BLACK HOLES AND BPS SOLUTIONS IN SIX-DIMENSIONAL SUPERGRAVITY

ROY STEGEMAN

SUPERVISOR:
PROF. DR. STEFAN VANDOREN

Utrecht University

INSTITUTE FOR THEORETICAL PHYSICS
FACULTY OF SCIENCE
UTRECHT UNIVERSITY

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SUPERVISOR:
Prof. Dr. Stefan Vandoren

SECOND EXAMINER:
Dr. Thomas Grimm

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‘Then you should say what you mean,’ the March Hare went on.
‘I do,’ Alice hastily replied; ‘at least–at least I mean what I say–that’s the same thing, you know.’
‘Not the same thing a bit!’ said the Hatter. ‘You might just as well say that "I see what I eat" is the same thing as "I eat what I see"!’.

— *Alice’s Adventures in Wonderland*, Lewis Carrol

ABSTRACT

We study supersymmetric solutions of minimal six-dimensional supergravity, which can be uplifted to F-theory. We focus in particular on the class of solutions defined on a Gibbons-Hawking base space, which are fully characterized by a set of six independent harmonic functions. We generate new solutions of this theory using transformations in the group $Sp(6, \mathbb{R})$ thereby mapping solutions to solutions. We analyze this group by acting on flat space background solutions, as well as a plane wave solution. We classify the generated solutions, and find a subgroup preserving flatness asymptotically when acting on flat space. Moreover, we discuss a solution with an $AdS_3 \times S^3$ event horizon geometry, and calculate its entropy. The solution changes in a nontrivial way by adjusting certain parameters, and we perform an initial analysis of the implications of such changes.

CONTENTS

1 INTRODUCTION

Shortly after Einstein published his theory of general relativity, a search for solutions to the field equations started. This resulted in a number of mysterious predictions, including that of the black hole and gravitational waves. Recent years have seen a breakthrough in the observation of gravitational waves, providing an entirely new method of observing astrophysical phenomena. However, as a result of the existence of black holes, the theory of general relativity has its own breaking point build into it. This is a result of the fact that general relativity provides the possibility for a smooth initial system to create singularities in spacetime [1]. Classically this is not so much of a problem, as these singularities are hidden behind an event horizon [2, 3]. But it is this property which eventually led Hawking to formulate the black hole 'information paradox', motivating, together with thermodynamic properties of black holes, the study of black hole microstates which is currently still an active field of research.

Another interesting, albeit it somewhat mysterious, property of general relativity, is that the field equations are not specific to any dimension. So far the solutions of four and five dimensional general relativity have been thoroughly analyzed, and in chapter 2 of this thesis we will discuss some of the interesting solution that have been found. In four dimensions the uniqueness theorems state that black holes in general relativity are fully characterized by three observables, these are mass, angular momentum and electric charge. Nevertheless, in five dimensions we find that there is not only an object with horizon topology S^3, analogue to a five dimensional black hole, but also an object with horizon topology $S^2 \times S^1$ called a black string. This is an example showing that, in general, physics is richer in higher dimensions. But other than simple curiosity of what solutions we can find for the field equations, the study of higher dimensions is also relevant in the context of string theory, which are theories of quantum gravity living in ten dimensions.

Where we just discussed general relativity in the context of the most massive objects in our universe, there is another theory describing the physics of fundamental particles. This is the standard model of particle physics, which is described in terms of quantum field theories, and this currently provides our most fundamental understanding of the smallest particles. Around same period when the standard model was being finalized and experimentally confirmed, another framework was being developed, that of (global) supersymmetry. Supersymmetry will be discussed in chapter 3 of this thesis, but allow us to mention here that the premise of supersymmetry is that it predicts a bosonic 'superpartner' for each fermion, and a fermionic superpertner for each boson, which lead to some promising results. Among the most significant of these results are that utraviolet divergence of supersymmetric theories is better behaved than that of the standard model as a result of cancellation between particles and their superpartner in loop diagrams, this is a purely technical argument. However, it can also be used to explain how the three gauge couplings of the standard model obtain the same value at high energy (also known as the Grand Unified Theory (GUT) scale), and the extra particles predicted by supersymmetry contain possible candidates for dark matter particles.

One of the key concepts underlying the standard model is gauge symmetry. This concept has been applied to supersymmetry, and resulted in an extension of general relativity called supergravity. Just as general relativity, supergravity still breaks down at the Planck scale, which is at an energy $E_P \approx 10^{19}\text{GeV}$. At this scale we need a theory of quantum gravity. String theory is currently the prime candidate to describe such a theory of quantum gravity, while also unifying all fundamental forces in nature in a single theory. Ten-dimensional supergravity is the low energy limit of string theory, and since quantum effects only become relevant beyond the Planck scale, for many purposes supergravity is studied in the context of string theory.

In addition to these properties, supergravity simplifies the classification of solutions to the field equations. As we will see in chapter 3 of this thesis, the symmetry parameter of supergravity admits a covariantly constant spinor, and this turns out to be such a strong constraint that it is often possible to determine general solutions by satisfying this condition. Such a characterization has been made for a number of supergravity theories in four and five dimensions, as well as for minimal six-dimensional supergravity by Gutowksi Martelli and Reall in [4]. In particular they found that the so called Gibbons-Hawking class of solutions in this theory are fully characterized by a set $\mathbb{V}$ of six harmonic functions. Subsequently Crichigno, Porri and Vandoren realized in [5], that the group $\mathrm{Sp}(6,\mathbb{R})$ provides a linear map $\mathbb{V} \to \mathbb{V}$, transforming solutions into solutions. Where the symplectic property of these transformations prevents singularities in the fields of the theory.

The main goal of this thesis is to provide more insight into the meaning and consequences of the group $\mathrm{Sp}(6,\mathbb{R})$ in the context of six-dimensional minimal supergravity. Previous work on this has been done by Porri [6], and Duaso [7], and this thesis is an extension of their work.

OUTLINE

In chapter 2 we will discuss solutions to Einstein's field equations. Most of our time will be spent on the discussion of the Reissner-Nordström black hole, but we will also spent some time on the motivating the study of higher dimensional solutions of general relativity, as well as plane wave solutions to the field equations.

In chapter 3, we introduce some basic aspects of supergravity. We introduce global supersymmetry and gauge the resulting spinor field (the gravitino) to obtain supergravity. The chapter ends with a discussion of p-branes and D-branes, and a describe how supergravity arises as the low energy limit of string theory.

Subsequently, in chapter 4, we introduce the theory of minimal six-dimensional supergravity following [4], which will be the main focus of this thesis. We then provide some examples of known solutions to the field equations within the context of this theory, and describe a way of generating more solutions using the group $\mathrm{Sp}(6,\mathbb{R})$.

Afterward, in chapter 5, we discuss some properties of this group in the context of the six-dimensional theory.

Chapters 6 and 8 provide examples of generated solutions by transforming flat space and plane wave solutions using the group $\mathrm{Sp}(6,\mathbb{R})$, respectively.

Finally, in chapter 8 we study nontrivial transformations of a BTZ-black hole. Among other things, we identify a horizon, and calculate the macroscopic entropy.

Part I

SUPERGRAVITY

This part introduces supergravity and the specific supergravity theory of interest in this thesis. Before delving right into supersymmetry, we will first introduce some well known solutions to Einstein's field equations and motivate the study of higher dimensions than the 3+1 commonly studied in general relativity. After this, (global) supersymmetry will be introduced, and we will discuss how gauging supersymmetry leads to a theory of supergravity. Finally, the theory of six dimensional minimal supergravity is introduced closely following [4].

2 SOLUTIONS TO EINSTEIN'S FIELD EQUATIONS

In this thesis we commit to the study of suprsymmetric solutions to Einsteins field equations in six dimensions. However, in order to gain understanding for the physical implications of these solutions it is useful to first understand some of the most important solutions to general relativity in four dimensions, which we will introduce here. We will provide a discussion of the Reissner-Nordström black hole, as well as the plane wave solutions, which will both aid the understanding of some of the more exotic solutions we will encounter in this thesis. Good pedagogical introductions to general relativity include [8–10], or as a rich resource with extensive bibliography, albeit it somewhat outdated in some parts, [11] is widely regarded the standard textbook in the field.

2.1 BLACK HOLES

Among the most remarkable features of General Relativity, is its prediction of black holes. In the four-dimensional theory of general relativity a stationary, asymptotically flat black hole coupled to electromagnetism is uniquely determined by its mass M, electric and magnetic charges Q and P and angular momentum J [12–15]. This kind of charged, rotating black hole is called a Kerr-Newmann black hole and the uniqueness theorems are commonly referred to as the no-hair theorem. This uniqueness does however not hold in higher dimensional solutions to general relativity, which we will briefly touch upon later in this section.

2.1.1 *Charged black holes*

In this subsection we will discuss Reissner-Norström black hole, which is a black hole solution coupled to electromagnetism. It will serve as a toy-model for the more complicated objects studied later on in this thesis.

Starting from the Einstein-Hilbert action in four spacetime dimensions coupled to matter fields

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} R + S_m, \tag{2.1}$$

we can obtain the equations of motion of general relativity coupled to matter by varying the action with respect to the metric. The resulting equations are Einstein's field equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{2.2}$$

where $R_{\mu\nu}$ is the Ricci tensor, R the Ricci scalar and G_N the Newton constant in four spacetime dimensions.

The energy-momentum tensor of the matter fields can be obtained from S_m via

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{2.3}$$

Since we are discussing the Reissner-Norström black hole, we will be interested in gravity coupled to electromagnetism. In this case S_m is the Maxwell action given by

$$S_m = -\frac{1}{4} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} \int d^4x F \wedge \star F. \tag{2.4}$$

Using (2.3) we can calculate the energy-momentum tensor of the electromagnetic field, which reads

$$T_{\mu\nu} = \frac{1}{2}\left(F_{\mu\sigma}F^{\sigma}_{\nu} - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\right). \tag{2.5}$$

Besides Einstein's equations, we also have to take into account the homogeneous Maxwell equations, they are

$$dF = 0, \quad d\star F = 0. \tag{2.6}$$

The electromagnetic field strength F is defined as the exterior derivative of the electromagnetic potential:

$$F \equiv dA. \tag{2.7}$$

The electromagnetic potential can be given a geometric interpretation in terms of a five dimensional theory, in which case electromagnetism is gauge theory on a fibre bundle which is topologically S^1, with $U(1)$ gauge symmetry. This is the main idea behind the Kaluza-Klein reduction discussed in appendix E, and is one of the reasons to study higher dimensions.

A solution satisfying both Einstein's and Maxwell's equations is given by the Reissner-Nordsröm metric

$$ds^2 = -\Delta(r)dt^2 + \Delta^{-1}(r)dr^2 + r^2 d\Omega_2^2, \tag{2.8}$$

where

$$\Delta(r) = 1 - \frac{2G_n M}{r} + \frac{G(Q^2+P^2)}{r^2}, \tag{2.9}$$

and $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on a two-sphere S^2.

The corresponding field strength is

$$F = -\frac{Q}{r^2}dt \wedge dr^2 + P\sin\theta d\theta \wedge d\phi. \tag{2.10}$$

Setting $Q = P = 0$ in this solution, we obtain a black hole solution without any matter content, this is the Schwarzschild solution [16].

When $\Delta(r) = 0$ at $r_\pm = G_N M \pm \sqrt{G_N^2 M^2 - G(Q^2+P^2)}$, g_{tt} vanishes, meaning there appears to be a horizon. However, g_{rr} blows up, indicating a singularity in the line element. To see that these are not actual curvature singularities but rather just artifact of our choice of coordinates we can calculate curvature scalars:

$$R = 0, \quad R_{\mu\nu}R^{\mu\nu} = \frac{4G_N^2(P^2+Q^2)^2}{r^8}. \tag{2.11}$$

From these scalars it becomes clear that the only curvature singularity is at $r = 0$.

Depending on the values of parameter Q, P and M we distinguish three different cases:

- $G_N M^2 < Q^2 + P^2$ In this case $\Delta(r)$ is positive for all $r > 0$, and as a result there is no horizon shielding the singularity at $r = 0$. Such a naked singularity violates the cosmic censorship conjecture, which implies that such solutions cannot be the result of gravitational collapse, and the result is therefore generally considered nonphysical. More intuitively the statement $GM^2 < Q^2 + P^2$ means that the total energy of the solution is less than the energy of the electromagnetic field, which would be the case when the mass of the matter that carries the charge is negative.

- $G_N M^2 > Q^2 + P^2$ In this case we do not have the properties that led us to consider the previous solution to be nonphysical. This solution has two null surfaces at $r = r_\pm > 0$, thus the singularity at $r = 0$ is hidden behind the event horizons.

- $G_N M^2 = Q^2 + P^2$ If this equality is satisfied, the solution describes what is called the extremal Reissner-Nordström black hole. In this case there is only a single horizon at $r = r_+ = r_- = G_N M$ when $\Delta(r) = 0$. The extremal Reissner-Norström black hole is an example of an object which can preserve certain supersymmetries, and are therefore of interest in the study of black holes in quantum gravity. An interesting property of an extremal black hole is that the mass is balanced by its charge, and as a result the electromagnetic and gravitational forces between multiple black holes cancel. This enables us to write stable solutions containing multiple black holes.

Let us now focus on this last case, the extremal Reissner-Nordström black hole. Here the metric is

$$ds^2 = -\left(1 - \frac{G_N M}{r}\right)^2 dt^2 + \left(1 - \frac{G_N M}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2, \tag{2.12}$$

with the horizon at $r = G_N M$. To study the solution near the horizon we shift the radial coordinate as

$$\rho \equiv r - G_N M, \tag{2.13}$$

resulting in the metric in isotropic form:

$$ds^2 = -H^{-2}(\rho) dt^2 + H^2(\rho)\left(d\rho^2 + \rho^2 d\Omega_2^2\right), \tag{2.14}$$

where

$$H(\rho) = 1 + \frac{G_N M}{\rho}, \tag{2.15}$$

which is a Harmonic function with respect to the coordinates on $\mathbb{R}^3$. One can check that H obeys the Poisson equation

$$\nabla^2 H = \nabla^2 \left(1 + \frac{G_N M}{\rho}\right) = 0, \quad \rho > 0. \tag{2.16}$$

Taking the limit $\rho \to 0$ in this metric, yields the near horizon metric, which reads

$$ds^2 = -\left(\frac{\rho}{G_N M}\right)^2 dt^2 + \left(\frac{G_N M}{\rho}\right)^2 d\rho^2 + G_N^2 M^2 d\Omega_2^2. \tag{2.17}$$

For a description of Anti-de Sitter (AdS) space see appendix C.

Here we can recognize that the near-horizon geometry is $AdS_2 \times S^2$. It might not be immediately clear that the Ricci curvature scalar still vanishes, but since the AdS_2 and S^2 both have nonzero Ricci curvature of equal magnitude but different sign, the total Ricci curvature vanishes.

The field strength of the solution (2.10), now allows us to calculate the charges contained inside the horizon:

$$P = \frac{1}{2\pi}\int_{S^2} F, \tag{2.18}$$

$$Q = \frac{1}{4\pi}\int_{S^2} \star F. \tag{2.19}$$

Solutions of this kind, are known as Bertotti-Robinson solutions [17, 18].

2.1.2 *Dirac string*

If we so desire, we are free to add an electric charge to the Maxwell equations, resulting in

$$dF = 0, \quad d \star F = \star J. \tag{2.20}$$

However, introducing a magnetic monopole is not as straightforward since we want the field strength to be a closed form. Dirac proposed in [19], that one could introduce a magnetic monopole if we were to exclude a single point from the integration area in (2.18). In this situation $dF = 0$ could hold anywhere, but on this single point on the integration surface. In order to introduce an actual magnetic monopole, this would require defining a line of such points stretching to infinity since we can choose to integrate over any two-surface around the monopole, and we should always find such a point. This line of points is called a Dirac string. Since the vector potential A_μ is not well defined on the string, such a string can only exist if it is not observable. This is the case if the wave-function of a particle circulating the string, only acquires as a phase change a multiple of 2π. The acquired phase by a particle moving in a loop around the string is equal to the magnetic flux through this loop, and thus through the Dirac string. From this it can be concluded that the restriction on the phase change after such a loop, quantizes the magnetic flux through the Dirac string. This has led to the Dirac quantization condition, which states that the existence of a magnetic monopole would imply that the magnetic and electric charges must be quantized.

2.1.3 *Electromagnetic duality*

Let us have another look a the Lagrangian (2.4). The equations of motion (2.6) obtained from this Lagrangian are invariant under transformations of the form

$$\begin{pmatrix} F \\ \star F \end{pmatrix} \to g \begin{pmatrix} F \\ \star F \end{pmatrix}, \tag{2.21}$$

where $g \in GL(2, \mathbb{R})$. Note that even though the equations of motion are invariant, the action itself will be scaled by a factor equal to the determinant of G.

In a Lorentzian manifold there is a restriction to the subgroup $Sp(2, \mathbb{R}) \subset GL(2, \mathbb{R})$. To see this, consider the transformation

$$\begin{pmatrix} F' \\ (\star F)' \end{pmatrix} = g \begin{pmatrix} F \\ \star F \end{pmatrix}. \tag{2.22}$$

Transforming with $g \in GL(2, \mathbb{R})$ can result in $F' \propto \star F$ and $(\star F)' \propto F$, and combining these expressions leads to

$$F' \propto \star(\star F)'. \tag{2.23}$$

When the Hodge operator is defined in a Lorentzian manifold, acting with the Hodge star twice changes sign: $\star\star A = -A$. This means that (2.23) can be rewritten as

$$\star F' \propto -(\star F)'. \tag{2.24}$$

However, for consistency we require $(\star F)' = \star(F')$, which is in disagreement with (2.24). Limiting g to be an element of the group $Sp(2, \mathbb{R})$, resolves this problem due to the anti-symmetric property of the symplectic group. This result generalizes to the transformation of n field strengths corresponding to n different one-form gauge fields, so essentially to a case with n different charges. In this case the duality transformation are elements of the group $Sp(2n, \mathbb{R})$ [20, Chapter 4].

2.1.4 *Rotating black holes*

The Reissner-Norström metric discussed above is fully described by the mass and charge parameters. However, if we were to study the astrophysical black holes observed in our universe, we would notice that angular momentum is another parameter needed to describe these black holes.

The metric of a solution of a rotating mass requires axial symmetry instead of spherical symmetry, this proved to be a considerably more challenging task than finding the metric that describes black holes with only mass and charges, and it was not until 1963 when Kerr had managed to find the solutions for a rotating black hole [21]. Shortly after, this result was generalized to include charge, the resulting charged, rotating black hole is described by the Kerr-Newman metric [22]. In Boyer-Lindquist coordinates [23], the metric reads

$$\begin{aligned} ds^2 =& -\left(1 - \frac{2G_N Mr - G_N M(Q^2+P^2)}{\rho^2}\right) dt^2 \\ & - \frac{2(2G_N Mr - G_N M(Q^2+P^2))a\sin^2\theta}{\rho^2} dt d\phi + \frac{\rho^2}{\Delta} dr^2 \\ & + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right] d\phi^2, \end{aligned} \tag{2.25}$$

where

$$\Delta(r) = r^2 - 2GMr - G_N M(Q^2+P^2) + a^2, \tag{2.26}$$

$$\rho(r,\theta) = r^2 + a^2\cos^2\theta, \tag{2.27}$$

and

$$a = J/M, \tag{2.28}$$

with J the angular momentum of the black hole.

2.1.5 *Higher dimensions*

As the title of this thesis already suggests we will be interested in solutions of more than four dimensions, here we will briefly discuss some higher-dimensional, asymptotically flat vacuum solutions important in the study of black holes, explicitly solutions in five spacetime dimensions. Reviews of these type of solutions are [24, 25].

A solution which might be among the most obvious generalizations of four dimensional black holes, is the Myers-Perry black hole [26]. The Myers-Perry black hole is the higher-dimensional analogue of the four dimensional Kerr solutions, it is parametrized by its mass and angular momentum, just as the Kerr solution, however since we are in higher dimensions, there is the possibility for rotation in several independent planes [27].

Another higher dimensional solution parametrized by mass and angular momentum is the black ring [28]. This is a solution with $S^1 \times S^2$ horizon topology. A black ring is a loop of matter, where the gravitational force acting to collapse the ring, is counteracted by the angular momentum, which results in a stable solution. If we were to find such an object in asymptotically Taub-NUT space, which is topologically $\mathbb{R}^3 \times S^1$, we can reduce the fifth dimension on a circle and obtain a solution in four dimensions with an S^2 event horizon, a Kerr black hole.

Appendix E provides an introduction to dimensional reduction on a circle.

We have just provided two objects parametrized by the same parameters of mass and angular momentum (and without charge), which is impossible in four dimensions by virtue of the uniqueness theorems. This shows the richness of solutions in higher dimensions, which motivates further research of these spacetimes. As an example we will discuss the explicit solution of a higher dimensional black hole.

The Tangherlini black hole and black rings

The Tangherlini black hole is a generalization to five-dimensions of the Reissner-Nordström black hole. It turns out that the five-dimensional analogue is simpler to construct from a brane perspective than the four-dimensional black hole, and it is therefore commonly studied in the context of string theory. The Tangherlini metric in Einstein frame reads [29]

$$ds^2 = -H^{-2}dt^2 + H\left(dr^2 + r^2 d\Omega_3^2\right), \tag{2.29}$$

where H is a harmonic function with respect to the spacelike coordinates.

Considering only a single black hole, the corresponding single centered solution has

$$H = 1 + \frac{Q}{r^2}, \tag{2.30}$$

where Q is the electric charge originating from the only nonvanishing component in the gauge field $F_{\mu\nu}$, $F_{tr} = -Q/r^3$.

This solution is similar to the solution of the Reissner-Nordström black hole, with the major difference being that there is a different power of r in the harmonic function.

Above we also mentioned the existence of black ring solutions. A heuristic way to construct such a ring is as the direct product of $D-1$ dimensional Tangherlini black holes and a circle S^1. This results in a black object with horizon topology $S^{D-3} \times S^1$. In essence such a solution will contract along the S^1 direction as a result of tension and gravitational interaction with itself. However, as mentioned above, we can introduce angular momentum along this direction to counteract this collapse.

2.1.6 *The BTZ black hole*

In three spacetime dimensions the graviton does not have any polarization modes, and hence the concept of gravity is we know it form dimensions $D \geqslant 4$ does not apply in three dimensions. Nonetheless, there exists a solution known as the BTZ black hole, named after Bañados, Teitelboim and Zanelli, which is a black hole solution in three dimensions [30, 31]. This solutions shares many properties with the black hole in four dimensions, including the fact the fact that it can be the final state of a gravitational collapse [32]. These similarities, are the reason we call the solution a black hole, but there are of course also differences with the black holes in higher dimensions, where especially the origin of its horizon is very different.

The BTZ black hole is locally AdS_3. But other than geometrical aspects, the topology of a manifold is also of importance. Let us take ξ to be the generator of a rotation in AdS_3, and let us furthermore define

$$x \to e^{n2\pi\xi}x, \quad n \in \mathbb{Z}. \tag{2.31}$$

Because ξ is a Killing vector, and as such the orbits that are a result of the identification 2.31 describe isometries, the metric remains well-defined with constant negative curvature, and still describes a solution to Einstein's equations.

To ensure the absence of closed timelike curves, ξ must be spacelike [30], resulting in the condition

$$\xi \cdot \xi > 0. \tag{2.32}$$

To enforce this condition the part of spacetime where the condition is violated is removed from the solution. As a result there are no longer any geodesics that cross $\xi \cdot \xi = 0$, and therefore this point is interpreted as a singularity. The singularity is of a different nature than the ones we have seen so far, since it is not a curvature singularity but rather a singularity as a result of the causal structure of the solution.

The metric of a BTZ black hole reads

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 \left(N^{\phi} dt + d\phi\right)^2, \tag{2.33}$$

where

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \tag{2.34}$$

$$N^{\phi}(r) = -\frac{J}{2r^2}. \tag{2.35}$$

Here the coordinate ϕ corresponds to the rotations generated by ξ, and we have $0 \leqslant \phi \leqslant 2\pi$. The constants M and J are associated with mass and angular momentum, respectively.

$N(r)$ vanishes at two values of r, where the outer radius corresponds to the black hole horizon. For the horizon to exist, the following conditions must hold:

$$M > 0, \quad |J| \leqslant Ml. \tag{2.36}$$

If these conditions are not satisfied, the solution contains a naked singularity. In supergravity we are often interested in extremal solutions, this is the case when $|J| = Ml$. In the extremal case both radii for which $N(r)$ vanishes coincide.

It should be noted that the radius of curvature l, plays an important role in this solution. In this solution the mass is dimensionless, and l is the length scale used to define a horizon. Furthermore, it was shown in [32] that the BTZ black hole can be the result of collapsing matter.

2.1.7 *Black hole thermodynamics*

There are similarities between properties of black hole mechanics and the laws of thermodynamics, a review of the topic is presented in [33]. Here we shall limit ourselves to a quick description of the results.

They can be summarized as

- *Zeroth Law:* The surface gravity k, of a black hole is uniform over the black hole horizon.

- *First Law:* Variations in black hole energy E, are related to a change in angular momentum J, charge Q and area A as

$$dE = \Omega_{BH} dJ + \Phi_{BH} dQ + \frac{\kappa}{8\pi} dA, \tag{2.37}$$

  where Ω_{BH} is the angular velocity and Φ_{BH} is the electrostatic potential energy.

- *Second Law:* The area of the black hole horizon is non-decreasing as a function of time:

$$dA \geqslant 0. \tag{2.38}$$

- *Third Law:* $\kappa = 0$ cannot be reduced to zero by a finite sequence of operation.

Here the surface gravity at the horizon is defined as

$$k^{\mu} \nabla_{\mu} k^{\nu} = \kappa k^{\nu}, \tag{2.39}$$

where k^{μ} is a normalized Killing vector [8].

