The Rectilinear Crossing Number of a Graph

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Contents

Introduction

In this thesis we will treat the rectilinear crossing number, a specialization of the concept of the crossing number of a graph. In their paper “Bounds for Rectilinear Crossing Numbers” [1] published 1993, Daniel Bienstock and Nathiel Dean proved a number of interesting results. They characterise the behaviour of the rectilinear crossing number by the normal crossing number. We will showcase two of their results and undertake the effort to make the proofs of these results both more rigorous and readable. Both discussed results are presented in chapters 2 and 3, respectively.

If you have a basic introduction into graph theory including the crossing number, this will greatly ease your reading. For a more extended introduction on graph theory its advised to read the Chapters 1, 2, 3 and 10 of the book Graph Theory by Bondy and Murty . Especially the paragraphs 1 trough 4 of chapter 10, on the subject of planar graphs, are essential in understanding this thesis.

However, we concisely introduce all used theory. Throughout the thesis figures are added to visualize the proofs in the hope the thesis gains clarity for the reader.

Chapter 1

Basics of Graph Theory

A graph G is in its most basic form an set of *vertices* $V(G)$ that are connected by so called *edges* $e = \{v_1, v_2\}$. All these edges together form the *edge set* of G , $E(G)$. The graph is formally defined by only these two sets, i.e. $G = \{V(G), E(G)\}$. By the *edges* or *vertices* of a graph G we will mean its edge or vertex set.

A *subgraph* of a graph is a subset of its vertex and edge sets such that every edge in the edge subset is incident with two vertices in the vertex subset.

We will call a graph G a *subdivision* of another graph H when G can be obtained from H by repeatedly adding vertices in the middle of an edge. That is by replacing a path v_1ev_2 by $v_1e_1v_3e_2v_2$.

A graph is *simple* if every edge connects two distinct vertices (i.e. is not a loop) and every pair of vertices is connected by at most one edge. In this thesis we will only concern ourselves with simple graphs.

By a *path* we will mean a sequence of edges and vertices, e.g. $v_1e_1v_2e_2v_3e_3v_4$, since we're dealing with a simple graph this can be simplified without ambiguity to $v_1v_2v_3v_4$. If the begin and end vertices of such a path coincide, i.e. $v_1 = v_4$, the we will call such a path *closed*. A *cycle* is a closed path in which every vertex is visited at most one time, it hence contains no crossings.

Connectedness We call a graph *connected* if there is a path between every pair of vertices. A graph is 2-*connected* if this is still the case after the removal of one arbitrary vertex. Generally a graph is *k-connected* if the graph is still connected after we remove an arbitrary set of $k-1$ vertices.

A *vertex cutset*, or in this thesis also simply *cutset*[1] is a set S of vertices such that after their removal from G the remaining graph $G \backslash S$ is not connected. By a *k-cutset* we mean a cutset containing k vertices. A k-connected graph by definition has no $(k-1)$-cutsets.

[1]Please be aware that *cutset* can also refer to an edge cutset, which is the same concept but then for edges.

A *connected component* of a graph is an maximal but still connected subgraph of this graph.

In Figure 1.1 we see a connected graph G. This graph however is not 2-connected, removing u disconnects v from the rest of the graph. If we remove s and t we obtain a nonconnected graph $G\backslash\{s,t\}$ with 2 connected components, one on the left and the other on the right of the line $s-t$.

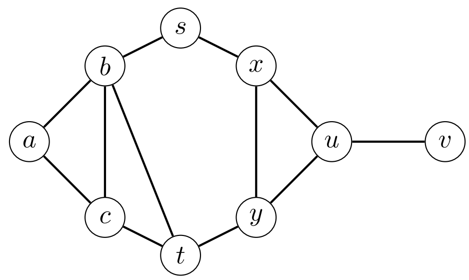


Figure 1.1: (A drawing of) a connected graph G

Drawing a graph We can represent graphs graphically in the plane (hence their name). If we do this we will say we have made a *drawing* of said graph. We do this by assigning a location to every vertex and subsequently connecting these with a curve if two vertices occur as endpoints for an edge of this graph. There is an endless amount of different drawings for the same graph. When confusion is not possible we will identify a graph with it's drawing.

The crossing number When two edges in a drawing occupy the same point in the plane (except if it's the endpoint of both vertices) we will say that these edges have a *crossing* in that drawing. The *crossing number* of a graph G, denoted by $\mathrm{cr}(G)$ is the minimal number of crossings for any drawing of that graph. In other words

$$\mathrm{cr}(G) = \min_{D \text{ drawing of } G} \{\text{number of crossings in } D\} .$$

If three edges cross in one point we will count this as three crossings, one crossing for every pair of edges that crosses. We will say that a graph is *maximal* with respect to the crossing number when it not possible to add an edge to this graph without increasing the crossing number.

We can now define the main subject of this thesis. The *rectilinear crossing number*, denoted by $\mathrm{cr}_1(G)$, is obtained by imposing the extra condition on D that all edges of G are drawn as straight lines. It is easy to see that $\mathrm{cr}_1(G)$ should be greater then $\mathrm{cr}(G)$, after all each straight-line drawing is also a normal drawing.

A rule on drawings Since for the purpose of the crossing number we are looking at drawings with a number of crossings that is as low as possible we can impose the following rule on the drawings we consider: We do not allow edges to cross twice.

This is because for such graphs there always is a drawing with two fewer crossings. This drawing is obtained by redrawing every pair of edges that crosses twice, we let the edges follow each others path from the first crossing to the second crossing.

1.1 Planar embeddings

We will call a drawing D of a graph G without any crossings a *planar embedding* or *planar drawing*. The crossing number of G in this case clearly equals 0, since there is a drawing of G without crossings. We will call such graphs G with $\mathrm{cr}(G) = 0$ *planar*.

We can define some interesting extra structure on planar embeddings. A embedding of such a planar graph G partitions the plane into a number of pathconnected open sets. We will call these sets the *faces* of G or D. Each face has a *boundary* consisting of all edges and vertices surrounding this face. We denote the boundary of a face F by ∂F. If a cycle is the boundary of a face we will call this a *facial cycle*. We define the *outer face* to be the one unbounded face, intuitively this is the face on the outside of the embedding.

Two planar embeddings will be considered equivalent when their face boundaries (as edge sets) are the same.

Convex embedding A convex embedding is a rectilinear, planar drawing where each face is bounded by a convex polygon whose corners are the vertices of that face.

Rotations In this paragraph we will assume that G is a 2-connected graph. By Theorem 10.7[2] in [2, p. 251] we know that such a graph has faces that have a cycle as boundary (unless G is K_1 or K_2, but we we will ignore these cases).

The *rotation* at a vertex v is the clockwise order in which other vertices occur as neighbours of v. For example the rotation of s in Figure 1.1 is *cby*, *byc* or *ycb*.

It is a result by Hefter [3] for the type of graphs that we treat in this paragraph that rotations at all the vertices determine the embedding, and v.v. that the embedding determines the rotation. Intuitively this result is easy to accept.

1.2 Whitney unique embedding theorem

As is said before, two embeddings of a graph G are equivalent if the face boundaries (as edge sets) are identical. We have the following theorem, as is stated in

[2]Being nonseprable is equivalent to being 2-connected

[2, p. 266].

Theorem 1 (Whitney Unique Embedding Theorem). *Every simple 3-connected planar graph has a unique plane embedding.*

We shall not proof this theorem here. Note also that by first using a *stereographic projection* (see [2, p. 247]) we can map this embedding to the sphere, by then rotating the sphere and applying the stereographic projection back to the plane we can let every face be the outer face.

Remark 1.1. *Every face in the can be the outer face of the above unique embedding.*

1.3 Tutte Spring Theorem

Tutte's Spring Theorem is a stronger result then Withney's unique embedding theorem. It probably is the most used theorem in this thesis. Originally due to Tutte [5], a shorter proof has now been given in [4]. We will reword (9.2) from [5].

Theorem 2 (Tutte Spring Theorem). *Let G be 3-connected, simple and planar. Let J be a facial cycle containing n nodes. Let Q be an n-sided convex polygon in the Euclidean plane. Then there is a unique barycentric representation of G on Q mapping the nodes of J on the vertices of Q in any arbitrary way preserving their cyclic order.*

An *barycentric representation* is a straight-line representation where every vertex (except those on the outer facial cycle) has an position that is the average of it's neighbours. Clearly such an representation is convex. After all, suppose one angle at a vertex v is larger then 180 degrees, then all the neighbours of v lie on one side of v and the position of v can never be their average.

1.4 Jordan Curve Theorem

The Jordan Curve theorem is primarily an topological theory about curves in the plane. It states that every *simple closed curve* divides the plane into an into the inside and outside of this curve. Here a *simple curve* is a curve that doesn't cross itself and a *closed curve* is one that's starts and ends at the same point. A simple closed curve is homeomorphic to the circle. By it's relation to this theorem we will sometimes call such a curve a *Jordan curve*.

We will now give the formal statement of the theory due to [2].

Theorem 3 (Jordan Curve Theorem). *Any simple closed curve C in the plane partitions the rest of the plane into two disjoint path-connected open sets.*

We will call the bounded open the interior of C and the other open the exterior of C. Since the interior and exterior are disjoint any path connecting them has at least a point in $\mathbb{R}^2 \setminus \operatorname{int}(C) \cup \operatorname{ext}(C) = C$.

Remark 1.2. *Every topological path connecting a point in* $\mathrm{int}(C)$ *with one in* $\mathrm{ext}(C)$ *must intersect the curve* C*.*

While a curve is in principle an topological construct and can lie anywhere in the plane, the most useful application of the theorem in graph theory is often given by considering a closed path as the closed curve. Suppose for example that we have a closed path $v_1v_2v_3$, this closed path is then also a closed curve and hence by the Jordan curve theorem we know that any other vertex v_i must either lie inside or outside the path $v_1v_2v_3$ (but not both).

The Jordan curve theorem is often used in disproving the planarity of graphs. An example of this can be seen in Theorem 10.2 of Graph Theory [2, p. 245].

Separating curves An use of the Jordan curve theorem is in the notion of separating curves. A *separating curve* between two mathematical objects (e.g. points, connected components) a and b is a simple closed curve (i.e. a Jordan curve) that has a in it's interior and b in it's exterior or vice versa. If there is a separating curve J between a and b any path connecting a and b must cross J by the above remark.

Chapter 2

Unboundedness of $\mathrm{cr}_1()$ w.r.t. $\mathrm{cr}()$

In this chapter we will show that for an arbitrary graph G with $\mathrm{cr}(G) \geq 4$ it's impossible to bound its rectilinear crossing number $\mathrm{cr}_1(G)$ by the ordinary crossing number $\mathrm{cr}(G)$. In other words, there is no constant $C \in \mathbf{R}^+$ such that for all graphs G the following holds

$$\mathrm{cr}_1(G) \leq C\,\mathrm{cr}(G).$$

Theorem 4 (Bienstock-Dean Theorem 1)**.** *For every $m > k \geq 4$ there exists a graph G with $\mathrm{cr}(G) = k$ but $\mathrm{cr}_1(G) \geq m$.*

To prove this statement we will first construct a graph G_m with $\mathrm{cr}(G_m) = 4$ and $\mathrm{cr}_1(G_m) \geq m$ and later generalize to the cases were $\mathrm{cr}(G) = k > 4$. We will see this generalization is not difficult.

2.1 The construction of G_m

In this section we will construct G_m, the graph used in this proof.

