Singular points in compactifications of type IIA string theory

Stijn Bruning

June 2019

Master's thesis

Supervised by prof. dr. Thomas Grimm

Institute of Theoretical Physics
Universiteit Utrecht

Contents

For, lo, the wicked bend their bow, they make ready their arrow upon the string, that they may privily shoot at the upright in heart. - Psalm 11:2 (King James Version)

Abstract

In this thesis, singular loci in Kähler moduli space of type IIA string theory on a Calabi-Yau threefold are considered. The compactification of type IIA string theory to 4D is described and the moduli spaces of the Calabi-Yau threefold are introduced. Then, singularities in the Kähler moduli space are discussed and classified using the theory of mixed Hodge structures and nilpotent orbits, which is introduced in some detail. We use this classification of singular loci to find constraints on the triple intersection numbers $\mathcal{K}_{IJK}$ that occur in the metric of type IIA Kähler moduli space. These are classified in terms of the various singularity types. We also try to find 'ficticious' $\mathcal{K}_{IJK}$ that violate the constraints, finding that these usually seem to yield moduli space metrics that degenerate somewhere on moduli space. This leads one to propose that this feature might be true in general. Lastly, we also describe the nilpotent orbits arising near the singular loci from a slightly different viewpoint, in which their nilpotent generators relate flux configurations of different kinds and are interpreted in terms of Freed-Witten anomalous branes attached to domain walls in the 4D theory.

1 Introduction

Auf, liebe Sänger! Greifet in die Saiten! — Richard Wagner,
Tannhäuser (act 2, scene 4).

1.1 String theory and compactification

Quantum mechanical theories have been extremely successful in predicting physics at the scale of fundamental particles. Hence, the laws of physics are believed to be ultimately written in the language of quantum mechanics. Incorporating gravity into quantum mechanics has, however, been a notoriously difficult task. The most naive procedure of turning general relativity into a quantum theory leads to severe mathematical inconsistencies when taken too seriously; hence, the theory obtained in this way cannot be fundamental. One thus needs a theory of quantum gravity that deals with these issues. Furthermore, one would like to extend this theory to include all of the other known forces, obtaining a unified framework that describes

String theory has proven to be an interesting theory of quantum gravity and a candidate for a consistent theory of everything. The starting point of string theory is the idea that fundamental particles are not pointlike, but consist of tiny loops of string, the vibrational modes of which determine the properties of the particle, such as their spin and mass. From this relatively simple idea, the laws of physics are supposed to emerge. String theory automatically includes the graviton, the particle that mediates the force of gravity. Hence, string theory includes gravity very naturally.

Apart from automatically including gravity, string theory has other appealing features that have turned it into an important focus of theoretical physics research. One important such feature is its high degree of uniqueness, which sets it apart from most frameworks in which one can invent theories of physics. For example, in quantum field theory, one can write down any renormalizable Lagrangian and use it as a physical theory. In contrast, string theory turns out to imply strong consistency requirements that severely limit the amount of possible theories. In fact, there turn out to be only five consistent superstring theories: the type I, type IIA, type IIB and two heterotic theories. These, in turn, are believed to all be different limits of a single larger theory, the precise formulation of which is not yet known. Furthermore, the five string theories specify the possible fields and interactions in much detail and specify the amount of spacetime dimensions to be precisely ten.

As noted before, in order for superstring theory to be mathematically consistent, the theory requires spacetime to be exactly ten-dimensional. Naively, this is a clear contradiction with everyday experience: the universe appears to be four-dimensional, having three spacelike dimensions and one timelike one. Hence, if string theory describes the real world, six out of ten dimensions should be hidden away from daily life. There are ways to achieve this: commonly, one assumes that the six 'invisible' dimensions comprise a compact manifold with a typical length scale so small that the extra dimensions are indeed invisible on 'regular' length and energy scales. The physics within the other six dimensions does, however, play an important role in the four-dimensional physics by influencing what appear to be the internal degrees of freedom of the theory from a 4D viewpoint.

1.2 Swampland criteria, the Swampland Distance Conjecture and nilpotent orbits

Although string theory, when viewed as a ten-dimensional theory, is highly unique, the resulting physics in the four large dimensions depends strongly on the choice of compactification manifold. Hence, if one wants to make contact between string theory and the real world, one needs a good understanding of the process of compactification. One can ask the question which theories can and cannot be obtained from string theory. In this context, a distinction is often made between the so-called 'string landscape', consisting of field theories that arise in some limit from string theory, and the 'swampland', consisting of theories that cannot be UV-completed using string theory[1].

Recent research activity has given rise to a host of different swampland criteria. These are (conjectured) properties that field theories should satisfy if they can be consistently embedded in string theory, that is, if they belong to the string landscape. For a comprehensive review of the various swampland criteria, we refer the interested reader to [2]. One specific conjecture that is relevant to this thesis is the so-called Swampland Distance Conjecture. This conjecture is concerned with certain special points P in field space that are at infinite field distance. That is, there exist no geodesics of finite length connecting P to any other point Q in that field space. Here, a 'field space' is understood to be the space in which the fields of the theory take their values. The statement of the Swampland Distance Conjecture is then that, when moving towards P in field space, towers of states appear that become massless exponentially fast in the field distance. Loosely speaking, the mass scale m of the states goes as $m \sim \exp(-d(P,Q))$, where $d(\cdot,\cdot)$ is the geodesic distance between two points[3].

In order to show that the Swampland Distance Conjecture holds in some setup, one first has to find infinite distance points, then identify a tower of states that becomes massless when approaching the infinite distance point. This was done for many types of singular points in the complex structure moduli space of type IIB string theory compactified on a Calabi-Yau threefold[4, 5], and in the mirror setting of the type IIA Kähler moduli space by the use of the mirror map[6]. These spaces are field spaces of 4D scalars that come from massless deformations of the metric on the Calabi-Yau space Y used for compactification. In these specific settings, applying a deep mathematical framework turned out to be a fruitful avenue to demonstrating rather general properties of the moduli spaces near singular points. This mathematical formalism is concerned with Hodge structures, their variation over moduli spaces and their degeneration into mixed Hodge structures[7, 8, 9, 10, 11].

The theory of degenerating Hodge structures describes (as a special case) the behavior of the middle cohomology $H^3(Y,\mathbb{C})$ of the Calabi-Yau threefold Y on which the theory is compactified. The middle cohomology $H^3(Y,\mathbb{C})$ is a vector space of differential three-forms that is topological in origin. $H^3(Y,\mathbb{C})$ decomposes into complex subspaces by a decomposition that depends on the complex structure of the Calabi-Yau threefold. This dependence is captured by the theory of degenerating Hodge structures. Furthermore, the theory allows one to analyze the theory in singular limits in which the middle cohomology becomes ill-behaved, by using the monodromy transformations of $H^3(Y,\mathbb{C})$ that arise when encircling the singularity.

In a realistic 4D theory resulting from a string theory compactification, the formalism of mixed Hodge structures operates 'under the hood'. One can then ask whether one can invent a theory formulated in 4D spacetime that looks realistic, but in which the requirements of mixed Hodge theory are not satisfied. In this way, one has a 4D theory that belongs to the swampland in a non-trivial way. In particular, there might be nontrivial constraints on coupling constants in the 4D theory.

In this thesis, the main focus will be on compactified type IIA superstring theory. Specifically, we will study the Kähler moduli space of type IIA supergravity, to which we will apply the formalism of mixed Hodge structures. In order to introduce this moduli space, we will treat the compactification of type IIA supergravity in some detail. Then, we will combine the formalism of mixed Hodge structures with explicit expressions for the monodromy matrices in order to search for a 4D theory that is inconsistent in the way described above.

1.3 Outline of this thesis

The structure of this thesis is as follows. In chapter 2, the compactification of type IIA superstring theory on a Calabi-Yau manifold is described, in order to provide some general context for the more detailed calculations that will follow. The low-energy effective action of type IIA superstring theory, type IIA supergravity, is introduced. Then, the geometry of Calabi-Yau manifolds and their moduli spaces is discussed. The compactification of the IIA supergravity action is then carried out, resulting in a 4D effective supergravity action. Finally, the relevance of orientifolds and flux parameters is described. Since the geometry of the Kähler moduli space of type IIA superstring theory is not profoundly affected by the orientifold procedure, we will not go into too much detail.

In chapter 3, we specialize to studying the type IIA Kähler moduli space and the monodromy that can arise on it. First, the framework of mixed Hodge structures will be introduced in the context of the mirror setting of type IIB complex structure moduli space and then applied to type IIA Kähler moduli space. The monodromy transformations arising near singular points and the nilpotent matrices generating these transformations are introduced, together with some explicit expressions in the context of certain large-moduli limits in type IIA Kähler moduli space. Then, the classification of singular points in moduli space will be introduced and the possible ways in which a singularity can degenerate will be discussed. The properties of the different singularity types are connected to the properties of the nilpotent monodromy generators, which turn out to be useful and interesting objects.

In chapter 4, we present partially new results regarding constraints on a certain tensor in 4D supergravity, the triple intersection tensor $\mathcal{K}_{IJK}$. For a 4D supergravity theory to descend from 10-dimensional supergravity compactified on a Calabi-Yau manifold, it is necessary that the elements of this tensor obey certain constraints called the 'polarization conditions', coming from the theory of degenerating Hodge structures as presented in the previous chapter. For low-dimensional moduli spaces, we will try to classify the possible $\mathcal{K}_{IJK}$ that can occur, depending on the singularity types of the various large-moduli limits that they give rise to. We will try to violate the polarization conditions, but find that this seems to result in the moduli space metric not being positive-definite. One suspects that this result might be general, but we will not attempt to prove this.

Finally, in chapter 5, we discuss a slightly different subject, that is nevertheless simply connected to the topic of nilpotent orbits. We treat a slightly different interpretation of the nilpotent matrices described in chapter 3 in terms of Freed-Witten anomalous branes in flux compactifications. Specifically, domain walls between different regions of space-time in which there are different flux configurations can decay by emitting Freed-Witten anomalous 4D strings, and the transformations that relate the different configurations on both sides of the domain wall turn out to be the monodromy transformations familiar from chapter 3. This more physical interpretation was found in references [12, 13] based on [14], where the scalar potential in type IIA supergravities compactified in the presence of fluxes was analyzed. Here, we will resort to a flying tour of the physics of Freed-Witten anomalies, D-branes and the way in which these give rise to nilpotent matrices.

We end this thesis with some concluding remarks, in which we summarize the discussion and the results.

2 Dimensional reduction of type IIA supergravity

String theory is a theory, formulated in ten spacetime dimensions, which is supposed to give a consistent description of quantum gravity phenomena at arbitrarily high energies. However, most household phenomena take place at low energies with respect to typical quantum gravity energy scales. Furthermore, daily life takes place in only four of the ten spacetime dimensions. Therefore, in order to make contact with mundane physics, one would like to have an effective description of the low-energy physics of string theory. Additionally, one needs to have a way to remove six of the ten spacetime dimensions, which can be done by a process called compactification.

In this chapter the low-energy action of type IIA string theory, type IIA supergravity, will be introduced. Then, after the theory of deformations of Calabi-Yau metrics and the corresponding moduli spaces is discussed, the compactification of IIA supergravity to 4D using a Calabi-Yau threefold is described. Finally, flux parameters and orientifold projections will be concisely discussed.

2.1 IIA superstring theory and supergravity

Superstring theory is a version of string theory in which there are fermionic excitations of the string as well as bosonic ones. These excitations are related to each other by supersymmetry, both on the worldsheet and in spacetime. Famously, there are five different (but dual) superstring theories, called type I, type IIA, type IIB, heterotic $SO(32)$ and heterotic $E8 \times E8$. These theories differ mainly in their field content. Here, we will not be concerned with the heterotic and type I superstring theories at all; we will mainly focus on type IIA string theory and refer to the type IIB string theory on a few occasions. Furthermore, we will almost entirely focus on closed strings, although D-branes will be introduced in chapter 5.

Superstring theory is often considered from a worldsheet perspective, in which the string is considered as a (1+1)-dimensional object on which one can do quantum field theory. However, the massless excitations of the superstring serve double duty as fields in (9+1)-dimensional spacetime that provide a background for the string to propagate in. Hence, it is convenient to have a spacetime description of the dynamics of these fields. This desription is provided by the low-energy effective action of the superstring theory[15, 16, 17].

Like the type IIA superstring theory itself, the type IIA effective action is formulated in 10 spacetime dimensions. It is given by the so-called type IIA supergravity action, which will be introduced below. Whereas the low-energy effective action of bosonic string theory can be derived by analyzing the beta functions of the corresponding worldsheet model[16], for superstrings, the form of the effective action is actually dictated by supersymmetry. The low-energy effective action of type IIA string theory (or rather, of its bosonic fields) will be the starting point of this thesis.

The massless bosonic excitations of the closed string sector of type IIA string theory are the metric $\hat{g}$, a dilaton $\hat{\phi}$ and a two-form $\hat{B}_2$ in the NSNS sector and the one-form $\hat{C}_1$ and three-form $\hat{C}_3$ in the RR sector. The low-energy effective action of these fields is given by the type IIA bosonic supergravity action[18, 19, 15]:

$$\begin{aligned} S_{\rm IIA} = \int & -\frac{1}{2}e^{-2\hat{\phi}}\hat{R} *_{10} 1 + 2e^{-2\hat{\phi}}d\hat{\phi}\wedge *_{10}d\hat{\phi} - \frac{1}{4}e^{-2\hat{\phi}}\hat{H}_3\wedge *_{10}\hat{H}_3 - \frac{1}{2}\hat{F}_2\wedge *_{10}\hat{F}_2 \\ & -\frac{1}{2}\hat{F}_4\wedge *_{10}\hat{F}_4 + \mathcal{L}_{\rm top}, \end{aligned} \tag{1}$$

with topological terms $\mathcal{L}_{\rm top}$ given by

$$\mathcal{L}_{\rm top} = -\frac{1}{2}\left[\hat{B}_2\wedge d\hat{C}_3\wedge d\hat{C}_3 - \left(\hat{B}_2\right)^2\wedge d\hat{C}_3\wedge d\hat{C}_1 + \frac{1}{3}\left(\hat{B}_2\right)^3\wedge d\hat{C}_1\wedge d\hat{C}_1\right], \tag{2}$$

where the powers of $\hat{B}_2$ are shorthand for wedge products; for example, $\left(\hat{B}_2\right)^2 \equiv \hat{B}_2\wedge\hat{B}_2$. $\hat{R}$ is the Ricci scalar belonging to the metric $\hat{g}$ and the field strengths $\hat{H}_3$, $\hat{F}_2$ and $\hat{F}_4$ of the fields $\hat{B}_2$, $\hat{C}_1$ and $\hat{C}_3$ are given by

$$\begin{aligned} \hat{H}_3 &= d\hat{B}_2, \\ \hat{F}_2 &= d\hat{C}_1 \quad \text{and} \\ \hat{F}_4 &= d\hat{C}_3 - dC_1\wedge\hat{B}_2. \end{aligned} \tag{3}$$

We use a hat to denote 10-dimensional quantities; after compactification, we will end up with a 4D theory and the hats will be dropped. We denote the ten-dimensional Hodge star by $*_{10}$; the Hodge star on the resulting 4D space will be denoted by $*_4$ or simply by $*$ when there is no risk of confusion. The Hodge star on Y is denoted by $*_Y$.

The action of the fermionic fields of type IIA supergravity is related to the bosonic action by supersymmetry. Hence, we will not consider the fermionic fields here.

2.2 Compactification on a Calabi-Yau manifold

Type IIA superstring theory and type IIA supergravity are formulated in a spacetime that is ten-dimensional. In order to arrive at a four-dimensional theory, one needs to get rid of six of the spatial dimensions. A common way to do this is to assume that spacetime is a product $\mathbb{R}^{1,3}\times Y$, where $\mathbb{R}^{1,3}$ is standard 4D Minkowski spacetime and Y is a six-dimensional compact manifold that has a typical length scale l_c that is so small that it is not detectable in a direct way. One can then investigate the physics on energy scales much smaller than $1/l_c$ and compute the resulting effective action. This is what we will do in this chapter.

The fact that the typical length scale of Y is much smaller than the length scale does not mean that the properties of Y are irrelevant to the effective 4D physics. On the contrary, the choice of compactification manifold Y is crucial in determining the precise field content and couplings of the 4D theory.

A particularly well-known kind of compactification manifold is given by the so-called Calabi-Yau manifolds. These turn out to give the 4D theory a realistic amount of supersymmetry[20]. Furthermore, the theory of compactifications on these manifolds is by now well-established.

In order to describe what a Calabi-Yau manifold is, we first need to introduce so-called Kähler manifolds. A Kähler manifold is a complex manifold that admits a Hermitian metric h and a real (1,1)-form J that are compatible with each other and with the complex structure[21]. In such a setting, J is more commonly called the Kähler form and given by the imaginary part of the metric: $J = ih_{\mu\nu}dx^\mu d\bar{x}^\nu$.

A Calabi-Yau manifold, in turn, is a Kähler manifold in which the metric can be chosen to be Ricci-flat: that is, one can find a Kähler metric h of which the corresponding Levi-Civita connection has vanishing Ricci tensor, $R(h) = 0$. One of many other equivalent definitions is that a Calabi-Yau manifold is a Kähler manifold with vanishing first Chern class; yet another definition is that a Calabi-Yau manifold has $SU(N)$ holonomy.

A fact that will be relevant to our discussion is that the shape of the complex cohomology of a Calabi-Yau manifold is highly constrained. More precisely, if one writes down the general Hodge diamond of a Calabi-Yau manifold, in which the dimensions $h^{p,q}$ of the cohomology groups $H^{p,q}(Y)$ (the so-called Hodge numbers) are displayed, that Hodge diamond is given by[22]

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1. \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{4}$$

There are only two independent Hodge numbers: $h^{2,1}$ and $h^{1,1}$. Furthermore, the cohomology groups $H^{3,0}$ and $H^{0,3}$ are one-dimensional. They are spanned by harmonic forms respectively called Ω and $\bar{\Omega}$, where $\bar{\Omega}$ is the complex conjugate of Ω.

2.3 Calabi-Yau moduli spaces

Generically, on a given Calabi-Yau threefold Y, the Calabi-Yau metric h is not unique[23]. One can vary the metric by adding a small variation δh so that the new metric is $h + \delta h$. If this new metric satisfies the Ricci flatness condition $R(h + \delta h) = 0$ and is Hermitian with respect to some complex structure, it is a valid Calabi-Yau metric in its own right. Hence, Y comes with a parameter space of possible Calabi-Yau metrics, which is called the moduli space of Y.

If Y is the internal Calabi-Yau of a compactified supergravity theory, the moduli of Y turn out correspond to massless fields in the effective 4D theory. Hence, it is important to have a good understanding of Calabi-Yau moduli spaces if one wants to study compactified supergravity theories. We will here describe the deformation theory of Calabi-Yau manifolds, introduce and discuss the two distinct kinds of moduli spaces and introduce their natural metrics.

We are interested in the possible metrics on a Calabi-Yau manifold that do not destroy the Calabi-Yau property. To this end, we need the varied metric $h + \delta h$ to preserve Ricci flatness $R(h + \delta h) = 0$. Some such 'deformations' are the δh that arise due to coordinate transformations. These are not very interesting, since they do not change the physical properties of Y. One therefore removes such metric variations by demanding that the interesting δh obey the coordinate condition $\nabla^\mu \delta h_{\mu\nu} = 0$. Metric variations obeying this condition also obey $\int_Y \sqrt{h}\delta h^{\mu\nu}(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu)d^6x = 0$; hence, these δh are orthogonal to metric variations obtained by coordinate transformations[15, 23].

