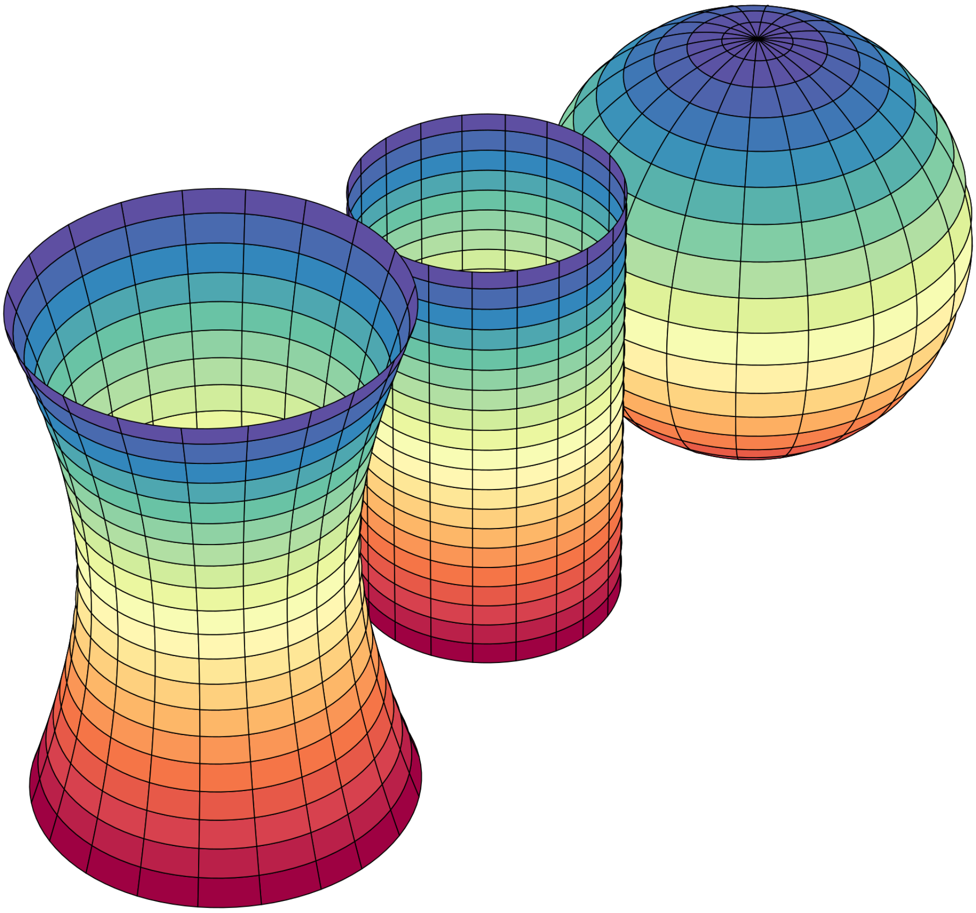
A Characterization of Flat Pseudo-Riemannian Manifolds

Jens de Vries

A thesis submitted to the Department of Mathematics at Utrecht University
in partial fulfillment of the requirements for the degree of:

Bachelor in Mathematics



Supervisor: dr. F. J. Ziltener

June, 2019

Utrecht University

Abstract

This thesis is concerned with the curvature of pseudo-Riemannian manifolds. A pseudo-Riemannian manifold is called flat when it can be covered by charts that intertwine the pseudo-metrics of the manifold and psuedo-Euclidian space. In general, the curvature of a manifold is described by an operator κ_∇, called the Riemann curvature. In the main theorem of this thesis, we will prove that a pseudo-Riemannian manifold is flat if and only if κ_∇ vanishes identically.

Acknowledgements

First and foremost I would like to Thank to my supervisor dr. F. J. Ziltener for his advice and support. He was always available for questions and discussion. I would also like to thank my friend A. J. van Aarssen for his help with some of the pictures. All the pictures without a reference to a source in the caption are created by him.

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1 Introduction

1.1 Main Result and Motivation

Compare the country Greenland and the continent Africa on a chart in an atlas and on a globe.

The Problem with our Maps [1]

On a chart in the atlas, Greenland has roughly the same size as Africa. On the globe, Greenland is much smaller in comparison to Africa. In general, one may wonder why such representations (charts in an atlas) are not scaled to reality (a globe):

Question 1: Why are (local) representations of the Earth's surface in an atlas distorted?

The answer is that it is impossible to cover the surface of the Earth (modeled as a Riemannian manifold) with charts that intertwine the metrics of the Earth's surface and the Euclidian metric on open subsets of the Euclidian plane.

To explain this, we need the theory of Riemannian geometry. A Riemannian manifold is a manifold M together with a choice of innerproduct g_p on each tangent space T_pM that varies smoothly with respect to $p \in M$. Such a collection of innerproducts is called a metric. A metric allows us to measure geometrical quantities such as angles, distances and (hyper)volumes on the manifold.

Let us model the globe as the manifold S^2. A chart in the atlas can then be viewed as a chart

$$\chi\colon U \to \widetilde{U}, \quad \chi(p) := (\chi^1(p), \dots, \chi^m(p))$$

of S^2 in the mathematical sense, i.e. a diffeomorphism. Here $U \subset S^2$ and $\widetilde{U} \subset \mathbb{R}^2$ are open subsets. To answer Question 1, we want to know when charts preserve geometrical quantities.

This introduces the main concept of flatness: charts that intertwine metrics. Our intuition tells us that (an open subset of) the Euclidian space $\mathbb{R}^m$ equipped with the Euclidian metric is a flat space. Now, one may wonder when a given Riemannian m-dimensional manifold M, such as the Earth's surface, is flat. Naively, one could ask the following question:

Question 2: Does there exist an open subset $\widetilde{U} \subset \mathbb{R}^m$ and a diffeomorphism $\chi\colon M \to \widetilde{U}$ (i.e. a global chart) that intertwines the metric of M and the Euclidian metric on $\widetilde{U}$?

If the answer is 'yes', it means that an 'm-dimensional being' is not able to distinquish between M and $\widetilde{U}$ by global and local measurements. However, the answer to this question is in general not very interesting, since often the answer is 'no' for a topological reason. As an example, no compact manifold [1] admits a global chart. In such cases the metric plays no role. A better question is the following:

Question 3: Can we cover M by local charts that intertwine the metric of M and the Euclidian metric?

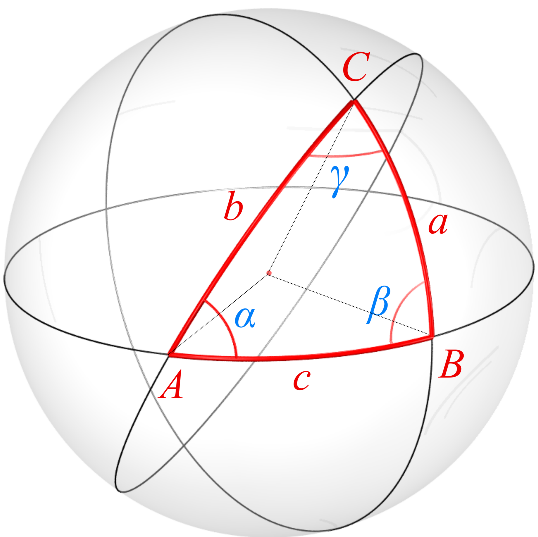
We call M flat if and only if the answer is 'yes'. It means that an 'm-dimensional being' can not distinguish between the manifold M and Euclidian space $\mathbb{R}^m$ by local measurements. To clarify the discussion above, we give two examples.

Suppose that we have a sheet of paper. Now roll the paper into a cylinder. Locally, this construction preserves distances, areas and angles on the surfaces. So the metric on the cylinder is locally preserved: around every point in the cylinder we can find a local chart that preserves the metric of the cylinder. However, the cylinder and sheet of paper are not homeomorphic, i.e. there exists no global chart. Hence, for the 2-dimensional cylinder the answer to Question 2 is 'no' and the answer to Question 3 is 'yes'.

Now consider a 2-dimensional sphere. The sphere is compact and hence we already know that the answer to Question 2 is 'no'. The answer to Question 3 is more subtle, but turns out to be 'no'. To verify this, we must show that there exists a point on the sphere such that every chart around that point distorts the properties of the metric. Suppose that there exists a chart $\chi\colon U \to \widetilde{U}$ of S^2 that intertwines the metrics of S^2 and $\widetilde{U}$. Let $\Delta \subset U$ be a spherical triangle. Suppose that the angles of Δ are given by α, β and γ. One can show (see [5]) that

$$\alpha + \beta + \gamma = \text{area}(\Delta) + \pi.$$

Since $\text{area}(\Delta) > 0$, it follows that the sum of the angles of a spherical triangle are strictly greater than π.



Spherical Triangle [2]

Since χ preserves angles and distances, it follows that $\chi_*\Delta \subset \widetilde{U}$ must be a planar triangle with the same angles and sides as Δ. But it is a well-known fact that the angles of a planar

[1] Throughout this thesis, all manifolds are assumed to be manifolds without boundary.

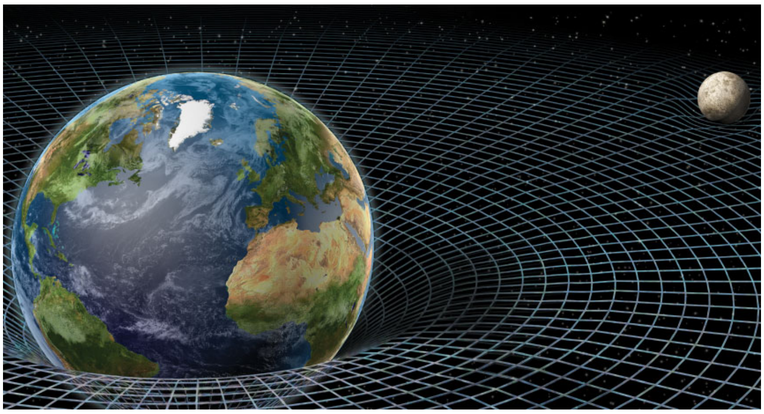
triangle precisely add up to π. Hence we find a contradiction. It therefore follows that χ can not exist.

More generally, the following question arises:

> **Question 4:** Does there exist an invariant that distinguishes flat Riemannian manifolds from curved ones?

The answer is 'yes' and the invariant is called Riemann curvature [2]. However, to understand the definition of Riemann curvature, we must first introduce a lot of technical machinery.

Moreover, as the title of this thesis suggests, we will study a more generalized form of Riemannian manifolds: pseudo-Riemannian manifolds. They play an essential role in Einstein's theory of relativity. Here, gravity is described by the Riemann curvature of such pseudo-Riemannian manifolds.



Gravity Modeled as the Curvature of Spacetime [3]

The precise answer to Question 4 is contained in the following theorem:

> **Main Theorem:** A pseudo-Riemannian manifold is flat if and only if its Riemann curvature vanishes identically.

The theorem also says that the Riemann curvature is sufficient for flatness, i.e. it is precisely the obstruction to being flat. In other words: we find a characterization of flat pseudo-Riemannian manifolds.

1.2 Organization

In Chapter 2 we discuss the notion of vector bundles. We also introduce the sections of a vector bundle. The tangent bundle is an example of a vector bundle and vector fields are the sections of the tangent bundle. We then study connections on vector bundles. A connection is a choice of additional structure which can be added to a vector bundle.

In Chapter 3 we give the precise definition of pseudo-Riemannian manifolds. We also introduce the Levi-Civita connection, which is a connection on the tangent bundle that is completely determined by the pseudo-metric on a pseudo-Riemannian manifold.

A connection on a vector bundle induces the covariant derivative operator along a curve on the manifold. In Chapter 4 we introduce and use this operator to study how a vector in the vector bundle changes when we move it along a curve in the base space. In particular, it allows us to define the notion of parallel transport. In Chapter 4 we also study the curvature of vector bundles, which is an operator defined in terms of the connection the vector bundle is

[2]There are different kinds of curvature (e.g. Ricci curvature, scalar curvature, Gaussian curvature and sectional curvature), but they can all be expressed in terms of the Riemann curvature (see [10]).

equipped with. The Riemann curvature of a manifold is then defined as the curvature of its tangent bundle.

Finally, in Chapter 5 we define flatness of pseudo-Riemannian manifolds and we prove the main theorem.

1.3 Preknowledge

This thesis requires basic knowledge of calculus, topology and manifolds. In particular the reader should be familiar with the definitions of smooth maps between manifolds, (co)tangent spaces and tensors.

2 Vector Bundles with Connection

2.1 Vector Bundles

A vector bundle is a special kind of manifold which plays an important role throughout this thesis. Intuively, a vector bundle is a manifold that is made of vector spaces which are 'smoothly glued together'. Let M be a manifold. A *vector bundle of rank n over M* is a manifold E together with a smooth surjective map $\pi\colon E \to M$ (called the *bundle projection*) that maps onto M (called the *base space*) such that the following conditions are satisfied:

- For each $p \in M$ the fibre $\pi^*\{p\}$ is endowed with a (real) n-dimensional vector space structure.

- Around every point in M there exists an open neighbourhood U together with a diffeomorphism $\Phi\colon \pi^*U \to U \times \mathbb{R}^n$ (called a *local trivialization*) which satisfies the following properties:

  - Let π_U be the map π but with its domain and codomain restricted to π^*U and U respectively. Then the following diagram commutes:

$$\begin{tikzcd} \pi^*U \arrow[r, "\Phi"] \arrow[d, "\pi_U"'] & U \times \mathbb{R}^n \arrow[dl, two heads] \\ U & \end{tikzcd}$$

  - For each $p \in U$ the map

$$\pi^*\{p\} \to \{p\} \times \mathbb{R}^n, \quad \varepsilon \mapsto \Phi(\varepsilon)$$

    is an isomorphism of vector spaces.

The following lemma introduces an important construction of vector bundles.

Lemma 2.1 (Vector Bundle Chart Lemma)**.** *Let M be a m-dimensional manifold and suppose that for each $p \in M$ we are given an n-dimensional vector space E_p. Define the set*

$$E := \bigcup_{p \in M} \{p\} \times E_p$$

and consider the map

$$\pi\colon E \to M, \quad \pi(p, u) := p.$$

Suppose that for every chart $\chi\colon U \to \widetilde{U}$ of M we are given a bijection $\Phi_\chi\colon \pi^*U \to U \times \mathbb{R}^n$ in such a way that the following conditions are satisfied:

- *For every chart $\chi\colon U \to \widetilde{U}$ of M and each $p \in M$, the bijection Φ_χ restricts to an isomorphism*

$$\Phi_\chi^{(p)}\colon \{p\} \times E_p \to \{p\} \times \mathbb{R}^m$$

  of vector spaces.

- *If $\chi\colon U \to \widetilde{U}$ and $\psi\colon V \to \widetilde{V}$ are two charts of M with overlapping domains, then the map*

$$U \cap V \times \mathbb{R}^n \to U \cap V \times \mathbb{R}^n, \quad (p, \vec{y}) \mapsto \Phi_\psi(\Phi_\chi^{-1}(p, \vec{y}))$$

is of the form

$$\Phi_\psi(\Phi_\chi^{-1}(p, \vec{y})) = (p, A_{\chi,\psi}(p)\vec{y})$$

for some smooth map $A_{\chi,\psi}\colon U \cap V \to \mathrm{Aut}(\mathbb{R}^n)$.

Then E admits a unique topology and smooth structure making it into a $(m+n)$-dimensional manifold and a vector bundle of rank n over M for which the map π is the bundle projection and the maps Φ_χ are local trivializations.

Proof. The map π is surjective and for each $p \in M$ the fibre $\pi^*\{p\}$ is precisely equal to the vector space $\{p\} \times E_p$. Also, because the maps $\Phi_\chi^{(p)}$ are isomorphisms of vector spaces, it follows that the following diagram commutes:

$$\begin{tikzcd} \pi^*U \arrow[rr, "\Phi_\chi"] \arrow[dr, "\pi_U"'] & & U \times \mathbb{R}^n \arrow[dl] \\ & U & \end{tikzcd}$$

We now construct a topology on the set E:

<u>Topology on E</u>: We use the topology on $M \times \mathbb{R}^n$ in order to create one for E. To be precise, we prove that the collection

$$\{\Phi_\chi^*\Omega_U \mid \chi\colon U \to \widetilde{U} \text{ chart of } M,\ \Omega_U \subset U \times \mathbb{R}^n \text{ open}\}$$

defines a base for a topology on E. It certainly covers E and hence it suffices to show that the intersection $\Phi_\chi^*\Omega_U \cap \Phi_\psi^*\Omega_V$ of two base sets is again a base set. To see this, first observe that the map

$$U \cap V \times \mathbb{R}^n \to U \cap V \times \mathbb{R}^n, \quad (p, \vec{y}) \mapsto \Phi_\psi(\Phi_\chi^{-1}(p, \vec{y}))$$

is a continuous bijection. This implies that $(\Phi_\chi)_*(\Phi_\psi^*(\Omega_V \cap (U \cap V \times \mathbb{R}^n)))$ is open in $U \cap V \times \mathbb{R}^n$ and hence also open in $U \times \mathbb{R}^n$. Therefore we conclude that

$$\Phi_\chi^*\Omega_U \cap \Phi_\psi^*\Omega_V = \Phi_\chi^*(\Omega_U \cap (\Phi_\chi)_*(\Phi_\psi^*(\Omega_V \cap (U \cap V \times \mathbb{R}^n))))$$

is also a base set, as claimed.

