On the internal logic of regular categories

Bachelor thesis

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Contents

1. Introduction

In this thesis basic knowledge of categories and first order model theory is assumed. In particular, we assume the reader is familiar with functors, mono-, epi and isomorphisms, natural transformations and limits. Moreover, the reader should know the statement of the soundness and completeness theorem for first order logic.

In mathematics we often analyse structures. Category theory can be used to study a collection of structures and structure preserving mappings between them.

In model theory we see formulas as a concatenation of symbols according to some rules. We then assign meaning to those symbols. Basic model theory does this by interpreting constants as elements of some set, function symbols as functions and relation symbols as relations.

Categorical logic is where category theory and model theory meet. We assign meaning to symbols by interpreting them in a category. This thesis will focus on a fragment of first order logic: regular logic. In a nutshell, this is a logic that has only equality, conjunction and the existential quantifier as logical operators. We can interpret this in a special type of categories that have some additional structure, which are called regular categories.

This thesis is a more detailed account of chapter 4 of dr. Jaap van Oosten's lecture notes [7]. We will follow its structure unless mentioned otherwise.

When choosing a language to frame a concept, it may be useful to define different sorts of variables. For example, if we wish to describe a vector space, we can define a sort for scalars and a sort for vectors.
As an object is the categorical version of a set, it is natural to assign an object to a sort. It represents the 'set' of those things of that sort. What a regular category allows us to do is define the image of a morphism, which is a subobject. This allows us to give an interpretation for the existential quantifier. We can define a formula as the subobject of all variables that satisfy that formula. We do this using the limits explored in chapter 2.

In chapter 2 we will talk about products, equalisers, pullback squares and coequalisers. We also give some useful statements about those limits and colimits.

Chapter 3 defines rigorously what a regular category is. This allows us to prove theorem 26, which may be considered the main theorem of this chapter. We then define the notion of a subobject.

In chapter 4 we will define regular logic and its interpretation in a regular category more precisely. A sequent is a statement of the form "if A, then B". Furthermore, we define when a sequent is true. Given the sequents "if A, then B" and "if B, then C", we can derive that "if A, then C". We formalise this concept with theories. A theory is a set of sequents, and can be taken intuitively as a set containing some sequents and all sequents we can derive from that. It is the analogue of a proof tree.

This creates a connection between sequents that are true and sequents that can be derived using theories. This naturally gives rise to a soundness theorem (if a sequent can be derived, it is true) and a completeness theorem (if a sequent is true, it can be derived). In chapter 5 we formulate these theorems precisely and give a proof of the soundness theorem. The completeness theorem is not proven due to time constraints.

One may wonder what applications of interpreting logic in a regular category is. It can be used to make reasoning in a regular category more intuitive. Treating this is beyond the scope of this thesis, so I refer the interested reader to chapter D1.3 of [6] for more information and examples.

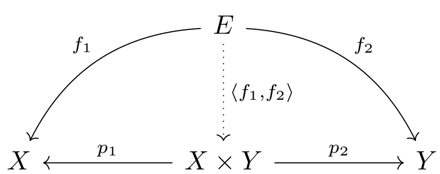
2. **Limits and a colimit**

In this chapter we define the following types of limits: products, equalisers and pullback squares. We show that products are associative in a sense and give an important lemma about pullbacks that is known as the pullback pasting lemma. Finally, we will give the definition of the colimit called a coequaliser.

2.1 Limits

The following limits are needed for the interpretation of formulas.

Definition 1 ([7, p.17], Product)**.** *Let X, Y be objects in $\mathcal{C}$. Consider the embedding of the subcategory consisting of objects X, Y and morphisms id_X and id_Y. Then a limit of this diagram is an object in $\mathcal{C}$ we call $X \times Y$ and two morphisms p_1, p_2 such that for every third object E and morphisms $f_1 : E \to X, f_2 : E \to Y$ we have a unique morphism we call $\langle f_1, f_2 \rangle$ such that the following diagram commutes.*

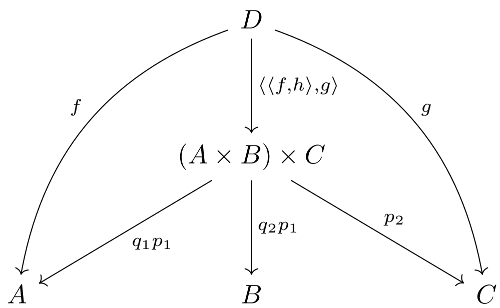


Example 2. This slightly suggestive notation leads us to believe this will be a Cartesian product if $\mathcal{C}$ is Set. This is in fact true. It is clear that (f_1, f_2) makes this diagram commute. Now for uniqueness, assume that $g : e \mapsto (g_1(e), g_2(e))$ satisfies this property. Then $p_1 \circ g(e) = f_1(e)$, so $g_1(e) = f_1(e)$ and similarly $g_2(e) = f_2(e)$, so $g = (f_1, f_2)$. Note that we can generalise for any finite number of objects n by defining $\mathcal{C}$ to consist out of objects $X_1, ..., X_n$ and morphisms $\mathrm{id}_{X_1}, ... \mathrm{id}_{X_n}$. The product for $n = 0$ is a terminal object.

We prove an easy but important theorem that says products are associative in a sense, and that projections compose in a natural manner. More precisely:

Theorem 3. *The limit $(A \times B) \times C$ with projections p_1, p_2 is isomorphic to $A \times B \times C$ with projections π_1, π_2, π_3. If we call the projections of $A \times B$ q_1 and q_2, we have that $\sigma q_1 p_1 = \pi_1$, $\sigma q_2 p_1 = \pi_2$, $\sigma p_2 = \pi_3$ where σ is the isomorphism between $(A \times B) \times C$ and $A \times B \times C$.*

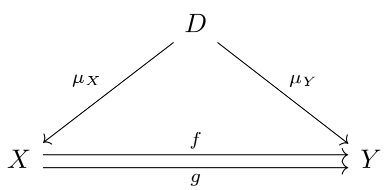
Proof. Consider the following diagram



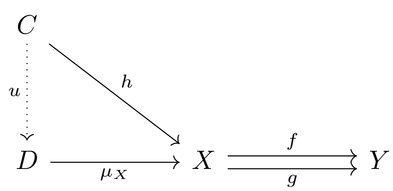
There is also a morphism h from D to B, but that does not fit in the diagram. As $q_1p_1\langle\langle f,h\rangle,g\rangle = q_1\langle f,h\rangle = f$ and $q_2p_1\langle\langle f,h\rangle,g\rangle = q_2\langle f,h\rangle = h$ and $p_2\langle\langle f,h\rangle,g\rangle = g$, we have that $(A\times B)\times C$ with projections q_1p_1, q_2p_1, p_2 satisfies the universal property of $A\times B\times C$ and must therefore be isomorphic to it by some iso σ. Composing this with the projections yields the equalities from the theorem. □

By applying this as many times as necessary, we can place and remove brackets wherever we want and the projections will compose in a natural way up to isomorphism.

Definition 4 ([7, p.17], equaliser)**.** *An equaliser is another special case of a limit. Let* **2** *denote the category consisting of two objects x, y and two morphisms $a, b : x \to y$ between them (and of course the identity morphisms). A functor is just two objects X and Y with two morphisms $f, g : X \to Y$. Now a cone with vertex D and natural transformation μ is of the form*



But as we required this diagram to commute, the morphism μ_Y is already given by $f\mu_X = g\mu_X$. This means we can give a limit of this diagram by an object D and a morphism $\mu_X : D \to X$ such that $f\mu_X = g\mu_X$, satisfying that for any other morphism $h : C \to X$ such that $fh = gh$, there is a unique map $u : C \to D$ satisfying $h = \mu_X u$.



Example 5. In Set, an equaliser of f, g is isomorphic to the inclusion of $\{x \in X \mid f(x) = g(x)\}$ into X. It is easy to verify that this makes the diagram commute. As for the universal property: suppose we have a morphism $h : C \to X$ such that $fh = gh$. Then the image of h is contained in $\{x \in X \mid f(x) = g(x)\}$, so h must factor via the inclusion of $\{x \in X \mid f(x) = g(x)\}$ into X. This is also where the term equaliser comes from: in Set the functions f and g are equal on C.

Theorem 6 ([7, 35])**.** *If $E \xrightarrow{e} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is an equaliser diagram, then e is iso if and only if $f = g$.*

Proof. Suppose e is an isomorphism with inverse k. As $fe = ge$, also $fek = gek$. So $f\,\mathrm{id}_X = g\,\mathrm{id}_X$, so $f = g$. Suppose $f = g$. The following diagram commutes because $f\,\mathrm{id}_X = f = g = g\,\mathrm{id}_X$.

$$X \xrightarrow{\mathrm{id}_X} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

If $a : A \to X$ is a morphism such that $fa = ga$, then there is a unique morphism $u : A \to X$ such that $\mathrm{id}_X\, u = a$, namely a itself. So the diagram above is an equaliser diagram as well.
We know that there is exactly one morphism u between E and X such that $\mathrm{id}_X\, u = e$ and that that is an isomorphism. As $e : E \to X$ satisfies this condition, it must be an isomorphism. □

We need the following theorem for well definedness of interpretations.

Theorem 7 ([7, 34])**.** *Every equaliser is a monomorphism.*

Proof. An arbitrary equaliser diagram looks like $E \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$. Suppose for $k, l : Z \to E$ we have that $ek = el$, which we will call h. Then $fh = fek = gek = gh$, so the following diagram commutes

$$\begin{array}{ccccc} Z & & & & \\ {\scriptstyle l}\downdownarrows{\scriptstyle k} & \searrow^{h} & & & \\ E & \xrightarrow{e} & X & \underset{g}{\overset{f}{\rightrightarrows}} & Y \end{array}$$

Because e is an equaliser, there is only one morphism $j : Z \to E$ such that $ej = h$. It follows that $k = l$. □

Definition 8 ([7, p.18], Pullback square)**.** *Let J denote the category* $\begin{array}{ccc} & & y \\ & & \downarrow{\scriptstyle b} \\ x & \xrightarrow{a} & z \end{array}$. *A functor $F : J \to \mathcal{D}$ is specified by giving two morphism in $\mathcal{D}$ with the same codomain, say $f : X \to Z$ and $g : Y \to Z$. Then a cone is given by a vertex W and morphisms p_X, p_Y, p_Z such that* $\begin{array}{ccc} W & \xrightarrow{p_Y} & Y \\ {\scriptstyle p_X}\downarrow & \searrow^{p_Z} & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$ *commutes.*

As $p_Z = gp_Y = fp_X$, we can also just give W and p_X, p_Y such that $gp_Y = fp_X$ if it is a limiting cone. The diagram $\begin{array}{ccc} W & \xrightarrow{p_Y} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$ *is called a pullback square, and a limit of J is called a pullback.*

Example 9. In Set, the object W is isomorphic to $\{(x, y) \in X \times Y \mid f(x) = g(y)\}$ with the obvious projections.

As two adjacent pullback squares form a square that again commutes, one might wonder whether it is again pullback. This and a partial converse are true and is called the pullback pasting lemma.

Theorem 10 ([7, 38], Pullback pasting lemma)**.** *Given two pullback squares:*

$$\begin{CD} A @>{b}>> B @>{c}>> C \\ @V{a}VV @V{f}VV @V{d}VV \\ X @>{g}>> Y @>{h}>> Z \end{CD}$$

the composite square

$$\begin{CD} A @>{cb}>> C \\ @V{a}VV @V{d}VV \\ X @>{hg}>> Z \end{CD}$$

is a pullback square as well.

Moreover, if

$$\begin{CD} B @>{c}>> C \\ @V{f}VV @V{d}VV \\ Y @>{h}>> Z \end{CD} \quad \text{and} \quad \begin{CD} A @>{cb}>> C \\ @V{a}VV @V{d}VV \\ X @>{hg}>> Z \end{CD}$$

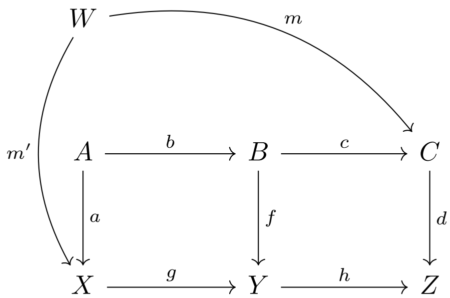
are pullback squares, also

$$\begin{CD} A @>{b}>> B \\ @V{a}VV @V{f}VV \\ X @>{g}>> Y \end{CD}$$

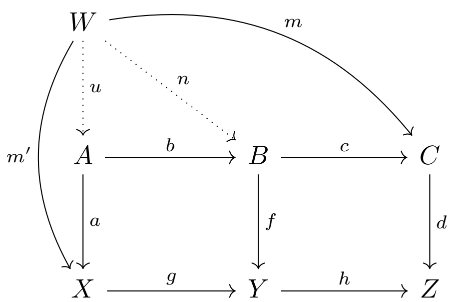
is a pullback square.

Proof. We start with the first statement.

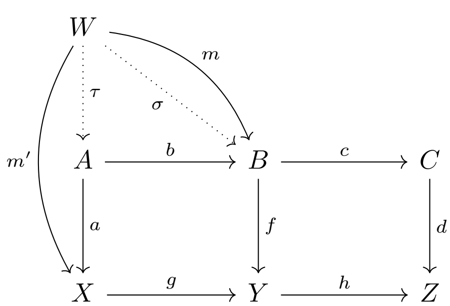
This new square commutes, so we only need to check its universal property. Suppose we have an object W and morphisms $m : W \to C, m' : W \to X$ such that $dm = hgm'$.



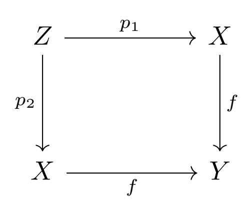
Then W, m and gm' satisfy $hgm' = dm$, so there is a unique $n : W \to B$ such that $cn = m$ and $fn = gm'$. But now W, n, m' satisfy $fn = gm'$, so there is a unique $u : W \to A$ such that $bu = n$ and $au = m'$. But then also $cbu = cn = m$, so the composite square is a pullback square.



Moving on to the second statement, suppose we have an object W and morphisms $m : W \to B$, and $m' : W \to X$ such that $fm = gm'$. Then $dcm = hfm = hgm'$. So because of the universal property of the small pullback square, there is a unique $\sigma : W \to B$ such that $f\sigma = gm'$ and $c\sigma = cm$. As m satisfies this, the morphisms m and σ are equal. But as $dcm = hgm'$, we have by the universal property of the large pullback square, that there is a unique $\tau : W \to A$ such that $cb\tau = cm$ and $a\tau = m'$. If we can prove that $b\tau = m$, we have shown the universal property of the left square, and then we are done. For this purpose, note that $cb\tau = cm$ and $fb\tau = ga\tau = gm'$. As m is the only morphisms satisfying this requirement, the morphisms m and $b\tau$ are equal.



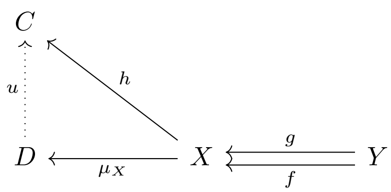
□

Definition 11 (Kernel pair)**.** *For a morphism $f : X \to Y$ and a pullback square*  *we* *call* $Z \underset{p_1}{\overset{p_0}{\rightrightarrows}} X$ *the kernel pair of* f.

2.2 Coequaliser

Recall that given a category $\mathcal{C}$ we can define a category $\mathcal{C}^{\text{op}}$ that has the same objects as $\mathcal{C}$, but the morphisms are reversed. A limit in $\mathcal{C}^{\text{op}}$ is called a colimit in $\mathcal{C}$. The only colimit we will need is the dual concept of an equaliser.

Definition 12 (Coequaliser). *A coequaliser is an equaliser with the arrows inverted, so the coequaliser for* $g, f : Y \to X$ *is an object* D *and a morphism* μ_X *such that* $\mu_X f = \mu_X g$*, and for any other object* C *and morphism* $h : X \to C$ *such that* $hg = hf$*, there must be a unique morphism* $u : D \to C$ *such that* $u\mu_X = h$.



Definition 13 (Regular epimorphism). *A regular epimorphism is an epimorphism that it the coequaliser for the kernel pair of some morphism.*

Remark 14. Regular epimorphisms are actually epimorphisms. The proof for this works the same as theorem 7, but with the arrows reversed.

3. Regular categories

We cannot interpret regular logic in just any category, because we need some structure. The type of category we use is called a regular category. The main point of a regular category is that theorem 26 holds. This allows us to define the image of a morphism, and can serve as a motivation for the definition of a subobject. We will be interpreting formulas as subobjects of suitable products.