One can then expand the Ricci tensor $R(h + \delta h)$ to first order in δh. Using the Ricci flatness condition as well as $\nabla^\mu \delta h_{\mu\nu} = 0$ one arrives at[15, 24]

$$\nabla^\rho \nabla_\rho \delta h_{\mu\nu} + 2R_{\mu\ \nu}^{\ \rho\ \sigma} \delta h_{\rho\sigma} = 0. \tag{5}$$

Depending on their index structure, the metric variations $\delta h_{\mu\nu}$ obeying the above condition split into variations of the Kähler class $\delta h_{i\bar{j}}$ and of the complex structure δh_{ij}. On a Calabi-Yau manifold, these two types of δh obey equation 5 separately. We will now describe both of these classes of variations in turn and express them in terms of harmonic forms on Y.

2.4 Complex structure moduli space

We first turn to metric variations of the form δh_{ij}. From the index structure of δh_{ij}, one concludes that the new metric $h + \delta h$ is not Hermitian with respect to the old complex structure anymore[15]. Hence, such variations are variations of the complex structure on Y.

The complex structure variations δh_{ij} are in one-to-one correspondence with (1,2)-forms $\bar{\chi}$ on Y [23] through

$$\delta h_{ij} = \frac{i}{||\Omega||^2} \bar{z}^K \left(\bar{\chi}_K\right)_{i\bar{\imath}\bar{\jmath}} \Omega^{\bar{\imath}\bar{\jmath}}{}_j \tag{6}$$

The basis χ_u is related to Ω by Kodaira's formula,

$$\partial_{z^u} \Omega = \left(-\partial_{z^u} K^{\text{cs}}\right) \Omega + i \chi_u, \tag{7}$$

where K^{cs} is a function that will turn out to be the complex structure Kähler potential, which will be introduced later. The z^u are called complex structure moduli and function as coordinates on the so-called complex structure moduli space $\mathcal{M}_{\text{cs}}$.

The space $\mathcal{M}_{\text{cs}}$ of complex structure moduli turns out to be a Kähler manifold, with the z^i as coordinates. One can investigate the geometry of $\mathcal{M}_{\text{cs}}$ in more detail. To do this, we first decompose the holomorphic three-form Ω into a real, symplectic basis (α_K, β^K) $(K = 0,\ \ldots,\ h^{2,1})$, having the properties

$$\begin{aligned} \int_Y \alpha_K \wedge \alpha_L &= 0, \\ \int_Y \beta^K \wedge \beta^L &= 0 \quad \text{and} \\ \int_Y \alpha_K \wedge \beta^L &= \delta_K^L. \end{aligned} \tag{8}$$

It is convenient to choose a real basis, since it does not depend on the complex structure of the manifold. Ω is then decomposed as

$$\Omega = Z^K \alpha_K - \mathcal{G}_K \beta^K. \tag{9}$$

where the periods Z^K and $\mathcal{G}_K$ are given by

$$\begin{aligned} Z^K &= \int_Y \Omega \wedge \beta^K, \\ \mathcal{G}_K &= \int_Y \Omega \wedge \alpha_K. \end{aligned} \tag{10}$$

Since there are $2h^{2,1}+2$ periods, whereas $\mathcal{M}_{\text{cs}}$ is only $h^{2,1}$-dimensional, not all of these periods can be independent. One can write

$$\mathcal{G}_K = \partial_{Z^K}\mathcal{G} \tag{11}$$

where $\mathcal{G}$ is a prepotential that is a homogeneous function of degree 2 in the Z^K periods[25]. Hence, the $\mathcal{G}$ periods are not independent, but are entirely determined by the Z^K, given the form of the prepotential.

One is left with $h^{2,1}+1$ periods, the last of which is removed by considering that Ω can be arbitrarily rescaled without changing the complex structure. Hence, one can arbitrarily choose one of the (nonzero) Z^K, which is conventionally denoted by Z^0, and set it to 1. The remaining $h^{2,1}$ periods $z^i = Z^i/Z^0$ $(i=1,\ \ldots\ h^{2,1})$ are the coordinates on $\mathcal{M}_{\text{cs}}$ given before.

The periods can be displayed in the form of a period vector Π given by

$$\Pi = \begin{pmatrix} Z^0 \\ Z^K \\ \mathcal{G}_K \\ \mathcal{G}_0 \end{pmatrix}. \tag{12}$$

This object transforms as a vector under symplectic rotations of the basis (α_K, β^K).

The tangent space of $\mathcal{M}_{\text{cs}}$ has a natural metric $g_{i\bar{\jmath}}$, called the Weil-Petersson metric, given by

$$g_{i\bar{\jmath}} = -i\frac{\int_Y \chi_i \wedge \bar{\chi}_{\bar{\jmath}}}{\int_Y \Omega \wedge \bar{\Omega}}. \tag{13}$$

It can be given in terms of a Kähler potential K^{cs} given by

$$\begin{aligned} K^{\text{cs}} &= -\log\left(i\int_Y \Omega\wedge\bar{\Omega}\right) \\ &= -\log\left(i\bar{Z}^K\mathcal{G}_K - Z^K\bar{\mathcal{G}}_K\right). \end{aligned} \tag{14}$$

Then the metric $g_{K\bar{L}}$ is obtained from K^{cs} by

$$g_{K\bar{L}} = \partial_{z^K}\partial_{\bar{z}^{\bar{L}}}K^{\text{cs}}, \tag{15}$$

with the χ_K from the basis given earlier. From these expressions, one concludes that a holomorphic rescaling of Ω does not affect the metric g_{ij}, as we anticipated earlier.

Furthermore, the Hodge duals of the (α_K, β^K) are traditionally written in terms of a coupling matrix $\mathcal{M}$ by using the annoying expressions

$$\begin{aligned} \int_Y \alpha_A \wedge *\alpha_B &= -\left(\text{Im}\ \mathcal{M}\right) - \left(\text{Re}\ \mathcal{M}\right)\left(\text{Im}\ \mathcal{M}\right)^{-1}\left(\text{Re}\ \mathcal{M}\right) \\ \int_Y \beta^A \wedge *\beta^B &= -\left(\text{Im}\ \mathcal{M}\right)^{-1} \\ \int_Y \alpha_A \wedge *\beta^B &= -\left(\text{Re}\ \mathcal{M}\right)\left(\text{Im}\ \mathcal{M}\right)^{-1}. \end{aligned} \tag{16}$$

Even worse, $\mathcal{M}$ is given in terms of the following awkward expression,

$$\mathcal{M}_{AB} = \bar{\mathcal{G}}_{AB} + \frac{2i}{Z^C \text{Im}\ \mathcal{G}_{CD} Z^D}\text{Im}\ \mathcal{G}_{AI}Z^I \text{Im}\ \mathcal{G}_{BJ}Z^J, \tag{17}$$

that is dictated by special geometry. We will not need this explicit expression, but we will use the matrix $\mathcal{M}$ in the 4D effective action.

The presentation of the complex structure moduli space given here is purely geometrical in nature. In type IIA string theory, the behavior of the complex structure sector is strongly modified by space-time (D-brane) instantons. Meanwhile, in type IIB string theory, the complex structure sector retains its classical structure.

2.5 Kähler moduli space

Now, we consider variations of the metric of the form $\delta h_{i\bar{\jmath}}$. Metric variations of this form preserve the Hermiticity condition on the metric automatically, as can be concluded from the index structure. The condition (5) now reads

$$\nabla^\rho \nabla_\rho \delta h_{i\bar{\jmath}} + 2R_{i\ \bar{\jmath}}^{\ k\ \bar{l}} \delta h_{k\bar{l}} = 0. \tag{18}$$

Reinterpreting $\delta h_{i\bar{\jmath}}$ as a (1,1)-form, this is the condition that its Laplacian $\Delta \delta h_{i\bar{\jmath}}$ vanishes. Hence the metric variations $\delta h_{i\bar{\jmath}}$ correspond to harmonic (1,1)-forms, making the space of such variations $h^{1,1}$-dimensional. Given a basis ω_A of $H^{1,1}(Y)$ $(A = 1, ..., h^{1,1})$, a general $\delta h_{i\bar{\jmath}}$ can be expanded as

$$\delta h_{i\bar{\jmath}} = \sum_{A=1}^{h^{1,1}} v^A (\omega_A)_{i\bar{\jmath}}. \tag{19}$$

Note that varying the metric in this way also changes the cohomology class of the Kähler form

$$J = i h_{i\bar{\jmath}} dz^i \wedge d\bar{z}^{\bar{\jmath}}. \tag{20}$$

Hence, the v^A are called Kähler moduli.

There is an additional constraint coming from the fact that the metric $h + \delta h$ should be positive-definite[15]. This restriction translates into the conditions on the Kähler form

$$\int_C J > 0; \quad \int_S J \wedge J > 0; \quad \int_Y J \wedge J \wedge J > 0, \tag{21}$$

for all irreducible proper curves C and surfaces S on Y. The space of $t^A \omega_A$ for which these conditions are satisfied is called the Kähler cone and one could check that it is indeed a cone. The Kähler cone of a Calabi-Yau threefold can be simplicial or non-simplicial; if it is simplicial, this means that the cone is spanned by exactly $h^{1,1}$ generators. If $h^{1,1} \geq 3$, the Kähler cone can also be non-simplicial, in which case more generators are needed to span the full cone.

In a string theory setting, one also has a B_2 field that can upon compactification on a CY threefold likewise be decomposed into the ω_A basis as

$$B_2 = ... + \sum_{A=1}^{h^{2,1}} b^A \omega_A, \tag{22}$$

where the dots indicate the external part, normally a two-form in Minkowski space. It turns out to be convenient to combine v^A and b^A into $h^{1,1}$ complex scalars, yielding the so-called complexified Kähler moduli $t^A = b^A + iv^A$. Equivalently, one can talk about a complexified Kähler form J_c given by

$$J_c = B_2 + iJ = t^A \omega_A. \tag{23}$$

The space of complexified Kähler moduli t^A is a manifold $\mathcal{M}_{\rm K}$ in its own right, called the Kähler moduli space.

Like $\mathcal{M}_{\rm cs}$, $\mathcal{M}_{\rm K}$ also has a natural metric G_{AB} that gives the kinetic terms of the t^A fields and comes naturally out of the compactification procedure. It is given by

$$G_{AB} = \frac{1}{4\mathcal{K}} \int_Y \omega_A \wedge *\omega_B, \tag{24}$$

where $\mathcal{K}$ is given by

$$\begin{aligned} \mathcal{K} &= \frac{1}{6} \int_Y J \wedge J \wedge J = \frac{1}{6} \int_Y v^A v^B v^C \omega_A \wedge \omega_B \wedge \omega_C \\ &\equiv \frac{1}{6} \mathcal{K}_{ABC} v^A v^B v^C \\ &= \frac{i}{48} \mathcal{K}_{ABC} \left(t^A - \bar{t}^A\right) \left(t^B - \bar{t}^B\right) \left(t^C - \bar{t}^C\right), \end{aligned} \tag{25}$$

where we defined the intersection numbers

$$\mathcal{K}_{ABC} = \int_Y \omega_A \wedge \omega_B \wedge \omega_C. \tag{26}$$

The intersection tensor represented by the $\mathcal{K}_{ABC}$ is a topological object, although its components depend on the basis ω_A. In particular, if one chooses an integral basis consisting of the generators of the Kähler cone, the intersection numbers are integers.

We can rewrite the expression for G_{AB} in another interesting form. The Hodge dual of a harmonic (1,1)-form η can be written as[15, 26]

$$*\eta = -J \wedge \eta + \frac{3}{2} \frac{\int \eta \wedge J \wedge J}{\int J \wedge J \wedge J} J \wedge J \tag{27}$$

Using this identity, the metric G_{AB} can alternatively be expressed as[19]

$$G_{AB} = -\frac{1}{4} \left(\frac{\mathcal{K}_{AB}}{\mathcal{K}} - \frac{1}{4} \frac{\mathcal{K}_A \mathcal{K}_B}{\mathcal{K}^2} \right). \tag{28}$$

Here, we use the traditional abbreviations

$$\mathcal{K}_A = \int_Y \omega_A \wedge J \wedge J, \qquad \mathcal{K}_{AB} = \int_Y \omega_A \wedge \omega_B \wedge J. \tag{29}$$

One can now check that the metric G_{AB} can be obtained as

$$G_{AB} = -\partial_{v^A} \partial_{v^B} \log\left(8\mathcal{K}\right). \tag{30}$$

Hence, G is a Kähler metric with Kähler potential K given by

$$K = -\log\left(8\mathcal{K}\right). \tag{31}$$

Actually, one can write this Kähler potential in terms of a prepotential as well[23]. One defines the coordinates $X^I = (1, t^A)$ and a prepotential $\mathcal{F}$ as

$$\mathcal{F} = -\frac{1}{6}\frac{\mathcal{K}_{IJK}X^IX^JX^K}{X^0}, \tag{32}$$

and, like in the complex structure sector, we can denote derivatives of the prepotential by suffixes as

$$\mathcal{F}_I \equiv \partial_{X^I}\mathcal{F}. \tag{33}$$

The Kähler potential is then given by

$$K = -\log i\left(\bar{X}^I\mathcal{F}_I - X^I\bar{\mathcal{F}}_I\right) \equiv -\log\left(i\bar{\Pi}\cdot\vartheta\cdot\Pi\right), \tag{34}$$

where

$$\Pi = \begin{pmatrix} X^0 \\ X^A \\ \mathcal{F}_A \\ \mathcal{F}_0 \end{pmatrix} = \begin{pmatrix} 1 \\ t^A \\ -\frac{1}{2}\mathcal{K}_{ABC}t^Bt^C \\ \frac{1}{6}\mathcal{K}_{ABC}t^At^Bt^C \end{pmatrix} \tag{35}$$

and the symplectic pairing ϑ is given by

$$\vartheta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \delta_{IJ} & 0 \\ 0 & -\delta_{IJ} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{36}$$

The above expressions for the period vector Π and the prepotential $\mathcal{F}$ are purely geometric quantities. In contrast to the complex structure moduli space, however, the Kähler moduli space undergoes perturbative α'-corrections coming from string theory. If one wants an integral basis for the cohomology, one needs to take into account these quantum corrections. The computations of these corrections is hard will not be presented here. A way to find the full form of the Kähler period vector, including α'-corrections, is to compute the complex structure periods on the mirror manifold[27]. These are given by [28, 29, 6]

$$\Pi = \begin{pmatrix} 1 \\ t^A \\ \frac{1}{2}\mathcal{K}_{ABC}t^Bt^C + \frac{1}{2}\mathcal{K}_{ABB}t^B - b_A \\ \frac{1}{6}\mathcal{K}_{ABC}t^At^Bt^C - \left(\frac{1}{6}\mathcal{K}_{AAA} + b_A\right)t^A + \frac{i\zeta(3)\chi}{8\pi^3} \end{pmatrix}. \tag{37}$$

The pairing ϑ is now given by

$$\vartheta = \begin{pmatrix} 0 & -\frac{1}{6}\mathcal{K}_{JJJ} - 2b_J & 0 & -1 \\ \frac{1}{6}\mathcal{K}_{III} + 2b_I & \frac{1}{2}(\mathcal{K}_{IIJ} - \mathcal{K}_{IJJ}) & \delta_{IJ} & 0 \\ 0 & -\delta_{IJ} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{38}$$

where $b_I = \frac{1}{24}\int_Y \omega_I \wedge c_2(Y)$, with $c_2(Y)$ the second Chern class of Y.

One can also compute the Kähler potential in the presence of quantum corrections, which is modified to

$$K = -\log\left(\frac{1}{6}\mathcal{K}_{ABC}v^Av^Bv^C + \frac{\zeta(3)\chi}{32\pi^3}\right) \equiv -\log\mathcal{V}_Y. \tag{39}$$

Finally, the Kähler sector also has its own coupling matrix $\mathcal{N}$, which is the analog of $\mathcal{M}$ from the complex structure sector and which is given by the analogous horrible-looking expression:

$$\mathcal{N}_{IJ} = \bar{\mathcal{F}}_{IJ} + \frac{2i}{X^M \text{Im } \mathcal{F}_{MN} X^N} \text{Im } \mathcal{F}_{IK} X^K \text{Im } \mathcal{F}_{JL} X^L. \tag{40}$$

Armed with this expression and the explicit form of the prepotential, one can also compute the explicit expressions for the real and imaginary parts of $\mathcal{N}$:

$$\begin{aligned} &\text{Re } \mathcal{N}_{00} = -\frac{1}{3}\mathcal{K}_{ABC} b^A b^B b^C, && \text{Im } \mathcal{N}_{00} = -\mathcal{K} + \left(\mathcal{K}_{AB} - \frac{1}{4}\frac{\mathcal{K}_A \mathcal{K}_B}{\mathcal{K}}\right) b^A b^B \\ &\text{Re } \mathcal{N}_{I0} = \frac{1}{2}\mathcal{K}_{IAB} b^A b^B, && \text{Im } \mathcal{N}_{I0} = -\left(\mathcal{K}_{IA} - \frac{1}{4}\frac{\mathcal{K}_I \mathcal{K}_A}{\mathcal{K}}\right) b^A \quad . \\ &\text{Re } \mathcal{N}_{IJ} = -\mathcal{K}_{IJA} b^A, && \text{Im } \mathcal{N}_{IJ} = \left(\mathcal{K}_{IJ} - \frac{1}{4}\frac{\mathcal{K}_I \mathcal{K}_J}{\mathcal{K}}\right) \end{aligned} \tag{41}$$

2.6 Compactification of type IIA supergravity on a Calabi-Yau threefold

Recall that the bosonic part of the ten-dimensional type IIA supergravity action is given in the string frame by[19]:

$$\begin{aligned} S_{\text{IIA}} = \int & -\frac{1}{2} e^{-2\hat{\phi}} \hat{R} *_{10} 1 + 2e^{-2\hat{\phi}} d\hat{\phi} \wedge *_{10} d\hat{\phi} - \frac{1}{4} e^{-2\hat{\phi}} \hat{H}_3 \wedge *_{10} \hat{H}_3 - \frac{1}{2}\hat{F}_2 \wedge *_{10} \hat{F}_2 \\ & - \frac{1}{2}\hat{F}_4 \wedge *_{10} \hat{F}_4 + \mathcal{L}_{\text{top}}, \end{aligned} \tag{42}$$

with

$$\mathcal{L}_{\text{top}} = -\frac{1}{2}\left[\hat{B}_2 \wedge d\hat{C}_3 \wedge d\hat{C}_3 - \left(\hat{B}_2\right)^2 \wedge d\hat{C}_3 \wedge d\hat{C}_1 + \frac{1}{3}\left(\hat{B}_2\right)^3 \wedge d\hat{C}_1 \wedge d\hat{C}_1\right]. \tag{43}$$

We compactify the action (42) on a Calabi-Yau three-fold Y_3. In order to do this, we decompose the fields in terms of modes on Y and assume that only the 4D massless modes contribute. These are the eigenmodes of the Laplacian corresponding to eigenvalue zero, that is, the harmonic forms on Y.

