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MASTER'S THESIS

Sieving on Projective Varieties

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Abstract

We consider a system of quadratic forms $F_1, \dots, F_R$ with integer coefficients. Generalizing the work of Myerson we find an asymptotic formula for the number of integral zeros within a growing box which also lie in suitable residue classes, i.e. fix $m, n \in \mathbb{N}$ and let Ω_p be a chosen subset of $(\mathbb{Z}/p^m\mathbb{Z})^n$ then we count all $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$ with $|x_i| < P$ such that $F_1(\boldsymbol{x}), \dots, F_R(\boldsymbol{x}) = 0$ and $\boldsymbol{x} \in \Omega_p$. This is done by using Selberg's sieve. As an application we study the number of rational points that lie in a thin set in the intersection of a system of quadratic forms $F_1, \dots, F_R$. Lastly we find a lower bound for the number of $\boldsymbol{x} \in \mathbb{Z}^n$ with $|x_i| < P$ such that $F_1(\boldsymbol{x}), \dots, F_R(\boldsymbol{x}) = 0$ and $x_1 \cdots x_n$ is a positive integer with at most r prime divisors outside some given finite set.

Contents

1 Introduction

A fundamental theme in mathematics is the study of rational points on projective algebraic varieties. Let $X \subset P^n_{\mathbb{Q}}$ be a projective variety that is the zero set of a finite system of homogeneous equations. A question we could ask ourselves is: when is $X(\mathbb{Q})$ non-empty? how large is $X(\mathbb{Q})$ when it is non-empty? There is a conjecture by Manin [1, Conjecture C'] which tries to to give an answer. We will state Manin's conjecture in the way Browning [6] does.

Points of $\mathbb{P}^n(\mathbb{Q})$ can be represented by vectors $\boldsymbol{x} = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$ with

$$\gcd(x_0, \ldots, x_n) = 1.$$

This can be done in two ways: having $\prod x_i > 0$ or having $\prod x_i < 0$. We define a height function $H : \mathbb{P}^n(\mathbb{Q}) \to \mathbb{Z}$ as

$$x \mapsto |\boldsymbol{x}| := \max_i(|x_i|),$$

where $\boldsymbol{x} = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$ represents $x \in \mathbb{P}^n(\mathbb{Q})$. The function H is also known as the standard exponential height function associated to the supremum norm.

Given a suitable Zariski open subset $U \subset X$, the goal is then to study the quantity

$$N(U, H, P) := \#\{x \in U(\mathbb{Q}) : H(x) \leq P\},$$

as $P \to \infty$. Suppose for simplicity that X is a non-singular complete intersection, with $X = W_1 \cap \ldots \cap W_R \subset X$, for hypersurfaces $W_i \subset \mathbb{P}^n$ of degree d_i. We assume X to be Fano, and therefore its Picard group is a finitely generated free $\mathbb{Z}$-module, and we denote its rank by $\rho(X)$. Then in this setting the Manin conjecture [1, Conjecture C'] takes the following shape.

Conjecture 1.1. *Suppose that $d_1 + \ldots + d_R \leq n$. Then there exists a Zariski open subset $U \subset X$ and a constant $c_{X,H}$ such that*

$$N(U, H, P) = c_{X,H} P^{n+1-d_1-\ldots-d_R} (\log P)^{\rho(X)-1} (1 + o(1)), \tag{1.1}$$

as $P \to \infty$ and where $\rho(X)$ is the rank of the Picard group of X.

It is not so difficult to see why we need to restrict the counting function to something open. If we look at the zeros in $\mathbb{P}^3(\mathbb{Q})$ of $F(x) = x_0^3 + x_1^3 - x_1^3 - x_3^3$ for which $|\boldsymbol{x}| \leq P$, it is easy to see that if $x_0 = x_2$ and $x_1 = x_3$, then $F(x) = 0$. So we have at least P^2 solutions. Now we can see why (1.1) won't hold if we don't restrict it to some open U, namely P^2 grows faster then $P(\log P)^{4-1}$ as $P \to \infty$.

The original version of Manin's conjecture (1.1) is false in general. This was first shown in 1996 by Batyrev and Tschinkel [2]. As Browning and Loughran [8] say, numerous authors have recently investigated a "thin" version of Manin's conjecture (see [21, §8], [17], [7] or [16]), where one is allowed to remove a thin subset of $X(\mathbb{Q})$, rather than just a Zariski closed set. See Definition 10.1 for the definition of thin subsets. A question is whether removing a thin subset could change the asymptotic behavior of the counting function $N(U, H, P)$. We will show that, under some conditions, this is not the case when the rational points are equidistributed and X is the zero set of some quadrics.

Theorem 1.2. *Let $X(\mathbb{Q}) \subset \mathbb{P}^n(\mathbb{Q})$ be a smooth variety defined by the quadratics $F_1, \ldots, F_R$ with integer coefficients. Suppose $\dim X = n - R$, $\dim \tilde{X}^* \leq n - 1$ where $\tilde{X}^*$ is defined as in §8.4.1. Further suppose $n + 1 - \sigma_{\mathbb{R}} > 8R$, where*

$$\sigma_{\mathbb{R}} = 1 + \max_{\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{\mathbf{0}\}} \dim \operatorname{Sing} V(\boldsymbol{\beta} \cdot \boldsymbol{F})$$

and $V(\boldsymbol{\beta} \cdot \boldsymbol{F})$ is the hypersurface cut out in $\mathbb{P}^n$ by $\beta_1 F_1 + \ldots + \beta_R F_R$. Let $\Upsilon \subset X(\mathbb{Q})$ be a thin set. Then there exists $\theta_n > 0$ such that

$$\#\{x \in \Upsilon : |x| \leq P\} \ll_{\Upsilon, X} P^{n+1-2R-\theta_n}.$$

This theorem is proved in section 10 by using another main result of this thesis, stated below. Let X be an integral model for the variety defined by $F_1, \ldots, F_R = 0$. Fix $m \in \mathbb{N}$ and let $\Omega_{p^m} \subset X(\mathbb{Z}/p^m\mathbb{Z})$ then define

$$N(P, \Omega) := \#\{x \in X(\mathbb{Q}) : |x| \leq P, x \bmod p^m \in \Omega_{p^m} \text{ for all } p\}.$$