As a direct consequence we see that the maps Φ_χ are homeomorphisms. Also, the set $\pi^*V \subset E$ is open whenever $V \subset M$ is open. Indeed, for any open subset $V \subset M$ we have

$$\pi^*V = \bigcup_{\substack{\chi\colon U \to \widetilde{U} \\ \text{chart of } M}} \Phi_\chi^*(U \cap V \times \mathbb{R}^n).$$

We now show that the topological space E is a topological manifold.

⋙ **Claim 1:** *The topological space E is Hausdorff.*

To see that E is Hausdorff, suppose that we have two distinct points $\varepsilon_1, \varepsilon_2 \in E$. Note that

$\varepsilon_1 \in \pi^*\{p_1\}$ and $\varepsilon_2 \in \pi^*\{p_2\}$ for some $p_1, p_2 \in M$. We distinguish the two cases $p_1 = p_2$ and $p_1 \neq p_2$. First assume that $p := p_1 = p_2$. We can find a chart $\chi\colon U \to \widetilde{U}$ of M such that $\pi^*\{p\} \subset \pi^*U$. In particular it follows that $\varepsilon_1, \varepsilon_2 \in \pi^*U$. The first case now follows from the fact that Φ_χ is a homeomorphism from π^*U to the Hausdorff space $U \times \mathbb{R}^n$. Now assume that $p_1 \neq p_2$. Since M is Hausdorff we can find two disjoint open neighbourhoods U and V around p_1 and p_2 respectively. The sets U and V yield, in their turn, two disjoint open neighbourhoods π^*U and π^*V around ε_1 and ε_2 respectively. This proves the second case. $\lhd_1$

$\ggg$ **Claim 2:** *The topological space E is second-countable.*

To verify the second-countability condition of E, observe that we can find a countable open subcover

$$\{U_i\}_{i\in\mathbb{N}} \subset \{U \mid \chi\colon U \to \widetilde{U} \text{ chart of } M\}$$

of M. The latter observation is a consequence of the fact that M is second-countable. Thus, the collection $\{\pi^*U_i\}_{i\in\mathbb{N}}$ of open sets forms a countable open cover of E. Each one of those open sets π^*U_i is homeomorphic to $U_i \times \mathbb{R}^n$ and is therefore second-countable. It follows that E is second-countable. $\lhd_2$

$\ggg$ **Claim 3:** *The topological space E is locally Euclidian.*

To see that E is locally Euclidian, just note that the compositions

$$\pi^*U \xrightarrow{\Phi_\chi} U \times \mathbb{R}^n \xrightarrow{\chi\times\mathrm{id}_{\mathbb{R}^n}} \widetilde{U} \times \mathbb{R}^n$$

are homeomorphisms between open subsets of E and open subsets of $\mathbb{R}^m \times \mathbb{R}^n$. $\lhd_3$

It follows that E is a topological manifold. We now construct a smooth structure on the $(m+n)$-dimensional topological manifold E:

Smooth structure on E: For each chart $\chi\colon U \to \widetilde{U}$ of M, define $\widetilde{\Phi}_\chi$ to be the homeomorphism we used to show that E is locally Euclidian. In order to prove that the collection

$$\{\widetilde{\Phi}_\chi \mid \chi\colon U \to \widetilde{U} \text{ chart of } M\}$$

defines a smooth atlas, we have to show that the maps in it are smoothly compatible. In other words, we must show that the transition map

$$\chi_*(U\cap V) \times \mathbb{R}^n \to \psi_*(U\cap V) \times \mathbb{R}^n, \quad (\vec{x},\vec{y}) \mapsto \widetilde{\Phi}_\psi(\widetilde{\Phi}_\chi^{-1}(\vec{x},\vec{y}))$$

is smooth for all charts $\chi\colon U \to \widetilde{U}$ and $\psi\colon V \to \widetilde{V}$ of M that satisfy $U \cap V \neq \emptyset$. Since

$$\widetilde{\Phi}_\psi(\widetilde{\Phi}_\chi^{-1}(\vec{x},\vec{y})) = (\psi(\chi^{-1}(\vec{x})), A_{\chi,\psi}(\chi^{-1}(\vec{x}))\vec{y}),$$

we conclude that the transition maps are indeed smooth. Any smooth atlas is contained in a unique smooth structure. Thus we have constructed a smooth structure on E.

It follows that E is a smooth $(m+n)$-dimensional manifold. The maps π and Φ_χ are smooth since both of the following diagrams commute:

$$\begin{tikzcd} \pi^*U \arrow[r, "\pi_U"] & U \arrow[d, "\chi"] \\ \widetilde{U}\times\mathbb{R}^n \arrow[u, "\widetilde{\Phi}_\chi^{-1}"] \arrow[r, two heads] & \widetilde{U} \end{tikzcd} \qquad \begin{tikzcd} \pi^*U \arrow[r, "\Phi_\chi"] & U\times\mathbb{R}^n \arrow[d, "\chi\times\mathrm{id}_{\mathbb{R}^n}"] \\ \widetilde{U}\times\mathbb{R}^n \arrow[u, "\widetilde{\Phi}_\chi^{-1}"] \arrow[r, "\mathrm{id}_{\widetilde{U}\times\mathbb{R}^n}"'] & \widetilde{U}\times\mathbb{R}^n \end{tikzcd}$$

Observe that the diagram on the right-hand side also commutes when we reverse the directions. We therefore deduce that the maps Φ_χ are diffeomorphisms and hence also local trivializations.

It is clear that this topology and smooth structure are the unique ones satisfying the conclusions of the lemma. ■

Let M be an m-dimensional manifold. Using Lemma 2.1, one can show that the tangent bundle TM and cotangent bundle $T'M$ are vector bundles. Recall that

$$T'_pM := \mathrm{Lin}(T_pM, \mathbb{R})$$

and also that

$$TM := \bigcup_{p\in M}\{p\}\times T_pM, \quad T'M := \bigcup_{p\in M}\{p\}\times T'_pM.$$

Lemma 2.2 (Tangent and Cotangent Bundle)**.** *Suppose that M is a m-dimensional manifold. Then the tangent bundle TM and cotangent bundle $T'M$ are both smooth vector bundles of rank m over M.*

Proof. Consider the surjective maps $\pi\colon TM \to M$ and $\pi'\colon T'M \to M$ defined by

$$\pi(p, X^p) := p, \quad \pi'(p, \omega_p) := p.$$

Let $\chi\colon U \to \widetilde{U}$ be a chart of M. Define $\Phi_\chi\colon \pi^*(U) \to U\times\mathbb{R}^m$ and $\Phi'_\chi\colon (\pi')^*(U) \to U\times\mathbb{R}^m$ by

$$\Phi_\chi\left(p, \sum_j y^j\cdot \frac{\partial}{\partial\chi^j}\bigg|^p\right) := (p,(y^1,\dots,y^m)), \quad \Phi'_\chi\left(p, \sum_j y^j\cdot \partial\chi^j|_p\right) := (p,(y^1,\dots,y^m)).$$

The maps Φ_χ and Φ'_χ are both bijective. Also, Φ_χ restricts to an isomorphism of vector spaces from $\{p\}\times T_pM$ to $\{p\}\times\mathbb{R}^m$ and Φ'_χ restricts to an isomorphism of vector spaces from $\{p\}\times T'_pM$ to $\{p\}\times\mathbb{R}^m$. Suppose we have two charts $\chi\colon U\to\widetilde{U}$ and $\psi\colon V\to\widetilde{V}$ of M such that $U\cap V\neq\emptyset$. Consider the smooth transition function

$$\tau_{\chi,\psi}\colon \chi_*(U\cap V)\to\psi_*(U\cap V), \quad \tau_{\chi,\psi}(\vec{x}) := \psi(\chi^{-1}(\vec{x}))$$

and its inverse $\tau_{\psi,\chi}$. For any index $1\le j\le m$ and any point $p\in U\cap V$ we know that

$$\frac{\partial}{\partial\chi^j}\bigg|^p = \sum_i \frac{\partial\tau^i_{\chi,\psi}}{\partial x^j}(\chi(p))\cdot\frac{\partial}{\partial\psi^i}\bigg|^p, \quad \partial\chi^j|_p = \sum_i \frac{\partial\tau^j_{\psi,\chi}}{\partial x^i}(\psi(p))\cdot\partial\psi^i|_p.$$

Thus for $(p,\vec{y})\in U\cap V\times\mathbb{R}^m$ we get

$$\Phi_\psi(\Phi_\chi^{-1}(p,\vec{y})) = (p, A_{\chi,\psi}(p)\vec{y}), \quad \Phi'_\psi((\Phi'_\chi)^{-1}(p,\vec{y})) = (p, A'_{(\chi,\psi)}(p)\vec{y}),$$

where the smooth maps $A_{\chi,\psi}\colon U\cap V\to\mathrm{Aut}(\mathbb{R}^m)$ and $A'_{\chi,\psi}\colon U\cap V\to\mathrm{Aut}(\mathbb{R}^m)$ are given by

$$A_{\chi,\psi}(p) = \left[\frac{\partial\tau^i_{\chi,\psi}}{\partial x^j}(\chi(p))\right]_{i,j}, \quad A'_{(\chi,\psi)}(p) = \left[\frac{\partial\tau^j_{\psi,\chi}}{\partial x^i}(\psi(p))\right]_{i,j}.$$

Usage of Lemma 2.1 finishes the proof. ■

Recall that a (k,l)-tensor on a d-dimensional vector space Λ is a multilinear map

$$V \times \overset{k}{\dots} \times V \times V' \times \overset{l}{\dots} \times V' \to \mathbb{R}.$$

The set of all (k,l)-tensors is denoted by $\mathrm{Ten}_{k,l}(\Lambda)$ and naturally comes with a d^{k+l}-dimensional vector space structure. Just as in the proof of Lemma 2.2, one can use Lemma 2.1 and show that the (k,l)-tensor bundle

$$\mathscr{T}_{k,l}M := \bigcup_{p \in M} \{p\} \times \mathrm{Ten}_{k,l}(T_pM)$$

is also a vector bundle of rank m^{k+l} over M.

2.2 Sections

One of the main objects on vector bundles are their (local) sections. We have already seen sections, namely, in the form of vector fields. Let E be a vector bundle over M and let π be the corresponding bundle projection. A *(global) section of* E is a smooth map $s\colon M \to E$ satisfying $\pi \circ s = \mathrm{id}_M$. We denote the set of all sections of E by $\Gamma(E)$. Now suppose that s^1 and s^2 are sections of E and that $f\colon M \to \mathbb{R}$ is a smooth real-valued map. Using the fact that the fibres of π are vector spaces, we can naturally define $f \cdot s^1 + s^2$ pointwise. One can verify that the map $f \cdot s^1 + s^2$ is indeed a section of E. Hence the set $\Gamma(E)$ is endowed with a $C^\infty(M)$-module structure. By identifying any real scalar with its corresponding constant map from M to $\mathbb{R}$, we see that any $C^\infty(M)$-module comes with a natural real vector space structure.

Similarly, a *local section of* E is a smooth map $s\colon U \to E$ defined on some open subset $U \subset M$ that satisfies $\pi \circ s = \iota_U$. The set of such local sections of E is denoted by $\Gamma_U(E)$ and admits the structure of a $C^\infty(U)$-module.

A *(local) vector field* is a local section of $\Gamma(TM)$ and a *(local) covector field* is a (local) section of $\Gamma(T'M)$. If $\chi\colon U \to \widetilde{U}$ is a chart of M, then for $1 \leq i \leq m$ it is easy to see that

$$\frac{\partial}{\partial \chi^i}\colon U \to TM, \quad \frac{\partial}{\partial \chi^i}(p) := \left(p, \left.\frac{\partial}{\partial \chi^i}\right|^p\right)$$

is a local vector field (called a *coordinate vector field*) and that

$$\partial \chi^i\colon U \to TM, \quad \partial \chi^i(p) := (p, \partial \chi^i|_p)$$

is a local covector field (called a *coordinate covector field*).

A *(local) (k,l)-tensor field* is a (local) section of $\mathscr{T}_{k,l}M$. Note that the definition of the tensor product $\otimes$ on tensors can be extended to tensor fields by pointwise evaluation. This extention of the tensor product preserves smoothness.

For technical reasons we often need global extensions of local sections. The following two lemmas discuss the existence of such extensions.

Lemma 2.3 (Extension Lemma Version 1)**.** *Let E be a vector bundle over M and let $U \subset M$ be an open subset. If $s\colon U \to E$ is a local section of E and $p_0 \in U$ a given point, then there exists an open neighbourhood $V \subset U$ around p_0 and a global section $\hat{s}\colon M \to E$ such that s and $\hat{s}$ coincide on V.*

Proof. Choose any open neighbourhood $V \subset U$ around the given point p_0 such that $\overline{V} \subset U$ and let $\eta_1, \eta_2 \in C^\infty(M)$ be a partition of unity subordinate to the open cover of M given by U and $M \setminus \overline{V}$. Because $\mathrm{supp}(\eta_1) \subset U$, it follows that $\eta_1|_U \cdot s$ vanishes on the open subset $U \setminus \mathrm{supp}(\eta_1)$. Thus we can extend $\eta_1|_U \cdot s$ by zero outside U. We obtain a global section $\hat{s}\colon M \to E$ that agrees with $\eta_1|_U \cdot s$ on U and vanishes outside $\operatorname{supp}(\eta_1)$. Since $\operatorname{supp}(\eta_2) \subset M \setminus \overline{V}$, we observe that $\eta_2|_{\overline{V}} \equiv 0$. The partition of unity satisfies $\eta_1 + \eta_2 \equiv 1$. So it follows that $\eta_1|_{\overline{V}} \equiv 1$. Consequently, for any $p \in V$ we get

$$\hat{s}(p) = \eta_1|_U \cdot s(p) = \eta_1(p) \cdot s(p) = s(p),$$

which finishes the proof. ■

Lemma 2.4 (Extension Lemma Version 2). *Let E be a vector bundle of rank n over a manifold M. Given $p_0 \in M$ and $\varepsilon_0 \in \pi^*\{p_0\}$, there exists a local section $s\colon U \to E$ such that $s(p_0) = \varepsilon_0$.*

Proof. First choose any local trivialization $\Phi\colon \pi^*U \to U \times \mathbb{R}^n$ around ε_0. Since Φ is bijective, there is a unique $\vec{y}_0 \in \mathbb{R}^n$ such that Φ sends ε_0 to $(p_0, \vec{y}_0)$. Now define a smooth map

$$s\colon U \to E, \quad s(p) := \Phi^{-1}(p, \vec{y}_0).$$

Since we have

$$\pi(\Phi^{-1}(p, \vec{y})) = p$$

for all $(p, \vec{y}) \in U \times \mathbb{R}^n$, we conclude that $\pi \circ s = \iota_U$. It follows that s is a local section with the desired property. ■

2.3 Local Frames

Let E be a vector bundle of rank n over an m-dimensional manifold M and let $U \subset M$ be an open subset. Suppose that we have an n-tuple $(s^1, \dots, s^n)$ consisting of local sections of E which are defined on an open set $U \subset M$. We say that $(s^1, \dots, s^n)$ is a *local frame of E* if and only if $(s^1(p), \dots, s^n(p))$ is a basis for the fibre $\pi^*\{p\}$ for every $p \in U$. The open set U is called the *domain* of the local frame.

For example, note that any chart $\chi\colon U \to \widetilde{U}$ of M induces local frames $(\partial/\partial\chi^1, \dots, \partial/\partial\chi^m)$ and $(\partial\chi^1, \dots, \partial\chi^m)$ of the tangent bundle TM and the cotangent bundle $T'M$ respectively. We refer to these local frames as the *coordinate frame* and *coordinate coframe* of χ. Their domains equal U, i.e. the domain of the given chart χ.