First, the dimensional reduction of the metric is sketched out; then, the other fields will be treated. We assume that the 10D metric $\hat{g}_{MN}$ can be written in the following diagonal form[30], inspired by the discussion from the previous section:

$$\hat{g}_{MN} dx^M dx^N = g(x)_{\mu\nu} dx^\mu dx^\nu + h_{i\bar{j}} dy^i dy^{\bar{j}} - iv^A(x)(\omega_A)_{i\bar{j}} dy^i dy^{\bar{j}} + \bar{z}^K(x)(\bar{b}_K)_{ij} dy^i dy^j, \tag{44}$$

where the coordinates on Y_3 are denoted by y^i and the coordinates in the resulting 4D space by x^μ. Furthermore, $g(x)$ is a 4D metric that can have any form, but which is assumed not to depend on the coordinates on the internal manifold Y, although one can generalise the following treatment to the case where it does. $h_{i\bar{j}}$ is a constant background metric on Y. v^a and $\bar{z}^u$ are 4D scalar fields that will turn out to be massless, at least in the absence of fluxes. As before, the ω_A are (1,1)-forms with components $(\omega_A)_{i\bar{j}}$; they form a basis of $H^{1,1}(Y)$. Finally, the $\bar{b}_K$ are in one-to-one correspondence with harmonic (2,1)-forms on Y by[30]

$$(\bar{b}_K)_{ij} = \frac{i}{||\Omega||^2}(\bar{\chi}_K)_{i\bar{a}\bar{b}}\Omega^{\bar{a}\bar{b}}{}_j, \tag{45}$$

with $||\Omega||^2 = \frac{1}{6}\Omega_{ijk}\bar{\Omega}^{ijk}$.

To perform the dimensional reduction of the Einstein-Hilbert term, we need to expand the Ricci scalar to second order in the moduli v^a and z^u. The result of this tedious calculation is

$$\begin{aligned}
\int_{10} \frac{1}{2} e^{-2\hat{\phi}} \hat{R} *_{10} 1 &= \int_4 -\frac{\mathcal{K}}{2} e^{-2\hat{\phi}} R *_4 1 + \frac{1}{4} e^{-2\hat{\phi}} dv^A \wedge *_4 dv^B \left(\int_Y \omega_A \wedge *\omega_B \right) \\
&\quad + \int_4 \frac{1}{2} e^{-2\hat{\phi}} dz^i \wedge *_4 d\bar{z}^j \left(\int_Y \chi_i \wedge *\chi_j \right) \\
&\equiv \int_4 \frac{1}{2} e^{-2\phi} R *_4 1 - e^{-2\phi} \left(G_{AB} dv^A \wedge *_4 dv^B \right) - e^{-2\phi} \left(g_{ij} dz^i \wedge *_4 d\bar{z}^j \right).
\end{aligned} \tag{46}$$

Here, the four-dimensional dilaton ϕ is given by

$$e^{-2\phi} = \mathcal{K} e^{-2\hat{\phi}}. \tag{47}$$

The field metrics G_{AB} and g_{ij} are the field space metrics given by equations 15 and 24.

We now turn to the dimensional reduction of the other fields. In the low-energy approximation, the relevant modes are supposed to be the massless ones. As stated before, these correspond to harmonic forms on the internal Calabi-Yau. Hence, we expand the bosonic fields in bases of harmonic forms on Y_3:

$$\begin{aligned}
\hat{\phi}(x, y) &= \hat{\phi}(x) \\
\hat{B}_2(x, y) &= B_2(x) + b^A(x)\omega_A(y) \\
\hat{C}_1(x, y) &= C_1(x) \\
\hat{C}_3(x, y) &= dC_3(x) + C_A^A(x) \wedge \omega_A(y) + \xi^K(x)\alpha_K(y) - \tilde{\xi}_L(x)\beta^L(y),
\end{aligned} \tag{48}$$

where ω_A is the basis of (1,1)-forms used before and (α_K, β^K) is a symplectic basis of real 3-forms that obeys

$$\begin{aligned}
&\int \alpha_K \wedge \alpha_L = 0, \int \beta^K \wedge \beta^L = 0 \\
&\int \alpha_K \wedge \beta^L = \delta_K^L.
\end{aligned} \tag{49}$$

$\hat{C}_1$ has no internal part, because there are no harmonic 1-forms on a Calabi-Yau manifold. Furthermore, the 4D three-form C_3 is not dynamical, since it is dual to a constant and hence does not have any physical degrees of freedom. However, it turns out that the 4D three-form C_3 does play a role as a flux parameter[19]. We will discuss flux parameters in a later section, but leave C_3 in for now; we can set the resulting parameter to zero afterwards.

The 10D field strengths are now given by

$$\begin{aligned}
&\hat{H}_3(x,y) = dB_2 + db^A \wedge \omega_A \\
&\hat{F}_2(x,y) = dC_1 \\
&\hat{F}_4(x,y) = dC_3 + \left(dC^A_A - dC_1 \wedge b^A\right) \wedge \omega_A + d\xi^K \wedge \alpha_K - d\tilde{\xi}_L \wedge \beta^L - dC_1 \wedge B_2.
\end{aligned} \tag{50}$$

One can use these expressions in the type IIA supergravity action and evaluate the integrals on the internal Calabi-Yau manifolds. This gives for the kinetic part of the action

$$\begin{aligned}
S_{\text{IIA}} = &\int_{10} \mathcal{L}_{\text{metric}} + 2e^{-2\hat{\phi}} d\hat{\phi} \wedge *_{10} d\hat{\phi} - \frac{1}{4} e^{-2\hat{\phi}} H_3 \wedge *_{10} H_3 \\
&- \frac{1}{4} e^{-2\hat{\phi}} db^A \wedge *_4 db^B \wedge \omega_A \wedge *_Y \omega_B - \frac{1}{2} dC_1 \wedge *_{10} dC_1 \\
&- \frac{1}{2} \left(dC^A_A - dC_1 \wedge b^A\right) \wedge *_4 \left(dC^B_A - dC_1 \wedge b^B\right) \wedge \omega_A \wedge *_Y \omega_B \\
&- \frac{1}{2} \left(d\xi^K \alpha_K - d\tilde{\xi}_K \beta^K\right) \wedge *_{10} \left(d\xi^L \alpha_L - d\tilde{\xi}_L \beta^L\right) \\
&- \frac{1}{2} \left(dC_3 - dC_1 \wedge B_2\right) \wedge *_{10} \left(dC_3 - dC_1 \wedge B_2\right) + \mathcal{L}_{\text{top}} \\
= &\int_4 \mathcal{L}_{\text{metric}} + 2e^{-2\phi} d\phi \wedge *_4 d\phi - \frac{1}{4} e^{-2\phi} H_3 \wedge *_4 H_3 \\
&- e^{-2\phi} G_{AB} db^A \wedge *_4 db^B - \frac{\mathcal{K}}{2} F^0 \wedge *_4 F^0 \\
&- 2\mathcal{K} G_{AB} \left(F^A - b^A F^0\right) \wedge *_4 \left(F^B - b^B F^0\right) \\
&+ \frac{1}{2} \left(\text{Im}\, \mathcal{M}^{-1}\right)^{AB} \left[d\tilde{\xi}_A + \left(\mathcal{M} \cdot d\xi\right)_A\right] \wedge *_4 \left[d\tilde{\xi}_B + \left(\mathcal{M} \cdot d\xi\right)_B\right] \\
&- \frac{\mathcal{K}}{2} \left(dC_3 - F^0 \wedge B_2\right) \wedge *_4 \left(dC_3 - F^0 \wedge B_2\right) + \mathcal{L}_{\text{top}}
\end{aligned} \tag{51}$$

where the 10D dilaton $\hat{\phi}$ has again been replaced by the 4D dilaton ϕ and the 4D field strengths are given by

$$\begin{aligned}
H_3 &= dB_2, \\
F^0 &= dC_1, \\
F^A &= dC^A_A.
\end{aligned} \tag{52}$$

The coupling matrix $\mathcal{M}$ was used to express the terms with the (α, β) forms and their Hodge duals, with $\left(\mathcal{M} \cdot d\xi\right)_A = \mathcal{M}_{AC} d\xi^C$.

Similarly, the compactification of the topological terms $\int \mathcal{L}_{\text{top}}$ gives

$$\begin{aligned}
\mathcal{S}_{\text{top}} = &\int_4 \frac{1}{2} H_3 \wedge \left(\tilde{\xi}_K d\xi^K - \xi^K d\tilde{\xi}_K\right) - \frac{1}{2} \mathcal{K}_{ABC} b^A F^B \wedge F^C + \frac{1}{2} \mathcal{K}_{ABC} b^A b^B \wedge F^C \wedge F^0 \\
&- \frac{1}{6} \mathcal{K}_{ABC} b^A b^B b^C F^0 \wedge F^0,
\end{aligned} \tag{53}$$

where the first term has been integrated by parts. The total compactified action is then

$$\begin{aligned} S_{\text{IIA}} = \int_4 & \frac{1}{2} e^{-2\phi} R * 1 - e^{-2\phi} G_{AB} dt^A \wedge * d\bar{t}^B - e^{-2\phi} g_{ij} dz^i \wedge * d\bar{z}^j \\ & + 2e^{-2\phi} d\phi \wedge * d\phi - \frac{1}{4} e^{-2\phi} H_3 \wedge * H_3 \\ & - \frac{\mathcal{K}}{2} F^0 \wedge * F^0 - 2\mathcal{K} G_{AB} \left(F^A - b^A F^0 \right) \wedge * \left(F^B - b^B F^0 \right) \\ & + \frac{1}{2} \left(\text{Im}\, \mathcal{M}^{-1} \right)^{AB} \left[d\tilde{\xi}_A + (\mathcal{M} \cdot d\xi)_A \right] \wedge * \left[d\tilde{\xi}_B + (\mathcal{M} \cdot d\xi)_B \right] \\ & - \frac{\mathcal{K}}{2} \left(dC_3 - F^0 \wedge B_2 \right) \wedge * \left(dC_3 - F^0 \wedge B_2 \right) + \frac{1}{2} H_3 \wedge \left(\tilde{\xi}_K d\xi^K - \xi^K d\tilde{\xi}_K \right) \\ & - \frac{1}{2} \mathcal{K}_{ABC} b^A F^B \wedge F^C + \frac{1}{2} \mathcal{K}_{ABC} b^A b^B \wedge F^C \wedge F^0 \\ & - \frac{1}{6} \mathcal{K}_{ABC} b^A b^B b^C F^0 \wedge F^0, \end{aligned} \tag{54}$$

where v^A and b^A were combined into the complexified Kähler variables $t^A = b^A + iv^A$.

In order to rewrite the resulting action in a standard form and arrange all the fields in standard supergravity multiplets, one has to first dualize C_3 into a constant e_0 and then the B_2 field into a scalar field a[19]. The C_3 field can be dualized by adding a Lagrange multiplier term $\frac{e_0}{2} dC_3$ in the action, such that

$$S_{\text{IIA}} \supset \int -\frac{\mathcal{K}}{2} \left(dC_3 - F^0 \wedge B_2 \right) \wedge *_4 \left(dC_3 - F^0 \wedge B_2 \right) - e_0 dC_3 \tag{55}$$

Solving this equation for dC_3 causes this field to be dualized into the parameter e_0, yielding

$$S_{\text{IIA}} \supset \int -\frac{1}{2\mathcal{K}} e_0^2 * 1 + e_0 H_3 \wedge C_1, \tag{56}$$

where the second term has been integrated by parts. Similarly, dualizing B_2 to the scalar a can then be done by adding the term $\frac{1}{2} H_3 \wedge da$ to the action:

$$S_{\text{IIA}} \supset \int -\frac{1}{4} e^{-2\phi} H_3 \wedge * H_3 + \frac{1}{2} H_3 \wedge \left(\tilde{\xi}_K d\xi^K - \xi^K d\tilde{\xi}_K \right) + e_0 H_3 \wedge C_1 + \frac{1}{2} H_3 \wedge da. \tag{57}$$

Solving the equation of motion for H_3 gives

$$S_{\text{IIA}} \supset \int -\frac{e^{2\phi}}{4} \left[Da + \left(\tilde{\xi}_K d\xi^K - \xi^K d\tilde{\xi}_K \right) \right] \wedge * \left[Da + \left(\tilde{\xi}_L d\xi^L - \xi^L d\tilde{\xi}_L \right) \right], \tag{58}$$

where $Da \equiv da + 2e_0 C_1$.

The last step in the compactification to 4D is to perform a Weyl rescaling $g_{\mu\nu} \to e^{2\phi} g_{\mu\nu}$ of the metric in order to get the right Einstein-Hilbert term. Under this rescaling, the determinant of the metric transforms as $g \to e^{8\phi} g$ and the inverse metric as $g^{\mu\nu} \to e^{-2\phi} g^{\mu\nu}$; furthermore, the Ricci scalar can be checked to transform as $R \to e^{-2\phi} R$. Using that for any differential form ω we can write $\omega \wedge * \omega = \sqrt{-g}\; \omega_{\mu\nu\ldots} \omega_{\rho\sigma\ldots} g^{\mu\rho} g^{\nu\sigma} \ldots$, the action of the Weyl rescaling is easily checked to yield the action

$$\begin{aligned}
S_{\mathrm{IIA}} = \int_4 & \frac{1}{2} R * 1 - G_{AB} dt^A \wedge * d\bar{t}^B - g_{ij} dz^i \wedge * d\bar{z}^j \\
& + 2 d\phi \wedge * d\phi - \frac{\mathcal{K}}{2} F^0 \wedge * F^0 \\
& - 2\mathcal{K} G_{AB} \left(F^A - b^A F^0\right) \wedge * \left(F^B - b^B F^0\right) \\
& + \frac{e^{2\phi}}{2} \left(\mathrm{Im}\, \mathcal{M}^{-1}\right)^{AB} \left[d\tilde{\xi}_A + \frac{1}{2} e^{-2\phi} \left(\mathcal{M} \cdot d\xi\right)_A\right] \wedge * \left[d\tilde{\xi}_B + \left(\mathcal{M} \cdot d\xi\right)_B\right] \\
& - \frac{\mathcal{K}}{2} \left(dC_3 - F^0 \wedge B_2\right) \wedge * \left(dC_3 - F^0 \wedge B_2\right) \\
& - \frac{1}{2} \mathcal{K}_{ABC} b^A F^B \wedge F^C + \frac{1}{2} \mathcal{K}_{ABC} b^A b^B \wedge F^C \wedge F^0 \\
& - \frac{1}{6} \mathcal{K}_{ABC} b^A b^B b^C F^0 \wedge F^0 - \frac{e^{4\phi}}{2\mathcal{K}} e_0^2 * 1 \\
& - \frac{e^{2\phi}}{4} \left[Da + \left(\tilde{\xi}_K d\xi^K - \xi^K d\tilde{\xi}_K\right)\right] \wedge * \left[Da + \left(\tilde{\xi}_L d\xi^L - \xi^L d\tilde{\xi}_L\right)\right]
\end{aligned} \tag{59}$$

One can write this as the following concise expression:

$$\begin{aligned}
S_{\mathrm{IIA}}^{(4)} = \int_4 & \frac{1}{2} R * 1 + \frac{1}{2} \mathrm{Im} \mathcal{N}_{AB} F^A \wedge * F^B + \frac{1}{2} \mathrm{Re} \mathcal{N}_{AB} F^A \wedge F^B \\
& - G_{AB} dt^A \wedge * d\bar{t}^B - h_{uv} d\tilde{q}^u \wedge * d\tilde{q}^v - \frac{e^{4\phi}}{2\mathcal{K}} e_0^2 * 1,
\end{aligned} \tag{60}$$

where the appearance of the coupling matrix $\mathcal{N}$ was recognized from the explicitly given forms of $\mathcal{N}$, and where the quaternionic sector $h_{uv} d\tilde{q}^u \wedge * d\tilde{q}^v$ is given by

$$\begin{aligned}
h_{uv} d\tilde{q}^u \wedge * d\tilde{q}^v = {} & d\phi \wedge * d\phi + g_{ij} dz^i \wedge * d\bar{z}^j \\
& + \frac{e^{4\phi}}{4} \left[Da + \left(\tilde{\xi}_L d\xi^L - \xi^L d\tilde{\xi}_L\right)\right] \wedge * \left[Da + \left(\tilde{\xi}_L d\xi^L - \xi^L d\tilde{\xi}_L\right)\right] \\
& - \frac{e^{2\phi}}{2} \left(\mathrm{Im}\, \mathcal{M}^{-1}\right)^{AB} \left[d\tilde{\xi}_A + \left(\mathcal{M} \cdot d\xi\right)_A\right] \wedge *_4 \left[d\tilde{\xi}_B + \left(\mathcal{M} \cdot d\xi\right)_B\right].
\end{aligned} \tag{61}$$

The standard form of compactified IIA supergravity is obtained by setting the parameter e_0, that was included here only to facilitate dualization of B_2, to zero. In the next section, we will loosely discuss more parameters like e_0 and their relevance.

The fields in the 4D theory can be arranged into the supergravity multiplets to which they belong. This means that the fields in one multiplet transform into each other under supersymmetry transformations, together with their fermionic counterparts, which we did not take into consideration. The resulting classification of the (bosonic) fields is shown in table 1.

2.7 Flux parameters

A generic supergravity compactification on a Calabi-Yau manifold contains many massless fields, as can be concluded from the action in equation 60. These massless fields are the

Name	Amount	Fields
Gravity multiplet	1	$(g_{\mu\nu}, C_1)$
Vector multiplet	$h^{1,1}$	(C_A^A, t^A)
Hypermultiplet	$h^{2,1}+1$	$(z^K, \xi^K, \tilde{\xi}_K)$ and $(a, \xi^0, \tilde{\xi}_0)$

Table 1: Field content of a 4D compactification of type IIA string theory on a Calabi-Yau threefold.

moduli t and z of the Calabi-Yau manifold and the dilaton ϕ. Such massless fields give rise to long-ranged forces, which are not observed in nature. Hence, it is desirable to have a mechanism in order to stabilize the vacuum expectation value of these fields by giving them a mass.

A way out is suggested by the term $e^{4\phi}e_0^2 * 1$ in the compactified action, which provides a potential for ϕ. Hence, including e_0 at least partially cures the problem of massless fields. There are several other constant parameters like e_0 that can be included in the theory and that also contribute to the potential for the scalars. These parameters are called 'flux parameters'; they correspond to 4D expectation values for the field strengths F_p ('RR fluxes') and H_3 ('NS-NS fluxes'), when these are given harmonic parts on the internal Calabi-Yau. This is possible when the corresponding potentials do not enter separately in the action.

Explicitly, two sets of RR flux parameters, e_A and m^A, can be introduced by replacing[19]

$$\begin{aligned} d\hat{C}_3 &\to d\hat{C}_3 + e_A \tilde{\omega}^A \qquad \text{and} \\ d\hat{C}_1 &\to d\hat{C}_1 - m^A \omega_A, \end{aligned} \tag{62}$$

where $\tilde{\omega}^A$ is the basis dual to ω_A. Hence, adding flux parameters amounts to adding harmonic parts to the field strengths. This is only consistent when the potentials C_3 and C_1 only enter the action in terms of their field strengths.

e_0, as well, has an interpretation as the background value of a field strength, namely the dual field strength $*F_4$. This interpretation is more clearly visible in an alternative formulation of type IIA string theory, called the 'democratic formulation', in which the field strengths F_2 and F_4 are treated on the same footing as their Hodge duals $*F_4 \equiv F_6$ and $*F_2 \equiv F_8$, and the Hodge duality condition is implemented separately[31, 32]. We will not discuss the democratic formulation here.

