Cluster Algebras

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1 Introduction

Cluster algebras, [...] are constructively defined commutative rings equipped with a distinguished set of generators (cluster variables) grouped into overlapping subsets (clusters) of the same finite cardinality (the rank of an algebra in question). [...] both generators and algebraic relations among them are not given from the outset but are produced by an iterative process of seed mutations.

This is how Zelevinsky starts his article [1], explaining what cluster algebras are. After explaining how these seed mutations work, he writes the following:

One of its main consequences - and one of the main reasons for introducing it - is the Laurent phenomenon: every cluster variable, which a priori is just a rational function in the elements of a given cluster, is in fact a Laurent polynomial with integer coefficients. [...] The cluster algebra machinery provides a unified explanation of several previously known phenomena of this kind. One example is the Somos-5 sequence discovered some years ago by M. Somos: its first five terms are equal to 1, and the rest are given by the recurrence relation $a_m a_{m-5} = a_{m-1}a_{m-4} + a_{m-2}a_{m-3}$*. The fact that all terms of this sequence are integers can be deduced from the Laurent phenomenon for cluster algebras.*

Reading this gave more than enough incentive to study these cluster algebras by writing a thesis on this subject. To write this thesis, we set out to find an answer for the following question:

Can we find a proof for the Laurent phenomenon?

To present our findings, we first look at a specific case of cluster algebras of rank 2. After which we look at the general definition of cluster algebras, for this we first define an ambient field for our setting, then we introduce the definitions needed to come to a general definition of a cluster algebra, which we also clarify with examples, and finally we conclude the thesis with a proof of the Laurent phenomenon.

2 Cluster algebras of rank two

In this section, we will look at cluster algebras of rank two. In a cluster algebra of rank two, we are dealing with clusters containing two cluster variables. Since these clusters must be overlapping subsets, it follows that we have a chain of cluster variables $\ldots, y_{m-1}, y_m, y_{m+1}, \ldots$, for $m \in \mathbb{Z}$, such that every consecutive pair of cluster variables in this chain forms a cluster. These cluster variables are connected via algebraic relations, which we can see in the following definition of a specific case of cluster algebras of rank two: (This definition follows the definition Zelevinsky gives in [1])

Definition 2.1: Given a pair of positive integers (b, c), we will denote by $\mathcal{A}(b, c)$ a cluster algebra of rank two, for which the cluster variables satisfy the following algebraic relation:

$$y_{m-1}y_{m+1} = \begin{cases} y_m^b + 1, & \text{if m is even,} \\ y_m^c + 1, & \text{if m is odd.} \end{cases} \tag{2.1}$$

Note that, when we iterate these exchange relations, we find that we can express each cluster variable in $\mathcal{A}(b, c)$ as a rational function of any two consecutive cluster variables y_m and y_{m+1}. Therefore, in particular, $\mathcal{A}(b, c)$ is the subring of the field of rational functions $\mathbb{Q}[y_1, y_2]$, generated by all cluster variables y_m.

We will now illustrate this definition with a few examples.

Example 2.1: First of all, lets have a look at the cluster variables of $\mathcal{A}(1, 1)$. We start with cluster variables y_1 and y_2, iterating the exchange relations in equation 2.1, we get:

$$y_1y_3 = y_2 + 1 \qquad \rightarrow \qquad y_3 = \frac{y_2+1}{y_1},$$

$$\begin{aligned} y_2y_4 &= y_3 + 1 \\ &= \frac{y_2+1}{y_1} + 1 \\ &= \frac{y_1+y_2+1}{y_1} \end{aligned} \qquad \rightarrow \qquad y_4 = \frac{y_1+y_2+1}{y_1y_2},$$

$$\begin{aligned} y_3y_5 &= y_4 + 1 \\ &= \frac{y_1+y_2+1}{y_1y_2} + 1 \\ &= \frac{y_1+y_2+y_1y_2+1}{y_1y_2} \\ &= \frac{y_2+1}{y_1} \cdot \frac{y_1+1}{y_2} \end{aligned} \qquad \rightarrow \qquad y_5 = \frac{y_1+1}{y_2},$$

$$
\begin{aligned}
y_4 y_6 &= y_5 + 1 \\
&= \frac{y_1+1}{y_2} + 1 \qquad\qquad \rightarrow \qquad\qquad y_6 = y_1, \\
&= \frac{y_1+y_2+1}{y_2}
\end{aligned}
$$

$$
\begin{aligned}
y_5 y_7 &= y_6 + 1 \\
&= y_1 + 1 \qquad\qquad \rightarrow \qquad\qquad y_7 = y_2.
\end{aligned}
$$

Clearly the chain of cluster variables forms in this case a loop, hence $\mathcal{A}(1,1)$ contains a finite number of different cluster variables. In this example you can already notice the Laurent phenomenon: All cluster variables can be expressed as Laurent polynomials in y_1 and y_2.

Example 2.2: Next, we look at the cluster variables of $\mathcal{A}(2,2)$. Again we start with the cluster variables y_1 and y_2, and now, after iterating the exchange relations in equation 2.1 a two times, we get:

$$
y_1 y_3 = y_2^2 + 1 \qquad\qquad \rightarrow \qquad\qquad y_3 = \frac{y_2^2+1}{y_1},
$$

$$
\begin{aligned}
y_2 y_4 &= y_3^2 + 1 \\
&= \frac{(y_2+1)^2}{y_1^2} + 1 \qquad\qquad \rightarrow \qquad\qquad y_4 = \frac{y_2^4 + 2y_2^2 + y_1^2 + 1}{y_1^2 y_2}. \\
&= \frac{y_2^4 + 2y_2^2 + y_1^2 + 1}{y_1^2}
\end{aligned}
$$

Iterating a third time, we already get a quite complicated polynomial:

$$
\begin{aligned}
y_3 y_5 &= y_4^2 + 1 \\
&= \left(\frac{y_2^4 + 2y_2^2 + y_1^2 + 1}{y_1^2 y_2}\right)^2 + 1 \\
&= \frac{y_2^8 + 4y_2^6 + 2y_1^2 y_2^4 + 6y_2^4 + y_1^4 y_2^2 + 4y_1^2 y_2^2 + 4y_2^2 + y_1^4 + 2y_1^2 + 1}{y_1^4 y_2^2} \\
&= \frac{y_2^2+1}{y_1} \cdot \frac{y_2^6 + 3y_2^4 + 2y_1^2 y_2^2 + 3y_2^2 + y_1^4 + 2y_1^2 + 1}{y_1^3 y_2^2}.
\end{aligned}
$$

However y_5 is still a Laurent polynomial in y_1 and y_2 since the first fraction in the equation above is equal to y_3, so we have:

$$
y_5 = \frac{y_2^6 + 3y_2^4 + 2y_1^2 y_2^2 + 3y_2^2 + y_1^4 + 2y_1^2 + 1}{y_1^3 y_2^2}.
$$

The following two iterations result in even larger polynomials:

$$
y_6 = \frac{y_2^8 + 4y_2^6 + 3y_1^2 y_2^4 + 6y_2^4 + 2y_1^4 y_2^2 + 6y_1^2 y_2^2 + 4y_2^2 + y_1^6 + 3y_1^4 + 3y_1^2 + 1}{y_1^4 y_2^3},
$$

$$y_7 = \frac{1}{y_1^5 y_2^4}\Big(y_2^{10} + 5y_2^8 + 4y_1^2y_2^6 + 10y_2^6 + 3y_1^4y_2^4 + 12y_1^2y_2^4 + 10y_2^4 + 2y_1^6y_2^2 + 9y_1^4y_2^2 + 12y_1^2y_2^2 + 5y_2^2 + y_1^8 + 4y_1^6 + 6y_1^4 + 4y_1^2 + 1\Big).$$

And in the following iterations the rational polynomial on the left hand side of the exchange relation $y_{m-1}y_{m+1} = y_m^2 + 1$ is always of a higher degree then the divisor y_{m-1}. Hence, its reasonable to assume we have an infinite chain of cluster variables. To prove this however, we need to prove that the coefficients of the cluster variables are positive, which is far from trivial and outside the scope of this thesis.

As one can see, the cluster variables displayed in the previous two examples are indeed Laurent polynomials. In the following lemma we will prove that for any pair of positive integers b and c, the Laurent phenomenon holds for the cluster algebra $\mathcal{A}(b,c)$.