Another important example of local frames of a general vector bundle involves local trivializations. Suppose we are given a local trivization $\Phi\colon \pi^*U \to U \times \mathbb{R}^n$ of E. For every $1 \le i \le n$ we define a smooth map

$$s^i_\Phi\colon U \to E, \quad s^i_\Phi(p) := \Phi^{-1}(p, \vec{e}_i).$$

Observe that $\pi \circ s^i_\Phi = \iota_U$. It follows that s^i_Φ is a local section. Note that Φ^{-1} restricts to an isomorphism of vector spaces from $\{p\} \times \mathbb{R}^n$ to $\pi^*\{p\}$ and that it takes the standard basis $((p, \vec{e}_1), \dots, (p, \vec{e}_n))$ of $\{p\} \times \mathbb{R}^n$ to $(s^1_\Phi(p), \dots, s^n_\Phi(p))$ for each $p \in U$. Hence $(s^1_\Phi(p), \dots, s^n_\Phi(p))$ is a basis for $\pi^*\{p\}$. We conclude that $(s^1_\Phi, \dots, s^n_\Phi)$ is a local frame of E. We say that this local frame is *associated with* Φ.

Lemma 2.5. *Let E be a vector bundle over M. Every point in M is contained in the domain of some local frame of E.*

Proof. By definition of a vector bundle, every point in M is contained in the domain of a local trivialization. Therefore, every point in M is contained in the domain of the local frame associated to a local trivialization. ■

Lemma 2.6. *Let E be a vector bundle of rank n over M. Every local frame of E is associated to a local trivialization.*

Proof. Suppose $(s^1, \dots, s^n)$ is a local section of E over a domain $U \subset M$. Let $\Phi\colon \pi^*U \to U \times \mathbb{R}^n$ be the inverse of the bijective map

$$U \times \mathbb{R}^n \to \pi^*U, \quad (p, \vec{y}) \mapsto \sum_i y^i \cdot s^i(p).$$

Then Φ restricts to an isomorphism of vector spaces from $\pi^*\{p\}$ to $\{p\} \times \mathbb{R}^n$ for each $p \in U$. Moreover, it satisfies

$$s^i(p) = \Phi^{-1}(p, \vec{e}_i)$$

for each $p \in U$ and every $1 \leq i \leq n$. Hence to prove that it is the desired local trivialization, we have to show that Φ and its inverse Φ^{-1} are smooth. It suffices to prove that Φ is a local diffeomorphism. Fix a point in U and choose a neighbourhood V over which there exists a local trivialization $L\colon \pi^*V \to V \times \mathbb{R}^n$. By shrinking V if necessary, we may assume that $V \subset U$. We now prove that the restriction

$$\Phi_V\colon \pi^*V \to V \times \mathbb{R}^n, \quad \Phi_V(\varepsilon) := \Phi(\varepsilon)$$

is a diffeomorphism. Since L is a diffeomorphism, we may aswell prove that $L \circ \Phi_V^{-1}$ is a diffeomorphism.

$$\begin{tikzcd}
V \times \mathbb{R}^n \arrow[dr, two heads] & \pi^*V \arrow[l, "\Phi_V"'] \arrow[r, "L"] \arrow[d, "\pi_V"'] & V \times \mathbb{R}^n \arrow[dl, two heads] \\
& V &
\end{tikzcd}$$

For all $(p, \vec{y}) \in V \times \mathbb{R}^n$ we calculate

$$L \circ \Phi_V^{-1}(p, \vec{y}) = \sum_i y^i \cdot L(s^i(p)).$$

For every $1 \leq i \leq n$ the map

$$V \to V \times \mathbb{R}^n, \quad p \mapsto L(s^i(p))$$

is smooth by the chain-rule. Thus, for every $1 \leq i \leq n$ there are $\alpha_{i,1}, \dots, \alpha_{i,n} \in C^\infty(V)$ such that

$$L(s^i(p)) = (p, (\alpha_{i,1}(p), \dots, \alpha_{i,n}(p)))$$

for all $p \in U$. We get a smooth map

$$A\colon V \to \mathrm{End}(\mathbb{R}^n), \quad A(p) := [\alpha_{i,j}(p)]_{i,j}$$

and it follows that

$$L \circ \Phi_V^{-1}(p, \vec{y}) = (p, A(p)\vec{y}),$$

which clearly depends smoothly on $(p, \vec{y}) \in V \times \mathbb{R}^n$. Since $L \circ \Phi_V^{-1}$ restricts to an isomorphism of vector spaces from $\{p\} \times \mathbb{R}^n$ to itself, we see that $A(p)$ is invertible for each $p \in V$. Consequently, we have shown that the inclusion $A_*V \subset \mathrm{Aut}(\mathbb{R}^n)$ holds. The matrix inversion map is a smooth map from $\mathrm{Aut}(\mathbb{R}^n)$ to itself. Hence, the chain-rule implies smoothness of the map

$$V \to \mathrm{Aut}(\mathbb{R}^n), \quad p \mapsto A(p)^{-1}.$$

Finally, we conclude that

$$\Phi_V \circ L^{-1}(p, \vec{y}) = (p, A(p)^{-1}\vec{y})$$

depends smoothly on $(p, \vec{y}) \in V \times \mathbb{R}^n$. ■

Let $(s^1, \ldots, s^n)$ be a local frame of E with domain $U \subset M$. Since $(s^1(p), \ldots, s^n(p))$ forms a basis for $\pi^*\{p\}$ for each point $p \in U$, any map $s\colon U \to E$ (not necessarily smooth) that satisfies $\pi \circ s = \iota_U$ can be written as

$$s = \sum_i f_i \cdot s^i,$$

where $f_i\colon U \to \mathbb{R}$ is a real-valued function for every $1 \leq i \leq n$. We call these functions the *component functions of s with respect to $(s^1, \ldots, s^n)$*. It turns out that we can use the component functions to verify whether such a map s is smooth or not.

Lemma 2.7 (Smoothness Criterion)**.** *Let E be a vector bundle of rank n over M and let $(s^1, \ldots, s^n)$ be a local frame of E on some domain $U \subset M$. Suppose that $s\colon U \to E$ is a map that satisfies $\pi \circ s = \iota_U$. Then s is a local section of E if and only if its component functions with respect to $(s^1, \ldots, s^n)$ are smooth.*

Proof. Using Lemma 2.6 we can find a local trivialization $\Phi\colon \pi^*U \to U \times \mathbb{R}^n$ that is associated with $(s^1, \ldots, s^n)$. Because Φ is a diffeomorphism, the map s is smooth if and only if

$$U \to U \times \mathbb{R}^n, \quad p \mapsto \Phi(s(p))$$

is smooth. If $f_1, \ldots, f_n$ are the component functions of s with respect to $(s_1, \ldots, s_n)$, then

$$\Phi(s(p)) = \sum_i f_i(p) \cdot \Phi(s^i(p)) = (p, (f_1(p), \ldots, f_n(p))).$$

So $\Phi(s(p))$ depends smoothly on $p \in U$ if and only if $f_i(p)$ depends smoothly on $p \in U$ for every $1 \leq i \leq n$. ∎

2.4 Connections

Essentially, a connection is a mechanism which allows us to differentiate local and global sections of a vector bundle. Let E be a vector bundle over M. Note that the set consisting of linear maps from $\Gamma(E)$ to itself, as usual denoted by $\operatorname{End}(\Gamma(E))$, admits a $C^\infty(M)$-module structure. Recall that for every vector field $X \in \Gamma(TM)$ the *Lie derivative with respect to X* is defined as the map

$$L_X\colon C^\infty(M) \to C^\infty(M), \quad L_X(f)(p) = X^p f$$

For any open subset $U \subset M$, the Lie derivative with respect to a local vector field $X \in \Gamma_U(TM)$ is defined as the map

$$L_X^U\colon C^\infty(U) \to C^\infty(U), \quad L_X^U(f)(p) := X^p \hat{f},$$

where $\hat{f}\colon M \to \mathbb{R}$ is some smooth function that agrees with f in a neighbourhood of p in U. For a chart $\chi\colon U \to \widetilde{U}$ of M and a smooth function $f\colon U \to \mathbb{R}$, we also write

$$\frac{\partial f}{\partial \chi^i} := L^U_{\partial/\partial\chi^i}(f).$$

A *connection on E* is any $C^\infty(M)$-linear map

$$\Gamma(TM) \to \operatorname{End}(\Gamma(E)), \quad X \mapsto \nabla_X$$

that satiesfies

$$\nabla_X(f \cdot s) = L_X(f) \cdot s + f \cdot \nabla_X(s)$$

for all $X \in \Gamma(TM)$, $s \in \Gamma(E)$ and $f \in C^\infty(M)$. We refer to the latter property as the Leibniz-rule. Furthermore, note that any $C^\infty(M)$-linear map between $C^\infty(M)$-modules is also linear over the real numbers. A connection is a choice of additional structure which we can add to a vector bundle. The following three lemmas reveal the local nature of connections.

Lemma 2.8 (Locality of Connections Version 1)**.** *Let E be a vector bundle over a manifold M and suppose that E is equipped with a connection. Suppose that $U \subset M$ is an open subset.*

(i) If $X \in \Gamma(TM)$ vanishes on U, then $\nabla_X(s)$ vanishes on U for every $s \in \Gamma(E)$.

(ii) If $s \in \Gamma(E)$ vanishes on U, then $\nabla_X(s)$ vanishes on U for every $X \in \gamma(TM)$.

Proof. We prove part (ii). The proof of part (i) is similar to the proof of part (ii). Fix a vector field $X \in \Gamma(TM)$. Let $p \in U$ be a point and let $0^{(p)}$ denote the zero-element of the vector space $\pi^*\{p\}$. Let the maps $\eta_1^{(p)}, \eta_2^{(p)} \in C^\infty(M)$ be a smooth partition of unity subordinate to the open cover of M given by U and $M \setminus \{p\}$. Since the support of $\eta_1^{(p)}$ is contained in U, it follows that $\eta_1^{(p)}$ vanishes outside U. Combining this with the fact that s vanishes on U yields that $\eta_1^{(p)} \cdot s$ must be the zero-section of E. Hence on the one hand we find that

$$\nabla_X(\eta_1^{(p)} \cdot s)(p) = 0^{(p)}$$

and on the other hand we use Leibniz-rule to find that

$$\nabla_X(\eta_1^{(p)} \cdot s)(p) = L_X(\eta_1^{(p)}) \cdot s(p) + \eta_1^{(p)} \cdot \nabla_X(s)(p) = \eta_1^{(p)}(p) \cdot \nabla_X(s)(p).$$

Note that $\eta_2^{(p)}(p) = 0$ and $\eta_1^{(p)} + \eta_2^{(p)} \equiv 1$. Thus $\eta_1^{(p)}(p) = 1$, which proves that

$$\nabla_X(s)(p) = 0^{(p)}.$$

Since p was chosen arbitrarily in U, it follows that $\nabla_X(s)$ must vanish on U. ■

The locality of connections is an important result. In particular, for any open subset $U \subset M$, we note that a connection on E induces a *local connection*

$$\Gamma_U(TM) \to \mathrm{End}(\Gamma_U(E)), \quad X \mapsto \nabla_X^U.$$

Let $X \in \Gamma_U(TM)$ be a local vector field. To see how ∇_X^U acts on a local section $s \in \Gamma_U(E)$, we have to show how $\nabla_X^U(s)$ acts on some point $p \in U$. Using Lemma 2.3 we can find a global vector field $\hat{X} \in \Gamma(TM)$, a global section $\hat{s} \in \Gamma(E)$ and an open neighbourhood $V \subset U$ of p such that $\hat{X}|_V = X|_V$ and $\hat{s}|_V = s|_V$. Indeed, Lemma 2.8 now implies that the assignment

$$\nabla_X^U(s)(p) := \nabla_{\hat{X}}(\hat{s})(p)$$

is independent of the choices of $\hat{X}$ and $\hat{s}$. Moreover, it is $C^\infty(U)$-linear and satisfies the Leibniz-rule. To summarize the preceding lemma, it tells us that we can compute $\nabla_X(s)(p)$ knowing only the values of X and s near p. However, we can prove a stronger result. The next lemma shows that we can compute $\nabla_X(s)(p)$ knowing the value of X at p and values of s near p.

Lemma 2.9 (Locality of Connections Version 2)**.** *Let E be a vector bundle over a manifold M and suppose that E is equipped with a connection. Let $p_0 \in M$ be a point. Suppose that $X \in \Gamma(TM)$ is a vector field that vanishes in the point p_0. Then $\nabla_X(s)$ vanishes in p_0 for each section $s \in \Gamma(E)$.*

Proof. Fix a section $s \in \Gamma(E)$ and choose a chart $\chi\colon U \to \widetilde{U}$ of M around p_0. Using this chart we can write X locally as

$$X|_U = \sum_i X_i^\chi \cdot \frac{\partial}{\partial \chi^i},$$

where $X_i^\chi\colon U \to \mathbb{R}$ is a smooth function that is zero in p_0 for every $1 \leq i \leq n$. We compute

$$\nabla_X(s)(p_0) = \nabla^U_{X|_U}(s|_U)(p_0) = \sum_i X_i^\chi(p_0) \cdot \nabla^U_{\partial/\partial\chi^i}(s|_U)(p_0) = 0^{(p_0)}$$

and finish the proof. ∎

Consequently, for each open subset $U \subset M$ and $p \in U$ we get a well-defined linear map

$$T_pM \to \mathrm{Lin}(\Gamma_U(E), \pi^*\{p\}), \quad u \mapsto \nabla^U_u.$$

Let $p \in M$ be a point and $u \in T_pM$ a tangent vector at p. Indeed, using Lemma 2.4 we are able to find a local vector field $X \in \Gamma_U(TM)$ such that X^p equals u. The map ∇^U_u acts on local sections $s \in \Gamma_U(E)$ by

$$\nabla^U_u(s) := \nabla^U_X(s)(p).$$

Lemma 2.9 shows that this definition is independent of the choice of X.

Lemma 2.10 (Locality of Connections Version 3)**.** *Let E be a vector bundle with a connection over a manifold M. Let $p_0 \in M$ be a point, $X \in \Gamma(TM)$ a vector field and $s \in \Gamma(E)$ a section. Suppose that there exists a smooth curve $\gamma\colon I \to M$ and a $t_0 \in I$ such that $\gamma(t_0) = p_0$, $\dot\gamma^{t_0} = X^{p_0}$ and $s \circ \gamma(t) = 0^{(\gamma(t))}$ for all $t \in I$. Then $\nabla_X(s)$ vanishes in p_0.*

Proof. By shrinking I if necessary, we may assume that $\gamma_* I$ is contained in the domain U of a local frame $(s^1, \ldots, s^n)$ of E. We can find functions $f_1, \ldots, f_n \in C^\infty(U)$ such that

$$s|_U = \sum_i f_i \cdot s^i.$$

Note that $f_i(p_0) = 0$ for all $1 \leq i \leq n$. Using the Leibniz-rule we find

$$\nabla_X(s)(p_0) = \nabla^U_{X|_U}(s|_U)(p_0) = \sum_i L^U_{X|_U}(f_i) \cdot s^i(p_0) + f_i \cdot \nabla^U_{X|_U}(s^i)(p_0) = \sum_i (X^{p_0}\hat{f}_i) \cdot s^i(p_0),$$

where $\hat{f}_i\colon M \to \mathbb{R}$ is some smooth map that coincides with f_i in a neighbourhood of p_0 in U for every $1 \leq i \leq n$. Using the chain-rule and the fact that $\hat{f}_i \circ \gamma$ vanishes in a neighbourhood around t_0 we observe that

$$X^{p_0}\hat{f}_i = (\partial \hat{f}_i)_{\gamma(t_0)}(\dot\gamma^{t_0}) = \partial(\hat{f}_i \circ \gamma)_{t_0}\left(\frac{d}{dt}\bigg|^{t_0}\right) = \frac{d}{dt}\bigg|^{t_0} \hat{f}_i \circ \gamma = 0.$$ ∎

3 Pseudo-Riemannian Manifolds

3.1 Pseudo-Innerproducts

In this subsection we discuss some linear-algebraic preliminaries. Let Λ be a real d-dimensional vector space and $b\colon \Lambda \times \Lambda \to \mathbb{R}$ a bilinear map. We say that b is *symmetric* if and only if $b(u,v) = b(v,u)$ for all $u, v \in \Lambda$. We say that b is *non-degenerate* if and only if $b(u,v) = 0$ for all $v \in \Lambda$ implies that $u = 0_\Lambda$. A *pseudo-innerproduct on* Λ is a symmetric non-degenerate bilinear function from $\Lambda \times \Lambda$ to $\mathbb{R}$. Two vectors $u, v \in \Lambda$ are said to be *orthonormal* if and only if $b(u,v) = 0$, $b(u,u) = \pm 1$ and $b(v,v) = \pm 1$. A basis $(u_1, \ldots, u_d)$ for Λ is orthonormal if and only if the vectors of the basis $(u_1, \ldots, u_d)$ are pairwise orthonormal, i.e. $b(u_i, u_j) = \pm\delta_{i,j}$ for all $1 \leq i, j \leq d$. For $u \in \Lambda$ we define the set

$$u^\perp := \{v \in \Lambda \mid b(u,v) = 0\}.$$

It is easy to see that $u^\perp$ is a linear subspace of Λ. We will use the following technical lemma to prove the existence of orthonormal bases.