Theorem 1.3. *Let $X \subset \mathbb{P}^n(\mathbb{Q})$ be as above, in Theorem 1.2. In particular*

$$n + 1 - \sigma_{\mathbb{R}} > 8R. \tag{1.2}$$

Let $m \in \mathbb{N}$ and let $\Omega_{p^m} \subset X(\mathbb{Z}/p^m\mathbb{Z})$ for each prime p. Assume that $0 \leq \omega_p < 1$, where

$$\omega_p = 1 - \frac{\#\hat{\Omega}_{p^m}}{\#\hat{X}(\mathbb{Z}/p^m\mathbb{Z})}, \tag{1.3}$$

and

$$\hat{\Omega}_{p^m} = \{\boldsymbol{x} \in (\mathbb{Z}/p^m\mathbb{Z})^{n+1} : p \nmid \boldsymbol{x}, (x_0 : \ldots : x_n) \in \Omega_{p^m}\}$$

and

$$\hat{X}(\mathbb{Z}/p^m\mathbb{Z}) = \{\boldsymbol{x} \in (\mathbb{Z}/p^m\mathbb{Z})^{n+1} : p \nmid \boldsymbol{x}, F_1(x_0, \ldots, x_n) = 0 \textit{ mod } p^m\}.$$

Then for every $\xi \geq 1$ and any $\epsilon > 0$, we have

$$N(P, \Omega) \ll_{X,\epsilon} \frac{P^{n+1-2R}}{J(\xi)} + P^{n+1-2R-\delta} \xi^{2m\delta_2 + 2 + \epsilon},$$

where

$$J(\xi) = \sum_{k < \xi} \mu^2(k) \prod_{p | k} \left(\frac{\omega_p}{1 - \omega_p} \right)$$

and δ and δ_2 are some positive constants.

For the proof of Theorem 1.3 we will use Selberg's sieve. This is a very useful tool, since then, as we will see in section 9, we only need to have a formula for

$$\tilde{N}(P, \Omega_M) := \#\{\boldsymbol{x} \in \mathbb{Z}^n : |\boldsymbol{x}| < P\mathcal{B}, f_i(x) = 0 \text{ for } 1 \leq i \leq R, \boldsymbol{x} \bmod p^m \in \Omega_{p^m} \text{ for all } p^m || M\},$$

where $\mathcal{B}$ is a box in $\mathbb{R}^n$ contained in the box $[-1, 1]^n$, and having sides of length at most 1 which are parallel to the coordinate axes, f_i are quadratics and where we write $p^m || M$ if p^m divides M, but p^{m+1} does not divide M. The assumptions of the following theorem requires some more notation, which can be found in section 8.1.

Theorem 1.4. *Let $f_i \in \mathbb{Z}[x_1, \dots, x_n]$ be quadratic forms with integer coefficients and $n \geq 2$ and $\dim(\tilde{X}^*) \leq n-1$, where $\tilde{X}^*$ is defined as in §8.4.1. Let $N_h^{aux}(B)$ as in Definition 8.2. Suppose that the f_i are linearly independent and that*

$$N_{\boldsymbol{\beta}\cdot\boldsymbol{f}}^{aux}(B) \leq C_0 B^{(d-1)n-2^d\mathscr{C}}$$

for some $C_0 \geq 1, \mathscr{C} > 2R$ and all $\boldsymbol{\beta} \in \mathbb{R}^R$ and $B \geq 1$, where $\boldsymbol{\beta} \cdot \boldsymbol{f} = \beta_1 f_1 + \ldots + \beta_R f_R$. Then for all $P \geq 1$ we have

$$\tilde{N}(P, \Omega_M) = \sum_{[\boldsymbol{v}]_M \in \Omega_M} (\mathfrak{J}\mathfrak{S}(\boldsymbol{v}; M) P^{n-2R} M^{-n} + O(P^{n-2R-\delta} M^{-n+\delta_2})), \tag{1.4}$$

where the implied constant depends at most on $C_0, \mathscr{C}$ and the f_i, and δ and δ_2 are positive constants depending at most on $\mathscr{C}$ and R. Here $\mathfrak{J}$ and $\mathfrak{S}$ are as in Lemma 8.11 and Lemma 8.10.

For the proof of the previous theorem we follow Myerson [20]. The dependence of M in (1.4) is my own contribution. Myerson found in 2017 the following formula for $\tilde{N}(P, \Omega)$ if $\Omega_{p^m} = X(\mathbb{Z}/p^m\mathbb{Z})$ for all p.

Theorem 1.5. *([20, Theorem 1.2.]). Let $F_1(\boldsymbol{x}), \dots, F_R(\boldsymbol{x})$ be homogeneous forms of degree 2, with integer coefficients in n variables. Let $\mathcal{B}$ be a box in $\mathbb{R}^n$, contained in the box $[-1,1]^n$, and having sides of length at most 1 which are parallel to the coordinate axes. For each $P \geq 1$, we write*

$$N(P) = \#\{\boldsymbol{x} \in \mathbb{Z}^n : \boldsymbol{x} \in P\mathcal{B},\ F_1(\boldsymbol{x}), \dots, F_R(\boldsymbol{x}) = 0\}.$$

If $\dim X(F_1, \dots, F_R) = n - 1 - R$ and

$$n - \sigma_{\mathbb{R}} > 8R,$$

then for all $P \geq 1$, some $\mathfrak{J}_1 \geq 0$ depending only on the coefficients of F_i and on $\mathcal{B}$, and some $\mathfrak{S}_1 \geq 0$ depending only on the coefficients of F_i, we have

$$N(P) = \mathfrak{J}_1 \mathfrak{S}_1 P^{n-2R} + O(P^{n-2R-\delta_1})$$

where the implied constant depends only on the forms F_i and δ_1 is a positive constant depending only on R.