Let us make the identifications

$$S_{BH} = \frac{A}{4G_N}, \tag{2.40}$$

and [34]

$$T_{BH} = \frac{\kappa}{2\pi G_N}. \tag{2.41}$$

If we now look back at the dynamic properties of black holes as listed above, we can observe an analogy between the laws of black holes and the laws of thermodynamics by interpreting S_{BH} as the black hole entropy, and T_{BH} as the black hole temperature.

S_{BH} is referred to as the Bekenstein-Hawkking entropy [35, 36]. Here the entropy is described as a macroscopic property, but as we know from statistical physics, the entropy of a system is related to the number of microstates consistent with a macrostate. The microstates of black holes are studied in the context of branes in string theory, which was first done in [37] for certain five-dimensional extremal black holes. Since then, it has been done for a number of other black objects, and it is still an active area of research.

2.2 PLANE AND PP-WAVES

So far we have devoted most of our attention to the description of black holes and various other black objects. Nevertheless, another class of solutions that are interesting from a physical perspective are plane waves. These solutions describe electromagnetic or gravitational radiation. With the relatively recent observation of gravitational waves, new possibilities of studying the universe have become available. Besides the measurement of gravitational waves, we would like gain better understanding from a theoretical perspective as well. Therefore, this subsection introduces pp-wave spacetimes, as well the subset of plane wave solutions.

The class of pp-waves are characterized by a covariantly constant null vector field $\vec{k}$ describing plane-fronted waves. This means that the covariant derivative of $\vec{k}$ vanishes

$$\nabla \vec{k} = 0. \tag{2.42}$$

We call $\vec{k}$ the wave vector, and there exist hypersurfaces orthogonal to $\vec{k}$ which may be interpreted as a planar wave surface [38].

The class of pp-waves with constant curvature tensor at the surfaces orthogonal to the wave vector, are what we call plane waves.

Pp-wave spacetimes are often expressed in terms of Brinkmann coordinates. Defining the Killing vector $\vec{k} = \partial_v$, yields a metric of the form

$$ds^2 = -2du dv + H(u, \vec{x}) du^2 + |d\vec{x}|^2, \tag{2.43}$$

with $H(u, \vec{x})$ any smooth function. If $H(u, \vec{x})$ is a harmonic function with respect to the coordinates $\vec{x}$, the metric is a solution of the vacuum Einstein equations.

In the case where $H(u, \vec{x}) = b_{ij}(u) x^i x^j$, the corresponding metric describes a plane wave. If we wish for the plane waves to correspond to the transverse polarizations of gravitational waves, some restrictions on b_{ij} are needed as discussed in [8].

Let us consider a plane-wave in four spacetime dimensions, in which case

$$H(u, x_1, x_2) = a(u)(x_1^2 - x_2^2) + 2b(u) x_1 x_2 + c(u)(x_1^2 + x_2^2), \tag{2.44}$$

with a, b and c arbitrary functions of u. For H of the form (2.44) in the metric (2.43), a and b correspond to the 'plus'-polarization and the 'cross'-polarization modes of gravitational waves, respectively. c does not correspond to any polarization of the gravitational waves, and should therefore be set to zero.

3 ASPECTS OF SUPERGRAVITY AND D-BRANES

Currently our understanding of the most fundamental physics is described by two separate theories, both of which have been confirmed by experimental tests to work really well. One is the standard model of particle physics, described as a quantum field theory. The other is Einstein's theory of general relativity describing gravity, which is a classical theory. One might wonder if the fundamental mechanisms behind gravity and the standard model are entirely disconnected, or whether they are both embedded in a more fundamental theory. So the question to ask is if there is some way to introduce gravity into the standard model. An important role in approaching this question is fulfilled by supersymmetry, and particularly its gauged version: supergravity.

This chapter is intended to introduce the reader to some of the basic aspects of supergravity needed to understand the solutions discussed in this thesis. This section is largely inspired by [20], other sources providing an overview of the subject are e.g. [39, 40].

3.1 SUPERSYMMETRY

The standard model is invariant under certain spacetime, as well as internal transformations. An attempt to make a non-trivial connection between these two sets of symmetry algebra's leads to supersymmetric generators, here we will see how.

Starting from the set of spacetime symmetries:

$$G_{\text{Poincaré}} = \mathbb{R}^{1,3} \ltimes SO(1,3), \tag{3.1}$$

where the $\mathbb{R}^{1,3}$ symmetries correspond to translations in four dimensional spacetime, which are combined in a semi-direct product with the set of rotations and boosts forming the group of Lorentz transformations $SO(1,3)$. This combined set of transformations form what is called the Poincaré group, and any Lagrangian we write down for the standard model should be invariant under these spacetime transformations.

Another type of transformation that is present in the standard model are the internal transformations that do not have anything to do with spacetime but rather they are acting only on the fields. And so for the standard model it turns out that the set of internal symmetries is related to transformations in the Lie groups:

$$G_{\text{Internal}} = \underbrace{SU(3)}_{\text{Strong force}} \times \underbrace{SU(2) \times U(1)_Y}_{\text{Electroweak}} \xrightarrow{\text{Higgs}} SU(3) \times U(1)_{\text{EM}}. \tag{3.2}$$

Forcing the Lagrangian to be invariant under these transformations is actually a source of the interactions in the standard model and as such we can identify the $SU(3)$ gauge symmetry with the strong interactions, and $SU(2) \times U(1)_Y$ is the unified electroweak interaction. Below a critical temperature well above the temperature natural to our current universe, this symmetry is broken by the Higgs mechanism, and as a result the symmetry group that we observe is the strong interactions and the $U(1)_{EM}$ which corresponds to electromagnetism.

If we look at the semidirect product of spacetime generators, they have a Lie algebra given by the following set of commutators:

$$[P_\mu, P_\nu] = 0, \tag{3.3}$$
$$[M_{\mu\nu}, P_\rho] = P_\mu \eta_{\nu\rho} - P_\nu \eta_{\mu\rho}, \tag{3.4}$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}, \tag{3.5}$$

where the generators $M_{\mu\nu}$ correspond to Lorentz transformations and the generators P_μ to spacetime translations. For the internal groups we have Lie algebras

$$SU(3): \quad [t_A, t_B] = f_{AB}{}^C t_C \tag{3.6}$$
$$SU(2): \quad [t_A, t_B] = \epsilon^{ABC} t_C \tag{3.7}$$
$$U(1): \quad [t_A, t_B] = 0. \tag{3.8}$$

The standard model in its current form was finalized in the 1970's, but even before that people were already wondering whether it would be possible to combine spacetime and internal symmetries. Writing out the standard model group in full reads:

$$G_{\text{Standard Model}} = G_{\text{Poincaré}} \times G_{\text{Internal}}. \tag{3.9}$$

All commutators of generators between the internal and spacetime transformations vanish. Hence the spacetime and internal transformations are completely disconnected. If we wanted them to be connected in some way, at least one of the commutators has to be non-vanishing. People were attempting this until Coleman and Mandula came up with their no-go theorem in [41]. In this paper they claim that it is impossible to cobine spacetime and internal symmetries in "any but a trivial way".

However, this no-go theorem is based on a set of assumptions, one of them being that the symmetries in question are based on commutators of operators. Another way to think about this would be that the Coleman-Mandula theorem assumes bosonic symmetries. However, in 1975 Haag Łopuszański and Sohnius showed in [42] that by generalizing the theorem to include anticommuting symmetry generators, there is the possibility of a nontrivial extension of the Poincaré algebra, which we now know as the supersymmetric algebra. These anti-commuting operators satisfy what is known as Clifford algebra instead of Lie algebra relations. This leads to the following extension of the standard model algebra relations:

$$\begin{aligned} [Q_\alpha, P_\mu] &= 0, \\ \{Q_\alpha, \bar{Q}^\beta\} &= -\frac{1}{2}(\gamma_\mu)_\alpha{}^\beta P^\mu, \\ [M_{\mu\nu}, Q_\alpha] &= -\frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta. \end{aligned} \tag{3.10}$$

where Q is a spinor supercharge generating supersymmetric transformations, and $\bar{Q}$ is its Dirac adjoint. Resulting in massless multiples containing fields with spins $(s, s-1/2)$. Given the Coleman-Mandula theorem this is the only way to extend the group of standard model symmmetries, meaning there is a uniqueness to this extension and we can only add something that corresponds to a fermionic transformation.

3.2 EXTENDED SUPERSYMMETRY

We can extend the supersymmetry algebra by adding more supersymmetry generators, this is what is referred to as 'extended supersymmetry'. The algebra of an extended supersymmetric theory with $\mathcal{N}$ generators is

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} \mathcal{Z}^{IJ}, \tag{3.11}$$

where $I, J \in [\mathcal{N}]$ and $\mathcal{Z}$ is an operator called the central charge of the algebra. The central charge is traceless and antisymmetric in the indices I and J. It is traceless because a supercharge has vanishing anticommutation relation with itself, this means that for $\mathcal{N} = 1$, the central charge vanishes. The 'central' charge obtained its name because it has vanishing commutation relations with all other operators.

Using the second equation in (3.10) we can determine the expectation value of $\{Q, \bar{Q}\}$ with respect to an arbitrary state. Moreover, in the massive representation in the rest frame we have $P_\mu = (M, \vec{0})$. Combining this with (3.11), results in Bogomol'nyi-Prasad-Sommerfield (BPS) bound $M \geqslant \frac{1}{2}|\mathcal{Z}^{IJ}|$ [43, 44]. It is common in literature to define $\mathcal{Z}^{IJ} \equiv \epsilon^{IJ}\mathcal{Z}$, in which case the BPS bound reads

$$M \geqslant |\mathcal{Z}|. \tag{3.12}$$

A detailed derivation of this bound will be present in any good set of notes on supersymmetry, see e.g. [45]. Solutions that saturate the bound (3.12), e.g. representations for which the mass is equal to one or more eigenvalues of the central charge, are known as BPS states. If the bound is saturated for all of the eigenvalues, the solution is called full-BPS and when the bound is saturated for half of the eigenvalues, it is called half-BPS. One should now be able to extrapolate the systematic approach to naming these solutions.

We know from the spin-statics theorem that a boson is an integer spin particle and a fermion is a particle with half-integer spin. The transformation with the operators Q, is actually acting with a half integer spin, meaning that it generates a symmetry between fermions and bosons. From the third equation in (3.10) we know that supersymmetry transformations are tied to translations, and thus gauging supersymmetry requires making the transformations by Q local. And again because of the relation between translation and supersymmetry transformations, gauging supersymmtry requires the gauging of translations as well. And this is where things become particularly interesting, since a theory of gauged translations is general relativity and thus gauging supersymmetry automatically results in a theory of gravity. This theory precisely what we call supergravity.

Because the supersymmetry transformation only changes spin, the mass and all the other quantum numbers have to be the same for a particle and its superpartner. However, so far we have not observed these superpartners. A possible explanation for this lack of observations can be the spontaneous breaking of supersymmetry, and if this breaking of supersymmetry happens together with the symmetry breaking of the standard model, the Higgs mechanism could give mass to the superpartners that are larger than the experimental bounds, of which the current state of the art is set by the Large Hadron Collider at Cern.

One might wonder if there is some limit to the number of supercharges a theory can have. And indeed there does turns out to be such a bound. The reasoning is roughly that no particles with spin $s > 2$ are known. This means that we wish the helicity λ of the particles to be bounded to values ranging from -2 till 2. In fact a supersymmetry transformation can be interpreted as changing the helicity of a state, rather than the spin, the difference being that helicity contains information on whether the spin is aligned with the momentum. Transformations as a result of acting with a supercharge change the helicity by $1/2$, and it will thus take eight supercharges to combine states with helicity -2 and 2. As a result we say that there is a bound on the number of supercharges $\mathcal{N} \leqslant 8$.

3.3 GAUGING SUPERSYMMETRY

We have thus far introduced the concept of supersymmetry, which is a global symmetry. However, we also hinted that an attempt to gauge a supersymmetric theory would lead us towards a theory of gravity. In this section we will go somewhat more in-depth on the formulation of a local theory of supersymmtetry: supergravity.

Introducing the gravitino

The parameter of the transformations of supersymmtery is a constant spinor which we will denote ϵ_α. An attempt to formulate a supersymmetric gauge theory leads to the requirement that the spinor has to be spacetime dependent $\epsilon_\alpha(x)$. The locality property also means that the fermionic supersymmteric parameters are gauged, and this introduces an associated spin 3/2 gauge field, $\psi_{\mu\alpha}$, this is called the Rarita-Schwinger field and was first introduced in [46]. In the context of supergravity it is however called the gravitino, since it is the superpartner of the graviton.

considering the free limit, in which there is no interaction, we take $\psi_{\mu\alpha}(x)$ to be a free field, transforming under a gauge transformation as

$$\psi_{\mu\alpha}(x) \to \psi_{\mu\alpha}(x) + \partial_\mu \epsilon_\alpha(x). \tag{3.13}$$

We now need to find an action which is invariant under this tranformation, as well as first order in derivatives since the gravitino is a spinor. The action for this field is the Rarita-Schwinger action:

$$S_{3/2} = -\int d^D x \bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho. \tag{3.14}$$

Fundamental spinor

The transformation parameter of supersymmetry is the spinor ϵ_α. In general the spinor associated with a certain supersymmetric theory in D dimensions, is the one with the least possible components in that dimension. In the six-dimensional theory, this is the symplectic Majorana-Weyl spinor. And since Weyl spinors are Majorana spinors with well-defined chiral projections (where the Weyl spinor can be either left-chiral or right-chiral), we sometimes denote the theory by (l, r). Here l and r are the number of right-chiral and left-chiral spinors, respectively. In this thesis we study minimal supergravity in six dimensions. The addition 'minimal', means that $\mathcal{N} = 1$, which is often denoted $\mathcal{N} = (1, 0)$ in the case of a six-dimensional theory. For more information on spinors and their representation in various dimensions, see appendix D.

3.4 $D = 4$, $\mathcal{N} = 1$ SUPERGRAVITY

In order to obtain a theory of supergravity we need some way to couple the spinors to curved spacetime. We first define the gamma matrices on curved spacetime by relating them to the gamma matrices in Minkowski spacetime by making use of the vielbein:

$$\gamma^\mu(x) \equiv e^\mu_\alpha(x)\gamma^\alpha. \tag{3.15}$$

For an introduction to the vielbein formalism, and how it is used to couple spinors to curved backgrounds, see appendix D.2

Furthermore, we replace the partial derivative in (3.14) by a covariant derivative, which results in the following equation of motion:

$$\gamma^{\mu\nu\rho} \nabla_\nu \psi_\rho = 0. \tag{3.16}$$

This covariant derivative acts as

$$\nabla_\mu \psi_\nu = \partial_\mu \psi_\nu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\psi_\nu - \Gamma^\rho_{\mu\nu}\psi_\rho, \tag{3.17}$$

where $\omega_{\mu ab}$ is the spin connection, and $\Gamma^{\rho}_{\mu\nu}$ is the affine connection (a coordinate independent notion of the Christoffel symbols). Because the affine connection is symmetric in its lower indices μ and ν, it will vanish upon contraction of indices with the antisymmetric gamma matrix. This allows us to simplify the equation of motion of the gravitino somewhat, and it has to obey

$$\gamma^{\mu\nu\rho}D_\mu\psi_\nu = \gamma^{\mu\nu\rho}\left(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\right)\psi_\nu = 0. \tag{3.18}$$

The field content of the $D = 4$, $\mathcal{N} = 1$ theory consists only of the graviton and the gravitino, of which the transformations rules are [47]

$$\delta e^a_\mu = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \tag{3.19}$$

$$\delta\psi_\mu = D_\mu\epsilon(x) = \partial_\mu\epsilon + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\epsilon. \tag{3.20}$$

The action invariant under these transformations is

$$S = S_2 + S_{3/2}, \tag{3.21}$$

with

$$S_2 = \int d^Dx e e^{a\mu}e^{b\nu}R_{\mu\nu ab}(\omega) = \int d^Dx e R(e,\omega), \tag{3.22}$$

and $S_{3/2}$ as given in (3.14), which in curved spacetime becomes

$$S_{3/2} = -\int d^Dx\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\mu\nu\psi_\rho. \tag{3.23}$$

Constructing a solution

Let us consider the simplest classical solution of $\mathcal{N} = 1$, $D = 4$ supergravity, which is Minkowski spacetime. Therefore, we have $g_{\mu\nu}|_0 = \eta_{\mu\nu}$, and vanishing gravitino field $\psi_\mu|_0 = 0$. This simplifies the transformations rules (3.20) and (3.20), which in this case read

$$\delta e^a_\mu = 0, \tag{3.24}$$

$$\delta\psi_\mu = \partial_\mu\epsilon, \tag{3.25}$$

since $\psi_\mu|_0 = 0$ and in Minkowski space the spin connection vanishes. The residual global supersymmetric algebra is determined by the vanishing of these variations, and the requirement $\delta e^a_\mu = 0$ and $\delta\psi_\mu = 0$ mean that the transformations do not change the background fields.

In this case the requirements for the preservation of supersymmetry have thus simplified to $\partial_\mu\epsilon = 0$, which is solved for all four linearly independent equations if ϵ_α is a set of four linearly independent Majorana spinors. In analogy with Killing vectors, which represent isomteries of spacetime, these spinors are called 'Killing spinors'.

Using a combination of two such Killing spinors ϵ and ϵ', we can construct the spinor bilinear $(\bar{\epsilon}'\gamma^\mu\epsilon)$, which transforms as a Killing vector under Lorentz translations. Spinor bilinears are useful objects in the construction of a metric of the supersymmetric solution.

The total number of real components of supercharges in a theory is given by the number of real components in a spinor supercharge Q_α multiplied by the number $\mathcal{N}$ of supercharges. This statement may seem superfluous, but the number of real components of supercharges in a theory is often used to express how much of the supersymmetry is conserved by considering the fraction of linearly independent Killing spinors in a theory relative to the total number of real components of supercharges in a theory. The solution of Minkowski space studied above is called fully supersymmetric, as there are four linearly independent Killing spinors and thus four out of four supercharges are conserved.

Black holes in extended supergravity

Extended supergravities in four dimensions contain $U(1)$ gauge fields, called graviphotons. These correspond to a transformation of the central charge. If we consider the graviphoton to fulfill the role of the electromagnetic potential in the Einstein-Maxwell theory, we have that the central charge is [29]:

$$G_n \mathcal{Z}^2 = Q^2 + P^2. \tag{3.26}$$

In subsection 2.1.1 we mentioned that the Reissner-Nordström black hole is extremal if $G_N M^2 = P^2 + Q^2$ is satisfied. If a solution is supersymmetric, the BPS bound (3.12) is satisfied. Combining the BPS bound with (3.26), allows us to conclude that supersymmetric black holes are extremal.

3.5 BRANES

We have previousely in subsection 2.1.1 introduced the extremal Reissner-Nordström black hole, and in subsection 2.1.5, we hinted at the existence of higher dimensional black hole solutions. In this section we will introduce the p-brane, which are p-dimensional surfaces one can think of as a generalized version of a black hole. These p-branes are supersymmetric solutions in supergravity, and they turn out to correspond to the low energy limit of D-branes in string theory [48]. We can also introduce fields into the action, here we will only discuss the bosonic field content since this is sufficient to develop the ideas, and it is a rather straightforward exercise to complement the bosonic fields with fermionic fields through supersymmetric transformations. In what follows we will limit our attention to massless modes of the string, since massive excitation are at the Planck scale. Many of the results in this subsection are more thoroughly discussed in e.g. [49–51].

3.5.1 *Elementary string theory*

The principle concept of string theory is that the particles described in the standard model are not the most fundamental particles of nature, but instead they are composed of one-dimensional objects we call strings. The dynamics of a string in a curved spacetime background with metric $G_{\mu\nu}(X)$ is described by the action

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X), \tag{3.27}$$

called the 'non-linear sigma model' [49]. Which is a rather straightforward generalization of the Polyakov action. Here the coordinates on the string world sheet Σ are $\sigma^a = (\tau, \sigma)$, $a = 0, 1$, and $h_{\alpha\beta}$ is the worldsheet metric. X^μ are the coordinates of the string in spacetime, and α' is the only independent dimensionful parameter in string theory equal to the string length squared. Hence this action provides a map from the worldsheet of the string into a spacetime with metric $G_{\mu\nu}(X)$.

When we say 'the only dimensionful parameter', we of course mean in addition to the natural units we set to 1 in our convention.

We also have to impose boundary conditions, for which we have two options. The first one is the Neumann boundary condition

$$\partial_\sigma X^\mu|_{\partial\Sigma} = 0, \tag{3.28}$$

which corresponds to open strings.

The second option, is to impose the Dirichlet boundary condition

$$\partial_\tau X^\mu|_{\partial\Sigma} = 0, \tag{3.29}$$

which fixes the endpoints of the string at some constant position in space. Notice however, that momentum is not conserved at the endpoints. This suggest that the strings are coupled to other objects, which are called D-branes, where D stands for Dirichlet. A p-dimensional D-brane is usually called a Dp-brane. The existence of these objects implies that string theory is not just a theory of strings in a vacuum background, but instead the theory also contains these higher dimensional branes.

3.5.2 *Type IIB supergravity*

Those who have studied string theory might know that we distinquish five different types of string theory: Type I, heterotic $SO(32)$, heterotic $E_8 \times E_8$, as well as type IIA and type IIB. Witten found that all these different theories can arise as different limits of what he called M-theory, and they are related through S-dualities or T-dualities [52].

In this thesis we are interested in solutions of six-dimensional $(1,0)$ supergravity. Which can be obtained though the compactification of F-theory on an elliptically fibered Calabi-Yau threefold [53]. F-theory is effectively a twelve dimensional theory which provides a geometric interpretation to type IIB string theory with D7-branes [54]. The reason to study this configuration in F-theory is that the back-reaction of the D7-brane is strong and as a result cannot be studied perturbatively in ten dimensional string theory. For this reason we will focus our discussion on type IIB solutions.

Type II string theories, are theories describing closed strings, and we distinguish furthermore type IIA (non-chiral) and type IIB (chiral) string theories. Both of which are maximally supersymmetric.

Open strings, such as they appear in type I theory, can either have anti-periodic boundary conditions, in which case we say it lives in the Ramond (R) sector, or it can have periodic boundary conditions, in which case we say it is in the Neveu-Schwarz (NS) sector. We can combine these open strings to obtain closed strings, which as a result can live in the R-R, R-NS, NS-R or NS-NS sector. It turns out that the R-R and NS-NS sectors describe bosonic strings, whereas the R-NS and NS-R sectors describe fermionic strings.

The NS-NS sector of the massless spectrum of type II theories contains the graviton $G_{\mu\nu}$, a two-form $B_{\mu\nu}$, and a dilaton ϕ, this holds for both type II supergravity and type II string theory. Furthermore, in the R-R sector type IIA theories contain p-forms A_p with an odd rank (p) whereas a type IIB theory contains forms with an even rank: A_0, A_2 and A_4.

The bosonic part of the type IIB supergravity action reads [55]

$$S_{IIB} = S_{NS} + S_{R(IIB)} + S_{CS(IIB)}, \tag{3.30}$$

$$2\kappa^2 S_{NS} = \int d^{10}x\sqrt{-g}e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2\right), \tag{3.31}$$

$$2\kappa^2 S_{R(IIB)} = -\frac{1}{2}\int d^{10}x\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right), \tag{3.32}$$

$$2\kappa^2 S_{CS(IIB)} = -\frac{1}{2}\int C_4 \wedge H_3 \wedge F_3, \tag{3.33}$$

where $F_1 = dA_0$, $H_3 = dB_2$, and the field strengths in the Chern-Simons term $S_{CS(IIB)}$ read

$$\tilde{F}_3 = dC_2 - C_0 \wedge H_3, \tag{3.34}$$

$$\tilde{F}_5 = dC_4 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3. \tag{3.35}$$

Finally, the self duality constraint

$$\tilde{F}_5 = \star\tilde{F}_5, \tag{3.36}$$

does not automatically follow from the action and have to be imposed as an addition condition.

3.5.3 *p-branes in supergravity*

An interesting set of solutions to the equations of motion of the type II action (3.30) are what we call p-branes, these are hypersurfaces which extend in p+1 spacetime dimensions. p-branes couple to (p+1)-rank gauge field A_{p+1} as

$$\int d^{p+1}\sigma \partial_{\alpha_1} X^{\mu_1} \dots \partial_{\alpha-p+1} X^{\mu_{p+1}} A_{\mu_1 \dots \mu_{p+1}} \epsilon^{\alpha_1 \dots \alpha_{p+1}}, \tag{3.37}$$

which is a generalization of the way in which a point particle couples to a one-form gauge field, or $B_{\mu\nu}$ couples to a string worldstheet. Again analogous to the case of a point particle, the electric charge Q of a p-brane can be determined using

$$Q \sim \int_{S^{D-p-2}} \star F_{p+2}, \tag{3.38}$$

and the magnetic charge P by

$$P \sim \int_{S^{p+2}} F_{p+2}. \tag{3.39}$$

The charges have to satisfy

$$QP = 2\pi n, \quad n \in \mathbb{Z}, \tag{3.40}$$

which is a generalization of the Dirac quantization condition for electric and magnetic monopoles discussed in subsection 2.1.2.

Extremal p-branes are 1/2 BPS, and their solution reads

$$\begin{cases} ds^2 = H_p^{-1/2}(dt^2 + dx_i dx^i) + H_p^{1/2} dy_j dy^j, \\ e^{2\phi} = H_p^{(3-p)/4}, \\ A_{0,\dots p} = H_p^{-1} - 1, \end{cases} \tag{3.41}$$

where the x^i coordinates lie parallel to the brane, and the y_j coordinates are transverse to the brane direction. Moreover, H_p is a harmonic function, which for $7 - p > 0$, is given by

$$H_p = 1 + \left(\frac{Q_p}{r}\right)^{7-p}. \tag{3.42}$$

Here Q_p is an integration constant related to the charge of the p-brane, and it reads

$$Q_p = (4\pi)^{(5-p)/2} g_s \alpha'^{(7-p)/2} N, \tag{3.43}$$

where N denotes the number of branes.