Lemma 3.1. *Suppose that Λ is a finite dimensional vector space equipped with a pseudo-innerproduct b. Let $u_0 \in \Lambda$ be a vector that satisfies $b(u_0, u_0) \neq 0$. Then $\Lambda = \mathrm{span}(u_0) + u_0^\perp$ and $\mathrm{span}(u_0) \cap u_0^\perp = \{0_\Lambda\}$.*

Proof. Consider the linear maps

$$\varphi_1\colon \Lambda \to \Lambda, \quad \varphi_1(u) := \frac{b(u_0, u)}{b(u_0, u_0)} \cdot u_0$$

and $\varphi_2 := \mathrm{id}_\Lambda - \varphi_1$. Clearly $\mathrm{id}_\Lambda = \varphi_1 + \varphi_2$ and therefore we know that $\Lambda = (\varphi_1)_*\Lambda + (\varphi_2)_*\Lambda$. Observe that $(\varphi_1)_*\Lambda \subset \mathrm{span}(u_0)$. Bilinearity of b implies that

$$b(u_0, \varphi_2(u)) = b\left(u_0, u - \frac{b(u_0,u)}{b(u_0,u_0)} \cdot u_0\right) = b(u_0, u) - \frac{b(u_0,u)}{b(u_0,u_0)} \cdot b(u_0, u_0) = 0$$

for each $u \in \Lambda$. Hence we have $(\varphi_2)_*\Lambda \subset u_0^\perp$. We conclude that $\Lambda = \mathrm{span}(u_0) + u_0^\perp$. We now prove that the intersection of $\mathrm{span}(u_0)$ and $u_0^\perp$ is trivial. Suppose that $u \in \mathrm{span}(u_0) \cap u_0^\perp$. Then $u = \alpha \cdot u_0$ for some $\alpha \in \mathbb{R}$ and $b(u_0, u) = 0$. We also find that

$$b(u_0, u) = b(u_0, \alpha \cdot u_0) = \alpha \cdot b(u_0, u_0).$$

Because $b(u_0, u_0) \neq 0$, we must have $\alpha = 0$ and thus $u = 0_\Lambda$. We conclude that $\mathrm{span}(u_0) \cap u_0^\perp = \{0_\Lambda\}$. ∎

Lemma 3.2. *Every finite-dimensional vector space equipped with a pseudo-innerproduct has an orthonormal basis.*

Proof. We will prove this lemma by using induction on the dimension d of vector spaces. The case $d = 0$ is obvious. So assume that the result holds for vector spaces of dimension d. Let Λ be a $(d+1)$-dimensional vector space together with a pseudo-innerproduct b. There must be a vector $u_0 \in \Lambda$ with $b(u_0, u_0) \neq 0$, because otherwise the polarization identity

$$b(u,v) = \frac{1}{4}(b(u+v, u+v) - b(u-v, u-v))$$

would imply that $b \equiv 0$. Using Lemma 3.1 we see that $\Lambda = \mathrm{span}(u_0) + u_0^\perp$ and $\mathrm{span}(u_0) \cap u_0^\perp = \{0_\Lambda\}$. Note that $\dim(u_0^\perp) = d$. We now prove that the restriction $b_0 := b|_{u_0^\perp \times u_0^\perp}$ is a pseudo-innerproduct on $u_0^\perp$. Suppose that $u \in u_0^\perp$ satisfies $b_0(u, v) = 0$ for all $v \in u_0^\perp$. If $w \in \Lambda$, then we can find a $\alpha \in \mathbb{R}$ and a $v \in u_0^\perp$ such that $w = \alpha \cdot u_0 + v$. Hence we have

$$b(u, w) = b(u, \alpha \cdot u_0 + v) = \alpha \cdot b(u_0, u) + b_0(u, v) = 0.$$

Since this is true for all $w \in \Lambda$ and b is non-degenerate, we conclude that $u = 0_\Lambda$. It follows that b_0 is non-degenerate. The induction hypothesis implies that there exists an orthonormal basis $(u_1, \ldots, u_d)$ for $u_0^\perp$. If we define $u_{d+1} := u_0$, then $(u_1, \ldots, u_{d+1})$ is easily seen to be an orthonormal basis for Λ. ■

Lemma 3.3 (Sylvester's Law of Inertia)**.** *Suppose that Λ is a d-dimensional vector space equipped with a pseudo-innerproduct b. Then there exists a basis $(u^1, \ldots, u^d)$ for Λ^* with respect to which b has the expression*

$$b = u^1 \otimes u^1 + \ldots + u^k \otimes u^k - u^{k+1} \otimes u^{k+1} - \ldots - u^d \otimes u^d$$

for some integer $0 \leq k \leq d$. This number k is independent of the choice of basis.

Proof. Because of Lemma 3.2 we are able to find an orthonormal basis $(u_1, \ldots, u_d)$ of Λ. We can rearrange the basis vectors. Therefore we may assume that

$$b(u_1, u_1) = 1, \quad \ldots, \quad b(u_k, u_k) = 1, \quad b(u_{k+1}, u_{k+1}) = -1, \quad \ldots, \quad b(u_d, u_d) = -1.$$

A straightforward computation shows that the dual basis $(u^1, \ldots, u^d)$ with respect to which b has the required expression. Now suppose that there are two bases $(u^1, \ldots, u^d)$ and $(v^1, \ldots, v^d)$ of V^* such that

$$\begin{aligned} b &= u^1 \otimes u^1 + \ldots + u^k \otimes u^k - u^{k+1} \otimes u^{k+1} - \ldots - u^d \otimes u^d, \\ &= v^1 \otimes v^1 + \ldots + v^l \otimes v^l - v^{l+1} \otimes v^{l+1} - \ldots - v^d \otimes v^d \end{aligned}$$

for some integeres $1 \leq k, l \leq d$. We will prove that $k = l$. Define the subspaces

$$S := \mathrm{span}(u_1, \ldots, u_k), \quad T := \mathrm{span}(v_{l+1}, \ldots, v_d).$$

If $u \in S \backslash \{0_\Lambda\}$ and $v \in T \backslash \{0_\Lambda\}$, then there are real numbers $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ and $\beta_{l+1}, \ldots, \beta_d \in \mathbb{R}$ such that

$$u = \alpha^1 \cdot u_1 + \ldots + \alpha^k \cdot u_k, \quad v = \beta^{l+1} \cdot v_{l+1} + \ldots + \beta^d \cdot v_d.$$

It follows that

$$b(u, u) = (\alpha^1)^2 + \ldots + (\alpha^k)^2 > 0, \quad b(v, v) = -(\beta^{l+1})^2 - \ldots - (\beta^d)^2 < 0.$$

Hence we see that $S \cap T = \{0_\Lambda\}$. Combining this result with the fact that $S + T \subset \Lambda$, we conclude that $\dim(S) + \dim(T) \leq \dim(\Lambda)$. Thus we get the inequality $k + (d - l) \leq d$, i.e. $k \leq l$. A similar argument shows that $k \geq l$. ■

The integer k introduced in Lemma 3.3 is called the *signature of* b.

3.2 Pseudo-Metrics

Let M be a manifold. A *pseudo-metric on M with signature r* is a smooth symmetric $(2,0)$-tensor field

$$g\colon M \to \mathscr{T}_{2,0}M, \quad g(p) := (p, g_p)$$

such that g_p is non-degenerate at each point $p \in M$ and has the same signature r everywhere. Hence g_p defines a pseudo-innerproduct on the tangent space T_pM for each $p \in M$. Just as the topology and smooth structure, the pseudo-metric g is a choice of additional structure on the manifold. If g_p is a regular innerproduct on T_pM for each $p \in M$, we say that g is a *metric on M*. If M is equipped with a (pseudo-)metric g (with signature r), we say that M is a *(pseudo-)Riemannian manifold (with signature r)*. It is easy to see that every innerproduct is also an pseudo-innerproduct and hence that every Riemannian manifold is also a pseudo-Riemannian manifold.

Important examples of pseudo-Riemannian manifolds are given by pseudo-Euclidian spaces: an *m-dimensional pseudo-Euclidian space with signature r* is an open subset $\widetilde{U} \subset \mathbb{R}^m$ equipped with the pseudo-metric

$$\tilde{g}^{m,r} := \partial x^1 \otimes \partial x^1 + \ldots + \partial x^r \otimes \partial x^r - \partial x^{r+1} \otimes \partial x^{r+1} - \ldots - \partial x^m \otimes \partial x^m.$$

Note that we use the same notation for Euclidian pseudo-metrics on any open subset of $\mathbb{R}^m$.

If $\chi\colon U \to \widetilde{U}$ is a chart of a pseudo-Riemannian manifold M, then the pseudo-metric g can locally be written as

$$g|_U = \sum_{i,j} g^{\chi}_{i,j} \cdot \partial\chi^i \otimes \partial\chi^j,$$

where the component functions are given by the formula

$$g^{\chi}_{i,j} = g\left(\frac{\partial}{\partial\chi^i}, \frac{\partial}{\partial\chi^j}\right)$$

for all $1 \leq i, j \leq n$. For any $p \in U$, we can identify g_p with the matrix $[g^{\chi}_{i,j}(p)]_{i,j}$ by the matrix equation

$$\left[g_p\left(\sum_i u^i \cdot \frac{\partial}{\partial\chi^i}\bigg|^p, \sum_j v^j \cdot \frac{\partial}{\partial\chi^j}\bigg|^p\right)\right] = \begin{bmatrix} u^1 & \cdots & u^m \end{bmatrix} \begin{bmatrix} g^{\chi}_{1,1}(p) & \cdots & g^{\chi}_{1,m}(p) \\ \vdots & \ddots & \vdots \\ g^{\chi}_{m,1}(p) & \cdots & g^{\chi}_{m,m}(p) \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^m \end{bmatrix}.$$

Lemma 3.4. *Let M be a pseudo-Riemannian manifold. For any chart $\chi\colon U \to \widetilde{U}$ and $p \in U$, the matrix $[g^{\chi}_{i,j}(p)]_{i,j}$ is non-singular.*

Proof. Suppose that $u^1, \ldots, u^m \in \mathbb{R}$ satisfy

$$\sum_j g^{\chi}_{i,j}(p) \cdot u^j = 0$$

for all $1 \leq j \leq m$. Let u be the tangent vector in T_pM that has $u^1, \ldots, u^m$ as components with respect to the coordinate frame of χ. Then for all $v^1, \ldots, v^m \in \mathbb{R}$ we have

$$g_p\left(u, \sum_j v^j \cdot \frac{\partial}{\partial\chi^j}\right) = \sum_{i,j} g^{\chi}_{i,j}(p) \cdot u^i \cdot v^j = 0.$$

Since g_p is non-degenerate, it follows that u must be the zero tangent vector. Hence

$$\ker[g^{\chi}_{i,j}(p)]_{i,j} = \{\vec{0}\}$$

and we conclude that $[g^{\chi}_{i,j}(p)]_{i,j}$ is invertible. ∎

We denote the inverse matrix of $[g^{\chi}_{i,j}(p)]_{i,j}$ by $[g^{i,j}_{\chi}(p)]_{i,j}$. The symmetry of g implies that both of these matrices are symmetric.

3.3 Raising and Lowering Indices

Let M be a pseudo-Riemannian manifold and consider the map

$$\flat^{(p)} \colon T_pM \to T'_pM, \quad \flat^{(p)}(u)(v) := g_p(u,v).$$

Lemma 3.5. *Let M be a pseudo-Riemannian manifold. Then for each $p \in M$ the map $\flat^{(p)}$ is an isomorphism of vector spaces.*

Proof. Let $p \in M$ be a point. The map $\flat^{(p)}$ is obviously linear. Since $\dim(T_pM) = \dim(T'_pM)$, it suffices to prove that $\flat^{(p)}$ is injective. Suppose that $\flat^{(p)}(u) = \flat^{(p)}(v)$. Then $g_p(u-v,w) = 0$ for all $w \in T_pM$. Since g_p is non-degenerate, it follows that $u - v$ must be the zero tangent vector at p. ■

We denote the inverse of $\flat^{(p)}$ by $\sharp^{(p)}$. Now consider the map

$$\flat \colon \Gamma(TM) \to \Gamma(T'M), \quad \flat(X)(p) := (p, \flat^{(p)}(X^p)).$$

We show that $\flat$ is well-defined, i.e. that $\flat(X)$ is smooth for every vector field $X \colon M \to TM$. Let $\chi \colon U \to \widetilde{U}$ be a chart of M and observe that

$$\flat^{(p)}\left(\frac{\partial}{\partial \chi^j}\bigg|^p\right) = \sum_i g^\chi_{i,j}(p) \cdot \partial\chi^i|_p$$

for all $p \in U$. It follows that

$$\flat(X)_p = \flat^{(p)}(X^p) = \sum_j X^j_\chi(p) \cdot \flat^{(p)}\left(\frac{\partial}{\partial \chi^j}\bigg|^p\right) = \sum_{i,j} g^\chi_{i,j}(p) \cdot X^j_\chi(p) \cdot \partial\chi^i|_p.$$

So the component functions of $\flat(X)$ with respect to the coordinate coframe $(\partial\chi^1, \ldots, \partial\chi^m)$ are given by

$$\flat(X)^\chi_i = \sum_j g^\chi_{i,j} \cdot X^j_\chi.$$

Therefore we see that $\flat(X)$ is smooth and hence that $\flat$ is indeed well-defined. We say that $\flat$ *lowers the index* of vector fields.

Lemma 3.6. *Let M be a pseudo-Riemannian manifold. Then the map $\flat$ is an isomorphism of $C^\infty(M)$-modules.*

Proof. The $C^\infty(M)$-linearity of $\flat$ follows from a straightforward calculation. The injectivity is a pointwise application of Lemma 3.5. To prove surjectivity, suppose that $\omega \in \Gamma(T'M)$ is given. Define a map

$$X \colon M \to TM, \quad X(p) := (p, \sharp^{(p)}(\omega_p)).$$

We claim that X is smooth, i.e. that $X \in \Gamma(TM)$. Let $\chi \colon U \to \widetilde{U}$ be a chart and $p \in U$ a point. On the one hand we find

$$\flat^{(p)}(X^p) = \sum_{i,j} g^\chi_{i,j}(p) \cdot X^j_\chi(p) \cdot \partial\chi^i|_p$$

and on the other hand we find

$$\flat^{(p)}(X^p) = \flat^{(p)}(\sharp^{(p)}(\omega_p)) = \omega_p = \sum_i \omega^\chi_i(p) \cdot \partial\chi^i|_p.$$

From this we read that

$$\omega_i^\chi = \sum_j g_{i,j}^\chi \cdot X_\chi^j$$

and therefore we conclude that

$$X_\chi^i = \sum_j g_\chi^{i,j} \cdot \omega_j^\chi.$$

It follows that X is smooth. Finally, it is obvious that $\flat$ maps X to ω. ■

We denote the inverse of the map $\flat$ by $\sharp$. Let $\omega\colon M \to T'M$ be a covector field. Then $\sharp(\omega)(p) = (p, \sharp^{(p)}(\omega_p)$ for each $p \in M$. In terms of a coordinate frame, the component functions of $\sharp(\omega)$ are given by

$$\sharp(w)_\chi^i = \sum_j g_\chi^{i,j} \cdot \omega_j^\chi.$$

We say that $\sharp$ *raises the index* of covector fields.

3.4 Levi-Civita Connection

Suppose that M is a pseudo-Riemannian manifold. We restrict our attention to connections on the tangent bundle TM. A connection on TM is said to be *compatible with the pseudo-metric* g if and only if

$$L_X(g(Y,Z)) = g(\nabla_X(Y), Z) + g(Y, \nabla_X(Z))$$

for all vector fields $X, Y, Z \in \Gamma(TM)$. It turns out that requiring a connection to be compatible with g is not enough to determine a unique connection on TM. However, because we are working with TM instead of general vector bundles, we can talk about the *torsion of a connection on* TM which is defined as the map

$$\tau_\nabla\colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM), \quad \tau_\nabla(X,Y) := \nabla_X(Y) - \nabla_Y(X) - [X,Y].$$

The torsion map τ_∇ is $C^\infty(M)$-linear in both entries. We say that a connection on TM is *torsion-free* if and only if τ_∇ vanishes identically. Note that both of the properties 'compatible with a pseudo-metric' and 'torsion-free' are defined for global vector fields. However, one can use Lemma 2.3 together with the fact that Lie derivatives and connections act locally to show that both properties imply a similar properties for local vector fields.

A *Levi-Civita connection* is a torsion-free connection on TM that is compatible with g. As the following theorem will show, requiring a connection on TM to be torsion-free is precisely the additional property that will guarantee uniqueness.