Lemma 2.1: Given a pair of positive integers (b,c), any cluster variable in the cluster algebra $\mathcal{A}(b,c)$ can be expressed as a Laurent polynomial in the cluster variables y_{m-1} and y_m, for any $m \in \mathbb{Z}$.

Proof. To prove this lemma, we first show that any cluster variable can be written as a polynomial in the previous or next four cluster variables. We show this for y_k, where $k \in \mathbb{Z}$ is odd (the other case is symmetric up to exchanging b and c):

$$\begin{aligned} y_k &= \frac{y_{k-1}^c + 1}{y_{k-2}} = \frac{(y_{k-2}^b + 1)^c + y_{k-3}^c}{y_{k-2}y_{k-3}^c} \\ &= \frac{(y_{k-3}^c + 1)(y_{k-2}^b + 1)^c - y_{k-3}^c((y_{k-2}^b + 1)^c - 1)}{y_{k-2}y_{k-3}^c} \\ &= \frac{(y_{k-3}^c + 1)}{y_{k-2}} \cdot \frac{(y_{k-2}^b + 1)^c}{y_{k-3}^c} - \frac{1}{y_{k-2}}\left(\prod_{i=1}^{c}\binom{c}{i} y_{k-2}^{bi}\right). \end{aligned}$$

Which gives us

$$y_k = y_{k-4}y_{k-1}^c - \prod_{i=1}^{c}\binom{c}{i} y_{k-2}^{bi-1}.$$

So we can indeed express y_k as a polynomial in the previous four cluster variables. In the same way we find that y_k is also expressible as a polynomial in the next four cluster variables:

$$y_k = y_{k+4}y_{k+1}^b - \prod_{i=1}^{b}\binom{b}{i} y_{k+2}^{ci-1}.$$

This means that we can express any cluster variable in $\mathcal{A}(b,c)$ as a polynomial in any four consecutive cluster variables in $\mathcal{A}(b,c)$. Combining this with the fact that any four consecutive cluster variables in $\mathcal{A}(b,c)$ can be written as Laurent polynomials of the two center cluster variables, which follows directly from the exchange relation (2.1), we can deduce that all cluster variables in $\mathcal{A}(b,c)$ as Laurent polynomials in any two consecutive cluster variables. □

3 Cluster algebras of rank n

Now that we have seen a specific case of cluster algebras of rank two, we are ready to give a general definition of a cluster algebra. In this section we will introduce a definition for cluster algebras of rank n, where n is a positive integer. To do this, we will start by defining an ambient field for our setting, then we will introduce some definitions, which we will clarify with some examples, to come to our definition of cluster algebra of rank n, and finally we will discuss and prove the Laurent phenomenon for such an cluster algebra.

3.1 Ambient field

To define the ambient field for our setting we repeat some crucial definitions and lemmas from the field of Abstract Algebra, just in case the reader needs to be reminded of them. If this is not the case, the reader may wish to skip to the final paragraph of this subsection.

We will be following the discussion in [2], Chapters 3 and 4, only discussing the definitions, lemmas and theorems which are used in this thesis. The proofs given in this subsection are directly adopted from [2].

Definition 3.1 ([2], 3.1.2 (4)): First of all, we start with the definition of an integral domain. A ring $\langle A, +, \cdot \rangle$, with identity 0 for '+', is called an *integral domain* if:

1. A contains an identity $1 \neq 0$, for '$\cdot$',
2. '$\cdot$' is commutative,
3. If the product of two non-zero elements of A cannot be equal to zero.

Let A be an integral domain. We now wish to construct a field containing A, which is as small as possible. This construction will generalize the relationship between $\mathbb{Z}$ and $\mathbb{Q}$.

Every rational number can be expressed as any one of infinitely many fractions, for example, the rational number $\frac{1}{2}$ can also be expressed as $\frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{35}{70}, \ldots$. The fraction $\frac{1}{2}$ is merely a way of denoting an ordered pair, in this case, $(2, 3)$. We therefore observe that each rational number is part of an infinite class of ordered pairs (a, b), $a \in \mathbb{Z}$, $b \neq 0$ in $\mathbb{Z}$, which all represent the same rational number. Two ordered pairs (a, b) and (c, d) represent the same rational number if and only if $ad = bc$.

Theorem 3.1 ([2], 3.8.1):

(1) Let A be an integral domain, and let X be the set of all ordered pairs (a, b), $a \in A$, $b \neq 0$ in A. (Thus, $X = A \times (A \setminus \{0\}$.) On X, define a binary relation $\sim$ by:

$$(a, b) \sim (c, d) \quad \text{if} \quad ad = cb \quad ((a, b), (c, d) \in X).$$

Then $\sim$ is an equivalence relation on X.

(2) Denote by $\frac{a}{b}$ the equivalence class of (a,b) ($(a,b) \in X$), and let $F = \{\frac{a}{b} \mid a \in A, b \neq 0 \text{ in } A\}$. On F, define '+', '·' by:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+cb}{bd},$$
$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad (a,c \in A, b \neq 0, d \neq 0 \text{ in } A).$$

Then $\langle F, +, \cdot \rangle$ is a field, which we will call *field of quotients of* A.

(3) Define $\phi : A \to F$ by: $\phi(a) = \frac{a}{1}$. Then ϕ is a ring monomorphism (ϕ is 1-1).

(4) If E is a field containing A as a subring, then E contains an isomorphic copy of F.

Proof. The proof of this theorem is precisely the same as given in [2], except for point 2 where we also show that $\langle F, +, \cdot \rangle$ is a commutative ring with multiplicative identity $\frac{1}{1}$, different from the 0-element $\frac{0}{1}$.

(1) $\sim$ is an equivalence relation on X, for: clearly, $(a,b) \sim (a,b)$, since $ab = ab$ (for each $a \in A$, $b \neq 0$ in A).

If $(a,b) \sim (c,d)$, then $ad = cb$; hence $cb = ad$, and so $(c,d) \sim (a,b)$ (for each $a,c \in A$, $b \neq 0$, $d \neq 0$ in A).

If $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$ ($a,c,e \in A$, $b \neq 0$, $d \neq 0$, $f \neq 0$ in A), then from $ad = cb$ and $cf = ed$, we have $(ad)f = (cb)f = (cf)b = (ed)b$; hence, $(af)d = (eb)d$, and so $af = eb$. But then $(a,b) \sim (e,f)$.

Hence $\sim$ is reflexive, symmetric and transitive, and is thus an equivalence relation on X.

(2) Denote by $\frac{a}{b}$ the equivalence class of (a,b) ($(a,b) \in X$), and let $F = \{\frac{a}{b} \mid a \in A, b \neq 0 \text{ in } A\}$. We need to show that the proposed binary operations '+', '·' are well-defined, i.e., every pair of elements of F has a unique sum and a unique product in F.

Let $(a,b),(c,d) \in X$. Then $(ad+cb, bd)$ and (ac, bd) are elements of X since $bd \neq 0$ if $b \neq 0$ and $d \neq 0$ in the integral domain A. Thus, for each $\frac{a}{b}, \frac{c}{d} \in F$ we have

$$\frac{ad+cb}{bd} \quad \text{and} \quad \frac{ac}{bd}$$

in F. Now suppose

$$\frac{a}{b} = \frac{a'}{b'}, \frac{c}{d} = \frac{c'}{d'}$$

($a, c, a', c' \in A, b, b', d, d'$ non-zero elements of A). Then

$$ab' = a'b \quad \text{and} \quad cd' = c'd.$$

Hence, $(ac)(b'd') = ab'cd' = a'bc'd = (a'c')bd$, and so

$$\frac{ac}{bd} = \frac{a'c'}{b'd'}.$$

Thus, the proposed operation '$\cdot$' is a (well-defined) binary operation on F. Furthermore, we have

$$\begin{aligned}(ad + cb)b'd' &= adb'd' + cbb'd' \\ &= a'bdd' + c'dbb' = (a'd' + c'b')bd,\end{aligned}$$

and so the proposed operation'+' ia a (well-defined) binary operation on F.