Remember this discussion of the p-brane is prefaced by the statement that they are used in the construction of black holes. One of the properties that enables us to do this, is the fact that a p-brane has an event horizon at $r = 0$, this is also the reason why p-branes are sometimes referred to as black branes.

D-branes in string theory

We previously claimed that extremal p-branes in supergravity, are the low energy limit of D-branes in string theory, and hence they actually describe the same object [56]. At first sight p-branes, which are solutions of supergravity, and D-branes, which are solutions of perturbative string theory may seem unrelated as they appear in two different places. However, D-branes are charged under the R-R fields of type II string theory [48], which is what inspired Polchinski to make the identification of D-branes with R-R charged extremal p-branes, and in doing so sparking (together with other works) the 'second superstring revolution'.

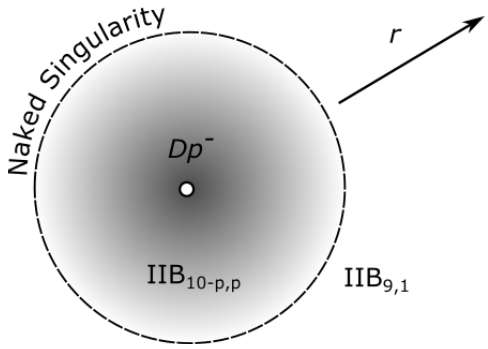


Figure 3.1: A Dp^- brane surrounded by a naked curvature singularity at a finite radius. The negative brane induces metric signature $(10-p,p)$ for transverse direction r smaller than the radius of the singularity.

3.5.4 *Negative branes*

The concept of negative branes is introduced in [57], and the basic idea of those branes is that the introduction of a stack of N Dp-branes results in a background solution similar to (3.41):

$$ds^2 = H(r)^{-1/2} ds^2_{p+1} + H^{1/2} ds^2_{9-p} \tag{3.44}$$

$$H(r) = 1 + \frac{(2\sqrt{\pi} l_s)^{7-p}\Gamma(\frac{7-p}{2})}{4\pi}\frac{N}{r}, \tag{3.45}$$

where we now have $N = N_+ - N_-$. If we consider a solution with $N_- > N_+$, $H(r) = 0$ at $r = r_s > 0$, resulting in a naked singularity at $r = r_s$. This is a result of the negative tension of a negative brane, and it is analogue to the naked curvature singularity that occurs for a Schwarzschild black hole with negative mass. However, we can analytically continue the metric as a function of H, avoiding the singularity by encircling $H(r) = 0$ in the complex plane. This results in the following metric beyond the singularity:

$$ds^2 = i^{-1}\bar{H}(r)^{-1/2} ds^2_{p+1} + i\bar{H}^{1/2} ds^2_{9-p}, \tag{3.46}$$

where $\bar{H} \equiv -H$. Now we can use a Weyl transformation to obtain a real metric:

$$ds^2 = -\bar{H}(r)^{-1/2} ds^2_{p+1} + \bar{H}^{1/2} ds^2_{9-p}, \tag{3.47}$$

up to an arbitrary overall sign. The resulting spacetime is real, but it means that for $r < r_s$, the metric has signature $(10-p,p)$ instead of the usual $(9,1)$. A schematic representation of the effect of a negative brane on the spacetime signature is presented in figure 3.1.

We denote the metric signature as (s,t), with s the number of spacelike dimensions and t the number of timelike dimensions.

4 MINIMAL SIX-DIMENSIONAL SUPERGRAVITY

In this thesis we are interested in solutions of minimal six-dimensional supergravity, which will be introduce in this chapter. First, the theory will be described in terms of its field content and equations of motion, followed by the discussion of the Gibbons-Hawking class of solutions where we discuss the results obtained in [4]. Finally, a number of examples of possible solutions are presented.

4.1 SETTING

In this thesis we are particularly interested in minimal ($\mathcal{N} = (1, 0)$) supergravity. In supergravity the number of degrees of freedom of the bosonic sector has to be equal to that of the fermionic sector [20, Chapter 6]. Previously we presented the $\mathcal{N} = 1$, $D = 4$ theory of which the field content was made up only of the graviton and the gravitino, which in four dimensions both have 2 on-shell degrees of freedom. However, when going to six dimensions, as we will do in this thesis, the on-shell degrees of freedom of the graviton and gravitino are nine and twelve, respectively. In the case of minimal supergravity in six dimensional supergravity the supermultiplet also contains a two-form field $B_{\mu\nu}$, with self-dual field strength $G \equiv dB$ (The self duality condition is $G = \star G$). If we were to repeat the counting of the on-shell degrees of freedom with this two-form included, one would see that both the bosonic and the fermionic sector count twelve degrees of freedom.

This two-form has a particularly interesting property. It can be considered as a generalization of the electromagnetic potential, which is a one-form giving charge to a point particle. To obtain the contribution to the action of the electromagnetic potential, we have to integrate over a one-dimensional worldline, which corresponds to a point particle. Analogously, to obtain the contribution to the action of the two-form potential we have to integrate over a two-dimensional worldsheet, which corresponds to an object that extends in one spatial dimension (see (3.37)). This means that the two-form potential couples to a one-dimensional string. In subsection 3.5.2 we mentioned that the field content of a type II theory includes a two-form, and following the reasoning above we can interpret the string from string theory as the source for this two-form field. Essentially this six-dimensional theory is the simplest example of a string theory.

The field content of minimal supergravity in six dimensions consists of the graviton $g_{\mu\nu}$, a two-form $B_{\mu\nu}$ with self-dual field strength $G \equiv dB$, and a gravitino ψ^A_μ. Here A is an index in the representation $Sp(1)$, and moreover the gravitino is a left handed symplectic Majorana-Weyl spinor [58]. Where left handed implies that the helicity is negative, or $\gamma^7\psi = -\psi$.

When discussing the example of $\mathcal{N} = 1$, $D = 4$ supergravity in subsection 3.4, we mentioned that the requirement to preserve supersymmetry is given by the Killing spinor equation. In six-dimensional supergravity we have to take into account the supersymmetry variation of the two-form field as well, and as a result the supersymmetry condition reads [4]

$$\nabla_\mu \epsilon + \frac{1}{4} G_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon = 0. \tag{4.1}$$

The equations of motion of the bosonic fields are

$$G = \star G \tag{4.2}$$
$$\nabla_\mu G^{\mu\nu\rho} = 0, \tag{4.3}$$
$$R_{\mu\nu} = G_{\mu\rho\sigma} G_\nu{}^{\rho\sigma}, \tag{4.4}$$

where, (4.2) and (4.3) can be rewritten as

$$dG = 0. \tag{4.5}$$

Using (4.4), we observe that the Ricci scalar is given by $R = R_\mu{}^\mu = G_{\mu\nu\rho} G^{\mu\nu\rho}$. In differential form we can write a similar expression:

$$G \wedge \star G = \frac{1}{3!} G_{\mu\nu\rho} G^{\mu\nu\rho} \mathrm{vol}_6 = \frac{1}{3!} R \, \mathrm{vol}_6, \tag{4.6}$$

where the left hand side vanishes due to the self duality of condition. Hence it follows that

$$R = 0, \tag{4.7}$$

in which case Einstein's field equations (2.2) read

$$R_{\mu\nu} = G_{\mu\rho\sigma} G_\nu{}^{\rho\sigma} = 8\pi G_N T_{\mu\nu}. \tag{4.8}$$

Given a spinor satisfying (4.1), one can construct spinor bilinears, which are [4]

$$V_\mu \epsilon^{AB} = \bar{\epsilon}^A \gamma_\mu \epsilon^B, \tag{4.9}$$
$$\Omega^{AB}_{\mu\nu\rho} = \bar{\epsilon}^A \gamma_{\mu\nu\rho} \epsilon^B, \tag{4.10}$$

obeying a number of constraints.

Using Fierz identities, some algebraic constraints can be obtained, specifically V has to be a null Killing vector field and the Killing spinor has to obey the projection

$$V \cdot \gamma \epsilon = 0. \tag{4.11}$$

Furthermore, the three forms Ω^{AB} induce an almost hyper-Kähler structure on a four-dimensional base space.

Supersymmetric solution in a coordinate dependent basis

The projection (4.11) and further differential constraints on the vector V and three-forms composing Ω^{AB}, imply that a basis can be chosen such that the equation for the Killing spinor reduces to

$$\partial_\mu \epsilon = 0. \tag{4.12}$$

As (4.11) is the only projection, a supersymmetric solution has to preserve either one half or all of the supersymmtry. Introducing coordinates, in which $V = \partial_v$, the six dimensional metric can be written as [4]

$$ds^2 = 2H^{-1}(du + \beta)\left(dv + \omega + \frac{F}{2}(du + \beta)\right) + H h_{mn} dx^m dx^n, \tag{4.13}$$

where h_{mn} will be referred to as the four dimensional base space, with β and ω one forms on this base space, and H and F are arbitrary functions. These elements can all depend on u and the base space coordinates x^i, but not on v since ∂_v is a Killing vector.

Similarly the expression for the three-form G can also be given in terms of these coordinates. However, as it is not very enlightening, we do not present it here.

4.2 THE GIBBONS-HAWKING CLASS OF SOLUTIONS

The general solutions simplify considerably when there is no dependence on u. Solutions of this type can be characterized by the existence of a Killing vector ∂_u. If we furthermore assume that ∂_u preserves the three forms Ω, there is no more u-dependence in our equations. In this case the base space is hyper-Kähler. The one-form β has self-dual curvature on the base, and metric reads

For a discussion on hyper-Kähler manifolds, and their application to supersymmetric solutions, see appendix B.3

$$ds^2 = -2H^{-1}(du+\beta)\left(dv+\omega-\frac{F}{2}(du+\beta)\right)+H ds^2_{HK_4}. \tag{4.14}$$

In order to find solutions satisfying the superposition principle expected of BPS objects, we study the most general hyper-Kähler manifold admitting a Killing vector field ∂_ψ. This are solutions of a hyper-Kähler base space with an extra $U(1)$ isometry called a Gibbons-Hawking space [59].

The metric of the Gibbons-Hawking base space reads

$$ds^2_{GH} = V_1^{-1}(d\psi+\chi)^2 + V_1 ds^2_{\mathbb{R}^3}, \tag{4.15}$$

and the expression for the three-form is [4]

$$\begin{aligned} G =&\frac{1}{2}\star_4 dH - \frac{1}{2}H^{-1}(du+\beta)\wedge\left(d\omega^- + \frac{1}{2}Fd\beta\right) \\ &+\frac{1}{2}(dv+\omega)\wedge d\left(H^{-1}(du+\beta)\right), \end{aligned} \tag{4.16}$$

where $\star_4$ is the Hodge star in the four-dimensional base space, and $d\omega^\pm = \frac{1}{2}(d\omega \pm \star_4 d\omega)$.

The coefficients and forms present in (4.14), (4.15) and (4.16), can be written in terms of six harmonic functions on $\mathbb{R}^3$ denoted by V_i, where $i \in [6]$. They are defined as [4]

$$\beta = \frac{V_2}{V_1}(d\psi+\chi)+\bar{\beta}, \tag{4.17}$$

$$\omega = \left(V_4 + \frac{V_6V_3+V_2V_5}{V_1} + \frac{V_2V_3^2}{V_1^2}\right)(d\psi+\chi)+\bar{\omega}, \tag{4.18}$$

$$F = 2V_5 + \frac{V_3^2}{V_1}, \tag{4.19}$$

$$H = V_6 + \frac{V_2V_3}{V_1}, \tag{4.20}$$

where

$$*_3 d\chi = dV_1, \quad *_3 d\bar{\beta} = -dV_2, \quad *_3 d\bar{\omega} = \langle \mathbb{V}, d\mathbb{V}\rangle. \tag{4.21}$$

Here $\mathbb{V} \in \mathbb{R}^6$ is a vector $\mathbb{V}_i \equiv V_i$, and

$$\langle A, B\rangle \equiv A^T\Omega B, \tag{4.22}$$

with Ω a skew-symmetric matrix, is the symplectic norm on $\mathbb{R}^6$. Properties of the symplectic group are discussed in section 5.

We will mostly be interested in solutions characterized by harmonic functions of the form $V_i = \alpha + \frac{\beta}{r}$, where α and β are real constants and $r^2 = x_i x^i$ with $i \in [3]$ is the radial coordinate in the $\mathbb{R}^3$ section of the base space. We will denote the vector

$$\mathbb{V} = \Gamma_\infty + \frac{\Gamma}{r}, \tag{4.23}$$

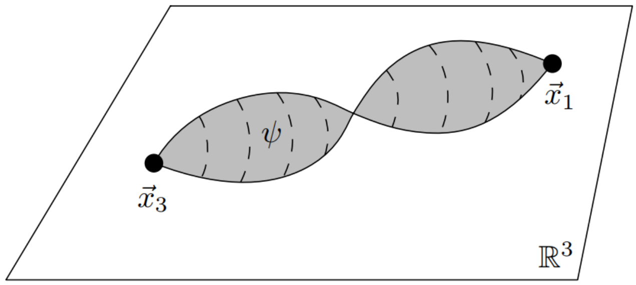


Figure 4.1: A schematic of the Gibbons-Hawking base space consisting of a Euclidean space $\mathbb{R}^3$ with a $U(1)$ fibre corresponding to the coordinate ψ, spanning curves between the sources of the function V_1 [5].

describing a point-like source at $r = 0$ with residues of the harmonic functions (charges)

$$\Gamma = (m, q, p, j, n, \mu) \tag{4.24}$$

and moduli specified by

$$\Gamma_\infty = (m_\infty, q_\infty, p_\infty, j_\infty, n_\infty, \mu_\infty). \tag{4.25}$$

In this thesis we are interested in single centered solution, however one can also aspire to analyze solutions with multiple centers. This has for example been done to study bound states of black holes in [5]. In this case (4.23) can be generalized to

$$\mathbb{V} = \Gamma_\infty + \sum_a \frac{\Gamma_a}{|\vec{x} - \vec{x_a}|}, \tag{4.26}$$

with $\vec{x}$ denoting points in the $\mathbb{R}^3$ section of the base space.

4.2.1 *Classifications of Gibbons-Hawking space*

The exact form of the Gibbons-Hawking metric is determined by the harmonic function V_1, which we will mostly consider to be single centered, which means that it is of the form $V_1 = m_\infty + \frac{m}{r}$. Close to a center the Gibbon-Hawking metric is $\mathbb{R}^4/\mathbb{Z}_{|m|}$, which means that $m \in \mathbb{Z}$. Possible spaces that can be obtained are flat space ($V_1 = \frac{1}{r}$, $V_1 =$ constant), Eguchi-Hanson space ($V_1 = \frac{m}{r}$) [60] and Taub-NUT space $\left(V_1 = m_\infty + \frac{m}{r}\right)$ [61, 62].

Eguchi-Hanson space is an example of asymptotically locally Euclidean (ALE) space, and Taub-NUT is an example of a asymptotically locally flat (ALF) space, these are asymptotic to $\mathbb{R}^4/\mathbb{Z}_m$ and $\mathbb{R}^3 \times S^1$, respectively. The S^1 fibre of an ALF solution is often a modulus of the solution, meaning that it can be altered freely. In the case where S^1 is large compared to the size of the area in which the sources are positioned, the base space can be considered locally flat. On the other hand, if S^1 is small the base is effectively three-dimensional. An example of a Gibbons-Hawking base corresponding to a description of two black holes, and a smooth center is shown in figure 4.1.

Below we list some explicit examples.

Flat space

Four dimensional Euclidean space $\mathbb{R}^4$ is obtained for $V_1 = \frac{1}{r}$, which gives the metric

$$ds^2_{\mathbb{R}^4} = r(d\psi + (1 + \cos\theta)d\phi)^2 + \frac{1}{r}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2). \tag{4.27}$$

Redefining the coordinates

$$r = \frac{\rho^2}{4}, \quad \theta = 2\theta', \quad \psi = 2\psi', \quad \phi = -\psi' - \phi', \tag{4.28}$$

results in the line element

$$ds^2_{\mathbb{R}^4} = d\rho^2 + \rho^2(d\theta'^2 + \sin^2 d\psi'^2 + \cos^2\theta' d\phi'^2), \tag{4.29}$$

which is a direct product $\mathbb{R}^3 \times S^1$.

Another choice of V_1 resulting in a (locally) flat base space is $V_1 = m_\infty$. In this case the metric reads

$$ds^2 = m_\infty^{-1} d\psi^2 + m_\infty |d\vec{x}|^2. \tag{4.30}$$

Which effectively describes a dimensional reduction of the ψ-coordinate on a circle.

Eguchi-Hanson space

Eguchi-Hanson space is an example of an asymptotically locally Euclidean (ALE) space. For $V_1 = \frac{m}{r}$, the metric is a metric on $\mathbb{R}^4/\mathbb{Z}_m$. We can see this by plugging $V_1 = \frac{m}{r}$ into the metric (4.15), which yields

$$ds^2 = \frac{r}{m}(d\psi + m\cos\theta)^2 + \frac{m}{r}\left(dr^2 + r^2 d\Omega_2^2\right). \tag{4.31}$$

Which we can rewrite to obtain

$$ds^2 = \frac{\rho^2}{4}\left(\frac{d\psi}{m} + \cos\theta\right)^2 + d\rho^2 + \frac{\rho^2}{4} d\Omega_2^2, \tag{4.32}$$

where we redefined $r = \frac{\rho^2}{4\sqrt{m}}$.

Taub-NUT space

Another interesting possibility for the base space is obtained when taking $V_1 = m_\infty + \frac{m}{r}$, in this case the base space metric is the Taub-NUT metric, where m is called the NUT charge. Taub-NUT space is an example of an asymptotically locally flat (ALF) space. The Taub-NUT metric reads

$$ds^2_{TN} = \left(m_\infty + \frac{m}{r}\right)^{-1}(d\psi + m\cos\theta d\phi)^2 \tag{4.33}$$

$$+ \left(m_\infty + \frac{m}{r}\right)(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2). \tag{4.34}$$

Close to the center ($r \to 0$) Taub-NUT locally looks like $\mathbb{R}^4/\mathbb{Z}_m$, which implies that $m \in \mathbb{Z}$ and that for $m = 1$ the metric locally describes $\mathbb{R}^4$.

Asymptotically ($r \to \infty$) the Taub-NUT metric is locally $\mathbb{R}^3 \times S^1$, which is an interesting property since this allows us to reduce the metric along this asymptotic S^1.

EXAMPLES OF GIBBONS-HAWKING CLASS SOLUTIONS

In later chapters we will be interested in classifying solutions of spacetimes. To do so one should of course be able to recognize different well-known spacetimes, and therefore we list a selection of several known classes of solutions below.

4.2.2 *Vacuum solutions*

An interesting class of solutions to Einstein's equations, are the vacuum solutions. These are Ricci-flat solutions for which the three-form (4.16) vanishes. For $G = 0$ we require

$$dH = d\beta = d\omega^- = 0, \tag{4.35}$$

which places some constraints on the harmonic functions [6]:

$$V_2 = \nu_1 V_1, \quad V_3 = \nu_3 - \frac{\nu_4}{\nu_6} V_1, \quad V_4 = \nu_4 - \nu_1 V_5, \quad V_6 = \nu_6 - \nu_1 \nu_3 + \nu_1 \frac{\nu_4}{\nu_6} V_1., \tag{4.36}$$

with constants $\nu_i \in \mathbb{R}$.

This description contains some gauge freedom as represented by the group element 5.26 discussed in section 5.4, which we can use to find that another set of restrictions on the harmonic functions describing a vacuum solution is [6]:

$$V_2 = 0, \quad V_3 = -\frac{\nu_4}{\nu_6} V_1, \quad V_4 = \nu_4, \quad V_6 = 1. \tag{4.37}$$

In which case $H = 1$ and $\beta = \omega = 0$, resulting in a metric of the form

$$ds^2 = -2du dv + F du^2 + V_1^{-1} (d\psi + \chi)^2 + V_1 |d\vec{x}|^2, \tag{4.38}$$

where

$$F = 2V_5 + \frac{\nu_4^2}{\nu_6^2} V_1. \tag{4.39}$$

These sets of constraints on the harmonic functions can be used to identify vacuum solutions just by looking at the harmonic functions defining the solution.

4.2.3 *Flat space*

Above we have seen how the harmonic function V_1 characterizes the Gibbons-Hawking base space, and among other things we found that $V_1 = \frac{1}{r}$ and $V_1 =$ constant yield flat spaces. One rather obvious choice for harmonic functions describing flat space, and one that we will come back to later, is

$$\mathbb{V}_{flat} = \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{4.40}$$

The corresponding metric is of the form (4.14), with $\beta = \omega = F = 0$, $H = 1$ and $V_1 = \frac{1}{r}$, which reads

$$ds^2 = -2du dv + ds_{\mathbb{R}^4}. \tag{4.41}$$

We can now ask ourselves what the most general vector of harmonic functions, $\mathbb{V}$, describing flat space looks like. In order to find such a general vector, we have to determine when all components of the Riemann tensor vanishes. This computation has been performed numerically using the *xAct* toolbox for Wolfram

Mathematica [63]. For the two different characterizations of a flat base space we found

$$\mathbb{V}_{flat} = \begin{pmatrix} m_\infty \\ q_\infty \\ p_\infty \\ j_\infty \\ n_\infty \\ \mu_\infty \end{pmatrix}, \quad \mathbb{V}_{flat} = \begin{pmatrix} \frac{m}{r} \\ 0 \\ p_\infty \\ 0 \\ n_\infty \\ \mu_\infty \end{pmatrix}. \tag{4.42}$$

These are invariant under the $SL(2, \mathbb{R})$ gauge transformations discussed in section 5.4.

4.2.4 $AdS_3 \times S^3$

By acting on flat space one can generate a set of harmonic functions describing $AdS_3 \times S^3$, which results in the set of harmonic functions [6]:

$$\mathbb{V}_{AdS_3 \times S^3} = \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ n \\ \frac{\mu}{r} \end{pmatrix}. \tag{4.43}$$

which corresponds to the metric

$$ds^2 = -2\frac{r}{\mu} du(dv - n du) + \frac{\mu}{r^2} dr^2 + 4\mu d\Omega_3^2, \tag{4.44}$$

and the three-from reads

$$\begin{aligned} G =& \frac{1}{2\mu} dr \wedge du \wedge dv + \frac{1}{2}\mu \sin(\theta) d\theta \wedge d\phi \wedge d\psi \\ =& \left(\frac{1 + \star_6}{2\mu}\right) du \wedge dv \wedge dr. \end{aligned} \tag{4.45}$$

Notice that the three-form does not depend on n, and that it can be removed from the metric by a coordinate transformation $v' = v + nu$. This means that n is a redundant parameter in the description of this solution.

Let us now define a coordinate ρ as

$$r = \frac{4\mu^2}{\rho^2}. \tag{4.46}$$

The metric in terms of the new coordinates v' and ρ is

$$ds^2 = \frac{4\mu}{\rho^2}\left(-2du dv' + dr'^2\right) + 4\mu d\Omega_3^2, \tag{4.47}$$

which is the metric on $AdS_3 \times S^3$ as we can see by defining $v = (t + x)/\sqrt{2}$ and $u = (t - x)/\sqrt{2}$, and comparing the resulting metric to (C.4).

4.2.5 *BTZ black hole*

By turning on just one parameter in (4.43) we can obtain

$$\mathbb{V}_{\mathrm{BTZ}_k} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ n \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ k \\ \mu \end{pmatrix} \frac{1}{r}. \tag{4.48}$$

As a result of this extra parameter, the metric is has changed and it now reads

$$ds^2 = -\frac{2r}{\mu} du dv' + \frac{2k}{\mu} du^2 + \frac{\mu}{r^2} dr^2 + 4\mu d\Omega_3^2. \tag{4.49}$$

Because of this, we can no longer remove the g_{uu} component of the metric via a coordinate transformation, as we have done for the $AdS_3 \times S^3$ solution.

It will now be helpful to define the following coordinates [6]:

$$r = \frac{w^2 - 4k}{4}, \quad u = \frac{t - 2\sqrt{\mu}\varphi}{\sqrt{2}}, \quad v = \frac{t + 2\sqrt{\mu}\varphi}{\sqrt{2}}, \tag{4.50}$$

which yields the metric

$$ds^2 = -h(w)^2 dt^2 + h(w)^{-2} dw^2 + w^2 \left(d\varphi - \frac{2k}{\sqrt{\mu} w^2} dt^2 \right)^2 + L^2 d\Omega_3^2, \tag{4.51}$$

where

$$h(w) = \frac{w^2 - 4k}{\sqrt{2}\mu w}. \tag{4.52}$$

Making the periodic identification $\varphi \sim \varphi + 2\pi$, this metric describes an extremal BTZ black holes in a direct product with a three-sphere with 4μ radius. The BTZ black hole itself is characterized by the radius L, mass M and angular momentum J:

$$L^2 = 4\mu, \quad M = \frac{2k}{\mu}, \quad J = \frac{4k}{\sqrt{\mu}}. \tag{4.53}$$

4.2.6 *The black string*

In subsection 2.1.5, we introduced black string solutions as a higher dimensional analogue of the black hole, with a topologically $S^{D-3} \times S^1$ horizon. In this subsection we will introduce the black string solution as a solution in the Gibbons-Hawking class, and we will express it in terms of Harmonic functions. The direction of the black ring will be along the u coordinate, and as such it proves a natural way to reduce the solution to five dimensions. The black string in six dimensions will have a $S^3 \times S^1$ horizon topology, and the near horizon geometry is a direct product of an extremal BTZ black holes and $S^3/\mathbb{Z}_m$.

