematics work?

3.3 Abstraction

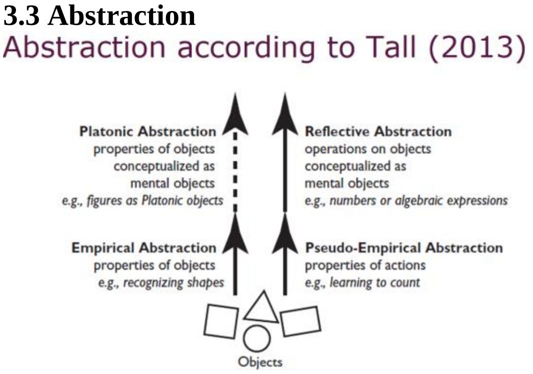
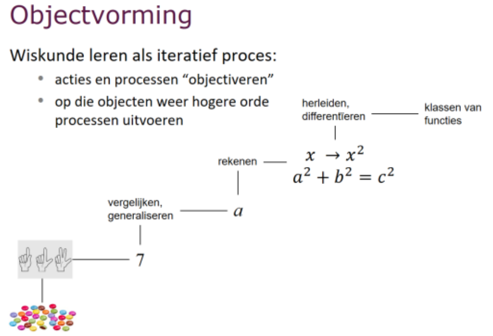


Figure 1.2. Piagetian and Platonic abstraction.



(In Dutch: Math as iterative process, make objects from actions and processes. Counting smarties -> numbers -> variables -> ….)
Do you recognize these steps?
How often do you do these steps during research? How often do you go a level of abstraction 'up' or 'down'?

End
Do you see similarities and differences between your mathematical thinking and doing and that of colleagues from other universities? How big are the differences? Is there a way of mathematical thinking and doing that is typically Utrecht?
[Closing, thanks]

Appendix B: Codebook

Name	Description
Abstraction	Use this code on explicit mention of abstraction or generalization. Also on references to ‘zooming out’ or looking at similar cases together.
Aesthetics of mathematics	When aesthetics of mathematics is mentioned. Posing the right problem has priority over this code.
Attend to precision	When having to work precisely is discussed.
Being cooperative	Code when the question ‘do you do math together or alone?’ is answered, or when the role of peers is discussed.
Being creative	Code when creativity is mentioned, or on mention of applying methods from one field in another, or using links between fields, or designing new procedures.
Conceptual thinking	Code when one discusses a way or type of reasoning, i.e. ‘I reason visually.’ Note that ‘I make a picture’ is not enough (that is visualizing), this code is about what a concept *is* (or means) to someone.
Criticizing the work of others	This code is used when one discusses giving feedback to the work of others. For this code, ‘the other’ is essential.
Discussing idea versus proof	Code when the relationship between a (creative) idea and the exact mathematical proof is discussed.
Interference	When dealing with uncertainty is discussed.
Knowing if you are right	Code on mention of ‘internal monitor,’ ‘verification,’ ‘check’ and processes that have to do with this. Also code when the question ‘how do you know when you are right?’ is answered.
Logical reasoning	On explicit reference to logical reasoning, or reasoning by building steps. Note that this can be a way of ‘knowing if you are right,’ but then that code is used.
Looking for similarities in repeated reasoning	When discussing similarities in methods. Note: when one uses these links, it is considered *creativity*.
Manipulating formulas	Code on explicit reference to dealing with formulas.
Modelling	Code when making a mathematical model is discussed. Also going back from a conclusion to the real world is part of this code.
Optimizing	Code when optimizing (or: improving, trying to do it as good as possible) a method is discussed or when optimizing as a goal in a problem is discussed.

Name	Description
Ordening	Only code when relations between fields or concepts are discussed. This code is used only for thinking about this ordering, and there are two exceptions: taking together things is considered *abstraction*. Using links is considered *creativity*.
Posing the right problems	Code when criteria are mentioned on when a problem is worth it to work on.
Problem solving	Only code when the entire process of solving problems, or a question about problem solving is explicitly answered. Parts of problem solving are always coded under another category.
Proving	When 'proving' is mentioned explicitly, or the process of precisely writing down mathematical reasoning. Do not use this when 'idea versus proof' could also be coded.
Reverse thinking	Only code on explicit reference to working 'from the conclusion backwards.'
Self-confidence	When discussing trust in own capacities.
Strategical thinking	Code on mention of strategical thinking, or for example discussing the process of choosing between methods in a smart way.
Using examples	When (the role of) examples is discussed. Do not code as answer to 'conceptual thinking.'
Using heuristics	Code on explicit reference, or when a way of remembering or thinking about a method is used.
Using schemata and representations	When going from one representation to another in a flexible way is discussed. Also code when choosing the right language is coded. 'Conceptual thinking' has priority over this code.
Visualising	Code when ways of visualizing are discussed. Do not code as answer to 'conceptual thinking.'