Viscosities of anisotropic quark-gluon plasma

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Abstract

We explore several methods for calculating the two shear viscosities and three bulk viscosities of magnetohydrodynamics for a quark-gluon plasma using improved holographic QCD. First we look at a method based on fluctuation equations. We work both with UV expansions and with a method based on the membrane paradigm. We calculate the shear viscosities, and describe some problems you stumble upon when calculating the bulk viscosities. Then we look at methods based on dually mimicking fluid equations, in this case Fick's law and the entropy current equations, we manage to extract several viscosities, but there seems to be a restriction on which viscosities you can extract based on the zero temperature boost symmetry breaking. Lastly, we calculate the a magnetic field dependent bulk viscosities for analytic two-dimensional model related to a gravity dual of the ABJM model.

Contents

Acknowledgements

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Chapter 1

Introduction

1.1 Quark-gluon plasma

In nature we find baryons. These are subatomic particles composed of an odd number of quarks, held together by the strong force. The medium of this strong force is a massless boson called a gluon. The most prominent examples of baryons are neutrons and protons, which together compose the atomic nuclei. The strong force has its name for a reason, and it is therefore very difficult to destroy the bond that holds the quarks in neutrons and protons together. However, in the Relativistic Heavy Ion Collider located at Brookhaven National Laboratory, USA as well as in the Large Hadron Collider located at CERN, Switzerland, gold ions are accelerated to a point where the collision temperature is so high, namely about 450-600MeV [45] ($\sim 10^{12}$ K), that this bond is broken (see figure 1.2). Again stressing how hot this is, this is a million times hotter than the core of the sun [12]. Because the 'pancake'-shaped ion clouds do not completely overlap, some baryons, which carry electric charge when they are protons, move around the collision site in a uniform way (see figure 1.3). This generates a very large magnetic field (about 10^8-10^9 Gauss). When the bond is broken, the quarks become deconfined and a quark-gluon plasma (QGP) is formed. Apart from being a phenomenon that can be studied in a laboratory under bizarre artificial circumstances, a QGP may have existed a few microseconds after the Big Bang and perhaps still is present in the inner core of neutron stars [60]. Studying this plasma is therefore of great scientific value. When studying this plasma, it is important to note that unlike baryonic matter, a quark gluon plasma can be treated as a fluid [73], and thus its fluid properties, such as the bulk and shear viscosity, can be studied, which is what will be done in this thesis. Lastly, both in the early universe and neutron stars, similar to heavy ion collisions, the magnetic effect on QGP needs to be considered [91], [50].

1.2 Holography

Studying a QGP is a difficult task. The dynamics of this plasma is quantum chromodynamics (QCD), which is the quantum field theory (QFT) which describes how gluons interact with quarks and with themselves. Because this fluid is strongly coupled, we cannot rely on perturbative QCD. Another route would be lattice QCD. However, because of the euclidean time nature of this method, incorporating dynamic effects comes with many systematic and statistical errors [45]. Yet another method would be holography, where we map a QFT to a higher dimensional semi-classical theory of gravity [5]. But also this method comes with disadvantages. The main problem is that it is often

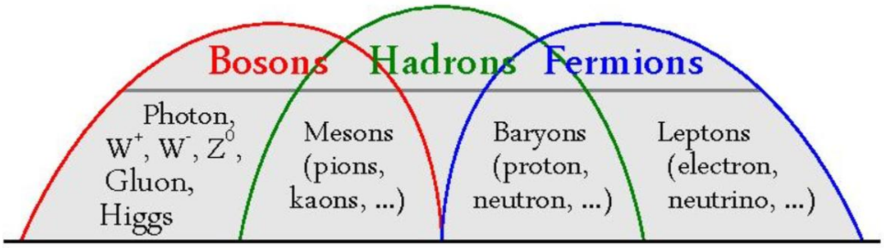

Figure 1.1: Venn diagram of subatomic particles in the Standard Model [88].

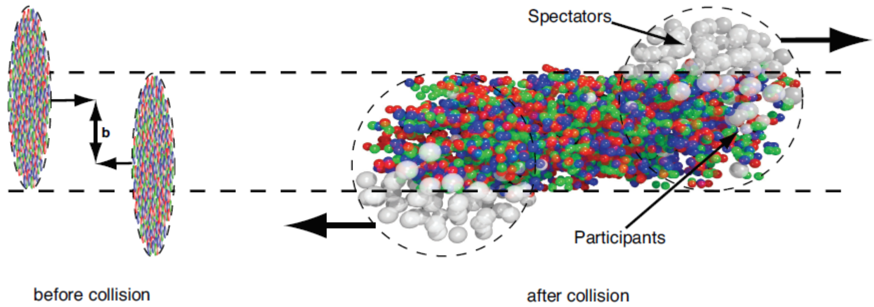

Figure 1.2: Schematic picture of a QGP formed by a heavy ion collision [45], because of deconfinement the baryons (zero color charge) decompose into quarks with red, blue and yellow color charge. Before the collision the ions have a 'pancake' shape due to special relativistic length contraction [22].

difficult to find a top down derivation of the gravitational theory. Such a top down derivation was first found by Maldacena in [74] for $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) Theory, a conformal theory with gauge group $SU(N)$. This theory is found to be dual, i.e. mathematically identical, to critical type IIB superstring theory $AdS_5 \times S^5$. Several constants of the two dual theories can be related to eachother in the following way:

$$\frac{\lambda}{N} = 2\pi g_s \tag{1.1}$$

$$2\lambda = \frac{L^4}{l_s^4} \tag{1.2}$$

Here λ and g_s are the 't Hooft coupling and string coupling constant and L and l_s are the radius of curvature and the string length respectively. The 't Hooft coupling is given by: $\lambda = g_{YM}^2 N$, with g_{YM} the Yang-Mills coupling constant. The duality of $\mathcal{N} = 4$ SYM to IIB superstring theory $AdS_5 \times S_5$ follows from comparing the open- and closed-string perspective. These perspectives become tractable in opposite limits for $g_s N$. Specifically, N $D3$-branes can be described by by open strings on a flat hyperplane (see figure 1.4) for $g_s N \ll 1$. The low energy limit is $\mathcal{N} = 4$ SYM. In addition, N $D3$-branes can be described by closed strings in curved space-time for $g_s N \gg 1$. In the low energy limit this becomes type IIB superstring theory $AdS_5 \times S^5$ [22].

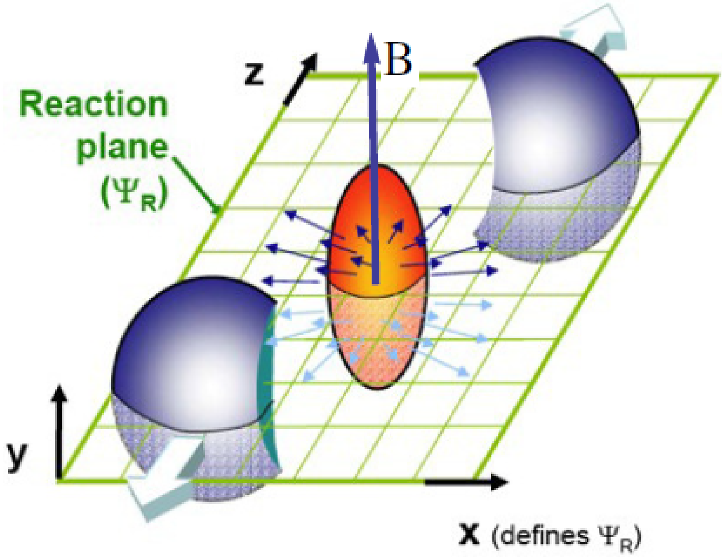

Figure 1.3: Schematic picture of a QGP formed by a heavy ion collision, during this collision a strong magnetic field is produced [45]. This is denoted by the B-arrow

1.3 'the Hooft limit

From equation 1.1 it follows that if we take the weak coupling limit on the string theory side, but keep $\frac{L^4}{l_s^4}$ constant, we get that only the tree level diagrams of string perturbation theory matter, i.e. we have removed the quantum degrees of freedom and the theory reduces to classical string theory. For this to hold it follows from equation 1.1 and 1.2 that N needs to be large. This is called the 't Hooft limit, and was first suggested by 't Hooft in [92]. Since λ is kept constant this means g_{YM} should be small. One could think that this means that we also work in the weak coupling limit for the QFT, which means that we might as well use perturbative QFT. However, since the number of components in the fields simultaneously becomes large [2], the significance of higher order diagrams remains invariant in this limit. Proceeding, we can also assume the radius of curvature to be large compared to the string length, which effectively turns strings into point-like particles, thus reducing classical string theory to classical supergravity. Contrary to perturbative QFT, this requires the coupling to be large on the field-theory side (but still small compared to N), which is the case for the running coupling constant of QCD for a specific regime with an energy scale that is not too far in the UV. Thus we conclude that when combining the 't Hooft limit with the strong coupling limit, that $\mathcal{N} = 4$ SYM, which is a conformal field theory, is dual to AdS_5 classical supergravity. It is also possible to use bottom up reasoning to arrive at this duality [22]. We do this by starting from that $\mathcal{N} = 4$ SYM is conformal, which means that the theory is unaffected by a change in the energy scale. We then say that this energy scale is holographically dual to the r-coordinate, the fifth dimension in the dual five-dimensional gravitational theory. From the requirement that this five-dimensional theory should be invariant under a transformation of the r-coordinate follows that the theory of gravity should be AdS_5. Lastly, we note that dual theories have the same symmetries. For example, AdS and the conformal group are both $SO(2, d)$. Where d are the spatial dimensions.

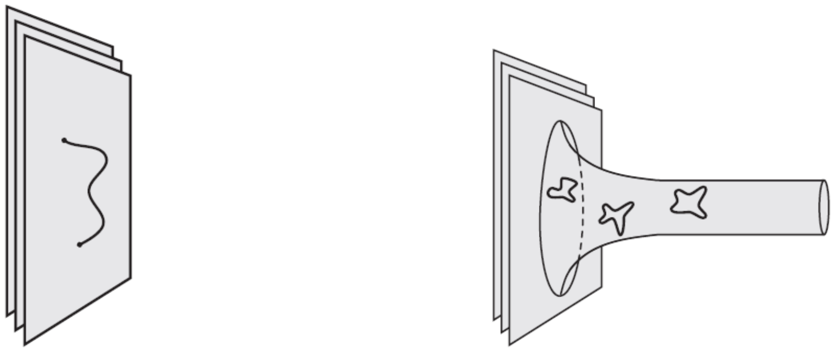

Figure 1.4: Excitations of the system in the open and closed string description [5]

1.4 Improved holographic QCD

There is no string theory which is proven to be dual to QCD and there are doubts whether such a theory exists. For example, in [79] Ooguri conjectures that all vacua spaces known to be used for holographic constructions, which are space that are asymptotically AdS and break supersymmetry, are unstable. Avoiding this troublesome exercise, we instead use a bottom up theory, which assumes that some 'low-lying' operators of QCD can be treated separately from the rest of the Hilbert space [45], because the other operators are $\mathcal{O}(g_s)$ and are therefore suppressed due to the 't Hooft limit. We then build a bottom up theory with these operators and their dual sources which mimicks the properties of QCD in a 5D gravitational theory. Whenever we now say 'dual' we no longer mean that two things are mathematically equivalent, but instead mean that two things correspond to each other based on the assumption that the bottom-up theory is a good holographic description of our QFT. The properties we want to mimick are asymptotic freedom, chiral symmetry breaking and confinement. With the holographic theory in place, we can use the Gubser-Klebanov-Polyakov-Witten (GKPW) rule [42] to extract information about the operators of the field theory from the sources at the boundary of the dual theory [58]:

$$\Big\langle \exp\Big[-i\int d^d x \phi_0 \mathcal{O}\Big]\Big\rangle = e^{-iS_{grav}[\phi(r\to\infty)=\phi_0} \tag{1.3}$$

The model that describes QCD with this method is called Improved Holographic QCD (ihQCD). This theory will be one of the main topics of this thesis. Table 1.1 shows all the operators and dual sources that we will work with in this thesis [45]. In this context the 't Hooft limit means taking the amount of colors N_c to be large [30]. Because the gluons transform in the adjoint representation, this limit affects the glue sector of our model. However, since magnetic field couples not to gluon but to quarks, which transform in the fundamental representation, we also need to consider the flavor sector. The way we add flavor to the model is by also taking the the amount of flavors to be large, but keep the ratio $x = N_c/N_f$ at a fixed value. This limit is called the Veneziano limit and the model that includes flavor this way is also called Veneziano QCD. This way of incorporating quarks has the advantage that, unlike methods where quarks are studied in the probe limit (see for example [35]), it considers the backreaction of the quarks and can therefore describe the behavior of quarks at high densities. The downside is that, as explained in [3], in the Veneziano limit, unlike for the 't Hooft limit, not all string loops are suppressed, and these contributions are extremely difficult to take into account, for one because the full string theory is unknown. When using ihQCD, we don't

Table 1.1: Operators and their dual sources.

boundary: operator	bulk: source
energy momentum tensor $T^{\mu\nu}$	metric tensor $g_{\mu\nu}$
current J_μ	gauge field A_μ
scalar glueball operator trF^2	dilaton ϕ
axionic glueball operator tr$F \wedge F$	axion χ
quark condensate $\langle q\bar{q}\rangle$	tachyon τ

have to rely on faith to know whether it is actually a quantitatively valid way of modeling QCD. For example, we can look the glueball spectrum [45], which are the gluon eigenvalues of QCD in the confined phase. We can compare the result that follows from ihQCD with results from lattice theory (see figure 1.5). In [6] the meson and glueball spectra of ihQCD with the Veneziano limit are fully analyzed. Since these methods are completely independent, it is a strong indication that ihQCD is a quantitatively valid way of modeling QCD.

J^{PC}	Lattice (MeV)	Our model (MeV)	Mismatch
0^{++}	*1475 (4%)*	*1475*	0
2^{++}	2150 (5%)	2055	4%
0^{++*}	*2755 (4%)*	*2753*	0
2^{++*}	2880 (5%)	2991	4%
0^{++**}	3370 (4%)	3561	5%
0^{++***}	3990 (5%)	4253	6%

Figure 1.5: Comparison of glueball spectra as found via lattice QCD and via ihQCD ('our model') [45].

1.5 Shear viscosity

Shear viscosity is the transport coefficient for the dissipation that occurs when two layers of fluid slide over each other (see figure 1.6) [77]. Shear viscosity explains why honey, which is a more viscous fluid than water, very slowly moves out of a jar when you hold it upside down, whereas water moves out almost instantly. In figure a schematic figure 1.7 the shear viscosities of several very non-viscous fluids, including a QGP, is given. In [82], Policastro, Son and Starinets (PSS) calculate the shear viscosity of $\mathcal{N} = 4$ SYM by assuming a large 't Hooft coupling, which allowed PSS to map the QFT to AdS_5. The result they find is:

$$\frac{\eta}{s} = \frac{1}{4\pi} \tag{1.4}$$

What is noteworthy is that this result is extremely low and temperature independent. Far from the UV where holographic QCD is valid, this result has been confirmed experimentally (see [12] and figure 7.2). Kovtun, Son and Starinets (KSS) then propose in [69] that this result could serve as a lower bound for a wide class of systems, such as the QGP. They argue that this holds because lower

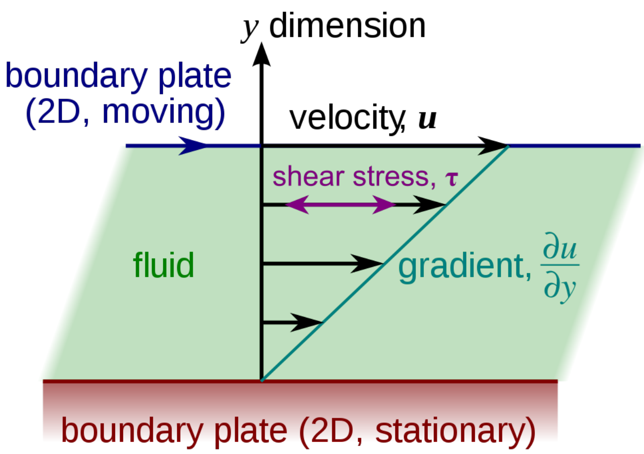

Figure 1.6: Non-zero shear viscosity causes shear stress when there is a velocity gradient between two layers of fluid [31]

values would violate Heisenberg's uncertainty principle. In [19], Buchel and Liu (BL) make an even stronger claim. They claim that this bound is saturated for any gauge field. The conjecture of both BL and KSS are proven to not be generally true. For example, Cremonini, Gursoy and Szepietowski, show in [26] that shear viscosities of theories with a non-trivial scalar profile no longer obey equation 1.4. Instead the shear viscosity becomes temperature dependent and can drop below the proposed universal bound. Also, anisotropic QFTs do not adhere necessarily to this lower bound. Examples of this are given in [34] and [84]. For the anistropic case, Rebhan and Steineder show that when the zero temperature $SO(3)$ rotational symmetry is broken in the z-direction by an axion linear in z the shear viscosity is defined by the horizon geometry in the following way [84] :

$$\frac{\eta_{\parallel}^{\chi}}{s} = \frac{1}{4\pi}\frac{g_{xx}(r_h)}{g_{zz}(r_h)} \tag{1.5}$$

Similarly, Critelli et al. show that when this is done by a magnetic field in the z-direction, we get the flipped result:

$$\frac{\eta_{\parallel}^{B}}{s} = \frac{1}{4\pi}\frac{g_{zz}(r_h)}{g_{xx}(r_h)} \tag{1.6}$$

$\eta_{\perp}$ is invariant under this symmetry breaking. In [23], In [59], Jain, Samanta and Trivedi generalised this by proposing that anisotropic shear viscosities might be determined by looking at the boost symmetry, which is the part of Lorentz group that relates time to the spatial dimensions. This symmetry is present for the isotropic theory of gravity at zero temperature. We define z the dimension for which the boost symmetry is left intact and x the dimension for which this symmetry is broken. A general formula, consistent with the previous results, for the shear viscosity parallel to the anisotropy direction would then be:

$$\frac{\eta_{\parallel}}{s} = \frac{1}{4\pi}\frac{g_{xx}(r_h)}{g_{zz}(r_h)} \tag{1.7}$$

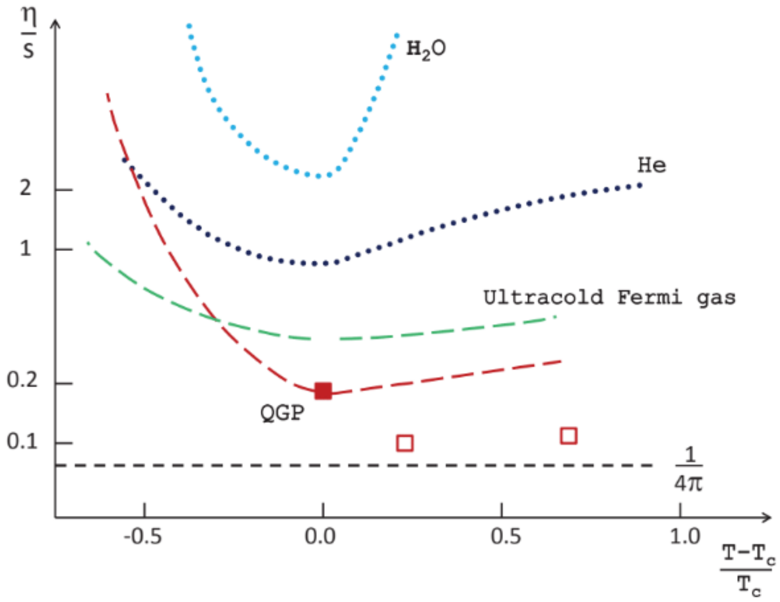

Figure 1.7: Schematic plotes of shear viscosities of several fluids [26], T_c means the temperature where vaporization for H_20 and He, superfluid transition for ultracold Fermi gas and deconfinement for QGP occurs respectively

1.6 Lieb-Robinson bound

In [51], Hartman, Hartnoll and Mahajan (HHM) argue in favor of the conjectured KSS bound in a more sophisticated way by deriving it from the Lieb-Robinson bound. The Lieb-Robinson bound follows from the observation that operators spread linearly in time, which means for operators A and B [64]:

$$||[A(x,t), B(0,0)]|| \leq ce^{-a(x-vt)} \tag{1.8}$$

Here v is the Lieb-Robinson velocity, which tells how fast an operator spreads. This bound establishes an effective light-cone for relativistic as well as non-relativistic theories. Based on this requirement HHM derive the following lower bound (see figure 1.8):

$$\frac{\eta}{s} \geq v^2 T \tau_{eq} \tag{1.9}$$

v is 1 in relativistic theories, T is the temperature and τ_{eq} is the local equilibration time, which is time it takes till we are in the diffusive regime. Using values for $T\tau_{eq}$ for tensorial fluctuations calculated in [76], HHM find $v^2 T\tau_{eq} \approx \frac{1}{4\pi}$. The violation of the KSS bound can perhaps be explained by looking at the way HHM formulated this bound. When you calculate $\eta_{\parallel}$, you no longer look at tensorial fluctuations when calculating $\tau_{eq}T$. Instead you look at vectorial fluctuations, which couple to other modes and therefore no longer give the universal $\frac{1}{4\pi}$ value. Because the imaginary frequency

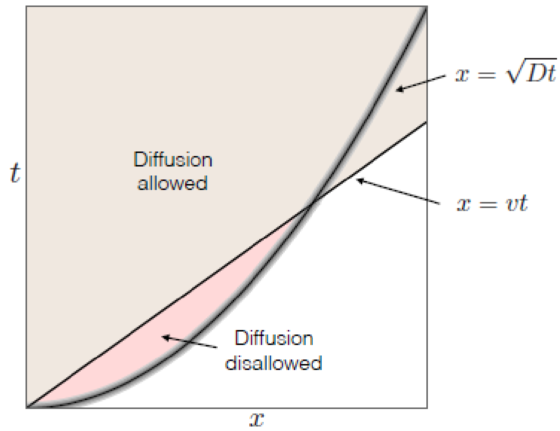

Figure 1.8: $t - x$ diagram which shows allowed and disallowed diffusion based on the Lieb-Robinson bound. Note that here diffusion is momentum diffusion, i.e. shear stress. From requiring that the diffusive regime starting from τ_{eq} is in the lightcone a lower bound for the shear viscosity can be derived [51]

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tells you how quickly a mode decays in time we have the following identity for τ_{eq}:

$$\tau_{eq} = \frac{1}{\mathrm{Im}\omega_{qnm}} \tag{1.10}$$

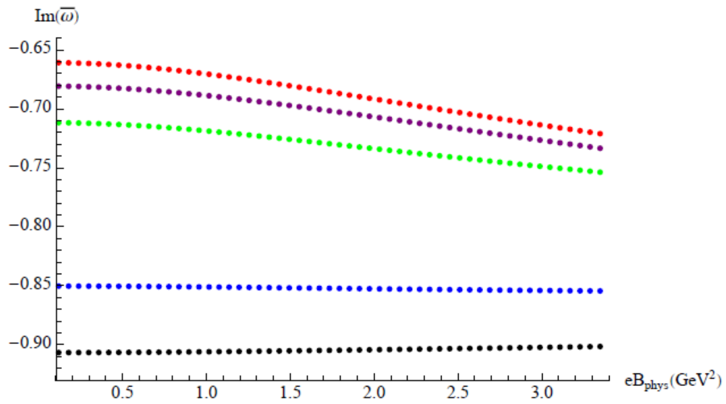
ω_{qnm} is the frequency of the quasinormal mode lowest to the real axis. A quasinormal mode is a solution to the holographic fluctuation equations. They are the holographic dual of the poles of the retar In [29], the quasinormal modes for the shear modes for a QGP with magnetic field are calculated as a function of magnetic field (see figure 1.9). From this it becomes clear that anisotropy tightens the bound for $\eta_{\|}$. It is plausible that for an action with an axion the quasinormal modes would increase, leading to a relaxation of the Lieb-Robinson bound which would allow for a violation of the KSS bound as is found by Rebhan and Steineder in [84].

1.7 Bulk viscosity

Bulk viscosity is the transport coefficient for the dissipation that occurs when a fluid is being compressed. For molecular fluids, it is related to the vibrational energy of the molecules [25]. Monatomic fluids such as He therefore have no bulk viscosity. Bulk viscosity is also zero for $\mathcal{N} = 4$ SYM, as well as all conformal field theories. In [43], GPR calculate the bulk viscosity for a theory of gravity which is dual to a non-conformal field theory, and they find that:

$$\frac{\zeta}{s} \propto \frac{1}{V}\frac{dV}{dr} \tag{1.11}$$

More details are provided in chapter 4. Here V is the dilaton potential which is the term in the gravitational action which causes deviation from AdS of the bulk, which is dual to non-conformality in the field theory. As will be explained in chapter 3, V can be related to the beta function. Unlike



(b)

Figure 1.9: Imaginary frequencies of quasinormal modes of the vectorial shear mode plotted as a function of magnetic field [29]

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the isotropic shear viscosity, the isotropic bulk viscosity is temperature dependent. Also for the bulk viscosity a lower bound for bulk viscosity is proposed:

$$\frac{\zeta}{\eta} \geq 2\Big(\frac{1}{3} - v_{♪}\Big) \tag{1.12}$$

Here $v_{♪}$ is the speed of sound of the fluid. This is called Buchel's bound [16]. However, Buchel himself shows that this bound can be violated in an anisotropic counterexample [17]. In [46], Gursoy et al. apply the GPR method to the ihQCD model and derive a plot for the bulk viscosity given in figure 1.10. Similarly, Critelli et al. also studied the bulk viscosity with a bottom-up model in [27], the result is in figure 1.11

1.8 Relativistic magnetohydrodynamics

Using ihQCD, we can calculate the transport coefficients of our QFT. As mentioned, we can treat the QGP as a fluid, and therefore we can use relativistic hydrodynamics to give an order by order prediction of which transport coefficients should emerge from the holographic calculations. As mentioned in section 1.1, a very strong magnetic field is produced during a heavy ion collision. Furthermore, neutron stars are known to have strong magnetic fields [15], [50], [75]. This motivates this thesis, which studies the viscosities of a QGP in a strong magnetic field. In [55], Hernandez and Kovtun (HK) make a hydrodynamic expansion for the entropy current in the presence of a magnetic field at first order. This result is elaborated on in chapter 2. KH find seven independent coefficients (which are, following from the fact that they contribute to the entropy current, dissipative): two conductivities, two shear viscosities and three bulk viscosities. There are two shear viscosities because shear-stress is a two-dimensional phenomenon (see figure 1.6). Consistent with the remaining zero temperature $SO(2)$ symmetry of the theory, $\eta_\perp$ describes shear stress where the x- and y-dimension are involved and $\eta_\perp$ describes shear stress where the z-direction and either the x- or y-

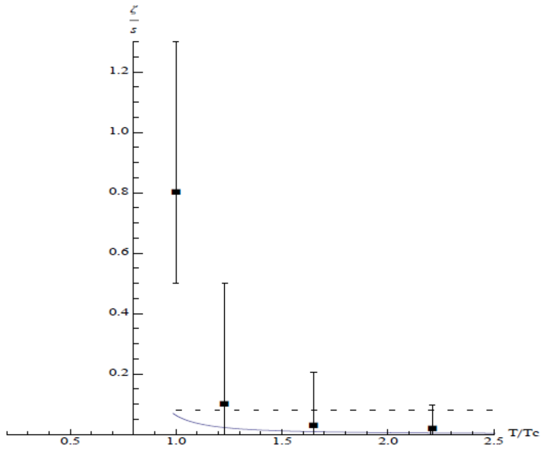

Figure 1.10: Plot of $\frac{\zeta}{s}$ as a function of temperature as found in [46]. The squares represent lattice data and the line comes from the GPR method applied to the ihQCD model

dimension is involved. For bulk viscosity the three bulk viscosities $\zeta_{\parallel}$,$\zeta_{\perp}$ and ζ_m relate to dissipation due to compression in the z-direction, the x- or y-direction and two both direction simultaneously respectively. As is be discussed in section 1.5, the values for the shear viscosities are very well established. Our main focus in this thesis is therefore to calculate the three bulk viscosities. The attempt made previously that comes closest to this goal is in the dissertation of Critelli [28], where the bulk viscosity for an Einstein-Maxwell action, i.e. an action with non-conformality induced by a magnetic field, is calculated. At this time it was not yet known that there where three bulk viscosities, and Critelli find that the that the two anisotropic bulk viscosities vanish for this action. From [43], we know that this does not happen when the non-conformality is induced by a dilaton potential, which is the case we will study in this thesis, where we will look for all five viscosities.

1.9 Calculating viscosities

There are several holographic routes towards the shear and bulk viscosities. One way is by looking at the dispersion relation of sound waves [83], [11], [61]. Hydrodynamics predicts the following dispersion relations for the shear and sound mode:

$$\omega_s = -i\frac{\eta}{\epsilon + P}k^2 + \mathcal{O}(k^3) \tag{1.13}$$

$$\omega_{♪} = v_{♪}k - i\frac{1}{\epsilon + P}(\zeta + \frac{4}{3}\eta)k^2 + \mathcal{O}(k^3) \tag{1.14}$$

$$v_{♪} = \frac{dP}{d\epsilon} \tag{1.15}$$

Here $v_{♪}$ is the speed of sound. Second, you can work with fluid-gravity correspondence [14], [56], [13], which is a theory where you take a family of solutions to your holographic Einstein equations and make an ultra-local expansion of the parameters belonging to that family for the space-time dimensions. You then solve the Einstein equations at first order, that arise from this expansion and calculate the stress-energy tensor at the boundary as is done in [8]. This stress energy tensor

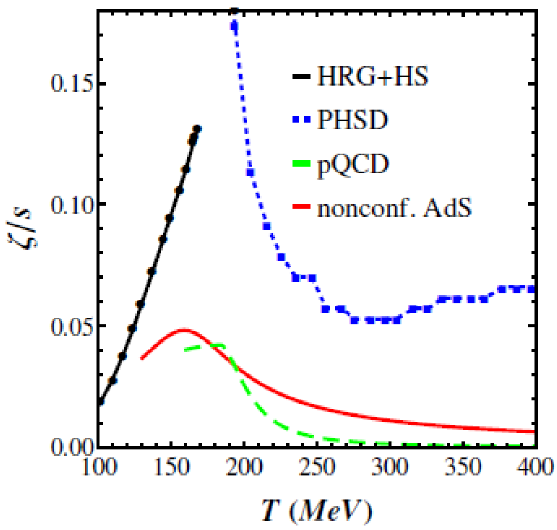

Figure 1.11: Several plots of the bulk viscosity [36] as a function of temperature using different methods. The solid red line uses the GPR method, the dashed green line is an extrapolation of perturbative QCD results [7], the blue squares come from parton-hadron-string dynamics (this method is similar to the one disccued in section 7.2) model [80], and the black points are found using the hadron resonance gas model [65].

is dual to the stress energy tensor of the fluid, which has the viscosities as coefficients of the first order terms. Another method is a Green's functions method where we look at the leading and sub-leading term of a boundary expansion of the dual source corresponding to a field-theory operator (in our case the energy-momentum tensor), which gives the transport coefficients of the relativistic fluid via the GKPW rule [52], [66]. This method is discussed in chapter 4. Similar to this, there is the Gubser-Pufu-Rocha (GPR) method which uses the membrane paradigm [58]. Specifically, it considers a Green's function at the horizon [43], instead of the boundary. For QCD this method can only be used in the deconfined phase, as there is a thermal gas solution without a horizon in the bulk when the field theory is confined [45]. This method is further explored in [46]. A method based on the GPR method is also discussed in chapter 4. Lastly, a completely different method is the Eling-Oz (EO) formula [33], which uses the Raychaudhuri equation. The Raychaudhuri equation, or null focusing equation, describes the evolution of the horizon entropy. This is dual to the entropy current equation of the fluid, and the coefficients of this equation give the viscosities. A new formula based on the EO-formula is discussed in chapter 5. In this thesis we will only use method based on the GPR method and the Eling-Oz formula to calculate the viscosities. We use both of these methods because that way we can check for consistency, and also confirm that the Eling-Oz formula, which is a relatively new and unexplored as well as a very simple formula, indeed can calculate the bulk viscosity. Such a comparison has already been done in [18] for an isotropic QGP, i.e. without a magnetic field. In chapter 3 we describe the gravitational method which is the basis for any attempt at holographically calculating viscosities.