Next we verify that the '$\cdot$' operation is commutative in F. Given any $a, c \in A$, $b \neq 0$, $d \neq 0$ in A, we have that for each $e \in A$, $f \neq 0$ in A holds: If $(ac, bd) \sim (e, f)$, then $acf = ebd$, which means $caf = edb$, since '$\cdot$' is commutative in A. Therefore $(ca, db) \sim (e, f)$, so

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}.$$

This means that indeed the '$\cdot$' operation is commutative in F.

The 0-element from F must be $\frac{0}{1}$, since

$$\frac{0}{1} + \frac{a}{b} = \frac{0 \cdot b + a \cdot 1}{1 \cdot b} = \frac{a}{b} \quad (a, c \in A, b \neq 0, d \neq 0 \text{ in } A).$$

That multiplicative identity for F must be equal to $\frac{1}{1}$ follows clearly from the fact that 1 is the multiplicative identity for A:

$$\frac{1}{1} \cdot \frac{a}{b} = \frac{1 \cdot a}{1 \cdot b} = \frac{a}{b} \quad (a, c \in A, b \neq 0, d \neq 0 \text{ in } A).$$

Finally, since $0 \neq 1$ in A, it follows that $(0, 1) \sim (1, 1)$ does not hold, which means that $\frac{0}{1} \neq \frac{1}{1}$. We can conclude that $\langle F, +, \cdot \rangle$ is a commutative ring with multiplicative identity $\frac{1}{1}$, different from the 0-element $\frac{0}{1}$. Since for any non-zero element $\frac{a}{b}$ in F we have $\frac{a}{b} \neq \frac{0}{1}$, it follows that $a \neq 0$ and therefore $\frac{b}{a} \in F$, and therefore F is indeed a field.

(3) Define $\phi : A \to F$ by

$$\phi(a) = \frac{a}{1}$$

for each $a \in A$. Since, for $a, b \in A$, $\phi(a + b) = \frac{a+b}{1} = \frac{a}{1} + \frac{b}{1} = \phi(a) + \phi(b)$, and $\phi(a \cdot b) = \frac{a \cdot b}{1} = \frac{a}{1} \cdot \frac{b}{1} = \phi(a) \cdot \phi(b)$, it follows ϕ is a ring homomorphism. But $\phi(a) = \frac{0}{1} \Leftrightarrow \frac{a}{1} = \frac{0}{1} \Leftrightarrow a = 0$; hence ϕ is a monomorphism.

(4) Let E be a field containing A as a subring, and let $\bar{F} = \{ab^{-1} \mid a \in A, b \neq 0 \text{ in } A\}$. Define $\psi : F \to E$ by: $\psi(\frac{a}{b}) = ab^{-1}$ for each $a \in A$, $b \neq 0$ in

A. ψ is well-defined, for: if $\frac{a}{b} = \frac{a'}{b'}$ ($a, a' \in A$, b, b' non-zero in A), then $ab' = a'b$; hence $ab^{-1} = a'(b')^{-1}$. ψ is a homomorphism, for: if $a, c \in A$, b, d non-zero in A, then

$$\psi\left(\frac{a}{b} + \frac{c}{d}\right) = \psi\left(\frac{ad + cb}{bd}\right) = (ad + cb)(bd)^{-1} = (ad + cb)b^{-1}d^{-1}$$
$$= ab^{-1} + cd^{-1} = \psi\left(\frac{a}{b}\right) + \psi\left(\frac{c}{d}\right),$$

and

$$\psi\left(\frac{a}{b} \cdot \frac{c}{d}\right) = \psi\left(\frac{ac}{bd}\right) = ac(bd)^{-1} = acb^{-1}d^{-1} = (ab^{-1})(cd^{-1}) = \psi\left(\frac{a}{b}\right) \cdot \psi\left(\frac{c}{d}\right).$$

Finally, ψ is a monomorphism, for: if $\psi(\frac{a}{b}) = ab^{-1} = 0$, then $a = 0b = 0$; hence $\frac{a}{b} = \frac{0}{b} = \frac{0}{1}$. Thus $\operatorname{Im}\psi = \bar{F}$ is a field isomorphic to F.

□

We now will now look at some definitions and lemmas concerning fields in general and introduce the notions of algebraicity and transcendence. If a subset, E, of a field F is a field with respect to the operations of F restricted to E, then E is a *subfield* of F and F is an *extension field* (or simply an extension) of E.

Every subset S of a field F generates a subring and a subfield of F in the sense of the following definition:

Definition 3.2 ([2], 4.1.1): Let F be a field and let S be a subset of F. Then the *subring of F generated by S* is the intersection of all subrings of F which contain S; and the *subfield of F generated by S* is the intersection of all subfields of F which contain S.

The subring and subfield from the definition above have, in some cases, an interesting relation to one and other, which we display in the following lemma:

Lemma 3.2 ([2], p. 238): Let F be a field, S a subset of F, which contains the multiplicative identity 1_F of F. If A is the subring of F generated by S, and K is the subfield of F generated by S, then A is an integral domain and K is the field of quotients of A in F.

Proof. Since fields are rings, the intersection of all subrings of F, containing S, is a subset of the intersection of all subfields of F, containing S. Thus, $A \subset K$. Since $1_F \in S$, we have $1_F \in A$; hence A is an integral domain. Let

$$\bar{K} = \{ab^{-1} \mid a, b \neq 0 \text{ in } A\},$$

the field of quotients of A in F. Since $A \subset K$, clearly $\bar{K} \subset K$. But $\bar{K}$ is a subfield of F, containing S; hence $\bar{K}$ contains the intersection of all such subfields of F, i.e., $\bar{K} \supset K$. It follows that $\bar{K} = K$, and so K is the field of quotients of A in F. □

Consider now a field K, a subfield F and a subset Y of K. From the lemma above we can deduce that the subfield of K generated by $F \cup Y$ is the field of quotients, in K, of the subring of K generated by $F \cup Y$.

Notation ([2], p. 242): We denote by '$F[Y]$' the subring of K generated by $F \cup Y$, and by '$F(Y)$' the subfield of K generated by $F \cup Y$. In the special case where $Y = \{\alpha\}$ ($\alpha \in K$), we write '$F[\alpha]$' and '$F(\alpha)$' respectively, for the subring and subfield of K generated by $F \cup \{\alpha\}$.

$F[\alpha]$ and $F(\alpha)$ are called the *subring* and the *subfield of K, obtained by adjoining α to F.*

We finally introduce the notions of algebraicity and transcendence in the following definition:

Definition 3.3 ([2], 4.2.1, 4.4.1, 4.14.1 and 4.15.2):
Let F be a subfield of a field K.

- Given $\alpha \in K$, we call α *algebraic over F* if there is some non-zero polynomial $f \in F[x]$, such that $f(\alpha) = 0$; we call α *transcendental over F* if there is no non-zero polynomial $f \in F[x]$ such that $f(\alpha) = 0$.
- We say K is an *algebraic extension* of F if every element of K is algebraic over F. Otherwise, we call K an *transcendental extension* of F. (In what follows we will write K/F is algebraic or transcendental)
- For $n \geq 1$, let $\{\alpha_1, \ldots, \alpha_n\}$ be a subset of K. Then $\{\alpha_1, \ldots, \alpha_n\}$ is *algebraically dependent* over F if, for some non-zero polynomial $f \in F[x_1, \ldots, x_n]$, we have $f(\alpha_1, \ldots, \alpha_n) = 0$. If $f(\alpha_1, \ldots, \alpha_n) \neq 0$ for all non-zero polynomials $f \in F[x_1, \ldots, x_n]$, then $\{\alpha_1, \ldots, \alpha_n\}$ is *algebraically independent* over F. In this case, $\{\alpha_1, \ldots, \alpha_n\}$ forms a *transcendence set* over F.
- Given a subset S of K. If S is a transcendence set over F, such that $K/F(S)$ is algebraic, then S is a *transcendence basis* for K/F.