3.1 The basics of regular categories

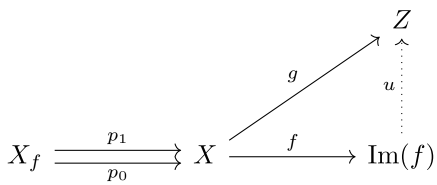
Definition 15 ([7, 4.1] Regular category)**.** *A category $\mathcal{C}$ is called regular if the following conditions hold:*

- *if J is a finite category, then any diagram $F : J \to \mathcal{C}$ has a limit.*
- *For every morphism f, if* $\begin{array}{ccc} Z & \xrightarrow{p_0} & X \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$ *is a pullback, the coequaliser of p_0, p_1 exists.*
- *Regular epimorphisms are stable under pullback, that is: in a pullback square* $\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle f} \\ \cdot & \longrightarrow & \cdot \end{array}$ *if f is a regular epimorphism, so is f. We say that a is a pullback of f.*

Example 16. In Set, the vertex of the kernel pair $f : X \to Y$ is isomorphic to $X_f = \{(x, x') \in X \times X \mid f(x) = f(x')\}$ with $p_0((x, x')) = x,\ p_1((x, x')) = x'$. The coequaliser of $X_f \underset{p_0}{\overset{p_1}{\rightrightarrows}} X$ is up to isomorphism

$$X_f \underset{p_0}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{f} \mathrm{Im}(f) \quad .$$

Proof. Suppose we have a set Z and a function $g : X \to Z$ such that $gp_0 = gp_1$. We can write any element of $\mathrm{Im}(f)$ as $f(x)$ for some $x \in X$ and define $u : \mathrm{Im}(f) \to Z$ by $u(f(x)) = g(x)$. This is well defined, for if $f(x) = f(x')$, then $g(x) = g(p_0((x, x')) = g(p_1(x, x')) = g(x')$. From the definition of u, it is clear that the diagram



commutes. Furthermore, the morphism u is unique, for if u_1, u_2 both have satisfy $u_i f = g$, then $u_1(f(x)) = g(x) = u_2(f(x))$. □

To make our lives a little easier when proving a category is regular, we prove the following theorem:

Theorem 17 ([7, 3.2])**.** *A category $\mathcal{C}$ has all finite limits if it has a terminal object, all binary products and all equalisers.*

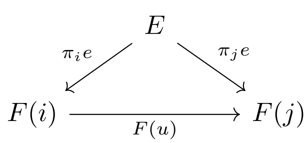
Proof. As all limits of a diagram are isomorphic, it suffices to show that every finite diagram has a limit in $\mathcal{C}$.

Let $F : \mathcal{E} \to \mathcal{C}$ be a diagram with $\mathcal{E}$ a finite category. For a morphism $m : A \to B$, we call A the domain of M - abbreviated to $\text{dom}(m)$ - and B the codomain of m - abbreviated to $\text{cod}(m)$. With $\pi_{\text{dom}(m)}$ we mean the projection to the domain of m and similarly for $\pi_{\text{cod}(m)}$.
Assume $\mathcal{C}$ has a terminal object 1, all binary products and all equalisers. With theorem 3 it is an easy induction to prove $\mathcal{C}$ contains all finite products: the category $\mathcal{C}$ has products of zero and two objects by assumption. It also has the product of one object, because that is just X with the identity as projection (the universal property is trivial). Suppose $\mathcal{C}$ has products of k objects. Then for objects $X_1, ..., X_{k+1}$, the product $X_1 \times ... \times X_k$ is a limit in $\mathcal{C}$. Because $\mathcal{C}$ has all binary products, the product $(X_1 \times ... \times X_k) \times X_{k+1}$ is a limit in $\mathcal{C}$, which is isomorphic to $X_1 \times ... \times X_{k+1}$ by theorem 3. This concludes the induction.

We construct

$$E \xrightarrow{\quad e \quad} \prod_{i \in \text{ob}(\mathcal{E})} F(i) \underset{\langle F(u)\pi_{\text{dom}(u)} | u \in \text{hom}(\mathcal{E})\rangle}{\overset{\langle \pi_{\text{cod}(u)} | u \in \text{hom}(\mathcal{E})\rangle}{\rightrightarrows}} \prod_{u \in \text{hom}(\mathcal{E})} F(\text{cod}(u))$$

as an equaliser diagram in $\mathcal{C}$. The family $(\mu_i = \pi_i e \ : \ E \to F(i))_{i \in \text{ob}(\mathcal{E})}$ with π_i the obvious projections, is a natural transformation between the constant functor Δ_E and F. This is true because for any morphism in $\mathcal{E}$, say $u : i \to j$, the diagram



commutes since $F(u)\pi_i e = F(u)\pi_{\text{dom}(u)} e = \pi_{\text{cod}(u)} e = \pi_j e$.

So (E, μ) is a cone for F. Let (D, ν) be another cone for F. Then $\langle \nu_i \mid i \in \text{ob}(\mathcal{E})\rangle$ precomposed with the two parallel morphisms from the equaliser diagram are equal to each other by definition, so there exists a unique morphism $u : D \to E$ such that $\mu_i u = \nu_i$, so (E, μ) is a limit. It is in $\mathcal{C}$ by construction, which concludes the proof.

$$\begin{array}{ccc} D & & \\ {\scriptstyle u}\,\vdots & \searrow^{\langle \nu_i \ \mid \ i \in ob(\mathcal{E})\rangle} & \\ E & \xrightarrow[\langle \mu_i \ \mid \ i \in \mathrm{ob}(\mathcal{E})\rangle]{} & \prod_{i\in\mathrm{ob}(\mathcal{E})} F(i) \end{array}$$

□

When proving a category $\mathcal{C}$ is regular we need to characterise pullbacks to check whether regular epimorphisms are stable under pullback. If we have used the previous theorem to prove $\mathcal{C}$ has all finite limits, we have found characterisations of binary products and equalisers. We can use those to characterise pullbacks using the following theorem.

Theorem 18 ([3], Construction of a pullback using binary products and equaliser)**.** *Given two morphism $A \xrightarrow{f} C \xleftarrow{g} B$, a binary product $A \times B$ with projections π_1, π_2 and $e : P \to A \times B$ the equaliser of $f \circ \pi_1$ and $g \circ \pi_2$*

$$\begin{array}{ccccc} P & & & & \\ & \searrow^{e} & & & \\ & & A \times B & \xrightarrow{\pi_1} & A \\ & & {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle f} \\ & & B & \xrightarrow{g} & C \end{array}$$

the square

$$\begin{array}{ccc} P & \xrightarrow{\pi_1 e} & A \\ {\scriptstyle \pi_2 e}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$$

is a pullback.

Proof. The square commutes because e equalises $\pi_1 f$ and $\pi_2 g$.
We now prove the universal property. Suppose there is an object Q and morphisms $a : Q \to A$ and $b : Q \to B$ such that $fa = gb$. Then $\langle a, b\rangle$ satisfies $f\pi_1\langle a, b\rangle = fa = gb = g\pi_2\langle a, b\rangle$. As e equalises $\pi_1 f$ and $\pi_2 g$, there is a unique morphism $u : Q \to P$ such that $eu = \langle a, b\rangle$. Consequently, we find that $\pi_1 eu = \pi_1\langle a, b\rangle = a$ and $\pi_2 eu = \pi_2\langle a, b\rangle = b$.

This means that $\begin{array}{ccc} P & \xrightarrow{\pi_1 e} & A \\ {\scriptstyle \pi_2 e}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{g} & C \end{array}$

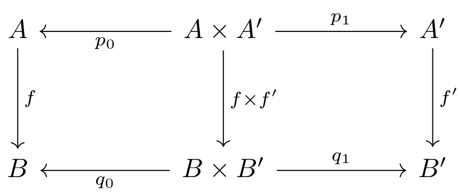
is a pullback square. □

Example 19. Set is a regular category. For the proof we refer you to page 8 of [5].

3.2 Details for the factorisation theorem

The proof of the factorisation theorem requires a lot of details to be checked. As this breaks the flow of the proof, I have collected them here. If the reader wants to see the proof for each detail, I advise to skip ahead to theorem 26 and come back to this section as the theorems are referenced.

Definition 20 (Product of morphisms)**.** *Let A, A', B, B' be objects and $f : A \to B$ and $f' : A' \to B'$ be morphisms. Let $p_0 : A \times A' \to A$, $p_1 : A \times A' \to A', q_0 : B \times B' \to B, q_1 : B \times B' \to B$ be the appropriate projections, then $f \times f'$ is the morphism making*



commute. We can tell from this diagram that $f \times f'$ is the unique morphism $\langle fp_0, f'p_1\rangle$.

Theorem 21. *Let $f : A \to B$ be a morphism and $\sigma : B \to C$ an iso. Then f is mono if and only if σf is mono. If $\delta : D \to A$ is iso, then f is mono if and only if $f\delta$ is mono.*

Proof. Throughout the proof, we take a, b to be suitable morphisms. Suppose f is mono and $\sigma fa = \sigma fb$. Then we can compose both sides with σ^{-1} and we get $fa = fb$, so $a = b$. Consequently, the morphism σf is mono.

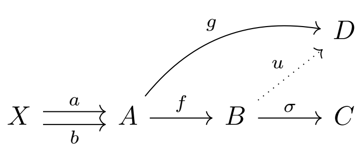
Suppose σf is mono and $fa = fb$. Then $\sigma fa = \sigma fb$, so $a = b$. This means that f is mono.

Suppose f is mono and $f\delta a = f\delta b$. Then $\delta a = \delta b$, so $\delta^{-1}\delta a = \delta^{-1}\delta b$, so $a = b$ and $f\delta$ is mono.

Suppose $f\delta$ is mono and $fa = fb$. Then $f\delta\delta^{-1}a = f\delta\delta^{-1}b$, so $\delta^{-1}a = \delta^{-1}b$. Composing with δ yields $a = b$, which means that f is mono. □

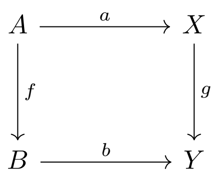
Theorem 22. *We work in a regular category. Let $f : A \to B$ be a morphism and $\sigma : B \to C$ an iso. Then f is regular epi if and only if σf is regular epi. If $\delta : D \to A$ is iso, then f is regular epi if and only if $f\delta$ is regular epi.*

Proof. Suppose f is regular epi, that is the coequaliser of some kernel pair $a, b : X \to A$. Then $\sigma fa = \sigma fb$ and for $g : A \to D$ such that $ga = gb$, we have a unique $u : B \to D$ such that



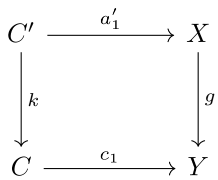
commutes. The morphism $u\sigma^{-1} : C \to D$ satisfies $u\sigma^{-1}\sigma f = uf = g$ and is the only morphism that satisfies this condition: suppose $k : C \to D$ satisfies $k\sigma f = g$, then $k\sigma = u$ because u is unique, and that means that $k = u\sigma^{-1}$. In conclusion, the morphism σf is a coequaliser for $a, b : X \to A$ as well, so σf is regular epi. The other direction and the case $f\sigma$ are analogous. □

Theorem 23. *In a regular category $\mathcal{C}$, if*

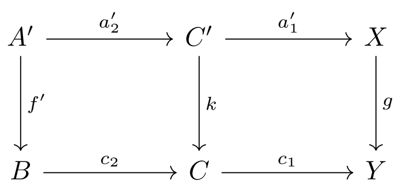


is a pullback diagram and $b = c_1 c_2$ for some regular epis $c_2 : B \to B$ and $c_1 : C \to Y$, then $a = a_1 a_2$ for some regular epis a_1, a_2.

Proof. As we work in a category with all finite limits, the pullback square of $C \xrightarrow{c_1} Y \xleftarrow{g} X$ exists, say

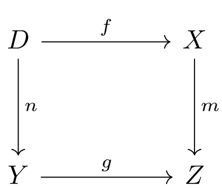


But similarly, the pullback of $C' \xrightarrow{k} C \xleftarrow{c_2} B$ exists as well. Say vertex A' and morphisms a'_2 to C' and f' to B. Drawing both these squares in one diagram yields



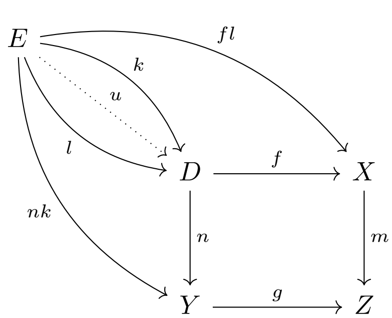
But the outer square is then a pullback as well (pullback pasting lemma), so by universal property there exists a unique isomorphism $h : A \to A'$ such that $f = f'h$ and $a = a'_1 a'_2 h$. Because regular epis are stable under pullback, the morphisms a'_1 and a'_2 are regular epis. Since h is an isomorphism, so is a'_2 (theorem 22). So we take $a_1 = a'_1$ and $a_2 = a'_2 h$. $\square$

Theorem 24. *If we have a pullback square*



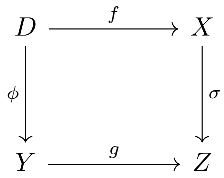
and m is mono, then so is n

Proof. Suppose $nk = nl$ for some $k, l : E \to D$. Then $gnk = gnl = mfl$, so there is a unique morphism $u : E \to D$ such that $mfu = gnu$. But both k, l satisfy this, so $k = l$. Therefore n is mono.



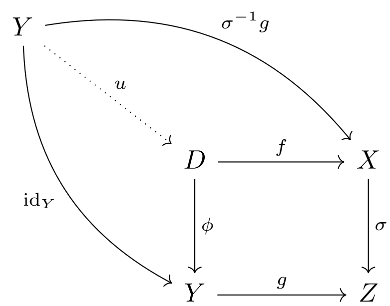
□

Theorem 25. *If we have a pullback square*



and σ is iso, then so is ϕ.

Proof. As $g \,\mathrm{id}_Y = g = \sigma\sigma^{-1}g$, there is a unique morphism $u : Y \to D$ such that the following diagram commutes:



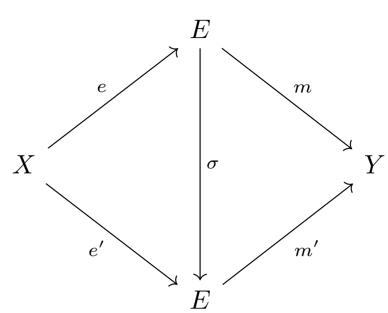
Actually the outer square is a pullback as well: suppose that morphisms $a : E \to X$ and $b : E \to Y$ satisfy $\sigma a = gb$. Then there is a unique morphism $v : E \to D$ such that $fv = a$ and $\phi v = b$. We claim that ϕv is the unique morphism from E to Y such that $\sigma^{-1}g\phi v = a$ and $\mathrm{id}_Y \,\phi v = b$. These equalities are true because $\sigma^{-1}g\phi v = \sigma^{-1}gb = \sigma^{-1}\sigma a = a$ and $\mathrm{id}_Y \,\phi v = \phi v = b$. Suppose c satisfies $\sigma^{-1}gc = a$ and $\mathrm{id}_Y \, c = b$, then $c = b = \phi v$, so ϕv is the only morphism that satisfies those equalities.

Consequently, the morphism u is an isomorphism. As the morphism ϕu is equal to id_Y, we have that $u\phi = u\phi u u^{-1} = u \,\mathrm{id}_Y \, u^{-1} = \mathrm{id}_D$, so ϕ is an isomorphism. □

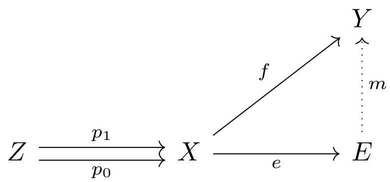
3.3 Factoring in regular categories

The following theorem motivates the definition of a subobject.

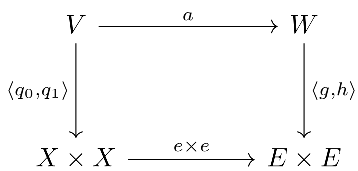
Theorem 26 ([7, 4.2])**.** *In a regular category, every arrow $f : X \to Y$ can be factored as $f = me \;:\; X \xrightarrow{e} E \xrightarrow{m} Y$ where e is regular epi and m is mono; and this factorisation is unique in the sense that if f can also be decomposed as $m'e' \;:\; X \xrightarrow{e'} E \xrightarrow{m'} Y$, with m' mono and e' regular epi, there is an iso $\sigma \;:\; E \to E'$ such that $\sigma e = e'$ and $m'\sigma = m$.*



Proof. Given $f : X \to Y$, we let $X \xrightarrow{e} E$ be the coequaliser of the kernel pair $Z \underset{p_0}{\overset{p_1}{\rightrightarrows}} X$ of f. By definition of a pullback, we have that $fp_0 = fp_1$. Consequently, the universal property of a coequaliser tells us that there is a unique $m : E \to Y$ such that $f = me$.

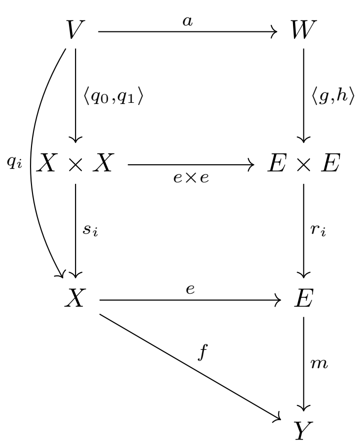


By construction e is regular epi; we must show that m is mono, and the uniqueness of the factorisation. Suppose that $mg = mh$ for $g, h : W \to E$; we prove that $g = h$. Let



be the pullback square of $\langle g, h\rangle$ and $e \times e$, which exists because we work in a regular category that has all finite limits.

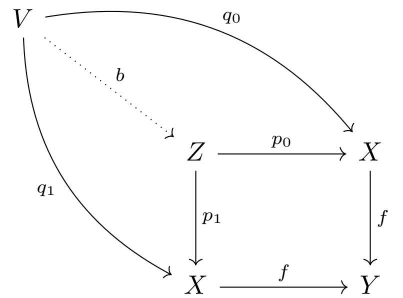
Underneath is a commuting diagram where s_1, s_2 are the projections on the first and second copy of X and note that $s_i\langle q_0, q_1\rangle = q_i$. We denote the projections of $E \times E$ on the first and second copy of E by r_1, r_2 and note again that $r_1\langle g, h\rangle = g$ and $r_2\langle g, h\rangle = h$. We write s_i and r_i instead of writing out the four arrows and neglect to write g and h to keep the diagram clear.