1.10 Outside holography

We will compare the results in this thesis to methods that do not utilize holography as well. We will look at experimental data in section 7.1 and at a computational non-holographic method based on the lowest Landau level approximation, which is discussed in section 7.2.

Chapter 2

Relativistic magnetohydrodynamics

2.1 Introduction

In [55], HK make a hydrodynamic expansion of the entropy current for a fluid in a magnetic field, which results in two conductivities, two shear viscosities and three bulk viscosities. In this chapter the parts of this paper useful for this thesis are summarized.

2.2 Physics of fluids

Relativistic hydrodynamics studies the physics of relativistic fluids in a magnetic field. First, it is useful to note what we mean when we talk about a fluid. A fluid is continuous system where every infinitesimal volume can be seen as close to local thermal equilibrium [60], with only the averages of the microscopic fluctuations affecting the macroscopic dynamics of zeroth order. Only the slowly varying quantities such as the conserved quantities contribute in this case. Rapidly fluctuating contributions, such as dissipative contributions, only affect the system at a perturbative level. This is the hydrodynamic limit. The quantitative requirement for calling something a fluid is a small Knudsen number [33].

2.3 Thermal equilibrium

As is consistent with our definition of a fluid, we start our expansion at zeroth order, where we require that our fluid is in thermodynamic equilibrium. Thermodynamic equilibrium can be enforced by requiring that there is a time-like Killing vector V, such that the Lie derivative of the thermodynamic sources and quantities with respect to V vanishes [63]. Phrased more simply, we have a time-independent system. This requirement yields the following equations:

$$T\partial_\lambda\left(\frac{\mu}{T}\right) + a_\lambda\mu - E_\lambda = 0 \tag{2.1}$$

$$a_\lambda + \frac{1}{T}\partial_\lambda T = 0 \tag{2.2}$$

$$\nabla_\mu u_\nu + u_\mu a_\nu + \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}u^\alpha\Omega^\beta = 0 \tag{2.3}$$

Here u_μ is the fluid velocity, a_μ is the fluid acceleration, Ω_μ is the fluid vorticity, T is the temperature and μ is the chemical potential. The external fields are the electric field E_μ and magnetic field B_μ, given by the following equations, in the fluid rest frame:

$$E_\mu = F_{\mu\nu} u^\nu \tag{2.4}$$
$$B_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\gamma} u^\nu F^{\rho\gamma} \tag{2.5}$$

Here $F_{\mu\nu}$ is the electromagnetic tensor. In this thesis we will always consider a strong electromagnetic field, i.e. $F_{\mu\nu}$ is a zeroth order tensor. However, E_μ does not affect the zeroth order free energy, because from equation 2.1 it follows that since E_μ is in an equation with the derivative of the zeroth order term μ, it will be screened at first order. In this thesis μ and E_μ will be kept zero, but for completeness it will be taken into account in this chapter. Whereas equation 2.1, 2.2 and 2.3 will no longer hold once thermal equilibrium is broken, the following two conservation equations will remain valid. They can be found by requiring gauge and diffeomorphism invariance of the free energy:

$$\nabla_\mu J^\mu = 0 \tag{2.6}$$
$$\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} J_\lambda \tag{2.7}$$

The same holds for the Bianchi identity, which upholds current conservation of the magnetic field lines [40].

$$\nabla_\nu \mathbb{J}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta}\nabla_\nu F_{\alpha\beta} = 0 \tag{2.8}$$

2.4 Hydrodynamic expansion

For the hydrodynamic expansion we start off by decomposing the energy-momentum tensor $T_{\mu\nu}$ and the current J_μ with respect to the fluid velocity.

$$T_{\mu\nu} = \mathcal{E} u_\mu u_\nu + \mathcal{P}\Delta_{\mu\nu} + \mathcal{Q}_\mu u_\nu + \mathcal{Q}_\nu u_\mu + \mathcal{T}_{\mu\nu} \tag{2.9}$$
$$J_\mu = \mathcal{N} u_\mu + \mathcal{J}_\mu \tag{2.10}$$

Here $\Delta_{\mu\nu}$ is the projector orthogonal to the fluid velocity:

$$\Delta_{\mu\nu} = \gamma_{\mu\nu} + u_{\mu\nu} \tag{2.11}$$

With $\gamma_{\mu\nu}$ the metric tensor. The coefficients of this decomposition have zeroth and first order component:

$$\mathcal{E} = \epsilon(T,\mu,B^2) + f_{\mathcal{E}} \tag{2.12}$$
$$\mathcal{P} = \Pi(T,\mu,B^2) + f_{\mathcal{P}} \tag{2.13}$$
$$\mathcal{N} = n(T,\mu,B^2) + f_{\mathcal{N}} \tag{2.14}$$
$$\mathcal{T}_{\mu\nu} = \alpha(T,\mu,B^2)(B_\mu B_\nu - \frac{1}{3}\Delta_{\mu\nu}B^2) + f_{\mathcal{T}\mu\nu} \tag{2.15}$$

Thermodynamics tells us that $\epsilon = -p + T(\partial p/\partial\mu)$, $\Pi = p - \frac{4}{3}\partial p/\partial B^2$ and $n = \partial p/\partial\mu$. The only vector that Q_μ and J_μ could be composed of at zeroth order is B_μ, but because B_μ has even parity (so it is actually a pseudovector) this is not allowed and therefore Q_μ and J_μ don't contribute at zeroth order. Now that the zeroth order is fully established, we can go to first order, were there are, among other terms, dissipative terms. Unlike for ideal fluids, the definition of local rest frame now becomes ambiguous [60]. We can choose to define the velocity by the flow of particles or by the flow of energy. These two frame choices are called the Eckart frame and the Landau-Lifshitz frame respectively. The Eckart frame leads to

$$J_\mu u^\mu = 0 \;\rightarrow\; \mathcal{J}_\mu = 0 \tag{2.16}$$

The Landau-Lifshitz frame requires:

$$T_{\mu\nu} u^\mu = \epsilon(T, \mu, B^2) u_\nu \;\rightarrow\; Q_\mu = 0 \tag{2.17}$$

We choose however yet another frame called the thermodynamic frame, which is introduced in [63]. For this frame choice, μ, T and u_μ remain unchanged when thermal equilibrium is broken. This requires:

$$T^{\text{non-eq}}_{\mu\nu} u^\mu = 0 \tag{2.18}$$

$$J^{\text{non-eq}}_{\mu} u^\mu = 0 \tag{2.19}$$

So all non-equilibrium contributions, which can be but are not necessarily dissipative, are absorbed by $f_{\mathcal{P}}$ and $f_{\mathcal{T}\mu\nu}$ and J_μ. We can then list all possible scalars, vectors and tensors that can appear in $\mathcal{E}$, $\mathcal{P}$, $\mathcal{Q}_\mu$, $\mathcal{T}_{\mu\nu}$, $\mathcal{J}_\mu$ and $\mathcal{N}$ at first order, and discard terms based on parity and because of dependence that follows from equation 2.6, 2.7 and 2.8. This is described in detail by KH in [68]. The entropy current is as follows:

$$\nabla_\mu S^\mu = \nabla_\mu\Big(\frac{p}{T}u^\nu\Big) = T^{\mu\nu}_{\text{non-eq}} \nabla_\mu \frac{u_\nu}{T} + J^\mu_{\text{non-eq}}\Big(\frac{E_\mu}{T} - \partial_\mu \frac{\mu}{T}\Big) \tag{2.20}$$

Here non-equilibrium is explicitly stated because subtraction of equilibrium contributions is necessary to make the entropy current well defined. We then find the following result:

$$\begin{aligned} T\nabla_\mu S^\mu = \sigma_\perp \frac{(B\cdot V)}{B^2} + \sigma_\parallel (\mathbb{B}_{\mu\nu} V^\mu)^2 + \frac{1}{2}\eta_\perp (\sigma^{\mu\nu})^2 + \eta_\parallel \Sigma^2 \\ \zeta_\parallel S_\parallel^2 + 2\zeta_m S_\parallel S_\perp + \zeta_\perp S_\perp^2 \end{aligned} \tag{2.21}$$

With $\mathbb{B}_{\mu\nu} = \Delta_{\mu\nu} - \frac{B_\mu B_\nu}{B^2}$, $V^\mu - T\Delta^{\mu\nu}\partial_\nu(\mu/T)$, $\sigma_{\perp\mu\nu} = \frac{1}{2}(\mathbb{B}^{\mu\lambda}\mathbb{B}_{\nu\rho} + \mathbb{B}^{\nu\lambda}\mathbb{B}_{\mu\rho} - \mathbb{B}^{\mu\nu}\mathbb{B}_{\lambda\rho})\sigma^{\lambda\rho}$ and $\Sigma_\mu = \mathbb{B}_{\mu\lambda}\sigma^{\lambda\rho} b_\rho$. From this we can conclude that magnetohydrodynamics is characterized by two conductivities, two shear viscosities and three bulk viscosities. Note that the result for the viscosities is independent of E_μ or μ. This is consistent with linear response theory. In linear response theory, viscosities are coefficients which tell us how source gives rise to an expectation value of the operator [40] [52]. We can write this down in the following way:

$$\langle \delta T_{12} \rangle = i\omega \eta_\perp \delta g_{12} \tag{2.22}$$

$$\langle \delta T_{13} \rangle = i\omega \eta_\parallel \delta g_{13} \tag{2.23}$$

$$\begin{pmatrix} \langle \delta T_{11} \rangle \\ \langle \delta T_{33} \rangle \end{pmatrix} = i\omega \begin{pmatrix} \zeta_\perp + \eta_\perp & \zeta_m \\ \zeta_m & \zeta_\parallel \end{pmatrix} \begin{pmatrix} \delta g_{11} \\ \delta g_{33} \end{pmatrix} \tag{2.24}$$

The $\eta_\perp$ enters in the bulk matrix because when you break the $SO(3)$ symmetry an ambiguity emerges when you want to couple modes to transport coefficients, we will elaborate on this in chapter 4.

Chapter 3

Gravitational action

3.1 Introduction

In this section we will work out the theory of gravity which we will use to mimick QCD. We will first work out the bulk geometry at zeroth order. At this order we have time independence, which is required for the dual QFT to be thermodynamically stable at zeroth order, and thus fluid-like. The starting point is writing out the action. The action that will be used in this thesis was constructed in [62] and [4]. [45], [30], [6] and [48], among others, further explore this action. The action consists of a glue sector and a flavor sector and gives the Einstein equations which fix the metric at zeroth order. For this metric $g_{\mu\nu}$ the following Ansatz will be used:

$$ds^2 = e^{2A(\phi)}(-f(\phi)dt^2 + dx^2 + dy^2 + e^{2W(\phi)}dz^2) + \frac{e^{2B(\phi)}}{f(\phi)}d\phi^2 \tag{3.1}$$

This metric is written in the $\phi = r$ gauge or Gubser gauge, as will be clarified in Appendix A. ϕ here roughly plays the role of the inverse energy scale [48]. $e^{2A(r)}$ incorporates the (non-)conformality and $e^{2A(r)}$ incorporates the anisotropy of the field theory. f is the blackening factor, which is characteristic for a theory of gravity dual to a deconfined fluid, because it means that the flux tubes that confine quarks dissolve when the tip of the string reaches the horizon. In the next section the glue and flavor sector of this action will be worked out.

3.2 Glue sector

The glue sector action is as follows:

$$S_g = M^3 N_c^2 \int dx^5 \sqrt{-g}(R - \frac{1}{2}(\partial\phi)^2 + V_g) \tag{3.2}$$

Here ϕ is the dilaton with its potential V_g. M is the five-dimensional Planck constant. We further have a Ricci scalar in the action, which is always there in a theory of gravity. To constrain V_g, the following mapping from the field theory to gravity is made [45]:

$$\sqrt{\frac{8}{3}} \ln\lambda = \phi \tag{3.3}$$

$$\ln E = A \tag{3.4}$$

Here E is the energy scale and λ is the 't Hooft coupling. The $\sqrt{\frac{8}{3}}$ is just convention. With this identification, the equations of motion (EOM) can be used to constrain the UV asymptoptics of V_g such that the renormalization group flow is QCD-like, i.e. shows asymptotic freedom. This means we get the following requirement from the beta function of QCD:

$$\beta(\lambda) = \frac{d\lambda}{d\ln E} = \frac{d\lambda}{dA} \tag{3.5}$$

The IR asymptotics of V_g are found by requiring that A should have a minimum somehwere in the bulk. This minimum makes sure that for low temperatures the field theory shows quark confinement, which in the dual setting means that a gluon flux tube is formed from the quark to the anti-quark at this minimum (see figure 3.1). Such a potential which satisfies these requirements, is one that will be used for our numerical calculations.

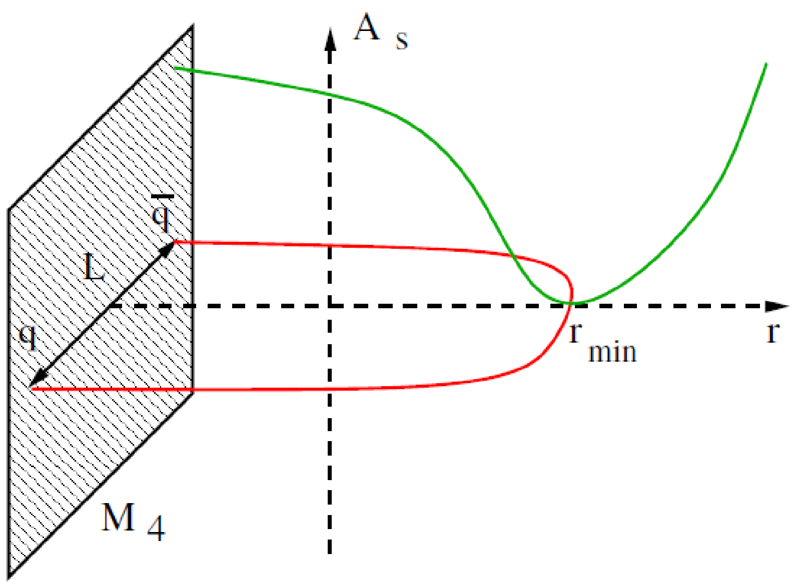

Figure 3.1: Holographic representation of confinement of a quark-antiquark pair mediated by a gluon flux tube [45].

3.3 Flavor sector

The flavor sector relates to quarks, and it should therefore mimic a phenomenon that is observed for quarks: chiral symmetry breaking. This is done with the following action:

$$S_f = -xM^3N_c^2 \int dx^5 V_f \sqrt{\det(g_{\mu\nu} + wF_{\mu\nu} + \kappa\partial_\mu\tau\partial_\nu\tau)} \tag{3.6}$$

Here $F_{\mu\nu}$ is the electromagnetic tensor and τ is the tachyon. This action is called Sen's action and is essentially a DBI-action. The derivation of this action therefore originates from string theory, where it was observed that brane-antibrane systems encode several features which are related to chiral symmetry breaking [57]. Specifically, when we take quarks to be massless they have a chiral $SU(N_f)_L \times SU(N_f)_R$ symmetry which is broken to $SU(N_f)_{L+R}$ because of the non-trivial vacuum expectation value of the quark condensate $\langle \overline{q}q \rangle$ [45], which is the order parameter of chiral symmetry breaking. This phenomenon is reproduced with N_f space-filling D4-$\overline{D}$4 brane-antibrane pairs in the theory of gravity. Dual to the quark-antiquark pair, the $SU(N_f)$ symmetry of the individual the branes and antibranes will be broken in the IR, which is achieved by properly tuning V_f, w and κ. Such tuned functions will be used for our numerical calculations. For magnetic field in the z-direction, the electremagnetic tensor is given by:

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} \tag{3.7}$$

$$A_\mu = \{0, -\frac{1}{2} B_m y, \frac{1}{2} B_m x, 0, 0\} \tag{3.8}$$

The magnetic susceptibility of our QGP is given by:

$$\chi_B = -\frac{1}{V_4}\frac{\partial}{\partial B^2}\Big(x M^3 N_c^2 \int dx^5 V_f \sqrt{\det(g_{\mu\nu} + w F_{\mu\nu} + \kappa \partial_\mu \tau \partial_\nu \tau)}\Big) \tag{3.9}$$

$$= -\frac{\partial}{\partial B^2}\Big(x M^3 N_c^2 \int_0^{r_h} dr V_f \sqrt{\det(g_{\mu\nu} + w F_{\mu\nu} + \kappa \partial_\mu \tau \partial_\nu \tau)}\Big) \tag{3.10}$$

Where we define $V_4 = \int dx^4$.

3.4 Phase diagram

In [62], three distinct phases were found for this model: the confined phase, and two deconfined phases, one with chiral symmetry and one with broken chiral symmetry. The phase affects the UV asymptotics of the tachyon [48]:

$$\tau(r) \approx m_q r(-\log \Lambda) + \langle \overline{q}q \rangle r^3 (-\log \Lambda r)^\rho \tag{3.11}$$

In this thesis we set quark mass m_q to zero. In the symmetric phase, which is the only phase we will consider in this thesis, we have $\langle \overline{q}q \rangle = 0$. From the tachyon EOM we know that if $\tau_h = 0$, τ_h' and τ_h'' are also zero. Combining this with that we have a zero m_q and $\langle \overline{q}q \rangle$ and that $\tau = 0$ everywhere is allowed we conclude that it is the only allowed solution. The phase diagram is given in figure 3.2. For $x = 0.1$ there is a fourth phase, but since we will not come close this this value for x it is not worth discussing in this thesis. As explained in [5], the field theory entropy is dual the Bekenstein-Hawking entropy, which is found integrating the extrinsic curvature over the boundary and regularizing by subtracting the entropy of the Minkowski vacuum. This leads to:

$$s = s_{BH} = 4\pi M^3 N_c^2 \sqrt{\frac{-g}{g_{00} g_{55}}}\Big|_{\phi=\phi_h} = 4\pi M^3 N_c^2 e^{3A(\phi_h) + W(\phi_h)} \tag{3.12}$$

Similarly, temperature is dual to the Hawking temperature of the black hole of our dual theory [43]:

$$T = T_H = -\frac{f'(\phi_h)}{4\pi} e^{A(\phi_h) - B(\phi_h)} \tag{3.13}$$

3.5 Background equations

With this action, and the Ansatz given by equation 3.1, we can find the background equations of our theory of gravity. This is done in Appendix A.

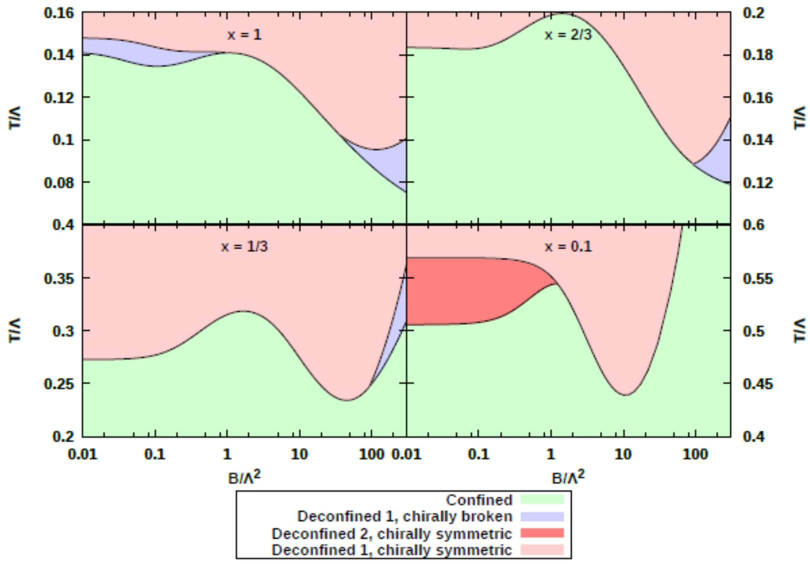

Figure 3.2: Phase diagram for QCD with magnetic field calculated by Gursoy, Iatrakis, Nijs and Jarvinen in [48]

3.6 Axion

In this thesis we will look at anisotropy induced by a magnetic field. However, a anisotropy can also be induced by an axion, and we will use this example several times for comparison. An axion is a source term for the axionic glueball operator on the field theory side. This term needs to be added if want to describe CP-odd phenomena [30] [47]. In its simplest form, the following term needs to be added to the action to include the axion:

$$S_a = -M^3 N_c^2 \int dx^5 \sqrt{-g} \frac{1}{2} Z(\phi)(\partial\chi)^2 \tag{3.14}$$

$$\chi = az \tag{3.15}$$

We have an 'anisotropic susceptibility' [49] (the axion equivalent of a magnetization), is given by:

$$\chi_a = -\frac{1}{V_4}\frac{\partial}{\partial a^2} M^3 N_c^2 \int dx^5 \sqrt{-g}\frac{1}{2} Z(\phi)(\partial\chi)^2 = -M^3 N_c^2 \int_0^{r_h} dx \sqrt{-g} Z(\phi)(g_{33})^{-1} \tag{3.16}$$

This is not entirely true as this term is still UV divergent but since we won't calculate the susceptibility we won't elaborate on this here. The important difference between an axion and a magnetic field is that when $SO(3)$ symmetry is broken to an $SO(2)$ symmetry for the x- and y-direction, an axion breaks zero temperature boost symmetry for the $z-$direction, whereas a magnetic field breaks it for the x- and y-direction [59]. This difference will turn out to be very important and we will mainly use the axion as a method for comparison so that we can better understand magnetic field induced anisotropy.

Chapter 4

Calculating viscosities: fluctuation equation

4.1 Introduction

In [43], Gubser, Pufu and Rocha (GPR) calculate the shear and bulk viscosity for an isotropic background. They do this by introducing time dependent fluctuations at first order. They then solve the fluctuation equations and find the Green's function located at the horizon which give the desired transport coefficients. This method is considerably more difficult for the bulk viscosity than for the shear viscosity, because the transformation properties of the shear modes under the $SO(2)$ symmetry guarantee that these modes do not couple to other modes, and we get massless EOM, which give trivial flow. In this chapter we will employ this method to calculate the viscosity for an anisotropic fluid with a magnetic field.

4.2 Green's function

To find a Green's function (or correlator, propagator or two-point function) for an operator $\mathcal{O}$, you need to take the functional derivative for our path integral $\left\langle \exp\left[i\int d^4x\phi_0\mathcal{O}\right]\right\rangle$ with respect to its source ϕ_0 twice [89]. For viscosities, the operator we are interested in is the energy momentum tensor $\frac{1}{2}T_{ij}$, and at first order, the corresponding source is the metric fluctuation h_{ij}. In Minkowski space-time, we can choose from the following Green's functions and all their linear combinations [87]:

$$G_R(k) = -i\int dx^4 e^{-ik\cdot x}\theta(t)\langle[\mathcal{O}(x), \mathcal{O}(0)]\rangle \tag{4.1}$$

$$G_A(k) = i\int dx^4 e^{-ik\cdot x}\theta(-t)\langle[\mathcal{O}(x), \mathcal{O}(0)]\rangle \tag{4.2}$$

$$G(k) = \int dx^4 e^{-ik\cdot x}\langle\{\mathcal{O}(x), \mathcal{O}(0)\}\rangle \tag{4.3}$$

With $\{...\}$ being anti-commutators. $\theta(t)$ is the Heaviside step function. As will be explained later, we will work with a Minkowski retarded function (equation 4.1). This is consistent with [43], where it is shown that this Green's function is mathematically equivalent to the Euclidean Green's function, for which there is only the Matsubara Green's function to choose from. Because work in the fluid rest frame we have $k = 0$. The following definition of the thermal Green's function is as general as is necessary for our computations:

$$G_{ijkl}(\omega) = -i \int dtd^3xe^{i\omega t}\theta(t)\langle[\frac{1}{2}T_{ij}, \frac{1}{2}T_{kl}]\rangle \tag{4.4}$$

This Green's function can be related to the expectation value of the linear response of a perturbation of the metric using a Kubo formula. In [43], GPR derive the following relation:

$$\delta\langle T_{ij}(\omega)\rangle = -\frac{1}{2}h_{kl}(\omega)G^{kl}_{ij}(\omega) \tag{4.5}$$

We will only need the imaginary part of this linear response to calculate the viscosities. GPR find the following equation for Im δT_{ij}.

$$\text{Im } \delta T_{ij}(\omega) = \frac{\omega}{2}h^{kl}(\omega)\Big(\eta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}) + \zeta\delta_{ij}\delta_{kl}\Big) \tag{4.6}$$

However, this is equation no longer holds when the magnetic field breaks the $SO(3)$ symmetry of the QGP into an $SO(2)$ symmetry. We can decompose δ_{ij} into a part parallel to the magnetic field b_i and a part orthogonal to the magnetic field β_{ij}:

$$\delta_{ij} = \beta_{ij} + b_ib_j \tag{4.7}$$

We get the following more general equation for the linear response consistent with the current symmetries:

$$\begin{aligned}\text{Im } \delta T_{ij}(\omega) = \frac{\omega}{2}h^{kl}(\omega)\Big(&\eta_\perp(\beta_{ki}\beta_{lj} + \beta_{li}\beta_{kj}) + \eta_\parallel(b_kb_i\beta_{lj} + b_lb_j\beta_{ki} + b_kb_j\beta_{li} + b_lb_i\beta_{kj}) \\ &+(\zeta_\perp - \frac{2}{3}\eta_\perp)\beta_{ij}\beta_{kl} + (\zeta_m - \frac{2}{3}\eta_\parallel)(\beta_{ij}b_kb_l + \beta_{kl}b_ib_j) + (\zeta_\parallel + \frac{4}{3}\eta_*)b_ib_jb_kb_l\Big)\end{aligned} \tag{4.8}$$

Considering the symmetries of our system, we could now turn the coefficients inside the brackets into single coefficients to find that this is consistent with what was predicted by KH in [55]:

$$\zeta_\perp - \frac{2}{3}\eta_\perp \to \zeta_\perp \tag{4.9}$$

$$\zeta_m - \frac{2}{3}\eta_\parallel \to \zeta_m \tag{4.10}$$

$$\zeta_\parallel + \frac{4}{3}\eta_* \to \zeta_\parallel \tag{4.11}$$

This is an arbitrary choice which is consistent with [40]. Note that the original definition of a bulk viscosity, i.e. the trace part of the first order dissipation, no longer has any meaning, and we don't get that turning off magnetic field gives that the result reduces to the isotropic result. Instead, we should think of these 'bulk viscosities' as the scalar or spin zero dissipative coefficients. 'Shear viscosities', which were previously the traceless part of the first order dissipation of the are then vectorial or tensorial dissipative coefficients. The extract transport coefficients can then be extracted in the following way:

$$-\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G_{s\perp} = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G_{1212} = \lim_{\omega\to 0}\frac{1}{\omega}\int dtd^3xe^{i\omega t}\theta(t)\langle[T_{12},T_{12}]\rangle = \eta_\perp \tag{4.12}$$

$$-\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G_{s\parallel} = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G_{1313} = \lim_{\omega\to 0}\frac{1}{\omega}\int dtd^3xe^{i\omega t}\theta(t)\langle[T_{13},T_{13}]\rangle = \eta_\parallel \tag{4.13}$$

$$\begin{aligned}&-\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G_{b\perp} = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\beta^{\mu\nu}\beta^{\rho\sigma}G_{\mu\nu\rho\sigma} = \\ &\lim_{\omega\to 0}\frac{1}{\omega}\int dtd^3xe^{i\omega t}\theta(t)\langle[\beta^{ij}\frac{1}{2}T_{ij},\beta^{kl}\frac{1}{2}T_{kl}]\rangle = \zeta_\perp + \eta_\perp\end{aligned} \tag{4.14}$$

$$\begin{aligned}&-\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G_{bm} = -\frac{1}{2}\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}b^\mu b^\nu\beta^{\rho\sigma}G_{\mu\nu\rho\sigma} - \frac{1}{2}\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\beta^{\mu\nu}b^\rho b^\sigma G_{\mu\nu\rho\sigma} = \\ &\frac{1}{2}\lim_{\omega\to 0}\frac{1}{\omega}\int dtd^3x(e^{i\omega t}\theta(t)\langle[b^ib^j\frac{1}{2}T_{ij},\beta^{kl}\frac{1}{2}T_{kl}]\rangle + e^{i\omega t}\theta(t)\langle[\beta^{ij}\frac{1}{2}T_{ij},b^kb^l\frac{1}{2}T_{kl}]\rangle) = \frac{1}{2}\zeta_m\end{aligned} \tag{4.15}$$

$$\begin{aligned}&-\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G_{b\parallel} = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}b^\mu b^\nu b^\rho b^\sigma G_{\mu\nu\rho\sigma} \\ &= \lim_{\omega\to 0}\frac{1}{\omega}\int dtd^3xe^{i\omega t}\theta(t)\langle[b^ib^j\frac{1}{2}T_{ij},b^kb^l\frac{1}{2}T_{kl}]\rangle = \frac{1}{4}\zeta_\parallel\end{aligned} \tag{4.16}$$

We have T_{12} and not $\frac{1}{2}T_{12}$, because we must treat h_{12} and h_{21} in a symmetric way (i.e. $h_{21} \equiv h_{12}$) which gives a factor two. These Green's functions can be found using holography. According to the GKPW rule, our four-dimensional path integral is equal to e^{-iS} in the dual five-dimensional theory of gravity if we move to the boundary. But we can use the membrane paradigm (see [58] and Appendix D), to justify that we can also look at the horizon. We can therefore take a functional derivatives on e^{-iS} and arrive at the Green's function this way. However, a functional derivative meant for a four-dimensional path integral is not well defined for a five-dimensional integral. To perform the functional differentiation properly we instead first solve the equations of motion and substitute them back into the action, making it an on-shell action. The Lagrangian in an on-shell action is a total derivative:

$$\mathcal{L}_{OS} = \partial_r J \tag{4.17}$$

Using Gauss' law we find that the action then becomes a boundary integral at the horizon. This boundary integral has to be canceled [58], yielding a term which can be subjected to a functional derivative in order to get our desired Green's function. However, we don't take the functional derivative with respect to the entire mode h_{ij}, only the part which depends on the space-time dimensions:

$$h_{ij}(r,t,\vec{x}) = (h(r)\xi(t,\vec{x}))_{ij} \tag{4.18}$$

In our case we have homogeneity for the spatial dimensions, so $\xi(t,\vec{x}) = \xi(t)$. Note that, from the structure of the Ricci scalar it follows that the Lagrangian contains only terms with no more than two derivatives. Furthermore, one can subtract the Euler-Lagrange equation from the original Lagrangian to get the on-shell Lagrangian [43]:

$$\mathcal{L}_{OS} = \partial_r J = \mathcal{L} - h^*_{ij}(\frac{\partial\mathcal{L}}{\partial h^*_{ij}} - \partial_r\frac{\partial\mathcal{L}}{\partial h^{*\prime}_{ij}}) \tag{4.19}$$

Since all other terms in the Lagrangian are real, we can combine these two facts to find that Green's function must have the following form:

$$\begin{aligned}\mathrm{Im}G(\omega)_{ijkl} &= -\mathrm{Im}\frac{\delta}{\delta\xi^{ij}(\omega)}\frac{\delta}{\delta\xi^{kl}(\omega)}e^{-iS_{OS}} \\ &= M^3N_c^2\mathcal{F} \\ &= M^2N_c^3P(r)(h^*_{ij}(r,\omega)h'_{kl}(r,\omega) - h^{*\prime}_{ij}(r,\omega)h_{kl}(r,\omega)) \\ &+ V(r)(h^*_{ij}(r,\omega)h_{kl}(r,\omega) - h^*_{ij}(r,\omega)h_{kl}(r,\omega))\end{aligned} \tag{4.20}$$

P will follow from the holographic computations, and $\mathcal{F}$ is the graviton flux. As the GKPW rule prescribes, the r-coordinate must be the boundary radius $r = 0$ in this computation, i.e. $G(\omega) = G(\omega, 0)$. However, because the graviton flux is conserved, we have [58]:

$$\partial_r\mathcal{F} = 0 \tag{4.21}$$

We can therefore choose any value for r we like. As will be shown in Appendix C, it turns out to be very convenient to look at the Green's function for $r = r_h$, which is what is done in the GPR method. As mentioned, the Green's function gives us the linear response to a source, in our case the metric perturbation. For our anisotropic case, these responses of the operators can be represented in the following matrix form.

$$\begin{pmatrix}\langle T_{11}\rangle \\ \langle T_{33}\rangle\end{pmatrix} = -i\begin{pmatrix}\mathrm{Im}G_{b\perp}(\omega) & 2\mathrm{Im}G_{bm}(\omega) \\ 2\mathrm{Im}G_{bm}(\omega) & 4\mathrm{Im}G_{b\|}(\omega)\end{pmatrix}\begin{pmatrix}\delta g_{11} \\ \delta g_{33}\end{pmatrix} \tag{4.22}$$

The can be related to the matrix in equation 2.24.