There is a similar upper bound like the one in Theorem 1.3 proven by Van Ittersum [13]. He requires

$$n + 1 - \dim X^* > 2R(R+1), \tag{1.5}$$

where X^* is the projective variety cut out in $\mathbb{P}^n(\mathbb{Q})$ by the condition that the $R \times (n+1)$ Jacobian matrix $(\partial F_i(x)/\partial x_j)_{ij}$ has rank less than R. Van Ittersum used the techniques of Birch [5]. Birch found in 1961/1962 a formula for $N(P, \Omega)$ if $\Omega_{p^m} = X(\mathbb{Z}/p^m\mathbb{Z})$ for all p. As mentioned earlier, Myerson [20] found also a formula for $N(P, \Omega)$ if $\Omega_{p^m} = X(\mathbb{Z}/p^m\mathbb{Z})$ and F_i are quadratics. If F_i are quadratics, the formula of Birch is weaker than the formula of

Myerson whenever $R \geq 4$. Likewise, (1.5) is weaker then (1.2) whenever $R \geq 4$. Namely, we have $\operatorname{Sing} X(\boldsymbol{\beta} \cdot F) \subset X^*$. So $\sigma_{\mathbb{R}} \leq 1 + \dim X^*$. Thus (1.5) is weaker than (1.2) whenever $2R(R+1) > 8R$, hence $R \geq 4$.

A main motivation in the work of Myerson was that Müller [19] and Bentkus and Götze [3], [4] in the period of 1997 to 2008 came with some ideas to find an upper bound for the integral of some function over any bounded measurable set. We will use these theorems in section 8.5, where the dependence of M is my own contribution.

Another main result of this thesis, which is proved by using Theorem 1.4, is the following.

Theorem 1.6. *Let $\boldsymbol{F}$ is a system of R linear independent quatrics in n variables and integer coefficients. Assume $\dim(\tilde{X}^*) \leq n-1$ and $n + 1 - \sigma_{\mathbb{R}} > 8R$. Let $\mathfrak{J}$ be as in Theorem 1.4, set*

$$\sigma_p = \lim_{k\to\infty} \frac{1}{p^{k(n+1-R)}} \#\{\boldsymbol{b} \in \{1, 2, \ldots, p^k\}^{n+1} : \boldsymbol{F}(\boldsymbol{b}) \equiv 0 \bmod p^k\}.$$

and

$$\Omega_p = \{\boldsymbol{x} \in (\mathbb{Z}/p\mathbb{Z})^n : \boldsymbol{F}(\boldsymbol{x}) \equiv 0, x_1 \cdots x_n \equiv 0 \text{ mod } p\}$$

and let $\hat{\Omega}_p$ be as in Theorem 1.3. Let B be the set of primes for which $\#\Omega_p = 0$ and $P_r(B)$ the set of positive integers with at most r prime divisors outside B. Assume (11.2), (11.3) and (11.4) hold. Let τ and β_κ be as in section 11. For any two reals numbers u and v satisfying

$$\frac{1}{\tau} < u \leq v, \quad \beta_\kappa < \tau\mu$$

we have

$$\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\cap P\mathcal{B} \\ \boldsymbol{F}(\boldsymbol{x})=0 \\ x_1\cdots x_n \in P_r(B)}} 1 \gg P^{n-2R}\mathfrak{J}\prod_p \sigma_p \prod_{p<(P^{n-2R}\mathfrak{J}\prod_p \sigma_p)^{1/v}} \frac{\#\hat{\Omega}_p}{\hat{X}(\mathbb{Z}/p\mathbb{Z})},$$

provided that

$$r > \tau\mu u - 1 + \frac{\kappa}{f_\kappa(\tau v)} \int_1^{v/u} F_\kappa(\tau v - s)(1 - \frac{u}{v}s)\frac{ds}{s}.$$

We first fix some notation in section 2. In section 3 we describe sieves in general, which will be used later. Section 4 is used to describes how we view points on varieties such that we can count them. After that, in section 5 and 6 we discuss $N(P, \Omega)$ if $R = 1$, for which we follow [8]. Then, in section 7 till 9 we prove Theorem 1.3 and 1.4. For that we follow mostly Myseron [20]. In section 10 we are then ready to prove Theorem 1.2. Lastely in section 11 we see a proof of Theorem 1.6.

2 Notation

For a point $\boldsymbol{x} \in \mathbb{R}^n$ we write $\boldsymbol{x} = (x_1, \ldots, x_n)$ with $x_i \in \mathbb{R}$ and we introduce the supremum norm

$$|\boldsymbol{x}| = \max(|x_1|, \ldots, |x_n|).$$

Here $|x_i|$ is the usual absolute value of $x_i \in \mathbb{R}$. For any $\boldsymbol{x} \in \mathbb{Z}^n$ and $d \in \mathbb{N}$ we write $[\boldsymbol{x}]_d$ for the reduction of $\boldsymbol{x}$ modulo d. We abbreviate $e^{2\pi i x}$ by $e(x)$ for $x \in \mathbb{R}$.

For $a, b \in \mathbb{N}$ we write (a, b) for $\gcd(a, b)$ and we write $\omega(a)$ for the number of distinct primes dividing a. For $m \in \mathbb{N}$ we write $p^m || M$ if p^m divides M but p^{m+1} does not divide M. We define

$$\mathbb{Z}^n_{\text{prim}} := \{\boldsymbol{y} \in \mathbb{Z}^n : \gcd(y_1, \ldots, y_n) = 1\}.$$

We let $\mathcal{B}$ be a box in $\mathbb{R}^n$, contained in the box $[-1, 1]^n$ and having sides of length 1 which are parallel to the coordinate axes.