Theorem 3.7 (Fundamental Theorem of Pseudo-Riemannian Geometry)**.** *There exists a unique Levi-Civita connection on the tangent bundle of a pseudo-Riemannian manifold.*

Proof. Let M be a pseudo-Riemannian manifold. First we prove uniqueness. Suppose that the connection on TM is a Levi-Civita connection. Using the fact that the connection is compatible with the pseudo-metric g and that it is torsion-free, one can verify that

$$\begin{aligned} 2 \cdot g(\nabla_X(Y), Z) = L_X(g(Y,Z)) + L_Y(g(Z,X)) - L_Z(g(X,Y)) \\ - g(Y,[X,Z]) - g(Z,[Y,X]) + g(X,[Z,Y]) \end{aligned}$$

for all $X, Y, Z \in \Gamma(TM)$. This equation is known as Koszul's equation. A straightforward calculation shows that the right-hand side of Koszul's equation is $C^\infty(M)$-linear in Z. By the tensor characterization lemma (see [9], Lemma 12.24), it follows that there exists a unique covector field $\omega^{X,Y}\colon M \to T'M$ such that the right-hand side of Koszul's equation is equal to $\omega^{X,Y}(Z)$. Hence Koszul's equation can be rewritten as

$$2 \cdot \flat(\nabla_X(Y)) = \omega^{X,Y}$$

for all $X, Y \in \Gamma(TM)$. Isolating $\nabla_X(Y)$ in this equation yields

$$\nabla_X(Y) = \frac{1}{2} \cdot \sharp(\omega^{X,Y}).$$

Since the right-hand side of Koszul's equation does not depend on the connection, it also follows that $\omega^{X,Y}$ does not depend on the connection. Therefore the connection must be the unique connection that is compatible with g and torsion-free. The proof of uniqueness suggests the formula we can use to prove existence: define $\nabla_X(Y)$ by raising the index of the covector field $\omega^{X,Y}$ and dividing the resulting vector field by 2. This expression is $C^\infty(M)$-linear in X, linear in Y and it satisfies the Leibniz-rule in Y (see [6]). ■

Whenever we speak of a pseudo-Riemannian manifold, we assume that its tangent bundle is equipped with the Levi-Civita connection. Now, let $\chi\colon U \to \widetilde{U}$ be a chart of M. For all $1 \leq i, j \leq m$ we can find smooth functions $\Gamma(\chi)_{i,j}{}^1, \ldots, \Gamma(\chi)_{i,j}{}^m \in C^\infty(U)$ such that

$$\nabla_{\partial/\partial\chi^i}\left(\frac{\partial}{\partial\chi^j}\right) = \sum_k \Gamma(\chi)_{i,j}{}^k \cdot \frac{\partial}{\partial\chi^k}.$$

We call these component functions the *Christoffel symbols with respect to* χ. The following lemma gives a formula for the Christoffel symbols in terms of the component functions of the pseudo-metric g.

Lemma 3.8. *Let M be a pseudo-Riemannian manifold. The Christoffel symbols with respect to a chart $\chi\colon U \to \widetilde{U}$ are given by*

$$\Gamma(\chi)_{i,j}{}^k = \frac{1}{2} \cdot \sum_\mu g_\chi^{k,\mu} \cdot \left(\frac{\partial g_{j,\mu}^\chi}{\partial\chi^i} + \frac{\partial g_{i,\mu}^\chi}{\partial\chi^j} - \frac{\partial g_{i,j}^\chi}{\partial\chi^\mu}\right)$$

for all $1 \leq i, j, k \leq m$.

Proof. Let $\chi\colon U \to \widetilde{U}$ be a chart of M. In the proof of Theorem 3.7 we have seen that the Levi-Civita connection satisfies Koszul's equation. Observe that this equation also holds for local vector fields. Since the Lie brackets of the coordinate frame of χ vanish, it follows that

$$2 \cdot g\left(\nabla^U_{\partial/\partial\chi^i}\left(\frac{\partial}{\partial\chi^j}\right), \frac{\partial}{\partial\chi^k}\right) = L^U_{\partial/\partial\chi^i}\left(g\left(\frac{\partial}{\partial\chi^j}, \frac{\partial}{\partial\chi^k}\right)\right) + L^U_{\partial/\partial\chi^j}\left(g\left(\frac{\partial}{\partial\chi^i}, \frac{\partial}{\partial\chi^k}\right)\right) - L^U_{\partial/\partial\chi^k}\left(g\left(\frac{\partial}{\partial\chi^i}, \frac{\partial}{\partial\chi^j}\right)\right).$$

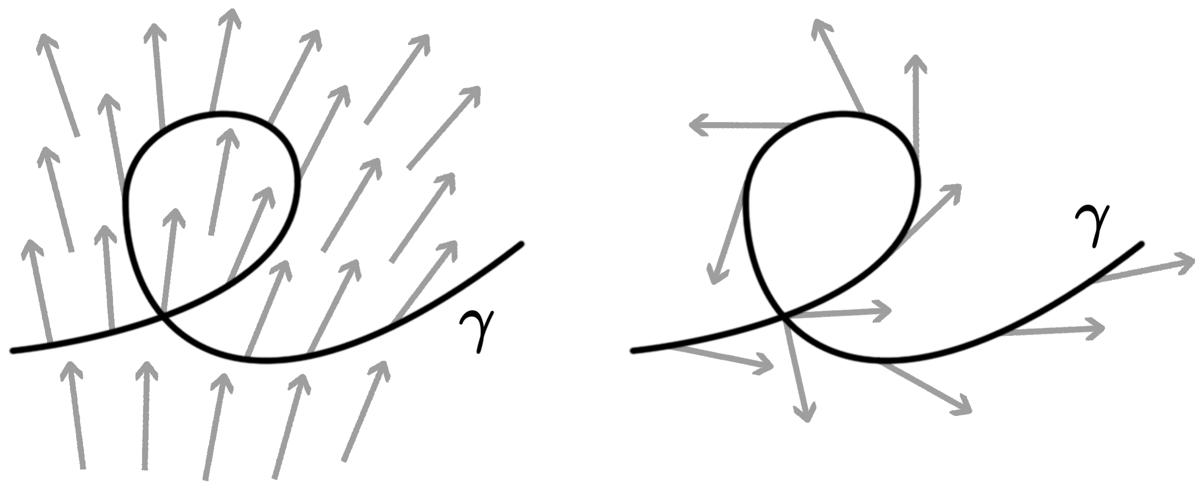
Using the fact that $g(\partial/\partial\chi^\mu, \partial/\partial\chi^\nu)$ is equal to $g_{\mu,\nu}^\chi$ for all $1 \leq \mu, \nu \leq m$, we find that

$$2 \cdot \sum_\mu \Gamma(\chi)_{i,j}{}^\mu \cdot g_{k,\mu}^\chi = \frac{\partial g_{j,k}^\chi}{\partial\chi^i} + \frac{\partial g_{i,k}^\chi}{\partial\chi^j} - \frac{\partial g_{i,j}^\chi}{\partial\chi^k}.$$ ■

4 Curvature

4.1 Covariant Derivative

Let $I \subset \mathbb{R}$ be an open interval, E a vector bundle over M and $\gamma\colon I \to M$ a smooth curve. A *section of E along γ* is a smooth map $\sigma\colon I \to E$ with the property that $\pi \circ \sigma = \gamma$. We say that σ is *extendible* if and only if there exists a local section $s\colon U \to E$ such that $\gamma_* I \subset U$ and $\sigma(t) = s(\gamma(t))$ for all $t \in I$.



An Extentible (Left) and a Non-extendible (Right) Section Along γ

An important example of a section of TM along a curve γ is the velocity vector field $\dot{\gamma}\colon I \to TM$ along γ.

We denote $\Gamma_\gamma(E)$ for the space of all sections of E along γ. The set $\Gamma_\gamma(E)$ comes with a natural $C^\infty(I)$-module structure and hence also a vector space structure.

A *covariant derivative along γ* is a linear map

$$\Gamma_\gamma(E) \to \Gamma_\gamma(E), \quad \sigma \mapsto \frac{\nabla^\gamma \sigma}{dt}$$

satisfying the following properties:

- For all $f \in C^\infty(I)$ and $\sigma \in \Gamma_\gamma(E)$ the following Leibniz-rule holds:

$$\frac{\nabla^\gamma (f \cdot \sigma)}{dt} = \frac{df}{dt} \cdot \sigma + f \cdot \frac{\nabla^\gamma \sigma}{dt}.$$

- Suppose that $\sigma \in \Gamma_\gamma(E)$ is extendible. If $s\colon U \to E$ is a local section with $\gamma_* I \subset U$ and $\sigma(t) = s(\gamma(t))$ for all $t \in I$, then

$$\frac{\nabla^\gamma \sigma}{dt}(t) = \nabla^U_{\dot{\gamma} t}(s).$$

The following lemma shows that there exists a covariant derivatives along γ and that it is unique.

Lemma 4.1 (Covariant Derivative Along a Curve)**.** *Suppose that E is a vector bundle over M of rank n with a connection. Then for any smooth curve $\gamma\colon I \to M$ there exists a unique covariant derivative along γ.*

Proof. First we prove uniqueness. Suppose that there exists a covariant derivative along the given curve γ. Note that it also induces the existence of a covariant derivative

$$\Gamma_{\gamma|_J}(E) \to \Gamma_{\gamma|_J}(E), \quad \sigma \mapsto \frac{\nabla^{\gamma|_J}\sigma}{dt}$$

along the restriction $\gamma|_J$, where $J \subset I$ is some open subinterval. Indeed, for any $t \in J$ we can argue as in the proof of Lemma 2.3 to find a section $\hat{\sigma} \in \Gamma_\gamma(E)$ that coincides with σ on an open subinterval $K \subset J$ around t. We can now define

$$\frac{\nabla^{\gamma|_J}\sigma}{dt}(t) := \frac{\nabla^\gamma\hat{\sigma}}{dt}(t)$$

and an argument similar to that of Lemma 2.8 shows that this definition is independent of the extension $\hat{\sigma}$. One can verify that this induced operator is a covariant derivative along $\gamma|_J$. Now fix a $\sigma \in \Gamma_\gamma(E)$ and a $t \in I$ and choose an open subinterval $J \subset I$ around t such that $\gamma_* J$ is contained in the domain U of a local frame $(s^1, \ldots, s^n)$ of E. For all $1 \leq i \leq n$ we define a section of E along $\gamma|_J$ by

$$\sigma^i \colon J \to E, \quad \sigma^i(t) := s^i(\gamma(t)).$$

With some minor adjustments to the proof of Lemma 2.7, we can use the proof to conclude that there exist smooth maps $f_1, \ldots, f_n \in C^\infty(J)$ such that

$$\sigma|_J = \sum_i f_i \cdot \sigma^i.$$

It follows that

$$\frac{\nabla^\gamma\sigma}{dt}(t) = \frac{\nabla^{\gamma|_J}\sigma|_J}{dt}(t) = \sum_i \frac{df_i}{dt} \cdot \sigma^i(t) + f_i \cdot \frac{\nabla^{\gamma|_J}\sigma^i}{dt}(t) = \sum_i \frac{df_i}{dt}(t) \cdot s^i(\gamma(t)) + f_i(t) \cdot \nabla^U_{\dot{\gamma}^t}(s^i).$$

Since this expression only on the connection on E, we see that the operator is unique. Observe that the proof of uniqueness suggests the formula we can use to prove existence of the covariant derivative along γ. For any open subinterval $J \subset I$ with $\gamma_* J$ contained in the domain U of a local frame $(s^1, \ldots, s^n)$ of E, one can check that

$$\Gamma_{\gamma|_J}(E) \to \Gamma_{\gamma|_J}(E), \quad t \mapsto \sum_i \frac{df_i}{dt}(t) \cdot s^i(\gamma(t)) + f_i(t) \cdot \nabla^U_{\dot{\gamma}^t}(s^i).$$

satisfies the conditions of the covariant derivative along $\gamma|_J$. Since we can find such an interval J around any $t \in I$, we can locally define the covariant derivative along γ by the same formula. Uniqueness ensures that the definitions agree when t lies in the overlap of multiple subintervals of which the images under γ are contained in the domains of local sections of E. ■

4.2 Parallel Transportation

Let $I \subset \mathbb{R}$ be an open interval, E a vector bundle over M and $\gamma \colon I \to M$ a smooth curve. Recall that for any point $p \in M$ we denote $0^{(p)}$ for the zero-element of $\pi^*\{p\}$. A section $\sigma \colon I \to E$ of E along a curve $\gamma \colon I \to M$ is said to be *parallel along* γ if and only if

$$\frac{\nabla^\gamma\sigma}{dt}(t) = 0^{(\gamma(t))}$$

for all $t \in I$, i.e. its covariant derivative along γ vanishes identically. A local section $s \colon U \to E$ is said to be *parallel* if and only if for every curve $\gamma \colon I \to M$ that satisfies $\gamma_* I \subset U$ the section

$$I \to E, \quad t \mapsto s(\gamma(t))$$

along γ is parallel along γ.

Lemma 4.2. *Let E be a vector bundle over M with a connection. A local section $s\colon U \to E$ is parallel if and only if the map*

$$\Gamma_U(TM) \to \Gamma_U(E), \quad X \mapsto \nabla^U_X(s)$$

vanishes identically.

Proof. First assume that s is parallel. Fix a point $p \in U$ and a local vector field $X \in \Gamma_U(TM)$. Choose an open interval I around 0 and let $\gamma\colon I \to M$ be a curve that satisfies $\gamma_* I \subset U$, $\gamma(0) = p$ and $\dot{\gamma}^0 = X^p$. Since s is parallel, it follows that

$$\sigma\colon I \to E, \quad \sigma(t) := s(\gamma(t)).$$

is parallel along γ. We therefore conclude that

$$\nabla^U_X(s)(p) = \nabla^U_{\dot{\gamma}^{t_0}}(s) = \frac{\nabla^\gamma \sigma}{dt}(0) = 0^{(p)}.$$

We now prove the converse. Let $\gamma\colon I \to M$ be a curve such that $\gamma_* I \subset U$ and define σ as before. We prove that σ is parallel along γ. Fix a time $t \in I$. Use Lemma 2.4 to choose a local vector field $X \in \Gamma_U(TM)$ with $X^{\gamma(t)} = \dot{\gamma}^t$. Then it follows that

$$\frac{\nabla^\gamma \sigma}{dt}(t) = \nabla^U_{\dot{\gamma}^t}(s) = \nabla^U_X(s)(\gamma(t)) = 0^{(\gamma(t))}.$$

Hence s is parallel. ■

The following lemma shows that parallelism of vector fields on a pseudo-Riemannian manifold preserves the pseudo-innerproducts on the tangent spaces. In particular, it tells us that the angles between vectors of two parallel vector fields remain constant on the connected components of the domain of the vector fields.

Lemma 4.3. *Let M be a pseudo-Riemannian manifold. If X^1 and X^2 are parallel vector fields defined on an open subset $U \subset M$, then $g(X^1, X^2)$ is constant on the connected components of U.*

Proof. Since the Levi-Civita connection is compatible with g, we see that

$$L^U_X(g(X^1, X^2)) = g(\nabla^U_X(X^1), X^2) + g(X^1, \nabla^U_X(X^2))$$

for all $X \in \Gamma_U(TM)$. By Lemma 4.2 it follows that $\nabla^U_X(X^1)$ and $\nabla^U_X(X^2)$ both vanish on U. Bilinearity of g shows that the Lie derivative of $g(X^1, X^2)$ with respect to every $X \in \Gamma_U(TM)$ vanishes on U. ■

In order to prove that parallel transport exists and is unique, we need the following result from the theory of linear ordinary differential equations.

Lemma 4.4. *Let $I \subset \mathbb{R}$ be an open interval and $t_0 \in I$. Suppose that $A\colon I \to \mathrm{End}(\mathbb{R}^n)$ is a smooth map. For every $\vec{y} \in \mathbb{R}^n$, there exists a unique smooth map $F^{(\vec{y})}\colon I \to \mathbb{R}^n$ that satisfies*

$$\frac{dF^{(\vec{y})}}{dt}(t) = A(t)F^{(\vec{y})}(t), \quad F^{(\vec{y})}(t_0) = \vec{u}$$

for all $t \in I$ and the map

$$I \times \mathbb{R}^n \to \mathbb{R}^n, \quad (t, \vec{y}) \mapsto F^{(\vec{y})}(t)$$

is smooth.