4.3 UV expansion

When we want to find the transport coefficients, we don't have to go to the horizon. The more conventional way is to look at the UV expansion of the perturbative modes. For example, in [39], this method is used to find the shear and Hall viscosity for a 2+1-dimensional holographic model for chiral superfluiditiy. To better understand this concept we will first work with a scalar field ϕ dual to $\mathcal{O}$, but this result for any p-form [5]. For ϕ, the bulk action is as follows:

$$S_{bulk} = -\frac{1}{2}\int drd^dx\sqrt{-g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2 + \text{interactions}) \tag{4.23}$$

At the boundary, we also have to include the Gibbon-Hawking term [45]:

$$S_{GH} = -\frac{1}{2}\int d^dx^4X\sqrt{\gamma}X^\mu\phi\partial_\mu\phi \tag{4.24}$$

$$\gamma_{\mu\nu} = g_{\alpha\beta}\partial^\alpha X_\mu\partial^\beta X_\nu \tag{4.25}$$

γ is the induced metric at the boundary. X is space-time dependent only. This term is needed to make sure that the boundary action only depends on first derivatives [37]. This term does not contribute to the equations of motion, but does contribute to the Greens function. Near the boundary our metric becomes AdS and looks as follows:

$$ds^2 \approx \frac{R^2}{r^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dr^2) \tag{4.26}$$

Here R is the AdS radius. The EOM that follows from this is:

$$r^{d+1}\partial_r(r^{1-d}\partial_r\Phi) - k^2r^2\Phi - m^2R^2\Phi = 0 \tag{4.27}$$

$$\phi(x^\mu, r) = \int \frac{d^dk}{(2\pi)^d} e^{ik\cdot x}\Phi(x^\mu, r) \tag{4.28}$$

This yields the following UV expansion for $k^2 \to 0$ (we will comment later on what happens when you don't work in this limit):

$$\Phi(r,k) \approx A(k)r^{d-\Delta} + B(k)r^\Delta \tag{4.29}$$

$$\Delta = \frac{d}{2} + \sqrt{m^2R^2 + \frac{d^2}{4}} \tag{4.30}$$

The two-point function or Green's function can be found from the action, after counterterms have been added to deal with UV divergences. We won't go into detail here (for this see [95]), but the result is:

$$\langle\mathcal{O}(\omega,k)\rangle = 2(\Delta - \frac{d}{2})A(k) \tag{4.31}$$

$$\langle\mathcal{O}(-\omega,-k)\mathcal{O}(\omega,k)\rangle = 2(\Delta - \frac{d}{2})\frac{B(k)}{A(k)} \tag{4.32}$$

We arrive at Green's functions for the field theory operators because we are at the boundary, where the GKPW rule (equation 1.3) holds. For a p-index function Γ, we can generalize this. At the bulk, the action then has the following structure:

$$S_{bulk} \propto \int drd^dx\sqrt{-g}(g^{\mu\nu}g^{\alpha\beta}...g^{\rho\sigma}\partial_\mu\Gamma_{\alpha...\rho}\partial_\nu\Gamma_{\beta...\sigma} + m^2\Gamma^2_{\alpha...\rho} + \text{interactions}) \tag{4.33}$$

The indices of Γ can also be anti-symmetrized like for the electromagnetic tensor but this would not change this derivation and this therefore holds for any two-derivative kinetic term. With the same method, this gives the following UV expansion [5]:

$$\Psi_{\mu_1...\mu_p} \approx A_{\mu_1...\mu_p}r^{d-p-\Delta} + B_{\mu_1...\mu_p}r^{\Delta-p} \tag{4.34}$$

$$m^2R^2 = (\Delta - p)(\Delta + p - d) \tag{4.35}$$

This works for the shear channel (i.e. the helicity one and two block), but as will become clear in section **??**, the situation for the scalar channel, or the helicity zero block, is more complicated than this.

4.4 Helicity blocks

With the magnetic field turned on, the $SO(3)$ symmetry that was there for the isotropic case, turns into an $SO(2)$ symmetry. As explained in [34], the metric fluctuations are characterized by their transformation properties under this symmetry. Based on this, we can divide 17 different perturbative modes in blocks where perturbative modes are only coupled to the other modes in the same block. Because of the $SO(2)$ symmetry, we can take $h_{1i} = h_{2i}$, for any i, except for the h_{12} mode. The three blocks we get then look as follows.

$$\text{helicity two} : h_{12} \tag{4.36}$$

$$\text{helicity one} : h_{01}, h_{13}, h_{15}, \delta A_1 \tag{4.37}$$

$$\text{helicity zero} : h_{00}, h_{05}, h_{03}, h_{11}, h_{33}, h_{35}, h_{55}, \delta\phi, \delta\tau, \delta A_0, \delta A_3, \delta A_5 \tag{4.38}$$

However, not all fluctuations have to be considered. For the helicity zero block, $\delta\tau$ can be ignored because we have $\tau = 0$ in all of our calculations. Since τ only appears quadratically in the action it doesn't appear in the first order EOM. The same holds for all gauge fields (see Appendix B). This is unlike the case where we have a non-zero baryon chemical potential, where coupling to the gauge field perturbations does happen [34]. Proceeding, we can also use diffeomorphism and gauge invariance of the perturbative modes to remove some fluctuations. The metric and dilaton fluctuations transform as follows [43]:

$$\delta\widetilde{\phi} = \delta\phi + \xi^{\mu}\partial_{\mu}\phi \tag{4.39}$$

$$\widetilde{h}_{\mu\nu} = h_{\mu\nu} + \nabla_{\mu}\xi_{\nu} \tag{4.40}$$

For helicity two, there are no transformations possible, so the h_{12} mode stays. For helicity one, the h_{01} mode can be gauged away (see Appendix B). The Einstein equation we will find for the h_{15} mode is $h_{15} = 0$, so we can ignore it as well. For helicity zero, we can gauge away three modes. We choose to throw away h_{03}, h_{05} and $\delta\phi$. We are left with h_{00}, h_{35}, h_{33} and h_{55}. As it turns out, one of the Einstein equations when including the h_{35} mode we get is $h_{35} = 0$, so we can dispose of this mode in advance also.

4.5 Shear viscosities

This section is about the helicity one and helicity two blocks of equation 4.38. These compuations are relatively simple and analogous to the computations done by Critelli et al. in [27]. The reason that it is simple is because the modes decouple from all the other modes and are massless. Formulated differently, they satisfy a 5D-wave equation [44]. The perturbative modes introduce time dependence to the EOM. Because this is the only time-dependence in the EOM and because the equation is linear for these modes, we know that there is harmonic time dependence [43]. We can therefore say $h_{ij}(\phi, t) = e^{i\omega t}h_{ij}(\phi)$. The rest of the computations are performed in Appendix C. The end result is:

$$\frac{\eta_{\perp}}{s} = \frac{1}{4\pi} \tag{4.41}$$

$$\frac{\eta_{\parallel}}{s} = \frac{e^{2W(\phi_h)}}{4\pi} \tag{4.42}$$

This is consistent with what Critelli et al. find in [27] for an Einstein-Maxwell-dilaton action.

4.6 Bulk viscosities

This section discusse the helicity zero block, which is the block with the bulk modes, we choose, similar to [43], the following perturbative metric $g_{\mu\nu} + \delta g_{\mu\nu}$.

$$\begin{aligned} ds^2 = e^{2A(\phi)}(-f(\phi)(1 + \frac{\lambda h_{00}(\phi)e^{i\omega t}}{2})^2 dt^2 + (1 + \frac{\lambda h_{11}(\phi)e^{i\omega t}}{2})^2(dx^2 + dy^2) \\ + (e^{2W(\phi)} + \frac{\lambda h_{33}(\phi)e^{i\omega t}}{2})^2 dz^2) + (1 + \frac{\lambda h_{55}(\phi)e^{i\omega t}}{2})^2 \frac{e^{2B(\phi)}}{f(\phi)} d\phi^2 \end{aligned} \tag{4.43}$$

Unlike the shear modes we study in Appendix C, the bulk modes cannot be made massless with a simple rescaling, which would give us trivial flow. As explained in Appendix D, the only requirement for this rescaling is that it becomes unity at the boundary. As a side note, one could think that a new method for understanding the flow of the modes is by deriving the rescaling which would make the mode massless. This concept is further elaborated in Appendix I, where it is also shown that this is of no use. The EOM are derived in Appendix F. In Appendix G, we work out how we the Green's function method at the horizon outlined by GPR in [43], would work for magnetic field induced anisotropy, though we will not use this method.

4.7 Scalar channel UV expansion

As mentioned in section 4.58, the UV expansion of the helicity zero fluctuation is more complicated than equation 4.35 and will be worked out in this section. In the UV, boost symmetry breaking induced by the black hole solution is undone. In the isotropic case, we therefore have three modes in the helicity zero block: h_{55}, ψ and $\delta\phi$. In r-variables, the metric then becomes:

$$ds^2 = e^{2A(r)}\Big\{(1+\lambda\psi(x_\mu, r))(-dt^2+dx^2+dy^2+dz^2)+(1+\lambda h_{55}(x_\mu, r))dr^2\Big\} \tag{4.44}$$

In [67], a very simple fluctuation equation is found for this channel in the UV:

$$\zeta'' + \Big(3A' + 2\frac{X'}{X}\Big)\zeta' + k^2\zeta = 0 \tag{4.45}$$

$$X \equiv \frac{\phi'}{A'} \tag{4.46}$$

$$\zeta \equiv \psi - \frac{\delta\phi}{X} \tag{4.47}$$

Here ζ is the gauge-invariant variable, which can be replaced by either $\delta\phi$ or ψ via a gauge transformation. Note that this source corresponds to the scalar glueball operator and the trace of the energy momentum tensor simultaneously, this follows from the fact that QCD has the following trace anomaly [9]:

$$\langle T^\mu_\mu\rangle = \frac{\beta(\lambda)}{2\lambda^2}\mathrm{tr}F^2 \tag{4.48}$$

In [6], the following UV asymptotics are found for ϕ' and A:

$$\phi' \approx -\frac{1}{r\log(r)} \tag{4.49}$$

$$A \approx -\log(r) \tag{4.50}$$

From this it follows that:

$$X \approx \frac{1}{\log(r)} \tag{4.51}$$

For $k^2 \to 0$, we can solve equation 4.44 to get:

$$\zeta = C + f(r) \tag{4.52}$$

$$\zeta' = Ke^{-\int dr'\Big\{3A'(r')+2\log(X(r'))'\Big\}} \tag{4.53}$$

$$= Ke^{-3A}X^{-2} = Kr^3\log^2(r) \tag{4.54}$$

Here C and K are constants. We thus conclude that at leading and sub-leading order, we have:

$$\zeta \approx A(k) + B(k)r^4 \log^2(r) \tag{4.55}$$

Specifically, for our anisotropic case where we look only at time dependence:

$$\zeta_{ii} \approx n - i\frac{\omega}{4\pi T} b_i^n r^4 \log^2(r) + \mathcal{O}(r^5) \tag{4.56}$$

$$G^R_{T_{jj}T_{ii}}(0,\omega) \propto -i\frac{\omega}{4\pi T} b_i^n \tag{4.57}$$

$$\zeta_i^n \propto \frac{1}{4\pi T} b_i^n \tag{4.58}$$

Here $n \in \{0,1\}$ depending on our boundary condition at the boundary, which follows from whether this mode is a source or not ($n = 1$ ($n = 0$) for $i = j$ ($i \neq j$)), so:

$$\zeta_{\parallel} = \zeta_3^1 \tag{4.59}$$

$$\zeta_{\perp} = \zeta_1^1 \tag{4.60}$$

$$\zeta_m = \zeta_1^0 = \zeta_3^0 \tag{4.61}$$

The particular choice for b^n is for convenience as will become clear in Appendix H. Note that the UV expansion for h_{ij} is actually more complicated than equation 4.58 suggests, because ζ is not space-time homogeneous, which gives rise to terms in the UV expansion of $\mathcal{O}(r^2)$ [49], this follows from the k^2 term in equation 4.27. When solving a decoupled fluctuation equation, these terms are real when ω is real and since the bulk viscosities are found by looking at the imaginary part of the UV expansion, it is still possible to extract them. For a coupled set of fluctuation equations, this is not necessarily the case, but since the contribution is proportional to ω we can still extract the bulk viscosities from the UV expansion in the $\omega \to 0$ limit as we will show in Appendix H. In equation 4.58 we only have a proportionality for the bulk viscosity, not an equality. We can find the precise equality through an extremely tedious computation analogous to [8] and [86], but we don't need it. Instead, we can simply use 'isotropic normalization', meaning that we only look at the ratio of the anisotropic bulk viscosities with respect to the isotropic one, which is very well establised [43]. This works as follows:

$$\frac{\zeta_{\parallel}}{\zeta_I} = \frac{b_3^1}{\mathbb{b}_3^1},\ \frac{\zeta_{\perp}}{\zeta_I} = \frac{b_1^1}{\mathbb{b}_1^1},\ \frac{\zeta_{\perp}}{\zeta_I} = \frac{b_1^0}{\mathbb{b}_1^1} = \frac{b_3^0}{\mathbb{b}_3^1} \tag{4.62}$$

Here ζ_I and $\mathbb{b}_i^n$ are the bulk viscosity and the sub-leading term with magnetic field turned off respectively. The particular choice for ζ_m is because in the isotropic case there is no notion of a mix bulk viscosity to normalize with.

Chapter 5

Calculating viscosities: mimicking fluid equations

5.1 Introduction

In this chapter we will discuss two methods to calculate viscosities. Here we don't look at thermal fluctuations flowing from the bouundary to the horizon like we did in chapter 4. Instead, the underlying theme is to rewrite the Einstein equations at the horizon in a form so that the equation can be dually equated to a fluid equation which describes our relativistic fluid. First, we will do this by using a Kaluza Klein reduction to arrive at Fick's law, from which we can extract the diffusion equation. Second, we will do this by calculating the null Raychaudhuri equation and equating it to the entropy current equation, from which we can calculate all viscosities.

5.2 Fick's law

One way to obtain the anisotropic shear viscosities is to mimick Fick's law. In [84], Rebhan and Steineder (RH) do this for a QGP with anisotropy induced by an axion with a method derived in [69]. Steineder provides more details in his dissertation [90]. Anisotropy due to an axion means that flavor sector studied in this thesis is replaced by an kinetic axion term like in section 3.6. The computation they perform is worked out in Appendix E. This eventually leads to the following formula for the shear viscosities:

$$\frac{\eta_\perp}{s} = T\frac{s}{4\pi M^2 N_c^3}\int_0^{r_h} dr'\,\frac{g_{00}(r')g_{55}(r')}{g_{11}(r')\sqrt{-g(r')}} \tag{5.1}$$

$$\frac{\eta_\parallel}{s} = T\frac{s}{4\pi M^2 N_c^3}\frac{g_{11}}{g_{33}}\Big|_{r=r_h}\int_0^{r_h} dr'\,\frac{g_{00}(r')g_{55}(r')}{g_{11}(r')\sqrt{-g(r')}} \tag{5.2}$$

This method works because when you perform the Kaluza Klein reduction in the x-direction, there is no coupling of the axion term to the Maxwell equations, unlike for a Kaluza Klein reduction in the z-direction which would yield a mass term. When you have anisotropy due to a magnetic field, we cannot use this method because the flavor action will couple to the Maxwell equations, yielding a mass term. We should therefore instead perform a Kaluza Klein reduction in the z-direction, because in this direction the magnetic field does not give a mass term. However, in this case it is only possible to acces the parallel shear viscosity, yielding:

$$\frac{\eta_{\|}}{\varepsilon + P_{\|}} = \frac{\sqrt{-g}}{g_{11}\sqrt{g_{00}g_{55}}g_{eff}^2}\Big|_{r=r_h} \int_0^{r_h} dr' \frac{g_{00}(r')g_{55}(r')g_{eff}^2(r')}{\sqrt{-g(r')}} \tag{5.3}$$

Our anisotropy is given by:

$$P_{\|} - P_{\perp} = \frac{1}{2}\chi_B B_m^2 \tag{5.4}$$

and the following thermodynamic identities [93]:

$$g = -P_{\perp} \tag{5.5}$$
$$f = -P_{\|} \tag{5.6}$$
$$\varepsilon = g + Ts - \frac{1}{2}\chi_B B_m^2 \tag{5.7}$$
$$= f + Ts \tag{5.8}$$

So we have: $P_{\|} = \varepsilon - Ts$. Concluding:

$$\frac{\eta_{\|}}{s} = T\frac{s}{4\pi M^2 N_c^3}\frac{g_{33}}{g_{11}}\Big|_{r=r_h} \int_0^{r_h} dr' \frac{g_{00}(r')g_{55}(r')}{g_{33}(r')\sqrt{-g(r')}} \tag{5.9}$$

We also have:

$$T = \frac{M^2 N_c^3}{s}\Big(\int_0^{r_h} dr' \frac{g_{00}(r')g_{55}(r')}{g_{33}(r')\sqrt{-g(r')}}\Big)^{-1} \tag{5.10}$$
$$= \frac{M^2 N_c^3}{s}\Big(\int_{-\infty}^{\phi_h} d\phi' \frac{g_{00}(\phi')g_{55}(\phi')}{g_{33}(\phi')\sqrt{-g(\phi')}}\Big)^{-1} \tag{5.11}$$
$$= \frac{e^{-3A-W}}{4\pi}\Big|_{\phi=\phi_h}\Big(\int_{-\infty}^{\phi_h} d\phi' \frac{e^{2A+2B}}{e^{2A+2W}\sqrt{e^{8A+2W+2B}}}\Big)^{-1} \tag{5.12}$$
$$= \frac{e^{-3A-W}}{4\pi}\Big|_{\phi=\phi_h}\Big(\int_{-\infty}^{\phi_h} d\phi' e^{B-3W-4A}\Big)^{-1} \tag{5.13}$$

We have confirmed this result numerically. For $\eta_{\perp}$ there's seems to not be a simple way to calculate it with a Kaluza Klein reduction, we would have to perform the Kaluza Klein-reduction from scratch and consider the presence of mass terms to see what the new Fick's law would look like. It is likely that similar to the shear computations in chapter 1.6 we can rescale the gauge fields to remove the mass term, but at the moment of making this thesis the computational power to perform such an analytical computation was not available. It is also unclear whether we can use a Kaluza Klein reduction to find the bulk viscosity.

5.3 Null Raychaudhuri equation

In [33], an equation is found for the null Raychaudhuri equation expanded hydrodynamically at second order, which is dual to the entropy current equation on the fluid side. By reading off the coefficients Eling and Oz (EO) find the famous $\frac{\eta}{s} = \frac{1}{4\pi}$ for the isotropic shear viscosity. More importantly, they find a very simple formula for the isotropic bulk viscosity, the EO-formula:

$$\frac{\zeta}{s} = \sum_i \left(s\frac{d\phi_i^h}{ds} + \rho^\alpha \frac{d\phi_i^h}{d\rho^\alpha} \right)^2 \tag{5.14}$$

This is a result for multiple scalar fields ϕ_i and a gauge field A_μ^α. This equation is an applied form of the null Raychaudhuri equation, which is an equation which describes the trace of the deviation of the null geodesic u_μ. In their article it is suggested that this equation can accurately describe the bulk viscosity in the high temperature limit and for an adiabatic approximation. However Buchel, Gursoy and Kritsitis (BGK) show that the result holds much more generally, because with this formula you don't need to look at the variation of ϕ at ϕ_h to get the bulk viscosity, but instead look at the variation of the constant ϕ_h which is a free parameter of your black hole solution. BGK find that the EO formula perfectly matches the GPR result, which unlike the EO-formula involves a numerically calculated constant c_{b11}. Ignoring charge and working with a single scalar field, they found the following match:

$$\frac{\zeta_{EO}}{s} = \frac{1}{36\pi(\frac{A(\phi)}{d\phi_h})^2}\bigg|_{\phi=\phi_h} = \frac{\zeta_{GPR}}{s} = \frac{1}{36\pi(\frac{A(\phi)}{d\phi})^2}|c_{b11}|^2\bigg|_{\phi=\phi_h} \tag{5.15}$$

The EO formula is also confirmed for using fluid gravity [94] for the Sakai-Sugimoto model, which is a holographic QCD model that includes flavours [85]. For this model the ratio between shear and bulk viscosity that is found with both methods is:

$$\frac{\zeta_{SS}}{\eta_{SS}} = \frac{4}{15} \tag{5.16}$$

In this chapter we will repeat Eling and Oz' computation, but now with an Ansatz which considers symmetry breaking caused by an external magnetic field.

5.4 Isotropic boosting

We have a bulk metric g_{AB}, and the metric associated with the dual hydrodynamic theory, which we call $\gamma_{\mu\nu}$. The indices are defined in the following way: $x^A = (r, x^\mu)$, i.e. capital letters are bulk indices and Greek letters are hypersurface or horizon indices. In regular coordinates, this metric is as follows:

$$ds^2 = e^{2A}(-fdt^2 + \frac{1}{f}dr^2 + \mathbf{dx}^2) \tag{5.17}$$

$$\mathbf{dx}^2 = dx^2 + dy^2 + e^{2W}dz^2 \tag{5.18}$$

For mathematical convenience we chose a different gauge for the fifth dimension coordinate:

$$ds^2 = -e^{2A}fdt^2 + e^{-2A}\frac{1}{f}du^2 + e^{2A}\mathbf{dx}^2 \tag{5.19}$$

Define u^* such that $du^* = \frac{e^{-2A}}{f}du$ We get:

$$ds^2 = -e^{2A}fdt^2 + dudu^* + e^{2A}\mathbf{dx}^2$$

To get to Eddington Finkelstein coordinates, we make the following changes of variables:

$$du^* = dv - dt$$
$$dt = dv - du^*$$

We get:

$$\begin{aligned} ds^2 &= 2dvdu + dudu^* + e^{2A}(-fdv^2 + \mathbf{dx}^2 - fdu^{2*}) \\ &= 2dvdu + e^{2A}(-fdv^2 + \mathbf{dx}^2) \end{aligned}$$

We then proceed to boosting our metric. In the isotropic case, the solution to the gravity action has four degrees of freedom. Firstly, we can choose the horizon radius, which is dual to choosing the temperature of the QFT. Secondly, we can go to a boosted frame for the space-time dimensions, which is, not unexpectedly, dual to considering a boosted frame for the fluid. For the isotropic case, we have the following identities for boosting [94]:

$$dv \to -u_\mu dx^\mu \tag{5.20}$$
$$x_i x^i \to \Delta_{\mu\nu} x^\mu x^\nu \tag{5.21}$$

Boosting like this is not valid for the anisotropic case, but we will will first work this out before we comment on how things change for the anisotropic case. We end up with:

$$ds^2_{EF} = -2dx^\mu u_\mu du + e^{2A}(-f(dx_\mu u^\mu)^2 + \Delta_{\mu\nu} dx^\mu dx^\nu)$$

Here $\Delta_{\mu\nu} = \gamma_{\mu\nu} + u_\mu u_\nu$ is the projector, which projects to the subspace orthogonal to the fluid four-velocity u_μ. We have:

$$u_\mu = \gamma \begin{pmatrix} -1 \\ \beta_i \end{pmatrix} \tag{5.22}$$
$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{5.23}$$

With β_i the fluid velocity and γ is the Lorentz factor. This is what we raise and lower our indices with in a boosted frame. Because we have a boosted black brane, which means we have to use projectors instead of metric tensors to raise and lower our hypersurface indices. Since we work with a Minkowski signature, we have the following constraint for the four-velocity magnitude:

$$\gamma^{\mu\nu} u_\mu u_\nu = -1 \tag{5.24}$$

Using this, we find that the projector satisfies the following identities:

$$\Delta^{\mu\nu} u_\nu = 0 \quad \Delta^{\mu\rho}\Delta_{\rho\nu} = \Delta^{\mu\rho}\gamma_{\rho\nu} = \Delta^\mu_\nu \quad \Delta_{\mu\nu}\Delta^{\mu\nu} = 3 \tag{5.25}$$

5.5 Four-dimensional metric

In the isotropic case, which is described in [33], $\gamma_{\mu\nu}$ is the Minkowski metric, but we will consider an anisotropic metric, which, as will become clear in section 5.8 gives some complications. When we have anisotropy induced by a magnetic field or an axion, one would assume that we have:

$$ds^2_z = -dt^2 + dx^2 + dy^2 + e^{2W(\phi_h)} dz^2 \tag{5.26}$$

However, using the freedom to choose our five-dimensional coordinate, we can also have:

$$ds^2_{xy} = -dt^2 + (dx^2 + dy^2)e^{-2W(\phi_h)} + dz^2 \tag{5.27}$$

To prove this, we start from:

$$ds^2 = e^{2A}(-fdt^2 + \mathbf{dx}^2) + \frac{e^{-2A}}{f}du^2 \tag{5.28}$$

$$\mathbf{dx}^2 = dx^2 + dy^2 + e^{2W}dz^2 \tag{5.29}$$

We perform the following transformations:

$$e^{2A} \to e^{2A^*-2W} \tag{5.30}$$

$$f \to f^* e^{2W} \tag{5.31}$$

$$du \to e^{-2W}du^* \tag{5.32}$$

And then redefine $A^* \equiv A$, $f^* \equiv f$ and $u^* \equiv u$.