Furthermore, one can write down a version of type IIA supergravity which contains a parameter m, called the Romans mass, making the two-form B_2 massive[33]. The action of this theory is modified from the original action in equation 1 by changing the RR field strengths to

$$\begin{aligned} \hat{F}_2 &= d\hat{C}_1 + m\hat{B}_2 \qquad \text{and} \\ \hat{F}_4 &= d\hat{C}_3 - dC_1 \wedge \hat{B}_2 - \frac{m}{2}\left(B_2\right)^2. \end{aligned} \tag{63}$$

A cosmological constant term $-\frac{1}{2}m^2 * 1$, is added to the action. Finally, the topological terms are changed to

$$\mathcal{L}_{\text{top}} = -\frac{1}{2}\left[\hat{B}_2 \wedge d\hat{C}_3 \wedge d\hat{C}_3 - \left(\hat{B}_2\right)^2 \wedge d\hat{C}_3 \wedge d\hat{C}_1 + \frac{1}{3}\left(\hat{B}_2\right)^3 \wedge d\hat{C}_1 \wedge d\hat{C}_1 \right. \\ \left. -\frac{m}{3}\left(\hat{B}_2\right)^3 \wedge d\hat{C}_3 + \frac{m}{4}\left(\hat{B}_2\right)^4 \wedge d\hat{C}_1 + \frac{m^2}{20}\left(\hat{B}_2\right)^5\right]. \tag{64}$$

Like e_0, the Romans mass m can be interpreted as the background value of an RR 'field strength'. In this case, the field strength would be a zero-form F_0, even though there is no such thing as a (-1)-form C_{-1} being the potential of F_0. Again, this description is more natural in the democratic formulation.

By turning on fluxes, it turns out that it is possible to stabilize all geometric moduli of the IIA theory, at least on the level of the classical theory[34, 35]. Hence, the study of flux compactifications has become important for finding phenomenologically realistic compactifications. Specifically, including these flux parameters generates a potential V for the scalars[19]. This potential can be written conveniently in the form of a superpotential W, given by[12]

$$W_{RR} = e_0 - e_A t^A + \frac{1}{2}\mathcal{K}_{ABC} m^A t^B t^C - \frac{m}{6}\mathcal{K}_{ABC} t^A t^B t^C. \tag{65}$$

The full scalar potential V, in the form in which it appears in the action $S = ... + \int V$, can then be obtained by

$$V = e^K \left(K^{\alpha\beta} D_\alpha W D_{\bar{\beta}} \bar{W} - 3|W|^2\right), \tag{66}$$

where $K = K_{\text{K}} + K_{\text{cs}}$ is the full Kähler potential, the Kähler derivative $D_\alpha = \partial_\alpha + (\partial_\alpha K)$ and the Greek indices run over all fields. $K^{\alpha\beta}$ is the inverse of the Kähler metric obtained from the potential K. This quirky way of writing the scalar potential is useful in the study of supergravity.

It is also possible to include NSNS fluxes in the compactification by some field redefinitions and adding a harmonic part to $d\hat{B}_2$ by

$$d\hat{B}_2 \to d\hat{B}_2 + p^A \alpha_A + q_A \beta^A. \tag{67}$$

Again, this is only possible if $\hat{B}_2$ only enters the action in terms of its field strength. We will not discuss NSNS fluxes in detail here.

2.8 Orientifolds and $N = 1$ supergravity

We arrived at the expression for type IIA supergravity compactified on a Calabi-Yau manifold to 4D:

$$S^{(4)}_{\text{IIA}} = \int -\frac{1}{2}R * 1 + \frac{1}{2}\text{Im}\mathcal{N}_{\hat{A}\hat{B}} F^{\hat{A}} \wedge *F^{\hat{B}} + \frac{1}{2}\text{Re}\mathcal{N}_{\hat{A}\hat{B}} F^{\hat{A}} \wedge F^{\hat{B}} \\ - G_{AB} dt^A \wedge *d\bar{t}^B - h_{uv} d\tilde{q}^u \wedge *d\tilde{q}^v. \tag{68}$$

This theory has two supersymmetry generators, so it is said to have $N = 2$ supersymmetry. Theories with $N = 2$ are interesting in themselves and have been the subject of much research. However, these theories have too much supersymmetry to yield realistic 4D compactifications. This is because $N = 2$ theories cannot contain chiral fermions; these can only appear in theories with $N = 1$ supersymmetry or without supersymmetry[15]. Since it is important to have chiral fermions if one wants to construct a theory that reproduces the Standard Model at low energies, one needs a technique to reduce the supersymmetry content of the above model.

This reduction is made possible by a procedure called orientifolding. This procedure amounts to taking the quotient of the compactified theory by a symmetry operation $\mathcal{O}$ and projecting out the field excitations that are not invariant under $\mathcal{O}$. Here, $\mathcal{O} = \Omega_p(-1)^{F_L}\sigma$, where Ω_p is the world-sheet parity operator, $(-1)^{F_L}$ is the space-time fermion number operator and σ is an anti-holomorphic involution of the Calabi-Yau three-fold Y [25] that acts on the differential forms on the Calabi-Yau. This orientifold projection reduces the $N = 2$ supersymmetry of the theory to $N = 1$.

In order to determine which modes are invariant under $\mathcal{O}$ and survive the orientifold projection, it is useful to first state the action of the operators Ω_p and $(-1)^{F_L}$. The RR fields C_1 and C_3 are odd under $(-1)^{F_L}$, whereas the other fields are even. Furthermore, under worldsheet parity Ω_p, B_2 and C_3 are odd, and the other fields are even. Hence the $\mathcal{O}$-invariant modes that survive the orientifold projection are the modes obeying

$$\begin{aligned} \sigma^*\hat{\phi} &= \hat{\phi} \\ \sigma^*\hat{g} &= \hat{g} \\ \sigma^*\hat{B}_2 &= -\hat{B}_2 \\ \sigma^*\hat{C}_1 &= -\hat{C}_1 \\ \sigma^*\hat{C}_3 &= \hat{C}_3. \end{aligned} \tag{69}$$

Since the 4D C_1 field only has external 4D indices, on which σ acts trivially, the C_1 field is fully projected out of the spectrum. Similarly, the 4D B_2 field which is the 4D two-form of $\hat{B}_2$ is projected out.

In the ansatzes for the compactification of the $\hat{B}_2$ and $\hat{C}_3$ fields, the orientifold procedure restricts the amount of internal harmonics on Y on which the fields are expanded. The space $H^p(Y)$ of p-forms on Y can be split into a part that is even under σ^* and a part that is odd:

$$H^p(Y) = H^p_-(Y) \oplus H^p_+(Y) \tag{70}$$

Hence, we can take a basis ω_A for the space $H^{1,1}(Y)$ that splits into even forms ω_i and odd forms ω_α. Similarly, the real symplectic basis (α_K, β^K) of $H^3(Y)$ splits into even forms (α_K, β^L) and odd forms (α_L, β^K). Furthermore, one can (by a relabeling $\alpha \leftrightarrow \beta$) choose these forms so that the forms (α_K, β^L) are absent, so that all the α are even and all the β are odd.

The fields $\hat{B}_2$ and $\hat{C}_3$ can now be expanded as

$$\begin{aligned} \hat{B}_2(x,y) &= b^\alpha(x)\omega_\alpha(y) \\ \hat{C}_3(x,y) &= c_3(x) + C^i_{(A)}(x) \wedge \omega_i(y) + \xi^K(x)\alpha_K(y), \end{aligned} \tag{71}$$

i.e. the $\hat{B}_2$ field is expanded into even modes, and the $\hat{C}_3$ field into odd modes. The 4D three-form from $\hat{C}_3$ was renamed to c_3, because C_3 is conventionally used for the last term of $\hat{C}_3$ in this context.

We can also look at the effect of the orientifold procedure on the Kähler and complex structure moduli spaces. We will start with the former. In order to preserve $N = 1$ supersymmetry, σ has to act on J as

$$\sigma^* J = -J, \tag{72}$$

In the original theory, the complexified Kähler form J_c is decomposed in a basis of harmonic (1,1)-forms ω_a:

$$J_c = B + iJ = (b^A + iv^A)\omega_A \equiv t^A \omega_A. \tag{73}$$

As noted before, under the orientifold projection, the space $H^{1,1}(Y)$ of harmonic (1,1)-forms decomposes as the direct sum of an even part $H^{1,1}_+(Y)$ and an odd part $H^{1,1}_-(Y)$. Hence, for the ω_A, one can take a basis that splits into even and odd forms.

The anti-holomorphic involution σ is required to obey $\sigma^* J = -J$ and B is odd under $(-1)^{F_L}\Omega_p$ due to being odd under the worldsheet parity operator Ω_p. Therefore, the $N = 1$ spectrum in the Kähler sector consists of the σ-odd modes v^a, where the index a now runs from 1 to the dimension of the σ-odd part of $H^{1,1}(Y)$. The new Kähler moduli space is a trivial truncation of the original Kähler moduli space that does not affect the structure and geometry of the moduli space significantly, apart from reducing the dimension of the space.

In contrast to the Kähler sector, the orientifold projection acts rather non-trivially on the complex structure moduli space. As stated before, the involution σ acts on Ω as

$$\sigma^*\Omega = e^{2i\theta}\bar{\Omega}, \tag{74}$$

where θ is some constant phase. This relation is implied by the action of σ^* on the Kähler form J,

$$\sigma^* J = -J, \tag{75}$$

which implies that the volume form is odd under σ^*[25].

The space of harmonic 3-forms on Y decomposes as the direct sum of the forms that are even under σ^*, contained in the space H^3_+, and the ones that are odd H^3_-. These two spaces have to be of equal dimension by Hodge duality and the fact that the volume form is odd.

The real symplectic forms (α_I, β^I) can be chosen to split up into even forms (α_L, β^L) and odd forms (α_K, β^K). Furthermore, one can (by a relabeling $\alpha \leftrightarrow \beta$) choose these forms so that the forms (α_K, β^L) are absent, so that all the α are even and all the β are odd. Hence,

$$\sigma^*\Omega = \sigma^*\left(Z^L\alpha_L - F_K\beta^K\right) = Z^L\alpha_L + F_K\beta^K; \tag{76}$$

for this to be equal to $e^{2i\theta}\bar{\Omega}$, it is required that

$$\mathrm{Im}\left(e^{-i\theta}Z^L\right) = 0 \tag{77}$$

and

$$\mathrm{Re}\left(e^{-i\theta}F_K\right) = 0. \tag{78}$$

These are conditions on the complex structure moduli that pick out a submanifold of the original complex structure moduli space.

In the $N = 2$ setting, one starts out with $h^{2,1} + 1$ complex variables Z^I, one of which does not correspond to an actual degree of freedom due to the scaling symmetry $\Omega \to \Omega e^{-h}$, leaving us with $h^{2,1}$ complex structure moduli z^a. Here we have to be a little bit more careful with counting the degrees of freedom. Initially, we have $h^{2,1} + 1$ complex variables Z^I again. One of the $h^{2,1} + 1$ real conditions in equation 77 can be trivially satisfied by rescaling Ω by a phase; the other equations remove $h^{2,1}$ real degrees of freedom from the complex structure moduli z^I. Hence, we are left with $h^{2,1} + 1$ real degrees of freedom. One of these could again be removed by using the rescaling symmetry of Ω. However, it is traditional to leave all the degrees of freedom and implement the rescaling symmetry by multiplying Ω by a 'compensator' C given by

$$C = e^{-D-i\theta} e^{K_{cs}/2} \tag{79}$$

The dilaton D provides the extra degree of freedom that is needed to get to $h^{2,1} + 1$ real degrees of freedom.

Eventually, the coordinates on the complex structure moduli space are given by the periods of the form Ω_c[25], which is given by

$$\Omega_c = C_3 + 2i\mathrm{Re}\,(C\Omega)\,, \tag{80}$$

where C_3 is now the fully internal part of the 10D RR three-form $\hat{C}_3$, given by

$$C_3 = \xi^K \alpha_K \tag{81}$$

in our basis. The complex coordinates N^K are then defined by

$$N^K = \frac{1}{2}\int \Omega_c \wedge \beta^K = \frac{1}{2}\xi^K + il^K. \tag{82}$$

The complex structure sector Kähler potential is given by

$$K^{\mathrm{cs}} = -2\log\left[2\int_Y \mathrm{Re}\,(C\Omega) \wedge *\mathrm{Re}\,(C\Omega)\right] = -\log e^{-4D}. \tag{83}$$

Just as for the $N = 2$ supergravity theory, one can write down an explicit effective action for the orientifolded theory. Since we will mostly focus on the Kähler sector, it is sufficient for us to note that the Kähler sector has the same structure as the original $N = 2$ Kähler sector, but with a reduced number of Kähler moduli.

3 Singular limits in moduli space and nilpotent orbits

The Kähler and complex structure moduli spaces $\mathcal{M}_{\mathrm{K}}$ and $\mathcal{M}_{\mathrm{cs}}$ contain singular points, at which the geometry of the Calabi-Yau behaves non-smoothly in some way. Examples of such limits include the large volume point in Kähler moduli space or its mirror, the large complex structure point.

As advertised in the introduction to this thesis, the geometry around these singular points turns out to be very interesting in the context of the Swampland Distance Conjecture. Recently, general singular points in the complex structure moduli space of type IIB string theory compactified on a Calabi-Yau threefold were studied in much detail and to a high degree of generality[4, 5]. It was found that a convenient handle on the geometry of type IIB complex structure moduli space is provided by the monodromy of the complex structure period vector Π_{cs} around singular points. The behavior of Π_{cs} around singular limits can be very well characterized using the advanced mathematical technology of mixed Hodge structures[7, 36, 37, 9] and the so-called nilpotent orbit theorem[8, 10], which can be applied very naturally to the complex structure moduli space. In this way, the existence of the towers of states required by the Swampland Distance Conjecture was proven quite generally for this specific setting. These findings and the formalism of nilpotent orbits were translated to the large-modulus limits of type IIA Kähler moduli space as well, by the use of the mirror map[5, 6].

The central idea behind the formalism outlined in this chapter is that the middle cohomology $H^3(Y,\mathbb{C})$ near singular points in moduli space can be structured using the (logarithm of the) monodromy transformations about the singular points as well as the usual Hodge decomposition into complex cohomology groups. In particular, when approaching the singularity, one obtains a refined splitting of the limit of $H^3(Y,\mathbb{C})$ into certain subspaces, called the Deligne splitting. This Deligne splitting can be used to classify the singularities into various different types. Furthermore, it yields certain nontrivial constraints on the Deligne splitting of $H^3(Y,\mathbb{C})$ at *intersecting* singular loci in moduli space.

As a side-effect of this, the period vector Π near a singular point can be well approximated by some constant vector $\mathbf{a}_0$ transformed by the exponential of a nilpotent matrix containing the dependence on the local coordinates on moduli space. This means that, for the period vector Π,

$$\Pi \approx \exp\left(t^I N_I\right) \mathbf{a}_0, \tag{84}$$

in some choice of local coordinates where the singular point is given by $t^I \to i\infty$. Here, $\mathbf{a}_0$ is a constant vector and N_I is a nilpotent matrix. This approximation for Π is called a nilpotent orbit; hence the name 'nilpotent orbit theorem'. It turns out that most of the information about the singularity types is also contained in the generators N_I of the monodromy around the singular loci. The period vector Π and the nilpotent generators N_I gives one convenient tools with which one can do more explicit computations.

In this section, the formalism of nilpotent orbits and mixed Hodge structures will be introduced in the setting of the type IIB complex structure moduli space and the mirror setting of the IIA Kähler moduli space. The classification of monodromy logarithms into singularity types will be given and some applications will be reviewed.

3.1 Mixed Hodge structures and Deligne splittings

Here, a short discussion of the abstract outlines of the theory of mixed Hodge structures will be given, mainly based on references [5] and [4], in which more details can be found. We mainly use the abstract theory discussed here in the specific setting of the complex structure moduli space of Calabi-Yau manifolds. After that, more concrete results will be treated, switching over to the quantum Kähler sector of type IIA moduli space, which is related by mirror symmmetry to the complex structure moduli space in type IIB string theory.

A pure Hodge structure $(H_{\mathbb{Z}}, H^{p,q})$ of weight n is defined as a lattice $H_{\mathbb{Z}}$ together with a decomposition of its complexification $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ into subspaces $H^{p,q}$ such that $H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$ and $H^{p,q} = \overline{H^{q,p}}$[38]. Such a decomposition can equivalently be described in terms of a decreasing filtration $\{F^p\}$ with

$$H_{\mathbb{C}} = F^0 \supset F^1 \supset ... \supset F^n \supset \{0\}, \tag{85}$$

with the condition that $H_{\mathbb{C}} \simeq F^p \oplus \overline{F^{n-p+1}}$. Then, $F^n \equiv H^{n,0}$, $F^{n-1} = H^{n,0} \oplus H^{n-1,1}$ and so forth.

A famous example of a pure Hodge structure is provided by the middle cohomology $H^3(Y, \mathbb{C})$ of a smooth Calabi-Yau threefold Y. This space decomposes as a direct sum of the Dolbeault cohomologies $H^{p,q}(Y)$ with $p + q = 3$, a decomposition that is called the Hodge decomposition. It is clear that this setting is a stereotypical example of a pure Hodge structure.

A polarized Hodge structure is defined to be a pure Hodge structure together with a polarization, which is an antisymmetric, nondegenerate bilinear form S such that

$$\begin{aligned} S(v, w) &= 0 \text{ for } v \in H^{p,q} \text{ and } w \in H^{r,s} \text{ with } (p, q) \neq (s, r) \\ i^{p-q} S(v, \bar{v}) &> 0 \text{ for } v \in H^{p,q} \text{ and } v \neq 0. \end{aligned} \tag{86}$$

On $H^3(Y, \mathbb{C})$ the form S is given by the intersection form,

$$S(v, w) = -\int_Y v \wedge w, \tag{87}$$

or in component notation,

$$S(v, w) = \mathbf{v}^{\mathrm{T}} \cdot \vartheta \cdot \mathbf{w}, \tag{88}$$

where

$$\vartheta_{IJ} = -\int Y \gamma_I \wedge \gamma_J. \tag{89}$$

Since the Hodge decomposition of $H^3(Y, \mathbb{C})$ depends on the complex structure moduli, one can study the behavior of $\{F^p\}$ as one varies the moduli. One uses the $\{F^p\}$ and not the $H^{p,q}$, since the $\{F^p\}$ turn out to vary holomorphically with respect to the moduli, whereas the $H^{p,q}$ do not[4]. We are particularly interested in points in moduli space where the geometry of Y becomes singular in some way, since these are possibly the infinite distance points which we are looking for. Around singular points, parallel transport of the Hodge structure of $H^3(Y, \mathbb{C})$ becomes path-dependent, and elements of $H^3(Y, \mathbb{C})$ undergo a monodromy transformation when parallel-transported around such points.

It turns out that the monodromy transformation can be written as the exponential of a nilpotent matrix N. This N can be used to give a finer split of $H^3(Y,\mathbb{C})$ near the singularity. To this end, one defines a mixed Hodge structure, which captures both the splits of the space $H^3(Y,\mathbb{C})$ on the basis of the Hodge decomposition and on the basis of the weight of elements of $H^3(Y,\mathbb{C})$ under the nilpotent generator N.

Near a singular locus, the splitting $\{F^p\}$ of $H^3(Y,\mathbb{C})$ becomes ill-behaved. However, one can 'divide out' the singular behavior from the spaces F^p. This is done as follows. Consider local coordinates t^I in which the singular locus is reached by sending the first n coordinates $t^1,...,t^n \to i\infty$. Then the spaces F^p_∞ given by

$$F^p_\infty = \lim_{t^1,...,t^n\to i\infty} \exp\left(-\sum_I t^I N_I\right) F^p, \tag{90}$$

with N_I the monodromy generators around the singular loci $t^I \to i\infty$, are in fact well-behaved[8]. Hence, these spaces can be used as a so-called limiting Hodge structure. Here, it is necessary to remark that this limiting Hodge structure is not an actual pure Hodge structure. Therefore, we need the notion of mixed Hodge structure in order to get a handle on $H^3(Y,\mathbb{C})$ at the singular locus. By using mixed Hodge structures, however, it will be possible to find certain flags of subspaces of the F^p_∞ that turn out to be pure Hodge structures, as we will soon describe.