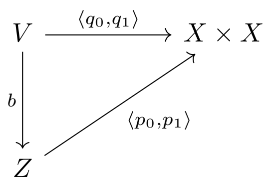
By diagram chasing we see that

$$fq_0 = meq_0 = mga = mha = meq_1 = fq_1$$

So there is a unique arrow $V \xrightarrow{b} Z$ such that $\langle q_0, q_1\rangle = \langle p_0, p_1\rangle b : V \to X \times X$ because of the kernel pair property:



We draw the following commutative diagram



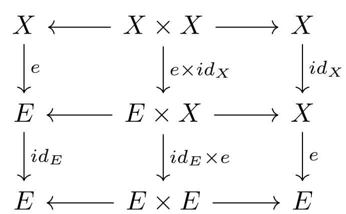
By chasing through this diagram and our large one, we find that

$$ga = eq_0 = ep_0 b = ep_1 b = eq_1 = ha$$

We claim that a is epi, from which it follows that $g = h$.
It is here that we use the requirement that regular epis are stable under pullback. Now $e \times e : X \times X \to E \times E$ is the composite

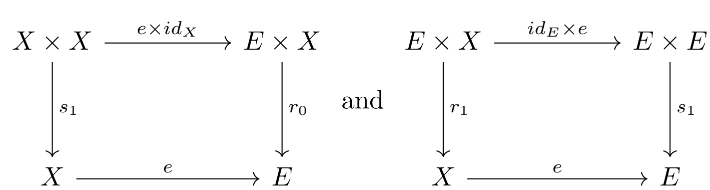
$$X \times X \xrightarrow{e \times id_X} E \times X \xrightarrow{id_E \times e} E \times E$$

because of the diagram



and the equalities $e = id_E e = e\,\mathrm{id}_X$.

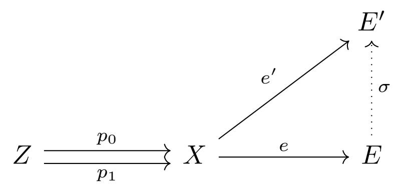
Both maps are regular epis since both squares below where r_i and s_i are the appropriate projections are pullbacks:



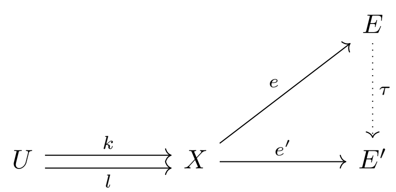
We prove this only for the first square, the second one is analogous. We see immediately that it commutes. For arrows $\langle \alpha_0, \alpha_1 \rangle : O \to E \times X$ and $\beta : O \to X$ such that $e\beta = r_0 \langle \alpha_0, \alpha_1 \rangle$ we have that $\langle \beta, \alpha_1 \rangle$ is the unique arrow $O \to X \times X$ such that $s_1 \circ \langle \beta, \alpha_1 \rangle = \beta$ and $(e \times \mathrm{id}_X) \circ \langle \beta, \alpha_1 \rangle = \langle \alpha_0, \alpha_1 \rangle$.

As a is the pullback of a composite of regular epis, it is itself the composite of regular epis by theorem 23. Then by theorem 14, we have that e is epi.

This proves m is mono, and we have our factorisation. As to uniqueness, suppose we had another factorisation $f = m'e'$ with $m' : E' \to Y$ mono and $e' : X \to E'$ regular epi (say the equaliser of a pair $k, l : U \to X$). Then $m'e'p_0 = fp_0 = fp_1 = m'e'p_1$, so since m' is mono, it follows that $e'p_0 = e'p_1$. Because e is the coequaliser of p_0 and p_1, there is a unique σ such that



commutes. Then $m'\sigma e = m'e' = f = me$ so since e is epi we conclude that $m = m'\sigma$.
Also, as $mek = m'e'k = m'e'l = mel$ and m is mono, it follows that $ek = el$, so there exists a unique $\tau : E \to E'$ such that



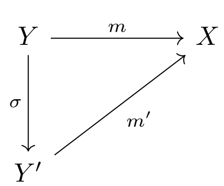
commutes. Then $m\tau\sigma e = m\tau e' = me$ and since m is mono and e epi, we have that $\tau\sigma = \mathrm{id}_E$. Similarly, the equality $\sigma\tau = \mathrm{id}_E$ holds. So σ is the required isomorphism. □

3.4 Subobjects

Definition 27 (Subobject)**.** *In any category $\mathcal{C}$ we define a subobject of an object X to be an equivalence class of monomorphisms* $Y \xrightarrow{m} X$ *where* $Y \xrightarrow{m} X$ *is equivalent to* $Y' \xrightarrow{m'} X$

if there is an isomorphism $\sigma : Y \to Y'$ such that $m'\sigma = m$. We say that $Y \xrightarrow{m} X$ *represents a*

smaller subobject than $Y' \xrightarrow{m'} X$ *if there is $\sigma : Y \to Y'$ such that $m'\sigma = m$.*



A subobject is the categorical version of a subset in set theory as illustrated by the following theorem:

Theorem 28. *In Set, two injective functions represent the same subobject if and only if their images are the same.*

Proof. Suppose $f : Y \to X$ and $g : Z \to X$ are two injective functions with the same image. Then for any $y \in Y$, there is a $z_y \in Z$ such that $g(z_y) = f(y)$ because f, g have the same image, and a unique such z_y because g is injective. So $\sigma(y) = z_y$ is well defined. If $\sigma(y) = \sigma(y')$, then $f(y) = g(\sigma(y)) = g(\sigma(y')) = f(y')$ and as f is injective, we find that $y = y'$. So σ is injective. For any $z \in Z$, there is a $y_z \in Y$ such that $f(y_z) = g(z)$ because f, g still have the same image, so $\sigma(y_z) = z$, which means that σ is surjective, so σ is bijective. We have that $g(\sigma(y)) = f(y)$, so $g\sigma = f$. In conclusion, the monomorphisms f and g represent the same subobject.
On the other hand, if f and g represent the same subobject, then there is a bijective function $\sigma : Y \to Z$ such that $g\sigma = f$. That means that for some $f(y) \in \mathrm{Im}(f)$, we have that $g(\sigma(y)) = f(y)$. So $\mathrm{Im}(f) \subseteq \mathrm{Im}(g)$. For some $g(z) \in \mathrm{Im}(g)$, we have that $g(z) = f(\sigma^{-1}(z))$, so $\mathrm{Im}(g) \subseteq \mathrm{Im}(f)$, so $\mathrm{Im}(g) = \mathrm{Im}(f)$. □

Because the factorisation $X \xrightarrow{f} Y$ as $X \xrightarrow{e} E \xrightarrow{m} Y$ with e regular epi and m mono, is determined uniquely up to isomorphism, we have that the equivalence class of m defines a subobject of Y. We call this $\mathrm{Im}(f)$ after the situation is Set.

We define a partial order $Sub(X)$ of subobjects of X ordered by the smaller than relation.

Remark 29. This is well defined.

Proof. We need to check whether this relation is transitive, reflexive and antisymmetric.

1. Suppose $Y_1 \xrightarrow{m_1} X$ is smaller than $Y_2 \xrightarrow{m_2} X$ by a morphism σ_1 and $Y_2 \xrightarrow{m_2} X$ is smaller than $Y_3 \xrightarrow{m_3} X$ by a morphism σ_2, then $Y_1 \xrightarrow{m_1} X$ is smaller than $Y_3 \xrightarrow{m_3} X$ by the morphism $\sigma_2\sigma_1$. As a result, this partial order is well defined.

2. The subobject $Y \xrightarrow{m} X$ is smaller than itself by the isomorphism id_Y.

3. Suppose $Y_1 \xrightarrow{m_1} X$ is smaller than $Y_2 \xrightarrow{m_2} X$ by some morphism σ_1 and the other way around by σ_2. Then $m_1 = m_2\sigma_1$ and $m_2 = m_1\sigma_2$, so $m_1 = m_1\sigma_2\sigma_1$, which means that $\sigma_2\sigma_1$ is the identity on Y_1. Similarly $\sigma_1\sigma_2$ is the identity on Y_2, which means that σ_1 and σ_2 are isomorphisms. So $Y_1 \xrightarrow{m_1} X$ and $Y_2 \xrightarrow{m_2} X$ represent the same subobject.

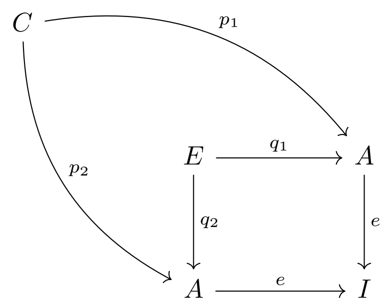
□

Example 30. If $A \xrightarrow{f} Y$ is a smaller subobject of Y than $B \xrightarrow{n} Y$, then $\mathrm{Im}(f)$ is a smaller subobject of Y than $B \xrightarrow{n} Y$.

Proof. As A is smaller than B, there is some morphism $g : A \to B$ such that $ng = f$. Taking the regular epi-mono factorisation of f, we get the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{e} & I \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle m} \\ B & \xrightarrow{n} & Y \end{array}$$

As e is regular epi, it coequalises the kernel pair $p_1, p_2 : C \to A$ for some morphism $h : A \to D$. Actually, the coequaliser e coequalises its own kernel pair as well. To see this, consider the diagram

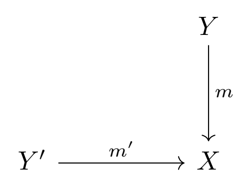


where q_1, q_2 is the kernel pair of e. As $ep_1 = ep_2$, there is a unique morphism $u : C \to E$ such that $q_1 u = p_1$ and $q_2 u = p_2$. If we have another morphism $e' : A \to I'$ such that $e' q_1 = e' q_2$, then $e' p_1 = e' q_1 u = e' q_2 u = e' p_2$. As e coequalises p_1, p_2, there is a unique morphism $\sigma : I \to I'$ such that $\sigma e = e'$, which means that e is the coequaliser of q_1, q_2.

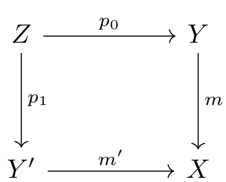
Since m is mono, we have that $meq_1 = meq_2$ if and only if $eq_1 = eq_2$, so the kernel pair of e is equal to the kernel pair of me. But $me = ng$, and the kernel pair of ng is the kernel pair of g by a similar argument. But then e is the coequaliser of the kernel pair of g, which means that there is a morphism $k : I \to B$ such that $ke = g$, which means precisely that $I \xrightarrow{m} Y$ represents a smaller subobject than $B \xrightarrow{n} Y$. □

Theorem 31. *In a category $\mathcal{C}$ with finite limits, each pair of elements of $Sub(X)$ has a greatest lower bound. We denote the greatest lower bound of $m : M \to X$ and $n : N \to X$ as $m \wedge n$. Moreover, the partial order $Sub(X)$ has a largest element.*

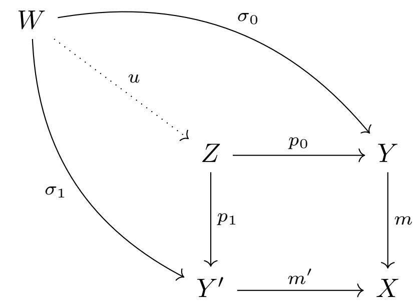
Proof. Let $Y \xrightarrow{\quad m \quad} X$ and $Y' \xrightarrow{\quad m' \quad} X$ represent two subobjects of X. As



is a finite category, we have a pullback

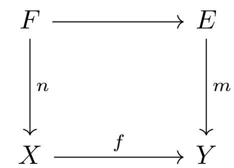


Give the morphism $mp_0 = m'p_1 : Z \to X$ the name n. We are going to prove that this represents $m \wedge m'$. We first prove that n is mono. If for some morphisms $f, g : W \to Z$ $ng = nf$, then $m'p_1 g = mp_0 f$. Consequently there is a unique morphism $u : W \to Z$ such that $p_0 u = p_0 f$ and $p_1 u = p_1 g$. As both f, g satisfy this requirement, we must have that $u = f = g$, so n is a monomorphism.
Next we claim that $n : Z \to X$ is the greatest lower bound of m, m'. As theorem 24 says that p_0 and p_1 are monomorphisms, we have that n is smaller than m and m'. Suppose $w : W \to X$ is smaller than m, m' as well, that is: there are $\sigma_0 : W \to Y$ and $\sigma_1 : W \to Y'$ such that $m\sigma_0 = w = m'\sigma_1$. That means that there is a unique morphism $u : W \to Z$ such that $p_i u = \sigma_i$.



As a result, the morphism nu is equal to w, so $w : W \to X$ is smaller than $n : Z \to X$, which means that n is the greatest lower bound of m, m'. The largest element of $Sub(X)$ is the identity on X, because for any monomorphism $m : Y \to X$, we have that $m = m\,\mathrm{id}_X$. ☐

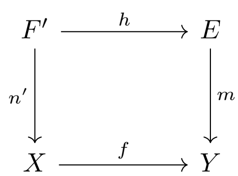
Definition 32 (The function f^*)**.** *Since monos and isos are stable under pullback by theorems 24 and 25, in any category $\mathcal{C}$ with pullbacks, any arrow $f : X \to Y$ determines an order preserving map $f^* : Sub(Y) \to Sub(X)$ by pullback along f: if $E \xrightarrow{m} Y$ represents the subobject M of Y and*



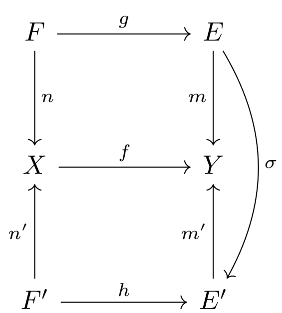
is a pullback, the mono $F \xrightarrow{n} X$ represents $f^*(M)$.

Theorem 33 ([7, 68])**.** *The map f^* is well defined and order preserving.*

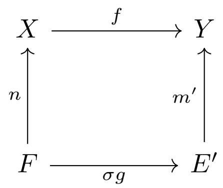
Proof. As monos are stable under pullback, we have that n is a mono. Suppose we have another pullback square of $X \xrightarrow{f} Y \xleftarrow{m} E$:



There exists a (unique) isomorphism $u : F' \to F$ such that $nu = n'$, so n and n' represent the same subobject. Therefore f^* is well defined. Carrying on to order preserving, suppose $E \xrightarrow{m} Y$ is smaller than $E' \xrightarrow{m'} Y$, that is: there exist a morphism $\sigma : E \to E'$ such that $m = m'\sigma$.



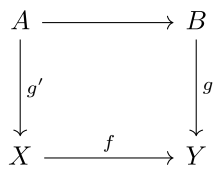
We want to find a $\sigma' : F \to F'$ such that $n = n'\sigma'$. Note that we also have a commutative square



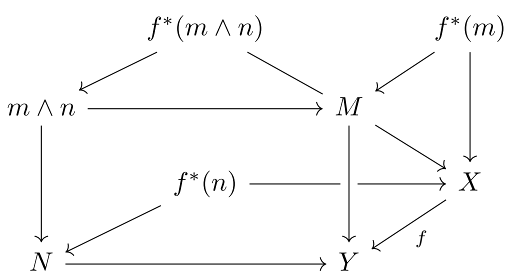
So there is a unique $\sigma' : F \to F'$ such that $n'\sigma' = n$ (and $h\sigma' = \sigma g$). This σ' satisfies the requirements. $\square$

This theorem is especially useful if f is some projection - say $X = Y_1 \times Y_2$ and $f = \pi_1$ - because it allows us to obtain a subobject of $Y_1 \times Y_2$ from an subobject of Y_1 in a natural way.

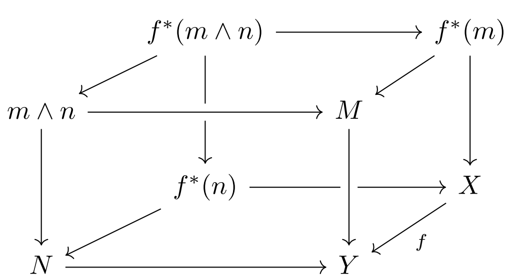
Theorem 34 ([7, 4.5])**.** *In a regular category, each f^* preserves greatest lower bounds and images, that is: for $f : X \to Y$,*

1. *for subobjects m, n of Y, it is true that $f^*(m \wedge n) = f^*(m) \wedge f^*(n)$*
2. *if*  *is a pullback, then $f^*(\mathrm{Im}(g)) = \mathrm{Im}(g')$.*

Proof. We start with 1. Let m and n be represented by $M \to X$ and $N \to X$. Using the description of greatest lower bound in the proof of theorem 31 we can make a commutative diagram with $m \wedge n$, $f^*(n)$, $f^*(m)$ and $f^*(m \wedge n)$ in it.



By the universal property of the pullback square of $M \to Y \overset{f}{\leftarrow} X$, there exists a unique morphism $f^*(m \wedge n) \to f^*(m)$ making the diagram commute (go from $f^*(m \wedge n)$ to $m \wedge n$ to M). We can do the same with $f^*(n)$ which gives the diagram



(the morphism $f^*(m \wedge n) \to X$ is already given by $f^*(m \wedge n) \to f^*(m) \to X$ which is equal to $f^*(m \wedge n) \to f^*(n) \to X$).
We will use the pullback pasting lemma several times implicitly. As the squares with vertices $f^*(m)$, X, Y, M and $f^*(m \wedge n)$, $m \wedge n$, Y, X are both pullback squares, the square with vertices $f^*(m \wedge n)$, $f^*(m)$, M and $m \wedge n$ is a pullback too. But then we can paste the squares with vertices $f^*(m \wedge n)$, $f^*(m)$, M and $m \wedge n$, M, N, Y, which gives a pullback square with vertices $f^*(m \wedge n)$, $f^*(m)$, N, Y. But as the square with vertices $f^*(n)$, N, Y, X is a pullback square by construction, the square with vertices $f^*(m \wedge n)$, $f^*(m)$, X, $f^*(n)$ is a pullback square as well. This means that $f^*(m \wedge n)$ and $f^*(m) \wedge f^*(n)$ are isomorphic, which is precisely what needed to be shown.