The graph J See also figure 2.1, we start this construction by taking the 8-cycle C_8 and numbering the vertices 1 trough 8 in a clockwise direction. We then add the edge $\{3, 7\}$, which we will call the *chord* of the cycle. The points that are the farthest away from the chord are called the *extremes*. We connect both the extremes with the neighbours of the other extreme (adding the edges $\{1, 4\}, \{1, 6\}, \{5, 2\}$ and $\{5, 7\}$). These edges we will refer to as *arcs* since they connect vertices over a significant distance and they recollect arcs in Figure 2.1. The graph we have now constructed will be referred to as J.

Lemma 1. *The (ordinary) crossing number of J is 2.*

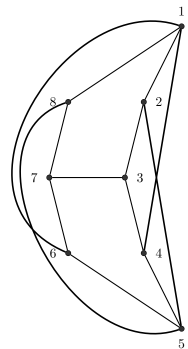


Figure 2.1: Graph J

Proof. There clearly is a drawing of J with only two crossings, one is given in Figure 2.1. Hence $\text{cr}(J) \leq 2$.

We will now proof this lemma by showing the crossing number can neither be 0 nor 1. We start with proving $\text{cr}(J) \neq 1$ this is the hardest case.

Suppose the crossing number of J is 1 then there must be a drawing D of J such that D has only one crossing. The removal of a single crossing edge e_c of D must yield an planar graph $J \backslash \{e_c\}$. The drawing D however has two crossing edges , and J must therefore have two edges such that their removal yields a planar graph. However, we will show that the chord $\{3, 7\}$ is the only edge with this property.[1] If we prove this fact we get a contradiction and we obtain $\text{cr}(J) \neq 1$.

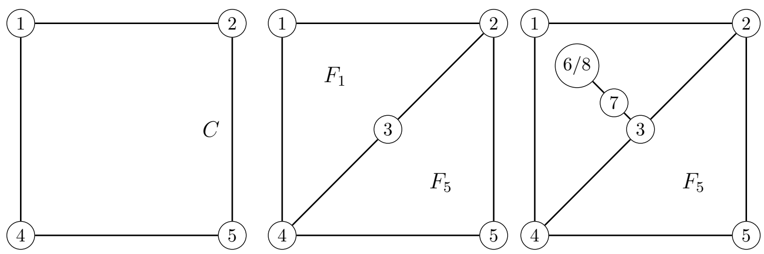
We will give a proof using the Jordan curve theorem that $J \backslash \{e_c\}$ is nonplanar (except, of course, for $e_c = \{3, 7\}$). Every case will follow a roughly similar setup. We will assume without loss of generality that the edge e_c that we are removing from J is one of $\{1, 8\}, \{8, 7\}, \{7, 6\}, \{6, 5\}$ (which we will call *cycle edges*) or $\{1, 6\}, \{5, 4\}$ (arcs). I.e. the missing edge is on the left side in the drawing of Figure 2.1. This can be done without loss of generality for the other edges we can switch the vertex numbers of 2 and 7, 3 and 7, and 4 and 6 and this will give us one of the cases we are about to threat.

In all cases we start by drawing the four-cycle given by $1-2-5-4$. We will call this cycle C, this cycle can be drawn as a closed simple curve. See Figure 2.2a. Subsequently we add the vertex 3 to the drawing. This vertex will, of course, lie on either the exterior or interior of C. We will assume without loss of generality that 3 is drawn on the interior of C. The exact same steps will lead to a contradiction in the case that 3 is in $\text{Ext}(C)$. This addition of the vertex 3 splits the interior of C into two faces F_1 and F_5, named after the unique vertex in their face boundaries. We now have the situation in Figure 2.2b.

When we now add 7 to the drawing it must be added in a face bordering 3

[1] It's easy to see $J \backslash \{3, 7\}$ is planar, you can draw the arcs incident with 1 in the interior of the cycle and the ones incident with 5 in the exterior.

(i.e. F_1 or F_5) to let the edge $\{3, 7\}$ be noncrossing. The choice that is made here is of no importance since there are paths from 7 to both 1 and 5 and one of these will offend the Jordan curve theorem. In our figures we will have made the choice that 7 is in F_1. In the case that the deleted edge e_c is $\{1, 8\}, \{8, 7\}$ or $\{5, 8\}$ we add the vertex 6 to our drawing and else (if $e_c \in \{\{6, 7\}, \{6, 5\}, \{1, 6\}\}$) we add 8. Whatever the choice was, either vertex must lie in the same face as 7, since they are connected to it by an edge. We now get the situation in 2.2c.



(a) (b) (c)

Figure 2.2: Various steps in the proof that $\text{cr}(J) = 2$

If we added 6 to the drawing in the previous step, the vertex 6 is adjacent to both 1 and 5. One of these edges will have a crossing by the Jordan curve theorem ($\{6, 5\}$ if $6 \in F_1$ and $\{6, 1\}$ if $6 \in F_5$). For example, in our drawing in Figure 2.2c the edge to $\{6, 5\}$ would cross. The same statements hold for 8 instead of 6 if we added 8 instead of 6 in the previous step. Hence $J \backslash e_c$ is nonplanar for every $e_c \neq \{3, 7\}$ therefore $\text{cr}(J) \neq 1$ as we presumed in the beginning of this proof.

The crossing number of J can't be zero, for then every subgraph of J should be planar. And we have just shown that, for example, $J \backslash \{1, 5\}$ is not so.

We hence conclude that $\text{cr}(J)$ must be 2.

□

The graph K We now create the graph K by glueing two copies of J at their extremes. That is, we take two edge-disjoint copies of J and let them meet at the extremes. We now completely color this graph by painting the cycles and the chords blue and the arcs red. See figure 2.3.

The graph $\mathbf{G_m}$ The final step in creating G_m is replacing all blue-coloured edges by an bundle of m pairwise internally disjoint paths. That is to say that every blue edge $e = \{u, v\}$ of K is replaced by an set $P(e)$ of m pairwise internally disjoint paths of length 2 with end-vertices u and v.

It is clear that $\text{cr}(G_m) \leq 4$, since graph K can be drawn with 4 crossings as is seen in Figure 2.3 and the bundle replacement doesn't add any crossings. Since $\text{cr}(J) = 2$ by Lemma 1 and G_m contains two edge disjoint copies of J we obtain that $\text{cr}(G_m) \geq 4$. By combining these inequalities we get $\text{cr}(G_m) = 4$.

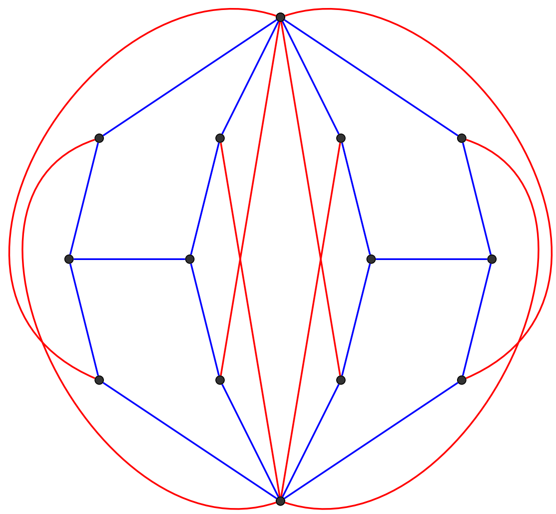


Figure 2.3: Graph K

2.2 Proof that $\mathrm{cr}_1(G_m) \geq m$

In this section we proof the following Lemma

Lemma 2. *The graph* G_m *has* $\mathrm{cr}_1(G_m) \geq m$

2.2.1 Proof outline

We will proof lemma 2 by contradiction. We will suppose there is a rectilinear drawing D of G_m with less then m crossings. We will then construct a drawing D' of K (i.e. one in which every blue edge isn't yet replaced by a bundle of edges) that is not necessary rectilinear. In this drawing blue edges won't cross and red edges will be straight. We will then proof a claim about the rotations at vertices 1 and 5 in the version of H drawn in D' whose 8-cycle is the boundary of the outer face. Subsequently we proof a claim on the positions of some vertices and finally deduce a contradiction. Leading us to the conclusion that the original drawing D must have been illegal.

Constructing $\mathbf{D'}$ Suppose there exists a drawing D of G_m with less then m crossings. Before we create the drawing D' let us remark that not all edges in a bundle $P(e)$ can be crossed by an edge not in the bundle. After all, if this were the case the drawing D would at least contain m crossings.

We then obtain D' by choosing from each bundle $P(e)$ a path $p(e)$ that is not externally crossed (i.e by an edge not in $P(e)$) and drawing it for the side e in K. We maintain the drawing of the red edges in D and use those in D'. Since D was a rectilinear drawing red edges will be drawn straight in D'.

From the way we constructed D' we can see that blue edges can't cross or, differently put, all crossings are between red edges. Hence the drawing of K_B, the blue subgraph of K, is without crossings in D'.

2.2.2 Configurations of the blue edges of K

Let us remark the following

Lemma 3. *There exist only two planar embeddings of the blue subgraph of K that admit no crossing. (We don't restrict ourselves to the rectilinear case)*

We remember from section 1.1 that the embedding of a graph is completely determined by the rotations at every vertex.

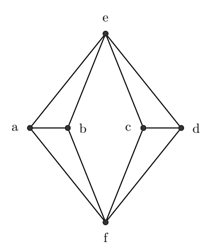


Figure 2.4: The graph D

Proof. To proof this we first consider that the blue subgraph of K is a subdivision of the graph D which is displayed in figure 2.4. The rotation at e determines the rotation at a, b, c and d by planarity. This in turn also determine the rotation in f.

We will now look at the different possible rotations at e. A rotation is about the order of occurrence of all, adjacent edges, hence we can without loss of generality suppose that a is the first vertex in the rotation at e. We can also without loss of generality assume that c comes before d in our rotation. After all suppose d is before c in the rotation at e in some drawing of D, then we can exchange the vertices c and d to get a drawing D'. The drawing D' is equivalent to D after we rename c with d and vice versa. The abstract structure of these drawings is the same.

We therefore only have to consider the following rotations at e: $(a, b, c, d), (a, c, b, d), (a, c, d, b)$. We will show the middle one is impossible, leaving only two different configurations. If you draw these, you will immediately see these are different. They are two embeddings can be seen in 2.5.

To see that the rotation (a, c, b, d) is impossible consider that $e - a - f - d$ in this case forms a closed curve with c in the interior and d in the exterior (by the rotation at e). By the Jordan curve theorem it's impossible to draw a crossing-free edge from c to d, which is required. Hence this rotation is impossible. □

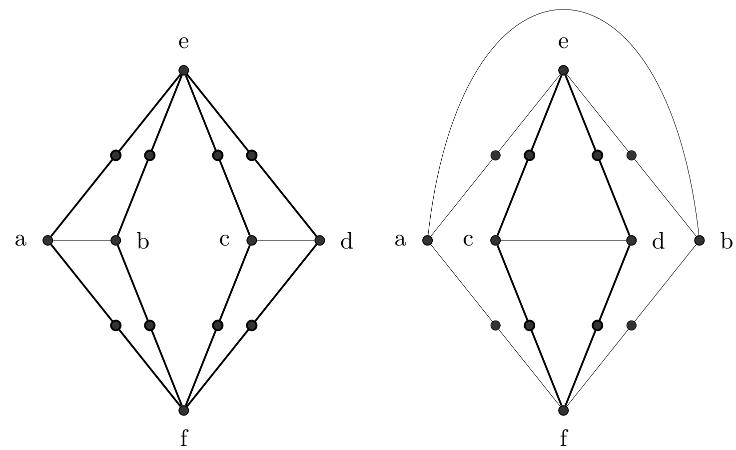


Figure 2.5: Configurations of K^B

We let J^B denote the blue subgraph of J.