A mixed Hodge structure is given by a vector space H together with two filtrations. The first filtration is a decreasing filtration $\{F^p\}$, as discussed above, which for the current setting of $H^3(Y,\mathbb{C})$ at a singular locus is given by the limiting Hodge filtration $\{F^p_\infty\}$. The second filtration is an increasing weight filtration $\{W_j\}$, called the Jacobson-Morosov filtration, for which one uses the properties of the nilpotent monodromy generators N. Note that every nilpotent matrix can induce a weight filtration on the space it acts on, in terms of the kernels and images of that matrix. Here, the monodromy weight filtration is specified by the conditions[4]

$$\begin{aligned} &W_{-1} \equiv 0 \subset W_0 \subset W_1 \subset ... \subset W_{2D} \equiv H, \\ &NW_i \subset W_{i-2}, \\ &\;N^j:\ Gr_{D+j} \to Gr_{D-j} \text{ is an isomorphism}, \end{aligned} \tag{91}$$

with the graded spaces $Gr_j \equiv W_j/W_{j-1}$.

Using these two filtrations, one can split the space $H^3(Y,\mathbb{C})$ into smaller spaces. A way to do this[9] is the so-called Deligne splitting $\{I^{p,q}\}$. The subspaces $I^{p,q} \subset H^3(Y,\mathbb{C})$ are given by the rather intimidating expression

$$I^{p,q} = F^p \cap W_{p+q} \cap \left(\bar{F}^q \cap W_{p+q} + \sum_{j\geq 1} \bar{F}^{q-j} \cap W_{p+q-j-1}\right). \tag{92}$$

Even though this expression looks annoying, the subspaces thus obtained turn out to have rather nice properties. Specifically, this splitting turns out to be the unique splitting[10] satisfying

$$F^p = \bigoplus_s \bigoplus_{r\geq p} I^{r,s}, \qquad W_l = \bigoplus_{p+q\leq l} I^{p,q}. \tag{93}$$

Hence, although the $I^{p,q}$ are given by a complicated expression, given the Deligne splitting $\{I^{p,q}\}$ it is easy to reconstruct the filtrations F^p and W_l, and there is an intuitive interpretation of the $I^{p,q}$ as the vectors of a certain Hodge weight p and monodromy weight $p+q$.

The dimensions of the $I^{p,q}$ are denoted by $i^{p,q}$, which are called the Deligne numbers. Like the Hodge numbers $h^{p,q}$, the $i^{p,q}$ can be displayed in a diamond-shaped diagram, here called a Hodge-Deligne diamond, which for $n=3$ takes the shape

$$\begin{array}{ccccccc} & & & i^{3,3} & & & \\ & & i^{3,2} & & i^{2,3} & & \\ & i^{3,1} & & i^{2,2} & & i^{1,3} & \\ i^{3,0} & & i^{2,1} & & i^{1,2} & & i^{0,3} \\ & i^{2,0} & & i^{1,1} & & i^{0,2} & \\ & & i^{1,0} & & i^{0,1} & & \\ & & & i^{0,0} & & & \end{array} \tag{94}$$

However, certain Deligne numbers are related by symmetries; one finds that $i^{p,q} = i^{q,p} = i^{3-p,3-q}$. Furthermore, in the middle cohomology of a Calabi-Yau threefold, $H^3(Y,\mathbb{C})$ and hence F^3 is one-dimensional; hence, only one of the $i^{3,q}$ can be nonzero.

Hence, there are only four possible Hodge-Deligne diamonds for $H^3(Y,\mathbb{C})$, which are given by the following shapes, with a dot representing that the corresponding vector space has zero dimension:

$$\mathrm{I}_a: \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & a & & \cdot & \\ 1 & & a' & & a' & & 1 \\ & \cdot & & a & & \cdot & \\ & & \cdot & & \cdot & & \\ & & & \cdot & & & \end{array} \qquad \mathrm{II}_b: \begin{array}{ccccccc} & & & \cdot & & & \\ & & \cdot & & \cdot & & \\ & 1 & & b & & 1 & \\ \cdot & & b' & & b' & & \cdot \\ & 1 & & b & & 1 & \\ & & \cdot & & \cdot & & \\ & & & \cdot & & & \end{array} \tag{95}$$

$$\mathrm{III}_c: \begin{array}{ccccccc} & & & \cdot & & & \\ & & 1 & & 1 & & \\ & \cdot & & c & & \cdot & \\ \cdot & & c' & & c' & & \cdot \\ & \cdot & & c & & \cdot & \\ & & 1 & & 1 & & \\ & & & \cdot & & & \end{array} \qquad \mathrm{IV}_d: \begin{array}{ccccccc} & & & 1 & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & d & & \cdot & \\ \cdot & & d' & & d' & & \cdot \\ & \cdot & & d & & \cdot & \\ & & \cdot & & \cdot & & \\ & & & 1 & & & \end{array} \tag{96}$$

These four types of Hodge-Deligne diamonds classify the possible types of singularities that can arise in the complex structure moduli space of a Calabi-Yau threefold, or its mirror.

The nilpotent matrix N acts as a morphism, with, for $v \in I^{p,q}$, $Nv \in I^{p-1,q-1}$. One can also consider the primitive subspaces $P^{p,q}$, consisting of the vectors not in the image of N, which are given by $P^{p,q} = I^{p,q} \cap \ker N^{p+q-2}$. In terms of these subspaces, one finds the following decomposition of the $I^{p,q}$:

$$\begin{array}{ccccccc} & & & P^{3,3} & & & \\ & & P^{3,2} & & P^{2,3} & & \\ & P^{3,1} & & P^{2,2} \oplus NP^{3,3} & & P^{1,3} & \\ P^{3,0} & & P^{2,1} \oplus NP^{3,2} & & P^{1,2} \oplus NP^{2,3} & & P^{0,3} \\ & NP^{3,1} & & NP^{2,2} \oplus N^2P^{3,3} & & NP^{1,3} & \\ & & N^2P^{3,2} & & N^2P^{2,3} & & \\ & & & N^3P^{3,3} & & & \end{array} \tag{97}$$

Crucially, whereas the full spaces $I^{p,q}$ do not constitute a Hodge structure, the $P^{p,q}$ for $p+q=k$ do in fact form a pure Hodge structure of weight k[5].

One also would like to have a notion of polarization on the mixed Hodge structure. This is provided by making use of the intersection form S on the original pure Hodge structure on $H^3(Y,\mathbb{C})$ as well as the nilpotent morphism N. Specifically, the primitive spaces $P^{p,q}$ are polarized by $S(\cdot, N^{p+q-3})$.

Let us summarize the above discussion. Near a singular point in moduli space, the Hodge structure on the middle cohomology $H^3(Y,\mathbb{C})$ ceases to be well-behaved; one can, however, consider the limiting Hodge structure given by the filtration F^p_∞. Refining this split by making use of the nilpotent monodromy generator N to create a mixed Hodge structure, one has a nicely structured space with a well-behaved polarization condition. Furthermore, one can start to ask interesting questions about the possible Deligne splittings into which $H^3(Y,\mathbb{C})$ can split up.

3.2 Monodromy around large-modulus limits in Kähler moduli space

So the ark of the Lord compassed the city, going about it once.
— Joshua 6:11 (King James Version)

We will now look at a more explicit treatment of the formal discussion from the previous section. To this end, we will treat singular loci arising in type IIA Kähler moduli space, which is mirror to the type IIB complex structure moduli space treated above. Specifically, we will discuss the singularities arising when sending one or multiple Kähler moduli to $i\infty$, since the monodromy transformations arising in these limits have well-known expressions[5, 6].

A particular result that fits within the formalism outlined above is that the period vector Π can be approximated as

$$\Pi \approx \lim_{t^1,\dots,t^n \to i\infty} \exp\left(\sum_I t^I N_I\right) \mathbf{a}_0. \tag{98}$$

Here, $\mathbf{a}_0$ is a nonsingular vector that depends holomorphically on the coordinates. We can write $\Pi = \exp\left(\sum_A t^A N A\right) \mathbf{a}_0$ as a consequence of Schmid's Nilpotent Orbit Theorem[8]. Physically, the nilpotent orbit approximation amounts to throwing away the exponential corrections caused by worldsheet instantons, which fall off when moving towards the large-volume point anyway.

As mentioned earlier, the large-volume period vector Π for Kähler moduli space is given by[5, 6]

$$\Pi = \begin{pmatrix} X^0 \\ X^A \\ \mathcal{F}_A \\ \mathcal{F}_0 \end{pmatrix} = \begin{pmatrix} 1 \\ t^A \\ \frac{1}{2}\mathcal{K}_{ABC} t^B t^C + \frac{1}{2}\mathcal{K}_{ABB} t^B - b_A \\ \frac{1}{6}\mathcal{K}_{ABC} t^A t^B t^C - \left(\frac{1}{6}\mathcal{K}_{AAA} + b_A\right) t^A + \frac{i\zeta(3)\chi}{8\pi^3} \end{pmatrix}, \tag{99}$$

where $b_I = 1/24 \int_Y c_2(Y) \wedge \omega_I$ as defined in the previous chapter, χ is the Euler number of Y.

We can investigate the action of shifting an axion by subtracting its unit period, $b^I \to b^I - 1$. We shift the axion by -1 in order to comply with the conventions outlined in reference [6]. This axion shift corresponds to circling the large modulus limit $t^I \to i\infty$, causing Π to undergo a monodromy transformation. Under this shift, Π changes to

$$\begin{pmatrix} 1 \\ t^A - \delta^{AI} \\ \frac{1}{2}\mathcal{K}_{ABC} t^B t^C - \mathcal{K}_{AIC} t^C + \frac{1}{2}\mathcal{K}_{ABB} t^B - b_A \\ \frac{1}{6}\mathcal{K}_{ABC} t^A t^B t^C - \frac{1}{2}\mathcal{K}_{IBC} t^B t^C + \left(\frac{1}{6}\mathcal{K}_{AAA} - \frac{1}{2}\mathcal{K}_{IIC} + b_A\right) t^A + b^I + \frac{i\zeta(3)\chi}{8\pi^3} \end{pmatrix}. \tag{100}$$

This expression can be written as the original Π transformed by a monodromy transformation matrix T_I:

$$\Pi(\dots b^I - 1 \dots) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\delta^{IA} & \delta_{AB} & 0 & 0 \\ 0 & -\mathcal{K}_{IAB} & \delta_{AB} & 0 \\ 0 & \frac{1}{2}\left(\mathcal{K}_{IIB} + \mathcal{K}_{IBB}\right) & -\delta^{IB} & 1 \end{pmatrix} \cdot \Pi(\dots b^I \dots) \equiv T_I \cdot \Pi(\dots b^I \dots). \tag{101}$$

Furthermore, T_I can in turn be written as $\exp N_I$, with N_I given by

$$N_I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\delta_{IA} & 0 & 0 & 0 \\ -\frac{1}{2}\mathcal{K}_{IIA} & -\mathcal{K}_{IAB} & 0 & 0 \\ \frac{1}{6}\mathcal{K}_{III} & \frac{1}{2}\mathcal{K}_{IBB} & -\delta_{IB} & 0 \end{pmatrix}. \tag{102}$$

Note now that the period vector Π can, as expected, actually be written as

$$\Pi = \exp\left(\sum_I t^I N_I\right) \cdot \begin{pmatrix} 1 \\ 0 \\ -b_I \\ \frac{i\zeta(3)\chi}{8\pi^3} \end{pmatrix} \equiv \exp\left(\sum_A t^A N_A\right) \cdot \mathbf{a}_0. \tag{103}$$

Hence, the N_I play a much bigger role than just generating monodromy transformations. The N_I capture the entire dependence of Π on the complex Kähler moduli t^I, at least in the regime where the given expressions for Π are valid. Since Π connects directly to the geometry of Kähler moduli space through the Kähler potential K, it should be no surprise that one can learn much about Kähler moduli space by studying the nilpotent generators N_I.

From now on, much of this discussion will take place in the case where the Kähler cone is simplicial. This means that it is spanned by exactly $h^{1,1}$ generators; in such a case, the $\mathcal{K}_{ABC}$ are all nonnegative in some choice of basis ω_A.

Note that one can also have 'combined' monodromy, in which one shifts multiple b^I at the same time. This corresponds to encircling a limit in which multiple t^I are sent to $i\infty$. The nilpotent matrices given by this monodromy are simply given by linear combinations of the original N_I. For instance, sending $b^1 \to b^1 + 2$ and $b^2 \to b^2 + 1$ corresponds to an N given by

$$N = 2N_1 + N_2. \tag{104}$$

Here, it is important to note that the N_I commute among themselves[4]. The generalization to sending an arbitrary amount of coordinates to $i\infty$ is obvious. For convenience, we define the nilpotent generator $N_{(n)}$ corresponding to encircling the first n coordinates, which is given by

$$N_{(n)} \equiv \sum_{i=1}^{n} N_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\sum_i^n \delta_{iA} & 0 & 0 & 0 \\ -\frac{1}{2}\sum_i^n \mathcal{K}_{iiA} & -\sum_i^n \mathcal{K}_{iAB} & 0 & 0 \\ \frac{1}{6}\sum_i^n \mathcal{K}_{iii} & \frac{1}{2}\sum_i^n \mathcal{K}_{iBB} & -\sum_i^n \delta_{iB} & 0 \end{pmatrix}. \tag{105}$$

3.3 Classification of singularities

One can classify the possible types of singularities in Kähler moduli space by characterizing the nilpotent matrices N_I. Specifically, a nilpotent matrix N can be characterized on the basis of its rank, the rank of its square N^2, the rank of its cube N^3 and the eigenvalues of $\vartheta \cdot N$ (where ϑ is the intersection form from the previous chapter) according to table 3.3[6, 5]. This classification into four types corresponds to the four different Deligne splittings of the mirror $H^3(Y, \mathbb{C})$, and one can check that the ranks of the matrices match

Type	Rank of N	Rank of N^2	Rank of N^3	Eigenvalues of $\vartheta \cdot N$
I_a	a	0	0	a negative
II_b	$2+b$	0	0	2 positive, b negative
III_c	$4+c$	2	0	not needed
IV_d	$2+d$	2	1	not needed

with the properties of N as a morphism within the mixed Hodge structures. For instance, as indicated previously, a IV_d type singularity corresponds to the Deligne splitting given by the following Hodge-Deligne diamond:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & \cdot & & \cdot & & \\ & \cdot & & d & & \cdot & \\ \cdot & & d' & & d' & & \cdot \\ & \cdot & & d & & \cdot & \\ & & \cdot & & \cdot & & \\ & & & 1 & & & \end{array} \tag{106}$$

Here N maps between subspaces that are aligned vertically as $NI^{p,q} \subset I^{p-1,q-1}$. From the diamond and the requirement that powers of N represent isomorphisms between spaces on different halves of the diamond, we conclude that N^3 has to have rank one, since it maps $N^3 I^{3,3} \subset I^{0,0}$. Furthermore, N itself has to have rank $d+2$, since it maps the d-dimensional space $I^{2,2}$ to $I^{1,1}$, it maps $I^{3,3}$ to a subspace of $I^{2,2}$ and it maps a one-dimensional subspace of $I^{1,1}$ to $I^{0,0}$.

One can apply this characterization immediately to the single-coordinate nilpotent generators N^I arising in the large-modulus limits $t^I \to i\infty$, with the expression for N^I given before. To this end, one computes N_I^2 and N^3:

$$N_I^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathcal{K}_{IIA} & 0 & 0 & 0 \\ 0 & \mathcal{K}_{IIB} & 0 & 0 \end{pmatrix} \quad \text{and} \quad N_I^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\mathcal{K}_{III} & 0 & 0 & 0 \end{pmatrix} \tag{107}$$

Hence, one can immediately conclude that a singularity $t^I \to i\infty$ is of type I or II if $\mathcal{K}_{IIA} = 0$ for all A. Actually, by computing $\vartheta \cdot N_I$ with the given expressions, one can conclude that type I singularities do not arise in this case. If $\mathcal{K}_{IIA} \neq 0$ for some A, but $\mathcal{K}_{III} = 0$, the singularity is of type III; if $\mathcal{K}_{III} \neq 0$, the singularity must be of type IV. The precise type can then be determined from the ranks of $\mathcal{K}_{IAB}$.

Similarly, one can characterize singularities that arise by sending multiple $t^I \to i\infty$. In this case, one finds

$$N_{(n)}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sum_{i,j=1}^n \mathcal{K}_{ijA} & 0 & 0 & 0 \\ 0 & \sum_{i,j=1}^n \mathcal{K}_{ijB} & 0 & 0 \end{pmatrix} \quad \text{and} \quad N_{(n)}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\sum_{i,j,k=1} \mathcal{K}_{ijk} & 0 & 0 & 0 \end{pmatrix}. \tag{108}$$

Armed with these expressions, the conditions to determine the corresponding singularity types follow straightforwardly. In particular, one can determine the singularity type of the large volume point, in which all of the $t^I \to i\infty$. There, one can conclude that the singularity type must be IV_d, since the single entry in N^3 is given by the volume of Y, which must be nonzero. Furthermore, $v^I \mathcal{K}_{IAB}$ is required to be full rank[23]; hence, $d = h^{1,1}$, making the singularity type at the large volume point $\mathrm{IV}_{h^{1,1}}$.

We can consider an enhancement chain, in which we send the coordinates t^I to infinity one after another. It is a fundamental and useful result that, in such an enhancement chain, the (Roman numeral) type of the singularity can only increase or stay the same[11]. There are also strong constraints on the subscript index of the enhanced singularity type. Hence, a general enhancement chain looks like[6]

$$\begin{aligned} &\mathrm{I}_{a_1} \to ... \to \mathrm{I}_{a_k} \to \mathrm{II}_{b_1} \to ... \to \mathrm{II}_{b_l} \\ &\qquad \to \mathrm{III}_{c_1} \to ...\mathrm{III}_{c_p} \to \mathrm{IV}_{d_1} \to ... \to \mathrm{IV}_{d_q}. \end{aligned} \tag{109}$$

The precise conditions on the sub-indices are called 'polarization conditions' and are displayed (for the relevant types of singularities) in table 3.3.

Possible starting type	Enhanced type	Conditions
II_b	$\mathrm{II}_{\hat{b}}$	$b \leq \hat{b}$
	$\mathrm{III}_{\hat{c}}$	$2 \leq b \leq \hat{c} + 2$
	$\mathrm{IV}_{\hat{d}}$	$1 \leq b \leq \hat{d} - 1$
III_c	$\mathrm{III}_{\hat{c}}$	$c \leq \hat{c}$
	$\mathrm{IV}_{\hat{d}}$	$c + 2 \leq \hat{d}$
IV_d	$\mathrm{IV}_{\hat{d}}$	$d \leq \hat{d}$

Table 2: Relevant polarization conditions for $h^{1,1} = 2$[5]. The conditions for type I singularities are not included, since these do not appear at the large-modulus loci and will not be considered here.

These polarization conditions can be motivated in the following way. First, one draws the Hodge-Deligne diamond for the starting Deligne splitting, using the 'initial' nilpotent generator N_1 corresponding to a limit in which one or possibly several of the t^I are sent to $i\infty$. As noted earlier, the primitive subspaces in this Deligne splitting constitute pure Hodge structures themselves. When the singularity enhances upon sending a new modulus $t^2 \to i\infty$, the corresponding nilpotent generator N_2 causes a further refined splitting of these pure Hodge structures into mixed Hodge structures. Hence, the only possible enhanced singularity types are the ones that are compatible both with the original mixed Hodge structure and with this refined splitting. Furthermore, this procedure turns out to be equivalent to reassembling the relevant Hodge-Deligne diamonds in a certain way. A precise description of this computation is given in appendix E of [5] based on reference [11].