Carrying on to 2, we give the regular epi-mono factorisation of g by $B \xrightarrow{c} C \xrightarrow{n} Y$.

First we take the pullback of $X \xrightarrow{f} Y \xleftarrow{n} C$.

$$\begin{array}{ccc} E & \longrightarrow & C \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ X & \xrightarrow[f]{} & Y \end{array}$$

As monos are stable under pullback (theorem 24), the morphism m is mono. We can also take the pullback of $E \to C \xleftarrow{c} B$. That gives us

$$\begin{array}{ccc} A' & \longrightarrow & B \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle c} \\ E & \longrightarrow & C \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Because regular epis are stable under pullback, the arrow e is regular epi. By the pullback pasting lemma, the outer square is a pullback as well. But by assumption,

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pullback of $X \xrightarrow{f} Y \xleftarrow{g} B$ as well, so there is a unique iso $u : A \to A'$ such that $meu = g'$. As eu is regular epi as well (theorem 22), we have a factorisation of g' and we see that '$f^*(n) = m$', or more precisely the subobjects represented by n and m, that is: $f^*(\mathrm{Im}(g)) = \mathrm{Im}(g)$. $\square$

The pullback function also behaves nicely with composition of morphisms:

Lemma 35. *For $A \xrightarrow{f} B \xrightarrow{g} C$ and subobject $D \to C$, we have that $f^*(g^*(D)) = (gf)^*(D)$.*

Proof. We take the pullback of $g : B \to C$ and $D \to C$, this yields

$$\begin{array}{ccc} g^*(D) & \longrightarrow & D \\ \downarrow & & \downarrow \\ B & \xrightarrow[g]{} & C \end{array}$$

Now taking the pullback of $f : A \to B$ and $g^*(D) \to B$ we get two adjacent pullback squares, and by the pullback pasting lemma, the outer square is a pullback as well.

$$\begin{array}{ccccc} f^*(g^*(D)) & \longrightarrow & g^*(D) & \longrightarrow & D \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

So $f^*(g^*(D)) \to A$ represents the same subobject as $(gf)^*(D)$. □

3.5 An example of a regular category

We give an example of a regular category. We first need to define some things.

Definition 36. *A frame Ω is a partially ordered set which has least upper bounds $\bigvee B$ for all subsets B. We give $\bigvee \Omega$ the name $\top$. It also has greatest lower bounds $\bigwedge A$ for all finite subsets A. For $A = \{x, y\}$ we denote this as $x \wedge y$. Moreover, the symbol $\wedge$ distributes over $\bigvee$ as follows:*

$$x \wedge \bigvee B = \bigvee \{x \wedge b \mid b \in B\}$$

for $x \in \Omega$, $B \subseteq \Omega$.

Example 37. An example of a frame is a topology. The order is $x \leq y$ if and only if $x \subseteq y$. Then the least upper bound is the union and greatest lower bound is the intersection. Hence the requirement that greatest lower bounds only exist for finite subsets.

Definition 38. *Given a frame Ω, we define the category $\mathcal{C}_\Omega$ as follows:*
Objects are pairs of a set X and a function $X \xrightarrow{E_X} \Omega$. Morphisms between (X, E_X) and (Y, E_Y) are functions $X \xrightarrow{f} Y$ satisfying the requirement $E_X(x) \leq E_Y(f(x))$ for all $x \in X$. The identity on X satisfies $E_X(x) \leq E_X(x) = E_X(\mathrm{id}(x))$ and for morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have that $E_X(x) \leq E_Y(f(y)) \leq E_Z(g(f(y)))$, so this is well defined.

We would like to prove $\mathcal{C}_\Omega$ is a regular category. We check the three requirements separately.

Theorem 39 ([7, 4.15])**.** *The category $\mathcal{C}_\Omega$ has all finite limits.*

Proof. We apply theorem 17. Consider a singleton $\{\bullet\}$ together with the function that sends $\bullet$ to $\top$. For any object (X, E_X), there is a unique function from $X \to \{\bullet\}$ and that satisfies $E_X(x) \leq \top$ for all $x \in X$ by definition of $\top$. So this is a terminal object in $\mathcal{C}_\Omega$.

Given (X, E_X) and (Y, E_Y), a product of the two is the object $(X \times Y, E_{X \times Y})$ where $E_{X\times Y}(x, y)$ is defined as $E_X(x) \wedge E_Y(y)$ with the usual projections π_1, π_2 on X and Y. To see this, suppose we have another object (Z, E_Z) together with morphisms $f : (Z, E_Z) \to (X, E_X)$ and $g : (Z, E_Z) \to (Y, E_Y)$. It is clear that (f, g) is the only function from Z to $X \times Y$ satisfying $\pi_1 \circ (f, g) = f$ and $\pi_2 \circ (f, g) = g$. It remains to be shown that (f, g) is a morphism. As $E_Z(z) \leq E_X(f(x))$ and $E_Z(z) \leq E_Y(g(y))$ by assumption, the element $E_Z(z)$ is smaller than $E_X(x) \wedge E_Y(y) =: E_{X\times Y}(x, y)$ as elements of the partial order Ω. This means that (f, g) is indeed a morphism from (Z, E_Z) to $(X \times Y, E_{X\times Y})$.

Given two morphisms $f, g : (X, E_X) \to (Y, E_Y)$ their equaliser is the inclusion of $(X', E_{X'})$ where $X' \subseteq X$ is $\{x \in X \mid f(x) = g(x)\}$ and E'_X is the restriction of E_X to X'. It is clear this is well defined. To see this is indeed the equaliser, suppose $h : (Z, E_Z) \to (X, E_X)$ is a morphism such that $f \circ h = g \circ h$. That means that the image of h is a subset of X'. So h factors via X', which is what needed to be shown.

We conclude with theorem 17 that $\mathcal{C}_\Omega$ has all finite limits. □

Remark 40. We can now use theorem 18 to characterise pullbacks in $\mathcal{C}_\Omega$. Suppose we have two morphisms $(X, E_X) \xrightarrow{f} (Z, E_Z) \xleftarrow{g} (Y, E_Y)$. Let us say π_1, π_2 are the projections of $(X \times Y, E_{X\times Y})$ on (X, E_X) and (Y, E_Y). Then $f \circ \pi_1$ is given by $(x, y) \mapsto f(x)$ and $g \circ \pi_2$ is given by $(x, y) \mapsto g(y)$. So the equaliser of $f \circ \pi_1$ and $g \circ \pi_2$ is given by the inclusion of $X' \times Y' := \{(x, y) \in X \times Y \mid f(x) = g(y)\}$ and $E_{X'\times Y'}$ which is $E_{X\times Y}$ restricted to $X' \times Y'$. So the pullback is given by

$$\begin{array}{ccc} (X' \times Y', E_{X'\times Y'}) & \xrightarrow{p_1} & (X, E_X) \\ \big\downarrow{\scriptstyle p_2} & & \big\downarrow{\scriptstyle f} \\ (Y, E_Y) & \xrightarrow{g} & (Z, E_Z) \end{array}$$

where $p_1(x, y) = x$ and $p_2(x, y) = y$.

Lemma 41 ([7, 4.15])**.** *A morphism $f : (X, E_X) \to (Y, E_Y)$ is regular epi if and only if the function f is surjective and for all $y \in Y$, $E_Y(y) = \bigvee\{E_X(x) \mid f(x) = y\}$.*

Proof. Suppose f is such and the morphisms p_1, p_2 are the kernel pair of f. We will prove f is the coequaliser of its kernel pair.

Assume $g : (X, E_X) \to (Z, E_Z)$ satisfies $gp_1 = gp_2$. Then $g(x) = g(x')$ whenever $f(x) = f(x')$. We build a map $h : Y \to Z$ as follows: f is surjective, so we can take an $x \in X$ such that $f(x) = y$. Then we define $h(y) = g(x)$ for that x. This is well defined because $g(x) = g(x')$ whenever $f(x) = f(x')$. Since $E_X(x) \leq E_Z(g(x))$, we have that for $f(x) = y$

$$E_Y(y) = \bigvee\{E_X(x) \mid f(x) = y\} \leq E_Z(g(x))$$

So h is actually a morphism between (Y, E_Y) and (Z, E_Z). It satisfies $g = hf$ by definition. Suppose $g = h'f$ as well, then $h'(y) = h'(f(x))$ for some $x \in X$ because f is surjective, so $h'(y) = h'(f(x)) = g(x) = h(f(x)) = h(y)$, so it is unique as well. Therefore f is the coequaliser of its kernel pair.

For the converse, assume $f : (X, E_X) \to (Y, E_Y)$ is regular epi, that is: the coequaliser for the kernel pair of some morphism. From the proof of example 30 we know that f is also the coequaliser of its own kernel pair p_1, p_2. By remark 14, the morphism f is an epimorphism. We will first prove that f is surjective with a slightly modified proof by contradiction taken from [4].

Suppose f is not surjective, that is: there is some $y' \in Y$ such that $f(x) \neq y'$ for all $x \in X$. Let $E'_Y : Y \cup \{Y\} \to \Omega$ be a function that sends $\{Y\}$ to $\top$ and is the same as E_Y restricted to Y.

We define $g, h : (Y, E_Y) \to (Y \cup \{Y\}, E'_Y)$ by $g(y) = y$ and $h(y) = \begin{cases} y & \text{if } y \neq y' \\ Y & \text{if } y = y' \end{cases}$

The function g is a morphism because $E_Y(y) = E'_Y(g(y))$. The function h is a morphism because $E_Y(y') \leq \top = E'_Y(h(y'))$ and $E_Y(y) = E_Y(h(y))$ for $y \neq y'$.

As y' is not in the image of f, we have that $gf = hf$, but $g \neq h$, contradiction. So f is surjective.

For the other requirement, it follows from the definition of morphisms that $E_Y(y')$ is an upper bound of $\{E_X(x) \mid f(x) = y'\}$. Suppose ω is another upper bound. We define $E'_Y : Y \to \Omega$ to be the function that sends y' to ω and all other y to $E_Y(y)$. Then f is also a morphism from (X, E_X) to (Y, E'_Y) because $E_X(x) \leq \omega = E'_Y(f(x))$ for x that satisfy $f(x) = y'$ and $E_X(x) \leq E_Y(f(x))$ for all other x because f is a morphism from (X, E_X) to (Y, E_Y). It is still true that $fp_1 = fp_2$, so there is a unique morphism $h : (Y, E_Y) \to (Y, E'_Y)$ such that $hf = f$. The only function satisfying this is id_Y, so we know that id_Y is a morphism from (Y, E_Y) to (Y, E'_Y). Therefore $E_Y(y') \leq E'_Y(y') = \omega$. We conclude that $E_Y(y')$ is the least upper bound of $\{E_X(x) \mid f(x) = y'\}$. □

Theorem 42. *All kernel pairs have a coequaliser.*

Proof. Let $f : (X, E_X) \to (Y, E_Y)$ be a morphism. We can define $E'_Y : f(X) \to \Omega$ by sending y to $\bigvee\{E_X(x) \mid f(x) = y\}$. This gives a restriction morphism $f_| : (X, E_X) \to (f(X), E'_Y)$ which is a coequaliser by the previous lemma. As $f_|$ is the coequaliser of its own kernel pair, and the kernel pair of f is equal to that of $f_|$ (see remark 40), we conclude that the kernel pair of f has a coequaliser. □

Theorem 43. *In $\mathcal{C}_\Omega$, regular epis are stable under pullback.*

Proof. We know from lemma 40 that a pullback looks like

$$\begin{array}{ccc} (X' \times Y', E_{X' \times Y'}) & \xrightarrow{p_1} & (X, E_X) \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f} \\ (Y, E_Y) & \xrightarrow{g} & (Z, E_Z) \end{array}$$

where $p_1(x, y) = x$ and $p_2(x, y) = y$.

Suppose f is regular epi. Then f is surjective. Consequently, for all $y \in Y$, there exists an $x \in X$ such that $f(x) = g(y)$. That means that $(x, y) \in X' \times Y'$, so $p_2(x, y) = y$ and we conclude that p_2 is surjective.

Now we need to prove that for all $y \in Y$, $E_Y(y) = \bigvee\{E_{X' \times Y'}(x', y') \mid p_2(x', y') = y\}$. Note that $\{E_{X' \times Y'}(x', y') \mid p_2(x', y') = y\} = \{E_{X' \times Y'}(x', y') \mid y' = y \text{ and } (x', y') \in X' \times Y'\} = \{E_{X' \times Y'}(x, y) \mid f(x) = g(y)\} = \{E_X(x) \wedge E_Y(y) \mid f(x) = g(y)\}$. As f is regular epi, we have that $E_Z(z) = \bigvee\{E_X(x) \mid f(x) = z\}$.

Using the distributivity of Ω, we find that $E_Z(g(y)) \wedge E_Y(y) = \bigvee\{E_X(x) \wedge E_Y(y) \mid f(x) = g(y)\}$. As $E_Y(y) \leq E_Z(g(y))$, the greatest lower bound of $E_Z(g(y))$ and $E_Y(y)$ is $E_Y(y)$. That means that $E_Y(y) = \bigvee\{E_X(x) \wedge E_Y(y) \mid f(x) = g(y)\} = \{E_{X' \times Y'}(x', y') \mid p_2(x', y') = y\}$. So it follows from lemma 41 that p_2 is regular epi as well.

□

Corollary 44 ([7, 4.15])**.** *The category $\mathcal{C}_\Omega$ is regular.*

4. Internal logic of regular categories

The fragment of first order logic we are going to interpret in regular categories is called regular logic. Interpreting structures in categories is a notion called internalisation, which is studied extensively in category theory, see [2] for a more detailed description. This is why interpreting a logic in a category is also called the internal logic of a category ([1]). The intuition behind definitions is drawn from the latter reference.

4.1 Regular logic

The logical symbols of regular logic are $=$, $\wedge$, $\exists$. A language consists of a set of sorts $S, T, ...$ and a denumerable set of variables $x_1^S, x_2^S, ...$ of sort S, for each sort; a collection of function symbols $(f : S_1, ..., S_n \to S)$ and relation symbols $(R \subseteq S_1, ..., S_m)$. We allow n to be zero, viewing a 0-ary function symbol as a constant and a 0-ary relation as an atomic proposition. The only new concept here is that of a sort. Think of a sort as a type of object. For example when treating vector spaces it can be useful to define a sort for scalars and a sort for vectors.

We start by defining the syntax of regular logic.

Definition 45 (Terms of sort S)**.** *We define terms of sort S inductively as follows:*

1. *x^S is a term of sort S if x^S is a variable of sort S.*
2. *if $t_1, ..., t_n$ are terms of sorts $S_1, ..., S_n$ respectively, and $(f : S_1, ..., S_n \to S)$ is a function symbol of the language, then $f(t_1, ..., t_n)$ is a term of sort S*

Definition 46 (Formulas)**.** *We define inductively like this:*

1. *$\top$ is a formula (the formula "true")*
2. *if t and s are terms of the same sort, then $t = s$ is a formula*
3. *if $(R \subseteq S_1, ..., S_m)$ is a relation symbol and $t_1, ..., t_m$ are terms of sort $S_1, ..., S_m$ respectively, then $R(t_1, ..., t_m)$ is a formula.*
4. *if ϕ and ψ are formulas then $\phi \wedge \psi$ is a formula.*
5. *if ϕ is a formula and x a variable of some sort, then $\exists x\phi$ is a formula.*

4.2 Interpretation of regular logic

An interpretation of such a language in a regular category $\mathcal{C}$ is given by choosing

- an object $[[S]]$ of $\mathcal{C}$ for each sort S. Think of such an object as a collection of things satisfying a certain requirement.

- a morphism $[[f]] : [[S_1]] \times ... \times [[S_n]] \to [[S]]$ for every function symbol $(f : S_1, ..., S_n \to S)$. In Set this is exactly how we define the interpretation of f in normal model theory.
- a subobject $[[R]]$ of $[[S_1]] \times ... \times [[S_m]]$ for $(R \subseteq S_1, ..., S_m)$. A subobject is the categorical notion of a subset, so we can view $[[R]]$ as the collection of elements of sort $S_1, ..., S_m$ satisfying a certain relation.

Limits are only defined up to isomorphism, so to be more precise, we have to choose a product $[[S_1]] \times ... \times [[S_n]]$.

Given this, we define interpretations $[[t]]$ for terms t and $[[\phi]]$ for formulas ϕ, as follows.
Write $FV(t)$ for the set of variables which occur in t and $FV(\phi)$ for the set of free variables in ϕ. If $FV(t) = \{x_1^{S_1}, ..., x_n^{S_n}\}$ we put $[[FV(t)]] = [[S_1]] \times ... \times [[S_n]]$ and the same for $FV(\phi)$. So in Set an element of $[[FV(\phi)]]$ would simply mean picking a constant for each of the free variables in ϕ, and $[[FV(\phi)]]$ is the set of all those possibilities. If $FV(\phi) = \emptyset$, then $[[FV(\phi)]]$ is the limit of the empty diagram, which is simply an object D such that for any other object E, there is precisely one morphism $u : E \to D$. There is only one such object up to isomorphism, namely the terminal object, which we will denote 1.