Remark 2.1. *In both configurations is at least one copy of J^B configured in such a way that the eight-cycle C_8 is the boundary of J^B's outer face. In Figure 2.5 we've drawn the 8-cycles we're considering with a thicker line.*

We will denote the drawing of a instance of J which has the eight-cycle as boundary of the outer face by D_1 and the drawing of the other instance of J (which may or may not have the 8-cycle as boundary of it's outer face) by D_2.

At this point we will also assume the vertices of the eight-cycle in D_1 are numbered in a clockwise order. We can make this assumption because we can send every drawing of K with the eight-cycle of D_1 numbered in the counter-clockwise direction to one where the eight-cycle of D_1 is numbered clockwise. We do this by mirroring the plane D' is drawn on in the line $1 - 5$. We now obtain a drawing D'' of K with the eightcycle numbered clockwise. If this drawing is leads to a contradiction the original drawing D' also would lead to a contradiction, hence we can without loss of generality assume the vertices of the eightcycle are numbered clockwise.

2.2.3 Rotations at 1 and 5

Although G_m is nonplanar we can determine the rotations at 1 and 5. That is, we can prove an certain ordering in the occurrence of incident edges at these vertices.

Claim 2.1. *The clockwise ordering of edges of* D_1 *incident with 1 is* $(\{1,4\},\{1,2\},\{1,8\},\{1,6\})$. *While that of 5 is* $(\{5,8\},\{5,6\},\{5,4\},\{5,2\})$.

Intuitively this claim states that the red edges go outside the graph H. The proof of this claim is clarified by Figure 2.6.

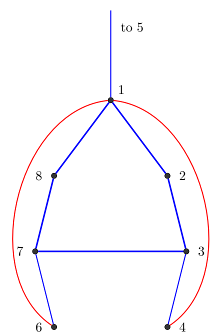


Figure 2.6: The situation in Claim 2.1

Proof. The edge $\{1,2\}$ precedes $\{1,8\}$ in a clockwise ordering of the eightcycle, which we just assumed for D_1. The edges $\{1,4\}$ and $\{1,6\}$ can't follow $\{1,2\}$ but precede $\{1,8\}$ by Jordan's theorem since they would then cross the closed curve $(1-2-3-7-8-1)$ (the bold path in Figure 2.6). This however is impossible since this curve is blue and therefore by construction crossing-free.

There also is a blue path in D_2 from 1 to 5, we can convince ourselves of this by looking at the embeddings of K_B in Figure 2.5. Both $\{1,4\}$ and $\{1,6\}$ must lie on a certain side of this path since they cannot cross it, the edge $\{1,6\}$ must precede this path while $\{1,4\}$ must follow it. Hence starting at the blue path and going clockwise we get the desired ordering $(\{1,4\},\{1,2\},\{1,8\},\{1,6\})$

The proof of the statement for 5 is analogous. This can be seen by renumbering 1 to 5, 2 to 6, 3 to 7, 7 to 3 and 8 to 4 in the above proof. □

2.2.4 The line L

We now draw a straight line L from 5 to 1. That is to say we let L go through 5 and 1 and we orient it from 5 to 1. By giving this orientation 'left of L' in the following claim is a sensible statement. We make the following claim about the positions of vertices 2,4,6 and 8.

Claim 2.2. *Vertices 2 and 4 are drawn strictly to the left of L while vertices 6 and 8 are drawn strictly to the right of it.*

Proof. The first thing to note is that the edges $\{1,4\}$ and $\{5,2\}$ cross at a point P. If we look at the blue cycle given by the path 1-2-3-4-5 and the blue path from 5 to 1 in D_2 , the other subgraph in G_m. Then we know that since only red edges can cross that $\{1,4\}$ and $\{5,2\}$ must lie completely in the inside of this cycle by the 'rotations' established in Claim 2.1,and hence must cross each other.

Since $\{1,4\}$ and $\{2,5\}$ are crossing straight segments we know that 2 will be drawn on the same side of L as 4. They can't be drawn on L since that would give a incidence of the edge $\{1,4\}$ and the vertex 2 and hence a illegal drawing. A drawing of this situation is given in Figure 2.7

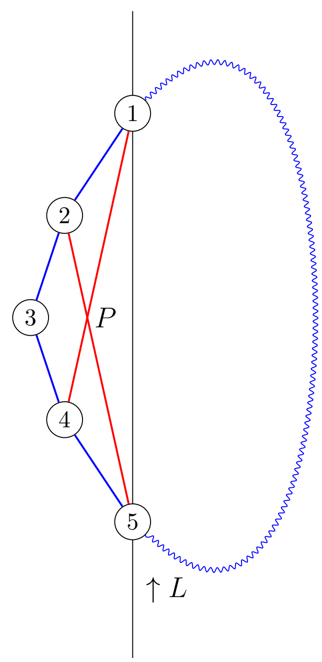


Figure 2.7: The way D must be drawn, with 2 and 4 left of L

We will now proof the Claim for vertices 2 and 4 by contradiction, let us assume they are drawn to the right of L. We now obtain a closed curve X that starts at 1 goes to P along $\{1,4\}$, then to 2 along $\{5,2\}$ and back to 1 by the edge $\{1,2\}$. By claim 2.1, the vertices 6 and 8 must lie in the interior of X. But now the blue path 6-5 will have a crossing with X (after all 5 is in the exterior of X). This gives us a contradiction, hence 2 and 4 must be drawn strictly to the left of L. In Figure 2.8 the curve X is drawn with a thicker line.

The rest of the statement follows by symmetry (This symmetry is given by

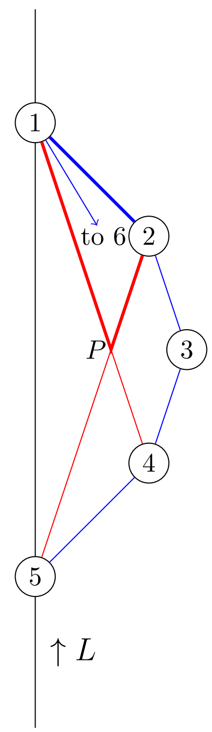


Figure 2.8: The way D can't be drawn

a renumbering in which we switch 8 and 2, 7 and 3, and 6 and 4). $\square$

We now have a contradiction with the numbering of the cycle being clockwise and this in principle concludes the proof. After all 2 is drawn left of 1 which is drawn left of 8.

If this is not yet convincing to the reader we can also finish the proof in what might be a more satisfying manner. We can denote the crossing point of $\{1, 6\}$ and $\{5, 8\}$ by P' (and recall that that of $\{1, 4\}$ and $\{5, 2\}$ is called P) and consider the quadrilateral Q given by segments of edges $(1 - P - 5 - P' - 1)$. The edge $\{3, 7\}$ is clearly in the exterior of Q and hence the exterior of Q is partitioned in a number of regions such that 1 and 5 lie in a different region, this for example by the path $(P - 2 - 3 - 7 - 8 - P')$.

We henceforth know that the drawing D_2 of the other instance of J must lie inside Q with the blue edges of course not crossing anything. There is after all a blue path from 1 to 5 in D_2. The 8-cycle must be drawn as the outer face otherwise the blue chord $\{3, 7\}$ of D_2 would cross Q.

But if we now repeat the steps we have done so far we get that D_2 has a quadrilateral inside of which must lie D_1. This is a contradiction and concludes the proof of Lemma 2.

2.3 Proving the theorem

We know from the previous sections that $\mathrm{cr}(G_m) = 4$ and $\mathrm{cr}_1(G_m) \geq m$. We can now proof Theorem 4.

To fulfil the requirement $\mathrm{cr}(G) = k$ for all $k \geq 4$, instead of only the case $k = 4$ we have currently proved. We simply add $k - 4$ fully disjoint copies of K_5 [2] to G_m. This completes the proof of Theorem 4.

[2] The crossing number of K_5 is 1. This proof is found in most basic books on graph theory, see for example [2, p. 245]

Chapter 3

Equality of crossing numbers

After the previous result, that characterises the behaviour $\mathrm{cr}_1(G)$ with respect to $\mathrm{cr}(G)$ if $\mathrm{cr}(G) \geq 4$ we will now look to smaller values of $\mathrm{cr}(G)$. Bienstock and Dean state in [1, Theorem 2] that the equality $\mathrm{cr}(G) = \mathrm{cr}_1(G)$ holds if $\mathrm{cr}(G) \leq 3$. Sadly they do not given the proof for the case $\mathrm{cr}(G) = 3$. We will prove the following result.

Theorem 5. *If* $\mathrm{cr}(G) \leq 2$ *then* $\mathrm{cr}_1(G) = \mathrm{cr}(G)$

3.1 Proof Outline

Central in our proof will be the correspondence between G, the graph under consideration, and G^D, the crossing-free version of this graph under a (minimal-crossing) drawing D. We formally define G^D as the graph that is obtained from G if all edges that cross in a drawing D, the *D-crossing edges*, are removed from it. We will constantly try to induce information about G, $\mathrm{cr}(G)$ and $\mathrm{cr}_1(G)$ from G^D. This will be done by making extensive use of case analysis. Most cases end with explicitly drawing G^D in a certain rectilinear manner that allows us to add the D-crossing edges back in without increasing the number of crossings.

3.1.1 Case structure

We have the following claims

Claim. *G^D is 2-connected*

Claim. *If G^D is 3-connected Theorem 5 holds.*

The interesting case is now when G^D is 2-connected but not 3-connected, in that case there must be at least one 2-cutset. In the following is s, t an arbitrary

2-cutset. We will proof several claims about this cutset. The final and most important one being.

Claim. *(a) Either there is a face F with s, t in its boundary that contains more then one crossing, or (b) there is an additional crossing face with $e = \{s, t\}$ being in the boundary of one of those faces.*

After that we show that the theorem holds in the second case by proving it's crossing faces are in a 3-connected part (a fragment at $\{s, t\}$) of the graph, and showing that hence the whole graph can be rectilinearly drawn. In the other case we analyse how the single crossing face must look in a topological manner and use a projection to create a rectilinear drawing.

3.2 Notions used in the proof

These notions are put here for two purposes; to be read now and after that to be used as a reference when you're reading the proof. The first remark will be a note on the style of our figures.

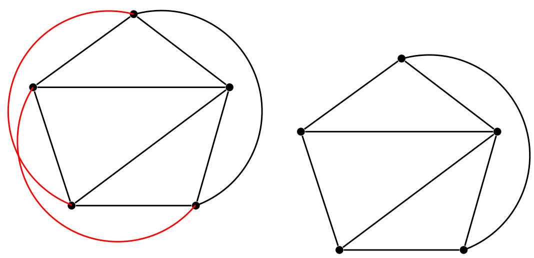
Note on figures During this proof we will make extensive use of figures. We will use some standard conventions that we will clarify below.

We shall with a *thick line* denote the edges that are important for the structure of our graph. This can for example be the boundary of a crossing face. Most of the times, if an graph has a low number of edges, all of them will be thick. The *thin edges* on the other hand will denote edges that must be drawn to make an example graph maximal, i.e. these edges are not important for the structure of the graph. Furthermore, we let *dashed lines* denote were the $D-$crossing edges of G are/were in a drawing. And if an area is *shaded* we mean it to be taken as a connected component. See Figure 3.9 as a prime example of this drawing style.