Now we are ready to define our ambient field. Let $\mathcal{P}$ be a torsion-free multiplicative abelian group of a finite rank m, with generators $g_1, \ldots, g_m$. (Torsion-free meaning that the multiplicative identity of $\mathcal{P}$ is the only element in $\mathcal{P}$ with finite order.) An ambient field of our setting is the field $\mathcal{F}$ of rational functions in n independent variables, which coefficients lie in $\mathbb{Z}(\mathcal{P})$, the field of quotients of the ring $\mathbb{Z}\mathcal{P} = \mathbb{Z}[\mathcal{P}]$. However, since $\mathcal{P}$, which we also will call the *coefficient group*, is torsion-free, we have that $\mathbb{Z}(\mathcal{P})$ is isomorphic to the field of rational functions in m independent variables with rational coefficients. So, if we take the n independent variables of $\mathcal{F}$ to be a transcendence basis over $\mathbb{Z}(\mathcal{P})$, we can think of $\mathcal{F}$ as the field of rational functions in $n+m$ independent variables with rational coefficients.

3.2 Definitions and examples

After defining an ambient field for our setting, we now continue with the definitions needed to come to the general definition of a cluster algebra of rank n.

Before we continue, some bibliographical notes are in order. In this subsection we follow the discussion in [3], Chapter 3, Section 1. Anything adopted from this discussion will be paired with a citation. (Some notations may differ from the notations used in [3].)

As we saw in the introduction, cluster algebras depend on an iterative process of seed mutations. A seed consists of a cluster and a matrix. The matrix is needed to define the iterative process of seed mutation and must satisfy some conditions. We therefore will start by introducing some properties of a square integer matrix.

Definition 3.4 ([3], 3.1): Let B be an $n \times n$ integer matrix. We say that B is

skew-symmetric if $b_{ij} = -b_{ji}$ for any $i, j \in [1..n]$ (here and in what follows, we denote with $[k..l]$, for integers k and l, the integer interval between k and l, including both k and l),

skew-symmetrizable if there exists a positive integer diagonal matrix D such that DB is skew-symmetric, in this case D is called the *skew-symmetrizer* of B, and B is called *D-skew-symmetrizable*.

sign-skew-symmetric if $\forall i, j \in [1..n]$ holds $b_{ij}b_{ji} < 0$ or $b_{ij} = b_{ji} = 0$.

To see what these properties look like, we look at a simple example.

Example 3.1: Lets look at the following matrices:

$$A = \begin{pmatrix} 0 & 6 & -10 \\ -6 & 0 & -12 \\ 10 & 12 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 6 & -10 \\ -1 & 0 & -2 \\ 5 & 6 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 6 & -10 \\ -1 & 0 & -2 \\ 10 & 2 & 0 \end{pmatrix}$$

Clearly matrix A is skew-symmetric. If we multiply matrix B with the matrix

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

we get A, so that means B is skew-symmetrizable. Finally, C is sign-skew-symmetric, since, for any pair $i, j \in [1..3]$ holds either $c_{ij}c_{ji} < 0$ or $c_{ij} = c_{ji} = 0$.

Remark 3.3: Looking at the matrices in the previous example, we can see that A is not only skew-symmetric but also skew-symmetrizable (using the identity matrix as skew-symmetrizer) and sign-skew-symmetric. Furthermore, we can see that B is also sign-skew-symmetric in addition to being skew-symmetrizable. In general holds that every skew-symmetric matrix is skew-symmetrizable, and that every skew-symmetrizable matrix is sign-skew-symmetric.

Definition 3.5 ([3], 3.2): A *seed* in $\mathcal{F}$ is a pair $\Sigma = (\mathbf{x}, \widetilde{B})$, where $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a transcendence basis of $\mathcal{F}$ over $\mathbb{Z}(\mathcal{P})$ (here $\mathcal{P}$ is the coefficient group defined at the end of Section 3.1), and $\widetilde{B}$ is an $n \times (n+m)$ integer matrix, whose principal part B (that is, the $n \times n$ submatrix formed by the columns $1, \ldots, n$) is a sign-skew-symmetric matrix.

Here $\mathbf{x}$ is the *cluster*, and its elements $x_1, \ldots, x_n$ are the *cluster variables*. We say that $\widetilde{\mathbf{x}} = \{x_1, \ldots, x_{n+m}\}$ is an *extended cluster*, where $x_{n+i} = g_i$, for $i \in [1, m]$ (remember, $g_1, \ldots, g_m$ are the generators of $\mathcal{P}$), which we will call *stable variables* since they will not be mutated like the cluster variables. The matrix B will be called the *exchange matrix*, and $\widetilde{B}$, the *extended exchange matrix*. We will denote the entries of $\widetilde{B}$ by b_{ij}, for $(i, j) \in [1..n] \times [1..n+m]$. Furthermore, we will say that $\widetilde{B}$ is skew-symmetric (skew-symmetrizable, sign-skew-symmetric) whenever B possesses this property.

Now that we have a definition for a seed in $\mathcal{F}$, we are going to define a way to mutate a seed. We will do this in three steps, first we will define a way to mutate a cluster, and after that we will define a way to mutate a matrix, to conclude with the definition for a seed mutation.

Definition 3.6 ([3], 3.3): Given a seed as defined above, the *adjacent cluster* to $\mathbf{x}$ in direction $k \in [1..n]$ (we will also call this the cluster achieved from mutating $\mathbf{x}$ in direction k) is defined by

$$\mathbf{x}^{(k)} = (\mathbf{x} \setminus \{x_k\}) \cup \{x_k^{(k)}\},$$

where the new cluster variable $x_k^{(k)}$ is defined by the *exchange relation*

$$x_k x_k^{(k)} = \prod_{\substack{1 \le i \le n+m \\ b_{ki} > 0}} x_i^{b_{ki}} + \prod_{\substack{1 \le i \le n+m \\ b_{ki} < 0}} x_i^{-b_{ki}}, \tag{3.1}$$

here the product over the empty set is, as usual, assumed to be equal to 1.

Example 3.2: Before we continue, let us look at a simple example to visualize the cluster mutation. Given a seed $\Sigma = (\mathbf{x}, \widetilde{B})$ with

$$\widetilde{B} = \begin{pmatrix} 0 & 2 & -2 & 2 & -2 & 2 & 2 \\ -2 & 0 & -1 & -1 & 1 & -3 & -2 \end{pmatrix}$$

and $\widetilde{\mathbf{x}} = (x_1, \ldots, x_7)$, we have

$$x_1^{(1)} = \frac{x_3^2 x_5^2 + x_2^2 x_4^2 x_6^2 x_7^2}{x_1} \text{ and } x_2^{(2)} = \frac{x_1^2 x_3 x_4 x_7^2 x_6^3 + x_5}{x_2},$$

Which are respectively the mutated cluster variables in the clusters $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$.

With the cluster mutation now out of the way, we can continue with the definition of the matrix mutation.

Definition 3.7 ([3], 3.5): Let $\widetilde{B}$ and $\widetilde{B}'$ be two $n \times (n+m)$ integer matrices. We say that $\widetilde{B}'$ is obtained from $\widetilde{B}$ by a *matrix mutation* in direction $k \in [1..n]$ and write $\widetilde{B}' = \widetilde{B}^{(k)}$ if

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise.} \end{cases}$$

If we look at $\widetilde{B}^{(k,k)}$ (which denotes the matrix $\widetilde{B}$ mutated twice in direction k), we find that $b_{ij}^{(k,k)}$ is equal to $-b_{ij}^{(k)} = b_{ij}$ if $i = k$ or $j = k$, otherwise we have:

$$\begin{aligned} b_{ij}^{(k,k)} &= b_{ij}^{(k)} + \frac{|b_{ik}^{(k)}|b_{kj}^{(k)} + b_{ik}^{(k)}|b_{kj}^{(k)}|}{2} \\ &= b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} + \frac{|-b_{ik}|(-b_{kj}) + (-b_{ik})|-b_{kj}|}{2} \\ &= b_{ij}. \end{aligned}$$

So we can conclude that $\widetilde{B}^{(k,k)} = \widetilde{B}$.

We say that two matrices $\widetilde{B}$ and $\widetilde{B}'$ are *mutation equivalent*, and write $B \simeq B'$, if each of them can be obtained from the other by a sequence of matrix mutations. That the property of a matrix to be sign-skew-symmetric is not necessarily preserved under matrix mutation equivalence can be seen from the following example: The matrices

$$\begin{pmatrix} 0 & -1 & 5 \\ 1 & 0 & -1 \\ -2 & 4 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & -5 \\ -1 & 0 & 4 \\ 2 & 2 & 0 \end{pmatrix}$$

are connected by a mutation in direction 1. However, where the first matrix is sign-skew-symmetric, the second is not. This leads to the following definition: a matrix is said to be *totally sign-skew-symmetric* if any matrix that is mutation equivalent to it is sign-skew-symmetric.