Definition 47. *The interpretation of a term t of sort S is a morphism $[[t]] : [[FV(t)]] \to [[S]]$ and defined by the following clauses. The motivation of interpreting a term t of sort S as a morphism $[[t]] : [[FV(t)]] \to [[S]]$ comes from Set: we view $[[S]]$ as the set of all things of sort S, and then t is a function that takes a substitution of constants for free variables as argument and assigns a thing of sort S to it.*

1. *$[[x^S]]$ is the identity on $[[S]]$ if x^S is a variable of sort S.*
2. *Given $[[t_i]] : [[FV(t_i)]] \to [[S_i]]$ for $i = 1, ..., n$ and a function symbol $(f : S_1, ..., S_n \to S)$ of the language, the interpretation $[[f(t_1, ..., t_n)]]$ is the morphism*

$$[[FV(f(t_1, ..., t_n))]] \xrightarrow{\langle \tilde{t} | i=1,...,n \rangle} \Pi_{i=1}^{n} [[S_i]] \xrightarrow{[[f]]} [[S]]$$

where $\tilde{t_i}$ is the composite

$$[[FV(f(t_1, ..., t_n)]] \xrightarrow{\pi_i} [[FV(t_i)]] \xrightarrow{[[t_i]]} [[S_i]]$$

where π_i is the projection corresponding to the inclusion $FV(t_i) \subseteq FV(f(t_1, ..., t_n))$.

Example 48 (In Set)**.** Let us have a look at how this works in Set.

- if f is a 0-ary function symbol of sort S, then $FV(f) = \emptyset$, so its interpretation is a singleton, say $\{\bullet\}$. Then the interpretation of f is a function from $\{\bullet\} \to S$, which we can completely determine by where it sends $\bullet$. This agrees to our notion of a constant, because we can obtain any element s of S, by interpreting f to be the function $\bullet \mapsto s$.
- the interpretation of x^S is the identity from S to S. We can view a variable as a substitution scheme for constants (for example, in $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$, the variable x acts as a placeholder for some real number). This interpretation of x^S agrees with our intuition, because we get constant c back when applying id_S to c. See also example 53 for a more concrete example.
- let $f : S_1 \times ... \times S_n \to S$ be a function symbol and let $t_1, ..., t_n$ be variables of the appropriate sorts. Then the interpretation of f is a function $\tilde{f}$ that sends $(s_1, ..., s_n)$ to $(s_1, ..., s_n)$ and then to $[[f]](s_1, ..., s_n)$.
- in general the terms $t_1, ..., t_n$ are not all going to be variables. They can be functions of variables, or functions of functions of variables and a variable, et cetera. If t_1 is a term of the form $[[g]](x, y)$ with x, y variables and t_2 a variable as well, then $[[f(t_1, t_2)]]$ is the function that first sends (x, y, t_2) to $([[g]](x, y), t_2)$ and then to $[[f]](g(x, y), t_2)$. This again agrees with our intuition.

We define formulas ϕ as subobjects $[[\phi]]$ of $[[FV(\phi)]]$. Intuitively $[[\phi(x_1, ..., x_n]]$ is the "subset" $\{(a_1, ..., a_n) \in A_1 \times ... \times A_n \mid \phi(a_1, ..., a_n)\}$. Informally we take an interpretation $[[\phi]] \to [[FV(\phi)]]$ to be true if it is the maximal subobject. After the definition we will look at what happens when we interpret the formulas in Set. Keep in mind we can compose projections naturally because of theorem 3.

Definition 49. *The interpretation $[[\phi]]$ as a subobject of $[[FV(\phi)]]$ is defined as follows:*

1. *$[[\top]]$ is the maximal subobject of $[[FV(\top)]] = 1$. That is, the identity on 1.*

2. *$[[t = s]] \to [[FV(t = s)]]$ is the equaliser of*
$$\begin{array}{ccccc} & & [[FV(t)]] & & \\ & \nearrow & & \searrow^{[[t]]} & \\ [[FV(t = s)]] & & & & [[T]] \\ & \searrow & & \nearrow_{[[s]]} & \\ & & [[FV(s)]] & & \end{array}$$
if t and s are of sort T; again, the left hand maps are projections corresponding to the inclusions of $FV(t)$ and $FV(s)$ into $FV(t = s)$.

3. *For $(R \subseteq S_1, ..., S_m)$ a relation symbol and terms $t_1, ..., t_m$ of sorts $S_1, ..., S_m$ respectively, let $\bar{t} : [[FV(R(t_1, ..., t_m))]] \to \Pi_{i=1}^m [[S_i]]$ be the composite map*
$$[[FV(R(t_1, ..., t_m)]] \xrightarrow{\langle \pi_1, \ldots, \pi_m \rangle} \Pi_{i=1}^m [[FV(t_i)]] \xrightarrow{\Pi_{i=1}^m [[t_i]]} \Pi_{i=1}^m [[S_i]]$$
where π_i are the appropriate projections and the second arrow is the generalised version of definition 20.

   Then $[[R(t_1, ..., t_m)]] \to [[FV(R(t_1, ..., t_m))]]$ is the subobject of $(\bar{t})^*([[R]])$, defined by pullback along $\bar{t}$.

4. *if $[[\phi]] \to [[FV(\phi)]]$ and $[[\psi]] \to [[FV(\phi)]]$ are given and*
$$\begin{array}{ccc} [[FV(\phi \wedge \psi)]] & \xrightarrow{\pi_1} & [[FV(\phi)]] \\ & \searrow^{\pi_2} & \\ & & [[FV(\psi)]] \end{array}$$
are again suitable projections, then $[[(\phi \wedge \psi)]] \to [[FV(\phi \wedge \psi)]]$ is the greatest lower bound in $Sub([[FV(\phi \wedge \psi)]])$ of $\pi_1^*([[\phi]])$ and $\pi_2^*([[\psi]])$.

5. *if $[[\phi]] \to [[FV(\phi)]]$ is given and $\pi : [[FV(\phi)]] \to [[FV(\exists x \phi)]]$ is the projection, let $[[FV'(\phi)]]$ be the product of the interpretations of the sorts of the variables in $FV(\phi) \cup \{x\}$ (so $[[FV'(\phi)]] = [[FV(\phi)]]$ if x occurs freely in ϕ; and $[[FV'(\phi)]] = [[FV(\phi)]] \times [[S]]$ if $x = x^S$ does not occur free in ϕ). Write $\pi' : [[FV'(\phi)]] \to [[FV(\phi)]]$ for the projection. Now take $[[\exists x \phi]] \to [[FV(\exists x \phi)]]$ to be the image of*
$$(\pi')^*([[\phi]]) \to [[FV'(\phi)]] \xrightarrow{\pi\pi'} [[FV(\exists \phi)]]$$
That is, the monomorphism $[[\phi]] \to [[FV(\exists x \phi)]]$ as defined above factorises as $[[\phi]] \xrightarrow{e} [[\exists \phi]] \xrightarrow{m} [[FV(\exists x \phi)]]$ with e regular epi and m mono.

Example 50 (In Set)**.** We again look at what happens in Set.

- The interpretation of $\top$ is the inclusion of $\{\bullet\}$ in $\{\bullet\}$. This is always true because $\{\bullet\}$ is the maximal subobject of $\{\bullet\}$.
- We study the interpretation of $t = s$, where t, s are of sort T and $FV(t = s) = \{x_1, ..., x_n\}$. Suppose $FV(t) \subseteq FV(t = s)$ consists of variables $y_1, ..., y_m$ and $FV(s) \subseteq FV(t = s)$ consists of variables $z_1, ..., z_k$. Letting $S_1, ..., S_n$ be of the appropriate sorts, the interpretation of $t = s$ is the inclusion of $\{(x_1, ..., x_n) \in [[S_1]] \times ... \times [[S_n]] \mid [[t]](y_1, ..., y_m) = [[s]](z_1, ..., z_k)\}$ into $[[S_1]] \times ... \times [[S_n]]$.
- A term is a nested string of variables and function symbols. To prevent a mess, we just look at a case with one function symbol. If we have more, we can just repeat the procedure.

  Suppose $(R \subseteq S_1, ..., S_m)$ is a relation symbol and $t_1, ..., t_{m-1}$ are variables of sort $S_1, ..., S_{m-1}$ and t_n is the term $f(a_n)$ where a_n is a variable of sort S_m, then we have defined the interpretation of $R(t_1, ..., t_n)$ along the pullback

$$\begin{array}{ccc} g^{-1}([[R]]) & \xrightarrow{\quad g \quad} & [[R]] \\ \downarrow & & \downarrow \\ [[S_1]] \times ... \times [[S_m]] & \xrightarrow[g]{} & [[S_1]] \times ... \times [[S_m]] \end{array}$$

  where the unmarked arrows are inclusions and g is the function $(s_1, ..., s_m) \mapsto (s_1, ..., s_{m-1}, [[f]](s_m))$. So then the interpretation of R is the inclusion of $\{(s_1, ..., s_m) \mid (s_1, ..., s_{m-1}, f(s_m)) \in R\}$ into $[[S_1]] \times ... \times [[S_m]]$.
- For a formula ϕ in free variables $a_1^{S_1}, ..., a_n^{S_n}$, we use the notation $\vec{a} \in [[FV(\phi)]]$ for $(a_1^{S_1}, ..., a_n^{S_n}) \in [[S_1]] \times ... \times [[S_n]]$ because of typographical reasons.

  If the interpretation of ϕ is $\{\vec{a} \in [[FV(\phi)]] \mid \phi(\vec{a})\}$ and the interpretation of ψ is $\{\vec{b} \in [[FV(\psi)]] \mid \psi(\vec{b})\}$, then the greatest lower bound of $\pi_1^*([[\phi]])$ and $\pi_2^*([[\psi]])$ is given by the following pullback square where all arrows are inclusions:

$$\begin{array}{ccc} \{\vec{ab} \in [[FV(\phi \wedge \psi)]] \mid \phi(\vec{a}) \text{ and } \psi(\vec{b})\} & \longrightarrow & \{\vec{ab} \in [[FV(\phi \wedge \psi)]] \mid \phi(\vec{a})\} \\ \downarrow & & \downarrow i \\ \{\vec{ab} \in [[FV(\phi \wedge \psi)]] \mid \psi(\vec{b})\} & \longrightarrow & [[FV(\phi \wedge \psi)]] \end{array}$$

  that is, the inclusion of $\{\vec{ab} \in [[FV(\phi \wedge \psi)]] \mid \phi(\vec{a}) \text{ and } \psi(\vec{b})\}$ into $[[FV(\phi \wedge \psi)]]$.

4.3 Sequents

In logic a sequent is a conditional assertion of the form

$$P_1, P_2, ..., P_n \vdash Q_1, ..., Q_m$$

for P_i, Q_i formulas. This means that assuming all of $P_1, ..., P_n$ is true, at least one of $Q_1, ..., Q_m$ is provable. Because we can formulate 'and' but not 'or' in regular logic, it does not make sense to have multiple Q's, and we can substitute $P_1, ..., P_n$ by $P_1 \wedge ... \wedge P_n$, so we define only sequents of the form $\psi \vdash \phi$. We also label the sequents with free variables.

Definition 51 (Labelled sequent)**.** *A labelled sequent is an expression of the form* $\psi \vdash_\sigma \phi$ *or* $\vdash_\sigma \phi$ *where* ψ *and* ϕ *are the formulas of the sequent and* σ *is a finite set of variables which includes all the variables which occur free in a formula of the sequent. If* σ *is empty we simply write* $\psi \vdash \phi$*.*

We also give the precise definition of truth of a labelled sequent.

Definition 52 (Truth)**.** *Let* $[[\sigma]] = [[S_1]] \times .... \times [[S_n]]$ *if* $\sigma = \{x_1^{S_1}, ..., x_n^{S_n}\}$*; there are projections* $[[\sigma]] \xrightarrow{\pi_\phi} [[FV(\phi)]]$ *and (in case* ψ *is there)* $[[\sigma]] \xrightarrow{\pi_\psi} [[FV(\psi)]]$*; we say that the sequent* $\psi \vdash_\sigma \phi$ *is true for the interpretation if* $(\pi_\psi)^*([[\psi]]) \leq (\pi_\phi)^*([[\phi]])$ *as subobjects of* $[[\sigma]]$*, and* $\vdash_\sigma \phi$ *is true if* $(\pi_\phi)^*([[\phi]])$ *is the maximal subobject of* $[[\sigma]]$*. We also say* ϕ *is true if* $\vdash_{FV(\phi)} \phi$ *is true.*

Example 53. Suppose ψ is the formula $(x = c) \wedge (y = c)$ and ϕ is the formula $x = y$ for x, y variables of sort S and c a constant of sort S. Interpret S as $\mathbb{N}$ and c as $\{\bullet\} \mapsto 5$. So $[[S]] = \mathbb{N}$ and $FV([[\phi]]) = [[FV(\psi)]] = \mathbb{N} \times \mathbb{N}$. Take $\sigma = \{x, y\}$.
We have that the interpretation of $(x = c)$ is the inclusion of $\{n \in \mathbb{N} \mid n = 5\} = \{5\}$ into $\mathbb{N}$, likewise with $(y = c)$. Calling the projections of $\mathbb{N} \times \mathbb{N}$ p_1 and p_2, we have that $p_1^*(\{5\}) = \{5\} \times \mathbb{N}$ included in $\mathbb{N} \times \mathbb{N}$ and $p_2^*(\{5\}) = \mathbb{N} \times \{5\}$ included in $\mathbb{N} \times \mathbb{N}$. The greatest lower bound of these two is $\{5\} \times \{5\}$, so the interpretation of ψ is the inclusion of $\{5\} \times \{5\}$ into $\mathbb{N} \times \mathbb{N}$. The interpretation of ϕ is the inclusion of $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x = y\}$ into $\mathbb{N} \times \mathbb{N}$. So $[[\psi]]$ is a smaller subobject than $[[\phi]]$ because we can first include $\{5\} \times \{5\}$ into $\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x = y\}$. So $\psi \vdash_\sigma \phi$ is true.

Example 54. We want to find the interpretation of $\exists x^S(x^S = x^S)$.
Let ϕ be $x^S = x^S$. Then $FV(\phi) = \{x^S\}$, so $[[FV(\phi)]] = [[S]]$. So $[[FV'(\phi)]] = [[S]]$ and $\pi : [[S]] \to [[S]]$ is just the identity. We have that $[[\phi]]$ is the equaliser of two identities on $[[S]]$, so just isomorphic to the identity on $[[FV(\phi)]] = [[S]]$ (theorem 6). The morphism $(\pi')^*([[\phi]])$ is defined along the pullback square

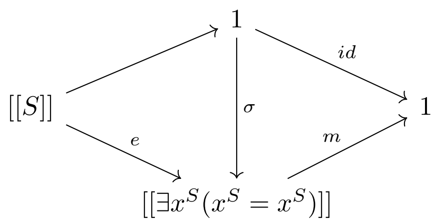
$$\begin{array}{ccc} [[S]] & \xrightarrow{id} & [[S]] \\ \Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle id} \\ [[S]] & \xrightarrow{id} & [[S]] \end{array}$$

so in this case $(\pi')^*([[\phi]])$ is $\mathrm{id}_{[[S]]}$. When we then apply π' and π we just get the unique morphism from $[[S]] \to 1$. Therefore $[[\exists x^S(x^S = x^S)]] \to 1$ is the image of this map.

Example 55. Is every epimorphism a regular epimorphism? The answer to this is no, and we will use definition 36 to find an epimorphism that is not regular. Let Ω be a frame and $a, b \in \Omega$ such that $a \vee b \neq \top$. Define $X = (\{0, 1\}, E_X)$ where $E_X(0) = a$ and $E_X(1) = b$. Then the unique map f from (X, E_x) to the terminal object $(\{\bullet\}, \bullet \mapsto \top)$ is an epimorphism for the following reason: morphisms in $\mathcal{C}_\Omega$ are functions in particular, and because f is surjective, the implication $gf = hf \implies g = h$ holds for functions, which means it also holds for morphisms in $\mathcal{C}_\Omega$. But it follows from lemma 41 that f is only regular epi if $\top = E_X(0) \vee E_X(1) = a \vee b$, which is false by assumption.

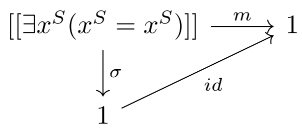
Example 56. We can now look at a requirement for a morphism to be regular epi: the sequent $\vdash \exists x^S(x^S = x^S)$ is true if and only if the unique map $[[S]] \to 1$ is a regular epimorphism.

Proof. As described above we have a regular epi-mono factorisation $[[S]] \xrightarrow{e} [[\exists x^S(x^S = x^S)]] \xrightarrow{m} 1$ of the unique map $[[S]] \to 1$. If $[[S]] \to 1$ is regular epi, then $[[S]] \to 1 \xrightarrow{\mathrm{id}_1} 1$ is a regular epi-mono factorisation, so there must be an isomorphism σ such that

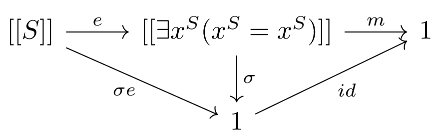


commutes. This means that $[[\exists x^S(x^S = x^S)]] \xrightarrow{m} 1$ represents the same subobject as $1 \xrightarrow{\text{id}} 1$ which is the maximal subobject of $[[\emptyset]] = 1$ (theorem 31).

For the other direction, suppose $[[\exists x^S(x^S = x^S)]] \xrightarrow{m} 1$ is the maximal subobject of $[[\emptyset]] = 1$. Then $[[\exists x^S(x^S = x^S)]] \xrightarrow{m} 1$ represents the same subobject as $1 \xrightarrow{id} 1$ (theorem 31). So there is an iso σ such that the following diagram commutes:



so



commutes as well. As e is regular epi and σ is iso, σe is regular epi as well (theorem 22). The morphism σe is the unique morphism from $[[S]] \to 1$ (as 1 is a terminal object). That concludes the other implication.