5.6 Null geodesics

We have the fluid velocity of u_A (capital letters are five-dimensional indices, Greek letters are four-dimensional indices), which is null in five dimensions. We define $l_A = \delta^5_A = dr$, which is also null. We change variables to the Eddington-Finkelstein metric and we boost:

$$ds^2 = -2dx^\mu u_\mu du - e^{2A}f(du_\mu x^\mu)^2 + e^{2A}\Delta_{\mu\nu}dx^\mu dx^\nu \tag{5.33}$$

Note again that this boosting is not actually necessarily allowed for an anisotropic bulk metric, we will ignore this for now and comment on this later. We have the following identities:

$$g^{AB}l_A u_B = g^{u\mu}u_\mu = u^\mu u_\mu = -1 \tag{5.34}$$

$$g^{AB}l_A l_B = 0 \tag{5.35}$$

$$g^{AB}u_A u_B = 0 \tag{5.36}$$

This first identity is necessary for the projector we want to define later on. It is important to note that $u^\mu u_\nu \neq u_A u^A$. Furthermore we have:

$$\begin{aligned} l^B\nabla_A u_B &= u^B\nabla_A u_B = l^B\nabla_A l_B \\ l^B\nabla_B u_A &= -l^B\nabla_A u_B + \mathcal{O}(\partial) = 0 + \mathcal{O}(\partial) \end{aligned} \tag{5.37}$$

Here we used Killing's equation, which holds for u_μ at first order and equation 5.34. We further have:

$$\begin{aligned} u^B\nabla_B l_A\Big|_{u=u_h} &= -u^B\Gamma^C_{BA}l_C\Big|_{u=u_h} = -\frac{1}{2}u^B g^{CD}(-\partial_D g_{AB} + \partial_A g_{BD} + \partial_B g_{AD})l_C\Big|_{u=u_h} \\ &= \frac{1}{2}u^B u^C(\partial_A g_{BC})\Big|_{u=u_h} = -\frac{1}{2}e^{2A(u_h)}f'(u_h)l_A = 2\pi T l_A \end{aligned} \tag{5.38}$$

We define the projector which projects to the subspace $T_\perp$ orthogonal to u_A and l_A:

$$Q_{AB} = e^{-2A(u_h)}g_{AB} + u_A l_B + u_B l_A \tag{5.39}$$

Since this projector is orthogonal to the r-direction we can also talk about Q^{μ}_{α} Looking at equation 5.33, we find that at the horizon this precisely becomes:

$$ds^2_{Q_h} = \Delta_{\mu\nu} dx^{\mu} dx^{\nu} \tag{5.40}$$

i.e. $Q_{\mu\nu}(u_h) = \Delta_{\mu\nu}$. Unlike in [21], we have:

$$u^A \nabla_A u_B \Big|_{u=u_h} \neq 0 \tag{5.41}$$

As would be required due to the geodesic equation. This is because the u_B is not an affine parameter at the horizon. We instead have:

$$u^A \nabla_A u_B \Big|_{u=u_h} = \kappa u_B \tag{5.42}$$

Here κ is the the surface gravity, which is the acceleration that is experienced at the horizon from a point of view at the boundary (from a point of view at the horizon this acceleration would be infinite). It measures the extent to which u_μ is not affinely parametrized at the horizon [10]. We can find $\kappa_{(0)}$ by using that u_μ is a Killing vector at zeroth order, and Killing's equation therefore holds [21]:

$$\nabla_{(A} u_{B)} = \mathcal{O}(\partial) \tag{5.43}$$

$$u^A \nabla_B u_A \Big|_{u=u_h} = -\kappa_{(0)} u_B \tag{5.44}$$

$$-u^A \Gamma^C_{BA} u_C \Big|_{u=u_h} = \tag{5.45}$$

$$-u^A g^{CD} (-\partial_D g_{AB} + \partial_A g_{BD} + \partial_B g_{AD}) u_C \Big|_{u=u_h} = \tag{5.46}$$

$$-u^A g^{rC} (-\partial_r g_{AB}) u_C \Big|_{u=u_h} = \tag{5.47}$$

We conclude:

$$\kappa_{(0)} = -(-1)^2 \frac{1}{2} e^{2A(u_h)} f'(u_h) = 2\pi T \tag{5.48}$$

We find, for a vector V_μ in the subspace $T_\perp$ orthogonal to u_μ and l_μ:

$$\begin{aligned} \mathcal{L}_u V^A &= u^B \nabla_B V^A \\ & u^B \nabla_B (Q^A_\mu V^\mu) \end{aligned} \tag{5.49}$$

5.7 Deviation equation

Having defined the projectors, we will now define the second rank tensor B_{AB}, which is the failure of the deviation of a geodesic to be parallel transported [21] (see figure 5.1). If the geodesic is u_A and the deviation is V_A, this means:

$$\mathcal{L}_u V^A = u^B \nabla_B V^A = B^A_B V^B \tag{5.50}$$

$$B^A_B = \nabla_B u^A \tag{5.51}$$

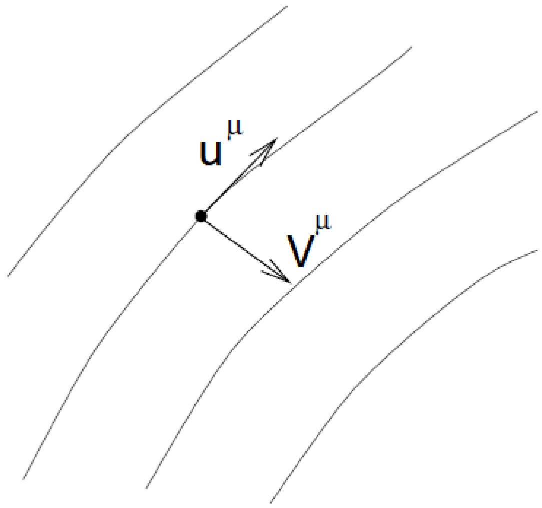

Figure 5.1: A geodesic u_μ (fluid velocity) and the deviation V_μ from this geodesic [21]

Here we used:

$$[u, V] = \nabla^B u_B V^A - u_B \nabla^B V^A = 0 \tag{5.52}$$

This follows from the fact that V_A has a constant orientation with respect to u_A (which is orthogonal, see figure 5.1). Use equation 5.38, and 5.42:

$$\begin{aligned}
\mathcal{L}_u V^A &= u^B \nabla_B V^A \\
&= u^B \nabla_B (Q^\mu_A V^A) \\
&= u^B Q^\mu_A \nabla_B V^A + u^B V^A \nabla_B (u^\mu l_A) + u^B V^A \nabla_B (l^\mu u_A) \\
&= Q^\nu_B u^B Q^\mu_A \nabla_B V^A \\
&= \hat{B}^\mu_\nu V^\nu
\end{aligned} \tag{5.53}$$

We thus know that $\hat{B}_{\mu\nu} = Q^A_\mu Q^B_\nu B_{AB}$. We can make the following decomposition for $\hat{B}_{\mu\nu}$:

$$\hat{B}_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu} + \frac{1}{3}\theta \Delta_{\mu\nu} + \omega_{\mu\nu} \tag{5.54}$$

Here $\omega_{\mu\nu}$ describes the rotation, which is zero in this system. $\sigma_{\mu\nu}$ and θ describe the traceless and trace part respectively:

$$\sigma_{\mu\nu} = \Delta^\rho_\mu \Delta^\sigma_\nu \nabla_{(\rho} u_{\sigma)} - \frac{2}{3}\Delta^{\rho\sigma}\nabla_\rho u_\sigma \tag{5.55}$$

$$\theta = \Delta^{\rho\sigma}\nabla_\rho u_\sigma \tag{5.56}$$

. Then we write down an equation for the evolution of $\hat{B}_{\mu\nu}$:

$$\begin{aligned}
\mathcal{L}_u \hat{B}_{\mu\nu} &= Q^B_\mu Q^C_\nu (u^A \nabla_A B_{BC}) = Q^B_\mu Q^C_\nu (u^A \nabla_A \nabla_B u_C) \\
&= Q^B_\mu Q^C_\nu (u^A \nabla_B \nabla_A u_C - R_{ABCD} u^D u^A) \\
&= Q^B_\mu Q^C_\nu (\nabla_B (u^A \nabla_A u_C) - (\nabla_B u^A)(\nabla_A u_C) - R_{ABCD} u^D u^A)
\end{aligned} \tag{5.57}$$

We use equation 5.42:

$$\begin{aligned} &= Q^B_\mu Q^C_\nu (\kappa B_{BC} - B^A_B B_{CA} - R_{ABCD} u^D u^A) \\ &= \kappa \hat{B}_{\mu\nu} - \hat{B}^\rho_\mu \hat{B}_{\nu\rho} - Q^B_\mu Q^C_\nu R_{ABCD} u^D u^A \end{aligned} \tag{5.58}$$

Taking the trace of this equation, the null Raychaudhuri equation becomes:

$$\frac{d\theta}{d\tau} = \mathcal{L}_u \theta = \kappa\theta - \hat{B}^{\rho\sigma}\hat{B}_{\rho\sigma} - R_{DA} u^D u^A \tag{5.59}$$

In $\hat{B}$ you can now substitute equation 5.54. We are allowed to ignore the projectors on the Riemann tensor because we take the trace with $Q_{\mu\nu}$. We can substitute the following identity for the Ricci tensor using the Einstein equation:

$$R_{AB} = \frac{1}{2} g_{AB}(R - \frac{1}{2}(\partial\phi)^2 + V - V_f\sqrt{D}) - V_f \frac{1}{\sqrt{D}}\frac{dD}{dg^{AB}} + \frac{1}{2}\partial_A\phi\partial_B\phi \tag{5.60}$$

$$D = 1 + w^2 \frac{1}{2} F^{AB} F_{AB} \tag{5.61}$$

Because $g_{AB}u^A u^B = 0$ the terms proportional to g_{AB} don't enter the null focusing equation. Furthermore, we have:

$$u^A u^B \frac{dD}{dg^{AB}} \propto E^2 \tag{5.62}$$

This part relates to conductivities, and therefore will not be considered. The Raychaudhuri equation is as follows as first order:

$$\theta_{(1)} = \mathcal{O}(\partial^2) \tag{5.63}$$

We use [21]:

$$\theta = \nabla_\mu u^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}u^\mu) = \frac{1}{a}\partial_\mu(a u^\mu) = \frac{1}{s}\partial_\mu(u^\mu s) \tag{5.64}$$

Where a is the area density. We have the following identity at second order [33]:

$$-\mathcal{L}_u\theta_{(1)} - \frac{1}{3}\theta^2_{(1)} + \kappa_{(0)}\theta_{(2)} + \frac{1}{4}\Delta^{\mu\alpha}\Delta^{\nu\beta}\sigma_{\mu\nu}\sigma_{\alpha\beta} = \frac{2\pi T}{s}\partial_\mu(u^\mu s) + \mathcal{O}(\partial^3) \tag{5.65}$$

The null focusing equation is now given by:

$$\partial_\mu(u^\mu s) = \frac{s}{8\pi T}\Delta^{\mu\alpha}\Delta^{\nu\beta}\sigma_{\mu\nu}\sigma_{\alpha\beta} + \frac{s}{4\pi T}(\mathcal{L}_u\phi_h)^2 \tag{5.66}$$

We then use the ideal entropy conservation law:

$$\partial_\mu(s u^\mu) = 0 \rightarrow u^\mu\partial_\mu s = -s\partial_\mu u^\mu \tag{5.67}$$

For magnetic field we use the zeroth order definition of the flux of the magnetic field lines found by Iqbal, Grozdanov and Hofman in [40] and in equation 2.8:

$$\mathbb{J}^{(0)}_{\mu\nu} = 2\rho u_{[\mu} b_{\nu]} \tag{5.68}$$

Here ρ is the conserved flux density. The Bianchi identity, i.e. equation 2.8, tells us that:

$$\partial^\mu J^{(0)}_{\mu\nu} = (b_\nu \mathcal{L}_u - u_\nu \mathcal{L}_b)\rho + \rho u_{[\mu}\partial^\mu b_{\nu]} - \rho b_{[\mu}\partial^\mu u_{\nu]} = 0 \tag{5.69}$$

We use $\rho b_\mu = B_\mu$:

$$\begin{aligned} u^\mu \partial_\mu B_\nu &= -B_\nu \partial_\mu u^\mu + B^\mu \partial_\mu u_\nu + u_\nu \partial_\mu B^\mu \\ &= -B_\nu \partial_\mu u^\mu + B^\mu \partial_\mu u_\nu \end{aligned} \tag{5.70}$$

This is the 'ideal conservation law for magnetic field'. The last step is because rightmost term vanishes due to Gauss's law for magnetism. Unlike the example that EO considers where ϕ_h depends on charge and entropy, it now depends on magnetic field and entropy. We can therefore rewrite the Lie derivative on ϕ_h in the following way:

$$\begin{aligned} \mathcal{L}_u \phi = u^\mu((\partial_\mu s)\frac{\partial \phi_h}{\partial s} + (\partial_\mu B_\nu)\frac{\partial \phi_h}{\partial B_\nu}) &= -(s\frac{\partial \phi_h}{\partial s} + B_\nu \frac{\partial \phi_h}{\partial B_\nu})\Delta^{\alpha\beta}\partial_\alpha u_\beta + B_\beta \frac{\partial \phi_h}{\partial B_\nu}\Delta^{\alpha\beta}\partial_\alpha u_\nu \\ = -(s\frac{\partial \phi_h}{\partial s} + B_\nu \frac{\partial \phi_h}{\partial B_\nu})\Delta^{\alpha\beta}\partial_\alpha u_\beta + B_\nu \frac{\partial \phi_h}{\partial B_\nu} S_\parallel \end{aligned} \tag{5.71}$$

$$= -(s\frac{\partial \phi_h}{\partial s}\Delta^{\alpha\beta}\partial_\alpha u_\beta + B_\nu \frac{\partial \phi_h}{\partial B_\nu} S_\perp) \tag{5.72}$$

We we have used that:

$$S_\parallel = \frac{B^\mu B^\nu}{B^2}\partial_\mu u_\nu \tag{5.73}$$

$$S_\perp = (\Delta^{\mu\nu} - \frac{B^\mu B^\nu}{B^2})\partial_\mu u_\nu \tag{5.74}$$

For a system with only u_μ as a vector available at zeroth order, it follows that when ϕ_h depends on the parameters p_i, we always have the following ideal conservation law:

$$\partial^\mu J^{(0)}_\mu \propto \partial^\mu(u_\mu p_i) = 0 \tag{5.75}$$

We thus have, for ϕ_h depending on any set of scalars, similar to the original result of EO:

$$\partial_\mu(u^\mu s) = \frac{s}{8\pi T}\Delta^{\mu\alpha}\Delta^{\nu\beta}\sigma_{\mu\nu}\sigma_{\alpha\beta} + \frac{s}{4\pi T}(\sum_i p_i \frac{\partial \phi_h}{\partial p_i})^2 \Big(\Delta^{\alpha\beta}\partial_\alpha u_\beta\Big)^2 \tag{5.76}$$

For the parameter χ it is unclear what the ideal conservation law is and thus we cannot find the bulk viscosities. In Appendix J we comment on how the anisotropic shear term in the entropy current would give new 'bulk viscosity terms'. An analogous derivation holds for magnetic field induced anisotropy.

5.8 Anisotropic boosting

As mentioned many times now in this thesis, anisotropy induced by an axion or a magnetic field breaks the zero temperature space-time boost symmetry for the theory of gravity in the x- and y-direction and the z-direction respectively. This means that not all boosts that previously were valid solutions to the Einstein equations still are. Therefore the projector we worked with $\Delta_{\mu\nu}$ can only take specific forms in order to still represent a valid result. For an axion, we can still boost with $\beta_i = \{\beta_x, \beta_x, 0\}$. For magnetic field, we can still boost with $\beta_i = \{0, 0, \beta_z\}$. From this we can immediately conclude for the shear viscosity that:

$$\eta_{\perp}^{\chi} = \frac{1}{4\pi} \tag{5.77}$$

$$\eta_{\parallel}^{\chi} = \frac{e^{-2W(u_h)}}{4\pi} \tag{5.78}$$

$$\eta_{\parallel}^{B} = \frac{e^{2W(u_h)}}{4\pi} \tag{5.79}$$

This is consistent with [59]. The reason that we can find $\eta_{\perp}$ for the axion case because it doesn't require boosting with β_z, but for finding $\eta_{\parallel}$ we do need to boost with β_x or β_y. Note that this situation is analogous to the one we had for Kaluza Klein reduction discussed in section 5.2 For the helicity zero sector, we can only conclude:

$$\zeta_{\parallel}^{B} = \frac{1}{4\pi} s \frac{\partial \phi_h}{\partial s} \tag{5.80}$$

This result is consistent with [28], where it was shown that a magnetic field induced breaking of conformal symmetry does not imply a non-zero bulk viscosity. To get a non-zero bulk viscosity you need a non-trivial dilaton profile. Clearly, we have not been able to extract all viscosities with this method.

Chapter 6

Bulk viscosity for 2+1 dimensional analytic model

6.1 Introduction

In [38], Gnecchi, Gursoy, Papadoulaki and Toldo (GGPT) find an analytic solution for a gravity dual based on 4D $\mathcal{N} = 2$ Fayet-Iloupolos gauged supergravity. This gravitational theory is dually related to the ABJM model [1]. This is a strongly coupled field theory, which is dual to $AdS_4 \times S^7$. 4D $\mathcal{N} = 2$ Fayet-Iloupolos gauged supergravity is used to understand quantum phase transitions at strong coupling in condensed matter systems and is therefore not related to a QGP, which is a high energy physics phenomenon. However, the same methods used to calculate the viscosities of a QGP can also be used to calculate the bulk viscosity of this system. In this thesis the conventions of GGPT are changed to ours, how this is done is shown in Appendix L.

6.2 The analytical solution

We work with the following action:

$$S = \frac{1}{\kappa^2} \int \sqrt{-g} d^4x \Big(R - \frac{(\partial\phi)^2}{2} - V(\phi) - \frac{V_b(\phi) F_{\mu\nu} F^{\mu\nu}}{4} \Big) \tag{6.1}$$

This is a version of the Einstein Maxwell dilaton action, as described in [30]. In order to use this solution, we cannot work in the $\phi = r$-gauge, because this would require us to analytically solve u as a function of ϕ which is not possible for this analytic solution. To be consistent with GGPT, we therefore work instead with u which satisfies: $e^{2A} dr = e^{-2A} du$. We get the following analytic results for $g_{\mu\nu}$, V_b and V:

$$ds^2 = -\frac{g}{\sqrt{H_0 H_1^3}}dt^2 + \sqrt{H_0 H_1^3}(\frac{du^2}{g} + u^2(dx^2 + dy^2)) \tag{6.2}$$
$$H_0 = 1 - \frac{3b}{u} \tag{6.3}$$
$$H_1 = 1 + \frac{b}{u} \tag{6.4}$$
$$g = -\frac{B_m^2}{2bu} + 3\frac{B_m^2}{2u^2} + u^2(1 - \frac{3b}{u})(1 + \frac{b}{u})^3 \tag{6.5}$$
$$\phi = \sqrt{\frac{3}{4}} \log\left(\frac{b+u}{u-3b}\right) \tag{6.6}$$
$$V = 6\frac{b-u}{b+u}\sqrt{\frac{4b}{u-3b} + 1} \tag{6.7}$$
$$V_b = 6\sqrt{\frac{u-3b}{u+b}} \tag{6.8}$$

6.3 EOM

Similar to Chapter 3, we have to start with the background equations. From the choice of the u-coordinate it follows that we have the following Ansatz:

$$ds^2 = e^{2A}(-fdt^2 + dx^2 + dy^2) + \frac{e^{-2A}du^2}{f} \tag{6.9}$$

For the GPR method, the Green's function calculation is slightly different with respect to [43], we now have the following matrix:

$$M^{\phi\phi} = e^{3A-B} f \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.10}$$

$$M_5^{\phi} = -e^{3A-B} \begin{pmatrix} A'f \\ \frac{1}{2}f' + 2A'f \\ 0 \end{pmatrix} \tag{6.11}$$

We use:

$$h'_{00} = \frac{2fh'_{11}\left(2A'f' + f\left(4A'^2 - 1\right)\right)}{8f^2A'^2} + \text{non-contributing terms} \tag{6.12}$$
$$h_{55} = \frac{h'_{11}}{A'} + \text{non-contributing terms} \tag{6.13}$$

Our Green's function is given by:

$$-\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\eta^{\mu\nu}\eta^{\rho\sigma}G_{\mu\nu\rho\sigma} = \lim_{\omega\to 0}\frac{1}{\omega}\int dtd^3xe^{i\omega t}\theta(t)\langle[\eta^{ij}\frac{1}{2}T_{ij}, \eta^{kl}\frac{1}{2}T_{kl}]\rangle = \zeta \tag{6.14}$$

This part is still in the $\phi = r$ gauge. We get: $\mathcal{F} = -\frac{ie^{3A-B}f}{8A'^2}(h^*_{11}h'_{11} - h^{*\prime}_{11}h_{11})$, and $s = \frac{4\pi}{\kappa^2}e^{3A-B}$. We have the following EOM at the horizon:

$$\{e^{4A}fh''_{11} + e^{4A}f'h'_{11} + \frac{\omega^2 h_{11}}{f}\}_{u\to u_h} = 0 \tag{6.15}$$

Solving for $h_{11} = C(u-u_h)^\rho$, we again find the same boundary expansion:

$$\rho = \frac{i\omega}{f'(u_h)} e^{-2A(u_h)} = \frac{i\omega}{4\pi T} \tag{6.16}$$

$$h_{11} \approx c_{11}\Big(1 + \frac{i\omega}{4\pi T}\log(u-u_h)\Big) \tag{6.17}$$

This leads to:

$$\begin{aligned}\frac{\zeta}{s} &= -\frac{1}{s}\lim_{\omega\to 0}\frac{1}{\omega}\text{Im}G(\omega) = \frac{1}{s}\lim_{\omega\to 0}\frac{1}{\kappa^2}\mathcal{F} = \frac{1}{s}\lim_{\omega\to 0}\frac{1}{4\kappa^2(\frac{A(\phi)}{d\phi})^2}|c_{b11}|^2\Big|_{\phi=\phi_h} \\ &= \frac{1}{16\pi(\frac{A(\phi)}{d\phi})^2}|c_{b11}|^2\Big|_{\phi=\phi_h}\end{aligned} \tag{6.18}$$

6.4 Result

In Appendix L and M the details of the compuation are worked out. The result is given in figure 6.1. We see that the bulk viscosity increases as a function of magnetic field and decreases as a function of temperature (from Appendix M we know that larger b means smaller T). The temperature dependence is similar to results found by GPR in [43]. Also, we can conclude that for $b \in [1, \infty)$, i.e. for low temperatures, the 'fluid' is not in the hydrodynamic limit because the bulk viscosity is ~ 1. For $b = 0.5$, the influence of magnetic field is very small.

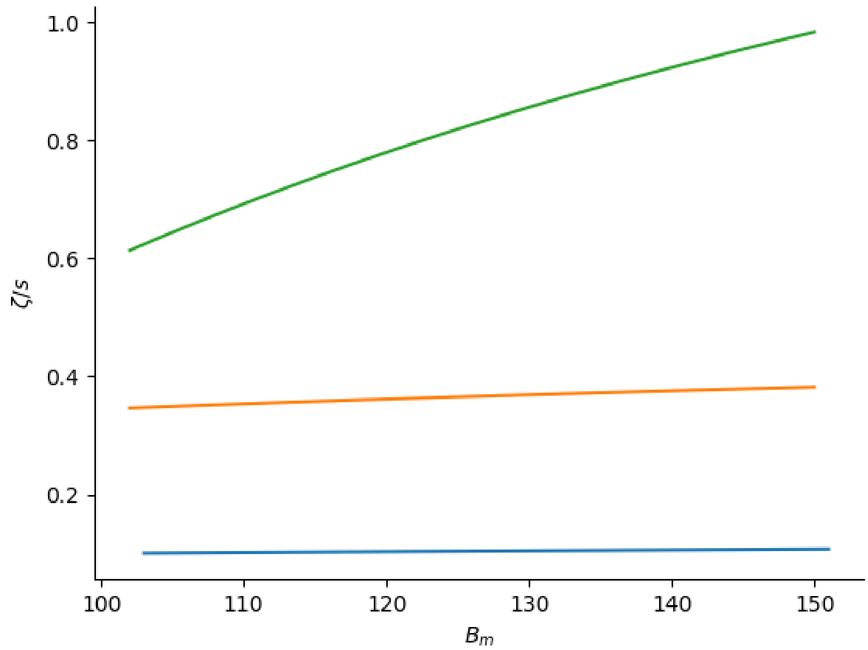

Figure 6.1: $\frac{\zeta}{s}$ found as a function of B_m for $b = 0.5$ (blue), $b = 1$ (yellow) and $b = 2$ (green)

Chapter 7

Comparison of results with literature

7.1 Experimental data

Experimental data on the transport properties can only come from the LHC and RHIC, as these are the only places on earth where a temperature can be reached that is high enough to produce a QGP. In the past decade, the main focus of research has been on the shear viscosity [81] as this transport property is strongly related to other measurements such as the azimuthal momentum distributions of charged hadrons. For the isotropic shear viscosity, holography and experimental physics are consistent and a value of $\eta/s \approx 1/4\pi$ is found [12]. Unfortunately, there is no data available on the magnetic field dependence of the shear viscosity. As for the bulk viscosity, one thing that is known is that for high temperatures (larger than 1 GeV), the bulk viscosity vanishes. This is consistent with theory as QCD becomes conformal in the high temperature limit. For temperatures near the deconfinement temperature, large bulk viscosities are possible. In a statistical analysis of data from both the LHC and the RHIC, Paquet et al. find that the peak bulk viscosity is ≈ 0.2. The LHC and the RHIC give mixed signals on this, but the temperature at which this maximum occurs is at around 0.2 GeV (see figure 7.1). In [12], using a Bayesian method for estimating the isotropic viscosities of

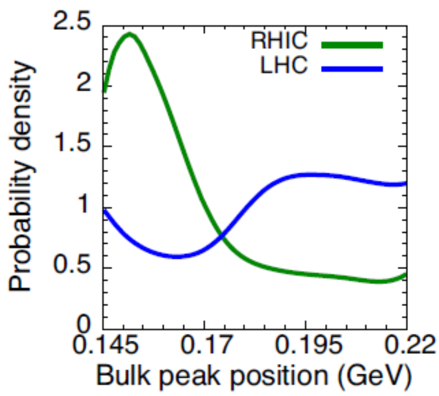

Figure 7.1: Probability density of the location of the maximum of bulk viscosity as a function of temperature for both ion accelerators [81].

a QGP, Bernhard finds the graphs of figure 7.2. This is the best experimental data available on bulk

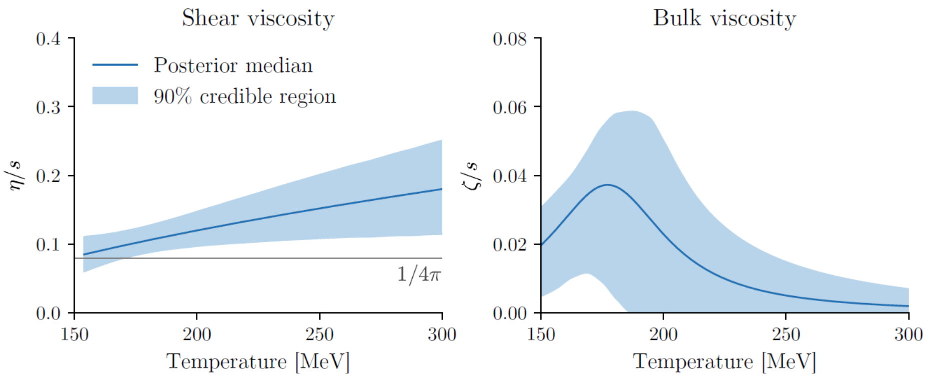

Figure 7.2: Estimations of bulk and shear viscosity of a QGP as a function of temperature as found in [12]

viscosity. Unfortunately, there is no data available on the magnetic field dependence of the bulk viscosity. As a last note, in a not yet published work [32], Nijs uses holographic QCD to describe the merging of neutron stars. The data that comes out of this computations could perhaps be compared to gravitational wave data. This link between holography and experiment could then possibly allows us to also learn more about the transport coefficients of QGP.

7.2 Lowest Landau level approximations

There is a method for calculating the bulk viscosities in a magnetic field that does not use holography, but instead uses a lowest Landau level (LLL) approximation. As shown by Landau [72], a uniform magnetic field induces harmonic oscillator-like quantization for charged particles. This is called Landau quantization. The LLL approximation means assuming that the magnetic field is so large that compared to temperature, i.e. $T^2 \ll eB$, that all charged particles, in our case quarks, are in the LLL because higher Landau levels are surpressed by a factor $e^{-\sqrt{eB}/T}$. The Landau levels are as follows [71]:

$$E_n = \hbar\omega_c(\frac{1}{2} + n) + \frac{\hbar^2 k_z^2}{2m} \tag{7.1}$$

Here $\omega_c = |e|B/m$ is the cyclotron frequency. The LLL approximation is different from Bose-Einstein condensation (and therefore does not violate the Pauli exclusion principle), because the system is highly degenerate due to the fact that the lowest Landau levels and thus also the LLL, are independent of k_x and k_y. Moving on from the LLL approximation, this method also requires that the quark self-energy does not affect the bulk viscosity, which requires $\alpha_s eB/T^2 \ll 1$. Combining these limits, we thus are allowed to use this approximation in the following limit [54]:

$$\alpha_s eB \ll T^2 \ll eB \tag{7.2}$$

As can be seen from equation 7.1, when all quarks are in the LLL, the quarks have a 1+1 dimensional dispersion relation, i.e. only movement parallel to the magnetic field is allowed. This means that the magnetic field induced contribution to the bulk viscosity will be to the parallel bulk viscosity. Using the Kubo formula, Hattori et al. analytically find the following result for the bulk viscosity:

$$\zeta \approx 0.12 \frac{m_f^4}{T} \to \zeta_{\parallel} \approx 0.031 \frac{|eB| m_f^2}{T \ln(T/M_g)} \tag{7.3}$$

In figure 7.3 the result of Hattori et al. is given. In the blue (red) region the LLL approximation is not valid because the left (right) part of equation 7.2 is violated. In [70] Kurian and Chandra (KC) also

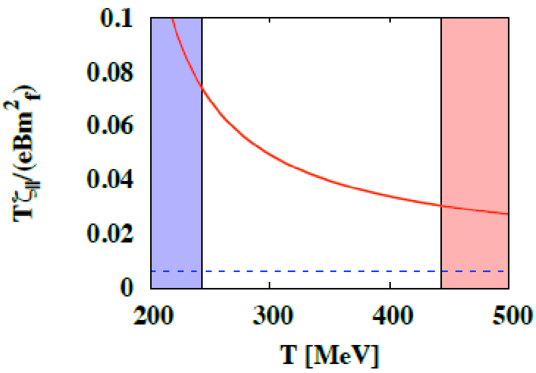

Figure 7.3: Temperature dependence of the longitudinal bulk viscosity as found in [53].

calculate the bulk viscosity using the LLL approximation. However, instead of the Kubo formula, the Boltzmann equation is used. Furthermore, a more general hot QCD medium is considered, with a numerical quark fugacity which serves as an input for the quark distribution function, this is called the effective fugacity quasiparticle model. This quark fugacity that comes from a lattice QCD equation of state (EoS). At high temperatures, the result is similar to Hattori et al. (see figure 7.4), as expected.

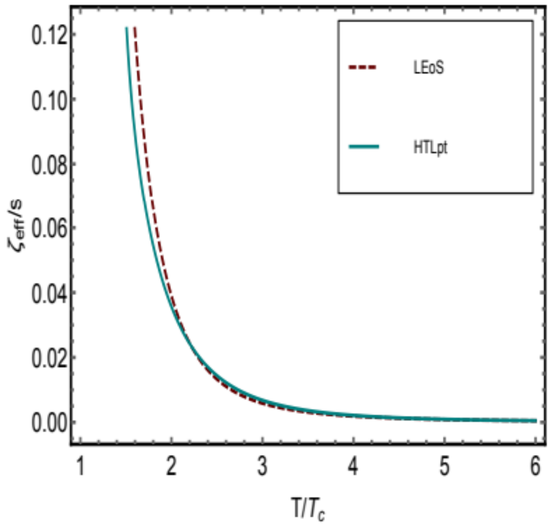

Figure 7.4: Temperature dependence of the longitudinal bulk viscosity for different lattice QCD EoS for $|eB|= 0.3$ GeV as found in [70].

Chapter 8

Conclusion

8.1 Results

In this thesis we used holography to study the viscosities an anisotropic relativistic fluid, in particular a QGP with anisotropy induced by a magnetic field. In chapter 1.1, we briefly walked through topics like QGP, holography, ihQCD and hydrodynamics. In chapter 2 we summarized [55] and explained how there for relativistic magnetohydrodynamics there are three bulk viscosities and two shear viscosities. In chapter 3, we described the gravity setup that is mainly used in this thesis and which is based on [62] and [4]. The results of our thesis are as follows:

1. We showed in chapter 4, that everything we know about shear viscosities for an EMD-action also holds for more complicated ihQCD actions which include magnetic field
2. We showed in chapter 5 how to compute $\eta_{\parallel}^{B}$ using a Kaluza Klein-reduction
3. We showed in chapter 4 how to compute $\eta_{\perp}^{\chi}$, $\eta_{\parallel}^{\chi}$ and $\zeta_{\parallel}^{B}$ using the EO formula
4. We found consistency with [28], were it was found that conformal symmetry breaking (in this case caused by a magnetic field) does not imply a non-zero bulk viscosity
5. In chapter 6, we calculated magnetic field dependent ζ for an analytic two-dimensional ABJM model
6. We made progress in calculating the three viscosities of relativistic magnetohydrodynamics in chapter 4 and 5.

In chapter 7 we looked at experimental data and predictions based on the lowest Landau level approximation relating to bulk viscosities in a magnetic field. Unfortunately, we did not get to the point where we could compare this to holographic predictions.

8.2 Outlook

8.2.1 Introduction

In this section we will discuss several things that could be studied proceeding the research that was done in this thesis. In the first paragraph we discuss some open problems in this thesis which could be studied further and then after that we will go through some more general topics related to this thesis which could be studied in the future.