The most general form of the harmonic functions corresponding to the black string solution is

$$\mathbb{V}_{bs} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{m}(p - q n_\infty) \\ n_\infty/2 \\ 1 \end{pmatrix} + \begin{pmatrix} m \\ q \\ p \\ j \\ n/2 \\ \mu \end{pmatrix} \frac{1}{r}. \tag{4.54}$$

This result has been obtained in [5] by restricting the metric to be asymptotically isometric to $\mathbb{R}^{1,4} \times S^1_u$.

The corresponding metric reads

$$ds^2 = -2\left(1+\frac{\tilde{Q}}{2\sqrt{2}mr}\right)^{-1}\left[dv + \frac{J_\psi}{8m^2 r}(d\psi + m\cos\theta d\phi) - \frac{1}{2}\left(n_\infty + \frac{Q}{4mr}\right)du'\right]du' + \left(1+\frac{\tilde{Q}}{2\sqrt{2}mr}\right)\left[\frac{r}{m}(d\psi + m\cos\theta d\phi)^2 + m\frac{dr^2}{r} + mr d\Omega_2^2\right], \tag{4.55}$$

with $u' = u + \frac{q}{m}\psi$. Due to the periodicity of u and ψ this leads to the restriction $\frac{4\pi q}{Lm} \in \mathbb{Z}$, with L the period of u, in order for the coordinate system to be globally defined.

The constants Q, $\tilde{Q}$ and J appearing in the metric are defined as

$$\begin{aligned} \tilde{Q} &= 4\sqrt{2}(\mu m + qp), \\ Q &= 4(nm + p^2), \\ J_\psi &= 8\left(qp^2 + \mu pm + \frac{1}{2}qnm + jm^2\right). \end{aligned} \tag{4.56}$$

The black string solution has a horizon at $r = 0$, and this can be used to calculate the Bekenstein-Hawking entropy:

$$S = \frac{A}{4G} = \frac{1}{4G_N}\int_{r=0} dA\sqrt{g} = \frac{2\pi}{|m|}\sqrt{\left(\frac{1}{2}Q\tilde{Q}^2 - J^2\right)}, \tag{4.57}$$

with g the determinant of the metric at $r = 0$ and working in conventions in which $G_N = \frac{1}{4}\pi L$.

Using methods that will be discussed in section 4.3, the black string solution can be reduced along the u direction resulting in a black hole in five dimensions with horizon topology S^3. In the context of these black holes we find that the charges (4.56) have a geometric meaning, from which we can deduce a physical interpretation. The scalars in (4.56) can be obtained through the integrals on a the horizon $\mathcal{H}$ of a black hole in five dimensions, with the Komar integrals [5]:

$$\tilde{Q} = -\frac{\sqrt{2}}{8\pi^2}\int_{\mathcal{H}} \star_5 d\tilde{A}, \tag{4.58}$$

$$Q = -\frac{1}{8\pi^2}\int_{\mathcal{H}} \star_5 dA \tag{4.59}$$

$$J_\psi = \frac{1}{4\pi^2}\int_{\mathcal{H}} \star_5 K^{(\psi)}, \tag{4.60}$$

(4.61)

where $K^{(\psi)}$ is the one form associated with the Killing vector ∂_ψ. This way we are able to make the identification of J_ψ as the angular momentum, while $\tilde{Q}$ and Q are two electric charges corresponding to the one-forms obtained through dimensional reduction as per the Ansatz (4.78) and (4.73).

Using the timelike Killing vector ∂_t, we can also calculate the total energy of the solution. However, it is already determined via the BPS condition and reads [5]

$$M = \frac{1}{4}\left(Q + \sqrt{2}\tilde{Q}\right). \tag{4.62}$$

4.2.7 *Plane and pp-waves*

In section 2.2 we introduced plane and pp-wave spacetimes, as solutions of the field equations characterized by a covariantly constant null Killing vector field. In this subsection we will discuss the possible plane waves in the Gibbons-Hawking class of solutions.

We have two possible choices to define this Killing vector, they are ∂_v and ∂_u, and each choice will lead to a different description in terms of harmonic functions. Here we will simply state the results, so that we can use them later. For a derivation of the results, see [6].

As said there are two different choices for the Killing vector, and since plane waves are defined on a flat base space, there are two more options for the harmonic function V_1: $V_1 = \frac{1}{r}$ and $V_1 = m_\infty$. Thus we will have four different descriptions of plane wave solutions.

Plane wave: wave vector ∂_v, base space $\mathbb{R}^4$

Starting with ∂_v as the covariantly constant vector, and with $V_1 = \frac{1}{r}$ leads to the set of harmonic functions [6]:

$$\mathbb{V} = \begin{pmatrix} 0 \\ 0 \\ a_0/c_6 \\ -c_3c_6 \\ 0 \\ c_6 - a_0c_2/c_6 \end{pmatrix} + \begin{pmatrix} 1 \\ c_2 \\ c_3 \\ c_2c_3^2/2 \\ -c_3^2/2 \\ -c_2c_3 \end{pmatrix} \frac{1}{r} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ c_6 - a_0c_2/c_6 \\ a_0/c_6 \\ 0 \end{pmatrix} \vec{a}\cdot\vec{x} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -c_2 \\ 1 \\ 0 \end{pmatrix} \vec{b}\cdot\vec{x}, \tag{4.63}$$

which results in a metric diffeomorphic to

$$ds_I^2 = -2du dv + \left(2\vec{b}\cdot\vec{x} - \frac{1}{4}|\vec{a}|^2|\vec{y}|^2\right) du^2 + |d\vec{y}|^2, \tag{4.64}$$

and the three-form reads

$$G = -\frac{a_i}{2} du \wedge \left(dx^i \wedge (d\psi + \chi) + \frac{1}{r} \star_3 dx^i\right). \tag{4.65}$$

The coordinates $x^i \in \mathbb{R}^3$ and $y^i \in \mathbb{R}^4$ are related via

$$\begin{aligned} 2x^1 &= y^1y^3 + y^2y^4 \\ 2x^2 &= y^2y^3 - y^1y^4, \\ 4x^3 &= (y^1)^2 + (y^2)^2 - (y^3)^2 - (y^4)^2. \end{aligned} \tag{4.66}$$

We furthermore note that the only physically relevant parameters in the harmonic functions are $\vec{a}$ and $\vec{b}$, as all the other do not appear in the metric or three-form. The parameters in $\vec{a}$ and $\vec{b}$ parametrize the gravitational and electromagnetic components of the wave, respectively. We recognize this again from the form of the quadratic function in (2.44), as well as from the fact that the three-form is characterized by $\vec{a}$.

In this case there are only three independent parametrizations of the gravitational wave, while a graviton in six dimensions without any restrictions has nine degrees of freedom. In this case the fact that ∂_ψ is Killing, has led to this restriction on the degrees of freedom. This is better understood by explicitly providing the mapping $\mathbb{R}^3 \to \mathbb{R}^4$, which reads

$$y_1 = 2\sqrt{r}\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\psi+\phi}{2}\right), \tag{4.67}$$

$$y_2 = 2\sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\psi+\phi}{2}\right), \tag{4.68}$$

$$y_3 = 2\sqrt{r}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\psi-\phi}{2}\right), \tag{4.69}$$

$$y_4 = 2\sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\psi-\phi}{2}\right). \tag{4.70}$$

In the case of a plane wave solution we want g_{uu} to be quadratic in the background coordinates, and the metric components have to be independent of ψ, since ∂_ψ is Killing. There are only three linearly independent combinations satisfying these solution, and this are the ones given in (4.66).

wave vector ∂_v, base space $\mathbb{R}^3 \times S^1$

Another possible class of plane wave solutions, is one where the wave vector is ∂_v and the base space is $\mathbb{R}^3 \times S^1$, as a result of choosing $V_1 = m_\infty$. In this case metric that can be obtained reads [6]:

$$ds^2 = -2dudv + \left(2b_{ij}x^ix^j - \frac{|\vec{c}|^2}{3}|\vec{x}|^2\right)du^2 + |d\vec{x}|^2 + m_\infty^{-1}d\psi^2, \tag{4.71}$$

and the three-form is given by

$$G = -\frac{1}{2}du \wedge (dx^i \wedge d\psi + \star_3 dx^i)c_i. \tag{4.72}$$

Here we can make a distinction between the vector $\vec{c}$ parametrizing electromagnetic waves, and the tensor b_{ij} parametrizing gravitational waves. And because b_{ij} corresponds to gravitational waves, there are some restrictions on the degrees of freedom since it should be in the transverse traceless gauge [64]. As a result, b_{ij} corresponds to five independent polarizations.

It is clear that this solution differs from the one discussed previously. The most notable difference are the number of independent polarizations of the plane waves. Where the previous solution only had three independent polarizations for the gravitational part of the wave, this solution has five. In the previous case it was intuitively hard to interpret the restrictions, in this case it is very clear. This solution describes a plane wave in five dimensions in a direct product with S^1_ψ. Here the counting of the polarizations works out, as the graviton in five dimensions has exactly five independent polarization modes.

wave vector $\partial_u + \frac{F}{2}\partial_v$

As mentioned in the beginning of this subsection there is another possibility for the covariant vector: ∂_u. However, ∂_u is not null, so instead we work with the linear combination $\partial_u + \frac{F}{2}\partial_v$, which is null.

We can be quick about solutions of this class. In the case where $V_1 = \frac{1}{r}$, the solutions we can obtain are isometric to a subclass of the solution with wave vector ∂_v and $V_1 = \frac{1}{r}$. This subclass are the solutions where $|a| = a_0$ and $b_i = 0$.

The case where $V_1 = m_\infty$ is even simpler, as it corresponds to flat space on a base $\mathbb{R}^3 \times S^1$.

4.3 REDUCTION TO FIVE DIMENSIONS

In section 2.1.5 we discussed some interesting properties of higher dimensional solutions to the field equations, and thereby motivated the study of those higher-dimensional solution. We have also mentioned the possibility of compactification in the context of Taub-NUT spaces in section 4.2. When considering higher-dimensional solutions there appears to be a disagreement with the physics of our universe where we observe only four dimensions, therefore when discussing such solution we are interested in the compactification on compact manifolds, which provides a method of relating higher dimensional solutions to solutions in four dimensions. In this section we will discuss the compactification of the six dimensional solution on u or ψ obtaining a five-dimensional solution.

Kaluza-Klein compactificatoin of minimal six-dimensional supergravity yields one Kaluza-Klein vector from the metric, one from the two-form potential and one from dualizing the three-form field strength. However, due to the self duality condition on the three-form, only two of these are independent.

Provided u is spacelike, i.e. provided F is positive, the $D = 5$ timelike class can be obtained by dimensional reduction of u-independent solutions using the Ansatz (E.6) in Einstein frame:

$$d\hat{s}^2 = e^{-\frac{2}{3}\phi} ds^2 + e^{2\phi}(dz + A)^2. \tag{4.73}$$

.

To do so it is convenient to rewrite the six-dimensional metric as

$$ds_6^2 = H^{-1}F\left(du + \beta - F^{-1}(dv + \omega)\right)^2 - H^{-1}F^{-1}(dv + \omega)^2 + H ds_4^2. \tag{4.74}$$

Using the Ansatz (4.73) we find the dilaton is given by

$$e^{2\phi} = H^{-1}F, \tag{4.75}$$

and the expression for the graviphoton is

$$A = \beta - F^{-1}(dt + \omega). \tag{4.76}$$

Here we renamed $v \to t$, because if we look at the five-dimensional metric as a result of this compactification

$$ds_5^2 = -H^{-4/3}F^{-2/3}(dt + \omega)^2 + H^{2/3}F^{1/3}ds_4^2, \tag{4.77}$$

we find that t is the timelike coordinate.

We can also reduce the three-form field strength, for which the Ansatz is

$$\hat{G} = G + \frac{1}{2} d\tilde{A} \wedge (du + A)\,. \tag{4.78}$$

We will not present the explicit reduction to obtain the three-form in five dimensions here, but this does provide insight into the origin of the second vector we claimed to obtain though Kaluza-Klein reduction.

Reduction to four dimensions

By reducing the six-dimensional solution along both the u and ψ coordinate, we obtain a four-dimensional theory characterized by six harmonic functions. As a result of this compactification the theory now contains two more vector potentials, besides the vector potential corresponding to the electric and magnetic field strengths already present in the six-dimensional theory. In this four-dimensional theory we can define the vector

$$(m, q, p, j, n, \mu)\,, \tag{4.79}$$

where m, q and p correspond to magnetic charges and j, n and μ to the electric charges [6]. The $Sp(6, \mathbb{R})$ group we consider acts linearly on these charges. This is important since any Lagrangian we write down for these charges is not invariant under just any transformation in $GL(6, \mathbb{R})$, but instead we have to restrict ourselves to elements of the symplectic group when transforming electric and magnetic charges. This is a generalization to multiple charges of the transformation we discussed in subsection 2.1.3 on electromagnetic duality. This is one way to gain some physical understanding of the significance of the group $Sp(6, \mathbb{R})$ in the context of a four-dimensional solution.

Part II

GENERATING SOLUTIONS

Having introduced the theory we are interested in, we now have the required knowledge to proceed with the analysis of the symplectic group, and the solutions generated using transformations in this group. We will begin with an introduction of the symplectic group itself, and discuss its algebra and several subgroups of interest. After this we will use the symplectic group to generate new solutions, and we aim to characterize and discuss interesting properties of these new solutions.

5 SYMPLECTIC GROUP

Since Einstein's equations are a set of ten non-linear partial differential equations, is is quite difficult to find solutions. And on top of that we are interested in solutions preserving supersymmetry. However, one way to find new solutions to the Einsteins equations is by employing symmetry relations to known solutions. In this thesis these symmetry relations correspond to transformations with $Sp(6, \mathbb{R})$, and this chapter is devoted to the analysis of this group.

5.1 BUBBLE EQUATIONS

As discussed in section 4.2, the Gibbons-Hawking class of solutions depends on a set of six harmonic functions. Thus acting linearly on these six harmonic functions maps a solution to a solution. However, although any transformation which is a result of acting with an element of $GL(6, \mathbb{R})$ on the six harmonic functions generates a new solution, these new solutions will typically contain Dirac-Misner string singularities [65]. These are similar to the Dirac string discussed in subsection 2.1.2, but where we use the term Dirac string for vector potentials of Maxwell fields, the term Dirac-Misner string is used when the vector is part of the metric. To obtain solutions which do not contain Dirac-Misner strings, the one-form $\bar{\omega}$ needs to be globally well-defined, which is only the case if

$$d^2\bar{\omega} = 0. \tag{5.1}$$

By inserting (4.21) into (5.1), this condition can be written as

$$d^2\bar{\omega} = d\left[*_3\langle \mathbb{V}, d\mathbb{V}\rangle\right] = 0. \tag{5.2}$$

Taking $\mathbb{V}$ of the form (4.26), results in [5]:

$$\sum_{a\neq b} \frac{\langle \Gamma_a, \Gamma_b\rangle}{r_{ab}} = \langle \Gamma_\infty, \Gamma_a\rangle. \tag{5.3}$$

Here r_{ab} is the distance between two centers and as a result of this equation, there is a constraint on these distances. These constraints are colloquially referred to as 'bubble equations', because they determine the behavior of the 'bubbles' in the Gibbon-Hawking base space, as is shown in figure 4.1.

As we have mentioned before, in this thesis we mostly restrict ourselves to single centered solutions described by harmonic functions of the form (4.23). Under this assumption (5.3) simplifies to

$$\langle \Gamma_\infty, \Gamma\rangle = 0. \tag{5.4}$$

This condition clearly does not hold for a general set of harmonic functions $\mathbb{V}$, but by starting from a solution which satisfy this condition, elements g in the subgroup of $GL(6, \mathbb{R})$ which preserves the symplectic product will send a solution free of Dirac-Misner strings to another solution with this same property. This leads us to consider elements obeying the relation

$$g^T \Omega g = \Omega, \tag{5.5}$$

which means $g \in Sp(6, \mathbb{R})$. We can thus use the symplectic group to generate solutions of six dimensional supergravity without Dirac-Misner strings. Under the condition that the starting solution is free of Dirac-Misner string singularities.

Example: Flat space to $AdS_3 \times S^3$

To show how such a transformation is performed we as an example send a solution describing flat space, to a solution describing $Ad_3 \times S^3$. For this, let us consider the matrix $A \in Sp(6, \mathbb{R})$:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\mu} & \frac{1}{\mu} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \mu & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.6}$$

Acting on flat space (as presented in (4.40)), with this matrix yields

$$A \cdot \mathbb{V}_{flat} = \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ 1 \\ \frac{\mu}{r} \end{pmatrix}, \tag{5.7}$$

where we recognize the set of harmonic functions corresponding to $AdS_3 \times S^3$, as given in 4.43.

5.2 PROPERTIES OF THE SYMPLECTIC GROUP

The symplectic group of degree $2n$ over the real numbers $\mathbb{R}$, denoted $Sp(2n, \mathbb{R})$, is the group of $2n \times 2n$ real matrices which preserve a skew-symmetric bilinear form on $\mathbb{R}^{2n}$. Hence, the group $Sp(2n, \mathbb{R})$ is defined as the set of matrices M obeying

It should be noted that what is here referred to as $Sp(2n, \mathbb{R})$*, is in some literature referred to as* $Sp(n, \mathbb{R})$.

$$M^T \Omega M = \Omega, \tag{5.8}$$

where Ω is a $2n \times 2n$ nonsingular anti-symmetric matrix. In this thesis we will work in the basis in which

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \tag{5.9}$$

where I_n is the $n \times n$ identity matrix.

Let us now consider $M \in Sp(2n, \mathbb{R})$ to be a block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \text{with } A, B, C, D \in \mathbb{R}^{n \times n}. \tag{5.10}$$

This yields the following restrictions on the matrices A, B, C and D:

$$A^T C = (A^T C)^T, \quad B^T D = (B^T D)^T, \quad A^T D - C^T B = I_n. \tag{5.11}$$

To determine the dimension of the group $Sp(2n, \mathbb{R})$, we start from the even dimensional general linear group denoted $GL(2n, \mathbb{R})$ and determine the degrees of freedom restricted when going to $Sp(2n, \mathbb{R})$. The first two equations in (5.11) restrict two $n \times n$ matrices to be symmetric, which means $\frac{1}{2}(n^2 - n)$ restrictions for each equations. The last equation results in another n^2 restrictions. From this we can determine the dimension of the symplectic group to be

$$\begin{aligned} \text{Dim}\left[Sp(2n,\mathbb{R})\right] &= \text{Dim}\left[GL(2n),\mathbb{R})\right] - \text{d.o.f. restricted} && (5.12)\\ &= (2n)^2 - (2n^2 - n) = n(2n+1). && (5.13)\end{aligned}$$

Taking the determinant on both sides of (5.8) yields

$$\det(M^T)\det(\Omega)\det(M) = \det(\Omega), \tag{5.14}$$

and thus $\det(M) = \pm 1$. A nice proof that it is in fact strictly positve, and thus

$$\det(M) = 1, \tag{5.15}$$

is given in [66].

GROUP ELEMENTS OF $Sp(2n,\mathbb{R})$

The group $Sp(2n,\mathbb{R})$ is generated by the matrices [67]

$$\begin{pmatrix} I & \lambda e_{ii} \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} I & 0 \\ \lambda e_{ii} & I \end{pmatrix}, \quad \begin{pmatrix} I & \lambda(e_{ij}+e_{ji}) \\ 0 & I \end{pmatrix}, \tag{5.16}$$

$$\begin{pmatrix} I & 0 \\ \lambda(e_{ij}+e_{ji}) & I \end{pmatrix}, \quad \begin{pmatrix} I+\lambda e_{ij} & 0 \\ 0 & I-\lambda e_{ji} \end{pmatrix}, \tag{5.17}$$

where $\lambda \in \mathbb{R}$, and $1 \leqslant i,j \leqslant n$ for $i \neq j$ and e_{ij} represent $n \times n$ matrices with 1 in the position (i,j) and 0 elsewhere. The 0 in the matrices above, denotes an $n \times n$ matrix with all elements equal to 0.

Considering that the microscopic description of any solutions found will rely on a integer number of branes, it is interesting to wonder if anything simplifies when considering the discrete group $Sp(2n,\mathbb{Z})$. The discrete group is generated by the matrices [68]:

translations

$$T = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}, \quad A^T = A, \tag{5.18}$$

rotations

$$R = \begin{pmatrix} B & 0 \\ 0 & (B^T)^{-1} \end{pmatrix}, \quad B \in GL(n,\mathbb{R}), \tag{5.19}$$

semi-involutions

$$S = \begin{pmatrix} C & I-C \\ C-I & C \end{pmatrix}, \tag{5.20}$$

where C are diagonal matrices with zeroes and ones on the diagonal.

Here we specify the group to $n = 3$ as apposed to keeping a general n, this is because there is no generalized description for the generators for any n. For example, if $n > 3$ the group is generated by only two matrices.

However, this does not constitute a minimal set of generators. In [69] it is shown that the matrices D, R and T below also generate the entire group $Sp(6,\mathbb{Z})$.

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.21}$$

$$R_{21} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.22}$$

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.23}$$

5.3 ON THE ALGEBRA $\mathfrak{sp}(2n, \mathbb{R})$

In order to find what restrictions an element X of the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ must obey, we define $M = e^X$. Plugging this into equation (5.8), we find that X should obey

$$(\Omega X)^T = \Omega X. \tag{5.24}$$

The result of (5.24) is more thoroughly derived in appendix F.2.

From which it can be observed that $X = \Omega^{-1}S$, where S is a symmetric matrix. A basis for these symmetric matrices S is the set of $n(2n+1)$ symmetric matrices $\{S_{ij}\}$, where the only nonzero entries of S_{ij} are the elements (i, j) and (j, i) of the matrix, which we will take to be equal to 1.

Using this we find that the corresponding set of generators $\{X_{ij}\}$ is

$$\{X_{ij}\} = \{\Omega^{-1}S_{ij}\}, \text{ with } i, j \in [2n]. \tag{5.25}$$

5.4 THE ENTROPY CONSERVING SUBGROUP

The entropy of a BPS object with an event horizon is given in (4.57). The set of entropy conserving matrices can be divided into six linearly independent generators of the group $Sp(6, \mathbb{R})$, these are discussed in [5]. These six generators can in turn be combined in pairs which correspond already known transformations. As we will see below, these matrices correspond to coordinate transformations called gauge transformations, spectral flow, and a rescaling. An analysis of the gauge transformations and spectral flow have been studied by Bena, Bobev and Warner in [70].

Gauge transformations

The matrix corresponding to gauge transformations reads [5]:

$$M_{gauge} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ g_2 & 1 & 0 & 0 & 0 & 0 \\ 2g_1 & 0 & 1 & 0 & 0 & 0 \\ 2g_1^2 g_2 & 2g_1^2 & 2g_1 g_2 & 1 & -g_2 & -2g_1 \\ -2g_1^2 & 0 & -2g_1 & 0 & 1 & 0 \\ -2g_1 g_2 & -2g_1 & -g_2 & 0 & 0 & 1 \end{pmatrix}. \tag{5.26}$$

Acting with a matrix of this form leaves the functions H, F, V_1 and the 1-form ω invariant, while the 1-form β is transformed: $\beta \to \beta - g_2 d\psi$, which can be undone by a transformation $u \to u + g_2\psi$. Here we thus find that when we set $g_2 = 0$, the metric is left unchanged.

The transformation of the harmonic functions as a result of M_{gauge} is

$$\begin{aligned} m &\to m \\ q &\to q + g_2 m \\ p &\to p + 2g_1 m \\ j &\to j + 2g_1^2 g_2 m - 2g_1\mu - g_2 n + 2g_1 g_2 p + 2g_1^2 g_2 q \\ n &\to n - 2g_1^2 m - 2g_1 p \\ \mu &\to \mu - 2g_1 g_2 m - g_2 p - 2g_1 q, \end{aligned} \tag{5.27}$$

which are redundancies in our description and leave the physics of the solution unchanged.

Spectral flow

The matrices corresponding to the spectral flow transformations are of the form [5]:

$$M_{sflow} = \begin{pmatrix} 1 & \gamma_2 & 2\gamma_1 & 2\gamma_1^2\gamma_2 & -2\gamma_1^2 & -2\gamma_1\gamma_2 \\ 0 & 1 & 0 & 2\gamma_1^2 & 0 & -2\gamma_1 \\ 0 & 0 & 1 & 2\gamma_1\gamma_2 & -2\gamma_1 & -\gamma_2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\gamma_2 & 1 & 0 \\ 0 & 0 & 0 & -2\gamma_1 & 0 & 1 \end{pmatrix}. \tag{5.28}$$

Matrices of this form induce some non-trivial transformations. Also note that $M_{sflow} = M_{gauge}^T$.

Rescaling

Finally there are matrices of the form [5]:

$$M_{resc.} = \begin{pmatrix} \beta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_1^2\beta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_2^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_1^{-2}\beta_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_1 \end{pmatrix}, \tag{5.29}$$

which simply rescales the harmonic functions.

Lie algebra

The matrices above can be written as an exponential mapping of six generators t_i as follows

$$\begin{aligned} M_{gauge} = e^{2g_1 t_1 + g_2 t_2}, \quad M_{sflow} = e^{2\gamma_1 t_3 + \gamma_2 t_4}, \\ M_{resc.} = e^{\ln(\beta_1) t_5 + \ln(\beta_1\beta_2) t_6}. \end{aligned} \tag{5.30}$$

Defining

$$e = \sqrt{2} t_3, \quad f = \sqrt{2} t_1, \quad h = 2t_6 \tag{5.31}$$

$$\tilde{e} = t_2, \quad \tilde{f} = t_4, \quad \tilde{h} = 2t_5 \tag{5.32}$$

The non-vanishing commutation relations of these generators are

$$[e, f] = h \quad [h, e] = 2e \quad [h, f] = -2f, \tag{5.33}$$

and similarly for the generators with a tilde.