Further, let $F_1, F_2, \ldots, F_R$ be homogeneous polynomials with integers coefficients. Let$X$ be an integral model for the zero set of these polynomials. Let $m, M \in \mathbb{N}$ be fixed and let $\Omega_M = \prod_{p^m||M} \Omega_{p^m}$, where Ω_{p^m} are chosen subsets of the variety X over $\mathbb{Z}/p^m\mathbb{Z}$. We use the notation $\hat{X}$ and $\hat{\Omega}$ for viewing X and Ω as affine spaces, i.e.

$$\hat{\Omega}_{p^m} = \{\boldsymbol{x} \in (\mathbb{Z}/p^m\mathbb{Z})^{n+1} : p \nmid \boldsymbol{x},\, (x_0 : \ldots : x_n) \in \Omega_{p^m}\}$$

and

$$\hat{X}(\mathbb{Z}/p^m\mathbb{Z}) = \{\boldsymbol{x} \in (\mathbb{Z}/p^m\mathbb{Z})^{n+1} : p \nmid \boldsymbol{x},\, F_1(x_0, \ldots, x_n) = 0 \bmod p^m\}.$$

Let f, g be functions on a subset of the real numbers. We write $f(x) \ll g(x)$ if $f(x) = O(g(x))$, that is there are constants $C > 0$ and x_0 such that $|f(x)| \leq Cg(x)$ for all $x > x_0$. We call C the implied constant of $f(x) \ll g(x)$. If we write $f(x) = O_R(g(x))$ or equivalently $f(x) \ll_R g(x)$ we mean that the implied constant and x_0 depends on R.

We use the notation $f(x) = o(g(x))$ as $x \to \infty$ for $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, i.e. $f(x)$ is of smaller order of magnitude than $g(x)$.

Lastely if $V \subset \mathbb{C}^n$ is a affine variety defined by $f_1, \ldots, f_R$, all of degree d, we define V^* as the Birch singular locus as the affine variety consisting of all points $\boldsymbol{x} \in \mathbb{C}^n$ for which

$$\operatorname{rank}\Big(\frac{\partial f_i^{[d]}}{\partial x_j}(\boldsymbol{x})\Big)_{i,j} < R,$$

where $f_i^{[d]}$ denotes the degree part of f_i.

3 Introduction to Selberg's Sieve

Sieves are used to estimate the size of a sifted sets of integers. There are a lot of different sieves and in this thesis we will use Selberg's sieve, also denoted as the Λ^2-sieve, which is a technique for estimating the size of sifted sets of positive integers which satisfy a set of conditions which are expressed by congruences. It was developed in the 1940s by Atle Selberg. First we will say something about sieves in general and then we discuss Selberg's sieve.

3.1 Sieves

We describe the basis ideas behind sieves for which we follow mostly Friedlander and Iwaniec [11, Chapter 5] and Iwaniec and Kowalski [14, Chapter 6].

We start with a sequence $\mathcal{A} = (a_n)$ of non-negative numbers. The ultimate question is how often these numbers are supported on primes. The basic input to any sieve mechanism comes via the subsequence $\mathcal{A}_d$ consisting of those a_n with $n \equiv 0 \pmod{d}$ and, in particular, from estimates for the congruence sums

$$|\mathcal{A}_d| = \sum_{n \equiv 0 (\mathrm{mod}\ d)} a_n.$$

Sometimes $\mathcal{A}$ is restricted to $n < x$, and then we write

$$A_d(x) = \sum_{\substack{n<x \\ n \equiv 0 (\mathrm{mod}\ d)}} a_n.$$

For applying a sieve we need an asymptotic formula for $A_d(x)$ of the following form

$$A_d(x) = g(d)A_1(x) + r_d(x) \tag{3.1}$$

where $g(d)A_1(x)$ is the expected main term and $r_d(x)$ is an error term which we think of as being relatively small. So $g(d)$ stands for the density of the masses a_n attached to $n \equiv 0 \pmod{d}$. Expecting that divisibility by distinct primes are independent events, we assume that the density function $g(d)$ is a multiplicative function and that

$$0 \leq g(p) < 1.$$

It is convenient to have a smooth approximation to $\mathcal{A}(x)$, which we label as X, and we replace (3.1) by

$$A_d(x) = g(d)X + r_d(x). \tag{3.2}$$

In practice we will choose X close enough to $A_1(x)$, so that (3.1) and (3.2) are equivalent. Let the sifting set of primes $\mathcal{P}$ be a set of all primes, $z \geq 2$ and

$$P(z) = \prod_{\substack{p<z \\ p \in \mathcal{P}}} p.$$

We seek seek estimates for the shifted sum

$$S(x,z) = \sum_{\substack{n<x \\ (n,P(z))=1}} a_n. \tag{3.3}$$

We refer to z as the sifting level. An instrumental role in the sifting process is played by the sifting function (3.3). The condition $(n, P(z)) = 1$ can be detected by the Möbius inversion formula

$$\sum_{\substack{d|n \\ d|P(z)}} \mu(d) = \begin{cases} 1 & \text{if } (n,P(z)) = 1 \\ 0 & \text{if } (n,P(z)) > 1. \end{cases}$$

Hence (3.3) becomes

$$S(x,z) = \sum_{n<x} \Big(\sum_{\substack{d|n \\ d|P(z)}} \mu(d) \Big) a_n.$$

If we change the order of summation we obtain

$$S(x,z) = \sum_{d|P(z)} \mu(d) A_d(x). \tag{3.4}$$

From this formula we see that in some sieve problems we do not need to sieve by all the primes. Indeed, we do not need in $\mathcal{P}$ any prime p for which all an with $n \equiv 0 \pmod p$ vanish, that is, such that $A_p = 0$. For example, if a_n is supported on the polynomial values $n = m^2 + 1$, then we can restrict $\mathcal{P}$ to primes $p \equiv 1 \pmod 4$. Since we use (3.2) only for $d|P(z)$, we are free to modify the multiplicative function $g(d)$ arbitrarily at d coprime to $P(z)$ and for notational simplicity we set

$$g(p) = 0 \text{ if } p \nmid \mathcal{P}.$$

Combining (3.4) with (3.2) gives

$$S(x,z) = XV(z) + R(x,z)$$

where

$$V(Z) = \prod_{p|P(z)} (1 - g(p))$$

and

$$R(x,z) = \sum_{d|P(z)} r_d(x). \tag{3.5}$$

3.2 Sifting Weights

The essence of sieve methods is the replacement of the Möbius function $\mu(d)$ by a function $\Lambda = (\lambda_d)$. This will give us a way to reduce the number of terms in (3.4) and, in particular the number of terms in the remainder term $R(x, z)$ (3.5). We choose Λ such that it has a finite support

$$\Lambda = (\lambda_d) \text{ for } d|P(z) \text{ and } d < D.$$

We refer to λ_d as sieve weights (or sifting weights) of level D. The price for the change to weights with finite support is that we no longer have the exact formula for the sifting function (3.4) but rather obtain the "sifted sum"

$$S^\Lambda(\mathcal{A}, z) = \sum_{d|P(z)} \lambda_d |\mathcal{A}_d|. \tag{3.6}$$