8.2.2 Follow-up

There are several things discussed in this thesis on which the book is not completely closed. In particular, we did not finish the computation of the bulk viscosities in chapter 4. Here we described how we could find the three bulk viscosities of magnetohydrodynamics for a QGP by looking at the fluctuation equations. This can be done by looking either at the horizon or at the boundary. Both methods have problems. For the UV expansion, it seems plausible that we should look at the ratio of the sub-leading over leading term, but it is not entirely clear what the right proportionality factor should be. In the membrane paradigm, the problem is that for ζ_m the term $\propto (h_{11}^* h_{33} - h_{11} h_{33}^*)_{\phi \to \phi_h}$ is present in the Green's function, which is divergent when using the Green-Kubo relations to find the bulk viscosity. Furthermore, in chapter 5, we looked at ways to mimic fluid equations so that viscosities can be extracted from it, but we encountered problems related to zero-temperature boost symmetry violation, which prevented us from looking at certain viscosities. It would be nice if these problem could be better understood so that we know how to get around this problem and still manage to extract all viscosities. In particular, we should find how we can boost our solution when magnetic field is introduced. We would also like to calculate the bulk viscosities with the EO formula for the axion, and for this we would need some ideal conservation law, more on this 8.2. Then there is also the problem, which holds generally for both chapter 4 and 5 (and is also elaborated in Appendix J), which is that it is hard to have one well-defined notion of anisotropic bulk viscosities. This will be needed if we eventually wish to find consistency between the results that could follow the methods described in chapter 5 and 4.

8.2.3 Relativistic hydrodynamics

Relativistic hydrodynamics is a very hot field in physics. New tools are made available to analyze the behavior of fluid, which uset he quasinormal modes as input [61], [41]. If we include momentum dependence, these methods can be applied to the fluctuation equations of this thesis. Furthermore, as disccused in section 1.6, we can learn more about hydrodynamics by seeing how [51] different forms of diffusion are related to eachother. In particular, it would be interesting to see how bulk viscosity fits into this picture of diffusion. Lastly, in chapter 2, we discussed in great detail a paper from Hernandez and Kovtun of all first order transport coefficients of magnetohydrodynamics. Something similar could perhaps be done for 'axiohydrodynamics', and this would help us understand how to deal with bulk viscosity in the case of axion. We assumed in this thesis that just like magnetohydrodynamics that there are three bulk viscosities for axiohydrodynamics, even though the steps decribed in section 2 and [55] clearly do not assume a general form a rotational symmetry breaking. Considering [84] it is extremely plausible that there are only two shear viscosities in a system with axion induced anisotropy, but perhaps there are more or less than three helicity zero transport coefficients.

8.2.4 Complications of the gravity background

One of the strengths of improved holographic QCD is that it is very easy to see how other physical phenomena can be included so that more phenomena of QCD can be studied. For example it is very clear how we could take chemical potential, conductivity (equation 5.62), Chern-Simons diffusion rate, quark mass and the chirally non-symmetric deconfined phase (section 3.2). Furthermore, it is also possible to get a more accurate holographic prediction by doing some more work. For example, you can get a more realistic temperature dependent prediction for shear viscosity by including a Gauss-Bonnet term [26] and, though very difficult, try to look at string loop contributions for the dual theory [3].

8.2.5 Expansion of the framework

Apart from improving the dual theory of gravity, it would also be nice to better understand the tools with which we can extract information from this dual theory. This is what this thesis was mainly about, but the work done here can be proceeded and other things can be looked at. As mentioned section 8.1, we could for look into Kaluza Klein reduction and how we could find all diffusion constants using this method. We could, similar to what was discussed in section 8.2, look at whether we could fit bulk viscosity into this picture. Furthermore, in Appendix N, we briefly discuss how could find general consistency with the EO formula and fluid gravity, something that has already been achieved for the Sagai-Sugimoto model, The reason this was not done in this thesis is because the computations were too heavy for Mathematica, but perhaps there is a way to simplify these computations.

8.2.6 Statistical analysis of experimental data

In [12], Bernhard uses a Bayesian method for finding the isotropic viscosities of quark-gluon plasma as a function of temperature (see figure 7.2). The method involves modelling the fluid behavior of a QGP and tuning the parameters of this model so that the output is consistent with data from the LHC and RHIC. A similar analysis where magnetic field dependence instead of temperature dependence is considered would be extremely useful to check the holographic predictions presented in this thesis and in previous works.

Appendix A

Zeroth order EOM

We work with the following Lagrangian:

$$\mathcal{L} = \sqrt{-g}(R - \frac{1}{2}(\partial\phi)^2 + V_g - xV_f\sqrt{D}) \tag{A.1}$$

With:

$$D = \det(\delta_\mu^\nu + wF_{\mu\alpha}g^{\alpha\nu}) \tag{A.2}$$

$$F_{\mu\nu} = \partial_{[\mu}A_{\nu]} \tag{A.3}$$

$$A_\mu = \{0, -\frac{1}{2}B_m y, \frac{1}{2}B_m x, 0, 0\} \tag{A.4}$$

This yields the following Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{4}g_{\mu\nu}(\partial\phi)^2 - \frac{1}{2}g_{\mu\nu}V_g + x\frac{V_f}{2}(g_{\mu\nu}\sqrt{D} - \frac{1}{\sqrt{D}}\frac{dD}{dg^{\mu\nu}}) \tag{A.5}$$

For $\tau = 0$, D becomes:

$$D = \det\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -e^{-2A}B_m & 0 & 0 \\ 0 & e^{-2A}B_m & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A.6}$$

$$= 1 + w^2B_m^2g^{11}g^{22} = 1 + w^2B_m^2e^{-4A} = 1 + w^2\frac{1}{2}F^{\mu\nu}F_{\mu\nu} \equiv Q^2 \tag{A.7}$$

We also have the dilaton EOM which is redundant but is useful as a check for possible errors in your calculations:

$$\frac{1}{\sqrt{-g}}\partial_r\left(\sqrt{-g}g^{rr}\partial_r\phi\right) + \partial_\phi V_g - x\partial_\phi V_f\sqrt{D} - xV_f\frac{\partial_\phi D}{2\sqrt{D}} = 0 \tag{A.8}$$

The Einstein equations consist of an Einstein tensor $G_{\mu\nu}$ and an energy-momentum tensor $T_{\mu\nu}$:

$$G_{\mu\nu} + T_{\mu\nu} = 0 \tag{A.9}$$

Because it is easier to work with our convention of stress-energy tensor $T_{\mu\nu}$ has a sign difference with respect to the standard energy-momentum tensor. Since the ϕ monotonously increases as a function of r, we can redefine our fifth dimension to be ϕ. This is called the $\phi = r$ gauge, also called the Gubser gauge [43]. From this follows an equation constraining B:

$$e^A dr = e^B d\phi \tag{A.10}$$

With this gauge, we get the following energy momentum tensors (i.e. the part of the Einstein equations that does not come from the Einstein Hilbert action):

$$T_{00} = -fe^{2A}x\frac{V_f}{2}Q + fe^{2A}V_g - \frac{f^2e^{2A-2B}}{4} \tag{A.11}$$

$$T_{11} = x\frac{V_f}{2}\frac{e^{2A}}{Q} - e^{2A}V_g + \frac{fe^{2A-2B}}{4} \tag{A.12}$$

$$T_{33} = e^{2A+2W}x\frac{V_f}{2}Q - e^{2A+2W}V_g + \frac{fe^{2A+2W-2B}}{4} \tag{A.13}$$

$$T_{55} = \frac{e^{2B}}{f}x\frac{V_f}{2}Q - \frac{e^{2B}}{f}V_g - \frac{1}{4} \tag{A.14}$$

We then get five EOM, with one being redundant:

$$\begin{aligned} A'' &= A'\left(B' + W'\right) \\ &+ \frac{1}{f}\Big(\frac{1}{3}f'W' - \frac{B_m^2 xe^{2B-4A}V_f w^2}{6Q}\Big) - \frac{1}{6} \end{aligned} \tag{A.15}$$

$$\begin{aligned} W'' &= W'\left(-4A' + B' - W'\right) \\ &+ \frac{1}{f}\Big(\frac{B_m^2 xe^{2B-4A}V_f w^2}{2Q} - f'W'\Big) \end{aligned} \tag{A.16}$$

$$f'' = f'\left(-4A' + B' - W'\right) + \frac{B_m^2 xe^{2B-4A}V_f w^2}{Q} \tag{A.17}$$

$$\begin{aligned} V =& e^{-2B}f'\left(3A' + W'\right) + \frac{1}{2}e^{-2B}f\left(12A'W' + 24A'^2 - 1\right) + \\ & xV_fQ \end{aligned} \tag{A.18}$$

$$\begin{aligned} 0 =& (4A' + W' - B' + \frac{f'}{f})fe^{-2B} + V' - xV_f'Q \\ & - xV_f e^{-4A}\frac{ww'B_m^2}{Q} \end{aligned} \tag{A.19}$$

At the horizon we find:

$$\frac{B_m^2 xe^{2B(\phi_h)-4A(\phi_h)}V_f(\phi_h)w^2(\phi_h)}{2Q(\phi_h)} = f'(\phi_h)W'(\phi_h) \tag{A.20}$$

$$\frac{1}{3A'(\phi_h) + W'(\phi_h)} = -\frac{V'(\phi_h) - xV_f'(\phi_h)Q - xV_f(\phi_h)e^{-4A(\phi_h)}\frac{w(\phi_h)w'(\phi_h)B_m^2}{Q(\phi_h)}}{V(\phi_h) - xV_f(\phi_h)Q(\phi_h)} \tag{A.21}$$

Appendix B

Transformations of perturbative modes

$\vec{\xi} = \{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5\}$ is the vector with which modes can be removed using gauge and diffeomorphism invariance. Because the modes depend only on r and t, $\vec{\xi}$ can only depend on these variables as well. For the helicity one block the modes transform in the following way:

$$\widetilde{h}_{01}(t,r) = h_{01}(t,r) - i\omega\xi_1(r,t) \tag{B.1}$$

So we can fix ξ_1 to put h_{01} to zero. For the helicity zero block the modes transform the following way:

$$\widetilde{h}_{03}(t,r) = h_{03}(t,r) - i\omega\xi_3(r,t) \tag{B.2}$$

$$\begin{aligned}\widetilde{h}_{05}(t,r) &= h_{05}(t,r) + \partial_r\xi_0(t,r) - i\omega\xi_5(r,t) - \Gamma^0_{05} \\ &= h_{05}(t,r) + \partial_r\xi_0(t,r) - i\omega\xi_5(r,t) + \frac{1}{2}f^2 e^{2A-2B}(2A' + \frac{f'}{f})\xi_0(r,t)\end{aligned} \tag{B.3}$$

$$\delta\widetilde{\phi}(t,r) = \delta\phi(t,r) + f e^{-2B}\xi_5\partial_r\phi(r) \tag{B.4}$$

We fix ξ_5 to put $\delta\phi$ to zero. We then have ξ_0 and ξ_3 left to put h_{03} and h_{05} to zero. Consider the Einstein equations at zeroth order:

The only way δA_i could affect the Einstein equations is via D. For helicity 1, $D + \delta D_0$ becomes:

$$D + \delta D_0 = \det\begin{bmatrix} 1 & 0 & 0 & -\delta\dot{A}_3(r,t) & -\delta A_0'(r,t) \\ 0 & 1 & -e^{-2A}B_m & 0 & 0 \\ 0 & e^{-2A}B_m & 1 & 0 & 0 \\ \delta\dot{A}_3(r,t) & 0 & 0 & 1 & -\delta A_3'(r,t) \\ \delta A_0'(r,t) & 0 & 0 & \delta A_3'(r,t) & 1 \end{bmatrix} \tag{B.5}$$

For helicity 0, $D + \delta D_1$ becomes:

$$D + \delta D_1 = \det\begin{bmatrix} 1 & -\delta\dot{A}_1(r,t) & 0 & 0 & 0 \\ \delta\dot{A}_1(r,t) & 1 & -e^{-2A}B_m & 0 & -\delta A_1'(r,t) \\ 0 & e^{-2A}B_m & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \delta A_1'(r,t) & 0 & 0 & 1 \end{bmatrix} \tag{B.6}$$

From this it follows that gauge field perturbations do not give first order contribution to the perturbative Einstein equations.

Appendix C

First order EOM for shear modes

C.1 Fluctuation equations

We start off with the following Ansätze:

$$ds^2 = e^{2A(\phi)}(-f(\phi)dt^2 + dx^2 + dy^2 + e^{2W(\phi)}dz^2 + 2\lambda h_{12}(\phi)e^{i\omega t}dxdy)) + \frac{e^{2B(\phi)}}{f(\phi)}d\phi^2 \tag{C.1}$$

$$ds^2 = e^{2A(\phi)}(-f(\phi)dt^2 + dx^2 + dy^2 + e^{2W(\phi)}dz^2 + 2\lambda h_{13}(\phi)e^{i\omega t}dxdz)) + \frac{e^{2B(\phi)}}{f(\phi)}d\phi^2 \tag{C.2}$$

Formulated differently, when considering a metric perturbation of $g_{\mu\nu} + \delta g_{\mu\nu}$, we work with the mode: $h_{\mu\nu} = g^{11}\delta g_{\mu\nu}$. The EOM have the following structure:

$$\delta G_{\mu\nu} + \delta T_{\mu\nu} = 0 \tag{C.3}$$

The first order contribution from the energy-momentum tensor $\delta T_{\mu\nu}$ to the first order EOM is as follows:

$$\delta T_{12} = \frac{1}{4}e^{2A}\lambda h_{12}(\frac{2xV_f}{Q} + e^{-2B}f - 2V_g) \tag{C.4}$$

$$\delta T_{13} = \frac{1}{4}e^{2A}\lambda h_{13}(2xV_fQ + e^{-2B}f - 2V_g) \tag{C.5}$$

Note that the V_f term is different because for h_{12} you also have to consider functional differentiation of the term $g^{12}g^{21}F_{12}F_{21}$.

$$-fe^{2A-2B}(4A'h'_{12} - B'h'_{12} + h''_{12} + h'_{12}W') - e^{2A-2B}f'h'_{12} - \frac{\omega^2 h_{12}}{f} = 0 \tag{C.6}$$

$$-fe^{2A-2B}(4A'h'_{13} - B'h'_{13} + h''_{13} - h'_{13}W') - e^{2A-2B}f'h'_{13} + e^{2A}V_f\frac{Q^2-1}{Q}h_{13} - \frac{\omega^2 h_{13}}{2f} = 0 \tag{C.7}$$

This result is consistent with [29].

C.2 Perpendicular shear viscosity

We will first calculate the shear viscosity $\eta_\perp$, which corresponds to the mode h_{12}, as the EOM for this mode are already massless. At the horizon the EOM reduce to:

$$fh_{12}'' + f'h_{12}' + \frac{\omega^2 h_{12} e^{-2A+2B}}{f} \approx 0 \tag{C.8}$$

This can be solved by plugging in the trial solutions $h_{12} = c_{12}^{\pm}(\phi_h - \phi)^{\pm\rho}$. Which gives: $\rho = \frac{i\omega}{4\pi T}$. This is a universal result for any perturbative mode breaking thermodynamic equilibrium [52]. We will therefore find the following solution to the EOM at the horizon for all the metric perturbations.

$$h_{ij} \approx c_{ij}^{+}(\phi_h - \phi)^{\frac{i\omega}{4\pi T}} + c_{ij}^{-}(\phi_h - \phi)^{-\frac{i\omega}{4\pi}} \tag{C.9}$$

Before we solve this EOM we will first write down the boundary conditions. Our horizon is a future horizon, dual to positive temperature, which requires in-falling boundary conditions for regularity at the horizon. This means that $c_{12}^{+} = 0$. For convenience we will now define $c_{12}^{-} \equiv c_{12}$. Our second boundary is simply the requirement that the mode converges to one near the boundary. For the h_{12} mode we can work this out analytically. We do this by solving the EOM twice. We solve it once for an infinitesimal distance from the horizon for $\omega \neq 0$, as we have already done. Then we solve it for the entire bulk for $\omega = 0$, which gives:

$$h_{12} = a_{12} + b_{12}\int_0^{\phi} d\phi' \frac{e^{-4A-W+B}}{f} \tag{C.10}$$

If we take the limit $\omega \to 0$, we can equate these two solutions near the horizon:

$$h_{12} \approx a_{12} + \frac{b_{12}}{f'(\phi_h)} e^{-4A(\phi_h)+B(\phi_h)-W(\phi_h)} \log(\phi_h - \phi) \approx c_{12}\Big(1 - \frac{i\omega}{4\pi T}\log(\phi_h - \phi)\Big) \tag{C.11}$$

From our boundary condition at the boundary we then know that $a_{12} = c_{12} = 1$. As explained in section 4.2, to get to the Green's function, we also need to find the flux. We can do this by expanding the Lagrangian up to second order:

$$\mathcal{L}_2 = \frac{1}{2}\frac{\partial^2}{\partial\lambda^2} e^{4A+B+W} R(\lambda)\ ,\ \lambda \to 0 \tag{C.12}$$

From equation 4.20 we know that we only need to look at double derivative terms, from the other terms do not contribute to the end result.

$$\mathcal{L}_{12} = \frac{1}{2f} e^{-B+5W}(e^{2(A+B}(3\dot{h}_{12}^2 + 4h_{12}\ddot{h}_{12}) + e^{4A}f^2(-3h_{12}'^2 - 4h_{12}h_{12}'')) + \text{non-contributing terms} \tag{C.13}$$

We subtract total derivatives:

$$\hat{\mathcal{L}}_{12} = \frac{1}{2f} e^{-B+5W}(-e^{4A}f^2 h_{12}'^2 + e^{2(A+B)}\dot{h}_{12}^2) + \text{non-contributing terms} \tag{C.14}$$

We thus get:

$$\mathcal{F}_{12} = -\frac{i}{2} f e^{4A-B+3W}(h_{12}^* h_{12}' - h_{12}^{*\prime} h_{12}) \tag{C.15}$$

We can confirm that this is the correct result by looking at whether the flux is conserved (see equation 4.21). At the horizon we can simplify in the following way:

$$\begin{aligned}
\mathcal{F}_{12} &= -\frac{i}{2} f e^{4A-B+3W}(h_{12}^{*} h_{12}' - h_{12}^{*\prime} h_{12}) \\
&= -\frac{i}{2} f e^{4A-B+3W}\Big(c_{12}^{*}\Big(1+\frac{i\omega}{4\pi T}\log(\phi_h-\phi)\Big)\Big)\partial_\phi\Big(c_{12}\Big(1-\frac{i\omega}{4\pi T}\log(\phi_h-\phi)\Big)\Big) \\
&+ \frac{i}{2}\partial_\phi\Big(c_{12}^{*}\Big(1+\frac{i\omega}{4\pi T}\log(\phi_h-\phi)\Big)\Big)\Big(c_{12}\Big(1-\frac{i\omega}{4\pi T}\log(\phi_h-\phi)\Big)\Big) \\
&= -\frac{i}{2} f e^{4A-B+3W}\Big(c_{12}^{*}\Big(1+\frac{i\omega}{4\pi T}\log(\phi_h-\phi)\Big)\Big)\Big(c_{12}\frac{i\omega}{4\pi T}\frac{1}{\phi_h-\phi}\Big) \\
&- \frac{i}{2}\Big(c_{12}^{*}\frac{i\omega}{4\pi T}\frac{1}{\phi_h-\phi}\Big)\Big(c_{12}\Big(1-\frac{i\omega}{4\pi T}\log(\phi_h-\phi)\Big)\Big) \\
&= |c_{12}|^2 \omega e^{3A+W}
\end{aligned} \tag{C.16}$$

We conclude:

$$-\lim_{\omega\to 0}\operatorname{Im}\frac{1}{\omega}G_\perp = M^2 N_c^3 \mathcal{F}_{12} \to \frac{\eta_\perp}{s} = \frac{1}{4\pi} \tag{C.17}$$

As expected, the shear viscosity perpendicular to the magnetic field is unaffected by the symmetry breaking. We can also derive this result, as is done in [59], [58], by writing equation C.6 in the following way:

$$\partial_\phi(\sqrt{-g}Pg^{55}\partial_\phi h_{13}) - \sqrt{-g}N\omega^2 g^{00} h_{12} = 0 \tag{C.18}$$

With P and N functions that depend on ϕ only, in this case we have $P = 1$ and $N = -1$. EOM that have this structure have a corresponding action that looks like this:

$$S = -M^3 N_c^2 \int dx^5 \sqrt{-g}\Big(\frac{1}{2}Pg^{rr}(\partial_\phi h_{12})^2 - \frac{1}{2}Ng^{tt}(\partial_t h_{12})^2\Big) \tag{C.19}$$

From this Lagrangian, you can immediately extract the shear viscosity via a Green's function the following way:

$$\eta_\perp = -\lim_{\omega\to 0}\frac{G_\perp}{i\omega} = \lim_{\omega\to 0}\frac{\Pi}{i\omega h_{12}}\Big|_{\phi\to-\infty} \tag{C.20}$$

Because of the way the action is written down you don't have to filter out the real part as is done in equation C.17. Π is the canonical momentum given by:

$$\Pi = \frac{\delta S}{\delta\partial_r h_{12}} = -M^3 N_c^2 \sqrt{-g} g^{55}\partial_\phi h_{12} \tag{C.21}$$

From equation C.18, we also find:

$$\partial_\phi \Pi = \frac{\sqrt{-g}}{M^3 N_c^2}\omega^2 g^{00} h_{12} \tag{C.22}$$

Equivalent to using the masslessness of our EOM in the previous method, we now use that from equation C.18 it follows that that $\partial_\phi \Pi = 0$, so we also have:

$$\eta_\perp = \lim_{\omega\to 0}\frac{\Pi(\phi_h)}{i\omega h_{12}(\phi_h)} = i\frac{M^3 N_c^2\sqrt{-g}Pg^{55}\partial_\phi h_{12}}{\omega h_{12}}\Big|_{\phi=\phi_h} \tag{C.23}$$

We can now write down a differential equation for $\eta_\perp$:

$$\partial_\phi \eta_\perp = i \lim_{\omega \to 0} \Big(-\frac{\partial_\phi \Pi(\phi_h)}{\omega h_{12}(\phi_h)} + \frac{\Pi(\phi_h)\partial_\phi h_{12}(\phi_h)}{\omega h_{12}^2(\phi_h)} \Big) \tag{C.24}$$

Substituting equation C.21 and C.22 gives:

$$\partial_\phi \eta_\perp = i \lim_{\omega \to 0} \Big(-\frac{\sqrt{-g}}{M^3 N_c^2} \omega g^{00} - \frac{M^3 N_c^2 \sqrt{-g} g^{55} (\partial_\phi h_{12})^2}{\omega h_{12}^2} \Big)\Big|_{\phi=\phi_h} \tag{C.25}$$

$$\partial_\phi \eta_\perp = i\omega \sqrt{\frac{g_{55}}{g_{00}}} \Big(-\frac{\sqrt{-g}}{\sqrt{g_{00} g_{55}}} M^3 N_c^2 + \sqrt{\frac{g_{00} g_{55}}{-g}} \frac{\eta_\perp^2}{M^3 N_c^2} \Big)\Big|_{\phi=\phi_h} \tag{C.26}$$

At the horizon, regularity requires that the part inside that brackets is zero, which gives:

$$\eta_\perp = \sqrt{\frac{-g}{g_{00} g_{55}}} M^3 N_c^2 \Big|_{\phi=\phi_h} \to \frac{\eta_\perp}{s} = \frac{1}{4\pi} \tag{C.27}$$

C.3 Parallel shear viscosity

For the $\eta_\parallel$, we can choose a new perturbative mode so that the same trick can be used as was used for the h_{12} mode:

$$h_{13} \to e^{2W} Z_{13} \tag{C.28}$$

As explained in Appendix D, because e^{2W} converges to unity at the boundary this does not change the end result. From this rescaling it follows that we work with the following Ansatz:

$$ds^2 = e^{2A(\phi)}(-f(\phi)dt^2 + dx^2 + dy^2 + e^{2W(\phi)}dz^2 + 2\lambda Z_{13}(\phi) e^{i\omega t} dxdz)) + \frac{e^{2B(\phi)}}{f(\phi)} d\phi^2 \tag{C.29}$$

So we have: $Z_{13} = \delta g_{13} g^{33}$. This gives the following EOM:

$$-e^{2A-2B}(f(4A' Z_{13}' - B' Z_{13}' + Z_{13}'' + 3Z_{13}' W') + f' Z_{13}') - \frac{\omega^2 Z_{13}}{f} \tag{C.30}$$

The second order expanded Lagrangian is as follows:

$$\hat{\mathcal{L}}_{13} = \frac{1}{2f} e^{4A-B+3W}(-f^2 Z_{13}'^2 + e^{2B-2A} \dot{Z}_{13}^2) + \text{non-contributing terms} \tag{C.31}$$

So we find:

$$\mathcal{F}_{13} = \omega c_{13}^2 e^{3A(\phi_h)+3W(\phi_h)} \to \frac{\eta_\parallel}{s} = \frac{e^{2W(\phi_h)}}{4\pi} \tag{C.32}$$

We could define this result in terms of the horizon geometry in the following way:

$$\frac{\eta_\parallel}{s} = \frac{1}{4\pi} \frac{g_{zz}}{g_{xx}} \Big|_{\phi=\phi_h} \tag{C.33}$$

When writing the EOM of equation C.30 in the form of equation C.18, we get $P = e^{2W(\phi_h)}$ and we get the identical result.

Appendix D

Rescaling invariance of perturbative modes

Consider the mode ψ and the rescaled mode $\psi^* = \rho\psi$, with ρ being a factor that converges to unity at the boundary. ψ and ψ^* is dual to $\mathcal{O}$ and $\mathcal{O}^*$. At the boundary we have $\psi(r_b) = \psi^*(r_b)$, so using the GKPW rule:

$$-\frac{\delta}{\delta\psi}\frac{\delta}{\delta\psi}e^{-iS_{grav}}\Big|_{r\to r_b} = G_{\mathcal{O}\mathcal{O}} = -\frac{\delta}{\delta\psi^*}\frac{\delta}{\delta\psi^*}e^{-iS_{grav}}\Big|_{r\to r_b} = G_{\mathcal{O}^*\mathcal{O}^*} \tag{D.1}$$

The transport coefficient σ that relates ψ to $\langle\mathcal{O}\rangle$ is given by:

$$\sigma = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G_{\mathcal{O}\mathcal{O}} = i\lim_{\omega\to 0}\frac{1}{\omega}K\mathcal{F}(r_b) = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G_{\mathcal{O}^*\mathcal{O}^*} = \lim_{\omega\to 0}\frac{1}{\omega}K^*\mathcal{F}^*(r_b) \tag{D.2}$$

K and K^* are constants. For the dual operator we can also conclude $\mathcal{O} = \mathcal{O}^*$. Because the flux is conserved, we have:

$$\frac{d\mathcal{F}}{dr} = \frac{d\mathcal{F}^*}{dr} = 0 \tag{D.3}$$

So:

$$\lim_{\omega\to 0}\frac{1}{\omega}K\mathcal{F}(r_h) = \lim_{\omega\to 0}\frac{1}{\omega}K^*\mathcal{F}^*(r_h) \tag{D.4}$$

So we can rescale our modes with any factor that becomes unity at the boundary, without it changing the result for the transport coefficient we wish to calculate.