This is the $\mathfrak{sp}(2, \mathbb{R})$ algebra. Thus the transformation in $Sp(6, \mathbb{R})$ which leave the entropy invariant are generated by the $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ algebra corresponding to the group

$$SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \subset Sp(6, \mathbb{R}). \tag{5.34}$$

A well known 2-to-1 homomorphism is $SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \simeq SO(2, 2)$ [71], which as is discussed in appendix C, describes the isomteries of AdS_3. The near-horizon geometry of a black string is the direct product of an extremal BTZ black hole and $S^3/\mathbb{Z}_m$, which in turn are locally isometric to $AdS_3 \times S^3$. It should come as no surprise that we find the entropy invariant under this subgroup of $Sp(6, \mathbb{R})$ since the entropy defined in (8.31) depends on the horizon area.

Reparametrization of u *and* ψ *by* $SL(2, \mathbb{R})$

The Gibbons-Hawking class of six-dimensional solutions studied in this theses are characterized by a $U(1) \times U(1)$ isometry, corresponding to the independence of the solution on the coordinates u and ψ. Because of this, the equations of motion can be reduced along $d\psi \wedge du$, which describes the reduction on a torus. This leads us to suspect that solutions of this type should contain some $SL(2, \mathbb{R})$ isometry relating to the u and ψ coordinates.

Let us show that this transformation is generated by the generators t_2 t_4 and t_5, which corresponds to transformations by the matrices (5.26), (5.28) and (5.29), with $g_1 = 0$, $\gamma_1 = 0$ and $\beta_1 = \beta_2^{-1}$. The product $M_{gauge}.M_{sflow}.M_{resc.}$ is

$$M = \begin{pmatrix} \frac{1}{\beta_1} & \beta_1\gamma_2 & 0 & 0 & 0 & 0 \\ \frac{g_2}{\beta_1} & \beta_1(g_2\gamma_2+1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta_1} & 0 & 0 & -\beta_1\gamma_2 \\ 0 & 0 & 0 & \beta_1(g_2\gamma_2+1) & -\frac{g_2}{\beta_1} & 0 \\ 0 & 0 & 0 & -\beta_1\gamma_2 & \frac{1}{\beta_1} & 0 \\ 0 & 0 & -\frac{g_2}{\beta_1} & 0 & 0 & \beta_1(g_2\gamma_2+1) \end{pmatrix} \in SL(2, \mathbb{R}). \tag{5.35}$$

Where for convencience we can define $a = \beta_1(g_2\gamma_2 + 1)$, $b = \beta_1\gamma_2$, $c = g_2\beta_1^{-1}$ and $d = \beta^{-1}$, resulting in

$$M = \begin{pmatrix} d & b & 0 & 0 & 0 & 0 \\ c & a & 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 & -b \\ 0 & 0 & 0 & a & -c & 0 \\ 0 & 0 & 0 & -b & d & 0 \\ 0 & 0 & -c & 0 & 0 & a \end{pmatrix}, \quad ad - bc = 1. \tag{5.36}$$

Which corresponds to the following change of coordinates [6]

$$\begin{pmatrix} \psi \\ u \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi \\ u \end{pmatrix}, \quad ad - bc = 1. \tag{5.37}$$

Thus we can conclude that the set of generators $\{t_2, t_4, t_5\}$, correspond to an $SL(2, \mathbb{R})$ transformation of the u and ψ coordinates.

Transformations by elements of the other $SL(2,\mathbb{R})$ *group*

It now remains to find out the implications of the other $\mathfrak{sl}(2,\mathbb{R})$ set of generators: $\{t_1, t_3, t_6\}$

Gauge transformations generated by t_1 do not change the solution, as all functions in the solution are invariant under these transformations.

Acting with the rescaling $\beta_1 = \beta_2$, transform the solution as [6]:

$$(ds^2)' = \beta_1^2 ds^2(\psi \to \beta_1^{-1}\psi, u \to \beta_1^{-3}u). \tag{5.38}$$

Finally all we have left are the spectral flow transformations with $\gamma_2 = 0$. These do change the solution in a nontrivial way, see for example (6.17), where flat space was transformed into a plane wave solution under this transformation.

5.5 A TRANSFORMATION PROVIDING AN EQUIVALENCE BETWEEN ANY TWO SOLUTIONS

An interesting element of $Sp(6,\mathbb{R})$ is

$$M = \begin{pmatrix} m & 0 & 0 & 0 & 0 & m_\infty \\ q & 1 & 0 & 0 & 0 & q_\infty \\ p & \frac{mn_\infty - nm_\infty}{\mu m_\infty - m\mu_\infty} & \frac{-m}{\mu m_\infty - m\mu_\infty} & \frac{-m_\infty}{\mu m_\infty - m\mu_\infty} & \frac{qm_\infty - mq_\infty}{\mu m_\infty - m\mu_\infty} & p_\infty \\ j & \frac{\mu n_\infty - n\mu_\infty}{\mu m_\infty - m\mu_\infty} & \frac{-\mu}{\mu m_\infty - m\mu_\infty} & \frac{-\mu_\infty}{\mu m_\infty - \mu_\infty m} & \frac{q\mu_\infty - \mu q_\infty}{\mu m_\infty - m\mu_\infty} & j_\infty \\ n & 0 & 0 & 0 & 1 & n_\infty \\ \mu & 0 & 0 & 0 & 0 & \mu_\infty \end{pmatrix}, \tag{5.39}$$

with $mj_\infty - jm_\infty + qn_\infty - nq_\infty + p\mu_\infty - \mu p_\infty = 0$.

Using a numerical method is has been confirmed that matrices of this form constitute a group, meaning that a product of two such matrices yields another matrix of the same form, and these matrices are invertible. The group is however still to be identified, but it should be noted that it is generated by matrices of the algebra $\{\mathfrak{sp}(6,\mathbb{R}) \setminus \mathfrak{sp}(4,\mathbb{R})\}$.

Furthermore, there are some restriction on the parameters in M, which should be obeyed in order for M to be an element of $Sp(6,\mathbb{R})$. A rather obvious restriction is that the determinant of the matrix has to be non-vanishing, and as a result not all variables in a row or column can be set to 0. This means that for example solutions with $m = m_\infty = 0$ can not be obtained from M.

Also, we observe that the parameters cannot be set in such a way that $\mu m_\infty - m\mu_\infty = 0$, as this will result in a matrix which is not well defined. However by acting with matrix D defined in (5.21), we obtain

$$D.M = \begin{pmatrix} q & 1 & 0 & 0 & 0 & q_\infty \\ p & \frac{mn_\infty - nm_\infty}{\mu m_\infty - m\mu_\infty} & \frac{m}{m\mu_\infty - \mu m_\infty} & -\frac{m_\infty}{\mu m_\infty - m\mu_\infty} & \frac{qm_\infty - mq_\infty}{\mu m_\infty - m\mu_\infty} & p_\infty \\ -j & \frac{n\mu_\infty - \mu n_\infty}{\mu m_\infty - m\mu_\infty} & \frac{\mu}{\mu m_\infty - m\mu_\infty} & \frac{\mu_\infty}{\mu m_\infty - m\mu_\infty} & \frac{\mu q_\infty - q\mu_\infty}{\mu m_\infty - m\mu_\infty} & -j_\infty \\ n & 0 & 0 & 0 & 1 & n_\infty \\ \mu & 0 & 0 & 0 & 0 & \mu_\infty \\ m & 0 & 0 & 0 & 0 & m_\infty \end{pmatrix}, \tag{5.40}$$

where we can redefine the variables as follows: $q \to m$, $p \to q$, $-j \to p$, $n \to j$, $\mu \to n$, $m \to \mu$, and similarly for the variables with ∞ subscript. This results in the matrix

$$(D.M)' = \begin{pmatrix} m & 1 & 0 & 0 & 0 & m_\infty \\ q & \frac{j\mu_\infty - \mu j_\infty}{\mu n_\infty - n\mu_\infty} & \frac{\mu}{\mu n_\infty - n\mu_\infty} & \frac{\mu_\infty}{\mu n_\infty - n\mu_\infty} & \frac{\mu m_\infty - m\mu_\infty}{\mu n_\infty - n\mu_\infty} & q_\infty \\ p & \frac{nj_\infty - jn_\infty}{\mu n_\infty - n\mu_\infty} & \frac{n}{n\mu_\infty - \mu n_\infty} & -\frac{1}{\mu - \frac{n\mu_\infty}{n_\infty}} & \frac{nm_\infty - mn_\infty}{n\mu_\infty - \mu n_\infty} & p_\infty \\ j & 0 & 0 & 0 & 1 & j_\infty \\ n & 0 & 0 & 0 & 0 & n_\infty \\ \mu & 0 & 0 & 0 & 0 & \mu_\infty \end{pmatrix}. \tag{5.41}$$

Acting on flat space with $(D.M)'$ has the same result as acting on flat space with M, because the first and last columns are the same. However, where (5.39) was not well defined if $\mu m_\infty - m\mu_\infty = 0$, (5.41) is not well defined if $\mu n_\infty - n\mu_\infty = 0$.

Operations with D, have an equivalence property. Explicitly this means that $D^1 = -D^7 = D^{13}$. Since we use D to transform M, and a sign change of M can be undone by a redefinition of the parameters, we say that the power of D has a modulus of six. Because of this we can write down six matrices of the form of M, but each one has to satisfy another condition to prevent elements from 'blowing up'. As a result, the only case in which we will not be able to write down a well defined matrix of the form $D^n.M$, is if all of these equations are satisfied

$$\begin{aligned} \mu m_\infty - m\mu_\infty &= 0, \\ qm_\infty - mq_\infty &= 0, \\ qp_\infty - pq_\infty &= 0, \\ jp_\infty - pj_\infty &= 0, \\ jn_\infty - nj_\infty &= 0, \\ n\mu_\infty - \mu n_\infty &= 0. \end{aligned} \tag{5.42}$$

Through some manipulation we find that this means that the only solution which can not be obtained from $\mathbb{V}_{flat}$ (besides of course solutions for which $\langle\Gamma_\infty, \Gamma\rangle \neq 0$), are solutions for which Γ is proportional to Γ_∞. Thus we can conclude that any two linearly independent vectors can be transformed into each other by acting with elements of the symplectic group.

On a final note it we should mention that in the case of single centered solutions, there are four different classes of solutions we can distinguish.

1. There is the class of orbits for solutions of the form $\Gamma \propto \Gamma_\infty$, which after acting with the same group element on both vectors will obviously remain proportional to each other with an unchanged proportionality factor.

2. Solutions for which $\Gamma = 0$, restricting the class of solutions to solutions with vanishing Riemann tensor, see (4.42).

3. Solutions for which $\Gamma_\infty = 0$, this includes $AdS_3 \times S^3$.

4. Solutions for which $\Gamma \neq 0$, $\Gamma_\infty \neq 0$ and Γ linearly independent of Γ_∞.

Here we will be most interested in solutions of the fourth type, as this is the only class of solutions in which acting with a matrix will induce nontrivial transformations on harmonic functions of the form $V = a + \frac{b}{r}$. However, since any combination of two linearly independent vectors with vanishing symplectic product can be transformed into any other combination of two vectors with vanishing symplectic product, this class of solutions can not be divided into different 'orbits'.

This result implies that systematically analyzing the transformations of this group will prove challenging, since a solution can be obtained through a symplectic transformation of any other solution.

6 TRANSFORMATIONS OF FLAT SPACE

In chapter 5 we found that the dimension of the symplectic group is $n(2n+1)$ (see (5.12)). Since we are interested in six dimensions, this means we are dealing with an algebra containing 21 generators. As this is a rather large number, a systematic approach of sending solutions to solutions would be preferred over randomly acting with each generator and then try to find out what the resulting spacetime is. In this chapter we will act on a specific example of a vector of harmonic functions, namely one corresponding to a solution for flat space, which we denote $\mathbb{V}_{\texttt{flat}}$. In our analysis we will furthermore separate solutions which are asymptotically (at large r) equal to the original vector $\mathbb{V}_{\texttt{flat}}$, and solutions for which the asymptotic limit has changed.

One can imagine that understanding the conjugacy classes of the symplectic group might be a good starting point when attempting a systemetic approach. A discussion of the conjugacy classes is given in F.1, but unfortunately did not provide helpful insight.

6.1 STABILIZER GROUP

A vector of harmonic functions that corresponds to flat space is given by (4.40):

$$\mathbb{V}_{\texttt{flat}} = \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{6.1}$$

To start off, we can determine what subgroup $g_1 \subset Sp(6, \mathbb{R})$ leaves $\mathbb{V}_{\texttt{flat}}$ unchanged. Elements $M_1 \in g_1$ of this subgroup should obey $M_1\mathbb{V}_{\text{flat}} = \mathbb{V}_{\text{flat}}$, corresponding to the requirement that a generator X_1 (defined as $M_1 \equiv e^{X_1}$) should satisfy $X_1\mathbb{V}_{\texttt{flat}} = \Omega^{-1}S_1\mathbb{V}_{\texttt{flat}} = 0$. This means that S_1 should be a symmetric matrix of the form

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & & & & & 0 \\ 0 & & & & & 0 \\ 0 & & & & & 0 \\ 0 & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.2}$$

where the empty elements are free scalars.

It can now be observed that the non-zero elements in the above matrix, when multiplied by Ω^{-1}, form the 10 dimensional set of generators $\mathfrak{sp}(4, \mathbb{R})$. And since the dimension of a Lie group is equal to the dimension of the corresponding Lie algebra, and the dimension of the group $Sp(4, \mathbb{R})$ is 10, it can be concluded that the empty elements in the matrix above correspond to generators for the entire group $Sp(4, \mathbb{R})$.

6.2 ASYMPTOTICALLY FLAT SOLUTIONS

Another property of interest is the asymptotic limit $r \to \infty$. In general we will study solutions of spacetime in which matter is placed on an empty background, the bending of spacetime as a result of this matter should be local, and thus in the asymptotic limit the background geometry should be recovered. Using this it can be determined that matrices $M_2 \in Sp(6, \mathbb{R})$ satisfying $M_2 \mathbb{V}_{\text{flat}} \overset{r\to\infty}{=} \mathbb{V}_{\text{flat}}$, are of the form

$$M_2 = \begin{pmatrix} 1 & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 0 \\ & & & & & 1 \end{pmatrix}, \tag{6.3}$$

where again the empty matrix elements represent 'free' scalars, with some restrictions as $M_2 \in Sp(6, \mathbb{R})$. We can thus conclude that the requirement to recover the flat background metric asymptotically results in seven more restrictions on the elements of the group we act with, as represented by the seven fixed parameters in M_2.

Flat space to Black String transformations

The dimension of the symplectic group in six dimension on real space is 21. In the section above we found that elements of $Sp(4, \mathbb{R})$ (which has dimension 10) preserve flat space in the form of equation (4.40), and we found that asymptotic flatness results in seven more restrictions on the generators. We thus expect to find $21 - 10 - 7 = 4$ generators which do not preserve flat space in its current form, but do respect asymptotic flatness. These are potentially interesting solutions.

Those four generators turn out to be

$$X_1 = \begin{pmatrix} 0 & 0 \\ \lambda_1 e_{11} & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 \\ (e_{12} + e_{21}) & 0 \end{pmatrix}, \tag{6.4}$$

$$X_3 = \begin{pmatrix} 0 & 0 \\ (e_{13} + e_{31}) & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} e_{21} & 0 \\ 0 & -e_{12} \end{pmatrix}, \tag{6.5}$$

where $\lambda \in \mathbb{R}$ and e_{ij} represent $n \times n$ matrices with 1 in the position (i, j) and 0 elsewhere.

These generators have non-vanishing commutation relation

$$[X_2, X_4] = -2X_1, \tag{6.6}$$

which forms a closed subalgebra isomorphic to the Heisenberg algebra.

It can be checked that the group elements given by the exponential mappings of X_1, X_2 and X_3, generate black string solutions, whereas the group elements corresponding to the exponential mapping of X_4 generates flat space. This flat space solution does have $V_2 = \lambda/r$, but the parameter λ can be removed through a coordinate transformation.

X_3 has only vanishing commutation relations implying isomorphism to the group $U(1)$.

The transformations corresponding to X_1, X_2 and X_3 are given in matrix form by

$$e^{\lambda_1 X_1} e^{\lambda_2 X_2} e^{\lambda_3 X_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \lambda_1 & \lambda_2 & \lambda_3 & 1 & 0 & 0 \\ \lambda_2 & 0 & 0 & 0 & 1 & 0 \\ \lambda_3 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.7}$$

with λ's real numbers. This is the product of a set of three fully commuting transformations. If we use these transformations to act on flat space or a black string solution, the angular momentum, momentum and electric charge are shifted as a result of acting with X_1, X_2 and X_3, respectively.

6.3 NOT ASYMPTOTICALLY FLAT

Now that we identified the asymptotically flat solutions, we move on to generating the solutions with the seven generators that have asymptotics which differ from $\mathbb{V}_{flat}$ as represented in equation (4.40). These are generated by acting on $\mathbb{V}_{flat}$ with group elements

$$M_{ij} = \exp(\lambda X_{ij}) = \exp\left(\lambda \Omega^{-1} S_{ij}\right). \tag{6.8}$$

Following the counting of generators above, there are seven generators left which we have not yet looked at. Here the transformations of those seven different generators will be given along with a classification of the resulting metric. We will preface this list by noting that six of the solutions found below are vacuum solutions, and one of them is a plane wave solution (see section 4.2.1 for a detailed discussion of these type of solutions).

1. The first transformation we look at is

$$M_{26} \cdot \mathbb{V}_{flat} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \lambda \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ \lambda \\ 1 \end{pmatrix}, \tag{6.9}$$

for which the metric reads

$$\begin{aligned} ds^2 &= -2du\,(dv + \lambda du) + ds^2_{\mathbb{R}^4} && (6.10) \\ &= -2du dv' + ds^2_{\mathbb{R}^4}, && (6.11) \end{aligned}$$

where we defined $v' = v + \lambda u$.

The obtained solution describes flat space.

2. The transformation

$$M_{36} \cdot \mathbb{V}_{flat} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ 0 \\ \lambda \end{pmatrix}, \tag{6.12}$$

yields the metric

$$ds^2 = -2\frac{1}{\lambda} du dv + \lambda ds^2_{\mathbb{R}^4}. \tag{6.13}$$

The solution we find is again diffeomorphic to flat space, we are free to redefine the coordinates such that the λ dependent factors disappear from the metric.

3. The transformation

$$M_{66} \cdot \mathbb{V}_{flat} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\lambda \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{r} \\ 0 \\ -\lambda \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{6.14}$$

yields the metric

$$ds^2 = -2dudv + \lambda^2 r du^2 - 2\lambda r(d\psi + \cos\theta d\phi)du + ds^2_{\mathbb{R}^4}, \tag{6.15}$$

of which it is not immediately clear what space this describes. However, since the form of the metric is invariant under reparametrization of u and ψ, we can perform a gauge transformation (5.36) with $a = 1$, $b = -\lambda$, $c = 0$ and $d = 1$ to generate the following transformation of the harmonic function

$$\begin{pmatrix} 1 & -\lambda & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{r} \\ 0 \\ -\lambda \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \mathbb{V}_{flat}. \tag{6.16}$$

Thus, it turns out that this solution also describes flat space.

4. The transformation

$$M_{56} \cdot \mathbb{V}_{flat} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\lambda \\ 0 & 0 & 1 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{r} \\ -\lambda \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{6.17}$$

gives the metric

$$ds^2 = -2\left(du - \lambda r(d\psi + \chi)\right) dv + r(d\psi + \chi)^2 + \frac{1}{r}|d\vec{x}|^2. \tag{6.18}$$

Let us now redefine $\psi \to \psi - \lambda v$, and we obtain

$$ds^2 = -2dudv - \lambda^2 r dv^2 + r(d\psi + \chi)^2 + \frac{1}{r}|d\vec{x}|^2, \tag{6.19}$$

which is a plane wave metric.

5. The transformation

$$M_{16} \cdot \mathbb{V}_{flat} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\lambda & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{r} \\ 0 \\ \frac{-\lambda}{r} \\ \lambda \\ 0 \\ 1 \end{pmatrix}, \tag{6.20}$$

Yields the metric

$$ds^2 = -2du dv + \frac{\lambda^2}{r} du^2 + ds^2_{\mathbb{R}^4}. \tag{6.21}$$

This is a Ricci-flat (vanishing Ricci tensor), pp-wave spacetime (∂_v is a covariantly constant Killing vector), but is not included in the class of plane waves. One of the requirements to have a plane-wave metric is that g_{uu} is at most quadratic in the Cartesian coordinates, in this case the requirement is not satisfied.

6. The transformation

$$M_{46} \cdot \mathbb{V}_{flat} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\lambda \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{r} - \lambda \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{6.22}$$

results in a change of the Gibbons-Hawking base space. The corresponding base space metric is a Taub-NUT metric, which has been discussed in subsection 4.2.

7.

$$M_{14} \cdot \mathbb{V}_{flat} = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{e^{-\lambda}}{r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{6.23}$$

Again, in this case V_1 is the only function to have changed under the transformation, and thus the base space has changed. The corresponding base space metric is an Eguchi-Hanson metric, the properties of which are discussed in subsection 4.2.

Here we have presented a number of solutions obtained through transformations of a flat space solution with different elements of the group $Sp(6, \mathbb{R})$. We have classified the obtained solutions, which were all examples of known solutions as discussed prior in this thesis. However, this does show the potential of these transformations, and motivates why it is interesting to pursue this line of research.

7 TRANSFORMATIONS OF A PLANE WAVE SOLUTION

With the relatively recent observation of gravitational waves, new possibilities of performing astrophysical measurements have arrived, and this provides an opportunity to learn even more about our universe. Because of this, we would also like to learn more about gravitational waves in supersymmetric solutions, which is what we look into in this chapter.

We will generate solutions by acting on the plane wave solution given in (4.63). However, we mentioned that the solution corresponding to (4.63) is independent of many of the constants present in the harmonic functions, for this reason we fix $c_6 = 1,\ c_2 = c_3 = a_0 = 0$ which still describes the same solution. To simplify the solution somewhat further, we set $\vec{a} = \vec{0}$, resulting in $G = 0$ and leaving only the gravitational part of the wave. Hence our starting point will be

$$\mathbb{V} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \frac{1}{r} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \vec{b} \cdot \vec{x}. \tag{7.1}$$

Where we now write the vector of harmonic functions as the sum

$$\mathbb{V} = \Gamma_\infty + \Gamma \frac{1}{r} + \Gamma_b \vec{b} \cdot \vec{x}. \tag{7.2}$$

At times we will generate solutions which are not contained in any known class of solutions. Often this will be explained by the fact that the solution violates some generally accepted physical condition. One way to check whether we are dealing with anomalous matter, is by checking if an energy condition is violated.

Most books on general relativity will contain information on the energy conditions, see e.g. [8, Chapter 4].

The energy conditions we can check include, the null energy condition (NEC), which states

$$T_{\mu\nu} l^\mu l^\nu \geqslant 0, \tag{7.3}$$

for l a null vector field. And the strong energy condition (SEC), which is

$$\left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right) t^\mu t^\nu \geqslant 0, \tag{7.4}$$

where t^μ is a timelike vector. However since the solutions studied have $T \propto R = 0$ (see (4.7)), the SEC simplifies to

$$T_{\mu\nu} t^\mu t^\nu \geqslant 0, \tag{7.5}$$

which is also referred to as the weak energy condition (WEC). In the cases we consider the Ricci scalar vanishes, and hence the energy-momentum tensor $T_{\mu\nu}$ is proportional to the Ricci tensor $R_{\mu\nu}$ as per the field equations, meaning that if the conditions hold for the Ricci tensor, they also hold for the energy-momentum tensor. The null vector fields of our solution are ∂_v and $\partial_v + \frac{F}{2}\partial_u$.

In this section we will act with generators of the form (5.16), which act non-trivially on the most right vector Γ_b in (7.1).

- This time, the first transformation we look at is

$$M_{15} \cdot \mathbb{V} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \mathbb{V} = \begin{pmatrix} \frac{1}{r} \\ -\frac{\lambda}{r} \\ 0 \\ \lambda \vec{b} \cdot \vec{x} \\ \vec{b} \cdot \vec{x} \\ 1 \end{pmatrix} \in \mathbb{V}_I. \tag{7.6}$$

Now, remember we fixed a number of the constants that appeared in the harmonic functions (4.63) to obtain (7.1). The solution we found here is described by the set of harmonic functions (4.63), with $c_2 = -\lambda$ and the other constants set to zero: $a_0 = c_2 = c_3 = c_6 = 0$. This means that this transformation did not change the corresponding metric or field strength characterizing the solution.

- The transformation

$$M_{25} \cdot \mathbb{V} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \mathbb{V} = \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ \lambda \vec{b} \cdot \vec{x} \\ 1 \end{pmatrix} \in \mathbb{V}_I, \tag{7.7}$$

leaves the solution unchanged as well. Also this set of harmonic functions is part of the class of harmonic functions (4.63), since we are free to redefine $\vec{b} \to \vec{b}/\lambda$. After this redefinition we obtain the set of harmonic functions given in (7.1).