The crossing-free plane graph G^D Given an arbitrary graph G and minimal drawing D (i.e. a drawing with the minimal number of crossings). We then denote by G^D the *crossing-free plane graph* of G and D. This is the drawing D with all its crossing edges removed, in this thesis we will frequently identify this drawing with its graph. For example, when we say in Claim 3.1 that G^D is 2-connected we mean to say that the graph of which G^D is a drawing is 2-connected.

The faces of G^D in which a crossing occurred before the deletion of edges are called the *crossing faces* of G^D, the other faces will be known as *noncrossing faces.* We refer to the crossing edges of drawing D as the *D-crossing edges.* And the crossing edges in a particular face F of G^D are the (D, F)*-crossing edges.*

For example, take $G = K_5$ and the drawing D of K_5 as given in Figure3.1a. G^D is then the graph drawn in Figure 3.1b. In this case the outer face F in Figure 3.1b is a crossing face and the red edges in figure 3.1a are the (D, F)-crossing edges (and in this case also all the crossing edges).



(a) A drawing D of K_5 (b) The graph ${K_5}^D$

A fragment at $\{s, t\}$ Let G be a 2-connected graph and $\{s, t\}$ be an arbitrary cutset of this graph. The removal of $\{s, t\}$ from this graph gives us a graph $G \backslash \{s, t\}$ that has multiple connected components H_i. One such connected component together with the $\{s, t\}$ and the edges with one end in $\{s, t\}$ and de other end H_i form a *fragment* of G at $\{s, t\}$.

In Figure 3.1 an example of a 2-connected graph with an cutset $\{s, t\}$ is given. Underneath we see the three fragments of this graph at $\{s, t\}$.

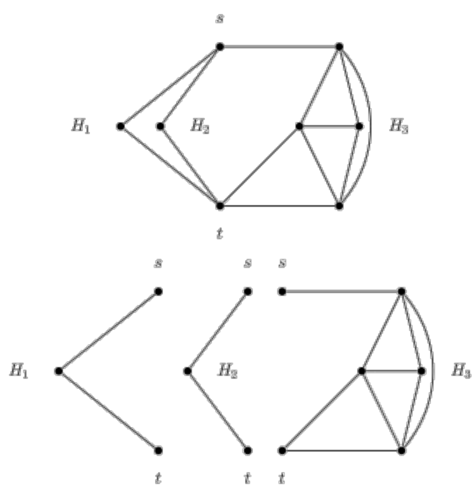


Figure 3.1: The fragments of a graph at $\{s, t\}$

Flipping a fragment If we let H be a fragment of a graph G in $\{s, t\}$. The edges of H are subsequent in the rotation of every vertex $v \in G$. We will by *flipping* this fragment mean the following operation giving us a new planar drawing of a graph: For every vertex v of G we obtain the new rotation in v by taking the old rotation and reversing the order of the edges of H.

For example, suppose the rotation in s is $g_1 g_2 h_1 h_2 h_3$ where h_i are vertices belonging to the fragment H and g_i are vertices not belonging to it. After flipping H the rotation in s will be $g_1 g_2 h_3 h_2 h_1$.

Intuitively this operation physically "flips" H around an imaginary axis through s and t .

A λ-dilatation with respect to r We let r be a straight line in the plane. The λ-*dilation w.r.t.* r is a function that maps every point x to the point y on the same halfplane as x w.r.t. r that is located on the line trough x orthogonal to r and that satisfies $d(y, r) = \lambda d(x, r)$. In which d is the distance function. A λ-dilatation makes a graph narrower (if $\lambda < 1$) or wider (if $\lambda > 1$) around the line r.

3.3 Proof

In the whole proof we will assume that G is a maximal graph with respect to the crossing number of G. That is, we can't add an edge to G without increasing the crossing number. This maximality will be very helpful during the proof.

We can assume G is maximal without loss of generality. Suppose that we have a graph G that is not maximal, in that case we can take an arbitrary maximization (i.e. a maximal graph w.r.t. the crossing number obtained by adding edges to G) $G_{\max}$ and assuming that Theorem 5 holds for $G_{\max}$ we get $\mathrm{cr}(G) = \mathrm{cr}(G_{\max}) = \mathrm{cr}_1(G_{\max}) \geq \mathrm{cr}_1(G)$, the last inequality holding because edge removals can only decrease the crossing number. But because we on the other hand know $\mathrm{cr}_1(G) \geq \mathrm{cr}(G)$, for every rectilinear embedding is also a plane embedding, we obtain $\mathrm{cr}(G) = \mathrm{cr}_1(G)$ for all graphs by proving it for maximal ones.

Furthermore we fix D and let it be a drawing of G with the minimum number of crossings.

3.3.1 When $\mathrm{cr}(G) = 0$

An maximal graph G with zero crossings is an triangulation (for an comprehensive explanation see Remark 3.1) by maximality. Hence it is 3-connected. We can now apply Tutte's Spring Theorem to obtain a convex embedding of G, and hence $\mathrm{cr}_1(G) = 0$.

3.3.2 Connectedness of G^D

We first show that G^D is 2-connected and that Theorem 5 holds in the case of 3-connectedness, leaving us only the case that G^D is two- but not three-connected.

Claim 3.1. *G^D is 2-connected*

Proof. We will prove this by showing G^D can't be disconnected or 1-connected and that hence G^D must be 2-connected. Suppose that G^D is disconnected. Then it must consist of at least two components, which we will refer to as A and B, bordering on the same (outer) face.

To get some insight in the current situation, let us first remark that due to its maximality G must be connected. For if it was not connected you could easily add an edge between the connected components without crossings. This is in contradiction with G being maximal. This implies that all edges that connected A and B in the drawing D are not part of G^D and are hence $D-$crossing edges.

Proving that G^D is not disconnected We can derive an even stronger condition on D from G's maximality, there must exist an closed simple curve γ consisting of crossing-edge segments that separates A and B. For if this curve wouldn't exist you could draw at least one additional edge e from A to B trough the "hole" in the encircling curve. That is to say, since A and B are not separated there must be a path connecting both topologically, this path can be extend to a graph theoretic path by following the boundary of A and B and connecting to the first vertex encountered. This 'following of the boundary' can't be prevented since graph theoretic paths can only start at vertices and we follow the boundary only so long as we do not encounter a vertex. Since e can be added to a crossing-minimal drawing D of G, G is not maximal. This is an contradiction and hence the separating curve γ must exist.

The curve γ can't exist of crossing-edge segments from only 1 or 2 edges. The first would require an edge to cross itself, the second two edges to cross twice. Both of which constitute illegal drawings. Hence γ consists of segment from at least three edges. Since changing from the segment of one edge to the segment of another edge can only take place at a crossing there must also be three crossings. This is in contradiction with the assumption in Theorem 5. Hence the curve γ can't exist and G^D therefore can't be disconnected.

Proving that G^D is not 1-connected The graph of G^D also can't be 1-connected due to a similar argument using properties of an hypothetical separating curve. Suppose that G^D is 1-connected, then it must posses at least one cutvertex u. And if u is removed from the graph then we obtain a graph $G^D \backslash \{u\}$ that has at least two components A and B. We now posit that there must be a curve γ of crossing-edge segments that separates A from B, then we show that such a curve leads to a contradiction.

Suppose in the meantime, however, that this curve γ does exist. Then it has to pass trough the vertex u, see also Figure 3.2 for a simplified view of G^D. To see that γ has to go trough u recall that the edges of G^D are the *non*crossing edges in D. A noncrossing edge can't be crossed (for we chose an path that was not crossed) and hence γ can't cross the edges of G^D, if it still needs to separate A and B it therefore has to go trough u.

The curve γ can't consist of one crossing-edge segment, since this segment would be an edge form u to u, which is in contradiction with G being simple. The curve γ also can't consist of two crossing-edge segments, since then both these segments would be part of an edge e_1, e_2 starting in u and terminating at the boundary of the same component[1], we take without loss of generality

[1]Otherwise they don't have to cross

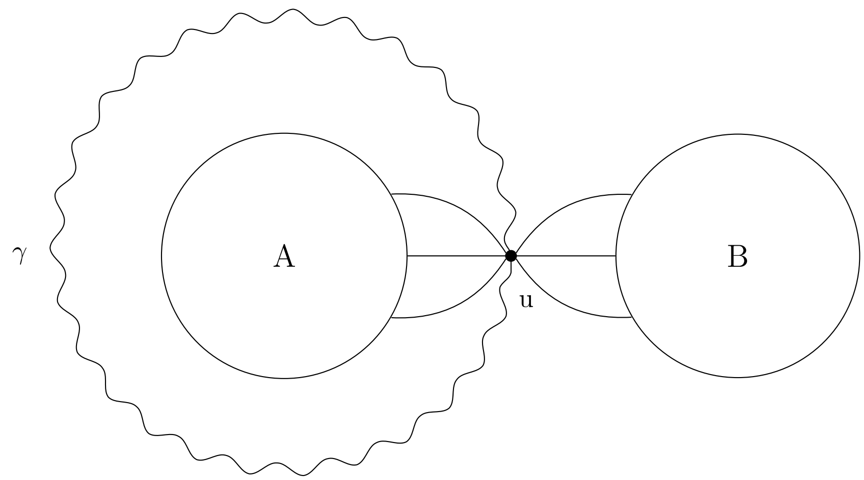


Figure 3.2: The graph G^D and the curve γ consisting of crossing-edge segments of D

component A, and let e_i terminate in a_i. Now we can draw the edges e_1, e_2 without crossing and hence D was not a minimal-crossing drawing, this is an contradiction. Therefore γ must consist of three segments. If it does then γ contains all the crossings by the assumption $\mathrm{cr}(G) \leq 2$.

Now we can make a drawing D' of G in which we can add an edge, showing that G is not maximal. The drawing D' is also depicted in Figure 3.3. We obtain D' by taking the component B in D that is only connected to the rest of the graph in u, we now rotate this component inside the curve γ. If we now let a be a neighbour of u in A and b be a neighbour in B both of them in the boundary of the outer face. We can now add, without increasing the number of crossings, the edge $\{a, b\}$. This is the green edge in Figure 3.3. Hence G is not maximal if G^D is 1-connected, thus G^D is at least 2-connected.

□

Claim 3.2. *If G^D is 3-connected then Theorem 5 is satisfied*

Proof. We first note that Theorem 5 clearly holds for a graph G with 4 or less vertices. This can be checked by showing the result for every such graph. K_4, for example can be drawn rectilinearly as a triangle with the fourth vertex in the center[2].

In the other case G^D, which has the same amount of vertices as G, has at least 5 vertices. We will prove that G^D has at least 3 faces.

[2] K_4 is the complete graph on 4 vertices, it contains 4 vertices all connected to each other by edges.

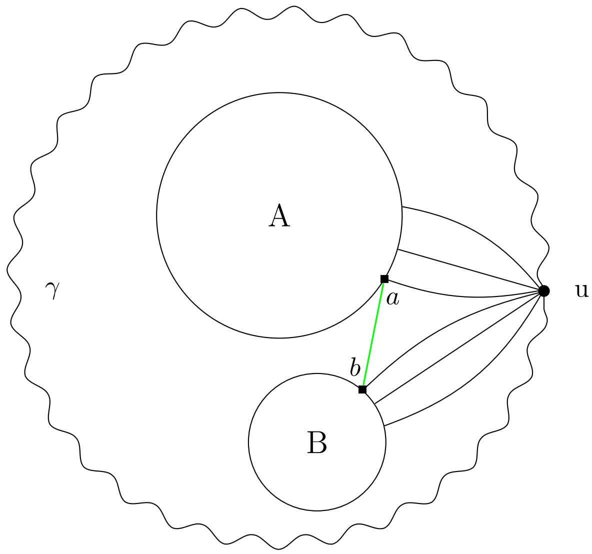


Figure 3.3: Schematic representation of the drawing D' of G.