Appendix E

Kaluza-Klein reduction

In [84], Rebhan and Steineder use a Kaluza Klein reduction of an gravitational background with anisotropy caused by an axion, similar to section 3.6. In this Appendix we summarize computations similar to [69] that leads to this result. With $G_{\mu\nu}$ being the non-compactified metric of our 5-dimensional theory, we perform the following Kaluza-Klein reduction:

$$\begin{aligned} ds^2 &= -G_{00}dt^2 + G_{11}(dx^2 + dy^2) + G_{33}dz^2 + G_{rr}dr^2 \\ &\rightarrow -g_{00}dt^2 + g_{11}dx^2 + g_{33}dz^2 + g_{55}dr^2 \end{aligned} \tag{E.1}$$

After this dimensional reduction, small metric perturbations now are gauge fields:

$$A_i = g^{11}h_{yi} \quad i \in \{0, 1, 3, 5\} \tag{E.2}$$

The Maxwell equations that will be used in this derivation, are the ones that follow from variation with respect to A_0 and A_5, components of the gauge fields are not coupled to the matter terms. This gauge field is also called a graviphoton, whose dynamics is governed by the Maxwell action:

$$S_{gauge} \propto \int d^5x\sqrt{-g}\Big(\frac{1}{g_{eff}^2}F^{\mu\nu}F_{\mu\nu}\Big) \tag{E.3}$$

g_{eff} is an r-dependent factor that follows from y-dependant part of the metric. This yields the following Maxwell equations:

$$\partial_\mu\Big(\frac{1}{g_{eff}^2}\sqrt{-g}F^{\mu\nu}\Big) = 0 \tag{E.4}$$

Because the background metric only depends on r, we can work with a plane wave solution with momentum oriented towards the x-direction: $A_\mu = A_\mu(t, r)e^{ikx}$. We also work with the radial gauge $A_r = 0$. From the Maxwell action the Poisson equation (variation with respect to A_5) follows. Using the Maxwell equation that follows from variation with respect to A_0 KSS show that if the distance from the horizon is not exponentially small compared to $\frac{T^2}{k^2}$ we have:

$$F_{01} = \partial_0 A_1 - \partial_1 A_0 \approx -\partial_1 A_0 \tag{E.5}$$

For $k \rightarrow 0$, the adiabatic approximation becomes exact and the Poisson equation then becomes:

$$\partial_r\Big(\frac{\sqrt{-g}g^{rr}g^{00}}{g_{eff}^2}\partial_r A_0\Big) = 0 \tag{E.6}$$

This equation has the following solution:

$$A_0(r) = C_0 \int_0^{r_h} dr' \frac{g_{00}(r')g_{rr}(r')g_{eff}^2(r')}{\sqrt{-g(r')}} \tag{E.7}$$

For the same limit as used in equation E.5, KSS show that at zeroth order a momentum dependent A_0 has the same form:

$$A_0(r,t,x) = C_0(t)e^{ikx} \int_0^{r_h} dr' \frac{g_{00}(r')g_{rr}(r')g_{eff}^2(r')}{\sqrt{-g(r')}} + \mathcal{O}\Big(\frac{T^2}{k^2}\Big) \tag{E.8}$$

We also find that:

$$F_{05} = -\partial_r A_0 \approx C_0(t)e^{ikx} \frac{g_{00}g_{rr}g_{eff}^2(r')}{\sqrt{-g}}\Big|_{r=r_h} \tag{E.9}$$

We thus get:

$$\frac{A_0}{F_{05}}\Bigg|_{r\approx r_h} \approx \frac{\sqrt{-g}}{g_{00}g_{55}g_{eff}^2}\Big|_{r=r_h} \int_0^{r_h} dr' \frac{g_{00}(r')g_{55}(r')g_{eff}^2(r')}{\sqrt{-g(r')}} \tag{E.10}$$

Going back to the Maxwell action, we can require gauge invariance and find that the membrane current at the horizon is given by [58]:

$$J^1_{mb} = -\frac{\sqrt{-g}}{g_{eff}^2}\sqrt{g_{55}}g^{55}g^{11}F_{51}\Big|_{r=r_h} \tag{E.11}$$

$$J^0_{mb} = -\frac{\sqrt{-g}}{g_{eff}^2}\sqrt{g_{55}}g^{55}g^{00}F_{50}\Big|_{r=r_h} \tag{E.12}$$

At the horizon, we require regularity, which means that there can be no divergences in the metric. Going to the Eddington-Finkelstein coordinates, one finds that this means that we want the following to hold for the Eddington-Finkelstein coordinate dv:

$$dv = dt + \sqrt{\frac{g_{55}(r_h)}{g_{00}(r_h)}}dr = 0 \tag{E.13}$$

From this we get:

$$J^1_{mb} = -\sqrt{-g}\sqrt{g_{55}}g^{11}g^{55}F_{51} \approx \sqrt{-g}\sqrt{g_{00}}g^{11}g^{55}\partial_1 A_0 \tag{E.14}$$

Using equation E.10:

$$J^1_{mb} \approx -\sqrt{-g}\frac{1}{\sqrt{g_{00}}}g^{11}\Big(\frac{A_0}{F_{05}}\Big)\partial_1 F_{05}\Big|_{r=r_h} \tag{E.15}$$

Using, equation E.12 We then get get the following equation for our membrane current, which is dual to Fick's law for the fluid:

$$\begin{aligned} J^1_{mb} &= -D_x \partial_x J^0_{mb} \\ D_x &= \frac{\sqrt{-g}}{g_{11}\sqrt{g_{00}g_{55}}g_{eff}^2}\Big|_{r=r_h} \int_0^{r_h} dr' \frac{g_{00}(r')g_{55}(r')g_{eff}^2(r')}{\sqrt{-g(r')}} = \frac{\eta_\perp}{\varepsilon + P_\perp} \end{aligned} \tag{E.16}$$

Here ε is the energy density and $P_\perp$ is the pressure orthogonal to the magnetic field. Since it this result is k independent, it is exact. For a perfect fluid, i.e. at zeroth order, we have:

$$\langle T_{\mu\nu} \rangle = \mathrm{diag}(\varepsilon, P_\perp, P_\perp, P_\parallel) \tag{E.17}$$

In units of energy density, the anisotropy is given by:

$$P_\perp - P_\parallel = \chi_a a^2 \tag{E.18}$$

Here χ_a is the anisotropic susceptibility [49]. We also have the following thermodynamic identities for an anisotropic fluid with zero chemical potential [20] [24]:

$$f = -P_\perp \tag{E.19}$$
$$g = -P_\parallel \tag{E.20}$$
$$\varepsilon = f + Ts \tag{E.21}$$
$$= g + Ts - \chi_a a^2 \tag{E.22}$$

Here f and g are the Helmholtz and Gibbs free energy density respectively. This leads to: $\varepsilon + P_\perp = Ts$.

$$\frac{\eta_\perp}{Ts} = \frac{\sqrt{-g}}{g_{11}\sqrt{g_{00}g_{55}}g_{eff}^2}\Big|_{r=r_h} \int_0^{r_h} dr' \frac{g_{00}(r')g_{55}(r')g_{eff}^2(r')}{\sqrt{-g(r')}} \tag{E.23}$$

Taking k in the z-direction, we find similarly:

$$\begin{aligned} J^z_{mb} &= -D_z \partial_x J^0_{mb} \\ D_z &= \frac{\sqrt{-g}}{g_{33}\sqrt{g_{00}g_{55}}g_{eff}^2}\Big|_{r=r_h} \int_0^{r_h} dr' \frac{g_{00}(r')g_{55}(r')g_{eff}^2(r')}{\sqrt{-g(r')}} = \frac{\eta_\parallel}{\varepsilon + P_\perp} \end{aligned} \tag{E.24}$$

Which yields:

$$\frac{\eta_\parallel}{Ts} = \frac{\sqrt{-g}}{g_{33}\sqrt{g_{00}g_{55}}g_{eff}^2}\Big|_{r=r_h} \int_0^{r_h} dr' \frac{g_{00}(r')g_{55}(r')g_{eff}^2(r')}{\sqrt{-g(r')}} \tag{E.25}$$

One can find g_{eff}, by expanding the non-Kaluza Klein reduced Einstein Hilbert action at at first order for $A_i = G^{11}h_{0i}$. KSS find:

$$\sqrt{-G}\mathcal{R} \to -\frac{1}{4}\frac{1}{g_{eff}^2}\sqrt{-g}F^{\mu\nu}F_{\mu\nu} + \text{tensor and scalar terms} \tag{E.26}$$

Defining the result for g_{eff} and rescaling to the non-reduced metric and using equation 3.12, we find:

$$\frac{\eta_\perp}{s} = \frac{Ts}{4\pi M^2 N_c^3} \int_0^{r_h} dr' \frac{G_{00}(r')G_{55}(r')}{G_{11}(r')\sqrt{-G(r')}} \tag{E.27}$$

$$\frac{\eta_\parallel}{s} = \frac{Ts}{4\pi M^2 N_c^3}\frac{G_{11}}{G_{33}}\Big|_{r=r_h} \int_0^{r_h} dr' \frac{G_{00}(r')G_{55}(r')}{G_{11}(r')\sqrt{-G(r')}} \tag{E.28}$$

Rebhan and Steineder confirm in [84] that these results are consistent with the results given in section 1.6. Using these results, we then have the following identity for the temperature:

$$T = \frac{M^2 N_c^3}{s}\Big(\int_0^{r_h} dr' \frac{G_{00}(r')G_{55}(r')}{G_{11}(r')\sqrt{-G(r')}}\Big)^{-1} \tag{E.29}$$

Appendix F

First order EOM for bulk modes

Since only the first order of $g_{\mu\nu} + \delta g_{\mu\nu}$ is relevant, it can also be written in a form that is simpler for these calculations:

$$
\begin{aligned}
ds^2 = e^{2A(\phi)}(&-f(\phi)(1+\lambda h_{00}(\phi)e^{i\omega t})dt^2 + (1+\lambda h_{11}(\phi)e^{i\omega t})(dx^2+dy^2) \\
&+ (1+\lambda h_{33}(\phi)e^{i\omega t})^2 e^{2W(\phi)}dz^2) + (1+\lambda h_{55}(\phi)e^{i\omega t})\frac{e^{2B(\phi)}}{f(\phi)}d\phi^2
\end{aligned}
\tag{F.1}
$$

Now we look at the contributions from the energy-momentum tensor. For this, we first focus on the flavor part $\delta T^f_{\mu\nu}$, i.e. the part of the perturbative energy-momentum tensor that comes from the flavor sector:

$$
\delta G_{\mu\nu} + \delta T^f_{\mu\nu} + \delta T^g_{\mu\nu} = 0 \tag{F.2}
$$

To keep it simple we will take $g_{11} = g_{22}$ for parts where there is no derivative with respect to g_{11}:

$$
\begin{aligned}
T^f_{\mu\nu} + \delta T^f_{\mu\nu} &= x\frac{V_f}{2}\Big((g_{\mu\nu} + \delta g_{\mu\nu})(\sqrt{D} + \delta\sqrt{D}) - \frac{1}{\sqrt{D} + \delta\sqrt{D}}\frac{\partial(\sqrt{D} + \delta\sqrt{D})}{\partial g^{\mu\nu}}\Big) \\
&= x\frac{V_f}{2}\Big((g_{\mu\nu} + \delta g_{\mu\nu})\sqrt{1 + w^2 B_m^2 g^{11} g^{11} + 2w^2 B_m^2 \delta g^{11} g^{11}} \\
&- \frac{1}{\sqrt{1 + w^2 B_m^2 g^{11} g^{11} + 2w^2 B_m^2 \delta g^{11} g^{11}}} \\
&\cdot \frac{d(w^2 B_m^2 g^{11} g^{22} + w^2 B_m^2 \delta g^{11} g^{22} + w^2 B_m^2 \delta g^{22} g^{11})}{dg^{\mu\nu}}\Big) \\
&= x\frac{V_f}{2}\Big((g_{\mu\nu} + \delta g_{\mu\nu})(\sqrt{1 + w^2 B_m^2 g^{11} g^{11}} + \frac{w^2 B_m^2 \delta g^{11} g^{11}}{\sqrt{1 + w^2 B_m^2 g^{11} g^{11}}}) \\
&- \Big(\frac{1}{\sqrt{1 + w^2 B_m^2 g^{11} g^{11}}} - \frac{w^2 B_m^2 \delta g^{11} g^{11}}{\sqrt{(1 + w^2 B_m^2 g^{11} g^{11})^3}}\Big) \\
&\cdot (w^2 B_m^2 g^{11} + w^2 B_m^2 \delta g^{11})(\delta^1_\mu \delta^1_\nu + \delta^2_\mu \delta^2_\nu)\Big) \\
&= x\frac{V_f}{2}\Big(\delta g_{\mu\nu}\sqrt{1 + w^2 B_m^2 g^{11} g^{11}} + g_{\mu\nu}\frac{w^2 B_m^2 \delta g^{11} g^{11}}{\sqrt{1 + w^2 B_m^2 g^{11} g^{11}}} \\
&- \Big(\frac{1}{\sqrt{1 + w^2 B_m^2 g^{11} g^{11}}} w^2 B_m^2 \delta g^{11} \\
&- \frac{w^2 B_m^2 \delta g^{11} g^{11}}{\sqrt{(1 + w^2 B_m^2 g^{11} g^{11})^3}} w^2 B_m^2 g^{11}\Big)(\delta^1_\mu \delta^1_\nu + \delta^2_\mu \delta^2_\nu)\Big)
\end{aligned}
\tag{F.3}
$$

Now we use $\delta g^{11} = -h_{11} e^{-2A}$ to write out the individual contributions to the Einstein equations at first order:

$$
\delta T^f_{00} = -x\frac{V_f}{2}(h_{00} f e^{2A}\sqrt{1 + w^2 B_m^2 e^{-4A}} - f e^{-2A}\frac{w^2 B_m^2 h_{11}}{\sqrt{1 + w^2 B_m^2 e^{-4A}}}) \tag{F.4}
$$

$$
\begin{aligned}
\delta T^f_{11} &= x\frac{V_f}{2}(h_{11} e^{2A}\sqrt{1 + w^2 B_m^2 e^{-4A}} - e^{-2A}\frac{w^2 B_m^2 h_{11}}{\sqrt{1 + w^2 B_m^2 e^{-4A}}} \\
&- (\frac{1}{\sqrt{1 + w^2 B_m^2 e^{-4A}}}(-w^2 B_m^2 h_{11}) e^{-2A} + \frac{w^2 B_m^2 h_{11}}{\sqrt{(1 + w^2 B_m^2 e^{-4A})^3}}(w^2 B_m^2 e^{-6A})))
\end{aligned}
\tag{F.5}
$$

$$
\delta T^f_{33} = x\frac{V_f}{2}(h_{33} e^{2A+2W}\sqrt{1 + w^2 B_m^2 e^{-4A}} - e^{-2A+2W}\frac{w^2 B_m^2 h_{11}}{\sqrt{1 + w^2 B_m^2 e^{-4A}}}) \tag{F.6}
$$

$$
\delta T^f_{55} = x\frac{V_f}{2}(\frac{1}{f} h_{55} e^{2B}\sqrt{1 + w^2 B_m^2 e^{-4A}} - \frac{1}{f} e^{2B-4A}\frac{w^2 B_m^2 h_{11}}{\sqrt{1 + w^2 B_m^2 e^{-4A}}}) \tag{F.7}
$$

Simplifying:

$$\delta T^f_{00} = -x\frac{V_f}{2}(fe^{2A}h_{00}Q - fe^{-2A}\frac{w^2B_m^2h_{11}}{Q}) \tag{F.8}$$

$$\delta T^f_{11} = x\frac{V_f}{2}\frac{h_{11}e^{2A}(2Q^2-1)}{Q^3} \tag{F.9}$$

$$\delta T^f_{33} = x\frac{V_f}{2}(h_{33}e^{2A+2W}Q - e^{-2A+2W}\frac{w^2B_m^2h_{11}}{Q}) \tag{F.10}$$

$$\delta T^f_{55} = x\frac{V_f}{2f}(h_{55}e^{2B}Q - e^{2B-4A}\frac{w^2B_m^2h_{11}}{Q}) \tag{F.11}$$

From the glue sector we get:

$$\begin{aligned}
\delta T^g_{00} &= \frac{f}{2}e^{2A}V_g - \frac{f^2}{4}e^{2A-2B}(h_{00}-h_{55}) \\
\delta T^g_{11} &= -\frac{1}{2}e^{2A}V_g + \frac{f}{4}e^{2A-2B}(h_{11}-h_{55}) \\
\delta T^g_{33} &= -\frac{1}{2}e^{2A+2W}V_g + \frac{f}{4}e^{2A+2W-2B}(h_{33}-h_{55}) \\
\delta T^g_{55} &= -\frac{1}{2}e^{2B}V_g
\end{aligned}$$

Now we can derive the fluctuation equations, the procedure is as follows: use the 05 Einstein equation to derive and expression for h_{55}, then use the 55 Einstein equation to get an expression for h'_{00}, and then use this to get two coupled differential equations with only h_{11} and h_{33} in it from the 11 and 33 Einstein equations. We solve for g''_{11} and g''_{33}. The 00 Einstein equation becomes trivial upon substitution of the equation for the h_{55} mode and the background equations. What we get is:

$$h_{55} = \frac{-2h_{11}f' - h_{33}f' + 4fh'_{11} + 2fh'_{33} + 2fh_{33}W'}{2f\,(3A'+W')} \tag{F.12}$$

$$\begin{aligned}
h'_{00} = \frac{2}{3A'+W'}(&-\frac{1}{8f^2(3A'+W')}(2f'(3A'+W') + f(12A'W' + 24A'^2 - 1))(2h_{11}f' \\
&+ h_{33}(f' - 2fW') - 2f(2h'_{11} + h'_{33})) \\
&- \frac{4B_m^2xfh_{11}V_fw^2e^{2B-4A}}{\sqrt{B_m^2e^{-4A}w^2+1}} + 2ff'(2h'_{11} + h'_{33}) - 3A'h'_{11} \\
&- \frac{3}{2}A'h'_{33} - \frac{\omega^2e^{2B-2A}(2h_{11}+h_{33})}{2f^2} - h'_{11}W')
\end{aligned} \tag{F.13}$$

This is the g''_{33} equation:

$$
\begin{aligned}
&\frac{1}{8fQ^3w(3A'+W')^3}(e^{-2B}(-72e^{2A+2B+2W}xfg_{11}V_fwA'^3Q^4+72e^{2A+2B}xfg_{33}V_fwA'^3Q^4 \\
&-24e^{2A+2B+2W}xfg_{11}V_fwA'W'^2Q^4-8e^{2A+2B}xfg_{33}V_fwA'W'^2Q^4 \\
&+4e^{2A+2B+2W}xfg_{11}wV_f'W'^2Q^4+4e^{2A+2B+2W}xfg_{11}V_fw'W'^2Q^4 \\
&+12e^{2A+2B+2W}xfg_{11}V_fwA'Q^4+6e^{2A+2B}xfg_{33}V_fwA'Q^4-24e^{2A+2B+2W}xg_{11}V_fwA'^2f'Q^4 \\
&-12e^{2A+2B}xg_{33}V_fwA'^2f'Q^4+48e^{2A+2B+2W}xfV_fwA'^2g_{11}'Q^4+24e^{2A+2B}xfV_fwA'^2g_{33}'Q^4 \\
&+36e^{2A+2B+2W}xfg_{11}wA'^2V_f'Q^4+36e^{2A+2B+2W}xfg_{11}V_fA'^2w'Q^4 \\
&-96e^{2A+2B+2W}xfg_{11}V_fwA'^2W'Q^4+4e^{2A+2B+2W}xfg_{11}V_fwW'Q^4 \\
&+2e^{2A+2B}xfg_{33}V_fwW'Q^4-8e^{2A+2B+2W}xg_{11}V_fwA'f'W'Q^4-4e^{2A+2B}xg_{33}V_fwA'f'W'Q^4 \\
&+16e^{2A+2B+2W}xfV_fwA'g_{11}'W'Q^4+8e^{2A+2B}xfV_fwA'g_{33}'W'Q^4 \\
&+24e^{2A+2B+2W}xfg_{11}wA'V_f'W'Q^4+24e^{2A+2B+2W}xfg_{11}V_fA'w'W'Q^4 \\
&+72e^{2B+2W}\omega^2g_{11}wA'^3Q^3-72e^{2B}\omega^2g_{33}wA'^3Q^3-4e^{2A}f^2g_{33}wW'^3Q^3 \\
&+16e^{2A}fg_{33}wA'f'W'^3Q^3+24e^{2A+2W}f^2wA'g_{11}'W'^3Q^3-96e^{2A}f^2g_{33}wA'^3W'^2Q^3+16e^{2A+2W}g_{11}wA'f'^2W'^2Q^3 \\
&+8e^{2A}g_{33}wA'f'^2W'^2Q^3+24e^{2B+2W}\omega^2g_{11}wA'W'^2Q^3-28e^{2A}f^2g_{33}wA'W'^2Q^3-4e^{2A}f^2g_{33}wB'W'^2Q^3 \\
&+96e^{2A+2W}fg_{11}wA'^2f'W'^2Q^3+96e^{2A}fg_{33}wA'^2f'W'^2Q^3+4e^{2A+2W}fg_{11}wf'W'^2Q^3+2e^{2A}fg_{33}wf'W'^2Q^3 \\
&-12e^{2A+2W}f^2wg_{11}'W'^2Q^3-24e^{2A+2W}f^2wA'B'g_{11}'W'^2Q^3-8e^{2A+2W}fwA'f'g_{11}'W'^2Q^3-24e^{2A}f^2wA'^2g_{33}'W'^2Q^3 \\
&+2e^{2A}f^2wg_{33}'W'^2Q^3-16e^{2A}fwA'f'g_{33}'W'^2Q^3-12e^{2B+2W}\omega^2g_{11}wA'Q^3-6e^{2B}\omega^2g_{33}wA'Q^3+2e^{2A+2W}fg_{11}wf'Q^3 \\
&+e^{2A}fg_{33}wf'Q^3-12e^{2A+2W}fg_{11}wA'B'f'Q^3-6e^{2A}fg_{33}wA'B'f'Q^3+288e^{2A+2W}f^2wA'^4g_{11}'Q^3 \\
&-48e^{2A+2W}f^2wA'^2g_{11}'Q^3-4e^{2A+2W}f^2wg_{11}'Q^3-72e^{2A+2W}f^2wA'^3B'g_{11}'Q^3+36e^{2A+2W}f^2wA'B'g_{11}'Q^3 \\
&+72e^{2A+2W}fwA'^3f'g_{11}'Q^3-12e^{2A+2W}fwA'f'g_{11}'Q^3-288e^{2A}f^2wA'^4g_{33}'Q^3-24e^{2A}f^2wA'^2g_{33}'Q^3 \\
&-2e^{2A}f^2wg_{33}'Q^3+72e^{2A}f^2wA'^3B'g_{33}'Q^3+18e^{2A}f^2wA'B'g_{33}'Q^3-72e^{2A}fwA'^3f'g_{33}'Q^3-6e^{2A}fwA'f'g_{33}'Q^3 \\
&+96e^{2B+2W}\omega^2g_{11}wA'^2W'Q^3-24e^{2B}\omega^2g_{33}wA'^2W'Q^3+48e^{2A+2W}g_{11}wA'^2f'^2W'Q^3+24e^{2A}g_{33}wA'^2f'^2W'Q^3 \\
&-4e^{2B+2W}\omega^2g_{11}wW'Q^3-2e^{2B}\omega^2g_{33}wW'Q^3+2e^{2A}f^2g_{33}wW'Q^3-12e^{2A}f^2g_{33}wA'B'W'Q^3 \\
&+192e^{2A+2W}fg_{11}wA'^3f'W'Q^3+96e^{2A}fg_{33}wA'^3f'W'Q^3-4e^{2A+2W}fg_{11}wA'f'W'Q^3-2e^{2A}fg_{33}wA'f'W'Q^3 \\
&-4e^{2A+2W}fg_{11}wB'f'W'Q^3-2e^{2A}fg_{33}wB'f'W'Q^3+72e^{2A+2W}f^2wA'^3g_{11}'W'Q^3-20e^{2A+2W}f^2wA'g_{11}'W'Q^3 \\
&-96e^{2A+2W}f^2wA'^2B'g_{11}'W'Q^3+12e^{2A+2W}f^2wB'g_{11}'W'Q^3-4e^{2A+2W}fwf'g_{11}'W'Q^3-72e^{2A}f^2wA'^3g_{33}'W'Q^3 \\
&+14e^{2A}f^2wA'g_{33}'W'Q^3+24e^{2A}f^2wA'^2B'g_{33}'W'Q^3+6e^{2A}f^2wB'g_{33}'W'Q^3-72e^{2A}fwA'^2f'g_{33}'W'Q^3 \\
&-2e^{2A}fwf'g_{33}'W'Q^3+72e^{2A+2W}f^2wA'^3g_{11}''Q^3+24e^{2A+2W}f^2wA'W'^2g_{11}''Q^3-12e^{2A+2W}f^2wA'g_{11}''Q^3 \\
&+96e^{2A+2W}f^2wA'^2W'g_{11}''Q^3-4e^{2A+2W}f^2wW'g_{11}''Q^3-72e^{2A}f^2wA'^3g_{33}''Q^3-6e^{2A}f^2wA'g_{33}''Q^3 \\
&-24e^{2A}f^2wA'^2W'g_{33}''Q^3-2e^{2A}f^2wW'g_{33}''Q^3-72e^{2A+2B}xfg_{33}V_fwA'^3Q^2+16e^{2A+2B+2W}xfg_{11}V_fwA'W'^2Q^2 \\
&+8e^{2A+2B}xfg_{33}V_fwA'W'^2Q^2-4e^{2A+2B+2W}xfg_{11}wV_f'W'^2Q^2-12e^{2A+2B+2W}xfg_{11}V_fwA'Q^2 \\
&-6e^{2A+2B}xfg_{33}V_fwA'Q^2+24e^{2A+2B+2W}xg_{11}V_fwA'^2f'Q^2+12e^{2A+2B}xg_{33}V_fwA'^2f'Q^2 \\
&-48e^{2A+2B+2W}xfV_fwA'^2g_{11}'Q^2-24e^{2A+2B}xfV_fwA'^2g_{33}'Q^2-36e^{2A+2B+2W}xfg_{11}wA'^2V_f'Q^2 \\
&+48e^{2A+2B+2W}xfg_{11}V_fwA'^2W'Q^2-4e^{2A+2B+2W}xfg_{11}V_fwW'Q^2-2e^{2A+2B}xfg_{33}V_fwW'Q^2 \\
&+8e^{2A+2B+2W}xg_{11}V_fwA'f'W'Q^2+4e^{2A+2B}xg_{33}V_fwA'f'W'Q^2-16e^{2A+2B+2W}xfV_fwA'g_{11}'W'Q^2 \\
&-8e^{2A+2B}xfV_fwA'g_{33}'W'Q^2-24e^{2A+2B+2W}xfg_{11}wA'V_f'W'Q^2+72e^{2A+2B+2W}xfg_{11}V_fwA'^3 \\
&+8e^{2A+2B+2W}xfg_{11}V_fwA'W'^2-4e^{2A+2B+2W}xfg_{11}V_fw'W'^2-36e^{2A+2B+2W}xfg_{11}V_fA'^2w' \\
&+48e^{2A+2B+2W}xfg_{11}V_fwA'^2W'-24e^{2A+2B+2W}xfg_{11}V_fA'w'W'))=0
\end{aligned}
\tag{F.14}
$$

The g''_{11} equation:

$$
\begin{aligned}
&\frac{1}{8fQ^3w(3A'+W')^3}(e^{-2B-2W}(36e^{2A+2B+2W}xfg_{11}V_fwA'^3Q^4 \\
&-36e^{2A+2B}xfg_{33}V_fwA'^3Q^4+12e^{2A+2B+2W}xfg_{11}V_fwW'^3Q^4 \\
&+4e^{2A+2B}xfg_{33}V_fwW'^3Q^4+60e^{2A+2B+2W}xfg_{11}V_fwA'W'^2Q^4 \\
&+4e^{2A+2B}xfg_{33}V_fwA'W'^2Q^4+4e^{2A+2B+2W}xg_{11}V_fwf'W'^2Q^4 \\
&+2e^{2A+2B}xg_{33}V_fwf'W'^2Q^4-8e^{2A+2B+2W}xfV_fwg'_{11}W'^2Q^4-4e^{2A+2B}xfV_fwg'_{33}W'^2Q^4 \\
&+4e^{2A+2B+2W}xfg_{11}wV'_fW'^2Q^4+4e^{2A+2B+2W}xfg_{11}V_fw'W'^2Q^4 \\
&+12e^{2A+2B+2W}xfg_{11}V_fwA'Q^4+6e^{2A+2B}xfg_{33}V_fwA'Q^4+12e^{2A+2B+2W}xg_{11}V_fwA'^2f'Q^4 \\
&+6e^{2A+2B}xg_{33}V_fwA'^2f'Q^4-24e^{2A+2B+2W}xfV_fwA'^2g'_{11}Q^4-12e^{2A+2B}xfV_fwA'^2g'_{33}Q^4 \\
&+36e^{2A+2B+2W}xfg_{11}wA'^2V'_fQ^4+36e^{2A+2B+2W}xfg_{11}V_fA'^2w'Q^4 \\
&+84e^{2A+2B+2W}xfg_{11}V_fwA'^2W'Q^4-36e^{2A+2B}xfg_{33}V_fwA'^2W'Q^4 \\
&+4e^{2A+2B+2W}xfg_{11}V_fwW'Q^4+2e^{2A+2B}xfg_{33}V_fwW'Q^4 \\
&+16e^{2A+2B+2W}xg_{11}V_fwA'f'W'Q^4+8e^{2A+2B}xg_{33}V_fwA'f'W'Q^4-32e^{2A+2B+2W}xfV_fwA'g'_{11}W'Q^4 \\
&-16e^{2A+2B}xfV_fwA'g'_{33}W'Q^4+24e^{2A+2B+2W}xfg_{11}wA'V'_fW'Q^4+24e^{2A+2B+2W}xfg_{11}V_fA'w'W'Q^4 \\
&-8e^{2A}fg_{33}wf'W'^4Q^3-12e^{2A+2W}f^2wg'_{11}W'^4Q^3-36e^{2B+2W}\omega^2g_{11}wA'^3Q^3+36e^{2B}\omega^2g_{33}wA'^3Q^3 \\
&+48e^{2A}f^2g_{33}wA'^2W'^3Q^3-8e^{2A+2W}g_{11}wf'^2W'^3Q^3-4e^{2A}g_{33}wf'^2W'^3Q^3-12e^{2B+2W}\omega^2g_{11}wW'^3Q^3 \\
&-48e^{2A+2W}fg_{11}wA'f'W'^3Q^3-56e^{2A}fg_{33}wA'f'W'^3Q^3-12e^{2A+2W}f^2wA'g'_{11}W'^3Q^3+12e^{2A+2W}f^2wB'g'_{11}W'^3Q^3 \\
&+4e^{2A+2W}fwf'g'_{11}W'^3Q^3+12e^{2A}f^2wA'g'_{33}W'^3Q^3+8e^{2A}fwf'g'_{33}W'^3Q^3+48e^{2A}f^2g_{33}wA'^3W'^2Q^3 \\
&-32e^{2A+2W}g_{11}wA'f'^2W'^2Q^3-16e^{2A}g_{33}wA'f'^2W'^2Q^3-60e^{2B+2W}\omega^2g_{11}wA'W'^2Q^3+12e^{2B}\omega^2g_{33}wA'W'^2Q^3 \\
&-16e^{2A}f^2g_{33}wA'W'^2Q^3-4e^{2A}f^2g_{33}wB'W'^2Q^3-144e^{2A+2W}fg_{11}wA'^2f'W'^2Q^3 \\
&-96e^{2A}fg_{33}wA'^2f'W'^2Q^3+8e^{2A+2W}fg_{11}wf'W'^2Q^3+4e^{2A}fg_{33}wf'W'^2Q^3-36e^{2A+2W}f^2wA'^2g'_{11}W'^2Q^3 \\
&-20e^{2A+2W}f^2wg'_{11}W'^2Q^3+60e^{2A+2W}f^2wA'B'g'_{11}W'^2Q^3+4e^{2A+2W}fwA'f'g'_{11}W'^2Q^3+48e^{2A}f^2wA'^2g'_{33}W'^2Q^3 \\
&-2e^{2A}f^2wg'_{33}W'^2Q^3-12e^{2A}f^2wA'B'g'_{33}W'^2Q^3+44e^{2A}fwA'f'g'_{33}W'^2Q^3-12e^{2B+2W}\omega^2g_{11}wA'Q^3 \\
&-6e^{2B}\omega^2g_{33}wA'Q^3+2e^{2A+2W}fg_{11}wf'Q^3+e^{2A}fg_{33}wf'Q^3-12e^{2A+2W}fg_{11}wA'B'f'Q^3-6e^{2A}fg_{33}wA'B'f'Q^3 \\
&-144e^{2A+2W}f^2wA'^4g'_{11}Q^3-48e^{2A+2W}f^2wA'^2g'_{11}Q^3-4e^{2A+2W}f^2wg'_{11}Q^3+36e^{2A+2W}f^2wA'^3B'g'_{11}Q^3 \\
&+36e^{2A+2W}f^2wA'B'g'_{11}Q^3-36e^{2A+2W}fwA'^3f'g'_{11}Q^3-12e^{2A+2W}fwA'f'g'_{11}Q^3+144e^{2A}f^2wA'^4g'_{33}Q^3 \\
&-24e^{2A}f^2wA'^2g'_{33}Q^3-2e^{2A}f^2wg'_{33}Q^3-36e^{2A}f^2wA'^3B'g'_{33}Q^3+18e^{2A}f^2wA'B'g'_{33}Q^3+36e^{2A}fwA'^3f'g'_{33}Q^3 \\
&-6e^{2A}fwA'f'g'_{33}Q^3-84e^{2B+2W}\omega^2g_{11}wA'^2W'Q^3+48e^{2B}\omega^2g_{33}wA'^2W'Q^3-24e^{2A+2W}g_{11}wA'^2f'^2W'Q^3 \\
&-12e^{2A}g_{33}wA'^2f'^2W'Q^3-4e^{2B+2W}\omega^2g_{11}wW'Q^3-2e^{2B}\omega^2g_{33}wW'Q^3+2e^{2A}f^2g_{33}wW'Q^3-12e^{2A}f^2g_{33}wA'B'W'Q^3 \\
&-96e^{2A+2W}fg_{11}wA'^3f'W'Q^3-48e^{2A}fg_{33}wA'^3f'W'Q^3+8e^{2A+2W}fg_{11}wA'f'W'Q^3+4e^{2A}fg_{33}wA'f'W'Q^3 \\
&-4e^{2A+2W}fg_{11}wB'f'W'Q^3-2e^{2A}fg_{33}wB'f'W'Q^3-180e^{2A+2W}f^2wA'^3g'_{11}W'Q^3-44e^{2A+2W}f^2wA'g'_{11}W'Q^3 \\
&+84e^{2A+2W}f^2wA'^2B'g'_{11}W'Q^3+12e^{2A+2W}f^2wB'g'_{11}W'Q^3-36e^{2A+2W}fwA'^2f'g'_{11}W'Q^3-4e^{2A+2W}fwf'g'_{11}W'Q^3 \\
&+180e^{2A}f^2wA'^3g'_{33}W'Q^3+2e^{2A}f^2wA'g'_{33}W'Q^3-48e^{2A}f^2wA'^2B'g'_{33}W'Q^3+6e^{2A}f^2wB'g'_{33}W'Q^3 \\
&+72e^{2A}fwA'^2f'g'_{33}W'Q^3-2e^{2A}fwf'g'_{33}W'Q^3-36e^{2A+2W}f^2wA'^3g''_{11}Q^3-12e^{2A+2W}f^2wW'^3g''_{11}Q^3 \\
&-60e^{2A+2W}f^2wA'W'^2g''_{11}Q^3-12e^{2A+2W}f^2wA'g''_{11}Q^3-84e^{2A+2W}f^2wA'^2W'g''_{11}Q^3-4e^{2A+2W}f^2wW'g''_{11}Q^3 \\
&+36e^{2A}f^2wA'^3g''_{33}Q^3+12e^{2A}f^2wA'W'^2g''_{33}Q^3-6e^{2A}f^2wA'g''_{33}Q^3+48e^{2A}f^2wA'^2W'g''_{33}Q^3-2e^{2A}f^2wW'g''_{33}Q^3 \\
&+36e^{2A+2B}xfg_{33}V_fwA'^3Q^2-8e^{2A+2B+2W}xfg_{11}V_fwW'^3Q^2-4e^{2A+2B}xfg_{33}V_fwW'^3Q^2 \\
&-32e^{2A+2B+2W}xfg_{11}V_fwA'W'^2Q^2-4e^{2A+2B}xfg_{33}V_fwA'W'^2Q^2 \\
&-4e^{2A+2B+2W}xg_{11}V_fwf'W'^2Q^2-2e^{2A+2B}xg_{33}V_fwf'W'^2Q^2+8e^{2A+2B+2W}xfV_fwg'_{11}W'^2Q^2 \\
&+4e^{2A+2B}xfV_fwg'_{33}W'^2Q^2-4e^{2A+2B+2W}xfg_{11}wV'_fW'^2Q^2-12e^{2A+2B+2W}xfg_{11}V_fwA'Q^2-6e^{2A+2B}xfg_{33}V_fwA'Q^2 \\
&-12e^{2A+2B+2W}xg_{11}V_fwA'^2f'Q^2-6e^{2A+2B}xg_{33}V_fwA'^2f'Q^2+24e^{2A+2B+2W}xfV_fwA'^2g'_{11}Q^2 \\
&+12e^{2A+2B}xfV_fwA'^2g'_{33}Q^2-36e^{2A+2B+2W}xfg_{11}wA'^2V'_fQ^2-24e^{2A+2B+2W}xfg_{11}V_fwA'^2W'Q^2 \\
&+36e^{2A+2B}xfg_{33}V_fwA'^2W'Q^2-4e^{2A+2B+2W}xfg_{11}V_fwW'Q^2-2e^{2A+2B}xfg_{33}V_fwW'Q^2 \\
&-16e^{2A+2B+2W}xg_{11}V_fwA'f'W'Q^2-8e^{2A+2B}xg_{33}V_fwA'f'W'Q^2+32e^{2A+2B+2W}xfV_fwA'g'_{11}W'Q^2 \\
&+16e^{2A+2B}xfV_fwA'g'_{33}W'Q^2-24e^{2A+2B+2W}xfg_{11}wA'V'_fW'Q^2-36e^{2A+2B+2W}xfg_{11}V_fwA'^3 \\
&-4e^{2A+2B+2W}xfg_{11}V_fwW'^3-28e^{2A+2B+2W}xfg_{11}V_fwA'W'^2-4e^{2A+2B+2W}xfg_{11}V_fw'W'^2 \\
&-36e^{2A+2B+2W}xfg_{11}V_fA'^2w'-60e^{2A+2B+2W}xfg_{11}V_fwA'^2W'-24e^{2A+2B+2W}xfg_{11}V_fA'w'W')) \\
&=0
\end{aligned}
\tag{F.15}
$$