Proof. See [10], Theorem 4.31. ■

Theorem 4.5 (Parallel Transport)**.** *Let E be a vector bundle of rank n over M. Suppose $\gamma\colon I \to M$ is a curve. Given $t_0 \in I$ and $\varepsilon_0 \in \pi^*\{\gamma(t_0)\}$, there exists a unique parallel section $\sigma\colon I \to E$ along γ that satisfies $\sigma(t_0) = \varepsilon_0$.*

Proof. First suppose that $\gamma_*(I)$ is contained in the domain $U \subset M$ of some local frame $(s^1, \ldots, s^n)$ of E. Then there exists a unique vector $\vec{y}_0 \in \mathbb{R}^n$ such that

$$\varepsilon_0 = \sum_j y_0^j \cdot s^j(\gamma(t_0))$$

and for any $1 \leq j \leq n$ we can find $\alpha_{1,j}, \ldots, \alpha_{n,j} \in C^\infty(I)$ such that

$$\nabla^U_{\dot{\gamma}^t}(s^j) = \sum_i \alpha_{i,j}(t) \cdot s^i(\gamma(t)).$$

Now consider the smooth map

$$A\colon I \to \mathrm{End}(\mathbb{R}^m), \quad A(t) := [-\alpha_{i,j}(t)]_{i,j}.$$

By Lemma 4.4 there exists a unique smooth real-valued map $F\colon I \to \mathbb{R}^n$ that satisfies

$$\frac{dF}{dt}(t) = A(t)F(t), \quad F(t_0) = \vec{y}_0$$

for all $t \in I$. Let $f_1, \ldots, f_n \in C^\infty(I)$ be the component functions of F and observe that

$$\sum_j \frac{df_j}{dt}(t) \cdot s^j(\gamma(t)) + f_j(t) \cdot \sum_i \alpha_{i,j}(t) \cdot s^i(\gamma(t)) = 0^{(\gamma(t))}$$

for all $t \in I$. Hence

$$\sigma\colon I \to E, \quad \sigma(t) := \sum_j f_j(t) \cdot s^j(\gamma(t))$$

is the unique section along γ that is parallel along γ.

Now suppose that $\gamma_* I$ is not covered by a single local frame of E. Choose a bounded open subinterval $J \subset I$ around t_0 such that $\gamma_* J$ is contained in the domain of a local frame of E.

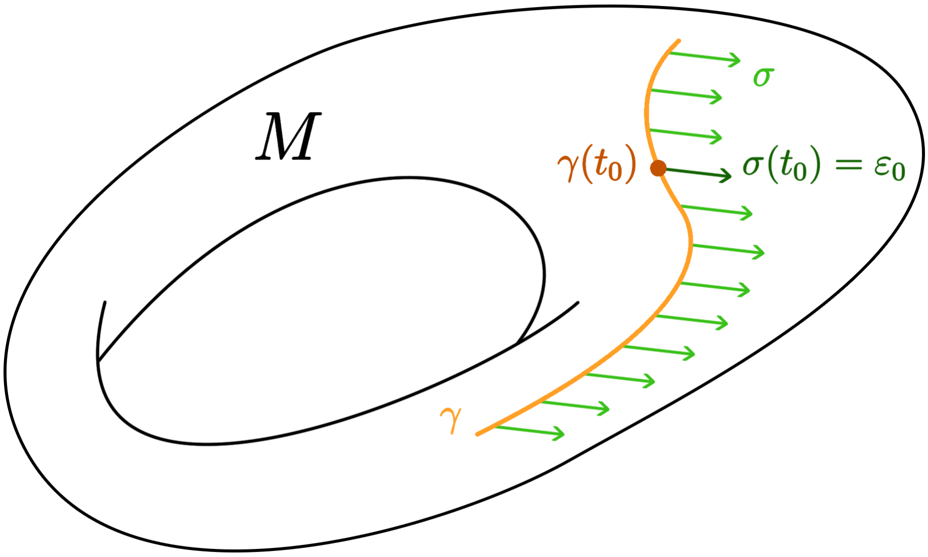
First suppose that I is bounded. We show that there exists a unique parallel section along the restriction of γ to $]\inf I, \sup J[$ that maps t_0 to ε_0. The proof for $]\inf J, \sup I[$ is analogous. Denote B for the set consisting of all $b < t_0$ such that $]b, \sup J[\subset I$ and there exists a unique parallel section along the restriction of γ to $]b, \sup J[$ that maps t_0 to ε_0. Note that $\inf I \leq \inf B \leq \inf J$. We may assume that $\inf B < \inf I$, because in case of equality we are done. There is a unique parallel section σ_1 along the restriction of γ to $]\inf B, \sup J[$ that satisfies $\sigma_1(t_0) = \varepsilon_0$. Now choose a bounded open subinterval $K \subset]\inf I, \inf J[$ around $\inf B$ such that $\gamma_* K$ is contained in the domain of a local frame of E. Then there exists a unique parallel section σ_2 along $\gamma|_K$ satisfying the initial condition

$$\sigma_2\left(\frac{\sup K - \inf B}{2}\right) = \sigma_1\left(\frac{\sup K - \inf B}{2}\right).$$

By uniqueness, we see that σ_1 and σ_2 must coincide on the overlap $]\inf B, \sup K[$ of their domains. We can therefore patch together the maps σ_1 and σ_2 into a unique parallel section along the restriction of γ to $]\inf K, \sup J[$ that maps t_0 to ε_0. By definition of B, it follows that $\inf B \leq \inf K$. This contradicts the fact that K is an open neighbourhood around $\inf B$.

The case where I is not bounded can be proven similarly. ■

Let $\gamma\colon I \to M$ be a curve. Fix $t_0 \in I$ and $\varepsilon_0 \in \pi^*\{\gamma(t_0)\}$. We define the *parallel transport of ε_0 along γ* as the unique parallel section $\sigma\colon I \to E$ along γ that satisfies $\sigma(t_0) = \varepsilon_0$.



Parallel Transport of ε_0 along γ

4.3 Curvature of Vector Bundles and Riemannian Curvature

In order to introduce curvature, we first consider the following example.

Example 4.6. Consider the trivial vector bundle $M \times \mathbb{R}$ of rank 1 over a manifold M. The single map

$$M \to M \times \mathbb{R}, \quad p \mapsto (p, 1)$$

defines a global frame of $M \times \mathbb{R}$. Hence, Lemma 2.7 implies that any smooth section of $M \times \mathbb{R}$ is of the form

$$s^f\colon M \to M \times \mathbb{R}, \quad s^f(p) := (p, f(p))$$

for some smooth map $f\colon M \to \mathbb{R}$. Using the properties of Lie derivative, it is easy to verify that the map

$$\Gamma(TM) \to \mathrm{End}(\Gamma(M \times \mathbb{R})), \quad X \mapsto \mathscr{L}_X$$

defined by the assignment

$$\mathscr{L}_X(s^f)(p) := (p, L_X(f)(p))$$

defines a connection on the vector bundle $M \times \mathbb{R}$. Again, by properties of the Lie derivative, this connection satisfies

$$\mathscr{L}_X\mathscr{L}_Y - \mathscr{L}_Y\mathscr{L}_X = \mathscr{L}_{[X,Y]}$$

for all vector fields $X, Y \in \Gamma(TM)$. $\lhd_{\text{Example}}$

One may wonder if all connections share this property. The answer is 'no': an important fact about connections on a vector bundle E over a manifold M is that, in general,

$$\nabla_X \nabla_Y - \nabla_Y \nabla_X \neq \nabla_{[X,Y]}.$$

A measure of this lack of commutativity is the *curvature of* E, which is defined as the map

$$\kappa_\nabla \colon \Gamma(TM) \times \Gamma(TM) \to \mathrm{End}(\Gamma(E)), \quad \kappa_\nabla(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

We say that E is *curvature-free* if and only if κ_∇ vanishes identically. In case of the Levi-Civita connection on the tangent bundle of a pseudo-Riemannian manifold, we call this operator the *Riemannian Curvature of* M.

Now let $U \subset M$ be an open subset. Then there is a natural local operator

$$\kappa_\nabla^U \colon \Gamma_U(TM) \times \Gamma_U(TM) \to \mathrm{End}(\Gamma_U(E)), \quad \kappa_\nabla^U(X,Y) := \nabla_X^U \nabla_Y^U - \nabla_Y^U \nabla_X^U - \nabla_{[X,Y]}^U.$$

Note that κ_∇^U also vanishes identically if E is curvature-free.

Lemma 4.7. *Let E be a vector bundle over M with a connection and $U \subset M$ an open subset.*

(i) The operator κ_∇^U is $C^\infty(U)$-bilinear.

(ii) For every $X, Y \in \Gamma_U(TM)$, the map $\kappa_\nabla^U(X,Y)$ is $C^\infty(U)$-linear.

Proof. First we prove part (i). Observe that

$$\kappa_\nabla^U(X,Y) = -\kappa_\nabla^U(Y,X)$$

for all local vector fields $X, Y \in \Gamma_U(TM)$. Therefore it suffices to show that the expression $\kappa_\nabla(X,Y)$ is $C^\infty(U)$-linear in X. For $f \in C^\infty(U)$ and $s \in \Gamma_U(E)$ we calculate

$$\kappa_\nabla^U(f \cdot X, Y)(s) = f \cdot \nabla_X^U(\nabla_Y^U(s)) - \nabla_Y^U(f \cdot \nabla_X^U(s)) - \nabla_{f\cdot[X,Y]-L_Y^U(f)\cdot X}^U(s).$$

By the Leibniz-rule we have

$$\nabla_Y^U(f \cdot \nabla_X^U(s)) = L_Y^U(f) \cdot \nabla_X^U(s) + f \cdot \nabla_Y^U(\nabla_X^U(s)).$$

Also note that

$$\nabla_{f\cdot[X,Y]-L_Y^U(f)\cdot X}^U(s) = f \cdot \nabla_{[X,Y]}^U(s) - L_Y^U(f) \cdot \nabla_X^U(s).$$

Combining the equations yields the desired result. Part (ii) can be proven in a similar manner: just use the Leibniz-rule successively and recall that

$$L_{[X,Y]}^U(f) = L_X^U(L_Y^U(f)) - L_Y^U(L_X^U(f))$$

by definition of the Lie bracket. ■

The following lemma plays a crucial role in the proof of the main theorem.

Lemma 4.8 (Parallel Extension)**.** *Let E be a vector bundle over an m-dimensional manifold M with a connection. Suppose that E is curvature-free. Given $p_0 \in M$ and $\varepsilon_0 \in \pi^*\{p_0\}$, there exists a parallel local section $s \colon U \to E$ defined on an open neighbourhood U of p_0 in M such that $s(p_0) = \varepsilon_0$.*

Proof. Choose a chart $\chi\colon U \to \widetilde{U}$ around p_0. By shrinking and translating the codomain of χ if necessary, we may assume that $\widetilde{U} = I^m$ for some open interval I around 0. We may also assume that $\chi(p_0) = \vec{0}$. For all $1 \leq k \leq m$ and each $p \in M$ we define a smooth curve by

$$\gamma_k^{(p)}\colon I \to M, \quad \gamma_k^{(p)}(t) := \chi^{-1}(\chi^1(p), \ldots, \chi^{k-1}(p), t, 0, \ldots, 0).$$

These curves satisfy

$$\gamma_{k+1}^{(p)}(0) = \gamma_k^{(p)}(\chi^k(p)).$$

Let $\sigma_1^{(p)}\colon I \to E$ be the parallel transport of $\varepsilon_0 \in \pi^*\{\gamma_1^{(p)}(0)\}$ along $\gamma_1^{(p)}$. Inductively define $\sigma_{k+1}^{(p)}\colon I \to E$ to be the parallel transport of $\sigma_k^{(p)}(\chi^k(p)) \in \pi^*\{\gamma_{k+1}^{(p)}(0)\}$ along $\gamma_{k+1}^{(p)}$. Observe that

$$\sigma_{k+1}^{(p)}(0) = \sigma_k^{(p)}(\chi^k(p)).$$

We now claim that

$$s\colon U \to E, \quad s(p) := \sigma_m^{(p)}(\chi^m(p))$$

is a parallel local section of E that maps p_0 to ε_0. We split this claim into smaller claims.

$\ggg$ **Claim 1:** *The map s is a local section.*

To prove that s is a section, we must show that s is a smooth map that satisfies $\pi \circ s = \iota_U$.

$\ggg$ **Claim 1.1:** *The map s is smooth.*

This follows from an inductive application of the part of Lemma 4.4 that states that solutions to linear ordinary differential equations depends smoothly on the initial data. $\lhd_{1.1}$

$\ggg$ **Claim 1.2:** *The map s satisfies $\pi \circ s = \iota_U$.*

Unwinding the definitions, we see that

$$\pi(s(p)) = \pi(\sigma_m^{(p)}(\chi^m(p))) = \gamma_m^{(p)}(\chi^m(p)) = p.$$

This shows that $\pi \circ s = \iota_U$. $\lhd_{1.2}$

Hence, s is indeed a local section of E. $\lhd_1$

$\ggg$ **Claim 2:** *The local section s is parallel.*

To prove that s is parallel, we can use Lemma 4.2 and show that $\nabla_X^U(s)$ vanishes for every $X \in \Gamma_U(TM)$. By $C^\infty(U)$-linearity, it suffices to prove that $\nabla_{\partial/\partial\chi^l}^U(s)$ vanishes for every index $1 \leq l \leq m$. For $p \in M$ and $t \in I$ we observe that

$$\begin{aligned} s \circ \gamma_k^{(p)}(t) = \sigma_m^{(\gamma_k^{(p)}(t))}(\chi^m(\gamma_k^{(p)}(t))) = \sigma_m^{(\gamma_k^{(p)}(t))}(0) = \sigma_{m-1}^{(\gamma_k^{(p)}(t))}(\chi^{m-1}(\gamma_k^{(p)}(t))) \\ = \ldots = \sigma_k^{\gamma_k^{(p)}(t)}(\chi^k(\gamma_k^{(p)}(t))) = \sigma_k^{(\gamma_k^{(p)}(t))}(t) \end{aligned}$$

and also, for each $f \in C^\infty(M)$, that

$$(\dot{\gamma}_k^{(p)})^t f = (d\gamma_k^{(p)})_t\left(\frac{d}{dt}\bigg|^t\right) f = \frac{d}{dt}\bigg|^t f \circ \gamma_k^{(p)} = \frac{\partial}{\partial x^k}\bigg|^{\chi(\gamma_k^{(p)}(t))} f \circ \iota_U \circ \chi^{-1} = \frac{\partial}{\partial \chi^k}\bigg|^{\gamma_k^{(p)}(t)} f.$$

For every $1 \leq k \leq m$ we define the set

$$U_k := \{p \in U \mid \chi^{k+1}(p) = \ldots = \chi^m(p) = 0\}.$$

Now, let $p \in U_k$ be a point. We see that

$$\gamma_k^{(p)}(\chi^k(p)) = p.$$

Therefore, it follows that

$$(\dot{\gamma}_k^{(p)})^{\chi^k(p)} = \frac{\partial}{\partial \chi^k}\bigg|^p.$$

Hence, by definition of parallel transport we have

$$\nabla^U_{\partial/\partial\chi^k}(s)(p) = \nabla^U_{(\dot{\gamma}_k^{(p)})^{\chi^k(p)}}(s) = \frac{\nabla^{\gamma_k^{(p)}}\sigma_k^{(p)}}{dt}(\chi^k(p)) = 0^{(\gamma_k^{(p)}(\chi^k(p)))} = 0^{(p)}.$$

This shows that $\nabla_{\partial/\partial\chi^k}(s)$ vanishes on U_k for every $1 \leq k \leq m$. We will now prove by induction on $1 \leq k \leq m$ that $\nabla^U_{\partial/\partial\chi^l}(s)$ vanishes on U_k for every $1 \leq l \leq k$. The case $k = 1$ immediately follows from the fact that $\nabla_{\partial/\partial\chi^1}(s)$ vanishes on U_1. So assume that the statement is true for some arbitrary k. We prove that the statement also holds for $k+1$. Since $\nabla^U_{\partial/\partial\chi k+1}(s)$ vanishes on U_{k+1}, we may assume that $0 \leq l \leq k$.