Let

$$\theta_n = \sum_{\substack{d|n \\ d|P(z)}} \lambda_d.$$

Then (3.6) becomes

$$S^\Lambda(\mathcal{A}, z) = \sum_{n|P(z)} a_n \theta_n.$$

If we choose $\Lambda^- = (\lambda^-)$ and $\Lambda^+ = (\lambda^+)$ such that

$$\theta_n^- = \sum_{\substack{d|n \\ d|P(z)}} \lambda_d^- \le \sum_{\substack{d|n \\ d|P(z)}} \mu(d) \le \sum_{\substack{d|n \\ d|P(z)}} \lambda_d^+ = \theta_n^+,$$

then $S^-(\mathcal{A}, z) \le S(\mathcal{A}, z) \le S^+(\mathcal{A}, z)$ where we used the notation S^+ (respectively, $S^-(\mathcal{A}, z)$) for the sieves with weight Λ^+ (respectively, Λ^-).

It is convenient to normalize Λ and $\Lambda^\pm$ such that $\lambda_1 = \lambda_1^\pm = 1$. It follows that $\theta_n^\pm = 1$ for $(n, P(z)) = 1$. Moreover we have $\theta_n^+ \ge 0$ for $(n, P(z)) > 1$ and $\theta_n^- \le 0$ for $(n, P(z)) > 1$.

A weight system having the properties stated above will be called a lower-bound sieve, upper-bound sieve, respectively. An example of a upper-bound sieve is Selberg's sieve, which will be discussed next paragraph. Of course, in the case of an upper-bound sieve, we should like to choose the weights λ_d^+ so as to make S^+ as small as possible.

In the case of the upper-bound sieve λ^+ and the lower-bound sieve λ^- we denote the corresponding sifted sums by $S^\pm(\mathcal{A}, z)$, their main term sums

$$V^+(D, z) = \sum_{\substack{d|P(z) \\ d<D}} \lambda_d^+ g(d), \qquad V^-(D, z) = \sum_{\substack{d|P(z) \\ d<D}} \lambda_d^- g(d) \tag{3.7}$$

and their remainder by

$$R^+(D, z) = \sum_{\substack{d|P(z) \\ d<D}} \lambda_d^+ r_d, \qquad R^-(D, z) = \sum_{\substack{d|P(z) \\ d<D}} \lambda_d^- r_d. \tag{3.8}$$

3.3 Selberg's Sieve

Selberg's sieve is used to find a upper bound for the sifted sum. We start with a sequence of real numbers λ_d^+. For notational simplicity hereafter we omit the superscript $+$. If we want an upper-bound sieve of level D the real numbers λ_d have to satisfy the conditions $\lambda_1 = 1$ and

$$\sum_{d|n} \lambda_d \geq 0 \text{ for } (n, P(z)) > 1. \tag{3.9}$$

For notational simplicity we omit the superscript $+$ in the rest of this section. The positivity condition is in general quite difficult to hold, but if we choose λ_d like Selberg, namely

$$\sum_{d|n} \lambda_d = \left(\sum_{d|n} \rho_d \right)^2, \tag{3.10}$$

where ρ_d is another sequence of real numbers, the square guarantees the positivity condition (3.9). From the condition $\lambda_1 = 1$, we see that we need to have

$$\rho_1 = 1. \tag{3.11}$$

If we have a closer look at the sum $\left(\sum_{d|n} \rho_d \right)^2$ in (3.10), we see that the term $\rho_{d_1}\rho_{d_2}$ appears if and only if $(d_1, d_2) \,|\, n$. By grouping the elements $\rho_{d_1}\rho_{d_2}$ for which $(d_1, d_2) = d$ for some $d|n$, we have

$$\left(\sum_{d|n} \rho_d \right)^2 = \sum_{d|n} \sum_{(d_1,d_2)=d} \rho_{d_1}\rho_{d_2}.$$

Hence

$$\lambda_d = \sum_{(d_1,d_2)=d} \rho_{d_1}\rho_{d_2}. \tag{3.12}$$

In order to control the level D we assume the coefficients ρ_d are supported on integers smaller than $\sqrt{D}$, i.e.

$$\rho_d = 0 \quad \text{if } d \geq \sqrt{D}. \tag{3.13}$$

Now the resulting sieve (λ_d) has level of support D.