□

4.4 Theories

We have just treated the analogue of a model, and will now turn to the analogue of a proof tree. The implications represent the reasoning steps we can do in regular logic.

Definition 57 (Theory)**.** *Given a language, a set T of labelled sequents of that language is called a theory if and only if the following conditions hold:*

1. $\vdash \top$ *is in T;* $\vdash_x x = x$ *is in T for every variable x;* $x = y \vdash_{\{x,y\}} y = x$ *is in T for variables x, y of the same sort;* $x = y \wedge y = z \vdash_{\{x,y,z\}} x = z$ *is in T for variables x, y, z of the same sort;* $R(x_1, ..., x_m) \vdash_{\{x_1,...,x_m\}} R(x_1, ..., x_m)$ *is in T;*
2. *if* $(\psi) \vdash_\sigma \phi$ *is in T then* $(\psi) \vdash_\tau \phi$ *is in T whenever* $\sigma \subseteq \tau$*;*
3. *if* $(\psi) \vdash_\sigma \phi$ *is in T and* $FV(\chi) \subseteq \sigma$ *then* $(\psi\wedge)\chi \vdash_\sigma \phi$ *and* $(\chi\wedge)\psi \vdash_\sigma \phi$ *are in T (when we place $\chi\wedge$ between parentheses we mean that it can be left out);*
4. *if* $(\psi) \vdash_\sigma \phi$ *and* $(\psi) \vdash_\sigma \chi$ *are in T, then* $(\psi) \vdash_\sigma \phi \wedge \chi$ *and* $(\psi) \vdash_\sigma \chi \wedge \phi$ *are in T;*
5. *if* $(\chi\wedge)\psi \vdash_\sigma \phi$ *is in T and x is a variable not occurring in ϕ or χ then* $(\chi\wedge)\exists x\psi \vdash_{\sigma\setminus\{x\}} \phi$ *is in T.*
6. *if x occurs free in ϕ and* $(\psi) \vdash_\sigma \phi[t/x]$ *is in T, then* $(\psi) \vdash_\sigma \exists x\phi$ *is in T. If x does not occur free in ϕ and* $(\psi) \vdash_\sigma \phi$ *and* $(\psi) \vdash_\sigma \exists x(x = x)$ *are in T, then* $(\psi) \vdash_\sigma \exists x\phi$ *is in T;*

7. *if* $(\psi) \vdash_\sigma \phi$ *is in* T *then* $(\psi[t/x]) \vdash_{\sigma \setminus \{x\} \cup FV(t)} \phi[t/x]$ *is in* T*;*

8. *if* $(\psi) \vdash_\sigma \phi[t/x]$ *and* $(\psi) \vdash_\sigma t = s$ *are in* T*, then* $(\psi) \vdash_\sigma \phi[s/x]$ *is in* T*;*

9. *if* $\psi \vdash_\sigma \phi$ *and* $\phi \vdash_\sigma \chi$ *are in* T*, then* $\psi \vdash_\sigma \chi$ *is in* T*.*

Example 58. The sequent $\phi \vdash_{FV(\phi)} \phi$ is in every theory, for every formula ϕ of the language.

Proof. Recall definition 46. As the set of formulas is the smallest set that contains 1, 2, 3 and is closed under the operations 4 and 5, it suffices to show that the theory contains $\top \vdash \top$ (*), $(t = s) \vdash_{FV(t) \cup FV(s)} (t = s)$ (**), $R(x_1, ..., x_m) \vdash_{x_1,...,x_m} R(x_1, ..., x_m)$ (***) and that the following implications hold:
If $\phi \vdash_{FV(\phi} \phi$ and $\psi \vdash_{FV(\psi)} \psi$ are in a theory, then $(\phi \wedge \psi) \vdash_{FV(\phi \wedge \psi)} (\phi \wedge \psi)$ is in that theory (****).
If $\phi \vdash_{FV(\phi)} \phi$ is in a theory and x is a variable, then $\exists x \phi \vdash_{FV(\phi) \setminus \{x\}} \exists x \phi$ is in that theory (*****).
We prove each of those statements. When we say rule i, we refer to point i in definition 57. Now * follows from rules 1 and 3, ** follows from rules 1 and 9, *** follows from rule 1, **** follows from rule 3 two times and then rule 4, ***** follows from rule 5.
This type of argument is called an inductive argument and is used extensively when we are proving a statement about a general formula. □

5. Soundness and completeness

As theories are defined using a list of closure conditions, a theory is a set of sequents closed under all those conditions. So the intersection of a collection theories is again a theory.

Definition 59. *Given a set of sequents S, the theory $Cn(S)$ generated by S is*

$$Cn(S) := \bigcap \{T \mid T \text{ is a theory and } S \subseteq T\}$$

In this chapter we treat an analogue of the normal soundness and completeness theorem in first order model theory. We will only prove the soundness theorem.

5.1 Soundness theorem

Theorem 60 ([7, 4.11] Soundness theorem)**.** *Suppose $T = Cn(S)$ and all sequents of S are true under the interpretation in the category $\mathcal{C}$. Then all sequents of T are true under that interpretation.*

Before we begin with the proof, we first prove key lemma:.

Lemma 61 ([7, 4.12])**.** *Suppose t is substitutable for x in ϕ. There is an obvious morphism*

$$[t] : [[FV(\phi)\backslash\{x\} \cup FV(t)]] = [[FV(\phi[t/x])]] \to [[FV(\phi)]]$$

induced by $[[t]]$; the components of $[t]$ are projections except for the factor of $[[\phi]]$ corresponding to x, where it is

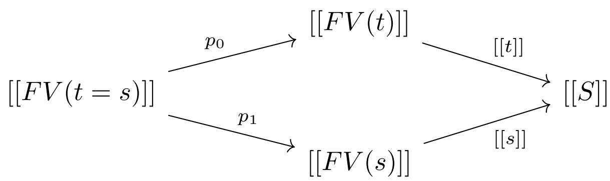
$$[[FV(\phi[t/x])]] \to [[FV(t)]] \xrightarrow{[[t]]} [[\{x\}]]$$

There is a pullback diagram

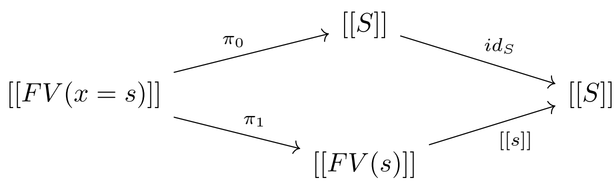
$$\begin{array}{ccc} [[\phi[t/x]]] & \longrightarrow & [[FV(\phi[t/x])]] \\ \big\downarrow & & \big\downarrow{\scriptstyle [t]} \\ [[\phi]] & \longrightarrow & [[FV(\phi)]] \end{array}$$

Proof. We prove this by induction on ϕ (recall definition 46).

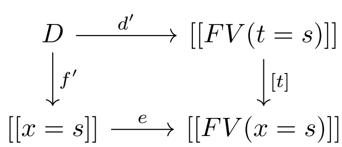
1. Suppose $\phi \equiv \top$, then there is nothing to substitute.
2. If $\phi \equiv (x = s)$ with x, s of sort S, then $[[t = s]] \to [[FV(t = s)]]$ is the equaliser of



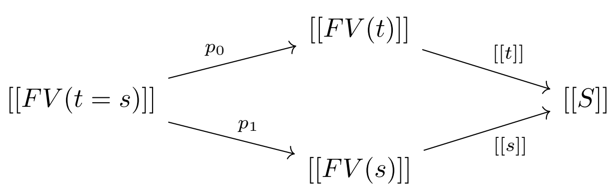
As $[[\{x\}]] = [[S]]$, we have that $e : [[x = s]] \to [[FV(x = s)]]$ is the equaliser of



We take a pullback of e and $[t]$ which yields

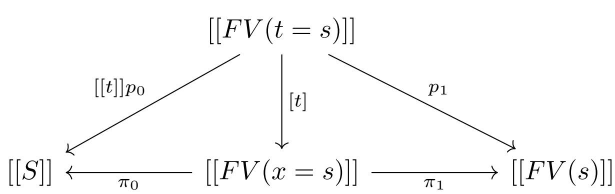


We now try to prove that $D \xrightarrow{d'} [[FV(t = s)]]$ defines the same subobject as $[[t = s]] \xrightarrow{d} [[FV(t = s)]]$. For this we need to prove that d' equalises



because then there would be an iso $\sigma : D \to [[t = s]]$ such that $d\sigma = d'$.

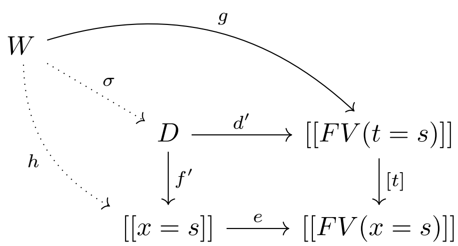
We have that $[[t]]p_0 = \pi_0[t]$ by the universal property of the product and the definition of $[t]$.



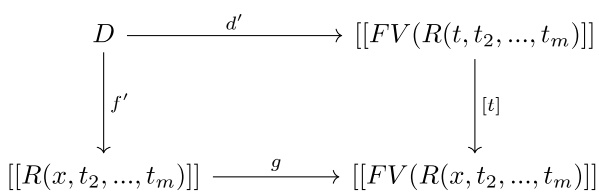
Because the pullback commutes, we can deduce that $[[t]]p_0d' = \pi_0[t]d' = \pi_0ef'$. Because e is an equaliser and $\pi_0 = id_S\pi_0$, we have that $\pi_0e = [[s]]\pi_1e$. So $[[t]]p_0d' = [[s]]\pi_1ef'$. Again using that the pullback commutes and definition of $[t]$, we get $[[t]]p_0d' = [[s]]\pi_1ef' = [[s]]\pi_1[t]d' = [[s]]p_1d'$.

We only need to show the universal property. Say we have a morphism $g : W \to [[FV(t = s)]]$ such that $[[t]]p_0g = [[s]]p_1g$. As $\pi_0[t] = [[t]]p_0$ and $\pi_1[t] = p_1$, we get $\pi_0[t]g = [[t]]p_0g = [[s]]p_1g = [[s]]\pi_1[t]g$.

It follows from the universal property of $[[x = s]]$ that there is a unique $h : W \to [[x = s]]$ such that $eh = [t]g$. Then there would also be a unique morphism $\sigma : W \to D$ as well such that $d'\sigma = g$ and $f'\sigma = h$ due to the universal property of a pullback square. That means that indeed $D \xrightarrow{d'} [[FV(t = s)]]$ is the sought equaliser and that it indeed defines the same subobject as $[[t = s]] \xrightarrow{d} [[FV(t = s)]]$.

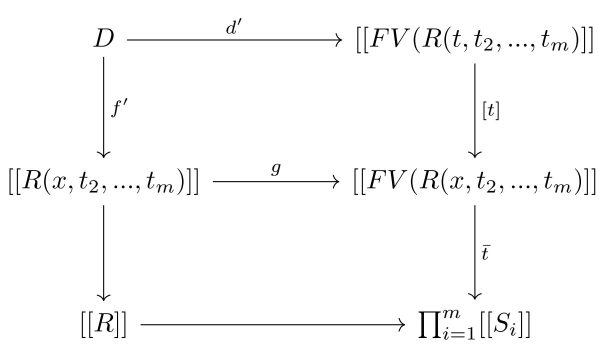


3. Suppose $(R \subseteq S_1, ..., S_m)$ is a relation symbol and $t, t_2, ..., t_m$ are terms of sorts $S_1, ..., S_m$ respectively and x a variable of sort S_1. Let $\phi \equiv R(x, t_2, ..., t_m)$. We will call the projections of $[[FV(\phi)]]$ on $[[S]]$ and $[[FV(t_i)]]$ $\pi_1, ..., \pi_m$ respectively. Similarly for $[[FV(\phi[t/x])]]$ we will call their projections $p_1, ..., p_m$. We take the pullback of $[[R(x, t_2, ..., t_m)]] \xrightarrow{g} [[FV(R(x, t_2, ..., t_m)]]$ and $[[FV(R(t, t_2, ..., t_m))]] \xrightarrow{[t]} [[FV(R(x, t_2, ..., t_m)]]$:



We again check whether $D \xrightarrow{d'} [[FV(\phi[t/x])]]$ represents the subobject $[[\phi[t/x]]] \to [[FV(\phi[t/x])]]$. Recall $\bar{t}$ of definition 49. The one corresponding to ϕ will be called $\bar{t}$ and the one corresponding to $\phi[t/x]$ will be called $\bar{t'}$. As projections compose naturally according to theorem 3 up to isomorphism, we have that $\bar{t}[t] = \bar{t'}$. Up to isomorphism is enough because we are working with subobjects, which are defined as an equivalence class where monos are related if and only if they are isomorphic as specified in definition 27.

So by the definition 49, the pullback pasting lemma and the following diagram, the monomorphism $D \xrightarrow{d'} [[FV(\phi[t/x])]]$ represents the subobject $[[\phi[t/x]]] \to [[FV(\phi[t/x])]]$.



4. Now suppose we have such a pullback for ϕ and for ψ. Let's call their $[t]$ $[t]_\phi$ and $[t]_\psi$. Recall that $[[\phi\wedge\psi]]$ as subobject of $[[FV(\phi\wedge\psi)]]$ is $\pi_1^*([[\phi]])\wedge\pi_2^*([[\psi]])$, with the projections as in definition 49. We call the projections of $[[FV((\phi\wedge\psi)[t/x])]]$ p_1 and p_2. We will prove that $[t]^*(\pi_1^*([[\phi]])\wedge\pi_2^*([[\psi]]))$ is the subobject $[[(\phi\wedge\psi)[t/x]]]\to[[FV((\phi\wedge\psi)[t/x])]]$. By theorem 34 we have that $[t]^*(\pi_1^*([[\phi]])\wedge\pi_2^*([[\psi]]))=[t]^*(\pi_1^*([[\phi]]))\wedge[t]^*(\pi_2^*([[\psi]]))$.

   By the pullback pasting lemma, the outer square is a pullback as well.

$$\begin{array}{ccc}
[t]^*(\pi_1^*([[\phi]])) & \longrightarrow & [[FV(\phi\wedge\psi[t/x])]] \\
\downarrow & & \downarrow{\scriptstyle [t]} \\
\pi_1^*([[\phi]]) & \longrightarrow & [[FV(\phi\wedge\psi)]] \\
\downarrow & & \downarrow{\scriptstyle \pi_1} \\
[[\phi]] & \longrightarrow & [[FV(\phi)]]
\end{array}$$

   Similarly

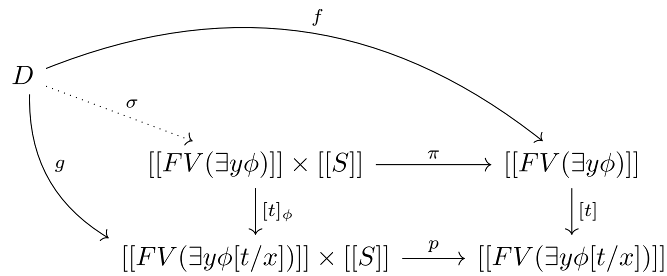
$$\begin{array}{ccc}
p_1^*([[\phi[t/x]]]) & \longrightarrow & [[FV(\phi\wedge\psi[t/x])]] \\
\downarrow & & \downarrow{\scriptstyle p_1} \\
[[\phi[t/x]]] & \longrightarrow & [[FV(\phi[t/x])]] \\
\downarrow & & \downarrow{\scriptstyle [t]_\phi} \\
[[\phi]] & \longrightarrow & [[FV(\phi)]]
\end{array}$$

   is a pullback. By noticing that $[t]_\phi p_1=\pi_1[[t]]$ and using the universal property of a pullback, we see that $[t]^*(\pi_1^*([[\phi]]))=p_1^*([[\phi[t/x]]])$. Mutatis mutandi we have a proof that $[t]^*(\pi_2^*([[\psi]]))=p_2^*([[\psi[t/x]]])$. So $[t]^*(\pi_1^*([[\phi]])\wedge\pi_2^*([[\psi]]))=[t]^*(\pi_1^*([[\phi]]))\wedge[t]^*(\pi_2^*([[\psi]]))=p_1^*([[\phi[t/x]]])\wedge p_2^*([[\psi[t/x]]])$ which is the subobject $[[(\phi\wedge\psi)[t/x]]]\to[[FV((\phi\wedge\psi)[t/x])]]$.

5. Lastly, suppose there is such a pullback for ϕ where we call the morphism $[t]$ for that pullback square $[t]_\phi$. We prove that there is such a pullback for $[[\exists y\phi]]$. First assume that y occurs freely in ϕ. Then $[[\exists y\phi]]\to[[FV(\exists y\phi)]]$ is the image of $[[\phi]]\to[[FV(\phi)]]\xrightarrow{\pi}[[FV(\exists y\phi)]]$. Let us call the projection from $[[FV(\phi[t/x])]]\to[[FV(\exists y\phi[t/x])]]$ p. We will prove that the subobject $[[\exists y\phi]\to[[FV(\exists y\phi)]]$ is also represented by $[t]^*([[\exists y\phi[t/x]]\to[[FV(\exists y\phi[t/x])]]])$. By theorem 34 it suffices to prove that $[[\phi]]\to[[FV(\phi)]]\xrightarrow{\pi}[[FV(\exists y\phi)]]=[t]^*([[\phi[t/x]]\to[[FV(\phi[t/x])]]\xrightarrow{\pi}[[FV(\exists y\phi[t/x])]]])$. Assuming the right square underneath is pullback, it follows from the pullback pasting lemma.