Suppose G^D has 1 face, then it can't have any cycles. But because there are more then 2 vertices. This means G^D has a cutvertex. This is in contradiction with the assumed 3-connectedness of G^D.

Suppose that G^D has two faces. One of these is the outer face and the other has a cycle as boundary. This is the only cycle of G^D, otherwise there would be a third face. This cycle must be a cycle containing all vertices, if a vertex would not be contained in this cycle, it can only be connected to one vertex v in the cycle, otherwise an additional face would be created. But now v is a cut vertex. In this cycle of more then 4 vertices we can however find a 2-cutset. And this again contradicts the 3-connectedness of G^D.

Since we have at least three faces and only at most two crossings, G^D must have a noncrossing face.

We now consider a convex drawing in which this (or any other) noncrossing face is the outer face, this drawing exist by Tutte's Spring Theorem. To this drawing we can add the $D-$crossing edges to the convex faces in a rectilinear manner without increasing the crossing number of G. This is because all the interior faces are convex. □

The only thing that remains for us to do is hence to proof the Theorem in the case that G^D is 2- ,but not 3-, connected. Hence G^D has at least one two-cutset. Choose an arbitrary one of these cutsets and call this cutset $\{s, t\}$.

3.3.3 Properties of G^D

We will now start to take a closer look on the structure of G^D. The following remark will give the reader a better image of the situation at hand.

Remark 3.1. *A noncrossing face of G^D is triangular. We take triangular to mean that the face has 3 edges and vertices in its boundary.*

Proof. We can easily see the remark is true since a face has at least three edges while a bigger face contradicts the maximality of G. Suppose we have a face F with $n \geq 4$ vertices, $\partial F = v_1 v_2 \ldots v_n$. We can then draw an edge e_c (c for cut) dividing the face, we can for example connect v_1 and v_3. Such an edge dividing a face F in a drawing D might be impossible if this edge already exist and is drawn in the exterior of F. If we would want to prevent an edge e_c dividing the face F, the drawing D needs to draw the edges $\{v_i, v_{i+2}\}, \{v_i, v_{i+3}\}, \ldots, \{v_i, v_{i+n-2}\}$(mod n) for every vertex v_i in ∂F in the exterior of F. This is bound to give crossings if $n \geq 4$. If we look for example at the case that $n = 4$ we see that $\{v_1, v_3\}, \{v_2, v_4\}$ need to be drawn outside F by D, but this will force a crossing, making D an invalid drawing since it should contain the minimum number of crossings. $\square$

Now we will continue the main proof, at this point the proof given in [1] makes another funny jump, we will try to rectify this by the following claim. Furthermore, Claim 3.3 in [1] is amalgamation of a lot of statements, here represented by Remark 3.2 and Claim 3.4 and 3.5. We will also use Remark 3.1 to simplify a number of proofs.

Claim 3.3. *There is at least one noncrossing face F of G^D such that $s, t \in \partial F$*

Proof. If we look at $G^D \backslash \{s, t\}$ we see this graph consists of a number $n \geq 2$ of connected components $A_1, \ldots, A_n$, laying in a outer face $\tilde{F}$. We number these components in the order they occur in the rotation of s in G^D.

When we add back the cutset $\{s, t\}$ this face $\tilde{F}$ is partitioned into several other faces. Some of these faces possibly between s or t and a component A_i, denoted by $\dot{F}_i, \ddot{F}_i$, etc. , but also for every connected component exactly one face with in it's boundary s, t, A_i and A_{i+1}, to be denoted by F_i. In the case that $i = n$ we take, instead of the nonexistent component A_{n+1}, A_1 as second component for the face F_n. See also figure 3.4.

We now differentiate into two cases. Suppose there are three or more connected components $A_1, \ldots, A_n$, then there are the same number of faces $F_1, \ldots, F_n$ containing $\{s, t\}$ in their boundary. Since only two faces can be a crossing face, by $\mathrm{cr}(G) \leq 2$, there must be a noncrossing face satisfying the claim.

Assume now there are only two connected components A_1 and A_2. If $\{s, t\}$ is an edge it will 'divide' one of the faces F_1, F_2 and there will be three faces and since only two of these can be a crossing face by $\mathrm{cr}(G) \leq 2$ we are finished in this case. We now assume that $\{s, t\}$ is not an edge. Then by the maximality of G there must be a separating curve between s and t in D. See Figure 3.5a.

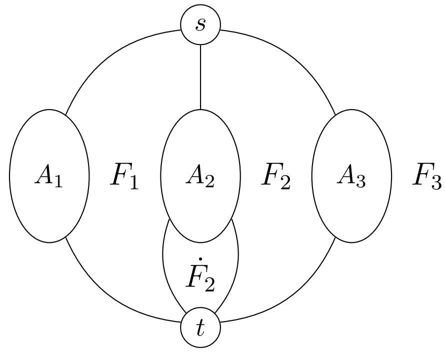


Figure 3.4: The faces of G^D

But also, since we assumed two connected components in G^D, there can't be a noncrossing edge between them in G. By G's maximality we now that there must be a separating curve between A_1 and A_2. Because the edges form s and t to A_1 and A_2 are noncrossing (since they are in G^D) this separating curve has to go trough s and t. We also know each of the faces F_1 and F_2 can contain only one crossing, hence the configuration in Figure 3.5b is the only one possible. However we see that $\{s, t\}$ is twice an edge in G (once in F_1 and once in F_2), this is of course illegal.

We have now obtained that in all cases there is a noncrossing face F of G^D such that $s, t \in \partial F$ $\square$

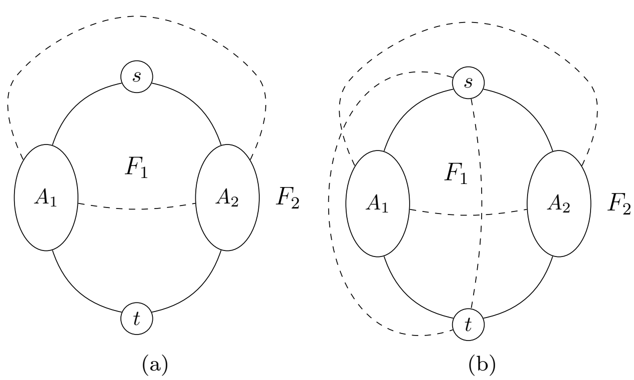


Figure 3.5: The case where there are 2 connected components

We now have seen that there always is a noncrossing face F such that $s, t \in \partial F$. Since a noncrossing face is triangular by Remark 3.1 such a face needs to contain $e \in \partial F$ else it will get a number of vertices that is too large, we can therefore make the following remark.

Remark 3.2. *The edge $e = \{s, t\}$ is in G^D and is therefore not a $D-$crossing edge.*

We will also prove that the number of these noncrossing faces is quite small.

Claim 3.4. *There are one or two noncrossing faces F of G^D such that $s, t \in \partial F$ and such faces contain $e \in \partial F$*

Proof. We know that there must be at least one such face by Claim 3.3. Recall that every noncrossing face is triangular, if they also must contain $s, t \in \partial F$ they need to contain $e \in \partial F$ else they get a number of vertices that is too large. Only two faces can have an edge in their boundary. Hence their can only be two noncrossing faces containing $s, t \in \partial F$. □

We can say even more about faces having $\{s, t\}$ in their boundary.

Claim 3.5. *There is a crossing face F' with $s, t \in \partial F'$*

Proof. Since $\{s, t\}$ is a two cutset it is in the boundary of at least two faces (in the case of 2 connected components), see Figure 3.4. Since e is an edge of G^D one of these faces will be split into two, therefore there must be at least three faces with s, t in their boundary. By Claim 3.4 one of these faces will be a crossing face. □

From this proof of Claim 3.5 we can also deduce the following useful remark

Remark 3.3. *There are three different faces F such that $s, t \in \partial F$*

Now we have the necessary results to proof the following claim. Note that the only requirement for the existence of the crossing face F' in Claim 3.5 is that $\{s, t\}$ is a 2-cutset. Since every 2-cutset forces a crossing face this on the other hand means that *every* 2-cutset is in the boundary of a crossing face.

Now we will prove claim 3.4i from [1]. Bienstock and Dean do not account for the case that e is in $\partial F'$ for F' a crossing face. We will fix this.

Claim 3.6. *Let F' be a crossing face with $s, t \in \partial F'$, there are distinct crossing edges starting in s and t which we will denote e_s and e_t respectively.*

Proof. There is a noncrossing face F containing $e = \{s, t\}$ in its boundary (Claim 3.4) and e is not a crossing edge (Remark 3.2).

We now differentiate between two cases, s and t are neighbours in the boundary of F' or they are not. In other words, $e \notin \partial F'$ or $e \in \partial F'$

Suppose that $e \notin \partial F'$. Let us note that the neighbours of s in the boundary of F' are not connected with an edge inside the face, for then s wouldn't be incident with the face. Suppose there is no crossing edge starting at s. Since G is maximal the neighbours of s must be connected, such an edge however can't be drawn in the exterior of the face F', due to the edge e existing and being a noncrossing edge. Therefore G is not maximal if no crossing edges start at s, thus there must start an crossing edges e_s at s. The same argument holds for t.

Now, for the other case suppose $e \in \partial F'$ and there's no crossing edge e_s starting at s. Let v be the other neighbour of s (i.e. not t) in the boundary of F'. We will show that G is not maximal. We note that $\{t, v\} = e_v$ must exist by maximality, it can be drawn in the face F' without obstruction. Furthermore it can't be drawn in in a way dividing the face F' since then this face wouldn't be a single face (a contradiction with the premise in the claim). Hence it is drawn in the exterior of F' and it can't be a crossing edge for then D wouldn't be a crossing-minimal drawing. We now have the situation in Figure 3.6, in which we note that every edge is an noncrossing edge in D, because e_v is one and $\partial F'$ is in G^D. We will call the outer cycle C. Since $\{s, t\}$ is a cutset of $G^D \backslash \{s, t\}$ must consist of at least two components, one of those is formed by the circuit C (minus t) and vertices connected to it. Another connected component can't be in the exterior of C, for then it can't be connected to s, and this would make t a 1-cutset. A contradiction with Claim 3.1 (2-connectedness). Another component also can't be in the interior F', this would contradict with our assumption that $e \in \partial F'$. Hence there must be at least one additional component J in the interior of C, but not in F'.

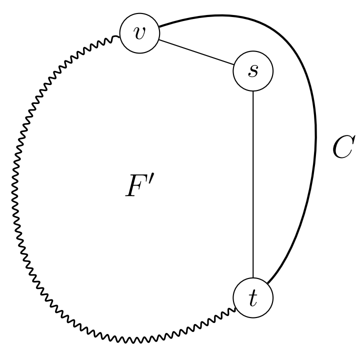


Figure 3.6: The situation when s and t are incident in $\partial F'$, the wiggled line is the rest of the face boundary

This component can be seen drawn in figure 3.7, we will now show that F'' must be a crossing face. Recall that the cycle C and the component H belong to different components of $G^D \backslash \{s, t\}$, hence v and H must not be connected in G^D. But then by the maximality of G there must be an arc of crossing-edge segments connecting s and t (in a Jordan curve theorem sense), to prevent such an edge from existing. The only way to make such an arc with only one crossing is in the way depicted in Figure 3.7. We have now determined all the crossing faces and crossings of the graph. By maximality only the vertices depicted in

Figure 3.7, can be on the boundary of the depicted components[3]. But now we can add the edge $\{y, s\}$ to G by drawing e_v in the interior of F'. Hence G is not maximal and we have obtained a contradiction.