- The transformation

$$M_{35} \cdot \mathbb{V} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 \end{pmatrix} \cdot \mathbb{V} = \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ \vec{b} \cdot \vec{x} \\ 1 + \lambda \vec{b} \cdot \vec{x} \end{pmatrix}, \tag{7.8}$$

results in a metric

$$ds^2 = -\frac{2du(dv - du\vec{b} \cdot \vec{x})}{1 + \lambda \vec{b} \cdot \vec{x}} + (1 + \lambda \vec{b} \cdot \vec{x}) ds^2_{\mathbb{R}^4}, \tag{7.9}$$

and three-form

$$G = \frac{\lambda}{2} \vec{b} \cdot \star_4 d\vec{x} + \frac{\lambda}{2(1 + \lambda \vec{b} \cdot \vec{x})^2} \vec{b} \cdot d\vec{x} \wedge dv \wedge du. \tag{7.10}$$

We have $H = 1 + \vec{b} \cdot \vec{x}$, this means that the curvature scalar

$$R_{\mu\nu} R^{\mu\nu} = \frac{3|\vec{b}|^2 r^2 \lambda^4}{2(1 + \lambda \vec{b} \cdot \vec{x})^6}, \tag{7.11}$$

blows up at $1+\vec{b}\cdot\vec{x}=0$. Where for the solutions we have encountered thus far $H=0$ was outside the range of our coordinate system, this is no longer the case.

It can be concluded that this solution has a two-dimensional planar singularity. Also note that the signature of the metric changes sign at the position of this singularity.

This metric has a rather simple form, but it is unlike any solution we have seen before. Before attempting to understand the physical implications of this solution and provide a characterization, we should check if any energy conditions are violated.

Let us start with the null energy condition (NEC), which should hold for both null vector fields (∂_v and $\partial_u + \frac{F}{2}\partial_v$). The relevant components of the Ricci tensor are

$$R_{vv} = 0, \tag{7.12}$$

$$R_{uv} = \frac{|b|^2 r\lambda^2}{2\,(\lambda b\cdot\vec{x}+1)^4} \geqslant 0, \tag{7.13}$$

$$R_{uu} = -\frac{|b|^2 r\lambda^2\,(b\cdot\vec{x})}{(\lambda b\cdot\vec{x}+1)^4} \leqslant 0, \tag{7.14}$$

From which it is immediately clear that the NEC holds for $l^\mu=\partial_v$, since $R_{vv}=0$. For $l^\mu = \partial_u + \frac{F}{2}\partial_v$ we have

$$T_{\mu\nu}l^\mu l^\nu \propto R_{uu}+FR_{uv} = \frac{|b|^2 r\lambda^2}{(\lambda b\cdot\vec{x}+1)^4}(F-b\cdot\vec{x}) = \frac{|b|^2 r\lambda^2}{(\lambda b\cdot\vec{x}+1)^4}b\cdot\vec{x}, \tag{7.15}$$

where we used that $F=2b\cdot\vec{x}$. This means the NEC only holds in the half-space $b\cdot\vec{x}\geqslant 0$. However, remember that at the singularity the signature of the metric flips. As a result of this signature change the NEC is also satisfied for $\vec{b}\cdot\vec{x}+1\leqslant 0$, and this leaves $-1/\lambda < \vec{b}\cdot\vec{x} < 0$ as the domain in which it is not satisfied.

Furthermore, since there is a curvature singularity at $b\cdot\vec{x} = \frac{-1}{\lambda}$ one could argue that there are no spacelike paths from the region $b\cdot\vec{x} > \frac{-1}{\lambda}$ into the region where the NEC fails.

In order to perhaps better understand this solution, we can go to a five-dimensional description by reducing along the u coordinate. This results in the metric

$$\begin{aligned} ds_5^2 =& -(1+\lambda\vec{b}\cdot\vec{x})^{-4/3}(2\vec{b}\cdot\vec{x})^{-2/3}dt^2 \\ &+(1+\lambda\vec{b}\cdot\vec{x})^{2/3}(2\vec{b}\cdot\vec{x})^{1/3}ds^2_{\mathbb{R}^4}, \end{aligned} \tag{7.16}$$

which has a Ricci scalar

$$R = \frac{-4|\vec{b}|^2|x|\left(1+(1+3\lambda\vec{b}\cdot\vec{x})^2\right)}{9(\vec{b}\cdot\vec{x})^2(1+\lambda\vec{b}\cdot\vec{x})^2}. \tag{7.17}$$

From this it can be concluded that the singularities we observe in (7.16), are not artifacts of our choice of coordinates, but they are actual curvature singularities. Which means we can interpret these two curvature singularities as two parallel planes. There appears not to be a horizon in (7.16) meaning that we are dealing with naked singularities.

The curvature tensor and three-form 'blow-up' at $1+\lambda\vec{b}\cdot\vec{x}$, meaning that this solution corresponds to one with infinite mass density at this singularity. Remember that we introduced the negative branes in subsection 3.5.4, which might explain a change in singularity but also do not quite fit this picture. Thus, for now we will not look into this solution any further as it does not seem to fit with any explanation.

- An interesting transformation is performed using the matrix below, which transforms our starting solution with ∂_v covariantly constant, into a type pp-wave spacetime with ∂_u covariantly constant. However, in order for this solution to describe a plane wave we require β to be preserved along the isometries of flat $\mathbb{R}^4$, this is however not the case.

  The transformation is as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{\lambda} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \mathbb{V} = \begin{pmatrix} \frac{1}{r} \\ \lambda b \cdot \vec{x} \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{7.18}$$

  Which corresponds to

$$H = 1, \quad \omega = F = 0, \tag{7.19}$$

$$\beta = \lambda r \vec{b} \cdot \vec{x} \, (d\psi + \cos\theta) \, d\phi - \frac{1}{2} \epsilon_{ijk} b^i x^j \, dx^k, \tag{7.20}$$

  yielding the metric

$$ds^2 = -2(du + \beta) dv + ds^2_{\mathbb{R}^4}, \tag{7.21}$$

  and three-form

$$G = \frac{1}{2} dv \wedge d\beta. \tag{7.22}$$

  Properties of this solution are a vanishing of the Kretschmann and $R_{\mu\nu}R^{\mu\nu}$ curvature scalars. R_{vv} is the only non-zero component of the Ricci tensor, and it scales with r^2. Since the Ricci tensor is proportional to the stress-energy tensor, this means that the physical interpretation of such a Ricci tensor corresponds to a spacetime with matter density growing with increasing r. This suggests a anomalous matter, and we will not proceed investigating this solution.

  Even though we do not proceed investigating this solution, it is important to note that this transformation maps a plane wave solution characterized by a covariantly constant killing vector ∂_v, to a pp-wave spacetime characterized by a covariantly constant Killing vector ∂_u.

- Another possibly interesting transformation is

$$M_{55} \cdot \mathbb{V} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \mathbb{V} = \begin{pmatrix} \frac{1}{r} \\ \lambda b \cdot \vec{x} \\ 0 \\ 0 \\ b \cdot \vec{x} \\ 1 \end{pmatrix}. \tag{7.23}$$

This corresponds to the functions

$$H = 1, \tag{7.24}$$

$$F = 2\vec{b}\cdot\vec{x}, \tag{7.25}$$

$$\beta = \lambda r \vec{b}\cdot\vec{x}\,(d\psi + \cos\theta) - \frac{1}{2}\epsilon_{ijk} b^i x^j d^k, \tag{7.26}$$

$$\omega = r\lambda(\vec{b}\cdot\vec{x})^2(d\psi + \cos\theta d\phi). \tag{7.27}$$

For this solution computation of the analytic expression using *Wolfram mathematica* has become too demanding. However, if we simplify the problem by setting $b_1 = b_2 = 0$, we have calculated the Ricci tensor. Some of the simplest expressions of the Ricci tensor are

$$R_{\nu\nu} = b_3^2 r^2 \lambda^2 (3\cos^2(\theta) + 1) \geqslant 0, \tag{7.28}$$

$$R_{u\nu} = b_3^2 r^2 \lambda^2 (3\cos^2(\theta) + 1) \geqslant 0, \tag{7.29}$$

$$R_{uu} = b_3^4 r^4 \lambda^2 \cos^2(\theta)(3\cos^2(\theta) + 2) \geqslant 0, \tag{7.30}$$

which can be used to check the validity of the NEC. One can immediately see that the NEC for the vector ∂_ν is satisfied. However, for the null vector $\partial_u + \frac{F}{2}\partial_\nu$ we require $\frac{F^2}{4}R_{\nu\nu} + FR_{u\nu} + R_{uu} \geqslant 0$. In this expression the $FR_{u\nu}$ term dominates at $r << 1$, and for $\vec{b}\cdot\vec{x} < 0$ the term becomes negative, meaning there is a small regime for $r << 1$ and $\vec{b}\cdot\vec{x} < 0$ where the NEC is not satisfied.

Furthermore, the Kretschmann scalar reads

$$K = \frac{1}{2} b_3^3 r^2 \lambda^2 \cos(\theta)\left(143 b_3^3 r^4 \lambda^2 \cos(\theta) + 33 b_3^3 r^4 \lambda^2 \cos(3\theta) + 128\right), \tag{7.31}$$

thus asymptotically the curvature scalar approaches infinity. This would imply that there is a region of infinite mass density in the large r limit, similar to the previous solution we studied the energy-momentum tensor also increases for larger r. For those reasons, as well as the violation of the NEC we will not proceed the analysis of this solution.

It should be noted that similar calculations have been performed by setting $b_1 = b_3 = 0$ and $b_2 = b_3 = 0$. Although the expressions are slightly different, the analysis, and as such the subsequent conclusion, remains the same.

Here we have performed a similar analysis as in chapter 6. However, in this case the obtained solutions were in some cases considerably more complex. We have found solutions with curvature scalars approaching infinity in the asymptotic limit, corresponding to infinite matter density in this limit. As it is clear that those solutions carry no physical explanation, we did not pursue analyzing them. We furthermore encountered solutions with planar singularities, and although this seems curious and we were not able to understand these solutions in terms of extended branes, it is not ruled out that further research might provide an explanation for these solutions.

8 SPECTRAL FLOW TRANSFORMATIONS OF THE BTZ BLACK HOLE

As discussed in subsection 4.2.5 it is possible to find solutions describing $BTZ \times S^3$, by acting on flat space with an element of the symplectic group. The Harmonic functions describing the vacuum $BTZ \times S^3$ solution are the same as for $AdS_3 \times S^3$. To obtain the BTZ solution from the Anti-de Sitter solution, a periodic identification of a Killing vector is introduced, as discussed in subsection 2.1.6.

Here we will act with M_{BTZ} on the vacuum BTZ solution, to 'switch on' an extra parameter in $BTZ \times S^3$ transforming the vacuum solution into a non-vacuum black hole. Acting with M^k_{BTZ} results in an extremal BTZ metric with mass $M = \frac{2k}{\mu}$ and angular momentum $J = \frac{4k}{\sqrt{\mu}}$. Then, we act with the spectral flow transformation (5.28) with $\gamma_2 = 0$, which we denote M_γ, in the hope that it will result in an interesting solution. The reason for choosing this specific matrix is that it was shown in [6], to result in a nontrivial transformation, namely it transformed flat space into a plane wave solution. In this section we will analyze the solution we find as a result of the aforementioned transformations.

Performing the transformations described above, one obtains harmonic functions of the form

$$M_\gamma M^k_{BTZ} \mathbb{V}_{AdS_3 \times S^3} \tag{8.1}$$

$$= \begin{pmatrix} 1 & 0 & 2\gamma & 0 & -2\gamma^2 & 0 \\ 0 & 1 & 0 & 2\gamma^2 & 0 & -2\gamma \\ 0 & 0 & 1 & 0 & -2\gamma & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2\gamma & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^k \begin{pmatrix} \frac{1}{r} \\ 0 \\ 0 \\ 0 \\ n \\ \frac{\mu}{r} \end{pmatrix} \tag{8.2}$$

$$= \begin{pmatrix} -2n\gamma^2 \\ 0 \\ -2n\gamma \\ 0 \\ n \\ 0 \end{pmatrix} + \begin{pmatrix} 1-2k\gamma^2 \\ -2\mu\gamma \\ -2k\gamma \\ 0 \\ k \\ \mu \end{pmatrix} \frac{1}{r}. \tag{8.3}$$

The expression of the corresponding metric tensor is not very illuminating, and is therefore omitted. The same goes for the explicit expression of the three-form. When discussing the Reissner-Nordström black hole in subsection 2.1.1, the relevance of the parameter H, in the metric became clear. In the context of the Reissner-Norström black hole solution the location of the black hole singularity corresponds to $H = 0$, and the black hole horizon corresponds to H blowing up. Therefore, analyzing point where H either vanishes or blows up, might prove useful in analyzing the solution. For the solution discussed here, we have

$$H = -\frac{\left(1 + 2\gamma^2(k + nr)\right)\mu}{r\left(-1 + 2\gamma^2(k + nr)\right)}. \tag{8.4}$$

Even though the metric and three-form of this solution appear rather incomprehensible, there are some aspects of the geometry we can check. To begin, we can calculate a curvature scalar

$$R_{\mu\nu}R^{\mu\nu} = \frac{3\left(4\gamma^4(k+nr)^2+4\gamma^2(k-nr)+1\right)^2}{2\mu^2\left(2\gamma^2(k+nr)+1\right)^6}, \tag{8.5}$$

from which it can be concluded that there is a curvature singularity when $H(r_s) = 0$, where

$$r_s = -\frac{1+2k\gamma^2}{2\gamma^2 n}. \tag{8.6}$$

Sign of H

The position r of the curvature singularity depends on the values of the parameters γ, k, μ and n. It is possible for the singularity to be at a point where $r < 0$, in which case it is outside the domain of the coordinate system. However, it is also possible for the singularity to be at a point where $r > 0$, in which case we cannot simply ignore the singularity.

The parameter μ, characterizes the curvature for the original $AdS_3 \times S^3$, therefore μ has to be positive. Then, to avoid negative mass terms, k should be non-negative. The vacuum solutions (for which $k = 0$), has been studied in [7], and therefore here we take $k > 0$. Remember also that $m \in \mathbb{Z}$, form which it follows that solutions in the regime of parameter space where $2k\gamma^2 < 1$, are not well-defined. We are thus interested in solutions for which $2k\gamma^2 > 1$. The parameter n is still free. We mentioned that analyzing the sign of H is helpful in understanding the Reissner-Norström metric, and for similar reasons we will analyze the sign of H for the new solution as well.

For $n > 0$

$$H > 0 \quad \text{for} \quad r < r_s,\ r_B < r < 0, \tag{8.7}$$
$$H < 0 \quad \text{for} \quad r_s < r < r_B,\ r > 0, \tag{8.8}$$

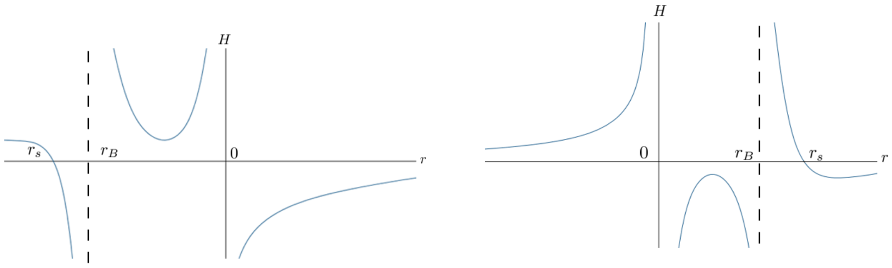
while for $n < 0$

$$H > 0 \quad \text{for} \quad r < 0,\ r_B < r < r_s, \tag{8.9}$$
$$H < 0 \quad \text{for} \quad 0 < r < r_B,\ r > r_s. \tag{8.10}$$

Here r_B is defined $V_1(r_B) = 0$, which gives $r_B = \frac{1-2k\gamma^2}{2n\gamma^2}$. A graphic representation of the above is given in figure 8.1. From the investigation of the sign of H, it immediately becomes clear that for $n > 0$, H approaches negative infinity at $r = 0$, while r_B and r_s are outside the domain of r. This is different for $n < 0$, where an observer approaching the black hole from infinity, first encounters r_s and r_B. We already know that r_s corresponds to a curvature singularity, and still wish to understand what happens at $r = r_B$ and $r = 0$.

$r = r_B$

When $r = r_B$ we have that $V_1 = 0$. However, it turns out that in most components of the metric, the V_1 terms cancel and thus nothing noteworthy happens at the point $r = r_B$. The only component in the metric where V_1 is still present the component g_{uv}, which vanishes when $r = r_B$. In solutions where the part of the metric depending on du and dv forms a direct product with the base space, this would mean that we might be dealing with an ergosphere. This follows from the fact that we can transform the lightcone coordinates to the standard coordinate



(a) $n > 0$ (b) $n < 0$

Figure 8.1: Graphs showint he form of H for positive and negative n. Points where H vanishes or blows up have been labeled r_B, r_s and 0.

system, where the vanishing of $g_{\mu\nu}$ would correspond to the vanishing of the timelike component. Here, the metric components are not simply a direct product, but one can still wonder whether there is something like an ergosphere at this point, perhaps just hidden behind the choice of coordinates. Ideally we would now go to the asymptotic limit, identify the time coordinate for a distant observer and see if this coordinate vanishes at $r = r_B$. In case it does, we would have confirmed that there is an ergosphere at this point. However, as we we discuss later in section 8.3 the asymptotics of this solution are not well understood. The timelike vector corresponds to energy conservation, and as such it should be a Killing vector. In the current coordinate system however, we were not able to identify a timelike vector in the asymptotic limit, and attempted coordinate transformations have also failed to provide the desired result. For this reason it seems likely that the vanishing of a metric component at the point $r = r_B$ is in fact just an artifact of our choice of coordinates. One way to confirm this would be by defining the killing vectors of a solution, and see if they describe a horizon at this location.

8.1 METRIC SIGNATURE

To determine the three-from field strength we have to calculate a Hodge dual on the base space, and thus we will have to know how the signature of the base space in different regimes of parameter space. This motivates the study of the metric signature below.

Let us start with the base space metric. One can calculate the eigenvalues of this metric and see that they have signature $(4,0)$ for $V_1 > 0$ and $(0,4)$ for $V_1 < 0$.

Let us take a closer look at the signature of the base space metric:

For $n > 0$ the signature of the metric is

$$(4,0) \quad \text{for} \quad r_B < 0 < r \tag{8.11}$$

$$(0,4) \quad \text{for} \quad r < r_B,\ r > 0. \tag{8.12}$$

While in the case where $n < 0$, the metric signature is

$$(4,0) \quad \text{for} \quad r < 0,\ r > r_B \tag{8.13}$$

$$(0,4) \quad \text{for} \quad 0 < r < r_B. \tag{8.14}$$

For the six-dimensional metric the calculation of the eigenvalues becomes less trivial, therefore these calculations were performed using Mathematica. It turns out that the metric changes signature only in the singularity. For $n > 0$ the metric signature is as follows:

$$(5,1) \quad \text{for} \quad r > r_s, \tag{8.15}$$

$$(1,5) \quad \text{for} \quad r < r_s. \tag{8.16}$$

And for $n < 0$ the metric signature is

$$(5,1) \quad \text{for} \quad r < r_s, \tag{8.17}$$

$$(1,5) \quad \text{for} \quad r > r_s. \tag{8.18}$$

Since the convention in this thesis is to work in the 'mostly plus' metric signature, we multiply the metric by an overall minus sign in the region where the metric signature is $(1,5)$, such that we recover a $(5,1)$ signature. This is a somewhat surprising result. Remember that in subsection 4.2.4 we noticed that in the solution of the BTZ black hole, n was a redundant parameter. Here n is clearly no longer redundant, and it would be nice to gains some understanding as to the physical significance of n in this case. Unfortunately, we have not been able to comprehend the meaning of this parameter and it remains an open problem.

8.2 HORIZON

Where we were unable to find a satisfying explanation for the point $r = r_B$, we have found that at $r = 0$ there is an event horizon. In this section we will discuss the topological and geometrical properties of this horizon.

Setting $r = 0$ yields the metric

$$\begin{aligned} ds^2|_{r=0} =& \frac{2k du^2}{\mu\left(1+2\gamma^2 k\right)} + \frac{\mu d\psi^2}{1+2\gamma^2 k} + \left(\mu + 2\gamma^2 k\mu\right)\left(d\theta^2 + d\phi^2\right) \\ &+ \frac{8\gamma k \mathrm{Cos}[\theta] du d\phi}{1+2\gamma^2 k} + \frac{2\left(1-2\gamma^2 k\right)\mu \mathrm{Cos}[\theta] d\phi d\psi}{1+2\gamma^2 k}, \end{aligned} \tag{8.19}$$

which extends in four spacelike dimensions. This corresponds to a hypersurface in six-dimensional spacetime.

The $r = 0$ surface has normal vector

$$N_\mu = \nabla_\mu r = \delta^r_\mu, \tag{8.20}$$

of which the norm is

$$|N|^2 = g^{\mu\nu}\delta^r_\mu \delta^r_\nu = g^{rr} = \frac{r^2}{\mu(1+2\gamma^2(k+nr))}. \tag{8.21}$$

The norm vanishes at the $r = 0$ surface, meaning it is a null hypersurface.

Furthermore, the Killing vector field $V^\mu = (\partial_\nu)^\mu$ can be used to calculate the surface gravity κ as defined in (2.39). We have

$$\kappa V^\nu = V^\mu \nabla_\mu V^\nu = V^\mu\left(\partial_\nu + \Gamma^\nu_{\mu\rho}\right)V^\rho = \Gamma^\rho_{\nu\nu} = 0, \tag{8.22}$$

for all ν. Thus it can be concluded that the surface gravity vanishes for this Killing horizon. Remember that vanishing surface gravity (or black hole temperature) is characteristic of an extremal black hole, and BPS objects are expected to be extremal.

8.2.1 *Horizon topology*

Let us now make the identification $u = u + L$, and define

$$\zeta = \frac{4\gamma k u + \mu(1-2\gamma^2 k)\psi}{\mu^2(1+2\gamma^2 k)}, \tag{8.23}$$

$$z = \frac{(2k\gamma^2 - 1)u + 2\gamma\mu\psi}{1+2k\gamma^2}, \tag{8.24}$$

where we require

$$\frac{L}{4\pi}\frac{4\gamma k}{\mu(1-2\gamma^2 k)} \in \mathbb{Z}, \quad \frac{L}{4\pi}\frac{2k\gamma^2-1}{2\gamma\mu} \in \mathbb{Z}, \tag{8.25}$$

for this coordinate change to be well defined. This leads to a quantization of the possible values of L, as well as an additional restriction

$$\frac{8\gamma^2 k}{(1-2\gamma^2 k)^2} \in \mathbb{Q}, \tag{8.26}$$

on the parameters characterizing the solution.

In these coordinates the metric reads

$$\begin{aligned} ds^2|_{r=0} =& \frac{2k}{\mu(1+2k\gamma^2)}dz^2 + \mu(1+2\gamma^2 k)(d\theta^2 + \sin^2\theta d\phi^2 + (d\zeta + \cos\theta d\phi)^2) \\ =& \frac{2k}{(1+2k\gamma^2)\mu}dz^2 + 4\mu(1+2\gamma^2 k)d\Omega_3^2, \end{aligned} \tag{8.27}$$

where we can that the horizon has $S^1_z \times S^3$ topology.

The charges as given in (4.56) are now

$$\tilde{Q} = 4\sqrt{2}\mu\left(1+2\gamma^2 k\right), \quad Q = 8k, \quad J = -32\gamma k\mu, \tag{8.28}$$

giving a non-vanishing entropy:

$$S = 16\pi\sqrt{2k\mu^2}, \tag{8.29}$$

independent of γ as expected since M_γ is an entropy conserving transformation.

As a consistency check we can also calculate the entropy by integrating over the area of the horizon $\mathcal{H}$ at $r = 0$:

$$A = \int_{\mathcal{H}} dA\sqrt{g} = \int_{\mathcal{H}} \sqrt{2k\mu^2\sin^2(\theta)du d\psi d\theta d\phi} = 16\pi^2 L\sqrt{2k\mu^2}. \tag{8.30}$$

One might think that we have to adjust the range of ψ in the integral since the coordinates are defined on a $\mathbb{Z}_m$ orbifold. However, in our coordinate system $\psi = \psi + 4\pi$, and changing this would induce a change of the metric components leaving the expression for the entropy S invariant.

We now substitute this area in the equation for the Bekenstein-Hawking entropy to obtain:

$$S = \frac{A}{4G_N} = 16\pi\sqrt{2k\mu^2}, \tag{8.31}$$

where we work in conventions the conventions of [72], in which $G_N = \frac{1}{4}\pi L$. Hence the results of (8.29) and (8.31), are in agreement as expected.

8.2.2 *Near-horizon geometry*

We can also look at the near horizon geometry of the $S^1_z \times S^3$ surface at $r = 0$. It has been checked and confirmed that going to the limit $r \to 0$ in the metric yields the same result as taking this limit in the harmonic functions. Thus the near horizon geometry is parametrized by the set harmonic functions

$$\mathbb{V} = \begin{pmatrix} 1-2k\gamma^2 \\ -2\mu\gamma \\ -2k\gamma \\ 0 \\ k \\ \mu \end{pmatrix} \frac{1}{r}, \tag{8.32}$$

corresponding to

$$H = \frac{1}{r}(\mu + 2\mu k\gamma^2), \quad F = \frac{1}{r}(2k + 4k^2\gamma^2), \tag{8.33}$$

$$\beta = \frac{2\mu\gamma}{2k\gamma^2 - 1}d\psi, \tag{8.34}$$

$$\omega = -\frac{1}{r}\left(4k\gamma\mu + 8\mu k^2\gamma^3\right)\left(d\psi + (1 - 2k\gamma^2)d\phi\right), \tag{8.35}$$

where it should be noted that the solution does not depend on n in this regime.