Applying the Λ^2-sieve to $\mathcal{A} = (a_n)$ in the shifting range $\{p \in \mathcal{P}, p < z\}$ we obtain

$$S(\mathcal{A}, z) = \sum_{(n,P(z))=1} a_n \leq S^+(\mathcal{A}, z), \tag{3.14}$$

where

$$\begin{aligned} S^+(\mathcal{A}, z) &= \sum_n a_n \left(\sum_{d|(n,P(z))} \rho_d \right)^2 \\ &= \sum_{d_1|P(z)} \sum_{d_2|P(z)} \rho_{d_1}\rho_{d_2} |\mathcal{A}_{(d_1,d_2)}| \\ &= XG + R^+(\mathcal{A}, \Lambda^2), \end{aligned} \tag{3.15}$$

and

$$G = \sum_{d_1|P(z)} \sum_{d_2|P(z)} g((d_1, d_2))\rho_{d_1}\rho_{d_2} \tag{3.16}$$

and

$$R^+(\mathcal{A}, \Lambda^2) = \sum_{d_1|P(z)} \sum_{d_2|P(z)} \rho_{d_1}\rho_{d_2} r_{(d_1,d_2)}. \tag{3.17}$$

It is easy to verify that these notation coincide with earlier notation (3.7) and (3.8). The goal of Selberg was to make the general inequality (3.14) optimal. We will first concern us about the main term G. We wish to minimize G with respect to the unknown numbers ρ_d subject to (3.11) and (3.13). The expression (3.16) is a quadratic form in ρ_d. In order to find the minimum of G it helps to diagonalize it. We assume that

$$\begin{aligned} 0 < g(p) < 1 &\quad \text{if } p|P(z) \\ g(p) = 0 &\quad \text{if } p \not| P(z). \end{aligned}$$

Let $h(d)$ be the multiplicative function defined by

$$h(p) = \frac{g(p)}{1 - g(p)}. \tag{3.18}$$

Rewriting gives

$$g(p) = \frac{h(p)}{1 + h(p)}. \tag{3.19}$$

We obtain

$$\begin{aligned} G &= \sum_{abc|P(z)} g(abc)\rho_{ac}\rho_{bc} \\ &= \sum_c g(c) \sum_{(a,b)=1} \sum g(a)g(b)\rho_{ac}\rho_{bc} \\ &= \sum_c g(c) \sum_d \mu(d)g(d)^2 \left(\sum_m g(m)\rho_{cdm} \right)^2 \\ &= \sum_{d|P(z)} h(d)^{-1} \left(\sum_{m\equiv 0 (\text{mod } d)} g(m)\rho_m \right)^2 \end{aligned}$$

If we now choose the following change of variables

$$y_d = \frac{\mu(d)}{h(d)} \sum_{m\equiv 0 (\text{mod } d)} g(m)\rho_m \tag{3.20}$$

we obtain

$$G = \sum_{d|P(z)} h(d)y_d^2, \tag{3.21}$$

which is a diagonal form.

Before we do something with this diagonal form, we will first look at the condition (3.11) and see what it is in terms of the new variables y_d. Using Möbius inversion we convert (3.20) into

$$\rho_l = \frac{\mu(l)}{g(l)} \sum_{d \equiv 0 (\operatorname{mod} l)} h(d) y_d. \tag{3.22}$$

In particular if we take $l = 1$ we then have

$$\sum_{d|P(z)} h(d) y_d = 1. \tag{3.23}$$

If we have a closer look at (3.20) and (3.22) we see that the support conditions for y_d are the same as ρ_d, hence

$$y_d = 0 \quad \text{if } d \geq \sqrt{D}. \tag{3.24}$$

Furthermore the convolution $1 * p$ in the new variables y_d becomes

$$\sum_{l|n} \rho_d = \sum_d \mu(d)((d,n)) h\left(\frac{d}{(d,n)}\right) y_d. \tag{3.25}$$

Having made this observations, we now wish to minimize (3.21) on the hyperplane (3.23). Applying Cauchy's inequality to (3.23) we derive

$$1 = \sum_{d|P(z)} h(d) y_d \leq \Bigg(\sum_{d|P(z)} h(d) y_d^2 \Bigg)^{\frac{1}{2}} \Bigg(\sum_{\substack{d|P(z) \\ d<\sqrt{D}}} h(d) \Bigg)^{\frac{1}{2}} = (GJ)^{\frac{1}{2}},$$

where we put

$$J = J(D) = \sum_{\substack{d|P \\ d<\sqrt{D}}} h(d). \tag{3.26}$$

Hence $JG \geq 1$. Since $h(d) \geq 0$, we have that $J \geq 0$. Hence G is greater of equal to J^{-1}. If we choose

$$y_d = J^{-1} \quad \text{if } d < \sqrt{D} \tag{3.27}$$

we have

$$G = \sum_{\substack{d|P(z) \\ d<\sqrt{D}}} h(d) y_d^2 \tag{3.28}$$

$$= \sum_{\substack{d|P(z) \\ d<\sqrt{D}}} h(d) J^{-2} = J^{-1}. \tag{3.29}$$

Hence the minimal value of G is J^{-1}. From now on we assume that y_d are the constants given by (3.27). We note that by just using the definition of $g(d)$ we could (3.26) also write as

$$J(D) = \sum_{d<\sqrt{D}} \mu^2(d) \prod_{p|d} \left(\frac{g(d)}{1-g(d)} \right). \tag{3.30}$$

Since we now know y_d we can compute p_l, by substituting (3.27) in (3.22). We have

$$\rho_l = \frac{\mu(l)}{g(l)} J^{-1} \sum_{\substack{d \equiv 0 \pmod{l} \\ d<\sqrt{D}}} h(d). \tag{3.31}$$

Since h is multiplcative and only supported on the square free number, we can rewrite $\sum_{\substack{d \equiv 0 \pmod{l} \\ d<\sqrt{D}}} h(d)$ as $\sum_{\substack{(d,l)=1 \\ d<\sqrt{D}/l}} h(d)h(l)$. So after pulling out this factor $h(l)$ we write (3.31) as

$$\rho_l = \mu(l) j(l) J_l(D) J^{-1} \tag{3.32}$$

where

$$j(l) = \frac{h(l)}{g(l)} = \prod_{p|l} \Big(1 - g(p)\Big)^{-1} \tag{3.33}$$

and

$$J_l(D) = \sum_{\substack{d<\sqrt{D}/l \\ (d,l)=1}} h(d). \tag{3.34}$$

and $J(D) = J_1(D)$. Now we know ρ_l, we use the definition of λ_d (3.12) to find

$$\lambda_d = \mu(d) j(d) \sum_{abc=d} \mu(c) j(c) J_{ac}(D) J_{bc}(D) J(D)^{-2}.$$