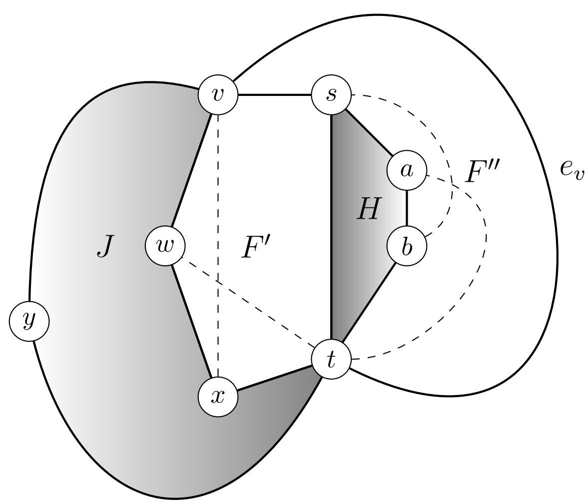


Figure 3.7

A very similar argument also works for t.

Since $e = \{s, t\}$ is a noncrossing edge, the crossing edge starting at s can't terminate at t, therefore $e_s \neq e_t$. □

Claim 3.7. *Let F' be a crossing face with $s, t \in \partial F'$ we then have either*

a) F' contains more then one crossing, or

b) There is an additional crossing face F'' with e in F' or F''

Proof. We will assume a) doesn't hold and will prove that in this case b) must hold. The face F' only contains one crossing, and by Claim 3.6, this must be the crossing between e_s and e_t. All the endpoints of the $(D, F')-$crossing edges

[3] These components could be empty except for edges $\{a, t\}, \{w, y\}$ and $\{x, y\}$

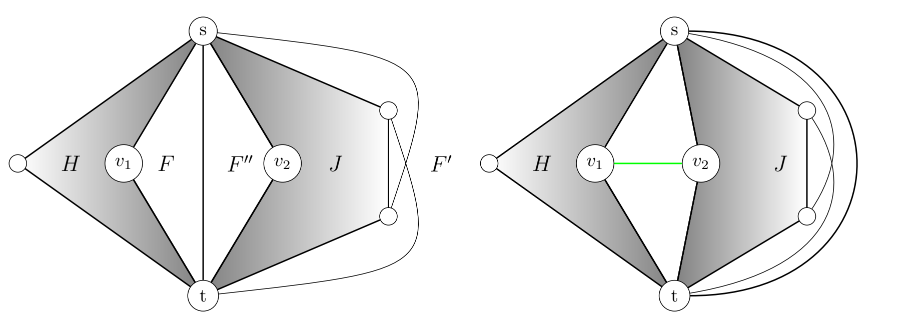
lie in $\partial F'$ and thus in a fragment J at $\{s, t\}$, we will for our convenience also include $e = \{s, t\}$ in J. By Remark 3.3 there are at least three faces containing s, t in their boundary. We will denote this third face with F''.

Either $e \in \partial F'$ or $e \notin \partial F'$. In the first case we only have to prove there is an additional crossing face. We let F'' be an arbitrary third face containing $\{s, t\}$ in its boundary. Assume F'' is a noncrossing face, then it must have $e \in \partial F''$ by remark 3.4. But now F, F' and F'' all have e in their boundary. This is impossible and hence F'' is a crossing face.

In the other case ($e \notin \partial F'$) we note that the following argument is true for any third face containing $s, t \in \partial F''$. This will therefore also hold for the particular face that has e in its boundary and is not the noncrossing face F. This face will then be the face F'' in the claim.

Now continuing the case that e *is* in the boundary of F'' we will, falsely, assume that F'' is not a crossing face. It then by Claim 3.4 has to have e in it's boundary and we get the situation in following picture 3.8a, here the shaded area's indicate the connected components. The interior of the connected components is not important. If we redraw e trough the crossing face F' we get Figure 3.8b, we see that we can add an edge$\{v_1, v_2\}$ (green in the Figure). Hence G is not maximal if F'' is not a crossing face, and therefore any third face containing s, t in its boundary must be a crossing face. □

At this point Bienstock and Dean prove claim 3.6 in their paper. This, however, is unnecessary since both cases of claim 3.7 imply a crossing number of at least 2. We can hence conclude that if the crossing number of a nontrivial, maximal graph $\mathrm{cr}(G) < 2$ the graph G^D must have been 3-connected and the case resolved.



(a) The structure of G^D drawn as G is by D (b) After redrawing e we can see that G was not maximal

Figure 3.8: Clarification on the proof of Claim 3.7

3.3.4 Finishing the proof in case b) of Claim 3.7

In the above proof we learned some things about the structure of G^D in case b), this will be summarised in the following remark. Remember that there can only be at most two crossing faces, and that an edge can only be in the boundary of two faces.

Remark 3.4 (Graph structure in case b))**.** *In the case of b) in the above proof we get the following structure. The edge e is bordered by a crossing face $\dot{F}$ and a noncrossing face F, furthermore there is a second crossing face $\ddot{F}$ containing s and t, but not e, in its boundary. These are all the crossing faces.*

For an example of a graph with the structure in this remark you can look ahead to Figure 3.9.

Claim 3.8. *If case b) in Claim 3.7 holds Theorem 5 holds.*

Proof. We will first make the structure of G^D more clear. G^D consists of two fragments at $\{s,t\}$, any more fragments will translate to more connected components of $G^D \backslash \{s,t\}$, but this is impossible because the face between two such components in G^D has to be a crossing face, else they will be connected to each other by the maximality of G and we only have two crossing faces.

On the endpoints of $(D,F)-$**crossing edges** Both the crossing faces are incident with $\{s,t\}$ by Remark 3.4. Their crossing edges will all terminate in the same fragment at $\{s,t\}$ as we will soon prove. We will call this fragment J and add the edge e to it (i.e. if we in the future refer to J we implicitly include e). We will call the other fragment of G^D at $\{s,t\}$ H.

To see all crossing edges terminate in the same fragment we use the following proof. The crossing edges of a face with one crossing, e_s and e_t, will have one end in {s,t} and the other end in the same fragment. Suppose the edges of the different crossing faces terminate in different fragments, we will call the fragment in which $(D,\dot{F})-$crossing edges terminates J and that in which those of $\ddot{F}$ terminate H. We get the the situation in Figure 3.9, in which the dashed lines are crossing edges.

We now see that flipping $J \cup \{e\}$, resulting in a physical flip around the $s-t$-axis, allows us to make the edge $\{x,y\}$. This is a contradiction with the maximality of G and hence such a configuration can't exist. All endpoints of crossing edges therefore must be in J.

On the shape of the crossing faces in J The are two crossing edges, e_s and e_t, in every crossing face F. We will define the vertices u and v by these edges, $e_s = \{s,v\}$ and $e_t = \{t,u\}$. We will place dots over these vertex-names in correspondence with the face-names. See Figure 3.10

The boundaries of the crossing faces $\dot{F}, \ddot{F}$ *in the fragment* J are given by $s,t,\dot{u},\dot{v}$ and $s,t,\ddot{u},\ddot{v}$, as is imaged in Figure 3.10. This can be seen in the following way, suppose there is an additional vertex j between t and v or v and

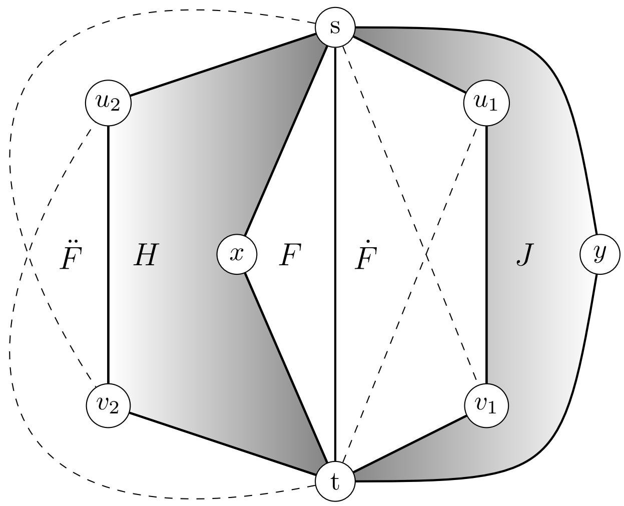


Figure 3.9: Crossing edges terminating in different fragments.

u or s and u. Then the edge x connecting them must still exist by maximality, there is after all no crossing edge in the face $\dot{F}$ or $\ddot{F}$ preventing this edge, for an example of such a situation we can look at Figure 3.11. But if we now draw j on the other side of this edge x it lies in a noncrossing face and it can hence form an edge, in our example this would be the edge $\{j, \dot{u}\}$.

Therefore if we have an additional vertex j in the boundary of the crossing face it must be between s and t. This vertex, that is connected to s and t, can however not be connected to any vertex in $J \backslash \{s, t\}$ by the crossing edges in both crossing faces. Therefore this vertex is not in the fragment J and it therefore doesn't effect the boundaries of the crossing faces in J

Showing J is 3-connected Since G^D is 2-connected (Claim 3.1) clearly J is also 2-connected. Suppose that J is not 3-connected then it must have a cutset $\{x, y\}$ that by Claim 3.5 lies in the boundary of a crossing face. We take this face, without loss of generality, to be $\dot{F}$. The cutset will not be $\{s, t\}$ since J is a fragment at $\{s, t\}$ and is thus connected if this is removed. Two other neighbouring vertices (i.e. not s and t) can't be a 2 cutset since the boundaries of the crossing faces stay connected to each other and every vertex not in this boundary is connected to boundaries of both crossing faces by a path, since the cutset is in the boundary of only one crossing face these other vertices stay connected.

A 2-cutset can't be a diagonal from a crossing face (e.g. $\{t, \dot{u}\}$ or $\{s, \ddot{v}\}$),

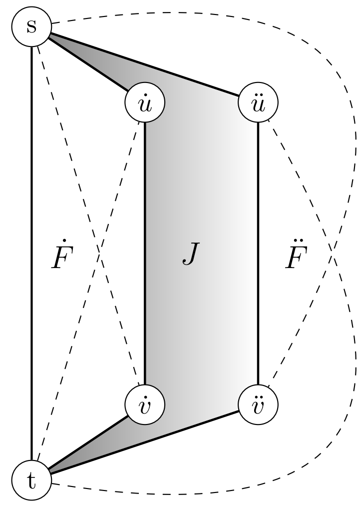


Figure 3.10: The fragment J and the crossing faces $\dot{F}, \ddot{F}$ in this fragment

since a cutset of J is also a cutset of G^D. This can be seen in the following way, the other fragment H at $\{s, t\}$ connects s and t. But if a cutset cuts J (that includes e) it also cuts $J \cup H = G^D$ since the only thing H adds is a connection from s to t. But this connection was already there in J thus this makes no difference. The diagonals of the crossing faces are the crossing edges and these cutsets would thus offend Remark 3.2. Hence there are no 2-cutsets and J is 3-connected.