⋙ **Claim 2.1:** *The section $\nabla^U_{\partial/\partial\chi^l}(\nabla^U_{\partial/\partial\chi^{k+1}}(s))$ vanishes on U_{k+1}.*

Let $p \in U_{k+1}$ be a point. We will use Lemma 2.10 to show that $\nabla^U_{\partial/\partial\chi^l}(\nabla^U_{\partial/\partial\chi^{k+1}}(s))$ vanishes at p. Define a smooth curve by

$$z_l^{(p)} \colon I \to M, \quad z_l^{(p)}(t) := \chi^{-1}(\chi^1(p), \ldots, \chi^{l-1}(p), t, \chi^{l+1}(p), \ldots, \chi^{k+1}(p), 0, \ldots, 0).$$

We note that

$$z_l^{(p)}(\chi^l(p)) = p$$

and also that

$$(\dot{z}_l^{(p)})^{\chi^l(p)} f = (dz_l^{(p)})_{\chi^l(p)}\left(\frac{d}{dt}\bigg|^{\chi^l(p)}\right) f = \frac{d}{dt}\bigg|^{\chi^l(p)} f \circ z_l^{(p)} = \frac{\partial}{\partial x^l}\bigg|^{\chi(p)} f \circ \iota_U \circ \chi^{-1} = \frac{\partial}{\partial \chi^l}\bigg|^p f$$

for all $f \in C^\infty(M)$. Observe that $(z_l^{(p)})_* I \subset U_{k+1}$. Since $\nabla^U_{\partial/\partial\chi^{k+1}}(s)$ vanishes on U_{k+1}, it follows that

$$\nabla^U_{\partial/\partial\chi^{k+1}}(s)(z_l^{(p)}(t)) = 0^{(z_l^{(p)}(t))}$$

for all $t \in I$. Hence, Lemma 2.10 implies that

$$\nabla^U_{\partial/\partial\chi^l}(\nabla^U_{\partial/\partial\chi^{k+1}}(s))(p) = 0^{(p)}.$$

This shows that $\nabla^U_{\partial/\partial\chi^l}(\nabla^U_{\partial/\partial\chi^{k+1}}(s))$ vanishes on U_{k+1}. $\lhd_{2.1}$

⋙ **Claim 2.2:** *The section $\nabla_{\partial/\partial\chi^l}(s)$ vanishes on U_{k+1}.*

Suppose that $p \in U_{k+1}$. We want to show that $\nabla^U_{\partial/\partial\chi^l}(s)$ vanishes at p. Since

$$\gamma_{k+1}^{(p)}(\chi^{k+1}(p)) = p,$$

it suffices to prove that the section

$$\varphi_l^{(p)}\colon I \to E, \quad \varphi_l(t) := \nabla^U_{\partial/\partial\chi^l}(s)(\gamma^{(p)}_{k+1}(t))$$

along $\gamma^{(p)}_{k+1}$ vanishes identically. By the induction hypothesis, we know that $\nabla_{\partial/\partial\chi^l}(s)$ vanishes on U_k. Hence, since $\gamma^{(p)}_{k+1}(0) \in U_k$, we see that $\varphi_l^{(p)}$ vanishes at 0. Because the unique parallel transport of the zero-vector along $\gamma^{(p)}_{k+1}$ must be identically zero along $\gamma^{(p)}_{k+1}$, it therefore suffices to prove that $\varphi_l^{(p)}$ is parallel along $\gamma^{(p)}_{k+1}$. We show that the covariant derivative of $\varphi_l^{(p)}$ along $\gamma^{(p)}_{k+1}$ vanishes on I. Note that the Lie bracket $[\partial/\partial\chi^{k+1}, \partial/\partial\chi^l]$ vanishes on U. So, because E is curvature-free, we observe that

$$\begin{aligned}\frac{\nabla^{\gamma^{(p)}_{k+1}}\varphi_l^{(p)}}{dt}(t) = \nabla^U_{(\dot\gamma^{(p)}_{k+1})^t}(\nabla^U_{\partial/\partial\chi^l}(s)) &= \nabla^U_{\partial/\partial\chi^{k+1}}(\nabla^U_{\partial/\partial\chi^l}(s))(\gamma^{(p)}_{k+1}(t)) \\ &= \nabla^U_{\partial/\partial\chi^l}(\nabla^U_{\partial/\partial\chi^{k+1}}(s))(\gamma^{(p)}_{k+1}(t))\end{aligned}$$

for all $t \in I$. Since $(\gamma^{(p)}_{k+1})_* I \subset U_{k+1}$, it follows that $\nabla^U_{\partial/\partial\chi^l}(s))$ vanishes on U_{k+1}. $\lhd_{2.2}$

The case $k = m$ shows that $\nabla_{\partial/\partial\chi^l}(s)$ vanishes on $U_m = U$ for all $1 \leq l \leq m$. Hence, it follows that s is parallel. $\lhd_2$

$\ggg$ **Claim 3:** *The parallel local section s sends p_0 to ε_0.*

Indeed, the calculation

$$\begin{aligned}s(p_0) = \sigma_m^{(p_0)}(\chi^m(p_0)) = \sigma_m^{(p_0)}(0) = \sigma_{m-1}^{(p_0)}(\chi^{m-1}(p_0)) &= \sigma_{m-1}^{(p_0)}(0) \\ &= \ldots = \sigma_1^{(p_0)}(\chi^1(p_0)) = \sigma_1^{(p_0)}(0) = \varepsilon_0\end{aligned}$$

shows that s sends p_0 to ε_0. $\lhd_3$

This finishes the proof of the lemma. ∎

Let M be a pseudo-Riemannian manifold and let $\chi\colon U \to \widetilde{U}$ be a chart of M. For all $1 \leq i, j, k \leq m$ we can find smooth functions $R(\chi)_{i,j,k}{}^1, \ldots, R(\chi)_{i,j,k}{}^m \in C^\infty(U)$ such that the Riemann curvature of M can locally be written as

$$\kappa^U_\nabla\left(\frac{\partial}{\partial\chi^i}, \frac{\partial}{\partial\chi^j}\right)\left(\frac{\partial}{\partial\chi^k}\right) = \sum_l R(\chi)_{i,j,k}{}^l \cdot \frac{\partial}{\partial\chi^l}.$$

We call these component functions the *Riemann symbols with respect to* χ. The following lemma gives a formula for the Riemann symbols in terms of the Christoffel symbols.

Lemma 4.9. *Let M be a pseudo-Riemannian manifold. The Riemann symbols with respect to a chart $\chi\colon U \to \widetilde{U}$ are given by*

$$R(\chi)_{i,j,k}{}^l = \frac{\partial\Gamma(\chi)_{j,k}{}^l}{\partial\chi^i} - \frac{\partial\Gamma(\chi)_{i,k}{}^l}{\partial\chi^j} + \sum_\mu \Gamma(\chi)_{j,k}{}^\mu \cdot \Gamma(\chi)_{i,\mu}{}^l - \Gamma(\chi)_{i,k}{}^\mu \cdot \Gamma(\chi)_{j,\mu}{}^l$$

for all $1 \leq i, j, k, l \leq m$.

Proof. Since the Lie brackets of coordinate vector fields vanish, we have

$$\begin{aligned}\kappa^U_\nabla\left(\frac{\partial}{\partial\chi^i},\frac{\partial}{\partial\chi^j}\right)\left(\frac{\partial}{\partial\chi^k}\right) &= \nabla_{\partial/\partial\chi^i}\nabla_{\partial/\partial\chi^j}\left(\frac{\partial}{\partial\chi^k}\right) - \nabla_{\partial/\partial\chi^j}\nabla_{\partial/\partial\chi^i}\left(\frac{\partial}{\partial\chi^k}\right) \\ &= \sum_l \nabla_{\partial/\partial\chi^i}\left(\Gamma(\chi)_{j,k}{}^l\cdot\frac{\partial}{\partial\chi^l}\right) - \nabla_{\partial/\partial\chi^j}\left(\Gamma(\chi)_{i,k}{}^l\cdot\frac{\partial}{\partial\chi^l}\right).\end{aligned}$$

By the Leibniz-rule it follows that

$$\begin{aligned}\sum_l \nabla_{\partial/\partial\chi^i}\left(\Gamma(\chi)_{j,k}{}^l\cdot\frac{\partial}{\partial\chi^l}\right) &= \sum_l \frac{\partial\Gamma(\chi)_{j,k}{}^l}{\partial\chi^i}\cdot\frac{\partial}{\partial\chi^l} + \Gamma(\chi)_{j,k}{}^l\cdot\nabla_{\partial/\partial\chi^i}\left(\frac{\partial}{\partial\chi^l}\right) \\ &= \sum_l\left(\frac{\partial\Gamma(\chi)_{j,k}{}^l}{\partial\chi^i} + \sum_\mu \Gamma(\chi)_{j,k}{}^\mu\cdot\Gamma(\chi)_{i,\mu}{}^l\right)\cdot\frac{\partial}{\partial\chi^l}\end{aligned}$$

and similarly that

$$\sum_l \nabla_{\partial/\partial\chi^j}\left(\Gamma(\chi)_{i,k}{}^l\cdot\frac{\partial}{\partial\chi^l}\right) = \sum_l\left(\frac{\partial\Gamma(\chi)_{i,k}{}^l}{\partial\chi^j} + \sum_\mu \Gamma(\chi)_{i,k}{}^\mu\cdot\Gamma(\chi)_{j,\mu}{}^l\right)\cdot\frac{\partial}{\partial\chi^l}.$$

Substraction yields the desired result. ∎

5 Proof of Main Theorem and Application

5.1 Frobenius Theorem

Let X^1 and X^2 be two local vector fields defined on some open subset $U \subset M$. Let $D_1 \subset U \times \mathbb{R}$ and $D_2 \subset U \times \mathbb{R}$ be the domains of the flows of X_1 and X_2 respectively. Recall that the flows $\varphi_1 \colon D_1 \to U$ and $\varphi_2 \colon D_2 \to U$ of X_1 and X_2 *commute* if and only if the following condition holds for every $p \in U$: whenever I_1 and I_2 are open intervals around 0 satisfying $\{p\} \times I_1 \subset D_1$ and $\{p\} \times I_2 \subset D_2$ such that either $(\varphi_2)_*(\{p\} \times I_2) \times I_1 \subset D_1$ or $(\varphi_1)_*(\{p\} \times I_1) \times I_2 \subset D_2$, then both inclusions hold and

$$\varphi_1(\varphi_2(p, t_2), t_1) = \varphi_2(\varphi_1(p, t_1), t_2)$$

for all $(t_1, t_2) \in I_1 \times I_2$.

Lemma 5.1. *Vector fields commute if and only if their corresponding flows commute.*

Proof. See [9], Theorem 9.44. ∎

The following theorem plays a crucial role in the proof of the main theorem.

Theorem 5.2 (Frobenius Theorem)**.** *A local frame $(E_1, \dots, E_m)$ of TM is locally expressible as a coordinate frame if and only if E_i and E_j commute for all $1 \le i, j \le m$.*

Proof. If $(E_1, \dots, E_m)$ is a coordinate frame, then it follows from a straightforward calculation that $[E_i, E_j]$ vanishes identically for every $1 \le i, j \le n$.

Conversely, assume that E_i and E_j commute for all $1 \le i, j \le m$. Let V be the domain of the given local frame. Fix a point $p_0 \in V$. Write $D_i \subset V \times \mathbb{R}$ for the domain of the flow of E_i and let $\varphi_i \colon D_i \to V$ be its flow for all $1 \le i \le m$. The fundamental theorem on flows (see [9], Theorem 9.12) implies that for each $t \in \mathbb{R}$ the set

$$D_i^t := \{p \in V \mid (p, t) \in D_i\}$$

is open in V and the map

$$\varphi_i^t \colon D_i^t \to D_i^{-t}, \quad \varphi_i^t(p) := \varphi_i(p, t)$$

is a diffeomorphism. Choose $\varepsilon_m > 0$ and a neighbourhood V_m of p_0 in V such that φ_m maps $]-\varepsilon_m, \varepsilon_m[\times V_m$ into V. Then, for $1 \le i \le m-1$, inductively choose $\varepsilon_i > 0$ and a neighbourhood V_i of p in V_{i+1} such that φ_i maps $]-\varepsilon_i, \varepsilon_i[\times V_i$ into V_{i+1}. Now define $\varepsilon := \min\{\varepsilon_1, \dots, \varepsilon_m\}$ and $I :=]-\varepsilon, \varepsilon[$. By construction, it follows that the smooth map

$$\Phi \colon I^m \to M, \quad \Phi(\vec{x}) := \varphi_m^{x^m}(\dots \varphi_1^{x^1}(p_0) \dots)$$

is well-defined. For every $1 \le i \le m$ the map

$$\gamma_i \colon I \to M, \quad \gamma_i(t) := \varphi_i^t(p_0)$$

is an integral curve of E_i. Unwinding the definitions, we see that $\gamma_i(0) = \Phi(\vec{0}) = p_0$. Because γ_i is an integral curve of E_i, it follows that

$$(d\Phi)_{\vec{0}} \left(\frac{\partial}{\partial x^i} \bigg|^{\vec{0}} \right) f = \frac{\partial}{\partial x^i} \bigg|^{\vec{0}} f \circ \Phi = \frac{d}{dt} \bigg|^{0} f \circ \gamma_i = (d\gamma_i)_0 \left(\frac{d}{dt} \bigg|^{0} \right) f = E_i^{p_0} f$$

for every $f \in C^\infty(M)$. It follows that $(d\Phi)_{\vec{0}}$ takes the basis $(\partial/\partial x^1|^{\vec{0}}, \ldots, \partial/\partial x^m|^{\vec{0}})$ for $T_{\vec{0}} I^m$ to the basis $(E_1^{p_0}, \ldots, E_m^{p_0})$ for $T_{p_0}M$. Consequently, we deduce that the differential $(d\Phi)_{\vec{0}}$ is an isomorphism of vector spaces. The inverse function theorem implies that Φ restricts to a diffeomorphism on a neighbourhood $\widetilde{U}$ of $\vec{0}$ in I^m. Denote $U := \Phi_* \widetilde{U}$ and let $\chi \colon U \to \widetilde{U}$ be the inverse of the diffeomorphism we just found. Then χ is a chart of M around p_0 satisfying

$$\frac{\partial}{\partial \chi^i}\bigg|^p f = (d\iota_U)_p \circ (d\chi)_p^{-1}\left(\frac{\partial}{\partial x^i}\bigg|^{\chi(p)}\right) f = \frac{\partial}{\partial x^i}\bigg|^{\chi(p)} f \circ \iota_U \circ \chi^{-1} = \frac{\partial}{\partial x^i}\bigg|^{\chi(p)} f \circ \Phi$$

for all $p \in U$ and $f \in C^\infty(M)$. By Lemma 5.1, vector fields commute if and only if their corresponding flows commute. Hence the map

$$\gamma_i^{(p)} \colon I \to M, \quad \gamma_i^{(p)}(t) := \varphi_i^t(\varphi_m^{\chi^m(p)}(\ldots \varphi_{i+1}^{\chi^{i+1}(p)}(\varphi_{i-1}^{\chi^{i-1}(p)}(\ldots \varphi_1^{\chi^1(p)}(p_0)\ldots))\ldots))$$

is well-defined and satisfies

$$\gamma_i^{(p)}(\chi^i(p)) = \Phi(\chi(p)) = p.$$

Since $\gamma_i^{(p)}$ is an integral curve, it follows that

$$\frac{\partial}{\partial x^i}\bigg|^{\chi(p)} f \circ \Phi = \frac{d}{dt}\bigg|^{\chi^i(p)} f \circ \gamma_i^{(p)} = (d\gamma_i^{(p)})_{\chi^i(p)}\left(\frac{d}{dt}\bigg|^{\chi^i(p)}\right) = E_i^p f$$

and therefore we conclude that

$$E_i|_U = \frac{\partial}{\partial \chi^i}. \qquad \blacksquare$$

5.2 Flat Pseudo-Riemannian Manifolds

Let m-dimensional pseudo-Riemannian manifold M with signature r. A chart $\chi \colon U \to \widetilde{U}$ *intertwines the pseudo-metrics of* M *and* $\widetilde{U}$ if and only if for each $p \in U$ the following diagram commutes:

$$\begin{tikzcd}
T_pM \times T_pM \arrow[r, "g_p"] \arrow[d, "(d\iota_U)_p^{-1} \times (d\iota_U)_p^{-1}"'] & \mathbb{R} \\
T_pU \times T_pU \arrow[r, "(d\chi)_p \times (d\chi)_p"'] & T_{\chi(p)}\widetilde{U} \times T_{\chi(p)}\widetilde{U} \arrow[u, "\tilde{g}^{m,r}_{\chi(p)}"']
\end{tikzcd}$$

Or, equivalently, if and only if $g^\chi_{i,j} \equiv 0$ whenever $i \neq j$ and

$$g^\chi_{1,1} \equiv 1, \quad \ldots, \quad , g^\chi_{r,r} \equiv 1, \quad g^\chi_{r+1,r+1} \equiv -1, \quad \ldots, \quad g^\chi_{m,m} \equiv -1.$$

We say that M is *flat* if and only if each point of M is contained in the domain of a chart $\chi \colon U \to \widetilde{U}$ that intertwines the pseudo-metrics of M and $\widetilde{U}$. Intuively, it means that an 'm-dimensional being' can not distinquish between a flat pseudo-Riemannian manifold M with signature r and pseudo-Euclidian space with signature r by local measurements.