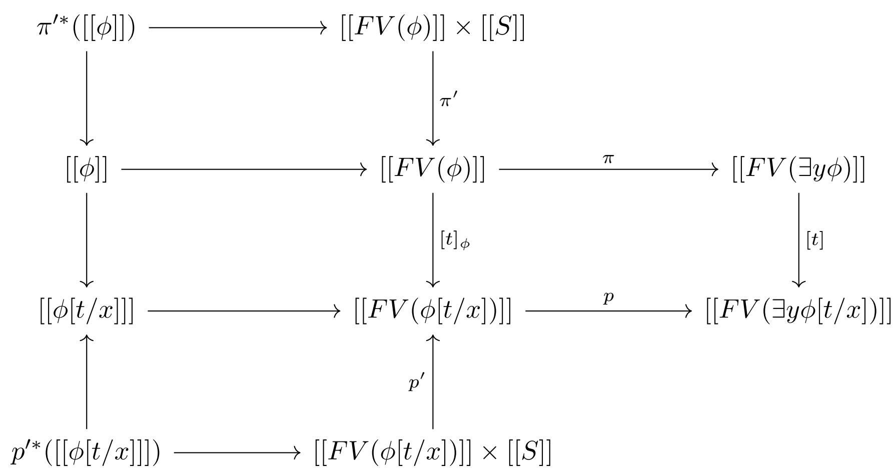
$$\begin{array}{ccccc}
[[\phi]] & \longrightarrow & [[FV(\phi)]] & \xrightarrow{\pi} & [[FV(\exists y\phi)]] \\
\downarrow & & \downarrow{\scriptstyle [t]_\phi} & & \downarrow{\scriptstyle [t]} \\
[[\phi[t/x]]] & \longrightarrow & [[FV(\phi[t/x])]] & \xrightarrow{p} & [[FV(\exists y\phi[t/x])]]
\end{array}$$

   We will now prove that the right square is pullback. It commutes because of the definition of $[t]_\phi$ and $[t]$. For the universal property, note that $[[FV(\phi)]]=[[FV(\exists y\phi)]]\times[[S]]$ where S is the sort of y and $[[FV(\phi[t/x])]]=[[FV(\exists y\phi[t/x])]]\times[[S]]$. Assume $f:D\to[[FV(\exists y\phi)]]$ and $g:D\to[[FV(\exists y\phi[t/x])]]$ satisfy $[t]f=pg$. We will prove that a $\sigma:D\to[[FV(\phi)]]$ is unique if it exists, and then it will be obvious that it exists.



We call g composed with the projection on $[[FV(\exists y\phi[t/x])]]$ g_1 and g composed with the projection on $[[S]]$ g_2 (so $g = \langle g_1, g_2\rangle$). The morphism σ is completely determined by σ composed with the projections on $[[FV(\exists y\phi)]]$ and $[[S]]$ - which we will call σ_1 and σ_2 - by definition of a product. As $[t]_\phi\sigma = g$ we have that $\sigma_2 = g_2$ and $[t]\sigma_1 = g_1$ (that $[t]_\phi$ composed with the projection on $[[FV(\exists y\phi[t/x]]]$ is $[t]$ follows from how we defined $[t]_\phi$ and $[t]$ and theorem 3). As $\pi\sigma = f$, we have that $\sigma_1 = f$. This completely determines σ, so $\sigma = \langle f, g_2\rangle$. We only need to check for well-definedness that $[t]f = g_1$, but this is true because we assumed $pg = [t]f$.

We carry on to the more difficult case where y does not occur freely in ϕ. In that case $[[FV'(\phi)]] = [[FV(\phi)]] \times [[S]]$ where S is the sort of y. We call the projections of ϕ π and π' as in definition 46 and the projections of $\phi[t/x]$ p and p'. This is analogous to the previous case, only now the commutative diagram is



This diagram commutes again because of the pullback pasting lemma and the observation that $[t]\pi = p[t]_\phi$, so the square with vertices $\pi'^*([[\phi]])$, $[[FV(\exists y\phi)]]$, $[[FV(\exists y\phi[t/x])]]$ and $p'^*([[\phi[t/x]]])$ commutes.

The universal property follows from using the universal properties of the smaller pullbacks (I trust we have done this enough to skip some details): let $g : W \to [[FV(\exists y\phi)]]$ and $h : W \to p'^*([[\phi[t/x]]])$ be suitable morphisms, then there exists a unique morphism $u_1 : W \to [[FV(\phi)]]$ making everything commute because we can compose g with $p'^*([[\phi[t/x]]]) \to [[FV(\phi[t/x])]] \times [[S]] \xrightarrow{p'} [[FV(\phi[t/x])]]$. But then there exists a unique arrow $u_2 : W \to [[\phi]]$ making everything commute using the pullback of $[t]_\phi$, so there is a unique $u_3 : W \to \pi'^*([[\phi]])$, which is the one we are looking for.

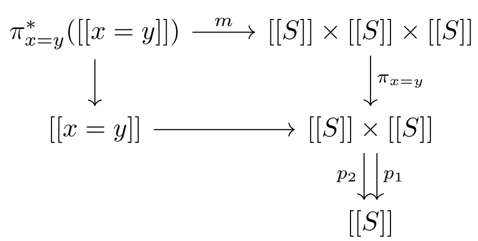
□

We can now start with the proof of the soundness theorem.

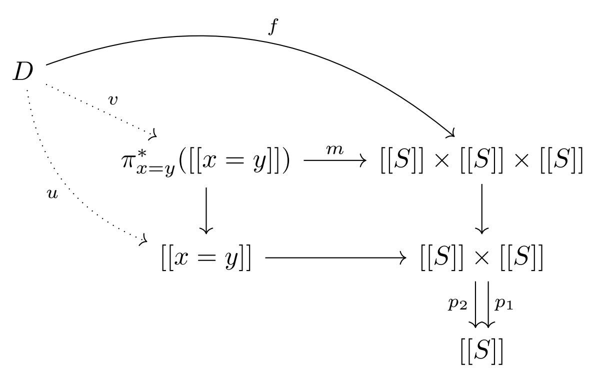
Proof. So we first need to check whether all sequents mentioned in 1 of definition 57 are true, and prove for 2 to 9 that if the premise is true, then the conclusion is true; in other words, that the set of true sequents is a theory. Often we will have an implication of the form "if $(\psi) \vdash_\sigma$ ... then $(\psi) \vdash_\sigma$ ...". We do not have to prove the case "if $\vdash_\sigma$ ..., then $\vdash_\sigma$ ..." separately for the following reason: if we take $\top$ for ψ, then $\top \vdash_\sigma \phi$ is true if and only if $\pi^*_\phi([[\phi]])$ as subobject of $[[\sigma]]$ is greater or equal than $[[\top]]$. As $[[\top]]$ is the maximal subobject of $[[\sigma]]$ by definition, this is equivalent to saying that $\pi^*_\phi([[\phi]])$ is the maximal subobject of $[[\sigma]]$, which is exactly the definition of $\vdash_\sigma \phi$ being true.

1. $\vdash \top$ is true because we define $[[\top]]$ to be the maximal subobject of $FV(\top) = 1$. As $[[x^S = x^S]]$ it the equaliser of two maps which are both the identity on $[[S]]$, the identity on $[[S]]$ equalises those, and limits are unique up to isomorphism: $[[x^S = x^S]]$ must be isomorphic to $[[S]]$. So it is the maximal subobject of $[[S]] = [[\sigma]]$, so $\vdash_x x = x$ is true. Let x, y, z be of sort S. Then $[[x = y]]$ and $[[y = x]]$ are both the equaliser of p_1 and p_2, the projections of $[[S]] \times [[S]]$. This means they represent the same subobject. So $\pi^*_{[[x=y]]}([[x = y]])$ is isomorphic to $\pi^*_{[[y=x]]}([[y = x]])$, so the sequent $y = x \vdash_{x,y} x = y$ is true.

   Carrying on to $(x = y) \wedge (y = z) \vdash_{x,y,z} x = z$. Call the projections of $[[S]] \times [[S]]$ on its components p_1 and p_2. We have that $[[\{x, y, z\}]] = [[\{x\}]] \times [[\{y\}]] \times [[\{z\}]] = [[S]] \times [[S]] \times [[S]]$ which has projections π_1, π_2, π_3 on $[[S]]$. Note that by theorem 3 lets us conclude that $p_1\pi_{x=y} = \pi_1,\ p_2\pi_{x=y} = \pi_2,\ p_1\pi_{x=z} = \pi_1,\ p_2\pi_{x=z} = \pi_3,\ p_1\pi_{y=z} = \pi_2$ and $p_2\pi_{y=z} = \pi_3$, which we will use freely. We draw the commutative diagram

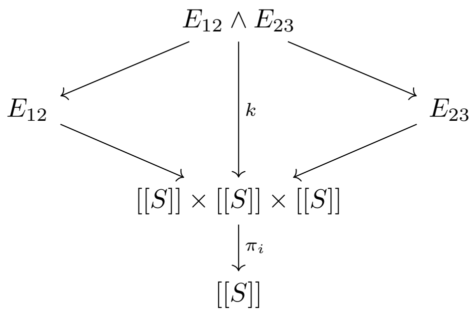
   

   Using the commutativity of the pullback square and the fact that $[[x = y]] \to [[S]] \times [[S]]$ is the equaliser of p_1, p_2, we see that $\pi^*_{x=y}([[x = y]]) \xrightarrow{m} [[S]] \times [[S]] \times [[S]]$, satisfies $\pi_1 m = \pi_2 m$. In fact, the morphism m is the equaliser of π_1, π_2. Suppose there was an object D and a morphism $f : D \to [[S]] \times [[S]] \times [[S]]$ such that $\pi_1 f = \pi_2 f$. Then $p_1\pi_{x=y} f = p_2\pi_{x=y} f$, so by the universal property of an equaliser, there is a unique arrow $u : D \to [[x = y]]$ such that the morphisms $D \xrightarrow{u} [[x = y]] \to [[S]] \times [[S]]$ and $D \xrightarrow{f} [[S]] \times [[S]] \times [[S]] \xrightarrow{\pi_{x=y}} [[S]] \times [[S]]$ are equal. So by the universal property of a pullback, there is a unique arrow $v : D \to \pi^*([[x = y]])$ such that $mv = f$, which means that m is the equaliser of π_1 and π_2.

   So if we give the equaliser for π_i, π_j the name E_{ij}, then $\pi^*_{x=y}([[x = y]])$ is E_{12}. Similarly $\pi^*_{y=z}([[y = z]])$

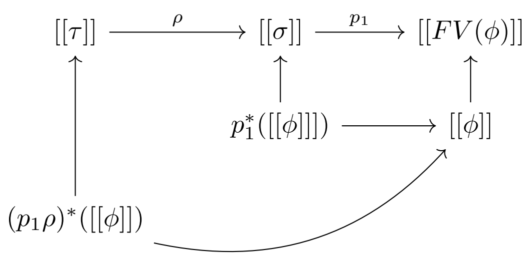
is E_{23} and $\pi^*_{x=z}([[x = z]])$ is E_{13}. We claim that the subobject $E_{12} \wedge E_{23}$ factors via E_{13}. For this purpose, consider the diagram



As the morphism $E_{12} \wedge E_{23} \to E_{12} \to [[S]] \times [[S]] \times [[S]]$ is equal to $E_{12} \wedge E_{23} \to E_{23} \to [[S]] \times [[S]] \times [[S]]$, let us call this morphism k, we see that $\pi_1 k = \pi_2 k = \pi_3 k$, so there must be a unique morphism $l : E_{13} \wedge E_{12} \to E_{23}$ such that kl is the morphism $E_{13} \to [[S]] \times [[S]] \times [[S]]$. This means that $x = y \wedge y = z \vdash_{x,y,z} x = z$ is true.

2. As $\sigma \subseteq \tau$, there is a projection $\rho : [[\tau]] \to [[\sigma]]$. Up to isomorphism the projections from $[[\tau]]$ to $[[FV(\phi)]]$ and $[[FV(\psi)]]$ are $p_1\rho$ and $p_2\rho$ where p_1 is the projection $[[\sigma]] \to [[FV(\phi)]]$ and p_2 the projection $[[\sigma]] \to [[FV(\psi)]]$ (theorem 3).

   Now the interpretation of ϕ as subobject of $[[\tau]]$ is defined to be $(p_1\rho)^*([[\phi]]) = \rho^*(p_1^*([[\phi]]))$ where $[[\phi]]$ is the interpretation of ϕ as subobject of $[[\sigma]]$. The equality is due to the lemma 35.



   Likewise with ψ. As the subobject $p_2^*([[\psi]])$ is smaller than $p_1^*([[\phi]])$ by assumption, the result follows from theorem 34.

3. Suppose $\pi^*_\psi([[\psi]]) \leq \pi^*_\phi([[\phi]])$ as subobjects of $[[\sigma]]$. By theorem 34 we have that $\pi^*_{\psi\wedge\chi}(\pi^*_\psi([[\psi]]) \wedge \pi^*_\chi([[\chi]])) = \pi^*_{\psi\wedge\chi}(\pi^*_\psi([[\psi]])) \wedge \pi^*_{\psi\wedge\chi}(\pi^*_\chi([[\chi]])))$ which is equal to $(\pi_\psi\pi_{\psi\wedge\chi})^*([[\psi]]) \wedge (\pi_\chi\pi_{\psi\wedge\chi})^*([[\chi]])$ by lemma 35. Because $FV(\chi) \subseteq \sigma$, we have that $\pi_\psi = \pi_\psi\pi_{\psi\wedge\chi}$ and $\pi_\chi = \pi_\chi\pi_{\psi\wedge\chi}$. So in conclusion $\pi^*_{\psi\wedge\chi}(\pi^*_\psi([[\psi]]) \wedge \pi^*_\chi([[\chi]])) = \pi^*_\psi([[\psi]]) \wedge \pi^*_\chi([[\chi]])$. By definition this is smaller than $\pi^*_\psi([[\psi]])$ which is smaller than $\pi^*_\phi([[\phi]])$ by assumption. Therefore $\psi \wedge \chi \vdash_\sigma \phi$ is true. The case $\chi \wedge \psi \vdash_\sigma \phi$ is analogous.

4. Suppose $\pi^*_\psi([[\psi]]) \leq \pi^*_\phi([[\phi]])$ and $\pi^*_\psi([[\psi]]) \leq \pi^*_\chi([[\chi]])$ as subobjects of $[[\sigma]]$. Analogously to the previous point, the subobject $\pi^*_{\phi\wedge\chi}(\pi^*_\phi([[\phi]]) \wedge \pi^*_\chi([[\chi]]))$ is equal to the subobject $\pi^*_\phi([[\phi]]) \wedge \pi^*_\chi([[\chi]])$. As by assumption $\pi^*_\psi([[\psi]])$ is a lower bound of $\pi^*_\phi([[\phi]])$ and $\pi^*_\chi([[\chi]])$, we have that $\pi^*_\psi([[\psi]]) \leq \pi^*_\phi([[\phi]]) \wedge \pi^*_\chi([[\chi]]) = \pi^*_{\phi\wedge\chi}(\pi^*_\phi([[\phi]]) \wedge \pi^*_\chi([[\chi]]))$ by definition. So $\psi \vdash_\sigma \phi \wedge \chi$ is true. The case $\psi \vdash_\sigma \chi \wedge \psi$ is again analogous.

5. Let

$$
\begin{array}{ccccccc}
 & & & & \rho^*([[\phi]]) & \longrightarrow & [[\phi]] \\
 & & & & \downarrow & & \downarrow \\
\mu^*([[\psi]]) & \longrightarrow & [[\sigma]] & \xrightarrow{\pi} & [[\sigma\backslash\{x\}]] & \xrightarrow{\rho} & [[FV(\phi)]] \\
\downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} & & \\
[[\psi]] & \longrightarrow & [[FV(\psi)]] & \xrightarrow{\tau} & [[FV(\exists x\psi)]] & &
\end{array}
$$

be the projections with some other details added in for later. As $[[\sigma]] = [[\sigma\backslash\{x\}]] \times [[S]]$ and $[[FV(\psi)]] = [[FV(\exists x\psi)]] \times [[S]]$, we conclude from theorem 3 that $\tau\mu = \nu\pi$. In fact,

$$
\begin{array}{ccc}
[[FV(\psi)]] & \xleftarrow[\mu]{} & [[\sigma]] \\
\downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \pi} \\
[[FV(\exists x\psi)]] & \xleftarrow[\nu]{} & [[\sigma\backslash\{x\}]]
\end{array}
$$

is a pullback. We check the universal property. Suppose $f : D \to [[FV(\psi)]]$ and $g : D \to [[\sigma\backslash\{x\}]]$ are such that $\tau f = \nu g$. We can split $[[\sigma]]$ into $[[FV(\exists x\psi)]] \times [[S]] \times R$ where R is the product of the sorts of the variables in σ that are not in ψ if we have those. Call the arrow f composed with the projection on $[[FV(\exists x\psi)]]$ f_1, the arrow f composed with the projection on $[[S]]$ f_2, the arrow g composed with the projection on $[[FV(\exists x\psi)]]$ g_1 and the arrow g composed with the projection on R g_2. As $\tau f = \nu g$, we have that $f_1 = g_1$. We will prove the uniqueness of a $u : D \to [[\sigma]]$ such that $\pi u = g$ and $\mu u = f$, from which it will be self evident that such a u exists as well. We can specify u by u composed with the projections on $[[FV(\exists x\psi)]]$, $[[S]]$ and R, which we will call $u_1,\ u_2,\ u_3$ respectively (so $u = \langle u_1, u_2, u_3\rangle$). As $\pi u = g$, we have that $\langle u_1, u_3\rangle = \langle g_1, g_2\rangle$, so $u_1 = g_1 = f_1$ and $u_3 = g_2$. As $\mu u = f$, we have that $\langle u_1, u_2\rangle = \langle f_1, f_2\rangle$, so $u_2 = f_2$, so $u = \langle f_1, f_2, g_2\rangle$, which is uniquely determined.