Making a rectilinear drawing Since J is 3-connected,we can by Tutte's spring theorem, make a convex drawing D' of J and force $\dot{F}$ and $\ddot{F}$ to be internal faces, we can add the crossing edges rectilinearly to this drawing while only adding two crossings. Since $H \cup \{e\}$ the other fragment at $\{s, t\}$ is has $\mathrm{cr}(H) = 0$ we can make a rectilinear drawing D'' of it (see Section 3.3.1). If we make this drawing "thin" enough we can fit it into D' without it crossing any of the crossing edges that we added back in. For an example of what the result of this procedure could look like see Figure 3.12, the small black vertices represent the fragment H, while the large white vertices represent J. We can also see the graph in this figure is maximal.

We have now produced a rectilinear drawing of G if case b) of Claim 3.7 holds. □

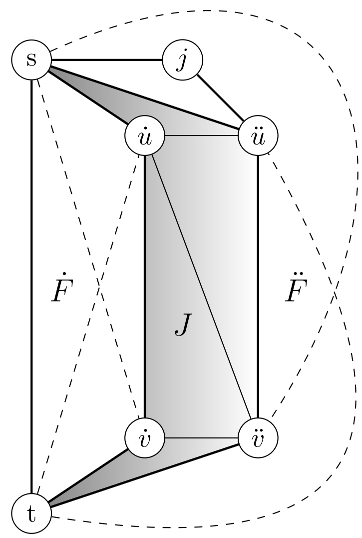


Figure 3.11: An extra vertex in the boundary of $\ddot{F}$, j is between s and $\ddot{u}$

3.3.5 Finishing the proof in case a) of Claim 3.7

Here we continue the proof in the case a) of Claim 3.7, this subsection is equivalent to Claims 3.7-3.9 from [1].

Let us remark that F' is the only crossing face of G^D. By case a) of Claim 3.7, this face must contain at least two crossings, while on the other hand $\mathrm{cr}(G) \leq 2$. Hence all the crossings of G must lie in F'. We will begin with two claims on the structure of the crossing edges in the only crossing face F', note that $\partial F'$ is the union of two internally disjoint paths from s to t in G^D, p_1 and p_2.

Claim 3.9. *There exists a topological path between s and t, contained in F', and made up of sections of (D, F')-crossing edges only.*

Claim 3.10. *There exist a (D, F')-crossing edge e_c (c for 'crossing') with one end in $V(p_1)\backslash\{s,t\}$ and the other in $V(p_2)\backslash\{s,t\}$.*

Here $V(p_i)$ denotes the vertex set of the path, i.e. all the vertices the edges of the path p_i are incident with.

Proof Claim 3.9. Let us first remark that s and t are not neighbours in $\partial F'$, since that are three faces containing $\{s,t\}$ in their boundary (Remark 3.3) and only F' is a crossing face. The other faces must contain e in their boundary by

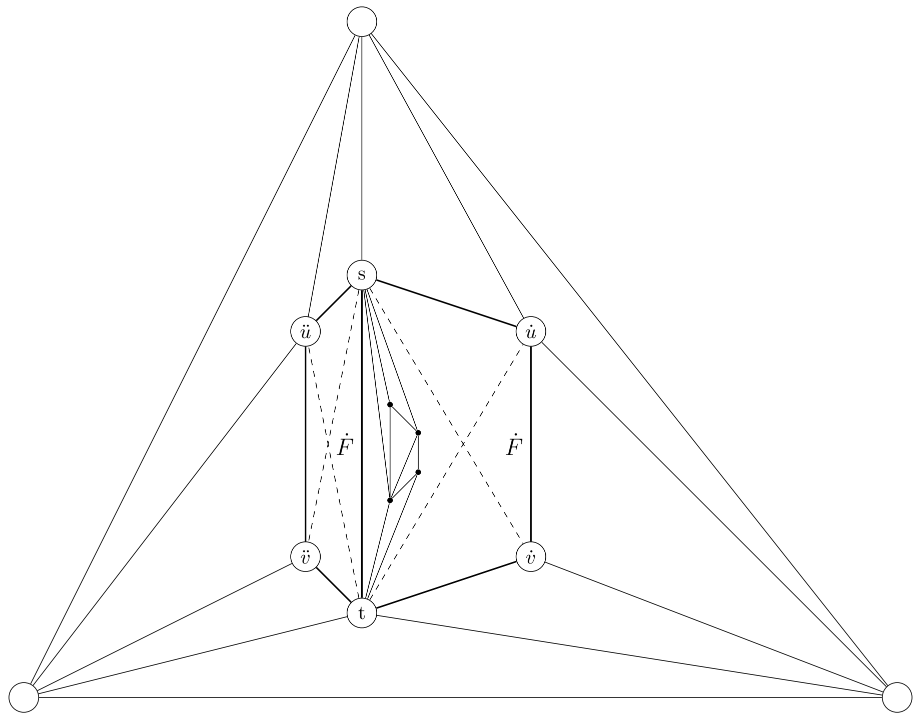


Figure 3.12: An example of a drawing the could be produced by the procedure that is given above.

Claim 3.4. Hence F' can't have e in it's boundary, and therefore s and t are not neighbours in $\partial F'$.

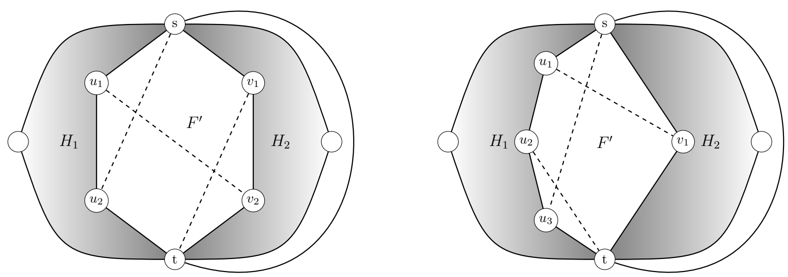
Suppose there is not such an topological path, in that case there we can add an edge from a vertex x in $V(p_1)\backslash\{s,t\}$ to a vertex y in $V(p_2)\backslash\{s,t\}$ (both sets are nonempty by the above consideration). Hence G is not maximal, this is a contradiction and hence such a path must exist. □

Proof Claim 3.10. We know $e \notin \partial F'$ as is shown in the proof of claim 3.9.

We will first show there must be an topological path of edge segments between $V(p_1)\backslash\{s,t\}$ and $V(p_2)\backslash\{s,t\}$ contained in F' and made up of sections of (D, F')-crossing edges. Suppose that there is not such a path, in that case we can draw the edge e 'in' the face F'. But now we have $e \in \partial F'$ and e in the boundary of one of the noncrossing faces. But now e can not be in the boundary of the other noncrossing face. This is in contradiction with Claim 3.4.

Suppose that e_c doesn't exist, then every edge has both ends in $V(p_1)$ or in $V(p_2)$, we can draw these two groups of edges without them ever crossing. While not increasing the number of mutual crossings. of these groups By the the crossing-minimality of D they will be drawn this way. It is now impossible to connect $V(p_1)\backslash\{s,t\}$ and $V(p_2)\backslash\{s,t\}$ by an path of edge segments. This is in contradiction with the above. Hence e_c must be an edge. □

We know the following about the crossing edges in F'. There are two distinct crossing edges e_s and e_t starting at s or t and terminating in an arbitrary vertex (Claim 3.6) and there is one additional crossing edge e_c from $V(p_1)\backslash\{s,t\}$ to $V(p_2)\backslash\{s,t\}$ (Claim 3.10). Hence there are, up to, refection symmetry in the $s-t$-axis and exchanging s and t, two possible patterns of crossing edges (in the sense of topological relations). These patterns are given in Figure 3.13.



(a) case a) (b) case b)

Figure 3.13: The two possible patterns of crossing edges in F'

With a proof similar to the paragraph "On the shape of crossing faces in J" we can see that the boundary must be as is pictured in Figure 3.13, and can't contain extra vertices. After all, suppose there is an additional vertex j between vertices x, y that are neighbours in Figure 3.13. Then the edge e_{xy} connecting them must still exist by maximality, for there is no crossing edge in the face F' preventing this edge. But if we now draw j on the other side of this edge e_{xy} it lies in a noncrossing face and it can hence form an edge with another vertex. Hence it must lie there by the maximality of G.

We will define the 'left' path, $p_1 = (s - u_1 - \ldots - u_n - t)$ and we the other path $p_2 = (s - v_1 - \ldots - v_n - t)$. Since F' is the only crossing face there can only be two fragments at $\{s,t\}$. We will denote these by H_1 and H_2, such that $V(p_i)\backslash\{s,t\}$ is contained in H_i see also Figure 3.13. We let $R_1 = H_1 \cup \{e, e_s\}$ and $R_2 = H_2 \cup \{e\}$.

General setup for Case a) of Figure 3.13

Claim 3.11. *Either R_2 is 3-connected or its only 2-cutset is $\{t, v_2\}$*

Proof. Since G^D is 2-connected we know that R_2 is also 2-connected. Suppose that R_2 has a 2-cutset $\{x,y\}$ then it must also be a cutset of G_D, after all, from the perspective of R_2 the only thing H_1 is in G^D is a connection from s to t and such a connection is already part of R_2 in the form of the edge e. Hence any 2-cutset of R_2 must also be a 2-cutset of G^D

Since $\{x, y\}$ is a 2-cutset of G_D it must satisfy Claim 3.5 and since F' is the only crossing face we get $\{x, y\} \subset V(p_2)$. By Remark 3.3 (a cutset is in three faces) and the fact that F' is the only crossing face, we know that $\{x, y\}$ can't consist of two neighbours in F' because in that case one of the noncrossing faces can't have the edge $\{x, y\}$ in its boundary. This offends Claim 3.4. The cutset $\{x, y\}$ of R_2 course can't equal $\{s, t\}$ since H_2 is a fragment in $\{s, t\}$.

Now the only remaining options for a 2-cutset are $\{s, v_2\}$ and $\{t, v_1\}$. However, it can't be $\{t, v_1\}$ since this is a crossing edge, hence this cutset would offend Remark 3.2. □

Now we will show that $\mathrm{cr}(G) = \mathrm{cr}_1(G)$ by explicitly drawing G rectilinearly in both of the above cases.

We let D_1 be a rectilinear drawing of R_1 on a halfplane P_1 in $\mathbb{R}^3$. By $\mathrm{cr}(R_1) = 0$ we can make a rectilinear drawing of it. By using a stereographic projection, rotating the sphere and again using a stereographic projection. We let the triangle su_2t be the boundary of the outer face of D_1. And thus such that e_s is part of the outer facial boundary. The lines tu_2 and u_2u_1 cross in u_2 and determine two plane angles in P_1, we let α be the plane angle on the outside of D_1. We let P_2 be another halfplane in $\mathbb{R}^3$, we will eventually draw R_2 on this plane. Let those two planes cross in the line st drawn on P_1 by D_1. Let p be a point in the plane angle α and let p' be a point 'above' it, i.e. a point perpendicular to P_1 on the opposite side of P_1 as P_2 is. We let $\pi : P_2 \mapsto P_1$ be the projection of P_2 towards p' on P_1. For an look at this configuration, see 3.14. Let us also note here that projections conserve rectilinearity on these halfplanes.

Case a) of Figure 3.13, $\{s, v_2\}$ is a 2-cutset In making a drawing in this case we will need the following claim.