An important class of totally sign-skew-symmetric matrices is expressed by the following proposition.

Proposition 3.4 ([3], 3.6): Skew-symmetrizable matrices are totally sign-skew-symmetric.

Proof. Given a skew-symmetrizable $n \times n$ matrix B, by Definition 3.4, we can find a positive $n \times n$ integer diagonal matrix D such that $A = DB$ is skew-symmetric. Let $k \in [1..n]$, then, for $(i,j) \in [1..n] \times [1..n]$,

$$b_{ij}^{(k)} = \begin{cases} \frac{a_{ij}}{d_i}, & \text{if } i = k \text{ or } j = k, \\ \frac{a_{ij}}{d_i} + \frac{|a_{ik}|a_{kj} + a_{ik}|a_{kj}|}{2d_i d_k}, & \text{otherwise .} \end{cases}$$

To verify that $B^{(k)}$ is sign-skew-symmetric, we have to show that, either $b_{ij}^{(k)}$ and $b_{ji}^{(k)}$ are both equal to 0, or $b_{ij}^{(k)}$ and $b_{ji}^{(k)}$ have different signs.

If $i = k$ or $j = k$, it is evident that $b_{ij}^{(k)} = -b_{ji}^{(k)}$. For all other cases we have

$$b_{ij}^{(k)} = \frac{a_{ij}}{d_i} + \frac{|a_{ik}|a_{kj} + a_{ik}|a_{kj}|}{2d_i d_k}, \quad \text{and} \quad b_{ji}^{(k)} = \frac{a_{ji}}{d_j} + \frac{|a_{jk}|a_{ki} + a_{jk}|a_{ki}|}{2d_j d_k}.$$

Since A is skew-symmetric, we can write

$$\begin{aligned} b_{ji}^{(k)} &= \frac{-a_{ij}}{d_j} + \frac{|-a_{kj}|(-a_{ik}) + (-a_{kj})|-a_{ik}|}{2d_j d_k} \\ &= \frac{1}{d_j} \cdot \left(-a_{ij} - \frac{|a_{kj}|a_{ik} + a_{kj}|a_{ik}|}{2d_k} \right) = \frac{d_i}{d_j} \cdot \left(-b_{ij}^{(k)} \right). \end{aligned}$$

From this we can conclude that, either $b_{ij}^{(k)}$ and $b_{ji}^{(k)}$ are both equal to 0, or $b_{ij}^{(k)}$ and $b_{ji}^{(k)}$ have different signs. Hence, $B^{(k)}$ is sign-skew-symmetric.

□

Notation: For a given seed $\Sigma = (\mathbf{x}, \widetilde{B})$ in our setting, we will write the right hand side of the exchange relation (3.1) sometimes as P_k or $p_k^+ + p_k^-$. In the latter case, p_k^+ denotes the first and p_k^- the second product in (3.1).

Furthermore, for $l \in [1..n]$, we write this for the seed $\Sigma' = (\mathbf{x}^{(l)}, \widetilde{B}^{(l)})$ as:

$$P_k^{(l)} = (p_k^+)^{(l)} + (p_k^-)^{(l)} = \prod_{\substack{1 \le i \le n+m \\ b_{ki}^{(l)} > 0}} (x_i^{(l)})^{b_{ki}^{(l)}} + \prod_{\substack{1 \le i \le n+m \\ b_{ki}^{(l)} < 0}} (x_i^{(l)})^{-b_{ki}^{(l)}},$$

where $x_i^{(l)}$ denotes the i-th variable in the cluster adjacent to $\mathbf{x}$ in direction l, which means that $x_i^{(l)}$ is equal to x_i, if i is not equal to l.

Remark 3.5: Now that we have a simplified notation for the right half of exchange relation (3.1), we can now observe that this half has some interesting properties. Given a seed as defined in Definition 3.5, we have for some $k, l \in [1..n]$ that

$$(p_l^+)^{(k)} + (p_l^-)^{(k)} = p_l^- + p_l^+,$$

if k is equal to l. If this is not the case, we have

$$(p_l^+)^{(k)} + (p_l^-)^{(k)} = \begin{cases} p_l^+ + p_l^-, & \text{if } b_{lk} = 0, \\[2ex] \dfrac{p_l^+ (p_k^+)^{b_{lk}} + (p_k^+ + p_k^-)^{b_{lk}} p_l^-}{p_{l,k} x_k^{b_{lk}}}, & \text{if } b_{lk} > 0, \\[2ex] \dfrac{(p_k^+ + p_k^-)^{-b_{lk}} p_l^+ + p_l^- (p_k^-)^{-b_{lk}}}{p_{l,k} x_k^{-b_{lk}}}, & \text{otherwise.} \end{cases}$$

Where

$$p_{l,k} = \begin{cases} \prod_{i=1}^{n+m} x_i^{\min[[-b_{li}]^+, [b_{lk}b_{ki}]^+]}, & \text{if } b_{lk} \geq 0, \\ \prod_{i=1}^{n+m} x_i^{\min[[b_{li}]^+, [b_{lk}b_{ki}]^+]}, & \text{otherwise.} \end{cases}$$

(Here we use $[a]^+$, for $a \in \mathbb{Z}$, to denote $\max[0, a]$.)

The first case is obvious since row l of B remains the same if we mutate in direction k when $b_{lk} = -b_{lk}^{(k)} = 0$. The last two cases are symmetric, so we only consider the second case. In this case we have $b_{lk} > 0$, which means that for $j \in [1, n+m]$ we have

$$b_{lj} = \begin{cases} -b_{lj}, & \text{if } j = k, \\ b_{lj}, & \text{if } j \neq k \text{ and } b_{kj} < 0, \\ b_{lj} + b_{lk}b_{kj}, & \text{otherwise.} \end{cases}$$

This is equivalent to dividing p_l^+ by $x_k^{b_{lk}}$ and multiplying it with $(p_k^+)^{b_{lk}}$ divided by its greatest common divisor with p_l^-, and to multiplying p_l^- with $(P_k/x_k)^{b_{lk}}$ and dividing it by its greatest common divisor with $(p_k^+)^{b_{lk}}$, which is expressed by the expression $(p_l^+(p_k^+)^{b_{lk}} + (p_k^+ + p_k^-)^{b_{lk}} p_l^-)/p_{l,k} x_k^{b_{lk}}$.

Note that $p_{l,k}$ does not contain x_k or x_l. This follows directly from the definition of $p_{l,k}$, and from the fact that $b_{ll} = 0 = b_{kk}$ and $b_{kl}b_{lk} < 0$.

We are now ready to define a seed mutation.

Definition 3.8 ([3], p. 39): Given a seed $\Sigma = (\mathbf{x}, \widetilde{B})$, we say that, for $k \in [1..n]$, $\Sigma^{(k)} = (\mathbf{x}^{(k)}, \widetilde{B}^{(k)})$ is *adjacent* to (or achieved from mutating) Σ in direction k. Two seeds are *mutation equivalent* if they can be connected by a sequence of pairwise adjacent seeds.

Remark 3.6: Given a seed $\Sigma = (\mathbf{x}, \widetilde{B})$, where $\widetilde{B}$ is skew-symmetrizable, mutating Σ in the same given direction k ($k \in [1..n]$) twice, gives us Σ again.

We already know that $\widetilde{B}^{(k,k)} = \widetilde{B}$. And since $(p_k^+)^{(k)} + (p_k^-)^{(k)} = p_k^- + p_k^+$, we have $x_k^{(k,k)} = (p_k^- + p_k^+)/x_k^{(k)} = x_k$. Hence $\Sigma^{(k,k)} = (\mathbf{x}^{(k,k)}, \widetilde{B}^{(k,k)}) = \Sigma$.

With that done, we are ready to give a definition for a cluster algebra.