At the horizon we have:

$$\begin{aligned}\frac{e^{2A-2B}}{2}(6A'W'+6A'^2-1)fh''_{11}+\frac{e^{2A-2B}}{2}(-2A'W'+6A'^2-1)f'h'_{11}+(6A'W'+6A'^2-1)\frac{\omega^2h_{11}}{2f}\\ -\frac{e^{2A-2B}}{4}(12A'^2+1)fh''_{33}-\frac{e^{2A-2B}}{4}(8A'W'+12A'^2+1)f'h'_{33}-(12A'^2+1)\frac{\omega^2h_{33}}{4f}\\ +\frac{B_m^2xe^{-2A}V_fw^2A'(2h'_{11}+h'_{33})}{Q}\\ +\frac{e^{-2(A+B)}}{f}(2W'f'A'e^{4A}+\frac{B_m^2xe^{2B}V_ff'w^2A'}{Q})(2h_{11}+h_{33})\approx 0\end{aligned} \tag{F.16}$$

Using equation A.20, this reduces to the following simple form:

$$\begin{aligned}e^{2A-2B}fh''_{11}+e^{2A-2B}f'h'_{11}+\frac{\omega^2h_{11}}{2f}\\ -\frac{12A'^2+1}{(6A'W'+6A'^2-1)}(fe^{2A-2B}h''_{33}+e^{2A-2B}f'h'_{33}+\frac{\omega^2h_{33}}{4f})\approx 0\end{aligned} \tag{F.17}$$

If you split this calculation up into two calculations, for which we take the boundary condition $g_{11}(\phi_h)=0$ and $g_{33}(\phi_h)=0$ respectively it reduces to the same simple horizon EOM for a fluctuation. For the 11 equation at the horizon we get same thing:

$$\begin{aligned}-(6A'W'+3A'^2+3W'^2+1)\frac{e^{2A-2B}}{2}fh''_{11}+\frac{e^{2A-2B}}{2}\\ -\frac{1}{2}(2A'W'+3A'^2-W'(r)^2+1)f'h'_{11}-(6A'W'+3A'^2+3W'^2+1)\frac{\omega^2h_{11}}{2f}\\ (6A'W'+6A'^2-1)\frac{e^{2A-2B}}{4}(12A'^2+1)fh''_{33}+\\ \frac{e^{2A-2B}}{4}(10A'W'+6A'^2+4W'^2-1)f'h'_{33}+(6A'W'+6A'^2-1)\frac{\omega^2h_{33}}{4f}\\ -\frac{B_m^2xe^{-2A}V_fw^2A'(2h'_{11}+h'_{33})}{2Q}\\ +\frac{e^{-2(A+B)}}{f}(2W'f'A'e^{4A}+\frac{B_m^2xe^{2B}V_ff'w^2A'}{Q})(2h_{11}+h_{33})\approx 0\end{aligned} \tag{F.18}$$

Again we get that using equation A.20, this reduces to a very simple form:

$$\begin{aligned}e^{2A-2B}fh''_{11}+e^{2A-2B}f'h'_{11}+\frac{\omega^2h_{11}}{2f}\\ -\frac{6A'W'+6A'^2-1}{6A'W'+3A'^2+3W'^2+1}(fe^{2A-2B}h''_{33}+e^{2A-2B}f'h'_{33}+\frac{\omega^2h_{33}}{4f})\approx 0\end{aligned} \tag{F.19}$$

Appendix G

Green's function

G.1 Introduction

The way we calculate the Green's function is similar to the way GPR do it [43]. We introduce the following vector: $\vec{h}^T = \begin{pmatrix} h_{00} & h_{11} & h_{33} & h_{55} \end{pmatrix}$, and decompose the second order expansion as follows:

$$S = M^3 N_c^2 \int d^5x \mathcal{L} \tag{G.1}$$

$$\mathcal{L} = \hat{\mathcal{L}} + \partial_t \hat{\mathcal{L}}^t + \partial_\phi \hat{\mathcal{L}}^\phi \tag{G.2}$$

$$\hat{\mathcal{L}} = \frac{1}{2} \vec{\dot{h}}^T M^{tt} \vec{\dot{h}} + \frac{1}{2} \vec{h}'^T M^{\phi\phi} \vec{h}' + \vec{h}'^T M^\phi \vec{h} + \frac{1}{2} \vec{h}^T M \vec{h} \tag{G.3}$$

It follows from section 4.2, we get the following equation for the flux $\mathcal{F}$:

$$\mathcal{F} = -\operatorname{Im} J = \frac{i}{2} \Big(\vec{h}'^{*T} (M^{\phi\phi} \vec{h} + M^\phi \vec{h}) - (\vec{h}'^{*T} M^{\phi\phi T} \vec{h} + \vec{h}^* M^{\phi T}) \vec{h} \Big) \tag{G.4}$$

$\mathcal{F}$ is related to the Green's function in the following way:

$$M^3 N_c^2 \mathcal{F} = -\lim_{\omega \to 0} \frac{1}{\omega} \operatorname{Im} \delta^{ij} \delta^{kl} G^R_{ij,kl}(\omega) \tag{G.5}$$

Where i is a spatial index. So $\mathcal{F}$ is the total flux, i.e. the sum over all possible Green's function contributions that are part of the helicity zero sector. For an isotropic fluid $\mathcal{F}$ is proportional to a Wronskian: $\mathcal{F}_{\text{iso}} \propto h^*_{11} h_{11} - h_{11} h^*_{11}$. For a coupled set of differential equations a Wronskian does not exist so the structure of $\mathcal{F}$ will be more complicated. We won't calculate all matrix elements for all matrices, but just the terms that contribute to the Green's function, this follows from equation 4.20 in section 4.2. We see that for fluxes relating to $\zeta_\perp$ and $\zeta_\parallel$, we have:

$$\mathcal{F}_\perp \propto h'^*_{11} h_{11} - h'_{11} h^*_{11}|_{\phi=\phi_h} \tag{G.6}$$

$$\mathcal{F}_\parallel \propto h'^*_{33} h_{33} - h'_{33} h^*_{33}|_{\phi=\phi_h} \tag{G.7}$$

We can thus use a method very similar to the GPR method. For ζ_m it becomes a lot more messy because we also have a contribution to the flux $\propto h_{11} h_{33} - h_{11} h^*_{33}$. Furthermore, $c_{b11} c^*_{b33}$ is not necessarily real, meaning that the expansion that was done earlier for shear in equation C.16, which allowed us to decompose the product of perturbative modes into real part constant for ω and an imaginary part linear for ω now looks as follows for the one-derivative part of the mixed flux:

$$\begin{aligned}
\mathcal{F}_{m1d} &\propto -\frac{i}{2}e^{A-B}f(h_{11}'^{*}h_{33} - h_{11}'h_{33}^{*}) + 1 \leftrightarrow 3 \\
&= -\frac{i}{2}\Big(c_{11}^{*}\Big(1+\frac{i\omega}{4\pi T}\log(\phi_h-\phi)\Big)\Big)\partial_\phi\Big(c_{33}\Big(1-\frac{i\omega}{4\pi T}\log(\phi_h-\phi)\Big)\Big) \\
&+ \frac{i}{2}\partial_\phi\Big(c_{33}^{*}\Big(1+\frac{i\omega}{4\pi T}\log(\phi_h-\phi)\Big)\Big)\Big(c_{11}\Big(1-\frac{i\omega}{4\pi T}\log(\phi_h-\phi)\Big)\Big) + 1 \leftrightarrow 3 \\
&= -\frac{i}{2}\Big(c_{11}^{*}\Big(1+\frac{i\omega}{4\pi T}\log(\phi_h-\phi)\Big)\Big)\Big(c_{33}\frac{i\omega}{4\pi T}\frac{1}{\phi_h-\phi}\Big) \\
&- \frac{i}{2}\Big(c_{33}^{*}\frac{i\omega}{4\pi T}\frac{1}{\phi_h-\phi}\Big)\Big(c_{11}\Big(1-\frac{i\omega}{4\pi T}\log(\phi_h-\phi)\Big)\Big) + 1 \leftrightarrow 3 \\
&= \frac{1}{2}(c_{11}^{*}c_{33} + c_{11}c_{33}^{*})\omega + \frac{i\omega^2}{4\pi T}(c_{11}^{*}c_{33} - c_{11}c_{33}^{*})\log(\phi_h-\phi) + 1 \leftrightarrow 3 \\
&= 2\mathrm{Re}(c_{11}^{*}c_{33})\omega
\end{aligned} \tag{G.8}$$

For ζ_m a much larger fraction of the work needs to be done numerically, we will comment on this later. For now we will will focus on $\zeta_\perp$ and $\zeta_\parallel$.

G.2 Orthogonal and parallel term

We can then write the $\hat{\mathcal{L}}$ with the terms that contribute for these bulk viscosities:

$$\hat{\mathcal{L}} = \frac{1}{2}\vec{h}'^{T}M^{\phi\phi}\vec{h}' + \vec{h}'^{T}M_5^{\phi}h_{55} + \text{non-contributing terms} \tag{G.9}$$

The second order Lagrangian is calculated as follows:

$$\mathcal{L} = \frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\Big(e^{4A+W+B}R(\lambda)(1+\frac{\lambda h_{00}}{2})(1+\frac{\lambda h_{11}}{2})^2(1+\frac{\lambda h_{33}}{2})(1+\frac{\lambda h_{55}}{2})\Big), \quad \lambda\to 0$$

We perform the computations in the $\phi = r$-gauge or Gubser gauge, however the end result will be independent of the gauge choice. We work with the following Ansatz:

$$\begin{aligned}
ds^2 &= e^{2A(\phi)}(-f(\phi)(1+\lambda h_{00}(\phi,t))dt^2 + (1+\lambda h_{11}(\phi,t))(dx^2+dy^2) \\
&+ (e^{2W(\phi)} + \lambda h_{33}(\phi,t))^2dz^2) + (1+\lambda h_{55}(\phi,t)\frac{e^{2B(\phi)}}{f(\phi)}d\phi^2
\end{aligned} \tag{G.10}$$

However it turns out that harmonic time dependence allows us to immediately discard all the terms which involve derivatives with respect to time. For the double derivative sector we get:

$$\begin{aligned}
\mathcal{L}_{\phi\phi} &= -\frac{e^{2A-B-W}}{4f}(2e^{2A}f^2(2h_{11}(e^{2W}h_{00}'' + e^{2W}h_{11}'' + h_{33}'') \\
&+ 2e^{2W}h_{00}'h_{11}' + h_{00}'h_{33}' + h_{00}''h_{33} - e^{2W}h_{00}'h_{55}' - e^{2W}h_{00}''h_{55} \\
&+ h_{00}(2e^{2W}h_{11}'' + h_{33}'') + 2h_{11}'h_{33}' + 2h_{11}''h_{33} - 2e^{2W}h_{11}'h_{55}' \\
&- 2e^{2W}h_{11}''h_{55} + e^{2W}h_{11}'^{2} - h_{33}'h_{55}' - h_{33}''h_{55}))
\end{aligned} \tag{G.11}$$

Subtracting total derivatives yields:

$$\hat{\mathcal{L}}_{\phi\phi} = \frac{1}{2} f e^{4A-B-W}(h'_{00}(2e^{2W}h'_{11} + h'_{33}) + h'_{11}(e^{2W}h'_{11} + 2h'_{33})) \tag{G.12}$$

For the 55 sector So:

$$M^{\phi\phi} = -fe^{4A+B+W}\begin{pmatrix} 0 & 1 & \frac{1}{2}e^{-2W} & 0 \\ 1 & 1 & e^{-2W} & 0 \\ \frac{1}{2}e^{-2W} & e^{-2W} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{G.13}$$

There is a factor -1 because as explained in section 4.2, we don't look to for the boundary term itself, but instead for the current that is induced at the horizon due to this boundary term so that this boundary term is canceled. The result is identical to what Critelli finds in [28]. Similarly, for the 55 sector:

$$\begin{aligned}\mathcal{L}_{55} = &-\frac{1}{4f}(h'_{55}e^{2A-B-W}(-8f^2e^{2A+2W}A'h_{00} - 16f^2e^{2A+2W}A'h_{11} - 8e^{2A}f^2A'h_{33} - fe^{2A+2W}f'h_{00} \\ &- 2fe^{2A+2W}f'h_{11} - e^{2A}ff'h_{33} - 2f^2e^{2A+2W}W'h_{00} - 4f^2e^{2A+2W}W'h_{11} + 2e^{2A}f^2W'h_{33})) \\ &- \frac{1}{4f}(h'_{55}e^{2A-B-W}(-8f^2e^{2A+2W}A'h_{00} - 16f^2e^{2A+2W}A'h_{11} - 8e^{2A}f^2A'h_{33} - fe^{2A+2W}f'h_{00} \\ &- 2fe^{2A+2W}f'h_{11} - e^{2A}ff'h_{33} - 2f^2e^{2A+2W}W'h_{00} - 4f^2e^{2A+2W}W'h_{11} + 2e^{2A}f^2W'h_{33}))\end{aligned} \tag{G.14}$$

Subtraction of the total derivatives gives:

$$\begin{aligned}\hat{\mathcal{L}}_{55} = &-\frac{1}{4f}h'_{55}e^{2A-B-W}(-8f^2e^{2A+2W}A'h_{00} - 16f^2e^{2A+2W}A'h_{11} - 8e^{2A}f^2A'h_{33} - fe^{2A+2W}f'h_{00} \\ &- 2fe^{2A+2W}f'h_{11} - e^{2A}ff'h_{33} - 2f^2e^{2A+2W}W'h_{00} - 4f^2e^{2A+2W}W'h_{11} + 2e^{2A}f^2W'h_{33})\end{aligned} \tag{G.15}$$

This term is different but the total derivative from the double derivative sector also changes:

$$\begin{aligned}\mathcal{L}_{TD} = &\frac{1}{4}f(4A' - B' + \frac{f'}{f} + W')(e^{2A-B+W}(2e^{2A}(h_{00}(-h_{55}) - 2h_{11}h_{55}))) \\ &- \frac{1}{4}(4A' - B' + \frac{f'}{f} - W')f(2e^{4A-B+W})(e^{-2W}h_{33}h_{55})\end{aligned} \tag{G.16}$$

So we find:

$$M_5^{\phi} = -\frac{e^{2A+B+W}}{4}\begin{pmatrix} 6A'h + 2W'h \\ 12A'h + 4W'h + 2h' \\ (6A'h + h')e^{-2W} \\ 0 \end{pmatrix} \tag{G.17}$$

We now have:

$$h_{55} = \frac{4fh'_{11} + 2fe^{-2W}h'_{33}}{2f\,(3A' + W')} + \text{non-contributing terms} \tag{G.18}$$

$$\begin{aligned}h'_{00} = &\frac{f'(3A' + W')(2h'_{11} + e^{-2W}h'_{33}) + f(2h'_{11}(6A'^2 - 2W'^2 - 1) + e^{-2W}h'_{33}(6A'W' + 6A'^2 - 1))}{2f(3A' + W')^2} \\ &+ \text{non-contributing terms}\end{aligned} \tag{G.19}$$

Yielding:

$$\mathcal{F} = \{\frac{fe^{4A-B+W}}{4(3A'+W')^2}\Big(h_{33}e^{-2W}((12A'^2+1)h'_{33}e^{-2W} - 2h'_{11}(6A'W'+6A'^2-1)) \\ +2h_{11}(2h'_{11}(6A'W'+3A'^2+3W'^2+1) + h'_{33}e^{-2W}(-6A'W'-6A'^2+1))\Big)\}_{\phi\to\phi_h} \tag{G.20}$$

Similar to Appendix C, we find:

$$\frac{\eta_\perp + \zeta_\perp}{s} = \frac{|c_{b11}^2|(6A'^2+3A'W'+3W'^2+1)}{4\pi(3A'+W')^2}\Big|_{\phi=\phi_h} \tag{G.21}$$

$$\frac{\zeta_\parallel}{s} = \frac{|c_{b33}^2|\left(12A'^2+1\right)}{4\pi\left(3A'+W'\right)^2}\Big|_{\phi=\phi_h} \tag{G.22}$$

As explained in section 4.2, this does not reduce to GPR's result in the zero magnetic field limit.

G.3 Mix term

As mentioned, for ζ_m the situation is more complicated. We will need the full matrix M_ϕ:

$$\hat{\mathcal{L}} = \frac{1}{2}\vec{h}'^T M^{\phi\phi}\vec{h}' + \vec{h}'^T M^\phi \vec{h} + \text{non-contributing terms} \tag{G.23}$$

With $\mathcal{F}$ given by [43]:

$$\mathcal{F} = -\text{Im } J = -\text{ Im }\left(\frac{1}{2}\vec{h}'^{*T}M^{\phi\phi}\vec{h} + \frac{1}{2}\vec{h}^{*T}M^{\phi\phi}\vec{h}' + \vec{h}^{*T}M^\phi\vec{h}\right) \tag{G.24}$$

$$= \frac{i}{2}\left(\vec{h}^{*T}M^{\phi\phi}\vec{h}' + \vec{h}'^{*T}M^{\phi\phi}\vec{h} + \vec{h}^{*T}M^\phi\vec{h}' + \vec{h}'^{*T}M^\phi\vec{h}\right) \tag{G.25}$$

With steps similar to the ones shown previously in this Appendix, we find:

$$M_\phi = e^{4A-B+W}\begin{pmatrix} 0 & -\frac{1}{2}f' & -\frac{1}{4}\left(f'+2fW'\right)e^{-2W} & -\frac{1}{2}f\left(3A'+W'\right) \\ 0 & 0 & -fW'e^{-2W} & -\frac{1}{2}\left(6fA'+f'+2fW'\right) \\ 0 & 0 & 0 & -\frac{1}{4}\left(6fA'+f'\right)e^{-2W} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{G.26}$$

If we perform the same operation of adding total derivatives to the result in [28], we again find consistency for the upperleft 3×3 matrix (Critelli only considers this part). Here we have used the fact that for a Lagrangian you can add total derivatives to eliminate all terms on the main diagonal as well as the lower triangle. We use the EOM:

$$h_{55} = -\frac{e^{-2W}\left(2h_{11}e^{2W}f' + h_{33}f' - 4fe^{2W}h'_{11} - 2fh'_{33} + 2fh_{33}W'\right)}{2f\left(3A' + W'\right)} \tag{G.27}$$

$$\begin{aligned}
h'_{00} = {} & \frac{e^{-2(A+W)}}{4f^2Q(3A'+W')^2}(Q(2e^{2A}f(f'(3A'+W')(2e^{2W}h'_{11} + h'_{33}) \\
& + f(2e^{2W}h'_{11}(6A'^2 - 2W'^2 - 1) + \\
& h'_{33}(6A'W' + 6A'^2 - 1))) - h_{33}(6A'(2e^{2A}ff'W' + e^{2A}f'^2 + 2\omega^2e^{2B}) \\
& - 24e^{2A}fA'^2(fW' - f') + 2W'(e^{2A}f'^2 + 2\omega^2e^{2B}) - e^{2A}ff' - 2e^{2A}f^2W')) \\
& - 2h_{11}e^{2W}(Q(6A'(2e^{2A}ff'W' + e^{2A}f'^2 + 2\omega^2e^{2B}) \\
& + 24e^{2A}fA'^2f' + 2W'(e^{2A}f'^2 + 2\omega^2e^{2B}) - e^{2A}ff') \\
& - 2xfQ^2V_fe^{2(A+B)}(3A'+W') + 2xfV_fe^{2(A+B)}(3A'+W')))
\end{aligned} \tag{G.28}$$

This leads to:

$$\mathcal{F}_m = \{\frac{i}{2}P(h^*_{11}h_{33} - h_{11}h^*_{33})\}_{\phi\to\phi_h} + \text{derivative term} \tag{G.29}$$

$$\begin{aligned}
P = {} & \frac{e^{2A-B-W}}{2fQ^2(3A'+W')^2}(Q^2(A'(-3e^{2A}f'^2 + 20e^{2A}f^2W'^2 - 12\omega^2e^{2B}) \\
& + 6e^{2A}fA'^2(4fW' - f') - 36e^{2A}f^2A'^3 - W'(e^{2A}f'^2 + 4\omega^2e^{2B}) \\
& + e^{2A}ff'(2W'^2 + 1) + e^{2A}f^2W'(4W'^2 + 1)) + xf(Q^2 - 1)QV_fe^{2(A+B)}(3A'+W'))
\end{aligned} \tag{G.30}$$

Because we won't use this result anyway it has not been subjected to extensive checks (one such check would be flux conservation, see equation 4.21) but it gives some confidence that h_{00} precisely cancels as it should. However, there is an $\frac{1}{f}$ term in the result which suggests that there may be some error, it could also be because $h^*_{11}h_{33} - h_{11}h^*_{33} \propto f'(\phi_h)(\phi - \phi_h)$. When you turn off the magnetic field, the result trivially reduces to GPR's result because then the imaginary part of the non-derivative term vanishes. If we were to use this method to obtain the bulk viscosity, we would have to take a $\omega \to 0$ limit for the entire thing, i.e. we would have to solve the perturbative EOM for many different ω values near $\omega = 0$ and look for the linear part of $\mathcal{F}_m$ as a function of ω, as is done by Critelli in [28]. In Appendix H we explain how we could have calculated c_{b11} and c_{b33} using linearity if we would have used this method.

Appendix H

Numerical method

Because this is the way the C++ code used works, we will work in A-coordinates, with A being the term in the exponential term of our Ansatz. At the horizon our equations of motion now become:

$$\{g_{33}'' + \frac{f'}{f}g_{33}' + \frac{\omega^2}{f^2}e^{-2A}q^2(A)g_{33}\}_{r\to 0} \approx 0 \tag{H.1}$$

So the BC are:

$$g_{33} \approx (A - A_h)^{-i\frac{q\omega}{f'e^A}} \qquad A \to A_h \tag{H.2}$$

$$g_{33}' \approx \frac{d(A - A_h)^{-i\frac{q\omega}{f'e^A}}}{dA} \qquad A \to A_h \tag{H.3}$$

$$\frac{f'e^A}{q}\Big|_{A\to A_h} = \frac{df}{dr}\Big|_{A\to A_h} = 4\pi T \tag{H.4}$$

Same for g_{11}. In order to calculate c_b we use the NDSolve tool from Mathematica and the numerically calculated background described in section 3. We have a linear second order differential equation with two boundary conditions, which follow from regularity at the horizon and normalization to unity at the boundary. Regularity at the horizon is imposed in the following approximate way:

$$h_{ij}(\phi_h - \delta\phi_h) = C_{ij}\delta\phi_h^{-\frac{i\omega}{4\pi T}} \tag{H.5}$$

$$h_{ij}'(\phi_h - \delta\phi_h) = -C_{ij}\frac{\partial\delta\phi_h^{-\frac{i\omega}{4\pi T}}}{\partial\delta\phi_h} \tag{H.6}$$

Here C_{ij} is a numerical constant and $\delta\phi_h$ is a small offset from the horizon; The second boundary condition will be imposed after solved the numerical calculation by rescaling, i.e. replacing C_{ij} with the properly fixed c_{ij}, for our numerical solution. For the bulk viscosity, we have two coupled differential equations, which, similar to [66], we solve twice, once for $C_{11} = 1,\ C_{33} = 0$ and once for $C_{11} = 0,\ C_{33} = 1$. After solving we can use linear algebra to make sure that the boundary conditions are satisfied. We require:

$$\begin{pmatrix} \alpha_1 & \widetilde{\alpha}_1 \\ \alpha_3 & \widetilde{\alpha}_3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ \widetilde{c} \end{pmatrix} \tag{H.7}$$

Where α_i, $\widetilde{\alpha}_i$ are the values at the boundary for the mode h_{ii} for the respective initial boundary conditions and c, $\widetilde{c}$ are the constants for the linear combination of these two solutions. We then finally get:

$$\begin{pmatrix} c \\ \widetilde{c} \end{pmatrix} = \frac{1}{\widetilde{\alpha}_3\alpha_1 - \widetilde{\alpha}_1\alpha_3} \begin{pmatrix} \widetilde{\alpha}_3 - \widetilde{\alpha}_1 \\ -\alpha_3 + \alpha_1 \end{pmatrix} \tag{H.8}$$

However, we will extract the transport coefficients by using a method which is different from the GPR method, instead we will use a method similar to the one described in [52], where we will study the boundary expansion the follows from numerically solving the fluctuations. As explained in 4.2, we have the following UV expansion for the perturbative modes:

$$h_{ij} \approx a_{ij} + \kappa r^2\omega^2 - i\frac{b_{ij}\omega r^4 \log^2(r)}{4\pi T} \tag{H.9}$$

The bulk viscosity is given by b_{ij}/a_{ij}. Here all coefficients are real. We can find transport coefficient by looking at how operators respond linearly to sources. A source of a metric perturbation is switched on when $a_{ij} = 1$, it is switched off when $a_{ij} = 0$. This is different from the GPR method where all sources are put on. When all sources are put off except for one, the Green's function follows from looking at the linear response, which is given by:

$$-\frac{1}{\omega}\mathrm{Im}G_{ijkl} \propto -\lim_{\omega\to 0}\mathrm{Im}\frac{\partial h_{ij}}{\partial\omega} = \frac{b_{ij}}{a_{kl}} \tag{H.10}$$

For one differential equation, we could just solve numerically and rescale afterwards so that $a_{ij} = 0$, and we would be sure that b_{ij} and κ were real. However, as mentioned, when we solve set of coupled differential equation we have to solve the differential equation multiple times and make linear combinations afterwards, which could cause mixing of real and imaginary terms. Therefore, we instead explicitly expand in ω order by order [78] so that contributions from the κ term as well as contributions from ω in the EOM are second order. We do this in the following way: we first solve the EOM twice for zeroth order, once for $C_{11} = 1, C_{33} = 0$ and once for $C_{11} = 0, C_{33} = 1$. The horizon boundary conditions also has to be expanded in ω:

$$h_{ij}(\phi_h - \delta\phi_h) = C_{ij}(1 - \frac{i\omega}{4\pi T}\log(\delta\phi_h)) + \mathcal{O}(\omega^2) \tag{H.11}$$

Our horizon boundary conditions then become:

$$a_{ij}(\phi_h - \delta\phi_h) = C_{ij} \tag{H.12}$$

$$a'_{ij}(\phi_h - \delta\phi_h) = 0 \tag{H.13}$$

Regardless of what kind of boundary conditions you take at the horizon, the EOM will always have the structure of equation H.9 at the boundary. Because we look at zeroth order in ω, the expansion reduces to:

$$a_{11} = a_{11}^{(0)} + a_{11}^{(1)} r^4 \log^2(r) \tag{H.14}$$

$$\widetilde{a}_{11} = \widetilde{a}_{11}^{(0)} + \widetilde{a}_{11}^{(1)} r^4 \log^2(r) \tag{H.15}$$

We do it again for first order. Because we don't want to deal with divergences we define $e^{b_{ij}} = B_{ij}$. We then have the following boundary conditions:

$$B_{ij}(\phi_h - \delta\phi_h) = C_{ij}\delta\phi_h \tag{H.16}$$

$$B'_{ij}(\phi_h - \delta\phi_h) = -C_{ij} \tag{H.17}$$

The boundary expansion for b_{11} is identical to the one for a_{11} because dependence of ω only comes in at second order:

$$b_{11} = b_{11}^{(0)} + b_{11}^{(1)} r^4 \log^2(r) \tag{H.18}$$

$$\widetilde{b}_{11} = \widetilde{b}_{11}^{(0)} + \widetilde{b}_{11}^{(1)} r^4 \log^2(r) \tag{H.19}$$

We fix the appropriate boundary conditions at the boundary as follows. When we look at the response of a h_{11} source we have:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11}^{(0)} - i\frac{\omega}{4\pi T} b_{11}^{(0)} & \tilde{a}_{11}^{(0)} - i\frac{\omega}{4\pi T}\tilde{b}_{11}^{(0)} \\ a_{33}^{(0)} - i\frac{\omega}{4\pi T} b_{33}^{(0)} & \tilde{a}_{33}^{(0)} - i\frac{\omega}{4\pi T}\tilde{b}_{33}^{(0)} \end{pmatrix} \begin{pmatrix} c_{11}(\omega) \\ c_{33}(\omega) \end{pmatrix} \tag{H.20}$$

When we look at the response of a h_{33} source we have:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{11}^{(0)} - i\frac{\omega}{4\pi T} b_{11}^{(0)} & \tilde{a}_{11}^{(0)} - i\frac{\omega}{4\pi T}\tilde{b}_{11}^{(0)} \\ a_{33}^{(0)} - i\frac{\omega}{4\pi T} b_{33}^{(0)} & \tilde{a}_{33}^{(0)} - i\frac{\omega}{4\pi T}\tilde{b}_{33}^{(0)} \end{pmatrix} \begin{pmatrix} c_{11}(\omega) \\ c_{33}(\omega) \end{pmatrix} \tag{H.21}$$

We note that $\zeta_\perp$ is the response of $\langle T_{11}\rangle$ to a h_{11} source, $\zeta_\parallel$ is the response of $\langle T_{33}\rangle$ to a h_{33} source and ζ_m is the response of $\langle T_{11}\rangle$ to a h_{33} source or the response of $\langle T_{33}\rangle$ to a h_{11} source. We then extract the transport coefficients in the following way:

$$\begin{pmatrix} \zeta_\perp + \eta_\perp & \zeta_m \\ \zeta_m & \zeta_\parallel \end{pmatrix} = -\mathrm{Im}\frac{d}{d\omega}\left(\begin{pmatrix} a_{11}^{(1)} - i\frac{\omega}{4\pi T} b_{11}^{(1)} & \tilde{a}_{11}^{(1)} - i\frac{\omega}{4\pi T}\tilde{b}_{11}^{(1)} \\ a_{33}^{(1)} - i\frac{\omega}{4\pi T} b_{33}^{(1)} & \tilde{a}_{33}^{(1)} - i\frac{\omega}{4\pi T}\tilde{b}_{33}^{(1)} \end{pmatrix} \begin{pmatrix} a_{11}^{(0)} - i\frac{\omega}{4\pi T} b_{11}^{(0)} & \tilde{a}_{11}^{(0)} - i\frac{\omega}{4\pi T}\tilde{b}_{11}^{(0)} \\ a_{33}^{(0)} - i\frac{\omega}{4\pi T} b_{33}^{(0)} & \tilde{a}_{33}^{(0)} - i\frac{\omega}{4\pi T}\tilde{b}_{33}^{(0)} \end{pmatrix}^{-1} \mathbb{1} \right)$$

$$\omega \to 0 \tag{H.22}$$

Checking that the off-diagonal bulk viscosity results match is a useful way to check that the calculations are done working properly.