Moreover, the convolution $1 * \rho$, as in (3.25) becomes

$$\sum_{l|n} \rho_l = \frac{1}{J(D)} \sum_{q<\sqrt{D}} \mu((n,q)) h\left(\frac{q}{(n,q)} \right).$$

We will now show that $|\rho_d| \leq 1$. By grouping the terms in (3.26) according to the greatest common divisor of d and l we get that

$$\begin{aligned} J(D) &= \sum_{k|l} \sum_{\substack{d|P(z) \\ d<\sqrt{D} \\ (d,l)=k}} h(d) \\ &= \sum_{k|l} h(k) \sum_{\substack{mk|P(z) \\ m<\sqrt{D}/k \\ (m,l)=1}} h(m) \\ &\geq \Big(\sum_{k|l} h(k) \Big) \sum_{\substack{mk|P \\ m<\sqrt{D}/l \\ (m,l)=1}} h(m) = j(l) J_l(D). \end{aligned}$$

Combing this with (3.32) gives

$$|\rho_d| = j(l)J_l(D)J^{-1} \le 1. \tag{3.35}$$

Again using the definition of λ_d (3.12) gives

$$\begin{aligned} |\lambda_d| &\le \sum_{(d_1,d_2)=d} 1 \\ &\le \sum_{d_1 d_2 d_3 = d} 1. \end{aligned} \tag{3.36}$$

We define

$$\tau_3(d) = \sum_{d_1 d_2 d_3 = d} 1. \tag{3.37}$$

Collecting the above results we conclude the following:

Theorem 3.1. *(Selberg's Λ^2-Sieve). Let $\mathcal{A} = (a_n)$ be a finite sequence of non-negative number and let P be a finite product of distinct primes. For every $d|P$ we write*

$$|\mathcal{A}_d| = \sum_{n \equiv 0\,(mod\ d)} a_n = g(d)X + r_d(\mathcal{A}). \tag{3.38}$$

where $X > 0$ and $g(d)$ is a multiplicative function with $0 < g(p) < 1$ for $p|P$. Let $h(d)$ be the multiplicative function given by $h(p) = g(p)(1 - g(p))^{-1}$ and $J = J(D)$ given by (3.26) for some $D > 1$. Then we have

$$S(\mathcal{A}, z) = \sum_{(n,P)=1} a_n \le XJ^{-1} + R^+(\mathcal{A}, \Lambda^2), \tag{3.39}$$

where

$$R^+(\mathcal{A}, \Lambda^2) = \sum_{d|P} \lambda_d r_d(\mathcal{A}), \tag{3.40}$$

with λ_d given by (3.12) and (3.32)-(3.34).

Using (3.36) and (3.37) we can estimate the remainder crudely by

$$|R^+(\mathcal{A}, \Lambda^2)| \le R(\mathcal{A}, \Lambda^2) \le \sum_{\substack{d|P \\ d<D}} \tau_3(d)|r_d(\mathcal{A})|. \tag{3.41}$$

4 Rational Points on Varieties

This section describes how we view points on varieties such that we can count them.

Let $X \subset \mathbb{P}(\mathbb{Q})$ be the zero set of some homogeneous polynomials $F_1, \dots, F_R$ with integer coefficients and suppose X is smooth. Points of $\mathbb{P}(\mathbb{Q})$ can be represented by vectors $\boldsymbol{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}_{\text{prim}}$, where $\mathbb{Z}^{n+1}_{\text{prim}} = \{\boldsymbol{y} \in \mathbb{Z}^{n+1} : \gcd(y_0, \dots, y_n) = 1\}$. Recall that we have defined the height function $H : \mathbb{P}^n(\mathbb{Q}) \to \mathbb{Z}$ as

$$x \mapsto |\boldsymbol{x}| = \max_i(|x_i|),$$

where $\boldsymbol{x} = (x_0, \dots, x_n) \in \mathbb{Z}^{n+1}_{\text{prim}}$ represent $x \in \mathbb{P}^n_{\mathbb{Q}}$. Note that the function H does not depend on the choice of representation. Further let $m \in \mathbb{N}$ be fixed once and for all. For each prime p we suppose that we are given a non-empty set of residue classes $\Omega_{p^m} \subset X(\mathbb{Z}/p^m\mathbb{Z})$. Put

$$\Omega_M = \prod_{p^m || M} \Omega_{p^m}. \tag{4.1}$$

We are interested in the following counting function

$$N(P, \Omega) := \#\{x \in X(\mathbb{Q}) : |\boldsymbol{x}| \leq P, [\boldsymbol{x}]_{p^m} \in \Omega_{p^m} \text{ for all } p\},$$

where $\Omega = (\Omega_{p^m})_p$ and $[\boldsymbol{x}]_{p^m}$ is the reduction of $\boldsymbol{x}$ mod p^m.

We start by making the observation that

$$N(P, \Omega) \leq \#\{\boldsymbol{x} \in \mathbb{Z}^{n+1}_{\text{prim}} \backslash \{\mathbf{0}\} : |\boldsymbol{x}| \leq P,\ F_1(\boldsymbol{x}) = 0, \dots, F_R(\boldsymbol{x}) = 0,\ [\boldsymbol{x}]_{p^m} \in \hat{\Omega}_{p^m} \text{ for all } p\}$$

where $\hat{\Omega}_{p^m} = \{\boldsymbol{x} \in (\mathbb{Z}/p^m\mathbb{Z})^{n+1} : p \nmid \boldsymbol{x},\ (x_0 : \dots : x_n) \in \Omega_{p^m}\}$. The condition $\boldsymbol{x} \in \hat{\Omega}_{p^m}$ for all p implies that $\boldsymbol{x} \in \mathbb{Z}^{n+1}_{\text{prim}}$. Hence

$$N(P, \Omega) \leq \#\{\boldsymbol{x} \in \mathbb{Z}^{n+1} : |\boldsymbol{x}| \leq P,\ F_1(\boldsymbol{x}) = 0, \dots, F_R(\boldsymbol{x}) = 0, [\boldsymbol{x}]_{p^m} \in \hat{\Omega}_{p^m} \text{ for all } p\} \tag{4.2}$$

One has to keep in mind that $N(P, \Omega)$ depends on our choice of $F_1, \dots, F_R$ and therefore also on R and the degree of the polynomials F_i. Our goal is to give an upper bound for $N(P, \Omega)$. Without an indication the implied constant may depend on n, R, d and the coefficients of the polynomials $F_1, \dots, F_R$. In section 5 and 6 we will discuss the case when $R = 1$. In section 7, 8 and 9 we will discuss the case when $R > 1$.