Example 5.3. To illustrate the definition of flatness, we show that the 2-cylinder

$$C^2 := \{\vec{y} \in \mathbb{R}^3 \mid (y^1)^2 + (y^2)^2 = 1, \ -1 < y^3 < 1\}$$

is flat. Let $\iota\colon C^2 \to \mathbb{R}^3$ denote the inclusion of C^2 into $\mathbb{R}^3$. We equip the manifold C^2 with the metric

$$g\colon C^2 \to \mathscr{T}_{2,0}C^2, \quad g(\vec{y}) := (\vec{y}, \tilde{g}^{3,3}_{\vec{y}} \circ ((d\iota)_{\vec{y}} \times (d\iota)_{\vec{y}})).$$

Observe that the four half cylinders determined by

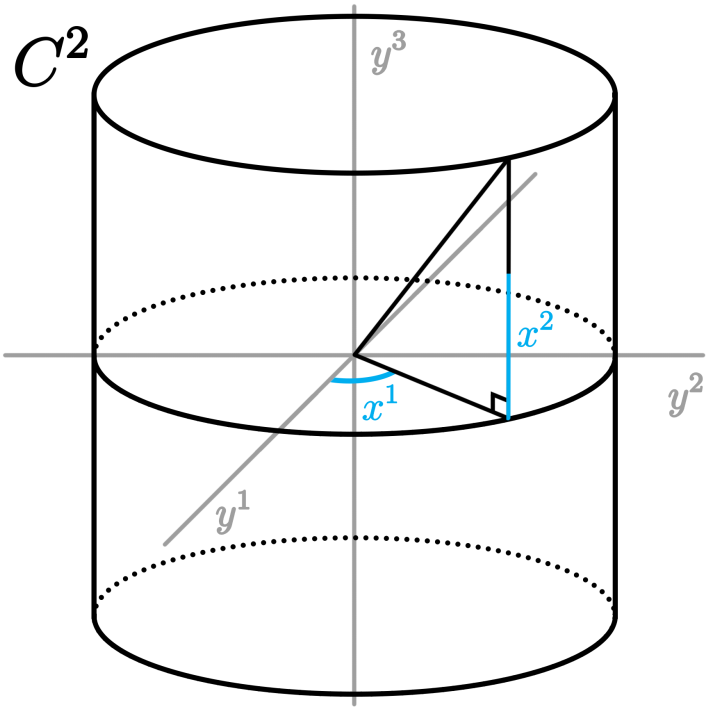
$$y^1 < 0, \quad y^1 > 0, \quad y^2 < 0, \quad y^2 > 0$$

form an open cover of the entire cylinder. We claim that cylindrical coordinates preserve the metric of C^2 on each of the four half cylinders. We treat the case in which $y^2 > 0$. The other cases can be proven analogously. Hence, consider the open subsets

$$U := \{\vec{y} \in C^2 \mid y^2 > 0\}, \quad \widetilde{U} := \{\vec{x} \in \mathbb{R}^2 \mid 0 < x^1 < \pi,\ -1 < x^2 < 1\}$$

of C^2 and $\mathbb{R}^2$ respectively and define a chart $\chi\colon U \to \widetilde{U}$ to be the inverse of the local parametrization of S^2 given by

$$\mathrm{par}\colon \widetilde{U} \to \mathbb{R}^3, \quad \mathrm{par}(\vec{x}) := (\cos(x^1), \sin(x^1), x^2).$$



Cylindrical Coordinates

Unwinding the definitions, we see that

$$g^{\chi}_{i,j}(\vec{y}) = g_{\vec{y}}\left(\frac{\partial}{\partial \chi^i}\bigg|^{\vec{y}}, \frac{\partial}{\partial \chi^j}\bigg|^{\vec{y}}\right) = \tilde{g}^{3,3}_{\vec{y}}\left((d\iota)_{\vec{y}}\left(\frac{\partial}{\partial \chi^i}\bigg|^{\vec{y}}\right), (d\iota)_{\vec{y}}\left(\frac{\partial}{\partial \chi^j}\bigg|^{\vec{y}}\right)\right)$$

for all indices $1 \leq i, j \leq 2$ and each $\vec{y} \in U$. For $1 \leq k \leq 2$ we have

$$\begin{aligned}(d\iota)_{\vec{y}}\left(\frac{\partial}{\partial \chi^k}\bigg|^{\vec{y}}\right) f &= \frac{\partial}{\partial x^k}\bigg|^{\chi(\vec{y})} f \circ \iota \circ \iota_U \circ \chi^{-1} = \frac{\partial}{\partial x^k}\bigg|^{\chi(\vec{y})} f \circ \mathrm{par} \\ &= \frac{\partial f}{\partial y^1}(\vec{y}) \cdot \frac{\partial\, \mathrm{par}^1}{\partial x^k}(\chi(\vec{y})) + \frac{\partial f}{\partial y^2}(\vec{y}) \cdot \frac{\partial\, \mathrm{par}^2}{\partial x^k}(\chi(\vec{y})) + \frac{\partial f}{\partial y^3}(\vec{y}) \cdot \frac{\partial\, \mathrm{par}^3}{\partial x^k}(\chi(\vec{y}))\end{aligned}$$

for all $f \in C^\infty(\mathbb{R}^3)$. Computing the partial derivatives of par^1, par^2 and par^3 with respect to the variable x^1 yields

$$(d\iota)_{\mathrm{par}(\vec{x})}\left(\frac{\partial}{\partial \chi^1}\bigg|^{\mathrm{par}(\vec{x})}\right) = -\sin(x^1)\cdot \frac{\partial}{\partial y^1}\bigg|^{\mathrm{par}(\vec{x})} + \cos(x^1)\cdot \frac{\partial}{\partial y^2}\bigg|^{\mathrm{par}(\vec{x})}$$

for all $x \in \widetilde{U}$. Similarly, differentiating with respect to the variable x^2 gives

$$(d\iota)_{\mathrm{par}(\vec{x})}\left(\frac{\partial}{\partial \chi^2}\bigg|^{\mathrm{par}(\vec{x})}\right) = \frac{\partial}{\partial y^3}\bigg|^{\mathrm{par}(\vec{x})}.$$

Straightforward calculations now show that

$$[g^\chi_{i,j}(\vec{y})]_{i,j} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This proves that χ intertwines the metrics of C^2 and $\widetilde{U}$ and we conclude that C^2 is indeed a flat space. $\lhd_{\text{Example}}$

Theorem 5.4 (Main Theorem)**.** *A pseudo-Riemannian manifold is flat if and only if its Riemann curvature vanishes identically.*

Proof. Let M be an m-dimensional pseudo-Riemannian manifold with signature r.

$\ggg$ **Claim 1:** *If M is flat, then the Riemann curvature of M vanishes identically.*

Suppose that M is flat. We want to prove that κ_∇ vanishes identically. Let $\chi\colon U \to \widetilde{U}$ be a chart that intertwines the pseudo-metrics of M and $\widetilde{U}$. Since

$$\kappa_\nabla(X,Y)(Z)|_U = \kappa_\nabla^U(X|_U, Y|_U)(Z|_U)$$

for all vector fields $X, Y, Z \in \Gamma(TM)$, it suffices to prove that κ_∇^U vanishes identically. We can restrict our attention even further: by $C^\infty(U)$-linearity it suffices to prove that the expression

$$\kappa_\nabla^U\left(\frac{\partial}{\partial\chi^i}, \frac{\partial}{\partial\chi^j}\right)\left(\frac{\partial}{\partial\chi^k}\right) = \sum_l R(\chi)_{i,j,k}{}^l \cdot \frac{\partial}{\partial \chi^l}$$

is the zero vector field on U for all $1 \le i, j, k \le m$. Because M is flat, the component functions of g with respect to χ satisfy $g^\chi_{\mu,\nu} \equiv \pm\delta_{\mu,\nu}$ for all $1 \le \mu, \nu \le m$. Hence, the Lie derivatives of the maps $g^\chi_{\mu,\nu}$ vanish on U. Thus Lemma 3.8 implies that the Christoffel symbols of the Levi-Civita connection also vanish on U. Consequently, Lemma 4.9 shows that the Riemann symbols are zero on U aswell. $\lhd_1$

$\ggg$ **Claim 2:** *If the Riemann curvature of M vanishes identically, then M is flat.*

Suppose that the Riemann curvature of M vanishes identically. Fix a point $p_0 \in M$ and choose an orthonormal basis $(u_1, \ldots, u_m)$ for $T_{p_0}M$. We may assume that this basis is in standard order, i.e.

$$g_{p_0}(u_1,u_1) = 1, \quad \ldots, \quad g_{p_0}(u_r,u_r) = 1, \quad g_{p_0}(u_{r+1},u_{r+1}) = -1, \quad ,\ldots, \quad g_{p_0}(u_m,u_m) = -1.$$

Since TM is curvature-free, we can use Lemma 4.8 to find parallel vector fields $E_1, \ldots, E_m$ defined on an open neighbourhood V of p_0 such that $E_i^{p_0} = u_i$ for each $1 \le i \le m$. By shrinking

V if necessary we may assume that V is connected. Because E_i and E_j are parallel, Lemma 4.3 implies that $g(E_i, E_j)$ is constant on V. Therefore we find that $g(E_i, E_j) \equiv g_{p_0}(u_i, u_j)$ on V. By virtue of Lemma 4.2 we see that the expression $\nabla^V_X(E_i)$ vanishes on V for all $1 \leq i \leq m$ and $X \in \Gamma_V(TM)$. Because the Levi-Civita connection is torsion-free, we find that

$$[E_i, E_j] = \nabla^V_{E_i}(E_j) - \nabla^V_{E_j}(E_i).$$

Hence we observe that $[E_i, E_j]$ vanishes on V. We conclude that $(E_1, \ldots, E_m)$ is a commuting frame of TM. Using Theorem 5.2 we can find a chart $\chi\colon U \to \widetilde{U}$ defined on some open neighbourhood $U \subset V$ of p_0 such that the induced coordinate frame $(\partial/\partial\chi^1, \ldots, \partial/\partial\chi^m)$ coincides with the local frame $(E_1|_U, \ldots, E_m|_U)$. Hence we find that

$$g^{\chi}_{i,j}(p) = g_p\left(\frac{\partial}{\partial\chi^i}\bigg|^p, \frac{\partial}{\partial\chi^j}\bigg|^p \right) = g_p(E^p_i, E^p_j) = g_{p_0}(u_i, u_j)$$

for each $p \in U$. It follows that χ intertwines the pseudo-metrics of M and $\widetilde{U}$. $\lhd_2$

This finishes the proof of the main theorem. ■

In other words, Theorem 5.4 shows that the Riemann curvature is precisely the obstruction to being flat

5.3 Application: Cartography of the Earth's Surface

We demonstrate how the main theorem can be used to show that a pseudo-Riemannian manifold is not flat.

Example 5.5. Consider the 2-dimensional manifold

$$S^2 := \{\vec{y} \in \mathbb{R}^3 \mid (y^1)^2 + (y^2)^2 + (y^3)^2 = 1\},$$

i.e. the 2-sphere. Let $\tilde{g}^{3,3}$ be the Euclidian metric on $\mathbb{R}^3$ and let $\iota\colon S^2 \to \mathbb{R}^3$ denote the inclusion of S^2 into $\mathbb{R}^3$. The *round metric* on S^2 is defined as

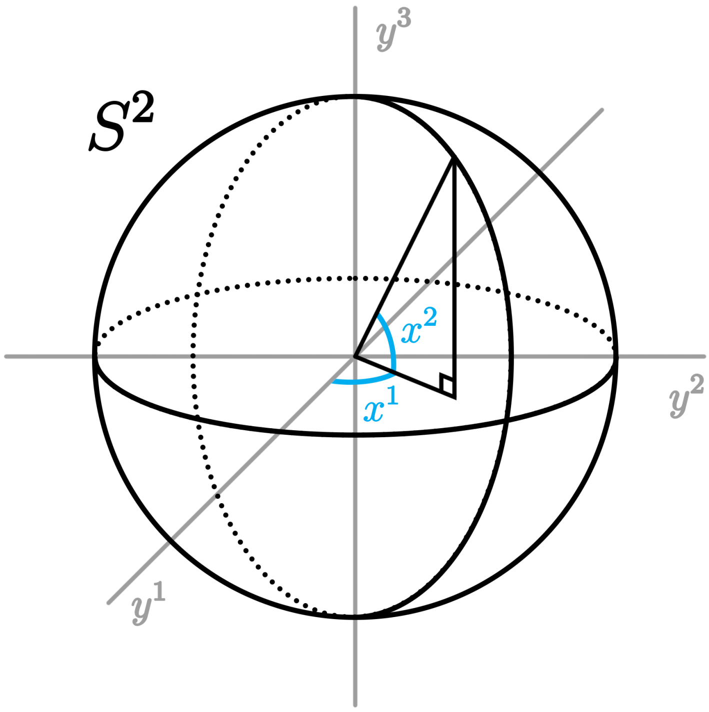
$$g\colon S^2 \to \mathscr{T}_{2,0}S^2, \quad g(\vec{y}) = (\vec{y}, \tilde{g}^{3,3}_{\vec{y}} \circ ((d\iota)_{\vec{y}} \times (d\iota)_{\vec{y}})).$$

In the introduction of this thesis we have shown that S^2 (equipped with the round metric) is not a flat space. We now prove the same result by partially calculating the Riemann curvature of S^2. To do so, we will use spherical coordinates. Consider the open subsets

$$U := \{\vec{y} \in S^2 \mid y^2 > 0\}, \quad \widetilde{U} := \{\vec{x} \in \mathbb{R}^2 \mid 0 < x^1, x^2 < \pi\}$$

of S^2 and $\mathbb{R}^2$ respectively and define a chart $\chi\colon U \to \widetilde{U}$ to be the inverse of the local parametrization of S^2 given by

$$\mathrm{par}\colon \widetilde{U} \to U, \quad \mathrm{par}(\vec{x}) := (\cos(x^1) \cdot \sin(x^2), \sin(x^1) \cdot \sin(x^2), \cos(x^2)).$$



Spherical Coordinates

We can use the same formula as for the 2-cylinder in the preceding subsection to compute

$$(d\iota)_{\mathrm{par}(\vec{x})}\left(\left.\frac{\partial}{\partial \chi^1}\right|^{\mathrm{par}(\vec{x})}\right) = -\sin(x^1)\cdot\sin(x^2)\cdot\left.\frac{\partial}{\partial y^1}\right|^{\mathrm{par}(\vec{x})} + \cos(x^1)\cdot\sin(x^2)\cdot\left.\frac{\partial}{\partial y^2}\right|^{\mathrm{par}(\vec{x})}.$$

Similarly, we calculate

$$\begin{aligned}(d\iota)_{\mathrm{par}(\vec{x})}\left(\left.\frac{\partial}{\partial \chi^2}\right|^{\mathrm{par}(\vec{x})}\right) = \cos(x^1)\cdot\cos(x^2)\cdot\left.\frac{\partial}{\partial y^1}\right|^{\mathrm{par}(\vec{x})} + \sin(x^1)\cdot\cos(x^2)\cdot\left.\frac{\partial}{\partial y^2}\right|^{\mathrm{par}(\vec{x})}\\ - \sin(x^2)\cdot\left.\frac{\partial}{\partial y^3}\right|^{\mathrm{par}(\vec{x})}.\end{aligned}$$

Using the definition of the Euclidian metric $\tilde{g}^{3,0}$, we easily verify that

$$[g^{\chi}_{i,j}(\mathrm{par}(\vec{x}))]_{i,j} = \begin{bmatrix}\sin(x^2)^2 & 0\\ 0 & 1\end{bmatrix}.$$

The inverse matrix of this diagonal matrix is given by

$$[g^{i,j}_{\chi}(\mathrm{par}(\vec{x}))]_{i,j} = \begin{bmatrix}1/\sin(x^2)^2 & 0\\ 0 & 1\end{bmatrix}.$$

We use Lemma 3.8 to calculate

$$[\Gamma(\chi)_{i,j}{}^{1}(\mathrm{par}(\vec{x}))]_{i,j} = \begin{bmatrix}0 & \cos(x^2)/\sin(x^2)\\ \cos(x^2)/\sin(x^2) & 0\end{bmatrix}$$

and also

$$[\Gamma(\chi)_{i,j}{}^{2}(\mathrm{par}(\vec{x}))]_{i,j} = \begin{bmatrix}-\cos(x^2)\cdot\sin(x^2) & 0\\ 0 & 0\end{bmatrix}.$$

To prove that the Riemann curvature of S^2 does not vanish identically, it suffices to find a combination of the indices $1 \leq i, j, k, l \leq 2$ such that the Riemann symbol ${R(\chi)_{i,j,k}}^l$ is not identically zero on U. We choose $i = k = 1$ and $j = l = 2$ and use Lemma 4.9 to see that

$${R(\chi)_{1,2,1}}^2(\mathrm{par}(\vec{x})) = -\sin(x^2)^2.$$

By Theorem 5.4 it now follows that S^2 is not flat. $\lhd_{\text{Example}}$

Distortion of Cylindrical Projection [4]

In particular, if we use S^2 as a mathematical model for the Earth's surface, this result shows that every smooth local Euclidian representation of the Earth's surface (i.e. a chart of S^2) necessarily distorts angles, areas or distances in some fashion.

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