Definition 3.9 ([3], 3.8): Let $\Sigma = (\mathbf{x}, \widetilde{B})$ be a seed with a $n \times (n+m)$ skew-symmetrizable matrix $\widetilde{B}$, $\mathbb{A}$ be a subring with unity in $\mathbb{Z}\mathcal{P}$ containing all coefficients in all products $p_k^{\pm}$ for all seeds mutation equivalent to Σ. The *cluster algebra* $\mathcal{A} = \mathcal{A}(\widetilde{B})$ over $\mathbb{A}$ associated with Σ is the $\mathbb{A}$-subalgebra of $\mathcal{F}$ generated by all cluster variables in all seeds mutation equivalent to Σ. The number of rows n in $\widetilde{B}$ is said to be the rank of $\mathcal{A}$. The ring $\mathbb{A}$ is said to be the *ground ring* of $\mathcal{A}$. If $m = 0$, we call $\mathcal{A} = \mathcal{A}(B)$ a *coefficient-free* cluster algebra.

Let us now examine this definition by looking at some examples.

Example 3.3 ([3], 3.9): We first look at a cluster algebra of rank 1. Given a seed $\Sigma = (\mathbf{x}, \widetilde{B})$ in this cluster algebra, the matrix $\widetilde{B}$ is actually a vector $(b_1, \ldots, b_{m+1})$ with $b_1 = 0$. The cluster $\mathbf{x}$ contains just one cluster variable, namely x_1, with the corresponding exchange relation $x_1 x_1^{(1)} = p_1^+ + p_1^-$, here $p_1^\pm$ are monomials in the stable variables $x_2, \ldots, x_{m+1}$. Clearly there is only one seed that is mutation equivalent to Σ, which is $\Sigma^{(1)}$. So the whole cluster algebra is generated by the polynomials x_1 and $(p_1^+ + p_1^-)/x_1$. Hence any cluster algebra of rank 1 is generated by the cluster variables in the initial seed Σ and the only mutation equivalent seed $\Sigma^{(1)}$. This concludes a general description of cluster algebras of rank 1.

Next, we look at some examples of cluster algebras of rank 2.

Example 3.4: As one might remember, we gave an informal definition of a specific case of cluster algebras of rank 2 in Section 2. We said $\mathcal{A}(b, c)$ to be the cluster algebra of rank 2, associated with the pair of positive integers b and c, for which the cluster variables (which, as we mentioned in Section 2, form a chain $\ldots, y_{m-1}, y_m, y_{m+1}, \ldots$, for $m \in \mathbb{Z}$) satisfy the algebraic relation

$$y_{m-1} y_{m+1} = \begin{cases} y_m^b + 1, & \text{if m is even,} \\ y_m^c + 1, & \text{if m is odd.} \end{cases} \tag{3.2}$$

If we try to express $\mathcal{A}(b, c)$ following our general definition of a cluster algebra, we first need an initial seed $\Sigma = (\mathbf{x}, \widetilde{B})$. Since each cluster in this cluster algebra contains only two cluster variables, we can take $\mathbf{x}$ to be equal to $\{y_1, y_2\}$. Furthermore, as we can see in the relation above no stable variables are used in the relations between consecutive cluster variables in $\mathcal{A}(b, c)$, hence $\mathcal{A}(b, c)$ must be coefficient-free. This means we must find a 2×2 skew-symmetrizable exchange matrix B for our initial seed Σ, such that $\mathcal{A}(B)$ is equal to $\mathcal{A}(b, c)$. Take B to be the matrix

$$\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}.$$

Mutating B in either direction 1 or 2 gives us $-B$, which again mutated in either direction gives us B again. So any seed, which is mutation equivalent to Σ has either B or $-B$ as its exchange matrix. This means we can express the exchange relation obtained from mutating a cluster $\{x_1, x_2\}$ in direction 1 as

$$x_1 x_1^{(1)} = x_2^b + 1,$$

and for mutating in direction 2 we can express the resulting exchange relation as

$$x_2 x_2^{(2)} = x_1^c + 1.$$

So, if we have a cluster $\mathbf{x}$ of the form $\{y_{k-1}, y_k\}$, for $k \in 2\mathbb{Z}$, we get adjacent clusters $\{y_{k+1}, y_k\}$ and $\{y_{k-1}, y_{k-2}\}$, from mutating $\mathbf{x}$ in directions 1 and 2, respectively. If we take $\mathbf{x}$ to be of the form $\{y_{k+1}, y_k\}$, for $k \in 2\mathbb{Z}$, we get

$\mathbf{x}^{(1)} = \{y_{k-1}, y_k\}$ and $\mathbf{x}^{(2)} = \{y_{k+1}, y_{k+2}\}$. From this we can conclude that, if we take our initial seed to be

$$\left(\{y_1, y_2\}, \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}\right),$$

we get the following chain of clusters from mutating in alternating directions (for $k \in \mathbb{Z}$):

$$\ldots \leftrightarrow^2 \{y_{k-1}, y_k\} \leftrightarrow^1 \{y_{k+1}, y_k\} \leftrightarrow^2 \{y_{k+1}, y_{k+2}\} \leftrightarrow^1 \ldots$$

Here the arrow between clusters denote that the two clusters are connected with a mutation in the direction of the number above the arrow. From this chain of clusters we can deduce that the cluster variables in $\mathcal{A}(B)$ also form a chain equal to $\ldots, y_{m-1}, y_m, y_{m+1}, \ldots$ (for $m \in \mathbb{Z}$), and therefore, also satisfy the algebraic relation (3.2). This means we can conclude that $\mathcal{A}(B)$ is equal to $\mathcal{A}(b, c)$.

Now we look at some specific examples of cluster algebras of rank 2. Where a cluster algebra of rank 1 is always generated by a finite number of cluster variables (namely 2), this is not necessarily the case for a cluster algebra of rank 2 (as we already conjectured in Section 2). This follows from the fact that given an initial seed Σ, we can mutate infinitely in alternating directions without necessarily ever returning to Σ. We will explore this further in the following examples.

Example 3.5: Given initial seed $\Sigma = (\mathbf{x}, \widetilde{B})$ with

$$\widetilde{B} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 & -1 & 1 & 1 \end{pmatrix},$$

mutating Σ in alternating directions starting with direction 1, gives us the following clusters

$$\begin{aligned}
\mathbf{x} &= \{x_1, x_2\}, \\
\mathbf{x}^{(1)} &= \left\{\frac{x_2x_3x_4 + x_6x_7}{x_1}, x_2\right\}, \\
\mathbf{x}^{(1,2)} &= \left\{\frac{x_2x_3x_4 + x_6x_7}{x_1}, \frac{x_2x_4x_3^2 + x_6x_7x_3 + x_1x_5}{x_1x_2}\right\}, \\
\mathbf{x}^{(1,2,1)} &= \left\{\frac{x_1x_5 + x_3x_6x_7}{x_2}, \frac{x_2x_4x_3^2 + x_6x_7x_3 + x_1x_5}{x_1x_2}\right\}, \\
\mathbf{x}^{(1,2,1,2)} &= \left\{\frac{x_1x_5 + x_3x_6x_7}{x_2}, x_1\right\}, \\
\mathbf{x}^{(1,2,1,2,1)} &= \{x_2, x_1\},
\end{aligned}$$

with corresponding extended matrices

$$\widetilde{B} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 & -1 & 1 & 1 \end{pmatrix},$$

$$\widetilde{B}^{(1)} = \begin{pmatrix} 0 & -1 & -1 & -1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & -1 & 0 & 0 \end{pmatrix},$$
$$\widetilde{B}^{(1,2)} = \begin{pmatrix} 0 & 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 \end{pmatrix},$$
$$\widetilde{B}^{(1,2,1)} = \begin{pmatrix} 0 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 0 & -2 & -1 & 0 & 0 & 0 \end{pmatrix},$$
$$\widetilde{B}^{(1,2,1,2)} = \begin{pmatrix} 0 & 1 & -1 & 0 & 1 & -1 & -1 \\ -1 & 0 & 2 & 1 & 0 & 0 & 0 \end{pmatrix},$$
$$\widetilde{B}^{(1,2,1,2,1)} = \begin{pmatrix} 0 & -1 & 1 & 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & -1 & -1 \end{pmatrix}.$$

Notice that the the first and last cluster contain the same variables (although they switched places), so these clusters are equivalent. Furthermore, notice that the exchange relations formed by the matrices corresponding to these variables also are the same, which means that the matrices are also equivalent. Therefore, the first and last seed are equivalent, and hence the number of non-equivalent seeds is in this case equal to 5.