In the setting of the complex structure moduli space of Calabi-Yau threefolds, it was shown for one-parameter degenerations that a singular point is at finite distance if and only if the singularity is of type I[39]. Consequently, type II, III and IV singularities are always at infinite distance. In a multi-parameter degeneration, it is hard to prove that all type II, III and IV singularities are at infinite distance; however, one can show the reverse direction, that infinite distance points are type II, III or IV singularities[5] by showing the equivalent condition that type I singularities are always at finite distance. Since the large-modulus limits are never of type I, these are all at infinite distance.

3.4 Moduli space metric and infinite distances

To illustrate the use of the classification of singularity types, one can look at the metric on Kähler moduli space and try to see that the large-modulus limits are indeed at infinite distance.

Recall that the Kähler potential on the Kähler moduli space was given in terms of Π by

$$\begin{aligned} K &= -\log\left(i\bar{\Pi}^t \cdot \vartheta \cdot \Pi\right) \\ &= -\log\left(\frac{1}{6}\mathcal{K}_{ABC}v^A v^B v^C + \frac{\zeta(3)\chi}{32\pi^3}\right) \equiv -\log \mathcal{V}_Y. \end{aligned} \tag{110}$$

The moduli space metric G_{AB} is given by

$$G_{A\bar{B}} = \partial_{t^A}\partial_{\bar{t}^B} K \tag{111}$$

The distance between two points $Q,\ P$ as measured along a path $\gamma(s)$ in moduli space is given by

$$d_\gamma(Q,P) = \int_\gamma \sqrt{2G_{A\bar{B}}\dot{t}^A\dot{\bar{t}}^B} ds, \tag{112}$$

where $\dot{t}^A$ is shorthand for $\partial t^A/\partial s$.

Due to a growth theorem by Cattani, Kaplan and Schmid[10], the leading growth of $\mathcal{V}_Y$ in a limit $v^1, ..., v^n \to \infty$ can be determined[5, 6]. Here, it is important to note that the v^i should first be ordered in such a way that the limit is taken in the growth sector

$$\left\{\frac{v^1}{v^2} > \lambda, ..., \frac{v^{n-1}}{v^n} > \lambda, v^n > \lambda\right\}, \tag{113}$$

for some positive λ. One can always relabel the v^i so that they lie in this growth sector. Then the leading term in $\mathcal{V}_Y$ as one approaches the singularity is given by

$$\mathcal{V}_Y \sim c\left(v^1\right)^{d_1}\left(v^2\right)^{d_2-d_1} ... \left(v^n\right)^{d_n-d_{n-1}}, \tag{114}$$

where c is some positive constant. Here, the d_n are integers determined by the singularity type corresponding to $N_{(n)}$, with $d = 1, 2, 3$ for $N_{(n)}$ having type $\mathrm{II}, \mathrm{III}, \mathrm{IV}$. Note that this must be the type of the *enhanced* singularity $N_1 + N_2 + ... + N_n$; hence, the d_i form a non-decreasing sequence. Since the d_i can nevertheless be at most 3, maximally three v^I can appear in equation 114. This is natural, given that $\mathcal{V}_q$ is cubic in the moduli. Furthermore, one can check that the leading term

From the above expression, it is clear that $\mathcal{V}_q$ becomes infinite in the large-modulus limit $v^I \to \infty$. Hence, since the Kähler metric is determined from $\mathcal{V}_q$ directly, it is plausible that the limits $v^I \to \infty$ are at infinite distance. One can use a Cauchy-Schwarz inequality to prove that this is indeed the case, by an argument given in reference [6].

4 Calculation of allowed singularity enhancements near large-modulus loci

In the previous chapter, the abstract formalism of mixed Hodge structures was discussed. Furthermore, some more concrete results in the specific setting of the large-modulus loci in Kähler moduli space were given. In this chapter we will utilize these results in order to find conditions on the intersection numbers $\mathcal{K}_{IJK}$.

Such conditions might be interesting in order to discover field theories that seem to be consistent, but cannot actually come from compactified string theories. In a supergravity theory compactified on a Calabi-Yau manifold Y, the intersection numbers $\mathcal{K}_{IJK}$ of Y appear in the Lagrangian of the compactified theory. In principle, one can make up a 4D theory in which one chooses one's favorite numbers for $\mathcal{K}_{IJK}$. However, these might not correspond to a valid compactification on a Calabi-Yau manifold. The most important constraint on the $\mathcal{K}_{IJK}$ is that they have to result in a positive-definite metric on the Kähler moduli space[40]. We want to see whether the polarization conditions result in additional conditions on the $\mathcal{K}_{IJK}$. In this light, it is interesting to try and construct $\mathcal{K}_{IJK}$ that look legit from any other viewpoint, but nevertheless cannot occur due to the polarization conditions.

Furthermore, given the explicit forms of the nilpotent generators N of monodromy around the large-volume loci, it is in principle possible to analyze all the possible singularity enhancements for a given dimension of the moduli space $h^{1,1}$. We will look at the cases $h^{1,1} = 2$ and $h^{1,1} = 3$. In the $h^{1,1} = 2$ case, all possible consistent singularity enhancements can be listed explicitly; in the $h^{1,1} = 3$ case, this task is too tedious to do by hand, but we can nevertheless analyze a few specific cases.

4.1 Two-variable case ($h^{1,1} = 2$)

We first consider the case in which there are two moduli t^1 and t^2. This case has already been analyzed in much detail by Chongchuo Li in a private communication[41]; here, we mostly follow his analysis, adding a few new results that correspond to slightly more subtle singularity enhancements. We believe the resulting list of possible cases is now exhaustive.

The two monodromy logarithms N_1 and N_2 are given by

$$N_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}\mathcal{K}_{111} & -\mathcal{K}_{111} & -\mathcal{K}_{112} & 0 & 0 & 0 \\ -\frac{1}{2}\mathcal{K}_{112} & -\mathcal{K}_{112} & -\mathcal{K}_{122} & 0 & 0 & 0 \\ \frac{1}{6}\mathcal{K}_{111} & \frac{1}{2}\mathcal{K}_{111} & \frac{1}{2}\mathcal{K}_{122} & -1 & 0 & 0 \end{pmatrix} \tag{115}$$

and

$$N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}\mathcal{K}_{122} & -\mathcal{K}_{112} & -\mathcal{K}_{122} & 0 & 0 & 0 \\ -\frac{1}{2}\mathcal{K}_{222} & -\mathcal{K}_{122} & -\mathcal{K}_{222} & 0 & 0 & 0 \\ \frac{1}{6}\mathcal{K}_{222} & \frac{1}{2}\mathcal{K}_{112} & \frac{1}{2}\mathcal{K}_{222} & 0 & -1 & 0 \end{pmatrix}. \tag{116}$$

Then, the monodromy logarithm N_V corresponding to encircling the large-volume point is given by

$$N_V \equiv N_1 + N_2. \tag{117}$$

As discussed before, requiring that the large moduli limits $t^1 \to i\infty$ and $t^2 \to i\infty$ have certain singularity types corresponds to certain criteria which should be fulfilled by the intersections $\mathcal{K}_{ABC}$ of the Calabi-Yau Y. The condition that the singularity type of the limit $t^A \to i\infty$ is II corresponds to the requirement that all intersection numbers of the form $\mathcal{K}_{AAJ}$ are zero; the singularity being of type III corresponds to the requirement that $\mathcal{K}_{AAA} = 0$, but at least one $\mathcal{K}_{AAJ}$ is nonzero.

In this case, there are only a few independent intersection numbers, namely $\mathcal{K}_{111}, \mathcal{K}_{112}, \mathcal{K}_{122}$ and $\mathcal{K}_{222}$. Hence, it is feasible to enumerate all the possible singularity enhancements explicitly. One can set various intersection numbers to zero or try to find any conditions on the intersection numbers that reduce the rank of the nilpotent generators N_1, N_2 and N_V. This yields a 'naive' list of the singularity enhancements that can be obtained by adjusting the intersection numbers. One then still needs to check whether these are actually compatible with the polarization conditions, which are repeated here in table 4.1. Furthermore, if all the intersections $\mathcal{K}_{AIJ}$ for a certain A are zero, the modulus v^A becomes unphysical.

We list the enhancements that are naively possible in table 4.1. Note that configurations which are related to the displayed ones by a trivial relabeling of moduli are omitted.

Possible starting type	Enhanced type	Conditions
II_b	$\mathrm{II}_{\hat{b}}$	$b \leq \hat{b}$
	$\mathrm{III}_{\hat{c}}$	$2 \leq b \leq \hat{c} + 2$
	$\mathrm{IV}_{\hat{d}}$	$1 \leq b \leq \hat{d} - 1$
III_c	$\mathrm{III}_{\hat{c}}$	$c \leq \hat{c}$
	$\mathrm{IV}_{\hat{d}}$	$c + 2 \leq \hat{d}$
IV_d	$\mathrm{IV}_{\hat{d}}$	$d \leq \hat{d}$

Table 3: Relevant polarization conditions for $h^{1,1} = 2$[5]. The conditions for type I singularities are not included, since these do not appear at the large-modulus loci.

Two types of invalid singularity enhancements have been marked in the table. The first type, marked by U, is the less interesting case. This corresponds to all of the intersection numbers belonging to a certain modulus being turned off. In this case, that modulus does not have any dynamics, since it does not enter into the Kähler potential at all. Hence, this type might be considered to be trivially uninteresting.

#	Nonzero intersections	$N_1 + N_2 \to N_V$	Notes
1	None	$\mathrm{II}_0 + \mathrm{II}_0 \to \mathrm{II}_0$	U
2	K_{112}	$\mathrm{III}_0 + \mathrm{II}_1 \to \mathrm{IV}_2$	
3	K_{111}	$\mathrm{IV}_1 + \mathrm{II}_0 \to \mathrm{IV}_1$	U, I
4	K_{112}, K_{122}	$\mathrm{III}_0 + \mathrm{III}_0 \to \mathrm{IV}_2$	
5	$K_{111},\ K_{112}$	$\mathrm{IV}_2 + \mathrm{II}_1 \to \mathrm{IV}_2$	
6	$K_{111},\ K_{122} \neq K_{111}$	$\mathrm{IV}_1 + \mathrm{III}_0 \to \mathrm{IV}_2$	
7	$K_{111},\ K_{122} = K_{111}$	$\mathrm{IV}_2 + \mathrm{III}_0 \to \mathrm{IV}_1$	I
8	$K_{111},\ K_{222}$	$\mathrm{IV}_1 + \mathrm{IV}_1 \to \mathrm{IV}_2$	
9	$K_{111},\ K_{112},\ K_{122}$ generic	$\mathrm{IV}_2 + \mathrm{III}_0 \to \mathrm{IV}_2$	
10	$K_{111} = \alpha K_{112} = \alpha^2 K_{122}$	$\mathrm{IV}_1 + \mathrm{III}_0 \to \mathrm{IV}_2$	
11	$K_{112} = \alpha K_{122}, K_{111} = (1 + \alpha^2 + \alpha) K_{122}$	$\mathrm{IV}_2 + \mathrm{III}_0 \to \mathrm{IV}_1$	I
12	$K_{111},\ K_{112},\ K_{222}$ generic	$\mathrm{IV}_2 + \mathrm{IV}_2 \to \mathrm{IV}_2$	
13	$K_{111} = (K_{112}^2/K_{222}) + K_{112},\ K_{112},\ K_{222}$	$\mathrm{IV}_2 + \mathrm{IV}_2 \to \mathrm{IV}_1$	I
14	$K_{111},\ K_{112},\ K_{122},\ K_{222}$ generic	$\mathrm{IV}_2 + \mathrm{IV}_2 \to \mathrm{IV}_2$	
15	$K_{111} = \alpha K_{112} = \alpha^2 K_{122},\ \alpha K_{222} \neq K_{122}$	$\mathrm{IV}_1 + \mathrm{IV}_2 \to \mathrm{IV}_2$	
16	$K_{111} = \alpha K_{112} = \alpha^2 K_{122} = \alpha^3 K_{222}$	$\mathrm{IV}_1 + \mathrm{IV}_1 \to \mathrm{IV}_1$	
17	$K_{111}\,(K_{122} K_{222}) + K_{112} K_{222} = K_{122}^2 + K_{112} K_{122} + K_{112}^2$	$\mathrm{IV}_2 + \mathrm{IV}_2 \to \mathrm{IV}_1$	I

Table 4: Singularity enhancements for $h^{1,1} = 2$ that are naively possible by only looking at the intersection numbers. In the first column, the intersection numbers that are not set to zero are listed. Intersection numbers used in the conditions are assumed to be nonzero, whereas intersection numbers not listed are assumed to be zero. In the second column, the resulting singularity types are listed. Configurations that either contain unphysical moduli (U) or are inconsistent with the polarization conditions (I) are marked in the last column.

$N_1 + N_2$	Amount	#
$\mathrm{II}_1 + \mathrm{III}_0$	2	2
$\mathrm{II}_1 + \mathrm{IV}_2$	10	5
$\mathrm{III}_0 + \mathrm{III}_0$	2	4
$\mathrm{III}_0 + \mathrm{IV}_1$	12	(6,) 10
$\mathrm{III}_0 + \mathrm{IV}_2$	3	9
$\mathrm{IV}_1 + \mathrm{IV}_1$	0	(8,) 16
$\mathrm{IV}_1 + \mathrm{IV}_2$	17	15
$\mathrm{IV}_2 + \mathrm{IV}_2$	2	(12), 14

Table 5: Statistics of different type enhancements as obtained from the KS list of Calabi-Yau threefolds. In the third list, the numbers of the possible (allowed) cases from the previous table corresponding to the singularity enhancements are given. The cases that yield non-positive definite metrics are enclosed in brackets.

The second type, consisting of the rows in the table marked by an I, is more interesting. These are the singularity enhancements that are naively possible, but that are in fact ruled out by the polarization conditions. Hence, we see that the polarization conditions can limit the possible intersection numbers that can appear.

As one can read off from the table, the singularity enhancements marked with an I all have IV_1 as their final type. However, this is in contradiction with the singularity type of the large volume limit always being $\mathrm{IV}_{h^{1,1}}$, as was noted in reference [5] based on [23]. Hence, in this case, there is still a rather obvious diagnostic that prevents these configurations from happening in practice.

Finally, one can compute if some of the cases in table 4.1 cannot give rise to positive-definite metrics at all, or if the metric cannot be positive definite in a large enough region of field space. One would desire the field space metric to be positive-definite in the region where $t^1, t^2 > 0$. This turns out to exclude cases 6, 8 and 12, as well as all the cases discarded before (1, 3, 7, 11, 13 and 17); one can compute the determinant of the field space metric, which turns out to always become zero somewhere within the allowed region $t^1, t^2 > 0$, indicating that the metric degenerates.

In table 4.1, the singularity configurations that occur in the CY threefolds with $h^{1,1} = 2$ from the KS list[42, 43] are listed, together with the labels from table 4.1 that give rise to these configurations. As a sanity check, the list of $h^{1,1} = 2$ was searched for the cases 6 and 8 from table 4.1; these were not found. We find, however, that all the possible singularity types actually occur in the KS list, except for the $\mathrm{IV}_1 + \mathrm{IV}_1$ enhancement, which only seems to occur in the highly constrained case 16.

4.2 Three-variable case ($h^{1,1} = 3$)

In this case we have another nilpotent generator N_3 and more intersection numbers. The new independent intersection numbers are $\mathcal{K}_{113}$, $\mathcal{K}_{133}$, $\mathcal{K}_{123}$, $\mathcal{K}_{223}$, $\mathcal{K}_{233}$ and $\mathcal{K}_{333}$; there are now 10 independent intersection numbers in total. The explicit expressions for the matrices N_1, N_2 and N_3 will not be presented here, since the big matrices now become rather unwieldy.

In the case $h^{1,1} = 3$, in contrast to the $h^{1,1} = 2$ case, it is a rather formidable task to enumerate all the possible singularity enhancements explicitly. It is easy to impose that the matrices should be of type II, III or IV by setting the corresponding intersection numbers to zero. In this way, one can also determine whether certain type enhancements are trivially impossible. However, the precise conditions on the ranks of the nilpotent matrices and their sums can typically be satisfied in multiple ways, except when there are few nonzero intersection numbers left.

A crude way of classifying the possible singularity enhancements in the $h^{1,1} = 3$ setting is to take the symbolic forms of N_1, N_2 and N_3 and all their sums, set several intersection numbers to zero and evaluate the resulting singularity types. This might be done using a computer, enumerating all the possible combinations of intersection numbers that are switched off. In principle, there are now $2^{10} = 1024$ possible ways of setting intersection numbers to zero. As in the previous case, many of these are equivalent by permuting the labeling of the three moduli.

It turns out that by doing this automated scan, one can already find several non-equivalent but related cases of singularity enhancements that are naively valid but actually forbidden by the polarization relations. Specifically, with one specific labeling of moduli, all of these cases have that $K_{111}, K_{112}, K_{113}, K_{122}, K_{123}$ and K_{222} are equal to zero. If the other intersection numbers are nonzero, one generically finds

$$\begin{aligned} \text{Type } N_1 &= \text{II}_1, \\ \text{Type } N_2 &= \text{III}_0, \\ \text{Type } N_3 &= \text{IV}_3, \\ \text{Type } N_1 + N_2 &= \text{III}_0, \\ \text{Type } N_1 + N_3 &= \text{IV}_3, \\ \text{Type } N_2 + N_3 &= \text{IV}_3, \\ \text{Type } N_1 + N_2 + N_3 &= \text{IV}_3, \end{aligned} \tag{118}$$

which contains a forbidden $\text{II}_1 \to \text{III}_0$ enhancement. This enhancement occurs by first sending $t^1 \to i\infty$ and then $t^2 \to i\infty$. With this configuration of intersection numbers, one can check that one can furthermore set K_{233} or K_{333} (or both) to zero. This changes the type of N_3 to III_1, but does not affect the offending enhancement. Notice that in this case, the singularity type of the large volume point is IV_3, as desired, in contrast to the illegal configurations in the $h^{1,1} = 2$ case. We will refer to this example as example A and discuss it below.

One can also try to find forbidden singularity enhancements by hand. A way to do this somewhat systematically is to choose the Roman numerals corresponding to the three singularity types belonging to N_1, N_2 and N_3, which give certain constraints on which intersection numbers should already be zero, then go through the remaining possibilities by hand. Even if this gives one a more or less systematic ritual by which one can compute all the enhancements, it is still much work to enumerate all the cases.

By this procedure, another interesting forbidden enhancement was found. This case occurs when all of the intersection numbers are set to zero, except $K_{122} = \alpha K_{123} = \alpha^2 K_{233}$. Now, one can check that the resulting singularity types are

$$\begin{aligned} \text{Type } N_1 &= \text{II}_2, \\ \text{Type } N_2 &= \text{III}_1, \\ \text{Type } N_3 &= \text{III}_0, \\ \text{Type } N_1 + N_2 &= \text{IV}_2, \\ \text{Type } N_1 + N_3 &= \text{III}_0, \\ \text{Type } N_2 + N_3 &= \text{IV}_3, \\ \text{Type } N_1 + N_2 + N_3 &= \text{IV}_3. \end{aligned} \tag{119}$$

This contains a forbidden $\text{II}_2 \to \text{IV}_2$ sub-enhancement, again upon first sending $t^1 \to i\infty$ and then $t^2 \to i\infty$. We will refer to this configuration as example B and treat it in some detail below.