If there are no variables in σ not in ψ, we ignore the R and everything still works ($u = \langle f_1, f_2\rangle$).

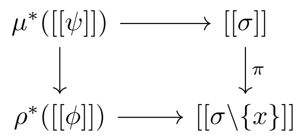
Since by assumption and theorem 3, the subobject $\mu^*([[\psi]])$ is smaller than $(\rho\pi)^*([[\phi]])$, there is a commutative diagram

$$
\begin{array}{ccc}
\mu^*([[\psi]]) & \longrightarrow & [[\sigma]] \\
\downarrow & \nearrow & \\
(\rho\pi)^*([[\phi]]) & &
\end{array}
$$

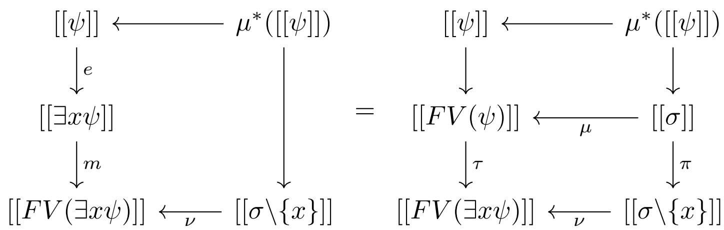
The following diagram commutes (lemma 35):

$$
\begin{array}{ccccc}
(\rho\pi)^*(\psi) & \longrightarrow & \rho^*([[\phi]]) & \longrightarrow & [[\phi]] \\
\downarrow & & \downarrow & & \downarrow \\
[[\sigma]] & \xrightarrow{\pi} & [[\sigma\backslash\{x\}]] & \xrightarrow{\rho} & [[FV(\phi)]]
\end{array}
$$

These two diagrams together gives us the commutative diagram

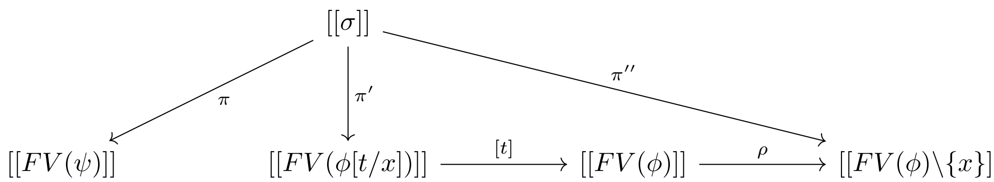


where $\mu^*([[\psi]]) \to \rho^*([[\phi]])$ is the composition $\mu^*([[\psi]]) \to (\rho\pi)^*([[\psi]]) \to \rho^*([[\phi]])$. By theorem 34 applied to

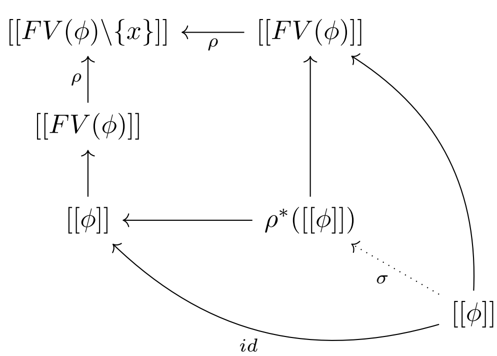


- which is pullback by the pullback pasting lemma - $\nu^*([[\exists x\psi]])$ is the image of $\mu^*([[\psi]]) \to [[\sigma\backslash\{x\}]]$. As $\mu^*([[\psi]]) \leq \rho^*([[\phi]])$ as subobjects of $[[\sigma\backslash\{x\}]]$, also $\nu^*([[\exists x\psi]]) \leq \rho^*([[\phi]])$ as subobjects of $[[\sigma\backslash\{x\}]]$ (example 30). So $\exists x\psi \vdash_{\sigma\backslash\{x\}} \phi$ is true.

6. Suppose x occurs free in ϕ. Consider the commutative diagram



with $[t]$ as in lemma 61 and the other maps projections. Recall that in this case $[[\exists x\phi]]$ is the image of $[[\phi]] \to [[FV(\phi)]] \xrightarrow{\rho} [[FV(\exists x\phi)]]$, so the subobject represented by $[[\phi]] \to [[FV(\phi)]] \xrightarrow{\rho} [[FV(\exists x\phi)]]$ is smaller than $[[\exists x\phi]] \to [[FV(\exists x\phi)]]$. From the following commutative diagram it follows that $\rho^*([[\phi]]) \to [[FV(\phi)]]$ and $[[\phi]] \to [[FV(\phi)]]$ represent the same subobject as the outer square is clearly a pullback, which means that σ is iso.



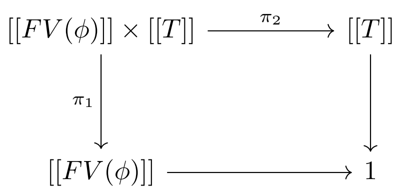
So from this and ρ^* being monotone (theorem 33) it follows that $[[\phi]] \leq \rho^*([[\exists x\phi]])$ as subobjects of $[[FV(\phi)]]$. So if $\pi^*([[\psi]]) \leq \pi'^*([[\phi[t/x]]])$, then with lemma 61,

$$\pi^*([[\psi]]) \leq \pi'^*([t]^*([[\phi]])) \leq \pi'^*([t]^*(\rho^*([[\exists x\phi]]))) = \pi''^*([[\exists x\phi]])$$

in $Sub([[\sigma]])$, which proves this case.

Now suppose x does not occur free in ϕ. We assume that ψ contains no more free variables than ϕ and say $\sigma = FV(\phi)$. In the general case we first have to view all subobjects as subobjects of $[[\sigma]]$ defined by pullback along projections, in the same way we did the previous points. Since this does not change the argument much, but makes the diagrams much more confusing, we will make this assumption.

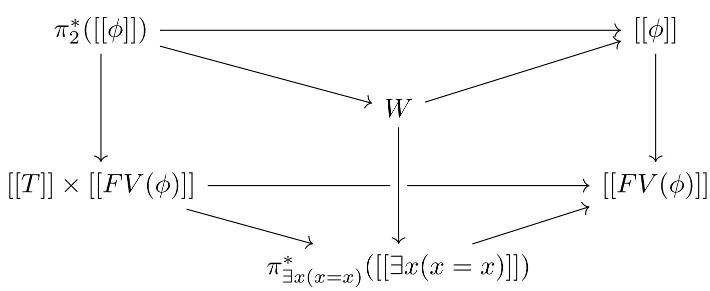
Assume that $\psi \vdash_\sigma \phi$ and $\psi \vdash_\sigma \exists x(x = x)$ are true. Let X be of sort T. We write π' for the projection of $[[T]] \times [[FV(\phi)]]$ to $[[FV(\phi)]]$. In this case, the subobject $[[\exists x\phi]]$ is represented by the image of $(\pi')^*([[\phi]]) \to [[T]] \times [[FV(\phi)]] \xrightarrow{\pi'} [[FV(\phi)]]$.
Recall that $[[\exists x(x = x)]]$ is the image of $[[T]] \to 1$. So we get that $\pi^*_{\exists x(x=x)}([[\exists x(x = x)]])$ is the image of the projection π_1 of $[[T]] \times [[FV(\phi)]]$ on $[[FV(\phi)]]$ by theorem 34 and the following pullback square.



This clearly commutes, and the universal property is easy to check because for morphisms $f : Z \to [[T]]$ and $g : Z \to [[FV(\phi)]]$, the morphism $\langle f, g\rangle : Z \to [[FV(\phi)]] \times [[T]]$ is the unique morphism we are looking for.

As $\psi \vdash_\sigma \phi$ and $\psi \vdash_\sigma \exists x(x = x)$ are true, we have that $\pi^*_\psi([[\psi]])$ is smaller than the greatest lower bound W of $[[\phi]]$ and $\pi^*_{\exists x(x=x)}([[\exists x(x = x)]])$ as subobjects of $[[FV(\phi)]]$.

We put everything in a commutative diagram.



The existence of an arrow from $\pi_2^*([[\phi]])$ to W follows from the universal property of the pullback square with vertices W, $[[\phi]]$, $[[FV(\phi)]]$, $\pi^*_{\exists x(x=x)}([[\exists x(x = x)]])$.

As the right front square and back square are pullback squares by definition and the diagram commutes, also

$$\begin{array}{ccc}
\pi_2^*([[\phi]]) & \longrightarrow & W \\
\downarrow & & \downarrow \\
[[T]] \times [[FV(\phi)]] & \longrightarrow & \pi^*_{\exists x(x=x)}([[\exists x(x=x)]])
\end{array}$$

is a pullback square by the pullback pasting lemma. As regular epis are stable under pullback, the morphism $\pi_2^*([[\phi]]) \to W$ is regular epi. So $W \to [[FV(\phi)]]$ is the image of $\pi_2^*([[\phi]])$ as well, which means that $[[\exists x\phi]]$ and W are isomorphic. As $\pi_\psi^*([[\psi]])$ is smaller than W as subobjects of $[[FV(\phi)]]$, the sequent $\psi \vdash_{FV(\phi)} \exists x\phi$ is true.

7. We apply lemma 61. Call the projections of $[[\sigma]]$ π_ψ and π_ϕ and the projections of $[[\sigma\backslash\{x\} \cup FV(t)]]$ p_ϕ and p_ψ. Suppose $\pi_\psi^*(\psi) \leq \pi_\phi^*(\phi)$ as subobjects of $[[\sigma]]$. Then $[[\phi[t/x]]] = [t]_\phi^*([[\phi]])$ and $[[\psi[t/x]]] = [t]_\psi^*([[\psi]])$, so $p_\phi^*([[\phi[t/x]]]) = ([t]_\phi p_\phi)^*([[\phi]])$ by lemma 35. Similarly $p_\psi^*([[\psi[t/x]]]) = ([t]_\psi p_\psi)^*([[\psi]])$.

   Define $[t]_\sigma : [[\sigma\backslash\{x\} \cup FV(t)]] \to [[\sigma]]$ to be projections except for the factor corresponding to $\{x\}$ where it is
$$[[FV(t)]] \xrightarrow{[[t]]} [[\{x\}]]$$

   We claim that $\pi_\phi[t]_\sigma = [t]_\phi p_\phi$ and similarly for ψ. As a morphism into $[[FV(\phi)]] = [[S_1]] \times ... \times [[S_n]] \times [[\{x\}]]$ is completely determined by its components, it suffices to check whether the morphisms $\pi_\phi[t]_\sigma$ and $[t]_\phi p_\phi$ agree there. For $[[S_i]]$ the morphism $\pi_\phi[t]_\sigma$ composed with the projection to $[[S_i]]$ is simply the projection of $[[\sigma\backslash\{x\} \cup FV(t)]]$ on $[[S_i]]$ and for $[t]_\phi p_\phi$ as well. For $[[\{x\}]]$ both morphisms composed with the proper projection are $[[FV(t)]] \xrightarrow{[[t]]} [[\{x\}]]$, so the claim is true.

   As $[t]_\sigma$ is monotone and $\pi_\psi^*([[\psi]]) \leq \pi_\phi^*([[\phi]])$, we can finally conclude that $p_\psi^*([[\psi[t/x]]]) = (\pi_\psi[t]_\sigma)^*([[\psi]]) = [t]_\sigma^*(\pi_\psi^*([[\psi]])) \leq [t]_\sigma^*(\pi_\phi^*([[\phi]])) = (\pi_\phi[t]_\sigma)^*([[\phi]]) = p_\phi^*([[\phi[t/x]]])$

   so $\psi[t/x] \vdash_{\sigma\backslash\{x\}\cup FV(t)} \phi[t/x]$ is true.

8. Suppose $\psi \vdash_\sigma \phi[t/x]$ and $\psi \vdash_\sigma t = s$ are true. Let t, s be of sort S. Call the projections of $[[FV(t = s)]]$ on $[[FV(t)]]$ and $[[FV(s)]]$ π_t and π_s respectively. Call the projection of $[[\sigma]]$ to $[[FV(\phi[t/x])]] = [[FV(\phi)\backslash\{x\}]] \times [[FV(t)]]$ π_ϕ and the projection of $[[\sigma]]$ to $[[FV(\phi[s/x])]] = [[FV(\phi)\backslash\{x\}]] \times [[FV(s)]]$ p_ϕ.

   As $\pi_{t=s}^*([[t = s]]) \xrightarrow{e'} [[\sigma]]$ factors via e, it is also true that $[[s]]\pi_s\pi_{t=s}e' = [[t]]\pi_t\pi_{t=s}e'$.

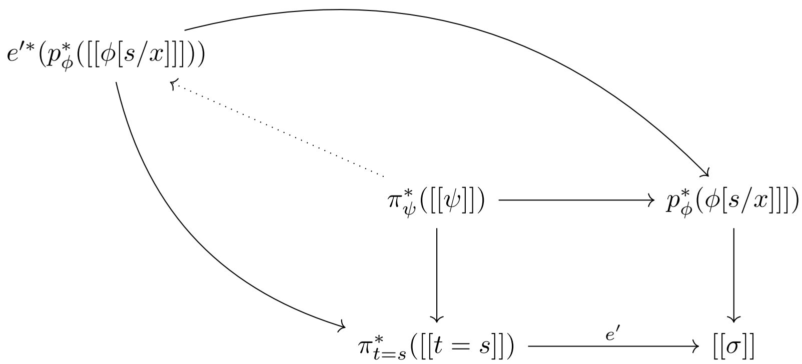
$$\begin{array}{ccccc}
\pi^*_{t=s}([[t=s]]) & \xrightarrow{e'} & [[\sigma]] & & \\
\downarrow & & \downarrow{\scriptstyle \pi_{t=s}} & & \\
[[t=s]] & \xrightarrow{e} & [[FV(t=s)]] & \underset{[[s]]\pi_s}{\overset{[[t]]\pi_t}{\rightrightarrows}} & [[S]]
\end{array}$$

   Using the definition of $[s]$, $[t]$, the composition of projections as described in theorem 3 and lemma 35 we get that $e'^*(p_\phi^*([[\phi[s/x]]])) = ([s]e')^*([[\phi]]) = ([t]e')^*([[\phi]]) = e'^*(\pi_\phi^*([[\phi[t/x]]]))$.

   By assumption the following diagram commutes

$$\begin{array}{ccc}
\pi^*_\psi([[\psi]]) & \longrightarrow & \pi^*_\phi(\phi[t/x]]]) \\
\downarrow & & \downarrow \\
\pi^*_{t=s}([[t=s]]) & \xrightarrow{e'} & [[\sigma]]
\end{array}$$

It follows from the universal property of a pullback that $\pi_\psi^*([[\psi]]) \leq e'^*(\pi_\phi^*([[\phi[t/x]]])) = e'^*(p_\phi^*([[\phi[s/x]]]))$ as subobjects of $\pi_{t=s}^*([[t = s]])$. So the diagram below commutes.



We see from that also $\pi_\psi([[\psi]]) \leq p_\phi^*([[\phi[s/x]]])$ as subobjects of $[[\sigma]]$, which is precisely what it means for $\psi \vdash_\sigma \phi[s/x]$ to be true.

9. This is because the relation $\leq$ on $Sub([[\sigma]])$ is transitive.

$\square$

5.2 Completeness theorem

We state the completeness theorem without proof:

Theorem 62 (Completeness theorem)**.** *Suppose that the sequent* $(\psi) \vdash_\sigma \phi$ *is true in every interpretation which makes all of the sequents from* T *true, then* $(\psi) \vdash_\sigma \phi$ *is in* T*.*

The proof of this theorem involves building a regular category $\mathcal{C}(T)$ for a theory T and a regular category $\mathcal{C}$, called a syntactic category, which allows an interpretation that makes precisely the sequents from T true.

6. **Conclusion**

To summarise, we can define a regular category by requiring it has all finite limits, all kernel pairs have a coequaliser and that regular epis are stable under pullback. This means we can define the image of a morphism which allows us to interpret the existential quantifier. We then interpret regular logic in regular categories by using various limits. We can define a notion of truth using sequents and define theories, which can be used to derive sequents. It follows from the soundness and completeness theorem that this supports the deduction rules from regular logic.

We did not get to treat all of chapter 4 of [7]. I would like to give a quick summary of what we skipped. Given a group G, we can find the closest abelian approximation by dividing the group by its commutator subgroup in the sense that any morphism f from G to an abelian group H factorises as

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle \pi}\downarrow & \nearrow_{f'} & \\ G/[G,G] & & \end{array}$$

for some unique morphism f'.

We can ask ourselves if there is a closest regular approximation to each category. It turns out there is in the case of small categories: we define a suitable theory and then the syntactic category $\mathcal{C}(T)$ (introduced in 5.2) is the closest approximation. This means that there is a functor $\eta : \mathcal{C} \to \mathcal{C}(T)$ such that any functor F from $\mathcal{C}$ to some small regular category $\mathcal{E}$ factorises as

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ {\scriptstyle \eta}\downarrow & \nearrow_{\tilde{F}} & \\ \mathcal{C}(T) & & \end{array}$$

where $\tilde{F}$ is a functor that preserves finite limits and regular epimorphisms (the structure of a regular category), determined uniquely up to isomorphism.

The category $\mathcal{C}(T)$ is also called the free regular category. This is exercise 85 of [7].

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