Appendix I

Finding the massless bulk mode

GPR find the constant c_b by solving the equation of motion for the bulk mode h_{11} numerically:

$$\begin{aligned} h''_{11} = -\frac{e^{-2A}}{6f^2A'}(&6e^{2A}fh_{11}A'B'f' - 18e^{2A}f^2A'B'h'_{11} + 6w^2e^{2B}h_{11}A' \\ &+ 6e^{2A}fA'f'h'_{11} + 24e^{2A}f^2A'^2h'_{11} - e^{2A}fh_{11}f' + 2e^{2A}f^2h'_{11}) \end{aligned} \tag{I.1}$$

They do this by using in-falling boundary conditions at the horizon and convergence to unity at the boundary. It is also possible to again use a method similar to section 4.5, where we looked at the mode choice which gave massless EOM, by performing the transformation:

$$h_{11} \to kZ_{11} \tag{I.2}$$

We can now fix k so that the EOM become massless for the mode Z_{11}, which we know gives trivial flow. We can prove analytically that this method is correct because the EOM for k that follows from that requirement rather trivially gives us the EOM we started with but with h_{11} replaced by k, which proves that the two approaches are identical and we thus do not gain anything. One could think that perhaps we could also use the same trick as a different and perhaps simpler way to calculate viscosities in the anisotropic case. However, in this case the EOM have the following structure when the modes have not been redefined yet to require masslessness:

$$\begin{aligned} &Mv = 0 \\ &M = \begin{pmatrix} A_{11} & 0 & C_{11} & D_{11} & (E_{11} + A_{11}\frac{e^{-2A+2B}}{f^2}\omega^2) & F_{11} \\ 0 & A_{33} & C_{33} & D_{33} & E_{33} & (F_{33} + A_{33}\frac{e^{-2A+2B}}{f^2}\omega^2) \end{pmatrix} \\ &v = \begin{pmatrix} h''_{11} & h''_{33} & h'_{11} & h'_{33} & h_{11} & h_{33} \end{pmatrix} \end{aligned} \tag{I.3}$$

With every term ϕ or r dependent. We then rescale as follows:

$$h_{11} \to kZ_{11} \tag{I.4}$$

$$h_{33} \to lZ_{33} \tag{I.5}$$

However, in order to get trivial flow one would have to fix k and l such that E_{11}, E_{33}, F_{11} and F_{33} vanish simultaneously, which is not always possible.

Appendix J

Anisotropic dissipative shear term for axion

We can decompose our projector in the following way, with b_u oriented parallel to the anisotropy direction (so in the z-direction for our metric Ansatz), and $\mathbb{B}_{\mu\nu}$ a tensor which projects to all directions apart from the anisotropy field direction:

$$\Delta_{\mu\nu} = \mathbb{B}_{\mu\nu} + b_\mu b_\nu e^{2W(\phi_h)} \tag{J.1}$$

Using what we know from chapter 5, we have the following fluid entropy balance law:

$$\partial_\mu(u^\mu s) = \frac{s}{8\pi T}\Delta^{\mu\alpha}\Delta^{\nu\beta}\sigma_{\mu\nu}\sigma_{\alpha\beta} + \text{bulk terms} \tag{J.2}$$

$$\sigma_{\mu\nu} = \Delta^\rho_\mu \Delta^\sigma_\nu \partial_{(\rho}u_{\sigma)} - \frac{2}{3}\Delta_{\mu\nu}\Delta^{\rho\sigma}\partial_\rho u_\sigma \tag{J.3}$$

Defining $S_\perp = \mathbb{B}^{\rho\sigma}\partial_\rho u_\sigma$ and $S_\parallel = b^\rho b^\sigma \partial_\rho u_\sigma$, we use the a decomposition of the shear tensor from [55], which is derived in Appendix K for an isotropic tensor. With the projector we use this shear tensor has the following form

$$\begin{aligned}\sigma_{\mu\nu} = {} & \sigma_{\perp\mu\nu} + b_\mu\Sigma_\nu + b_\nu\Sigma_\mu + \\ & b_\mu b_\nu\Big(\frac{4S_\parallel}{3} - \frac{2}{3}S_\perp e^{2W(\phi_h)}\Big) + \Big(-\frac{2}{3}\mathbb{B}_{\mu\nu}S_\parallel e^{-2W(\phi_h)} + \frac{\mathbb{B}_{\mu\nu}S_\perp}{3}\Big)\end{aligned} \tag{J.4}$$

Before we calculate $\Delta^{\mu\alpha}\Delta^{\nu\beta}\sigma_{\mu\nu}\sigma_{\alpha\beta}$, it will be useful to note that the following holds:

$$\mathbb{B}^{\mu\nu}\mathbb{B}_{\mu\nu} = 2 \tag{J.5}$$

$$\mathbb{B}^{\mu\nu}\sigma_{\perp\mu\nu} = 0 \tag{J.6}$$

The symmetry broken squared shear tensor looks as follows:

$$\begin{aligned}\Delta^{\mu\alpha}\Delta^{\nu\beta}\sigma_{\mu\nu}\sigma_{\alpha\beta} = {} & \sigma_{\mu\nu}\sigma_{\alpha\beta}(\mathbb{B}^{\mu\alpha} + b^\mu b^\alpha e^{-2W(\phi_h)})(\mathbb{B}^{\nu\beta} + b^\nu b^\beta e^{-2W(\phi_h)}) \\ = {} & \sigma_{\mu\nu}\sigma_{\alpha\beta}(\mathbb{B}^{\mu\alpha}\mathbb{B}^{\nu\beta} + 2\mathbb{B}^{\nu\beta}b^\mu b^\alpha e^{-2W(\phi_h)} + b^\mu b^\alpha b^\nu b^\beta e^{-4W(\phi_h)})\end{aligned} \tag{J.7}$$

The squared scalar terms are given by:

$$\begin{aligned}&\Big(-\frac{2}{3}\mathbb{B}_{\mu\nu}S_{\parallel}e^{-2W(\phi_h)}+\frac{\mathbb{B}_{\mu\nu}S_{\perp}}{3}\Big)^2\\&=\frac{8}{9}S_{\parallel}^2e^{-4W(\phi_h)}-\frac{8}{9}S_{\parallel}S_{\perp}e^{-2W(\phi_h)}+\frac{2}{9}S_{\perp}^2\end{aligned}\tag{J.8}$$

And:

$$\begin{aligned}&\Big(\frac{4b_\mu b_\nu S_{\parallel}}{3}-\frac{2}{3}b_\mu b_\nu S_{\perp}e^{2W(\phi_h)}\Big)^2\\&=\frac{16S_{\parallel}^2}{9}-\frac{16}{9}S_{\parallel}S_{\perp}e^{2W(\phi_h)}+\frac{4}{9}S_{\perp}^2e^{4W(\phi_h)}\end{aligned}\tag{J.9}$$

We finally arrive at:

$$\Delta^{\mu\alpha}\Delta^{\nu\beta}\sigma_{\mu\nu}\sigma_{\alpha\beta}=\sigma_{\perp}^2+2\Sigma^2e^{-2W(\phi_h)}+\frac{2S_{\perp}^2}{3}+\tag{J.10}$$

$$\frac{8}{3}S_{\parallel}^2e^{-4W(\phi_h)}-\frac{8}{9}S_{\parallel}S_{\perp}e^{-2W(\phi_h)}\tag{J.11}$$

Now we go back to the fluid entropy balance law:

$$\begin{aligned}\partial_\mu(u^\mu s)=\frac{s}{8\pi T}(\sigma_{\perp}^2+2\Sigma^2e^{-2W(\phi_h)}+\\\frac{8}{3}S_{\parallel}^2e^{-4W(\phi_h)}-\frac{8}{9}S_{\parallel}S_{\perp}e^{-2W(\phi_h)}+\frac{2S_{\perp}^2}{3})\\+\text{bulk terms}\propto\{S_{\parallel}^2,S_{\perp}S_{\parallel},S_{\perp}^2\}\end{aligned}\tag{J.12}$$

Similar to [55], we could now just read off the coefficients with the following general fluid entropy balance law.

$$\partial_\mu(u^\mu s)=\frac{1}{T}(\frac{1}{2}\eta_{\perp}\sigma_{\perp}^2+\eta_{\parallel}\Sigma^2)+\frac{1}{T}(\zeta_{\perp}S_{\perp}^2+\zeta_{\parallel}S_{\parallel}^2+2\zeta_m S_{\parallel}S_{\perp})$$

However, the new 'bulk viscosity terms' that come from decomposing the shear tensor are merely a product of the decomposition which was necessary for showing how the different terms are affected by anisotropy due to the magnetic field, they are not the product of the magnetic field itself because these terms are there without a magnetic field if you decompose the shear tensor for any direction (this is what is done in Appendix K). This again shows that, similar to what was described in section 4.2, there is not a clear map from the isotropic trace transport coefficient to the more three anisotropic helicity zero viscosities.

Appendix K

Decomposition of shear tensor

To get the decomposition given in [55], start off by writing the shear tensor as follows (i.e. project the shear tensor to everything, with the proper normalization):

$$\sigma_{\mu\nu} = (\frac{1}{2}(\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta_{\mu\beta} - \Delta_{\mu\nu}\Delta_{\alpha\beta}) + \frac{1}{2}\Delta_{\mu\nu}\Delta_{\alpha\beta})\sigma^{\alpha\beta} \tag{K.1}$$

Decompose the projector in a part that is parallel and orthogonal to the magnetic field:

$$\begin{aligned}\sigma_{\mu\nu} = (\frac{1}{2}&((\mathbb{B}_{\mu\alpha} + b_\mu b_\alpha)(\mathbb{B}_{\nu\beta} + b_\nu b_\beta) + (\mathbb{B}_{\nu\alpha} + b_\nu b_\alpha)(\mathbb{B}_{\mu\beta} + b_\mu b_\beta) \\ &-(\mathbb{B}_{\mu\nu} + b_\mu b_\nu)(\mathbb{B}_{\alpha\beta} + b_\alpha b_\beta)) + \frac{1}{2}(\mathbb{B}_{\mu\nu} + b_\mu b_\nu)(\mathbb{B}_{\alpha\beta} + b_\alpha b_\beta))\sigma^{\alpha\beta}\end{aligned} \tag{K.2}$$

Extract from this the identities $\sigma_{\perp\mu\nu} = \frac{1}{2}(\mathbb{B}^{\mu\lambda}\mathbb{B}_{\nu\rho} + \mathbb{B}^{\nu\lambda}\mathbb{B}_{\mu\rho} - \mathbb{B}^{\mu\nu}\mathbb{B}_{\lambda\rho})\sigma^{\lambda\rho}$ and $\Sigma_\mu = \mathbb{B}_{\mu\lambda}\sigma^{\lambda\rho}b_\rho$:

$$\sigma_{\mu\nu} = \sigma_{\perp\mu\nu} + \Sigma_\mu b_\nu + \Sigma_\nu b_\mu + (\frac{1}{2}(b_\mu b_\alpha b_\nu b_\beta + b_\nu b_\alpha b_\mu b_\beta) + \frac{1}{2}\mathbb{B}_{\mu\nu}\mathbb{B}_{\alpha\beta})\sigma^{\alpha\beta} \tag{K.3}$$

No we use $\sigma_{\mu\nu} = \Delta^\rho_\mu\Delta^\sigma_\nu\partial_{(\rho}u_{\sigma)} - \frac{2}{3}\Delta_{\mu\nu}\Delta^{\rho\sigma}\partial_\rho u_\sigma$ to decompose the last part of the tensor.

$$\sigma^{\alpha\beta} = \mathbb{B}^{\rho\alpha}\mathbb{B}^{\sigma\beta}\partial_{(\rho}u_{\sigma)} + 2b^\alpha b^\beta S_\parallel - \frac{2}{3}(\mathbb{B}^{\alpha\beta} + b^\alpha b^\beta)(S_\perp + S_\parallel) \tag{K.4}$$

We then get:

$$\sigma_{\mu\nu} = \sigma_{\perp\mu\nu} + \Sigma_\mu b_\nu + \Sigma_\nu b_\mu + \mathbb{B}_{\mu\nu}(\frac{1}{3}S_\perp - \frac{2}{3}S_\parallel) + b_\mu b_\nu(\frac{4}{3}S_\parallel - \frac{2}{3}S_\perp)) \tag{K.5}$$

Appendix L

Bulk viscosity for analytical GGPT solution

GGPT work with the following Lagrangian:

$$\begin{aligned}\mathcal{L} =&\sqrt{-g}\Big(\frac{R}{2} - \frac{(\partial\phi^*)^2}{2} - V^*(\phi^*) - 3e^{-\sqrt{\frac{2}{3}}\phi^*}F^*_{\mu\nu}F^{*\mu\nu}\Big) \\ =&\frac{1}{2}\sqrt{-g}\Big(R - (\partial\phi^*)^2 - 2V^*(\phi^*) - 6e^{-\sqrt{\frac{2}{3}}\phi^*}F^*_{\mu\nu}F^{*\mu\nu}\Big)\end{aligned} \tag{L.1}$$

$$F^*_{\mu\nu} =\frac{1}{2}B_m dx\wedge dy \tag{L.2}$$

$*$ indicates a different convention. They give the following analytic result:

$$ds^2 = -\frac{g}{\sqrt{H_0H_1^3}}dt^2 + \sqrt{H_0H_1^3}(\frac{du^2}{g} + u^2(dx^2+dy^2)) \tag{L.3}$$

$$H_0 = 1 - \frac{3b}{u} \tag{L.4}$$

$$H_1 = 1 + \frac{b}{u} \tag{L.5}$$

$$g = -\frac{(p^1)^2}{2bu} + 3\frac{(p^1)^2}{2u^2} + u^2(1-\frac{3b}{u})(1+\frac{b}{u})^3 \tag{L.6}$$

$$\phi^* = \sqrt{\frac{3}{8}}\log\frac{b+u}{u-3b} \tag{L.7}$$

$$V^* = -3\cosh(\sqrt{\frac{2}{3}}\phi^*) \tag{L.8}$$

The Lagrangian we want to work with:

$$\mathcal{L} = \sqrt{-g}\Big(R - \frac{(\partial\phi)^2}{2} - V(\phi) - \frac{V_b(\phi)F_{\mu\nu}F^{\mu\nu}}{4}\Big) \tag{L.9}$$

$$F_{\mu\nu} = B_m dx\wedge dy \tag{L.10}$$

The different conventions are therefore related in the following way:

$$\phi^* = \sqrt{\frac{1}{2}}\phi \tag{L.11}$$

$$V^* = \frac{1}{2}V \tag{L.12}$$

So in our convention, we have the following analytic result:

$$\phi = \sqrt{\frac{3}{4}}\log\left(\frac{b+u}{u-3b}\right) \tag{L.13}$$

$$V = -6\cosh(\sqrt{\frac{1}{3}}\phi) = -3(e^{\frac{1}{\sqrt{3}}\phi} + e^{-\frac{1}{\sqrt{3}}\phi}) \tag{L.14}$$

$$= 6\frac{b-u}{b+u}\sqrt{\frac{4b}{u-3b}+1} \tag{L.15}$$

$$V_b = 6B_m^2 e^{-\sqrt{\frac{1}{3}}\phi} = 6B_m^2\sqrt{\frac{u-3b}{u+b}} \tag{L.16}$$

To be consistent with GGPT we work with the following metric:

$$ds^2 = -fe^{2A}dt^2 + \frac{e^{-2A}du^2}{f} + e^{2A}(dx^2+dy^2) \tag{L.17}$$

Now we match the metric terms of the GPR convention and the GGPT solution:

$$fe^{2A}dt^2 = \frac{g}{\sqrt{H_0H_1^3}}dt^2 \tag{L.18}$$

$$e^{2A}dx^2 = \sqrt{H_0H_1^3}u^2dx^2 \tag{L.19}$$

From this we get:

$$f = \frac{g}{H_0H_1^3u^2} \tag{L.20}$$

$$A = \frac{1}{2}\log\sqrt{H_0H_1^3}u^2 \tag{L.21}$$

We can find u_h in the following way:

$$f(r_h) = \frac{g}{H_0(u_h)H_1^3(u_h)u_h^2} = 1 - \frac{B_m^2}{2b(b+u_h)^3} = 0 \tag{L.22}$$

$$u_h = -b + \sqrt[3]{\frac{B_m^2}{2b}} \tag{L.23}$$

With the same method we used in Appendix C, we get the following temperature:

$$T = |\frac{f'(u_h)}{4\pi}e^{2A(u_h)}| \tag{L.24}$$

For the entropy we have:

$$s = \sqrt{H_0H_1^3}u^2\Big|_{u=u_h} \tag{L.25}$$

Appendix M

Numerical calculations for analytical model

We start off with the following Ansatz:

$$\begin{aligned} ds^2 = e^{2A(u)}(-f(u)(1+\lambda h_{00}(u)e^{i\omega t})dt^2 + (1+\lambda h_{11}(u)e^{i\omega t})(dx^2+dy^2) \\ + (1+\lambda h_{55}(u)e^{i\omega t})\frac{e^{-2A(u)}}{f(u)}du^2 \end{aligned} \tag{M.1}$$

We define our perturbative Einstein equations as follows:

$$\delta G_{\mu\nu} + \delta T_{\mu\nu} = 0 \tag{M.2}$$

We then have:

$$\delta T_{00} =(h_{55}-h_{00})f^2\phi'^2\frac{e^{4A}}{4} - h_{00}\frac{fe^{2A}}{2}V + (2h_{11}-h_{00})B_m^2V_b\frac{fe^{-2A}}{4} \tag{M.3}$$

$$\delta T_{11} =f\frac{e^{4A}}{4}(h_{11}-h_{55})\phi'^2 + \frac{e^{2A}}{2}Vh_{11} + B_m^2V_b\frac{e^{-2A}}{4}h_{11} \tag{M.4}$$

$$\delta T_{55} =h_{55}\frac{e^{-2A}}{2f}V + (h_{55}-2h_{11})B_m^2V_b\frac{e^{-6A}}{4f}) \tag{M.5}$$

This leads to the following EOM.

$$h_{55} = \frac{2fh'_{11} - h_{11}f'}{2fA'} \tag{M.6}$$

$$h'_{00} = -\frac{e^{-6A}\left(8e^{6A}f^2A'h'_{11} + 2e^{6A}ff'h'_{11} + 2e^{4A}fh_{55}V + 4\omega^2e^{2A}h_{11} - 2fh_{11}Bm^2V_b + fh_{55}B_m^2V_b\right)}{4f^2A'} \tag{M.7}$$

$$\begin{aligned} h''_{11} = -\frac{e^{-6A}}{4f^2A'\phi'^2}(4e^{6A}fA'f'h'_{11}\phi'^2 - 4e^{6A}fh_{11}A'^2f'\phi'^2 - 4e^{6A}fh_{11}A'f'\phi'\phi'' \\ + 24e^{6A}f^2A'^2h'_{11}\phi'^2 + 8e^{6A}f^2A'h'_{11}\phi'\phi'' - 8fh_{11}A'^2V'_b \\ - 4fh_{11}V_bA'\phi'^2 + 4w^2e^{2A}h_{11}A'\phi'^2 - e^{6A}fh_{11}f'\phi'^4 + 2e^{6A}f^2h'_{11}\phi'^4) \end{aligned} \tag{M.8}$$

At the horizon, the fluctuation equation gets the following familiar form, yielding the boundary condition which follows from the requirement of regularity at the horizon:

$$\{\omega^2 h_{11}\frac{e^{-4A}}{f^2} + h'_{11}\frac{f'}{f} + h''_{11}\}_{u\to u_h} \approx 0 \tag{M.9}$$

We can also check whether the following background equations vanish to confirm that there are no errors in the computation:

$$A'' = \frac{1}{4}\left(-4A'^2 - \phi'^2\right) \tag{M.10}$$

$$f'' = e^{-6A}B_m^2 V_b - 4A'f' \tag{M.11}$$

$$V = \frac{1}{2}e^{-4A}\left(-e^{6A}\left(4A'f' + f\left(12A'^2 - \phi'^2\right)\right) - B_m^2 V_b\right) \tag{M.12}$$

$$\frac{V'}{\phi'} = e^{2A}\phi'\left(4fA' + f'\right) + fe^{2A}\phi'' - \frac{e^{-4A}B_m^2 V_b'}{2\phi'} \tag{M.13}$$

The near horizon solution is:

$$h_{11} \approx Ce^{-i\omega\frac{\log(u-u_h)}{4\pi T}} \tag{M.14}$$

$$T = |\frac{f'}{4\pi}e^{2A}| = \frac{1}{4\pi}\frac{3B_m^2\sqrt{\frac{u-3b}{(b+u)^5}}}{2b} \tag{M.15}$$

Before rescaling, we have the following boundary conditions:

$$h_{11}(u_h + \delta u_h) = \delta u_h^{-\frac{i\omega}{4\pi T}} \tag{M.16}$$

$$h'_{11}(u_h + \delta u_h) = \frac{\partial \delta u_h^{-\frac{i\omega}{4\pi T}}}{\partial \delta u_h} \tag{M.17}$$

δu_h is a small offset with respect to the horizon. In figure M.1 the numerical result found for c_b is given. It is important to note that the range in which we can compute c_b is bounded from below because the solution is not well defined when $r_h < 3b$, because in this case there is naked singularity. Physically this means that the black hole solution is unstable at low temperatures. In figure 6.1 the bulk viscosity result is given.

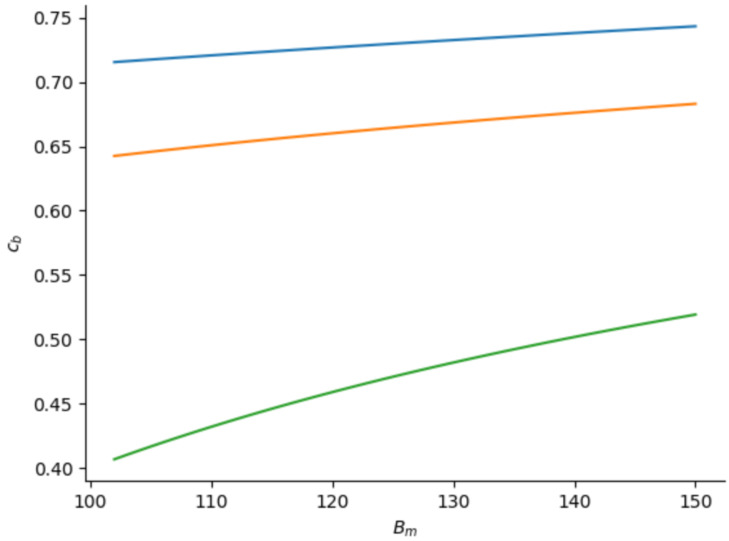

Figure M.1: The numerical constant c_b found as a function of B_m for $b = 0.5$ (blue), $b = 1$ (yellow) and $b = 2$ (green)

Appendix N

Fluid Gravity

N.1 Ansatz

In this Appendix we work out what a calculation of the isotropic shear and bulk viscosity for a general non-conformal non-zero temperature strongly coupled relativistic fluid roughly should look like, this is mainly based on a paper of Wu, Chen and Huang (WCH) [94], where WCH computation is performed for a non-conformal computation for compactified $D4$-branes. We work with the following boosted Eddington Finkelstein metric Ansatz:

$$ds^2 = -2dx^\mu u_\mu dr + e^{2A}(-f(du_\mu x^\mu)^2 + P_{\mu\nu}dx^\mu dx^\nu) \tag{N.1}$$

At zeroth order, we start with the following ultra-local expansion:

$$\begin{aligned} ds^2_{(0)} = 2e^{B-A}(dvdr - \lambda x^\mu \partial_\mu \beta_i^0 dx^i dr) + e^{2A}\Big(&- f^{(0)}dv^2 + dx_i dx^i \\ &+2\lambda x^\mu \partial_\mu \beta^{(0)}(1-f^{(0)})dx^i dv - \lambda x^\mu \partial_\mu r_h \frac{\partial f^{(0)}(r,r_h)}{\partial r_h} dv^2\Big) \end{aligned} \tag{N.2}$$

λ gives the order of expansion. Where we performed the following expansion:

$$u_\mu = -\delta_\mu^0 + \lambda x^\mu \partial_\nu \beta_i^0 \delta_\mu^i \tag{N.3}$$

$$f = f^{(0)} + \lambda x^\nu \partial_\mu r_h \frac{\partial f^{(0)}}{\partial r_h} \tag{N.4}$$

For the energy momentum tensor we have the following identity:

$$T_{\mu\nu} = \frac{1}{2}g_{\mu\nu}V_g(r) - \frac{1}{4}g_{\mu\nu}(\partial r)^2 + \frac{1}{2}\partial_\mu r \partial_\nu r \tag{N.5}$$

We can split up our Ansatz into three sectors [94], the scalar, vector and tensor sector. Our boundary conditions are: convergence to zero at the boundary and regularity at the horizon. Also we work in the Landau frame:

$$T^{(1)}_{\mu\nu}u^\mu = 0 \tag{N.6}$$

Our gauge is: $g^{(1)}_{\mu r} = 0$ and $g^{(1)}_{rr} = 0$. All these choices fully constrain our solution.

N.2 Scalar sector

We work with the following very general Ansatz:

$$ds^2_{(1)} = k(r)dv^2 + h(r)dx_i dx^i + j(r)dvdr \tag{N.7}$$

Constraint equations:

$$g^{rr}(E_{rv} - T_{rv}) + g^{rv}(E_{vv} - T_{vv}) = 0 \tag{N.8}$$
$$g^{rr}(E_{rr} - T_{rr}) + g^{rv}(E_{rv} - T_{rv}) = 0 \tag{N.9}$$

Dynamical equations:

$$E_{rr} = T_{rr} \tag{N.10}$$
$$E_{rv} = T_{rv} \tag{N.11}$$
$$E_{ii} = T_{ii}\ ,\ \ i \in \{1,2,3\} \tag{N.12}$$
$$E_{vv} = T_{vv} \tag{N.13}$$
$$\partial_\phi V_g = -\frac{1}{\sqrt{-g}}\partial_r\left(\sqrt{-g}g^{rr}\partial_r\phi\right) \tag{N.14}$$

N.3 Vector sector

We work with the following very general Ansatz:

$$ds^2_{(1)} = w_i(r)dx^i dv \tag{N.15}$$

The constraint equation:

$$g^{rv}(E_{vi} - T_{vi}) + g^{rr}(E_{ri} - T_{ri}) = 0 \tag{N.16}$$

The dynamical equation:

$$E_{ri} - T_{ri} = 0 \tag{N.17}$$

N.4 Tensor sector

We work with the following very general Ansatz:

$$ds^2_{(1)} = \alpha_{ij}(r)dx^i dr^i \tag{N.18}$$

The dynamical equation:

$$E_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}E_{kl} = T_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}T_{kl} \tag{N.19}$$

Solution

Having determined the first order metric, we calculate the boundary action:

$$S = S_{bulk} + S_{GH} + S_{ct} \tag{N.20}$$

S_{ct} is the counterterm action which removes UV divergences and S_{GH} is the Gibbon Hawking action which makes sure that the boundary action only depends on first order derivatives, as is required for the path-integral approach [37]. We then take the variation with the boundary metric to get the boundary stress-energy tensor from which we can extract the shear and bulk viscosities as follows:

$$T^b_{\mu\nu} = pP_{\mu\nu} + \epsilon u_\mu u_\nu - 2\eta\sigma_{\mu\nu} - \zeta\partial_\rho u^\rho P_{\mu\nu} \tag{N.21}$$

As mentioned in section 5.3, WCH already succesfully calculated the bulk viscosity for the Sakai-Sugimoto model in [94] which was found to be consistent with the Eling-Oz formula [33].

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