4.2.1 Example A

Recall that in example A, we had set $K_{111}, K_{112}, K_{113}, K_{122}, K_{123}, K_{222}, K_{233}$ and K_{333} equal to zero. We now want to investigate the properties of this configuration in some more detail. In particular, it is interesting to know whether this configuration of intersection numbers defines a genuine field theory. A simple check that we can do is to find out whether the resulting field space metric is positive definite.

The Kähler potential now has the structure

$$K = \log 4 \left(K_{133} v^1 v^3 v^3 + K_{223} v^2 v^2 v^3 \right), \tag{120}$$

with a factor 3 to compensate for the symmetry of the contraction $\mathcal{K}_{IJK} v^I v^J v^K$. This allows one to compute the field space metric, which is given by

$$g_{IJ} = -\frac{1}{2} \partial_I \partial_{\bar{J}} K. \tag{121}$$

It turns out that this metric always has negative determinant as long as the intersection numbers and the v^I are positive, which is the case if we consider a simplicial Kähler cone. Hence, the metric is never positive definite. Furthermore, the situation does not change when setting K_{233} or K_{333} to some nonzero value. Clearly, this combination of intersection numbers yields an invalid field theory to start with.

4.2.2 Example B

In example B the intersections $K_{111}, K_{112}, K_{113}, K_{133}, K_{222}, K_{223}$ and K_{333} are equal to zero. Furthermore, $K_{122} = \alpha K_{123} = \alpha^2 K_{233}$, with $\alpha > 0$. We now want to investigate in some more detail the properties of this configuration. Recall that the offending enhancement $\mathrm{II}_2 \to \mathrm{IV}_2$ occurred when first sending $t^1 \to i\infty$ and then sending $t^2 \to i\infty$.

The Kähler potential now has the general structure

$$K = \log 4 \left(\alpha^2 K_{233} v^1 v^2 v^2 + 2\alpha K_{233} v^1 v^2 v^3 + K_{233} v^2 v^3 v^3 \right). \tag{122}$$

With this Kähler potential, we can again compute the moduli space metric. This time, the metric can be positive definite, depending on the position in moduli space and the value of α.

To check whether the metric is positive definite, one can use Sylvester's criterion. This criterion states that a Hermitian matrix, in particular a real symmetric one, is positive definite if and only if all the leading principal minors are positive[44]. Computing the principal minors for the Kähler metric under consideration, one finds that the first two minors are given by sums of products of α and the v^I, which are all supposed to be positive. Meanwhile, the determinant of the full metric g is given by

$$\det g = \frac{\alpha^2 \left(\left(v^3\right)^2 + 2\alpha v^1 v^3 + \alpha^2 \left(v^1 - v^2\right) v^2 \right)}{2 \left(v^2\right)^2 \left(\left(v^3\right)^2 + 2\alpha v^1 v^3 + \alpha^2 v^1 v^2 \right)^3}. \tag{123}$$

This is positive if the numerator is positive. The boundary of the region where this is the case is given by

$$v^3 = -\alpha \left(v^1 \pm \sqrt{\left(v^1\right)^2 - v^1 v^2 + \left(v^2\right)^2} \right). \tag{124}$$

We can conclude from the above expressions that the metric is positive definite for any v^3 when $v^1 > v^2$. However, one can choose positive values for v^1, v^2 and v^3 for which the metric is not positive definite. Specifically, sending $v^2 \to \infty$ while keeping v^1 and v^3 at finite values, one crosses a locus at which the metric degenerates. This set-up of intersection numbers apparently corresponds to a basis of $H^{1,1}(Y, \mathbb{C})$ that is not adapted to the Kähler cone, since the metric should be positive-definite on the Kähler cone. As a result, the monodromy around the singularity $v^2 \to \infty$ is not physical, and hence the offending singularity enhancement is implausible to begin with.

It seems like it is hard to find intersection numbers that give a plausible, positive definite metric for t^1, t^2, $t^3 > 0$ while violating the polarization conditions. More examples like the above also give rise to degenerating metrics. It seems like this might be a general rule, and proving whether $\mathcal{K}_{IJK}$ that are forbidden by the polarization conditions always give rise to invalid metrics might be an interesting task that we, however, wish to leave for further work. In this context, it needs to be noted that it was already remarked[40] that randomly chosen $\mathcal{K}_{IJK}$ do not tend to give positive definite metrics anyway, especially for large $h^{1,1}$. This does not rule out entirely, however, that there might be exceptional $\mathcal{K}_{IJK}$ that do give a positive-definite metric while violating the polarization conditions.

4.2.3 General scan of singularity type enhancements in the $h^{1,1} = 3$ case

As described before, one can do a crude scan of the intersection numbers in which one sets all possible subsets of the set of intersection numbers at given $h^{1,1}$ to zero. This scan quickly becomes computationally expensive, as can be seen as follows. The number of independent intersection numbers for given $h^{1,1}$ is equal to

$$\#\mathcal{K}_{IJK} = \frac{h^{1,1}\,(h^{1,1}+1)\,(h^{1,1}+2)}{3!} \tag{125}$$

and the total number of subsets of intersection numbers is equal to

$$2^{\#\mathcal{K}_{IJK}}, \tag{126}$$

which goes as the exponential of the cube of $h^{1,1}$. Hence, the number of combinations that are taken into account increases drastically with $h^{1,1}$. Many of these combinations are still related by a relabeling of moduli and one could attempt to 'mod out' this relabeling symmetry to reduce the computational cost, but one needs to do nontrivial combinatorics to achieve this.

Since this scan only yields rather crude results and does not take into account more refined conditions on the ranks of the nilpotent matrices, it is not useful to perform it at high values of $h^{1,1}$. Nevertheless, at low enough $h^{1,1}$, one can extract some basic statistics.

In the table in appendix A, the amount of singularity types obtained by the scan are displayed. Configurations that are equivalent by a moduli relabeling symmetry are taken into account separately; for example, the situation with only $\mathcal{K}_{111}$ set to zero is counted separately from the situation with only $\mathcal{K}_{222}$ set to zero. However, the results have been sorted so as to display the lowest singularity type in the configuration first.

Fron the list one can see that some singularity configurations are much more likely to occur than others, as one would expect. Since in this scan all intersection numbers are treated as generic, the table is probably biased towards higher intersection types, since the singularity sub-index will be lower if the intersection numbers have certain special, non-zero values.

As in the two-moduli case, we can also evaluate the singularity patterns for CY manifolds with $h^{1,1} = 3$ that exist 'in real life' by using the KS list. Here, we restrict to manifolds with simplicial Kähler cones in order to compare to the computed scan. The singularity patterns are also displayed in the table. We see that the results from the crude scan described above does not quite correspond to the statistical patterns in the real world. Hence, these statistics are perhaps not too meaningful. There might be more merit in a full list of all the possible cases in the spirit of table 4.1. Even though such a list is tedious to obtain by hand, it might be obtained by clever computer programming. One can then try to find out whether there are forbidden enhancements that do give rise to positive definite metrics.

5 Flux vacua and Freed-Witten anomalous branes

In the third chapter, the formalism of nilpotent orbits was introduced. These nilpotent orbits give one a handle on certain properties of the theory near singular loci in moduli space. They have an interpretation in terms of the degenerating Hodge structure of the Calabi-Yau near those singular loci (or its image under the mirror map).

The appearance of nilpotent matrices was also discovered in the context of flux scalar potentials in type IIA orientifolds[12, 13]. In that paper, a more physical argument for the nilpotent matrices and the resulting monodromy transformations was given. We will here review this interpretation.

The starting point is the fact that the monodromies discussed in chapter 3 also manifest themselves as relations between various flux vacua. That is, vacua with configurations of fluxes that are naively different can nevertheless be equivalent if there is a corresponding shift in the background value of one of the axions if the flux configurations are related by the corresponding monodromy transformation.

Such gauge equivalences become relevant if one has a spacetime in which the flux configuration differs in different spatial regions. Generically, in such spacetimes, there are domain walls separating the different flux domains, consisting of D-branes or NS5-branes; however, these domain walls can sometimes be unstable and can decay by emitting another brane[14]. The conditions that have to be satisfied for this decay to happen are related to the Freed-Witten consistency conditions on the emitted brane, which themselves depend on the flux configurations at both sides of the domain wall.

5.1 D-branes and NS5-branes

In this chapter, a role will be played by D-branes and NS5-branes. In string theory, branes are objects that can have any dimension, but are usually supposed to be different from the fundamental string. They can be described as solitonic solutions to supergravity theories; they also exist in string theories, where they correspond to nonperturbative excitations[45].

D-branes are the objects that are charged under the RR potentials C_1 and C_3[46]. There are the natural electric RR couplings

$$Q_1 \int_{\mathcal{M}_1} C_1 \text{ and } Q_3 \int_{\mathcal{M}_3} C_3 \tag{127}$$

showing that type IIA should contain (0+1)-dimensional and (2+1)-dimensional D-branes, called D0 and D2-branes respectively. In general, a $(p+1)$-dimensional D-brane is called a Dp-brane. There are also D4 and D6 branes that couple magnetically to C_3 and C_1 respectively, or electrically to the fields C_5 and C_7 given by the Hodge dualities $dC_5 = *dC_3$ and $dC_7 = *dC_1$. Lastly, it turns out that there is also a D8-brane[47], a fact that does not follow obviously from the above discussion. Its presence will not be relevant in what follows.

D-branes can be described by an effective action called the Dirac-Born-Infeld action. However, we will not need this dynamical description of D-branes here, since it is not necessary to understand the dynamical behavior of D-branes in order to understand the effects that are presented in this chapter. Rather, we will resort to using the D-branes as RR charges and only address their influence on the flux configuration in nearby vacua.

Like D-branes, NS5-branes are extended objects that can be described as supergravity solitons. They are the magnetic duals of fundamental strings: while the fundamental string carries electric charge under the NSNS 2-form $B_{\mu\nu}$, the NS5-brane is magnetically charged under this same field. NS5-branes are somewhat more poorly understood than D-branes[16]. However, many properties of NS5-branes can be investigated indirectly by using various dualities between string theories, that relate properties of D-branes and fundamental strings to properties of NS5-branes. Again, we will not need to treat the precise dynamical behavior of NS5-branes.

5.2 Flux vacua and domain walls

As described earlier in this thesis, string theory contains flux degrees of freedom, that correspond to nontrivial backgrounds of the gauge fields on the internal Calabi-Yau manifold Y. Specifically, the equations of motion for the RR and NS-NS form fields admit solutions in which the integral of their field strengths over cohomologically nontrivial cycles is nonzero and given by some (integer) flux parameter. Given a string theory compactified on a Calabi-Yau manifold, the choice of flux parameters in order to single out a specific vacuum state.

It is also possible to have a space-time in which different regions contain different fluxes[48]. In such settings, there should be some wall-like object separating these regions. Natural candidates for such domain walls are D-branes[49]. The fact that D-branes can separate regions of different flux can be seen by a simple argument. Consider a D4-brane wrapping a two-cycle Π_a in Y_3, furthermore spanning x_0, x_1 and x_2 and located at $x_3 = 0$. In 4D space, this object looks like a domain wall separating two regions that will turn out to differ by one unit of F_4 flux wrapped on the dual form of Π_a. The D4-brane acts as a magnetic RR source for F_4 as

$$dF_4 = \delta_4(\Pi_a) \wedge \delta(x^3)dx^3, \tag{128}$$

where $\delta_4(\Pi_a)$ is the Poincaré dual form (within Y) to Π_a and $\delta(x^3)dx^3$ is a delta-function valued one-form. The difference in $F_4|_{\Pi_a}$ flux is given by

$$\begin{aligned} \int_{\Pi,x^3=+\epsilon} F_4 - \int_{\Pi,x^3=-\epsilon} F_4 &= \int_{\Pi,x_3=(-\epsilon,+\epsilon)} dF_4 \\ &= \int_\Pi \delta_4(\Pi) \int_{x^3} \delta_0(x^3)dx^3 \\ &= 1. \end{aligned} \tag{129}$$

Hence, domain walls this separate regions of different flux content. By similar arguments, domain walls made out of other types of D-branes wrapped on certain cycles of Y cause shifts in other fluxes on the duals of those cycles.

5.3 Freed-Witten anomalies

The Freed-Witten anomaly condition provides a consistency condition on the allowed branes and fluxes in a string theory compactification. It restricts the possible fluxes that can be supported on a D-brane or NS5-brane. According to the FW condition, a D-brane cannot support nontrivial NSNS-flux on its worldvolume, modulo a topological class[50, 51].

The physics of Freed-Witten anomalies is most easily formulated in terms of D-branes and NSNS-flux. Hence, we consider a D-brane which is wrapped on a cycle Q of the Calabi-Yau space. In the presence of nontrivial NSNS flux, the condition for the system to be free of anomalies turns out to be

$$W_3(Q) + [H]|_Q = 0, \tag{130}$$

where $W_3(Q)$ is the third Stiefel-Whitney class of the tangent space of Q. We will ignore $W_3(Q)$ for simplicity; then the anomaly condition can be derived from the equation of motion of H and F on the D-brane, which reads

$$H = d(F + B). \tag{131}$$

This implies immediately that H should be trivial in cohomology. However, if there is a nonzero flux on the brane, $[H]|_Q \neq 0$, so H is not cohomologically trivial anymore, and the brane is Freed-Witten anomalous.

The anomaly can be cured by adding a magnetic source on the brane on a cycle Q' determined by

$$PD(Q' \subset Q) = [H]|_Q. \tag{132}$$

Here, the Poincaré dual $PD(Q' \subset Q)$ is taken to be the Poincaré dual of Q' within the homology of Q. By the definition of a magnetic source, $[dF] = PD(Q' \subset Q)$, which is harmonic and hence not trivial in cohomology anymore. Hence, with the magnetic source attached, one has the equation of motion

$$[dF] = [H]|_Q, \tag{133}$$

which is consistent.

Like D-branes, NS5-branes can also be Freed-Witten anomalous[14]. Since NS5-branes are more difficult to describe by themselves, this is most easily seen by making use of various duality transformations. Specifically, NS5-branes are related to D5-branes by S-duality[15], and the H_3 flux that drives D5-branes anomalous is converted to F_3 flux by S-duality[14]. Hence, NS5-branes are Freed-Witten anomalous in the presence of RR flux.

5.4 D-branes ending on other D-branes

We have not yet introduced the specific magnetic source that we can use in order to cure Freed-Witten anomalous D-branes or NS5-branes by the procedure suggested in the previous section. We can use the endpoints of other D-branes, which turn out to be magnetic sources of F [52, 53, 54].

For example, consider a D4-brane that wraps an FW anomalous 3-cycle $[\pi_3^K] \in H_3(Y, \mathbb{Z})$, the dual of which carries NSNS flux. We need to provide a magnetic source on the Poincaré dual of $[\pi_3^K]$ within the D4-brane, which is a 2-cycle. Therefore, we should attach a D2-brane, the boundary of which is the required source.

Again, using various dualities one can transfer this picture to Freed-Witten anomalous NS5-branes. It is known that the endpoints of D-branes can attach to NS5-branes[55]. One thus expects D-branes to be the sources curing Freed-Witten anomalies on anomalous NS5-branes.

5.5 Flux catalysis

Specific Freed-Witten anomalous branes that one is interested in in the context of 4D compactifications of string theory are branes that look like strings in 4D[14]. These can be wrapped on nontrivial cycles of Y and become anomalous if the duals of these cycles carry nontrivial fluxes. To cure the anomalies, one adds other branes to the anomalous strings. These can be the domain walls which we described earlier, with a boundary of their worldvolume attaching to the string.

Running this argument in reverse, we see that the appearance of these strings provides a decay mechanism for the domain walls[14, 56]. If one has a set of p domain walls in the presence of p units of a certain flux, those domain walls can nucleate a hole encircled by a 4D string. This 4D string would be FW anomalous if it were not attached to the domain walls, but the boundary of these walls provide exactly the right magnetic sources to cure the anomaly. This process of domain wall decay was termed 'flux catalysis' and studied in detail in reference [14].

The different possible types of FW-anomalous strings are displayed in table 6, together with the fluxes that drive them to be anomalous, the domain walls that they are attached to and the flux jumps over these domain walls. This table is adapted from reference [12].

FW-anomalous string	Flux	Domain wall	Jumping flux
NS5 on $\pi_4^I \in H_4(Y,\mathbb{Z})$	$F_0 = m$	D6 on π_4^I	F_2 on P.D. (π_4^I)
NS5 on $\pi_4^I \in H_4(Y,\mathbb{Z})$	$F_2 = m^J\omega_J$	D4 on $\pi_2 \in$ P.D. $(F_2 \wedge \omega_I)$	F_4 on P.D. (π_2)
NS5 on $\pi_4^I \in H_4(Y,\mathbb{Z})$	$F_4 = e_J$	D2	F_6
D4 on $\pi_3^K \in H_3(Y,\mathbb{Z})$	$H_3 = h_K\beta^K$	D2	F_6

Table 6: Different types of Freed-Witten anomalous strings that can appear in 4D string theory compactifications together with the fluxes that generate the anomaly and the domain walls that cure them[14, 12]. In the last column, the flux of which the value jumps over the domain wall is displayed.

We can see in the table that the FW-anomalous strings can be seen as linear maps sending the fluxes that create the FW anomaly to the fluxes that are caused to jump by the domain walls that cure the anomaly, a fact first noted in [12]. One can display the information from the table in two nilpotent matrices corresponding to the two types of FW-anomalous strings. These nilpotent matrices are given by

$$N_I^T = \begin{pmatrix} 0 & \delta_I^A & 0 & 0 & 0 \\ 0 & 0 & \mathcal{K}_{IJK} & 0 & 0 \\ 0 & 0 & 0 & \delta_I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } M_K^T = \begin{pmatrix} 0 & 0 & 0 & 0 & \delta_K^A \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{134}$$

in a basis ordered as $(H^{3,3}, H^{2,2}, H^{1,1}, H^{0,0}, H^3)$. Here, the first matrix corresponds to the wrapped NS5 brane and the second matrix corresponds to a wrapped D4 brane. Now, the first matrix, N_I^T, turns out to be the transpose of the the monodromy generator N_I considered in chapter 3, but with loop corrections discarded. The second matrix is new and involves the odd cohomology.

5.6 Equivalent flux vacua

The fact that sets of 4D domain walls can decay by emitting strings has consequences for the treatment of the flux configurations on both sides of the domain wall. In particular, flux vacua that are naively different can actually be related through gauge transformations, as advertised earlier.

As an example, consider a stack of p domain walls consisting of D6-branes wrapped on π_4^I described above. These domain walls separate two flux vacua in which the flux F_2 on the Poincaré dual of π_4^I differs by one unit. After turning on p units of F_0 flux, these domain walls become unstable with respect to nucleation of a 4D string loop consisting of an NS5 brane wrapped around the cycle π_4^I.

Moving through the 4D string also causes the axion $b^I = \int_{\pi_4^I} B_2$ to shift by one unit[56, 57]:

$$b^I \to b^I + 1.$$

Hence, one has now two vacua, in which the values of the background flux $\bar{F}_2$ and the axion b^I are different, but which are connected to each other. These vacua should therefore in fact be equivalent.

One can order the values of the RR fluxes in a flux vector $\mathbf{q}$ given by

$$\mathbf{q} = \begin{pmatrix} e_0 \\ e_I \\ m^I \\ m \end{pmatrix}. \tag{135}$$

We have found that the combined operation of shifting the background value of the axion $b^I \to b^I + 1$ and transforming the RR fluxes by the monodromy transformation $\mathbf{q} \to T_I \cdot \mathbf{q}$ yields a gauge equivalent vacuum. This is because the flux vector appears in the superpotential in the combination

$$\Pi^T \cdot \mathbf{q}, \tag{136}$$

which is an inner product in the even cohomology and which transforms under this combined operation as

$$\Pi(...b^I + 1...) \cdot T_I \cdot \mathbf{q}^T = \mathbf{q}^T \cdot \Pi(...b^I...). \tag{137}$$

Hence, the superpotential and the resulting physics are fully invariant under the combined transformation, which is just a monodromy transformation from the moduli space viewpoint.