Now that we have seen an example of a cluster algebra of rank 2 with a finite number of non-equivalent seeds, we now will examine a case with an infinite number of non-equivalent seeds.

Example 3.6: Let us start with the seed $\Sigma = (\mathbf{x}, \widetilde{B})$, where

$$\widetilde{B} = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 0 \end{pmatrix}.$$

If we mutate $\widetilde{B}$ in alternating directions starting with direction 1 we get the following matrices

$$\widetilde{B} = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 0 \end{pmatrix}, \qquad \widetilde{B}^{(1)} = \begin{pmatrix} 0 & -2 & 2 \\ 2 & 0 & -4 \end{pmatrix},$$
$$\widetilde{B}^{(1,2)} = \begin{pmatrix} 0 & 2 & -6 \\ -2 & 0 & 4 \end{pmatrix}, \qquad \widetilde{B}^{(1,2,1)} = \begin{pmatrix} 0 & -2 & 6 \\ 2 & 0 & -8 \end{pmatrix},$$
$$\widetilde{B}^{(1,2,1,2)} = \begin{pmatrix} 0 & 2 & -10 \\ -2 & 0 & 8 \end{pmatrix}, \ldots$$

Which gives us a chain of matrices $\widetilde{B}_0, \widetilde{B}_1, \ldots$, where $\widetilde{B}_n$, for n a non-negative integer, denotes the matrix obtained from mutating $\widetilde{B}$ in alternating directions, starting with direction 1, for a total of n mutations. Note that if n is odd, the last mutation was in direction 1 and if n is even, the last mutation, given $n > 0$, was in direction 2, hence the number of times we mutated in direction 1 is given by $\lceil n/2 \rceil$ and the number of times we mutated in direction 2 is given by $\lfloor n/2 \rfloor$. We now can deduce that $\widetilde{B}_n$ satisfies the following relation:

$$\widetilde{B}_n = (-1)^n \begin{pmatrix} 0 & 2 & -2 - 4 \cdot \lfloor n/2 \rfloor \\ -2 & 0 & 4 \cdot \lceil n/2 \rceil \end{pmatrix}$$

This can be easily proven using induction. Since $\widetilde{B}_0 = \widetilde{B}$, the relation holds for $n = 0$. Now we assume the relation holds for $n = k$, for k an positive integer. If k is odd, it means that the next matrix in the chain is obtained from mutating $\widetilde{B}_k$ in direction 2, which gives us:

$$\begin{aligned}\widetilde{B}_k^{(2)} &= \begin{pmatrix} 0 & -2 & 2+4\cdot\lfloor k/2\rfloor \\ 2 & 0 & -4\cdot\lceil k/2\rceil \end{pmatrix}^{(2)} \\ &= \begin{pmatrix} 0 & 2 & 2+4\cdot\lfloor k/2\rfloor - 2\cdot 4\cdot\lceil k/2\rceil \\ -2 & 0 & 4\cdot\lceil k/2\rceil \end{pmatrix}.\end{aligned}$$

Since k is odd, $\lfloor k/2\rfloor = \lfloor (k+1)/2\rfloor - 1$ and $\lceil k/2\rceil = \lceil (k+1)/2\rceil = \lfloor (k+1)/2\rfloor$. This gives us:

$$\begin{aligned}\widetilde{B}_k^{(2)} &= \begin{pmatrix} 0 & 2 & 2+4\cdot(\lfloor (k+1)/2\rfloor - 1) - 2\cdot 4\cdot\lfloor (k+1)/2\rfloor \\ -2 & 0 & 4\cdot\lceil (k+1)/2\rceil \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 & -2-4\cdot\lfloor (k+1)/2\rfloor \\ -2 & 0 & 4\cdot\lceil (k+1)/2\rceil \end{pmatrix} = \widetilde{B}_{k+1}.\end{aligned}$$

In a similar way, we find for k is even, that the next matrix in the chain, which is obtained from mutating $\widetilde{B}_k$ in direction 1, is equal to $\widetilde{B}_{k+1}$. Therefore, the relation holds, which means that this chain of matrix mutations will never reach the same matrix twice. Since there are an infinite number of non-equivalent matrices which are mutation equivalent to $\widetilde{B}$, this means that there are also an infinite number of non-equivalent seeds which are mutation equivalent to Σ.

It turns out that for cluster algebras with rank greater than 1, the number of non-equivalent seeds is generally infinite, however, as we will prove in the next section, every cluster algebra is generated by a finite number of cluster variables.

3.3 Laurent phenomenon

In this subsection we will prove the Laurent phenomenon, to do this, we derived this proof from the proof given in [3], Chapter 3, Section 1.

Theorem 3.7 ([3], 3.14): Given a cluster algebra $\mathcal{A}$ of rank $n \geq 1$ with $m \geq 0$ stable variables, any cluster variable in $\mathcal{A}$ can be expressed via the cluster variables from the initial (or any other) cluster as a Laurent polynomial with coefficients in $\mathbb{Z}\mathcal{P} = \mathbb{Z}[\mathcal{P}]$. (Here $\mathcal{P}$ is the coefficient group defined at the end of Section 3.1, which is the torsion-free multiplicative abelian group generated by the stable variables $x_{n+1}, \ldots, x_{x_n+m}$.)

To prove this theorem we first introduce a number of lemmas. We will look at a cluster algebra of rank $n > 1$. (The proof for a cluster algebra of rank 1 follows directly from our observations in Example 3.3.)

Given a seed $\Sigma = (\mathbf{x}, \widetilde{B})$ in this cluster algebra, let us look at the ring of Laurent polynomials in the variables $x_1, \ldots, x_{n+m}$ with coefficients in $\mathbb{Z}\mathcal{P}$. The theorem claims that the whole cluster algebra is contained in the intersection of such rings over all seeds mutation equivalent to Σ. We will prove that it is already contained in the polynomial ring

$$\mathcal{U}(\mathbf{x}) = \mathbb{Z}\mathcal{P}[\mathbf{x}^{\pm 1}] \cap \mathbb{Z}\mathcal{P}[(\mathbf{x}^{(1)})^{\pm 1}] \cap \cdots \cap \mathbb{Z}\mathcal{P}[(\mathbf{x}^{(n)})^{\pm 1}],$$

called the *upper bound* associated with the cluster $\mathbf{x}$. (Here and in what follows, we will denote by $\mathbf{x}^{\pm 1}$ and $x^{\pm 1}$ the sets $\{x_1^{\pm 1}, \ldots x_n^{\pm 1}\}$ and $\{x, x^{-1}\}$, respectively.)

To prove that the cluster algebra is contained in the polynomial ring $\mathcal{U}(\mathbf{x})$, we just have to prove that $\mathcal{U}(\mathbf{x}) = \mathcal{U}(\mathbf{x}^{(k)})$ for some $k \in [1..n]$.

Lemma 3.8 ([3], 3.15): For some $k \in [1..n]$, we have

$$\mathcal{Q}_k[x_k^{\pm 1}] \cap \mathcal{Q}_k[(x_k^{(k)})^{\pm 1}] = \mathcal{Q}_k[x_k, x_k^{(k)}],$$

where $\mathcal{Q}_k = \mathbb{Z}\mathcal{P}[\mathbf{x}^{\pm 1} \setminus \{x_k^{\pm 1}\}]$.