Using the coordinate transformation defined in (8.23), we can describe the resulting metric describing the near-horizon geometry as

$$\begin{aligned}\lim_{r\to 0} ds^2 =& \frac{2k}{(1+2k\gamma^2)\mu}dz^2 + \frac{2r}{\mu(1+2k\gamma^2)}dzdv \\ &+ \frac{\mu}{r^2}(1+2k\gamma^2)dr^2 + 4\mu(1+2\gamma^2 k)d\Omega_3^2.\end{aligned} \tag{8.36}$$

This metric is locally $AdS_3 \times S^3$ which can be noticed by observing that it is of the same form as (C.3), which after making the necessary identifications as described in subsection 4.2.4, can be recognized as a direct product $BTZ_k \times S^3/\mathbb{Z}_m$.

8.3 ASYMPTOTICS

We will now study the asymptotic limit of the metric by taking $r \to \infty$ in the harmonic functions. This yields

$$\mathbb{V} = \begin{pmatrix} -2n\gamma^2 \\ 0 \\ -2n\gamma \\ 0 \\ n \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2\mu\gamma \\ 0 \\ 0 \\ 0 \\ \mu \end{pmatrix}\frac{1}{r}, \tag{8.37}$$

giving the corresponding metric

$$\begin{aligned}ds^2|_{r\to\infty} =& \frac{2dvd\psi}{\gamma n} + \frac{d\psi^2\mu}{2\gamma^2 nr} + \frac{2\gamma^2 n\mu dr^2}{r} + \frac{2dudvr}{\mu} \\ &+ 2\gamma^2 n\mu r d\theta^2 + 4dvd\phi\gamma \mathrm{Cos}[\theta]r + 2d\phi^2\gamma^2 n\mu r \mathrm{Sin}[\theta]^2,\end{aligned} \tag{8.38}$$

and three-form

$$\begin{aligned}G =& -\frac{dr\wedge du\wedge dv}{2\mu} + \gamma\mathrm{Cos}[\theta]dr\wedge dv\wedge d\phi \\ &+ \gamma r\mathrm{Sin}[\theta]dv\wedge d\theta\wedge d\phi + \frac{\sqrt{n^2}\mu\mathrm{Sin}[\theta]d\theta\wedge d\phi\wedge d\psi}{2n} \\ =& \left(\frac{1+\star_6}{2\mu}\right) du\wedge dv\wedge dr \\ &+ \gamma\mathrm{Cos}[\theta]dr\wedge dv\wedge d\phi + \gamma r\mathrm{Sin}[\theta]dv\wedge d\theta\wedge d\phi,\end{aligned} \tag{8.39}$$

where in the second step it was assumed that $n > 0$. Here the first term in the three-form is equal to the three-form corresponding to $AdS_3 \times S^3$ as presented in (4.45).

From our metric or three-form it is not immediately clear how to interpret our asymptotic space. Acting on the harmonic functions with the element of the symplectic group corresponding to gauge transformations (5.26), does not simplify the solution either.

We do have a constant Ricci tensor in the asymptotic limit, meaning that the energy-momentum tensor is constant, which implies that the solution might correspond to pp-waves in this limit. As describe in section 2.2 a pp-wave is characterized by a covariantly constant null Killing vector field, the metric (8.38) contains the null vectors ∂_v and ∂_u. However, they are not covariantly constant, since $d(g_{v\mu}dx^\mu) \neq 0$ and $d(g_{u\mu}dx^\mu) \neq 0$. Thus it can be concluded that the interpretation of the solution in the asymptotic limit as a pp-wave spacetime is not correct.

8.4 NEGATIVE BRANES

We might be able get some understanding in terms of a microscopic description in string theory, of the signature change we found in our solution by introducing the negative branes discussed in subsection 3.5.4. Because, as mentioned there one of the main properties of negative brane solutions is that they contain a singularity at some $r > 0$ where the signature changes.

We have to compactify four dimensions of the type IIB string theory to obtain a six dimensional theory, this leaves several options for the D-brane configuration. The options we have are introducing $D1^-$ branes, $D3^-$ branes wrapped on a 2-cycle, or $D5^-$ branes wrapped on a 4-cycle. In ten dimensions these would result in different metric signature for $r < r_s$, since the metric signature is $(10-p, p)$, but we will compactify the solution on the wrapped cycles and thus the lower dimensional metric signature does not depend on this choice of brane.

Regardless of the choice of brane, to obtain the six-dimensional solution from ten-dimensional string theory we have to compactify four dimensions. This will result in the required $(5, 1)$ metric signature on both sides of the singularity.

The brane direction should correspond to Killing vectors in our six-dimensional solutions. Let us denote therefore denote the directions of the brane that are not compactified as

$$p = p_1 \partial_u + p_2 \partial_v. \tag{8.40}$$

Because the brane directions change signature at the singularity, we need the brane to be positioned such that $|p|^2$ changes sign at the singularity. We have

$$|p|^2 = p^\mu g_{\mu\nu} p^\nu = \frac{2p_1}{\mu + 2\gamma^2 \mu(k+nr)} \Big(p_1(k+nr) + p_2 r(2\gamma^2(k+nr) - 1) \Big). \tag{8.41}$$

When crossing the singularity the above norm changes sign as a result of the change of signature of the metric. However, in general, the denominator in of the norm also changes sign at this point, resulting both sign flips canceling out at this singularity. To avoid this we would like to choose p_1 and p_2 such that the norm does change sign at the singularity, which is the case if $p_1(k+nr) + p_2 r(2\gamma^2(k+nr)-1) \sim 1 + 2\gamma^2(k+nr)$. Unfortunately, the only way to achieve this is for the scalars p to be dependent on r. This means the direction parallel to the brane is not described by a Killing vector in the metric we are trying to understand, and hence the metric we found does not have a microscopic description in terms of negative branes.

Besides ∂_u and ∂_v, ∂_ψ is also a Killing vector of the solution. However, including this vector in the search for a direction parallel to the brane, did not resolve the issue.

9 CONCLUSIONS AND DISCUSSION

In this thesis we have presented a review of black holes and higher dimensional solutions in general relativity. This review is extended to that of supersymmetric solutions in supergravity, the study of which has gained interest because of its relevance in the context of string theory.

We inspected properties of the group $Sp(6, \mathbb{R})$ in the context of minimal six-dimensional supergavity, and in particular explored the phase space of solutions by transforming solutions into solutions using elements of this group.

Transformations induced by the symplectic group are studied in the context of flat space, where we find the stabilizer group to be $Sp(4, \mathbb{R})$ and provide a classification of the generated solutions, while separately studying solutions which are asymptotically flat and those which are not.

A similar approach was taken in the context of plane wave solutions. In this case, the solutions we found included solutions with a planar curvature singularity, although we were not able to construct a physical argument for their existence, they are an interesting appearance, and provide an example of a property of a solution, which has by our best knowledge not been study before. Further research is needed to gain a better understanding of these singularities. Moreover, continuing the study of the symplectic group acting on plane wave solutions deserves our attention, and might lead to more intriguing solutions and insights.

Finally, we studied the tranformation of a BTZ black hole using a spectral flow transformation, which is an element of the entropy conserving subgroup of $Sp(6, \mathbb{R})$. We found that the solution had a horizon at $r = 0$, which is topologically $S^1 \times S^3$ and its geometry is locally isomorph to $AdS_3 \times S^3$. The asymptotic limit is however still not well understood, and more work on this is needed. Furthermore, we found that a parameter n, which is redundant in the description of the BTZ black hole, turned out to have a significant impact on the solution after this transformation. When $n < 0$ the solution contains a naked singularity, and at this point the metric signature flips. These properties are similar to that of negative branes, but we showed that a microscopic explanation could not be provided by these negative branes. We are not aware of any other objects with similar properties, and understanding physical role of n is still an open problem.

At the very least it can be said that the group $Sp(6, \mathbb{R})$ provides a promising set of transformations worth studying. In the best case the $Sp(6, \mathbb{R})$ could have an origin in the microscopic theory giving meaning to the group. But regardless, it is able to generate supersymmetric solutions of objects which have not been studied before, and one can be hopeful to find new solutions of spacetime by proceeding to explore the solutions to the theory using the elements of this group.

For further research I propose to attempt a more careful approach. A strong physical intuition tells us that in the asymptotic limit we should be able to recover the vacuum background solutions. In the case of minimal six-dimensional supergravity, we know that there are three distinct maximally supersymmetric solutions, those being flat $\mathbb{R}^{1,5}$, $AdS_3 \times S^3$ and CW_6 [4]. An approach for further research would be to find the most general set of harmonic functions describing these backgrounds, this has already been done for $\mathbb{R}^{1,5}$ in this thesis, for the results see (4.42). Deviation from these spacetimes in the asymptotic limit does not guarantee the solution solution is not worth studying, but considering the algebra consists of 21 generators, placing thought through restrictions on the solutions is advisable. Later on one could loosen this constraint by allowing for pp-waves.

Even if this method does not provide any 'new' solutions, it will enable us to make the statement that all solutions in the class obeying certain constraints, are already known, and the existence of possibly undiscovered solutions can then be ruled out.

Having found a potentially interesting solution in terms of a number of parameters in the harmonic functions, one should investigate the region in this parameter space where the solution is regular, meaning no closed timelike curves or other anomalies occur.

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Part III

APPENDIX

A CONVENTIONS

Some natural units are set to one:

$$c = \hbar = k_B = 1. \tag{A.1}$$

The spacetime metric used has sign convention 'mostly plus':

$$\eta_{\mu\nu} = (-, + \cdots +). \tag{A.2}$$

The (anti-)symmetrization of indices has weight 1, i.e.

$$T_{(\mu_1\mu_2\ldots\mu_n)} = \frac{1}{n!}\left(T_{\mu_1\mu_2\ldots\mu_n} + \text{ sum over permutations of indices }\right), \tag{A.3}$$

and similarly for anti-symmetric indices, except there the sum of indices is alternating, meaning that terms corresponding to an odd number of permutations obtain negative sign.

The Levi-Civita symbol is defined by

$$\epsilon_{\mu_1\mu_2\ldots\mu_n} = \begin{cases} +1, & \text{if } (\mu_1, \mu_2, \ldots, \mu_n) \text{ is an even permutation of } (1, 2, \ldots, n) \\ -1, & \text{if } (\mu_1, \mu_2, \ldots, \mu_n) \text{ is an odd permutation of } (1, 2, \ldots, n) \\ 0, & \text{otherwise} \end{cases} \cdot \tag{A.4}$$

Differential p-from components are defined by

$$\phi_p = \frac{1}{p!}\phi_{\mu_1\mu_2\ldots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p} \tag{A.5}$$

The exterior derivative acts form the left, hence we denote

$$dA = \partial_\nu A_\mu dx^\nu \wedge dx^\mu, \quad A = A_\mu dx^\mu. \tag{A.6}$$

We denote the set of integers $\{1, 2, \ldots, n\}$ as $[n]$.

γ's with multiple indices are antisymmetric products of gamma matrices

$$\gamma^{\mu_1\cdots\mu_p} = \frac{1}{p!}\left(\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_p} \pm \text{permutations of } \mu_1 \ldots \mu_p\right). \tag{A.7}$$

Unless otherwise specified, all the spinors are symplectic Majorana

$$\chi^i = \epsilon^{ij}(\chi^j)^c, \quad (\chi_i)^c = (\chi^i)^\dagger\gamma_0. \tag{A.8}$$

B

GEOMETRY

Since we are working in curved spacetime, it is convenient to introduce some of the concepts and tools that will prove useful in describing such spacetimes, and which return throughout this thesis. Some aspects are considered prerequisites, these include the metric, affine connection, Riemann tensor, and other concepts fundamental to understanding general relativity. Usually books on general relativity will discuss these things, e.g. [8, 10].

For those not familiar with differential forms, the vielbein formalism or complex geometry, this appendix provides an elementary introduction. For a more in-depth discussion of these subjects and other geometrical objects used in physics, see e.g. [51, 73, 74].

B.1 DIFFERENTIAL FORMS

Differential forms provide a method for constructing coordinate invariant expressions, simplifying certain calculations. For example that of the curvature tensor. Differential forms also play a central role in differential topology, such as De Rham cohomology.

To begin, a differential p-form $A^{(p)}$ is simply a $(0, p)$ tensor that is completely antisymmetric. And a p-from in D dimensions has

$$\binom{D}{p} = \frac{D!}{p!(D-p)!} \tag{B.1}$$

independent components, which is taken into account for the normalization of the forms.

Differential forms have some interesting properties, which show when we consider products, derivatives or integrals of forms. These three cases will be discussed below.

WEDGE PRODUCT

We can 'multiply' two forms and thereby obtain another form, if we are careful to preserve antisymmetry. Taking the product of a p-, and a q-form is done as follows:

$$A^{(p)} \wedge B^{(q)} = C^{(p+q)}, \tag{B.2}$$

which in terms of components is written as

$$C_{\mu_1 \dots \mu_{p+q}} = (A \wedge B)_{\mu_1 \dots \mu + p + q} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \dots \mu_p} B_{\mu_{p+1} \dots \mu_{p+q}]}. \tag{B.3}$$

Please be aware that often in differential-forms the wedge product symbol is not explicitly written. For example, we often write the basis of a p-form as $A^{(p)} = \frac{1}{p!} A_{\mu_1 \dots \mu_p} dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_p}$, where the differential forms dx, still anticommute.

EXTERIOR DERIVATIVE

Forms can vary over spacetime, so when we write $A_{\mu\nu}$, we really mean $A_{\mu\nu}(x)$. It would of course be useful to describe how forms vary, and this is where a derivative comes in. We could just use the partial derivative, but the result is generally difficult to work with. However, by carefully involving anti symmetry, we are able to define a derivative such that the derivative of a form gives another form.

Let us define $dA^{(p)}$, which is a (p+1)-form with components

$$dA_{\mu_1 \dots \mu_{p+1}} = (p+1)! \partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]}, \tag{B.4}$$

where is an arbitrary p-form.

This is what we call the exterior derivative (of $A^{(p)}$), and it has three especially helpful properties:

1. It satisfies a modified Leibniz product rule.
   For $A^{(p)}$ and $B^{(q)}$ we have $d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB$.

2. Due to the anti symmetric properties of the exterior derivative, as opposed to the regular partial derivative, the Christoffel connections cancel. As a result we can take these derivatives without having to worry about a metric, this means that exterior derivatives are only dependent of the topological aspects of the spacetime.

3. The symmetric property of the partial derivative used in the definition of the exterior derivative as given in (B.4), results in the vanishing of a 'second order' exterior derivative:

$$d(dA) = d^2 A = 0. \tag{B.5}$$

This feature is one of the keys to how exterior calculus of differential forms leads to topological invariants.

INTEGRATION

Once you have a derivative, it is natural to follow up with an attempt at integrating. Here the anticummuting properties of forms also turns out to be very convenient. As an example consider $dxdy$, and transform the variables as $x \to x'(x, y)$ and $y \to y'(x, y)$. which results in the transformation

$$\begin{aligned} dxdy \to dx'dy' &= \left(\frac{\partial x'}{\partial x}dx + \frac{\partial x'}{\partial y}dy\right)\left(\frac{\partial y'}{\partial x}dx + \frac{\partial y'}{\partial y}dy\right), \\ &= \frac{\partial x'}{\partial x}\frac{\partial y'}{\partial x}dxdx + \frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y}dxdy + \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x}dydx + \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial y}dydy. \end{aligned} \tag{B.6}$$

However, we know that $dxdy$ should transform with the Jacobian:

$$dx'dy' = \left(\frac{\partial x'}{\partial x}\frac{\partial y'}{\partial y} - \frac{\partial x'}{\partial y}\frac{\partial y'}{\partial x}\right)dxdy, \tag{B.7}$$

meaning that terms containing the products $dxdy$ should vanish in (B.6). This is exactly what happens when dx and dy anticommute.

From this we can observe that the basis of a p-from is actually an integration measure over a p-dimensional oriented volume. This means that an expression like $\int_{\Sigma_p} A^{(p)}$ is perfectly well defined and coordinate invariant. The physical significance of this is that there is a natural coupling between p-form fields and the p-dimensional world-surfaces swept out by (p-1)-dimensional objects.

B.2 HODGE STAR OPERATOR

Let us define the set of p-forms $\Omega(M)$ on a D-dimensional manifold M, with metric g. Because p-forms and q-forms have the same number of components if $p + q = D$ (see (B.1)), $\Omega^p(M)$ is isomorphic to $\Omega^{D-q}(M)$. This enables us to define an isomorphism between p-forms and (D-p)-forms, which we will call the Hodge operation, denoted by the Hodge star operator $\star$. The Hodge star is defined as

$$\langle\alpha, \beta\rangle \mathrm{vol}_D = \alpha \wedge \star\beta, \tag{B.8}$$

where vol_D is the volume form of M.

The action of the Hodge operator on a p-form α

$$\alpha = \frac{1}{p!}\alpha_{\mu_1\mu_2\ldots\mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}, \tag{B.9}$$

yields the Hodge dual of α, defined by

$$\star\,\alpha = \frac{\sqrt{-g}}{p!(D-p)!}\alpha_{\mu_1\mu_2\ldots\mu_p}\epsilon^{\mu_1\mu_2\ldots\mu_p}{}_{\nu_{p+1}\ldots\nu_D} dx^{\nu_{p+1}} \wedge \ldots \wedge dx^{\nu_D}. \tag{B.10}$$

Here it should be noted that the volume form vol_D, can also be written as

$$\star\, 1 = \sqrt{-g} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^D = \mathrm{vol}_D. \tag{B.11}$$

When the manifold M is 2n-dimensional, we can define the anti-self duality and self duality conditions of n-forms as

$$\star\alpha = -\,\alpha \tag{B.12}$$

$$\star\alpha = \alpha, \tag{B.13}$$

respectively.

Repeatedly acting with $\star$ on α yields

$$\star\star\alpha = (-1)^{p(D-p)}\alpha, \tag{B.14}$$

if the metric is Riemannian, and

$$\star\star\alpha = (-1)^{1+p(D-p)}\alpha, \tag{B.15}$$

if the metric is Lorentzian.

B.3 COMPLEX GEOMETRY

In this thesis we discuss solutions on four dimensional base space, where we restrict ourselves to solutions for which the base space is a Gibbons-Hawking space, which we claim is a hyper-Kähler space with an extra U(1) isometry. We are intersted in hyper-kähler manifolds because of the hypermultiplet which arises in supersymmetric theories. A hypermultiplet is made up of two half-hypermultiplets and hence contains four real scalars and two Weyl fermions. We can combine these four scalars into complex fields in three different ways, which corresponds to the existence of three complex structures. Here we will explain in some more detail what a hyper-Kähler space is. This section is largely based on [75, 76].

B.3.1 *Manifolds*

General relativity is defined on a manifold, this is a space which locally looks like $\mathbb{R}$. Take for example a sphere S^2, if we then take all distances in a local region on this sphere to be very small compared to the radius of the sphere, this enables us to make use of the theory of analysis as developed on $\mathbb{R}^n$. This does not mean that the metric on a curved space is the same as the metric on flat space, but that notions such as functions an coordinates work in a similar way. The entire manifold is then described by describing a continuous mapping form each of the local regions to its neighbors, often called the 'sewing' together of the local regions.

Imagine an n-dimensional manifold on which we can define two local regions A and B, for which there is some non-vanishing $A \cap B$. We then have mappings

$$\phi_A : A \to \mathbb{R}^n, \tag{B.16}$$
$$\phi_B : B \to \mathbb{R}^n, \tag{B.17}$$
$$\tag{B.18}$$

and on the region $A \cap B$ we have

$$\phi_A \circ \phi_B^{-1} : \mathbb{R}^n \to \mathbb{R}^n. \tag{B.19}$$

Which just describes a coordinate transformation form the region B to region A, by applying the two consecutive maps.

B.3.2 *Complex manifolds*

As the name already suggests, a complex manifold is related to the concept of complex numbers. Let take the complex number $c = x + iy$, where $x, y \in \mathbb{R}$, which we can define on the complex plane $\mathbb{C}$. Hence we understand there should be some map between an n dimensional complex space and an $2n$ dimensional real space: $\mathbb{R}^{2n} \to \mathbb{C}^n$.

Consider a 2n-dimensional manifold, on which we now define the mappings

$$\phi_A : A \to \mathbb{C}^n, \tag{B.20}$$
$$\phi_B : B \to \mathbb{C}^n, \tag{B.21}$$

and this time the intersection of a 2n-dimensional manifold enables us to define the transformation

$$\phi_A \circ \phi_B^{-1} : \mathbb{C}^n \to \mathbb{C}^n. \tag{B.22}$$

In this case we require the maps to only depend on the coordinates z^i, with $i \in [n]$, defined on the complex manifold, and not on their complex conjugates z^{i*}. Maps of these kind are called holomorphic. Thus an important difference between real manifolds and complex manifolds is that coordinate transformations are preformed using differential and holomorphic maps on real and complex space, respectively.

Let us consider a function $f : \mathbb{C} \to \mathbb{C}$, where

$$f(x + iy) = u(x, y) + iv(x, y). \tag{B.23}$$

Taking this mapping and imposing holomorphisity, we find the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{B.24}$$

We can think of a complex manifold as a real manifold with a certain different structure, this leads one to wonder in what cases we can think of a real manifold as a complex manifold with added structure. There are two restrictions on the real manifold for us to be able to make such an identification. The first one is rather obvious, and that is that the real manifold should be even dimensional.

The second requirement is not quite trivial. For this, let us consider a (1,1) tensor $J^{\mu}{}_{\nu}$ that satisfies $J^{\mu}_{\nu}J^{\lambda}_{\mu} = -\delta^{\lambda}_{\nu}$. Here $J^{\mu}{}_{\nu}$ is called an almost complex structure. Now we want to use this almost complex structure to define a complex manifold, to do so we define the Nijenhuis tensor

$$N_{\mu\nu}{}^{\rho} = (\partial_{\mu}J_{\nu}{}^{\sigma})J_{\sigma}{}^{\rho} - J_{\mu}{}^{\sigma}(\partial_{\sigma}J_{\nu}{}^{\rho}) - (\mu \leftrightarrow \nu). \tag{B.25}$$

Here we have not given an explicit expression for the almost complex structure with respect to local complex coordinates, and therefore the following argument may seem to appear out of thin air. Nevertheless, it should be mentioned that if the Nijenhuis tensor vanishes, $J_{\mu}{}^{\nu}$ is a complex structure and the manifold is a complex manifold. We will elaborate on this shortly.

General Tensors

When acting on a vector V^{μ}, $J^{\mu}{}_{\nu}V^{\nu} = V_{\mu}$ transforms a vector in tangent space to $\mathcal{M}$ at point p, into another vector in tangent space as

$$J^{\mu}_{\nu} : T_p\mathcal{M} \to T_p\mathcal{M}. \tag{B.26}$$

Here we have that $T_p\mathcal{M}$ is a real vector space at point p. To get a complex vector space, we use $T_c\mathcal{M} \equiv T_p\mathcal{M} \otimes \mathbb{C}$. Acting with $J^{\mu}{}_{\nu}$ twice gives

$$J^{\nu}\mu J^{\mu}{}_{\lambda}V^{\lambda} = -V^{\mu}. \tag{B.27}$$

From which it follows that the eigenvalues of J are $+i$ or $-i$. So we can take this complex vector space and split it up in a set of vectors in $T_p^{(1,0)}$ with eigenvalue $+i$ under the application of the complex structure, and a second set in $T_p^{(0,1)}$ with eigenvalue $-i$ under the application of the complex structure. Each of these eigenspaces are isomorphic to $\mathbb{C}^n$, complex conjugate to each other, and we can decompose

$$T_{\mathbb{C}}\mathcal{M} = T^{(1,0)}\mathcal{M} \oplus T^{(0,1)}\mathcal{M}, \tag{B.28}$$

where the (1,0) part is the holomorphic tangent bundle, and the (0,1) part is anti-holomorphic tangent bundle. Similar results can be obtained analogously for the cotangent space

$$T^*_{\mathbb{C}}\mathcal{M} = T^{*(1,0)}\mathcal{M} \oplus T^{*(0,1)}\mathcal{M}, \tag{B.29}$$

We can now consider vectors V^a in the holomorphic part of tangent space and vectors $V^{\bar{a}}$ in the anti-holomorphic part of tangent space, with the decomposition of a general vector $V^{\mu} = V^a \oplus V^{\bar{a}}$.

For $x \in T_{\mathbb{C}}\mathcal{M}$ we can form the set of holomorphic and antiholomorphic tangent space elements as

$$T^{(1,0)}\mathcal{M} = \{X - iJX\} \tag{B.30}$$

$$T^{(0,1)}\mathcal{M} = \{X + iJX\}. \tag{B.31}$$

Acting with J on both yields

$$J(X - iJX) = JX + iX = i(X - iJX) \tag{B.32}$$

$$J(X + iJX) = JX - iX = -i(X + iJX), \tag{B.33}$$

where we observe that indeed J can be used to split $T_{\mathbb{C}}\mathcal{M}$ into eigenvalues of J with eigenvalues $\pm i$.

Alternatively we can think of the projection

$$T^{(1,0)}\mathcal{M} = \frac{1}{2}(1 - iJ)X, \tag{B.34}$$

where we define $P_- \equiv \frac{1}{2}(1 - iJ)$. One can check that $P_-^2 = P_-$, and hence P_- is a projection operator.

Previously we mentioned that an almost complex manifold is a complex manifold if the Nijenhuis tensor vanishes, here we will show that this is equivalent to the statement that the transition on the overlaps are holomorphic.

An almost complex structure is also a complex structure when

$$\left[T^{(1,0)}\mathcal{M}, T^{(1,0)}\mathcal{M}\right] \subseteq T^{(1,0)}\mathcal{M}, \tag{B.35}$$

meaning when the Lie bracket of holomorphic vectors (which measures the rate of change of one vector field along the flow induced by the other) remains holomorphic. If $v = v^a\partial_a$, $w = w^a\partial_a$ then $[v, w] = v^a\partial_a w^b\partial_b - w^b\partial_b v^a\partial_a$.