Likewise, in references [12] and [13], nilpotent matrices were also found for the complex structure moduli (as noted above) and for D-brane moduli. It is interesting whether these nilpotent matrices also have a deep mathematical interpretation in terms of Hodge structures. We leave this question for future work.

6 Conclusions

In this thesis, type IIA string theory compactified on a Calabi-Yau threefold was discussed. The type IIA Kähler moduli space and the singular loci within this moduli space were analyzed using the formalism of mixed Hodge structures, which allows for a classification of singular loci in different subtypes. These were used to gain control over the triple intersection numbers $\mathcal{K}_{IJK}$ that occur in the Kähler moduli space metric. The theory of enhancements of mixed Hodge structures gives rise to important constraints on the $\mathcal{K}_{IJK}$. While we attempted to violate these constraints in order to produce a theory that nontrivially belongs to the swampland, it turned out that the constraints imposed by the polarization conditions were not stronger than the requirement of positive-definiteness of the metric in the examples considered here. We suspect that it is a general rule that $\mathcal{K}_{IJK}$ that violate the polarization conditions also give rise to non-positive definite metrics, but we did not attempt to prove this in general. Either way, the classification of Calabi-Yau manifolds on the basis of their intersection numbers and the resulting singularity types gives the zoo of Calabi-Yau manifolds an interesting structure.

7 Acknowledgements

First and foremost, I want to thank my supervisor Thomas Grimm, for giving me the opportunity to work on this nice project and learn much about string theory. It was fun chatting about nilpotent matrices and intersection numbers.

I also want to thank Damian van de Heisteeg, Chongchuo Li and Kilian Mayer for helping me with various things and chatting about string theory a couple of times. Furthermore, as noted in the main text, Chongchuo's list of singularity enhancements in the $h^{1,1} = 2$ case contained useful information that was used and elaborated on in this thesis.

Thanks be to the master students from the 'Ivoren Toren' as well: to Rik, Lotte, Floris and Joren for many useful and useless discussions about various subjects, for the elongated lunch and coffee breaks and for being my 'debugging rubber ducks', Ruud for being our conscience at a distance, Harm for discussing some string theory fundamentals with me, Barna for patiently allowing me to practice my Hungarian on a few occasions and all the others that made the student's room a nice place to be in.

Last but not least, I want to thank my family, my friends and Madeleine, for the moral support.

References

[1] C. Vafa, "The string landscape and the swampland," 2005.

[2] E. Palti, "The swampland: introduction and review," 2019.

[3] H. Ooguri and C. Vafa, "On the geometry of the string landscape and the swampland," *Nucl. Phys. B.*, vol. 766, pp. 21–33, 2007.

[4] T. W. Grimm, E. Palti, and I. Valenzuela, "Infinite distances in field space and massless towers of states," 2018.

[5] T. W. Grimm, C. Li, and E. Palti, "Infinite distance networks in field space and charge orbits," *JHEP*, vol. 2019, 2019.

[6] P. Corvilain, T. W. Grimm, and I. Valenzuela, "The Swampland Distance Conjecture for Kähler moduli," 2018.

[7] P. A. Griffiths, "Periods of integrals on algebraic manifolds, I. (Construction and properties of the modular varieties)," *American Journal of Mathematics*, vol. 90, no. 2, pp. 568–626, 1968.

[8] W. Schmid, "Variations of Hodge structures: the singularities of the period mapping," *Inventiones mathematicae*, vol. 22, pp. 211–319, 1973.

[9] P. Deligne, "Théorie de Hodge: III," *Publications Mathématiques de l'IHÉS*, vol. 44, pp. 5–77, 1974.

[10] E. Cattani, A. Kaplan, and W. Schmid, "Degenerations of Hodge structures," *Annals of Mathematics*, vol. 123, no. 3, pp. 457–535, 1986.

[11] M. Kerr, G. Pearlstein, and C. Robles, "Polarized relations on horizontal SL(2)s," 2017.

[12] A. Herráez, L. E. Ibáñez, F. Marchesano, and G. Zoccarato, "The type IIA flux potential, 4-forms and Freed-Witten anomalies," *hep-th/1802.05771*, 2018.

[13] D. Escobar, F. Marchesano, and W. Staessens, "Type IIA flux vacua and α'-corrections," 2018.

[14] M. Berasaluce-González, P. Cámara, F. Marchesano, and A. Uranga, "z_p charged branes in flux compactifications," *JHEP*, 2013.

[15] R. Blumenhagen, D. Lüst, and S. Theisen, *Basic concepts of string theory.* Springer-Verlag Berlin Heidelberg, 2013.

[16] D. Tong, "String theory," 2012.

[17] C. Callan, D. Friedan, E. Martinec, and M. J. Perry, "Strings in background fields," *Nucl. Phys. B.*, vol. 262, pp. 593–609, 1985.

[18] T. Grimm, "The effective action of type II Calabi-Yau orientifolds," *Fortsch.Phys.*, vol. 53, pp. 1179–1271, 2005.

[19] J. Louis and A. Micu, "Type II theories compactified on Calabi-Yau threefolds in the presence of background fluxes," *Nucl. Phys. B*, vol. 632, pp. 395–431, 2002.

[20] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, "Vacuum configurations for superstrings," *Nucl. Phys. B.*, vol. 258, pp. 46–74, 1985.

[21] S. Vandoren, "Lectures on Riemannian geometry, part II: Complex manifolds," 2008.

[22] P. Candelas, "Lectures on complex manifolds," 1987.

[23] P. Candelas and X. C. De la Ossa, "Moduli space of Calabi-Yau manifolds," *Nucl. Phys. B.*, vol. 355, 1991.

[24] A. Font and S. Theisen, "Introduction to string compactification," 2003.

[25] T. W. Grimm and J. Louis, "The effective action of type IIA Calabi-Yau orientifolds," *Nucl. Phys. B*, vol. 718, pp. 153–202, 2005.

[26] A. Strominger, "Yukawa couplings in superstring compactification," *Phys. Rev. Lett.*, vol. 55, no. 23, 1985.

[27] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, "A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory," *Nucl.Phys. B*, vol. 359, pp. 21–74, 1991.

[28] J. Halverson, H. Jockers, J. M. Lapan, and D. R. Morrison, "Perturbative corrections to Kähler moduli spaces," *Commun. Math. Phys.*, vol. 333, pp. 1563–1584, 2015.

[29] A. Gerhardus and H. Jockers, "Quantum periods of Calabi-Yau fourfolds," *Nucl.Phys. B*, vol. 913, pp. 425–474, 2016.

[30] S. Gurrieri, *$N = 2$ and $N = 4$ supergravities as compactifications from string theories in 10 dimensions.* PhD thesis, Universite de la Mediterranee - Aix-Marseille II, 2003.

[31] E. Bergshoeff, R. Kallosh, T. Ortín, D. Roest, and A. V. Proeyen, "New formulations of $D = 10$ supersymmetry and D8–O8 domain walls," *Classical and Quantum Gravity*, vol. 18, no. 17, p. 3359, 2001.

[32] R. Kallosh, "Democratic formulation of $d = 10$ supersymmetry," *Theoretical and Mathematical Physics*, vol. 128, p. 1193–1206, 2001.

[33] L. J. Romans, "Massive N=2A supergravity in ten dimensions," *Phys. Lett. B.*, vol. 169, pp. 374–380, 1986.

[34] O. DeWolfe, A. Giryavets, S. Kachru, and W. Taylor, "Type IIA moduli stabilization," *JHEP*, vol. 7, 2005.

[35] J.-P. Derendinger, C. Kounnas, P. M. Petropoulos, and F. Zwirner, "Superpotentials in iia compactifications with general fluxes," *Nucl. Phys. B*, vol. 715, pp. 211–233, 2005.

[36] P. A. Griffiths, "Periods of integrals on algebraic manifolds, II: (Local study of the period mapping)," *American Journal of Mathematics*, vol. 90, no. 3, pp. 805–865, 1968.

[37] P. A. Griffiths, "Periods of integrals on algebraic manifolds, III (Some global differential-geometric properties of the period mapping)," *Publications Mathématiques de l'IHÉS*, vol. 38, pp. 125–180, 1970.

[38] S. A. Filippini, H. Ruddat, and A. Thompson, "An introduction to Hodge structures," *Calabi-Yau Varieties: Arithmetic, Geometry and Physics*, pp. 83–130, 2015.

[39] C.-L. Wang, "On the incompleteness of the Weil-Petersson metric along degenerations of Calabi-Yau manifolds," *Math. Res. Lett.*, vol. 4, pp. 157–171, 1997.

[40] T. Rudelius, "On the possibility of large axion moduli spaces," *JCAP*, vol. 1504, p. 049, 2014.

[41] C. Li. Private communication, 2019.

[42] R. Altman, J. Gray, Y.-H. He, V. Jejjalaf, and B. D. Nelson, "A Calabi–Yau database: Threefolds constructed from the Kreuzer–Skarke list," 2014.

[43] M. Kreuzer and H. Skarke, "Complete classification of reflexive polyhedra in fourdimensions," *Adv.Theor.Math.Phys.*, vol. 4, pp. 1209–1230, 2002.

[44] G. T. Gilbert *The American Mathematical Monthly*, vol. 98, no. 1, pp. 44–46, 1991.

[45] J. Polchinski, "TASI lectures on D-branes," 1996.

[46] J. Polchinski, "Dirichlet-branes and Ramond-Ramond charges," *Phys. Rev. Lett.*, vol. 75, pp. 4724–4727, 1995.

[47] E. Bergshoeff, M. de Roo, M. Green, G. Papadopoulos, and P. Townsend, "Duality of type II 7-branes and 8-branes," *Nucl. Phys.B*, vol. 470, pp. 113–135, 1996.

[48] A. Ceresole, G. Dall'Agata, A. Giryavets, R. Kallosh, and A. Linde, "Domain walls, near-BPS bubbles and probabilities in the landscape," *Phys. Rev. D*, vol. 74, p. 086010, 2006.

[49] L. E. Ibáñez, *String Theory and Particle Physics: an introduction to string phenomenology.* Cambridge University Press, 2012.

[50] D. S. Freed and E. Witten, "Anomalies in string theory with D-branes," *Asian J. Math*, vol. 3, pp. 819–851, 1999.

[51] M. G. Maldacena, Juan and N. Seiberg, "D-brane instantons and K-theory charges," *JHEP*, vol. 11, 2001.

[52] C. P. Bachas, “Lectures on D-branes,” 1998.

[53] A. Strominger, “Open p-branes,” *Phys.Lett. B*, vol. 383, pp. 44–47, 1996.

[54] P. K. Townsend, “Brane surgery,” *Nucl. Phys. B*, pp. 163–175, 1997.

[55] R. Argurio, F. Englert, and L. Houart, “Intersection rules for p-branes,” *Phys. Lett. B.*, vol. 398, pp. 61–68, 1997.

[56] M. G. Alford, K.-M. Lee, J. March-Russell, and J. Preskill, “Quantum field theory of nonabelian strings and vortices,” *Nuclear Physics B*, vol. 384, pp. 251–317, 1991.

[57] F. Marchesano, G. Shiu, and A. M. Uranga, “F-term axion monodromy inflation,” 2014.

Appendix A: list of singularity enhancements from automated scan

$N_1 + N_2 + N_3$	Amount (%)	Amount in KS list (%)
$\mathrm{II}_0 + \mathrm{II}_0 + \mathrm{II}_0$	1 (0.10)	U
$\mathrm{II}_0 + \mathrm{II}_0 + \mathrm{IV}_1$	3 (0.29)	U
$\mathrm{II}_0 + \mathrm{II}_1 + \mathrm{III}_0$	6 (0.59)	U
$\mathrm{II}_0 + \mathrm{II}_1 + \mathrm{IV}_2$	6 (0.59)	U
$\mathrm{II}_0 + \mathrm{III}_0 + \mathrm{III}_0$	3 (0.29)	U
$\mathrm{II}_0 + \mathrm{III}_0 + \mathrm{IV}_2$	12 (1.17)	U
$\mathrm{II}_0 + \mathrm{IV}_1 + \mathrm{IV}_1$	3 (0.29)	U
$\mathrm{II}_0 + \mathrm{IV}_2 + \mathrm{IV}_2$	9 (0.88)	U
$\mathrm{II}_1 + \mathrm{II}_1 + \mathrm{III}_0$	3 (0.29)	0
$\mathrm{II}_1 + \mathrm{II}_1 + \mathrm{IV}_2$	3 (0.29)	0
$\mathrm{II}_1 + \mathrm{III}_0 + \mathrm{III}_1$	12 (1.17)	0
$\mathrm{II}_1 + \mathrm{III}_0 + \mathrm{IV}_1$	6 (0.59)	0
$\mathrm{II}_1 + \mathrm{III}_0 + \mathrm{IV}_2$	6 (0.59)	0
$\mathrm{II}_1 + \mathrm{III}_0 + \mathrm{IV}_3$	12 (1.17)	6 (1.25)
$\mathrm{II}_1 + \mathrm{III}_1 + \mathrm{IV}_2$	12 (1.17)	19 (3.97)
$\mathrm{II}_1 + \mathrm{IV}_1 + \mathrm{IV}_2$	6 (0.59)	0
$\mathrm{II}_1 + \mathrm{IV}_2 + \mathrm{IV}_2$	6 (0.59)	0
$\mathrm{II}_1 + \mathrm{IV}_2 + \mathrm{IV}_3$	12 (1.17)	55 (11.48)
$\mathrm{II}_1 + \mathrm{IV}_3 + \mathrm{IV}_3$	0	1 (0.21)
$\mathrm{II}_2 + \mathrm{II}_2 + \mathrm{II}_2$	1	9 (1.88)
$\mathrm{II}_2 + \mathrm{II}_2 + \mathrm{III}_0$	6 (0.59)	10 (2.09)
$\mathrm{II}_2 + \mathrm{II}_2 + \mathrm{III}_1$	3 (0.29)	1 (0.21)
$\mathrm{II}_2 + \mathrm{II}_2 + \mathrm{IV}_2$	0	7 (1.46)
$\mathrm{II}_2 + \mathrm{II}_2 + \mathrm{IV}_3$	12 (1.17)	23 (4.80)
$\mathrm{II}_2 + \mathrm{III}_0 + \mathrm{III}_0$	9 (0.88)	9 (1.88)
$\mathrm{II}_2 + \mathrm{III}_0 + \mathrm{III}_1$	18 (1.76)	2 (0.42)
$\mathrm{II}_2 + \mathrm{III}_0 + \mathrm{IV}_2$	6 (0.59)	18 (3.76)
$\mathrm{II}_2 + \mathrm{III}_0 + \mathrm{IV}_3$	42 (4.10)	16 (3.34)
$\mathrm{II}_2 + \mathrm{III}_1 + \mathrm{III}_1$	12 (1.17)	0

$N_1 + N_2 + N_3$	Amount (%)	Amount in KS list (%)
$\mathrm{II}_2 + \mathrm{III}_1 + \mathrm{IV}_2$	6 (0.59)	10 (2.09)
$\mathrm{II}_2 + \mathrm{III}_1 + \mathrm{IV}_3$	42 (4.10)	4 (0.84)
$\mathrm{II}_2 + \mathrm{IV}_2 + \mathrm{IV}_2$	3 (0.29)	0
$\mathrm{II}_2 + \mathrm{IV}_2 + \mathrm{IV}_3$	6 (0.59)	33 (6.89)
$\mathrm{II}_2 + \mathrm{IV}_3 + \mathrm{IV}_3$	51 (4.98)	2 (0.42)
$\mathrm{III}_0 + \mathrm{III}_0 + \mathrm{III}_1$	18 (1.76)	0
$\mathrm{III}_0 + \mathrm{III}_0 + \mathrm{IV}_1$	3 (0.29)	11 (2.30)
$\mathrm{III}_0 + \mathrm{III}_0 + \mathrm{IV}_2$	0	12 (2.51)
$\mathrm{III}_0 + \mathrm{III}_0 + \mathrm{IV}_3$	27 (2.64)	4 (0.84)
$\mathrm{III}_0 + \mathrm{III}_1 + \mathrm{III}_1$	3 (0.29)	0
$\mathrm{III}_0 + \mathrm{III}_1 + \mathrm{IV}_2$	30 (2.93)	0
$\mathrm{III}_0 + \mathrm{III}_1 + \mathrm{IV}_3$	42 (4.10)	1 (0.21)
$\mathrm{III}_0 + \mathrm{IV}_1 + \mathrm{IV}_2$	12 (1.17)	12 (2.51)
$\mathrm{III}_0 + \mathrm{IV}_1 + \mathrm{IV}_3$	0	12 (2.51)
$\mathrm{III}_0 + \mathrm{IV}_2 + \mathrm{IV}_2$	3 (0.29)	0
$\mathrm{III}_0 + \mathrm{IV}_2 + \mathrm{IV}_3$	36 (3.52)	13 (2.71)
$\mathrm{III}_0 + \mathrm{IV}_3 + \mathrm{IV}_3$	57 (5.57)	15 (3.13)
$\mathrm{III}_1 + \mathrm{III}_1 + \mathrm{III}_1$	33 (3.22)	0
$\mathrm{III}_1 + \mathrm{III}_1 + \mathrm{IV}_2$	3 (0.29)	0
$\mathrm{III}_1 + \mathrm{III}_1 + \mathrm{IV}_3$	111 (10.84)	1 (0.21)
$\mathrm{III}_1 + \mathrm{IV}_1 + \mathrm{IV}_2$	0	74 (15.45)
$\mathrm{III}_1 + \mathrm{IV}_1 + \mathrm{IV}_3$	0	2 (0.42)
$\mathrm{III}_1 + \mathrm{IV}_2 + \mathrm{IV}_2$	21 (2.05)	23 (4.80)
$\mathrm{III}_1 + \mathrm{IV}_2 + \mathrm{IV}_3$	12 (1.17)	3 (0.63)
$\mathrm{III}_1 + \mathrm{IV}_3 + \mathrm{IV}_3$	147 (14.36)	2 (0.42)
$\mathrm{IV}_1 + \mathrm{IV}_1 + \mathrm{IV}_1$	1 (0.10)	0
$\mathrm{IV}_1 + \mathrm{IV}_2 + \mathrm{IV}_2$	9 (0.88)	0
$\mathrm{IV}_1 + \mathrm{IV}_2 + \mathrm{IV}_3$	0	32 (6.68)
$\mathrm{IV}_1 + \mathrm{IV}_3 + \mathrm{IV}_3$	0	1 (0.21)
$\mathrm{IV}_2 + \mathrm{IV}_2 + \mathrm{IV}_2$	3 (0.29)	0
$\mathrm{IV}_2 + \mathrm{IV}_2 + \mathrm{IV}_3$	24 (2.34)	24 (5.01)
$\mathrm{IV}_2 + \mathrm{IV}_3 + \mathrm{IV}_3$	9 (0.88)	6 (1.25)
$\mathrm{IV}_3 + \mathrm{IV}_3 + \mathrm{IV}_3$	82 (8.01)	6 (1.25)

Table 7: Statistics of large modulus singularity patterns in the $h^{1,1} = 3$ case obtained by setting sets of intersection numbers to zero and treating the others as generic. The total amount of possibilities is 1024. In the last column, the corresponding amount of CY manifolds with the specified singularity patterns is given; here, only the $h^{1,1} = 3$ manifolds with simplicial Kähler cones were considered.