Proof. We can rewrite the left-hand side of the equation as

$$(\mathcal{Q}_k[x_k, x_k^{-1}] \cap \mathcal{Q}_k[x_k/(p_k^+ + p_k^-)])[(p_k^+ + p_k^-)/x_k].$$

Since for any $f \in \mathcal{Q}_k[x_k/(p_k^+ + p_k^-)]$ holds that f is either an element of $\mathcal{Q}_k[x_k]$ or f can not be expressed as a Laurent polynomial in the variables $x_1, \ldots, x_{n+m}$, it follows that $\mathcal{Q}_k[x_k, x_k^{-1}] \cap \mathcal{Q}_k[x_k/(p_k^+ + p_k^-)] = \mathcal{Q}_k[x_k]$. We therefore can rewrite the expression above as $\mathcal{Q}_k[x_k][(p_k^+ + p_k^-)/x_k]$, which is equal to $\mathcal{Q}_k[x_k, x_k^{(k)}]$, and that is exactly what we needed to prove. □

In what follows, let $\mathcal{Q}_{k,l} = \mathbb{Z}\mathcal{P}[\mathbf{x}^{\pm 1} \setminus \{x_k^{\pm 1}, x_l^{\pm 1}\}]$, for some $k, l \in [1..n]$, with $k \neq l$. We will now prove that the following holds:

Lemma 3.9: For some $k, l \in [1, n]$, $k \neq l$, we have

$$\mathbb{Z}\mathcal{P}[\mathbf{x}^{\pm 1}] \cap \mathcal{Q}_{k,l}[x_k^{\pm 1}, (x_l^{(l)})^{\pm 1}] \cap \mathcal{Q}_{k,l}[(x_k^{(k)})^{\pm 1}, x_l^{\pm 1}] = \mathcal{Q}_{k,l}[x_k, x_k^{(k)}, x_l, x_l^{(l)}].$$

Proof. Note that by using Lemma 3.8 we can rewrite the left-hand side of the equation as:

$$\mathcal{Q}_{k,l}[x_k^{\pm 1}, x_l, x_l^{(l)}] \cap \mathcal{Q}_{k,l}[x_k, x_k^{(k)}, x_l^{\pm 1}],$$

which is equivalent to

$$(\mathcal{Q}_{k,l}[x_k^{-1}] \cap \mathcal{Q}_{k,l}[x_l^{-1}])[x_k, x_k^{(k)}, x_l, x_l^{(l)}].$$

Since x_k^{-1} and x_l^{-1} are not elements of $\mathcal{Q}_{k,l}$, we can conclude that the expression above is equal to $\mathcal{Q}_{k,l}[x_k, x_k^{(k)}, x_l, x_l^{(l)}]$, which is what we needed to prove. □

The last equality we need to prove before we can finish the proof of the Laurent phenomenon, is the following:

Lemma 3.10 ([3], 3.17): For some $k, l \in [1..n]$, $k \neq l$, we have

$$\mathcal{Q}_{k,l}[x_k, x_k^{(k)}, x_l, x_l^{(l)}] = \mathcal{Q}_{k,l}[x_k, x_k^{(k)}, x_l, x_l^{(k,l)}]$$

Proof. To prove this, it is enough to show that $x_l^{(k,l)} \in \mathcal{Q}_{k,l}[x_k, x_k^{(k)}, x_l, x_l^{(l)}]$.

First of all, let us look at the case that $b_{kl} = 0 = b_{lk}$, then $P_l = P_l^{(k)}$ by Remark 3.5. This means that $x_l^{(k,l)} = x_l^{(l)}$, which implies that indeed $x_l^{(k,l)} \in \mathcal{Q}_{k,l}[x_k, x_k^{(k)}, x_l, x_l^{(l)}]$.

Now assume that $b_{kl}, b_{lk} \neq 0$. Without loss of generality, we may assume that $b_{lk} > 0$ (the other case is symmetric to this one). By Remark 3.5, we have

$$\begin{aligned} x_l^{(k,l)} &= \frac{p_l^+(p_k^+)^{b_{lk}} + (p_k^+ + p_k^-)^{b_{lk}} p_l^-}{p_{l,k} x_k^{b_{lk}} x_l} \\ &= \left(\frac{p_l^+ + p_l^-}{x_l} \frac{(p_k^+ + p_k^-)^{b_{lk}}}{x_k^{b_{lk}}} - \frac{p_l^+((p_k^+ + p_k^-)^{b_{lk}} - (p_k^+)^{b_{lk}})}{x_k^{b_{lk}} x_l} \right) \frac{1}{p_{l,k}} \\ &= \left(x_l^{(l)} (x_k^{(k)})^{b_{lk}} - \frac{p_l^+}{x_k^{b_{lk}}} \frac{p_k^-}{x_l} \prod_{i=1}^{b_{lk}} \binom{b_{lk}}{i} (p_k^+)^{b_{lk}-i} (p_k^-)^{i-1} \right) \frac{1}{p_{l,k}}. \end{aligned}$$

That this last expression belongs to $\mathcal{Q}_{k,l}[x_k, x_k^{(k)}, x_l, x_l^{(l)}]$, follows from the fact that p_l^+ contains $x_k^{b_{lk}}$, p_k^- contains $x_l^{-b_{lk}}$, and $p_{k,l}$ does not contain x_k or x_l (by Remark 3.5). Therefore, $x_l^{(k,l)}$ must be an element of $\mathcal{Q}_{k,l}[x_k, x_k^{(k)}, x_l, x_l^{(l)}]$, which is exactly what we needed to prove. ☐

Now we are ready to prove Theorem 3.7.

Proof. Consider the upper bound associated with cluster $\mathbf{x}$:

$$\begin{aligned} \mathcal{U}(\mathbf{x}) &= \mathbb{Z}\mathcal{P}[\mathbf{x}^{\pm 1}] \cap \mathbb{Z}\mathcal{P}[(\mathbf{x}^{(1)})^{\pm 1}] \cap \cdots \cap \mathbb{Z}\mathcal{P}[(\mathbf{x}^{(n)})^{\pm 1}] \\ &= \bigcap_{i=1}^{n} (\mathbb{Z}\mathcal{P}[\mathbf{x}^{\pm 1}] \cap \mathbb{Z}\mathcal{P}[(\mathbf{x}^{(i)})^{\pm 1}]). \end{aligned}$$

For some $k \in [1..n]$ we can write this intersection as:

$$\bigcap_{\substack{i=1 \\ i \neq k}}^{n} (\mathbb{Z}\mathcal{P}[\mathbf{x}^{\pm 1}] \cap \mathbb{Z}\mathcal{P}[(x_k^{(k)})^{\pm 1}, \mathbf{x}^{\pm 1} \setminus \{x_k^{\pm 1}\}] \cap \mathbb{Z}\mathcal{P}[(x_i^{(i)})^{\pm 1}, \mathbf{x}^{\pm 1} \setminus \{x_i^{\pm 1}\}]).$$

Applying Lemma 3.9 to this expression, gives us:

$$\bigcap_{\substack{i=1 \\ i \neq k}}^{n} \mathbb{Z}\mathcal{P}[x_k, x_k^{(k)}, x_i, x_i^{(i)}, (\mathbf{x} \setminus \{x_k, x_i\})^{\pm 1}]$$

With use of Lemma 3.10, we can now find:

$$\bigcap_{\substack{i=1\\i\neq k}}^{n} \mathbb{Z}\mathcal{P}[x_k, x_k^{(k)}, x_i, x_i^{(k,i)}, (\mathbf{x} \setminus \{x_k, x_i\})^{\pm 1}].$$

Since $\mathbf{x} \setminus \{x_k, x_i\} = \mathbf{x}^{(k)} \setminus \{x_k^{(k)}, x_i^{(k)}\}$, and $x_k = x_k^{(k,k)}$, we can write

$$\bigcap_{\substack{i=1\\i\neq k}}^{n} \mathbb{Z}\mathcal{P}[x_k^{(k)}, x_k^{(k,k)}, x_i^{(k)}, x_i^{(k,i)}, (\mathbf{x}^{(k)} \setminus \{x_k^{(k)}, x_i^{(k)}\})^{\pm 1}].$$

Which can be rewritten, by reversing our steps from the point at which we applied Lemma 3.9, to the following:

$$\bigcap_{i=1}^{n} (\mathbb{Z}\mathcal{P}[(\mathbf{x}^{(k)})^{\pm 1}] \cap \mathbb{Z}\mathcal{P}[(\mathbf{x}^{(k,i)})^{\pm 1}]) = \mathcal{U}(\mathbf{x}^{(k)}).$$

As we saw earlier in this section, this is precisely what we needed to prove.

□

So now we have proved that cluster variables can be expressed as Laurent polynomials, is there more to say about the structure of these Laurent polynomials? Can we for instance give a general expression for them? Sadly, no. However, there are some conjectures that seem to be true, but which are not yet proven. For instance the conjecture that the coefficients of these Laurent polynomials are positive. This would explain how the cluster variables are structured and shed more light on the concept of cluster algebras as a whole, however, as I already mentioned, no one has of yet proven this conjecture.

References

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