For

$$Z = [X - iJX, Y - iJY] = [X, Y] - [JX, JY] - i([X, JY] + [JX, Y]), \tag{B.36}$$

we have that $Z \subseteq T^{(1,0)}\mathcal{M}$ if $JZ = iZ$. Actions on (B.36) should thus give the same result as multiplying the expression by i, which implies the Nijenhuis tensor

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY], \tag{B.37}$$

vanishes. One can check that this is indeed the coordinate independent description of (B.25).

B.3.3 *Kähler manifolds*

Kähler manifolds are essentially those for which parallel transport of a holomorphic vector remains holomorphic.

More precisely, let us consider a hermitian metric g on a complex manifold $\mathcal{M}$, for any two vector fields X and Y we then have

$$g(JX, JY) = g(X, Y). \tag{B.38}$$

Let us now define a two-form ω as

$$\omega(X, Y) \equiv g(JX, Y). \tag{B.39}$$

Using the hermitian property and the almost complex structure it can be shown that

$$\omega(X, Y) = -\omega(Y, X), \tag{B.40}$$

where we can go to local coordinates to find $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

A Kähler manifold is defined as a manifold on which ω is closed, i.e $d\omega = 0$.

Recall that starting with a real manifold which locally looks like $\mathbb{R}^{2n}$, we added a complex structure to obtain a space which is locally $\mathbb{C}^n$. The real tangent space was $\mathbb{R}^{2n}$ itself which we copmlexified to $\mathbb{C}^{2n}$, we then split it into a holomorphic and anti-holomorphic subspace, each of complex dimension n.

So for a Kähler manifold of real dimension $2n$, parallel transport maps $\mathbb{C}^n \to \mathbb{C}^n$ preserve the length of a vector. These transformations form the group $U(n)$, i.e. $\mathrm{Hol}(\mathcal{M}) \subseteq U(n)$. The real space $\mathbb{R}^{2n}$ we started with has $\mathrm{Hol}(\mathcal{M}_{\mathbb{R}}{}^{2n}) \in SO(2n)$ as the most general group conserving the length of a vector in $\mathbb{R}^{2n}$. However, in the case of a Kähler manifold we have $\mathrm{Hol}(\mathcal{M}_{\mathcal{K}}{}^{n}) \in U(n)$.

So the Kähler condition states that we only use half of the tangent space (since it does not allow us to go from holomorphic to anti-holomorphic vector space or vice-versa). The holonomy group of a Kähler manifold is the set of transformations which takes something in one half of tangent space, and after transformation it is still in this half. Hence the symmetry group performing these operations is $U(1)$.

We know $SO(2n)$ has $n(2n-1)$ generators and $U(1)$ has n^2 generators, so for $n > 1$ (which is generally the case), there are more generators in $SO(2n)$. Here we have quantized the degrees of freedom restricted when 'promoting' a real manifold to a Kähler manifold.

B.3.4 *Hyper-Kähler manifolds*

A hyper-Kähler manifold $\mathcal{M}_{Hk}$ is a $4n$ dimensional manifold with $\mathrm{Hol}(\mathcal{M}_{Hk}) \in Sp(n)$. Hyper-kähler manifolds are a special class of Kähler manifolds that admit three distinct complex structures I, J and K, satisfying the quaternion relations

$$I^2 = J^2 = K^2 = IJK = -1. \tag{B.41}$$

In our case these complex structures are related to the non-zero bilinears (4.10), explaining why the base space of the theory discussed in this thesis is hyper-Kähler.

B.3.5 *Calabi-Yau manifolds*

In the context of string theory the Calabi-Yau manifold serves as the manifold on which ten-dimensional string theory is compactified to four dimensions. Therefore we often compactify a string theory on a six-dimensional Calabi-Yau threefold, although a Calabi-Yau manifold can be defined for any number of even dimensions.

Note that it while it is trivial to prove that a vanishing Ricci tensor leads to vanishing of the first Chern class, the prove in the other direction is highly nontrivial.

In 1954 Calabi proposed that any compact Kähler manifold with vanishing first Chern class for the holomorphic tangent bundle, i.e. $c_1(\mathcal{M}) = \frac{1}{2\pi}\mathrm{Tr}(R_{\mu\nu})$, admits a unique metric for which the Ricci tensor $R_{\mu\nu} = 0$ [77]. Calabi was able to prove this [78], and in 1977 Yau was able to prove existence of such a metric [79].

Since the existence prove, a lot of examples have been found. Currently estimates have it that $\sim 10^{723}$ standard models can be obtained from string theory through compactification on different Calabi-Yau manifolds [80]. Finding our universe among the myriad of possibilities is an open problem in string theory.

C ANTI-DE SITTER SPACETIME

Anti de Sitter spacetime in D dimensions (AdS_D) is the maximally symmetric solution of Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{C.1}$$

with with negative cosmological constant Λ, resulting in a spacetime with constant negative curvature. AdS_D is defined as a hyperboloid

$$-x_1^2 - x_2^2 + \sum_{i=3}^{D+1} x_i^2 = -l^2, \quad l = \sqrt{\frac{-1}{2\Lambda}(D-1)(D-2)} \tag{C.2}$$

embedded in $D+1$ dimensional spacetime $\mathbb{R}^{2,D-1}$.

AdS_D has the isometries that preserve the hyperboloid structure of the embedding. The group of rotations and boosts in a geometry corresponding to $\mathbb{R}^{p,q}$ is $SO(p,q)$, and upon determining the algebra via the killing vectors one indeed finds $SO(2,D-1)$ [64, 71] as the isometries of AdS_D.

The metric reads

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\Omega_{D-2}^2. \tag{C.3}$$

In Poincaré coordinates, which cover only part of the AdS-space, the metric is

$$ds^2 = \frac{1}{y^2}\left(-dt^2 + dy^2 + \sum_i dx_i^2\right). \tag{C.4}$$

From this it follows that Anti-de Sitter metric in D spacetime dimensions has a nonzero curvature scalar

$$R = \frac{2D}{D-2}\Lambda. \tag{C.5}$$

D

SPINORS

The Dirac equation is the relativistic Schrödinger equation based on the so called spinor representations, and reads

$$(\gamma^\mu \partial_\mu + m)\,\psi(x) = 0, \tag{D.1}$$

where the symbol γ^μ represents a set of matrices acting on the spinor indices of the wave function ψ.

In this appendix we will discuss the gamma matrices and their symmetries, as well as irreducible spinors, and the vielbein formalism. This appendix is inspired by [20, 47], which provide a more in-depth account of the matter discussed here.

D.1 GAMMA MATRICES AND SYMMETRIES

Acting with another $\gamma^\mu \partial_\mu$ on (D.1), yields

$$\gamma^\mu \partial_\mu \gamma^\nu \partial_\nu \psi(x) = m^2 \psi(x) \tag{D.2}$$

$$\frac{1}{2}\left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\right) \partial_\mu \partial_\nu \psi(x) = m^2 \psi(x), \tag{D.3}$$

and since we wish for the Dirac equation to be a generalization of a plane wave equation, the operator on the left hand side should be equal to the d'Alembert operator. This means the matrices should satisfy the relation

$$\{\gamma^\mu, \gamma^\nu\} \equiv \left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\right) = 2\eta^{\mu\nu} I, \tag{D.4}$$

which defines the Clifford algebra associated with the Lorentz group.

The Clifford algebra for a d dimensional Euclidean matrix can be constructed in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{D.5}$$

which is a basis for the algebra $\mathfrak{su}(2)$.

We then construct the gamma matrices as

$$\gamma^1 = \sigma_1 \otimes I \otimes I \otimes \ldots, \tag{D.6}$$

$$\gamma^2 = \sigma_2 \otimes I \otimes I \otimes \ldots, \tag{D.7}$$

$$\gamma^3 = \sigma_3 \otimes \sigma_1 \otimes I \otimes \ldots, \tag{D.8}$$

$$\gamma^4 = \sigma_3 \otimes \sigma_2 \otimes I \otimes \ldots, \tag{D.9}$$

$$\gamma^5 = \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \ldots, \tag{D.10}$$

$$\ldots . \tag{D.11}$$

To instead construct a Clifford algebra for a spacetime with Lorentzian signature (t, d), the first t matrices are multiplied by i.

For $D = t + m$ even, the representation is of dimension $2^{D/2}$, and since for $D = t + d + 1$ we do not need the last σ_1 in the construction of γ^{d+t+1}, the dimension remains $2^{D/2}$.

The representation presented here is unique up to

$$\gamma' = U^{-1} \gamma U, \tag{D.12}$$

where U is a unitary matrix.

To proceed we will have to note that a basis for the even-dimensional Clifford algebra is given by

$$\{\Gamma^A = I, \gamma^\mu, \gamma^{\mu_1\mu_2}, \dots, \gamma^{\mu_D}\}. \tag{D.13}$$

The highest ranked tensor element of the Clifford algebra is of special importance because of its properties relating odd and even dimensions, and it is related to the chirality of fermions. Let us define

$$\gamma_* = (-i)^{d/2+t}\gamma_1\gamma_2\dots\gamma_D, \quad (\gamma_*)^2 = 1. \tag{D.14}$$

There exists a unitary matrix C, called the charged conjugation matrix, such that each matrix $C\Gamma^A$ is either symmetric or antisymmetric. Whether the matrix is symmetric or antysimmetric depends on the rank r of Γ^A:

$$\left(C\Gamma^{(r)}\right)^T = -t_r C\Gamma^{(r)}, \quad t_r = \pm 1. \tag{D.15}$$

Here the sign of t_r depends on the spacetime dimension D modulo 8, and the rank r modulo 4.

Using γ_* as presented in (D.14), we define the right and left chiral projectors

$$P_L = \frac{1}{2}(I + \gamma_*), \quad P_R = \frac{1}{2}(I - \gamma_*), \tag{D.16}$$

respectively.

Since the representations are hermitian and as such obey (**??**), combining the symmetry property (D.15) and (**??**) can be used to determine the complex conjugate of γ as

$$\gamma^{\mu *} = -t_0 t_1 B\gamma^\mu B^{-1}, \quad B = it_0 C\gamma^0. \tag{D.17}$$

Because of the hermitian property of both the gamma matrices and the charge conjugation matrix makes we do not define complex conjugates of spinor fields. Instead, we will define the simpler charge conjugation operation, which is effectively the same operation for any spinor whose indices are all contracted. The charge conjugate of a spinor λ is defined as

$$\lambda^C = B^{-1}\lambda^*. \tag{D.18}$$

And using (D.17) we then have

$$\bar{\lambda^C} = (-t_0 t_1) i\lambda^\dagger \gamma^0. \tag{D.19}$$

IRREDUCIBLE SPINORS

The transformation parameter of supersymmetry is the spinor ϵ_α. In general the spinor associated with a certain supersymmetric theory in D dimensions, is the one with the least possible components in that dimension. Table D.1 shows the possible spinors in different spacetimes, where there is only one timelike dimensions. The conditions corresponding to the different spinors are listed below.

Weyl spinors

Spinors satisfying either of the projection condition

$$P_L\psi = \psi, \quad P_R\psi = \psi, \tag{D.20}$$

are referred to as left chiral, or right chiral Weyl spinors, respectively. This condition is restricted to even dimensions where we have γ_*, as given in (D.14).

D	spinor	min # components
2	MW	1
3	M	2
4	M	4
5	S	8
6	SW	8
7	S	16
8	M	16
9	M	16
10	MW	16
11	M	32

Table D.1: The irreducible spinors of D dimensional theories. The spinors can obey constraints corresponding to Majorana (M), Majorana-Weyl (MW), symplectic (S) or symplectic Weyl (SW) spinors [20].

Majorana spinors

Let us define the reality condition

$$\psi = \psi^C = B^{-1}\psi^*, \tag{D.21}$$

which using (D.19), can be written as

$$\psi^* = B\psi. \tag{D.22}$$

This is the reality condition. However, this condition does not hold in all dimensions. We can see this by taking the complex conjugate of (D.22), followed by inserting (D.22) back into the resulting expression, which leads us to $B^*B\psi = \psi$. This means that the condition to have Majorana spinors is $B^*B = I$, which means $t_1 = -1$ and $t_0 = \pm 1$. $B^2 = I$ if $t_0 = 1$, which is the case for $D = 2, 3, 4$ mod 8. The spinors in theories in these dimensions satisfying (D.22), are called Majorana spinors. If $t_0 = -1$, which is the case in $D = 8, 9$ mod 8, the spinor satisfying (D.22) is called a pseudo-Majorana spinor.

Majorana-Weyl spinors

In dimensions $D = 2$ mod 8, a spinor can satisfy both the projection condition (4.11) and the reality condition (D.22). These spinors are called Majorana-Weyl, are the most fundamental spinors in theories of dimension $D = 2$ mod 8.

Symplectic Majorana spinors

We just mentioned that $t_1 = -1$ is a requirement to have Majorana spinors. However, in dimensions $D = 5, 6, 7$ mod 8 we have $t_1 = 1$, which means that the corresponding spinors do not satisfy the reality condition. Instead we now have $\left(\psi^C\right)^C = -\psi$, in which case we can introduce a spinor doublet consisting of an even number of spinors χ^i. The spinors defined in theories with $t_1 = 1$ are symplectic Majorana spinors. Symplectic Majorana spinors satisfy an alternative reality conditions, namely

$$\chi^i = \varepsilon^{ij}\left(\chi^j\right)^C \tag{D.23}$$

where ε^{ij} is an invertible skew-symmetric matrix.

Symplectic Majorana-Weyl spinors

If $D = 6 \bmod 8$, both the symplectic Majarona constraint, as well as the chiral projection condition can be applied. Spinors satisfying both constraints are called symplectic Majorana-Weyl spinors.

D.2 SPINORS IN CURVED SPACETIME

Here we introduce how spinor fields are coupled to curved spacetime. To do so we will discuss the Vielbein $e^\mu_a(x)$, leading to the spin connection $\omega^{ab}_\mu(x)$ and we discuss the vielbein formalism used to couple spinors to curved spacetime.

Usually when coupling bosonic fields to curved spacetime, we make use of the metric tensor $g_{\mu\nu}$, as well as first and second order derivatives to provide a description of curvature in terms of the Christoffel connections $\Gamma^\lambda_{\mu\nu}$ and Riemann tensor $R^\mu_{\nu\rho\sigma}$. However, we cannot simply couple spinor fields to gravity using metric tensors, because there is no covering group for the group of general coordinate transformations. We do, on the other hand, know how to work with spinors in flat space. So taking the Clifford algebra in flat space $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$, and replacing the Minkwoski metric with by a general metric $g_{\mu\nu}(x)$, the Clifford algebra reads

$$\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x). \tag{D.24}$$

Usually the dependence of the metric on spacetime coordinates is not explicitly written, but it is exactly this dependence which requires spacetime dependence on the left-hand side as well. Expanding $\gamma^\mu(x)$ in terms of the constant Dirac matrices γ^a yields

$$\gamma^\mu(x) = e^\mu_a(x)\gamma^a. \tag{D.25}$$

Plugging (D.25) into (D.24), we obtain

$$g_{\mu\nu}(x) = e^a_\mu(x)e^b_\nu(x)\eta_{ab}. \tag{D.26}$$

We define the inverse e^a_μ of e^μ_a, as $e^\mu_a e^a_\nu = \delta^\mu_\nu$ is a complex structure and the space is a complex manifold.

Notice that here we distinguish the Greek indices $(\mu, \nu, \ldots)$, and the Latin indices $(a, b, \ldots)$, called 'global' and 'local' indices, respectively. Tensor fields $A_a(x) = e^\mu_a(x)A_\mu(x)$, with local indices transform under the local Lorentz transformations, as well as general coordinate transformations.

If we wish to write down an action of spinor fields invariant under local Lorentz transformations, we require a gauge field, this is the spin connection ω^{ab}_μ hinted at in the first few lines. As opposed to the Christoffel connection $\Gamma^\mu_{\nu\rho}$, which is symmetric in its lower indices, the spin connection is anti-symmetric: $\omega^{ab}_\mu = -\omega^{ba}_\mu$.

All the tools introduced above now enables us to couple a spinor ψ to spacetime. To do so we define the covariant derivative

$$D_\mu\psi(x) = \left(\partial_\mu + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab}\right)\psi. \tag{D.27}$$

The Lagrangian invariant under the Lorentz as well as general coordinate transformations can be written as

$$\mathcal{L} = ie\bar{\psi}_\mu\gamma^{mu\nu\rho}D_\mu\psi_\rho, \tag{D.28}$$

where $e = \det(e^a_\mu) = \sqrt{-g}$.

E

DIMENSIONAL REDUCTION ON S^1

Here we will provide a short review of the Kaluza-Klein reduction. For a more in depth overview of the subject, see e.g. [81], which this section is largely based on.

Kaluza-Klein theory is a way of explaining the existence of higher dimensions in string theory, but the idea already stems from before string theory and was a first attempt of explaining electromagnetism in general relativity. The main idea of the theory is that even though we only observe 3+1 dimensions, higher spatial dimensions can exist under the assumption they are curled up to a radius smaller than the Planck length, which would explain why they have not been observed.

An example often used to explain the principle of Kaluza-Klein compactification is to consider a water hose. A water hose is a two dimensional object (a cylinder), and for an ant walking on the outer surface of the tube, it will certainly seem to be a two dimensional surface. However, if a human were to look at the water hose from a large enough distance, it would look like a one dimensional line.

This theory can be used to obtain lower or higher dimensional expressions of space time, in which case the process is referred to as compactification or oxidation, respectively. The same concept can now be used in string theory to compactify a higher dimensional theory in an attempt to obtain a description in four spacetime dimensions in order to relate the theory to observables.

In the context of compactification it is assumed that spacetimes can be written in the form $\mathcal{M}_4 \times X$, where $\mathcal{M}_4$ corresponds to the four dimensional universe we observe and X is some compact manifold. In this section we study compactification of a single dimension to a circle: $\mathcal{M}_{D+1} = \mathcal{M}_D \times S^1$.

In the discussion of Kaluza-Klein reduction below we will denote coordinates in $D+1$ spacetime dimensions with a 'hat' $x^{\hat{\mu}} = (x^\mu, z)$, where z is the compact coordinate. As a result of the S^1 topology of the compact coordinate all fields in $D+1$ dimensions have to satisfy the boundary condition

$$\hat{\phi}(x^\mu, z) = \hat{\phi}(x^\mu, z + n2\pi R), \tag{E.1}$$

where R is the radius of the circle S^1 and $n \in \mathbb{Z}$. As a result of this periodicity all fields may be written as a Fourier expansion of the form

$$\hat{\phi}(x^\mu, z) = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu) e^{inz/R}. \tag{E.2}$$

Inserting this expansion in the massless Klein-Gordon equation gives

$$\hat{\Box}\hat{\phi}_n = 0 \quad \Rightarrow \quad \left(\Box + \left(\frac{n}{R}\right)^2\right)\phi_n = 0, \tag{E.3}$$

which implies that the fields have masses

$$M = |n|/R. \tag{E.4}$$

This infinite set of fields ϕ_n is typically called the tower of massive Kaluza-Klein modes.

since $n \in \mathbb{Z}$ the momentum in the z direction becomes quantized. An important aspect of Kaluza-Klein theory is that the momentum becomes a quantized charge, where we refer to n as a charge resulting from some potential A_μ.

As a result of (E.4), one can conclude that for small radius R all particles described by the fields ϕ_n have a large mass except for the massless mode $n = 0$. These massive modes can be neglected when probing energies much smaller than $1/R$, or distance scales much larger than R, in these limits theory would effectively be D dimensional.

REDUCTION OF THE METRIC

The $D+1$ dimensional metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ can be decomposed into the components: $\hat{g}_{\mu\nu}$, $\hat{g}_{\mu z}$ and $\hat{g}_{zz}$. Naively one could argue that these look like a metric, vector field and scalar field in D dimensions. However, if we identify these components as the corresponding D dimensional fields directly, they will not behave well under general coordinate transformations. Therefore, a better Ansatz for the metric is

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e^{2\alpha\phi}g_{\mu\nu} + e^{2\beta\phi}A_\mu A_\nu & e^{2\beta\phi}A_\mu \\ e^{2\beta\phi}A_\nu & e^{2\beta\phi} \end{pmatrix}, \tag{E.5}$$

with $\alpha,\ \beta \in \mathbb{R}$. Upon dimensional reduction we can identify the following D dimensional fields: $g_{\mu\nu}$ becomes the D dimensional metric, ϕ is a scalar field called the dilaton field and A_μ is a $U(1)$ gauge field called graviphoton. This ansatz corresponds to a line element of the form

$$d\hat{s}^2 = e^{2\alpha\phi}ds^2 + e^{2\beta\phi}(dz + A_\mu dx^\mu)^2. \tag{E.6}$$

Here α and β are arbitrary constants. Using this Ansatz we can obtain a lower-dimensional Einstein-Hilbert Lagrangian $\mathcal{L}_{EH} = e^{(\beta+(D-2)\alpha)\phi}\sqrt{-g}R$. In order to obtain a Lagrangian in the Einstein frame we thus require $\beta = -(D-2)\alpha$, which is fine as long as we do not reduce to two dimensions.

Another frame that can be useful is the string frame, which is obtained by setting $\alpha = 0$ and $\beta = 1$. Hence the Kaluza-Klein Ansatz in string frame is

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{\mu\nu} + e^{2\phi}A_\mu A_\nu & e^{2\phi}A_\mu \\ e^{2\phi}A_\nu & e^{2\phi} \end{pmatrix}. \tag{E.7}$$

REDUCTION OF THE DILATON

From Ansatz for the Kaluza-Klein reduction in string frame one can conclude that the square root of the determinants of the metric in different dimensions are related via the dilaton scaling as $\sqrt{-\hat{g}} = e^{\phi}\sqrt{-g}$. Now, if we wish to compactify more than just one dimension on a circle using this Ansatz, we obtain an expression for the lower-dimensional dilaton in string frame

$$e^{-2\phi_{(D)}} = \sqrt{g^{(D')}_{DD} \cdots g^{(D')}_{D'-1\,D'-1}}\, e^{-2\phi_{(D')}}. \tag{E.8}$$

F MORE PROPERTIES OF THE SYMPLECTIC GROUP

F.1 CONJUGACY CLASSES

As our aim is to classify solutions obtained through transformations of solutions by acting with elements of the group $Sp(6,\mathbb{R})$. As such, one can wonder if any vector can be obtained from any other vector through a transformation in the symplectic group, or whether there are certain non-overlapping 'orbits'. Here the concept of a conjugacy class will be usefull:

Definition: *Conjugacy Class.*
Let G be a group. Two elements $g_1, g_2 \in G$ are conjugate, if there exists an element $h \in G$ such that $hg_1h^{-1} = g_2$. A conjugacy class of the element g_1 is defined as

$$Cl(g_1) = \left\{ g_2 \in G : \exists h \in G,\ g_1 = hg_2h^{-1} \right\}. \tag{F.1}$$

The class number of G is the number of nonequivalent conjugacy classes, this might prove useful in determining the number of orbits of the BPS solutions discussed in these notes.

Using the definition of conjugacy one can observe that trace of all elements in a conjugacy class should be equal

$$\mathrm{Tr}\, g_1 = \mathrm{Tr}\, hg_2h^{-1} = \mathrm{Tr}\, g_2. \tag{F.2}$$

Since F-theory is quantized in the microscopic theory we might limit ourselves to conjugacy classes of $Sp(6,\mathbb{Z})$ instead of $Sp(6,\mathbb{R})$.

Thus for 0 trace we find matrices of the form

$$\begin{pmatrix} 0 & A \\ -A^{-1} & 0 \end{pmatrix}, \tag{F.3}$$

where $A \in SL(3,\mathbb{Z})$. Elements of $Sp(6,\mathbb{Z})$ with nonzero trace contain group elements of the following form

$$\begin{pmatrix} 0 & I_3 \\ -I_3 & A \end{pmatrix}, \tag{F.4}$$

where $A = A^T$. Since A can be any symmetric matrix, it can be concluded that an infinite number of conjugacy classes exists for the group $Sp(6,\mathbb{Z})$.

F.2 DERIVATION OF THE GENERATORS

The symplectic group is defined in (5.8): $M^T\Omega M = \Omega$. Defining $M = e^X$, it can be shown that elements X of the algebra should obey (5.24): $(\Omega X)^T = \Omega X$.

$$\begin{aligned} M^T\Omega M =& \Omega && \text{(F.5)} \\ \exp(X)^T \Omega \exp(X) =& \Omega && \text{(F.6)} \\ \exp(X^T)\Omega =& \Omega \exp(-X) && \text{(F.7)} \\ (I + X^T + \dots + \frac{(X^T)^n}{n!} + \dots)\Omega =& \Omega(I - X + \dots + \frac{(-1)^n X^n}{n!} + \dots) && \text{(F.8)} \end{aligned}$$

For the first order of both sides to satisfy the equality,

$$(\Omega X)^T = \Omega X \tag{F.9}$$

is required. This leaves the algebra with dimension $\frac{1}{2}n(n+1)$, which is the same as the dimension as the group, thus this equality should fully specify the generators. Thus if (F.9) indeed fully defines the algebra, this means that it should generalize for the higher order terms to

$$\frac{(X^T)^n}{n!}\Omega = (-1)^n \frac{\Omega X^n}{n!}. \tag{F.10}$$

Which can be shown to hold as follows

$$\frac{(X^T)^n}{n!}\Omega = \frac{(X^T)^{n-1}}{n!}X^T\Omega = -\frac{(X^T)^{n-1}}{n!}\Omega X = -\frac{(X^T)^{n-2}}{n!}X^T\Omega X \tag{F.11}$$

$$= -\frac{(X^T)^{n-2}}{n!}\Omega X^2 = \cdots = (-1)^n \frac{\Omega X^n}{n!}. \tag{F.12}$$

It can thus be concluded that the set of matrices X which obey (F.9), form the generators of the symplectic group.

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COLOPHON

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