Claim 3.12. *Suppose H is a 2-connected plane graph, containing an internal face with as boundary abcd, such that $\{b, d\}$ is the only 2-cutset of H and $\{b, d\}$ is an edge of H. Let q_b and q_d be arbitrary points in the plane. Then there exist a point z on the segment $[q_b, q_d]$ such that for every $\epsilon > 0$ there is a rectilinear plane drawing of H satisfying.*

i) b and d are drawn at q_b and q_d,respectively

ii) The segment $[a, c]$ does not cross any edges

iii) z is the projection of c onto the line-segment q_bq_d and the distance between c and z is less then ϵ

Proof. We will employ Tutte's theorem in this proof to obtain the desired result. By H_a we will denote the union of the fragment of H at $\{b, d\}$ containing a and the edge $\{b, d\}$. Similarly, with H_c we will mean the union of the fragment

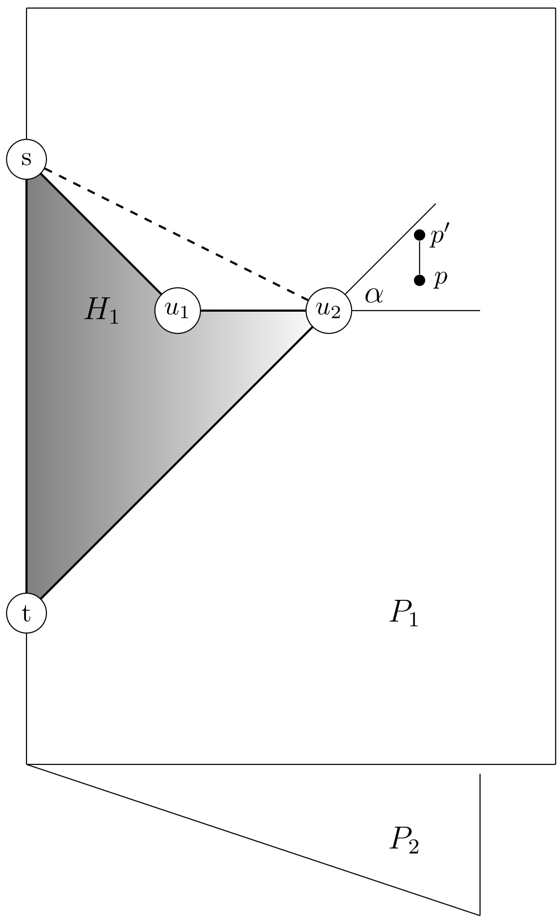


Figure 3.14: The general setup in case a)

containing c and $\{b, d\}$. Note that H_a has an rectilinear drawing since $\mathrm{cr}(H_a) = 0$ (see Section 3.3.1), by a transformation of the plane of this drawing we can make b lie at q_b and d lie at q_d. Furthermore, H_c is 3-connected, for suppose not, then there must be a cutset S of H_c with two vertices. However by the structure of the graph H_c this must then also be a cutset of H. From the local perspective of H_c, H_a functions as a connection from b to d but H_c already has this connection by virtue of the edge $\{b, d\}$. Hence a cutset of H_c is a cutset H, but now $\{b, d\}$ is not the only cutset and this is a contradiction.

Now we apply Tutte's Spring Theorem to H_c with as convex outer boundary face contained within the half strip given by the two lines perpendicular to the line $q_b - q_d$ in q_b and q_d. Now the projection of every vertex, and in particular c, lies in $[q_b, q_d]$. Let us denote by λD_c the λ-dilation of D_c in the line $q_b - q_d$. If we make λ small enough D_a and λD_c will not cross, and if we make $\lambda < \frac{\epsilon}{d_{D_c}(c,z)}$ then $d_{\lambda D_c}(c, z) = \lambda \cdot d_{D_c}(c, z) < \epsilon$ [4].

If we take the union $D = D_a \cup D_c$ this drawing satisfies all the requirements of the lemma. An example of such a drawing is given in Figure 3.15. The shaded areas denote the connected components. $\square$

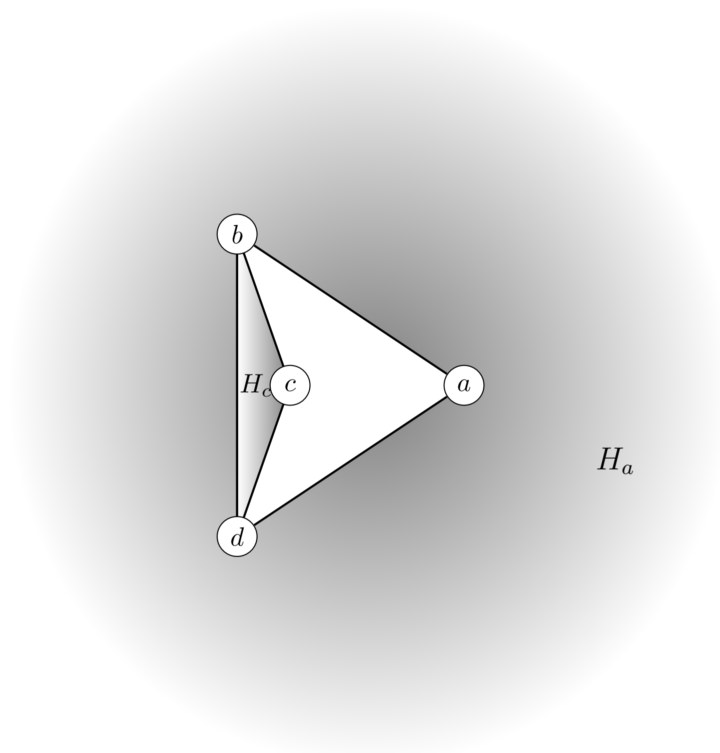


Figure 3.15: A drawing of H according to procedure in the proof of Claim 3.12.

We will now show the procedure Bienstock and Dean follow and show where it might go wrong.

They now apply the claim with $a = t, b = s, c = v_1$ and $d = v_2$. To create a rectilinear drawing of H_2, we transform it by rotating and rescaling such that s and t coincide with the s and t of drawing D_1 of R_1 on P_1. However there is no guarantee that all vertices will lie on the halfplane P_2, there might be vertices on the other side of the line $S - t$. They now let p' lie so close to p such that if we project by π we get that both $\pi(v_2)$ and $\pi(z)$ lie in α. If we now let ϵ go to zero, we see that $\pi(v_1)$ tends to $\pi(z)$ and hence $\angle \pi(v_1) t \pi(v_2)$ tends to zero. This implies that $\{u_2, \pi(v_2)\}$ will not cross $\{t, \pi(v_1)\}$. Next to that,

[4]In which d_D denotes the distance function in a drawing D.

since $\pi(z) \in \alpha$ we know that $\pi(v_1)$ will be in alpha for small enough epsilon. This projection, however, will not work for points that are not in the halfplane P_2. Certain points might not even be projected onto P_1 (if the line between the point and p' lies parallel to P_1). I've unfortunately not been able to fix this issue.

They now look at union $D = D_1 \cup \pi(D_2)$ and say that this is a rectilinear drawing of $G^D \cup e_s$. They add the other crossing edges e_t, e_c to this drawing to get a rectilinear drawing of G with two crossings. See Figure 3.16

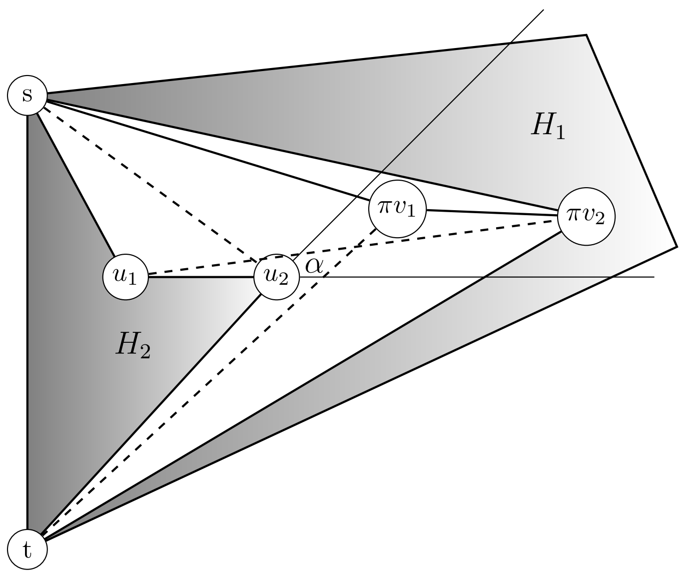


Figure 3.16: The rectilinear drawing when $\{s, v_2\}$ is a 2-cutset

Case a) of Figure 3.13, R_2 is 3-connected Since R_2 is 3-connected it has by Tutte's theorem a convex drawing on P_2 with sv_1v_2t as an internal face and s and t matching the positions of s and t on P_1. If p' is close enough to p, the projections $\pi(v_1)$ and $\pi(v_2)$ are inside α. If we now take $D = D_1 \cup \pi(D_2)$ we have obtained a rectilinear drawing of $G^D \cup e_s$. If we add the missing crossing edges e_t and e_c back into this drawing we get a rectilinear drawing of G with two crossings. See Figure 3.17

Case b) of Figure 3.13 Case b) isn't treated in [1] by Bienstock and Dean. But instead a remark is made on how similar methods can be used to proof the existence of a rectilinear drawing in this case. This, however, is not entirely true.

We can of course change the roles of H_1 and H_2 in the proof and draw the triangle sv_1t in the fragment H_2 as outer face on P_1. However, there is no guar-

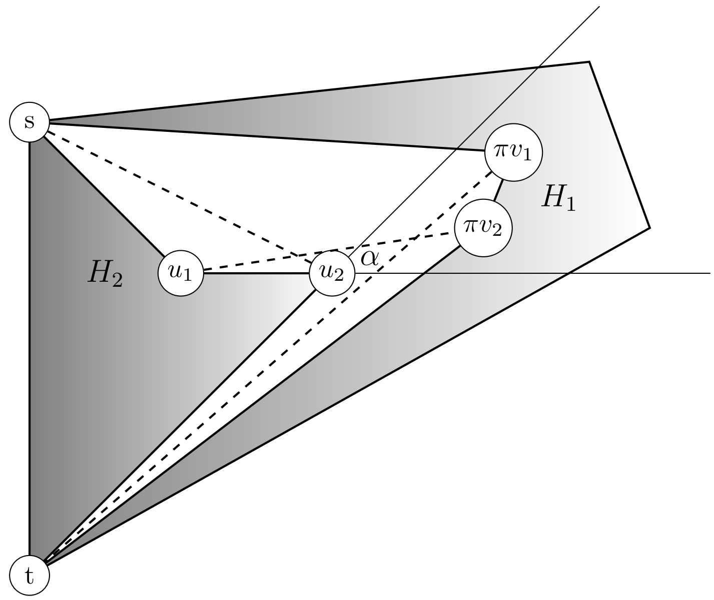


Figure 3.17: The rectilinear drawing when R_2 is 3-connected

antee that H_1 has only one cutset. A quick inspection allows $\{s, u_2\}, \{u_1, u_3\}$ and $\{t, u_1\}$ by the same reasoning as in Claim 3.11.

Even if only one of these set can actually be a cutset of a graph that satisfies all the assumptions in the proof (e.g. G being maximal, D being a minimal crossing drawing) at the same time. Which I find likely, the approach of drawing this graph can't be copied directly from the paragraph "Case a) of Figure 3.13, $\{s, v_2\}$ is a 2-cutset" above. Furthermore, the Claim 3.12 doesn't apply easily to these drawings.

For these reasons I have unfortunately decided that Case b) of Figure 3.13 is outside the scope of this thesis.

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