5 The Sieve Problem for One Quadratic

We describe how we can rewrite $N(P,\Omega)$ in such a way that we can use Selberg's sieve, where we follow Browning and Loughran [8]. For the remainder of this section let $X \subset \mathbb{P}^n_{\mathbb{Q}}$ be a smooth projective variety defined by one homogeneous polynomial $F_1 \in \mathbb{Z}[x_0, \dots, x_n]$. What we do in this section you could also do for $F_1, \dots, F_R$ where $R > 1$, but we don't need that, so we won't discuss that. Recall from (4.2) that

$$N(P,\Omega) \leq \#\{\boldsymbol{x} \in \mathbb{Z}^{n+1} : |\boldsymbol{x}| \leq P,\ F_1(\boldsymbol{x}) = 0, [\boldsymbol{x}]_{p^m} \in \hat{\Omega}_{p^m} \text{ for all } p\}$$

Consider the function $\omega_0 : \mathbb{R} \to \mathbb{R}_{\geq 0}$, given by

$$\omega_0(x) = \begin{cases} e^{-(1-x^2)^{-1}} & \text{if } |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

We will work with the weight function $\omega : \mathbb{R}^{n+1} \to \mathbb{R}$, given by

$$\omega(\boldsymbol{x}) = \omega_0(\tfrac{5}{2}|\boldsymbol{x}| - 2).$$

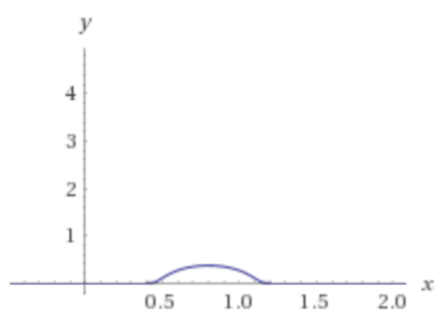

Figure 1: A plot of the weight function ω.

We have $\omega(\boldsymbol{x}) = 0$ unless $\frac{2}{5} < |\boldsymbol{x}| < \frac{6}{5}$. Moreover if $j \in \mathbb{N}$ and $1/2 < |2^j\boldsymbol{x}/P| \leq 1$, then $\omega(2^j\boldsymbol{x}/P) \geq \min\{\omega(1/2), \omega(1)\}$. Hence we can break the sum (4.2) into intervals, finding that

$$\begin{aligned} N(P,\Omega) &\leq \sum_{j=0}^{\infty} \#\left\{\boldsymbol{x} \in \mathbb{Z}^{n+1} : \substack{2^{-j-1}P < |\boldsymbol{x}| \leq 2^{-j}P \\ F_1(\boldsymbol{x})=0,\, [\boldsymbol{x}]_{p^m} \in \hat{\Omega}_{p^m} \text{ for all p}}\right\} \\ &\ll \sum_{j=0}^{\infty} \sum_{\substack{\boldsymbol{x} \in \mathbb{Z}^{n+1},\, F_1(\boldsymbol{x})=0 \\ [\boldsymbol{x}]_{p^m} \in \hat{\Omega}_{p^m} \text{ for all p}}} \omega(2^j\boldsymbol{x}/P). \end{aligned} \tag{5.1}$$

For every $\boldsymbol{x} \in \mathbb{Z}^{n+1}\{0\}$ we have $|\boldsymbol{x}| \geq 1$. Moreover if $|2^j\boldsymbol{x}/P| \geq \frac{6}{5}$, then $\omega(\boldsymbol{x}) = 0$. Hence if $j \geq \log_2(\frac{6}{5}P)$ then $\omega(2^j\boldsymbol{x}/P)$ is zero and thus the sum $\sum_j$ in (5.1) is finite. Since we can change P it is enough to find a upper bound for

$$\sum_{\substack{\boldsymbol{x} \in \mathbb{Z}^{n+1},\, F_1(\boldsymbol{x})=0 \\ [\boldsymbol{x}]_{p^m} \in \hat{\Omega}_{p^m} \text{ for all p}}} w(\boldsymbol{x}/P).$$

We define the density function ω_p as in Browning and Loughran [8],

$$\omega_p = 1 - \frac{\#\hat{\Omega}_{p^m}}{\#\hat{X}(\mathbb{Z}/p^m\mathbb{Z})} \in [0, 1]. \tag{5.2}$$

The notation $\hat{X}$ and $\hat{\Omega}$ stands for that we view the varieties X and Ω as affine space, i.e.

$$\hat{\Omega}_{p^m} = \{\boldsymbol{x} \in (\mathbb{Z}/p^m\mathbb{Z})^{n+1} : p \nmid \boldsymbol{x}, (x_0 : \ldots : x_n) \in \Omega_{p^m}\}$$

and

$$\hat{X}(\mathbb{Z}/p^m\mathbb{Z}) = \{\boldsymbol{x} \in (\mathbb{Z}/p^m\mathbb{Z})^{n+1} : p \nmid \boldsymbol{x}, F_1(x_0, \ldots, x_n) = 0 \bmod p^m\}.$$

Note that $\omega_p = 0$ implies that $\hat{\Omega}_{p^m} = \hat{X}(\mathbb{Z}/p^m\mathbb{Z})$. Since these primes, for which $\omega_p = 0$, do not play a role in the inclusion-exclusion process, we are not interested in those primes. Let $P, \xi \geq 1$ and let $\mathcal{P}$ denote the product over distinct primes $p < \xi$ for which $\omega_p > 0$, i.e. $\#\hat{\Omega}_{p^m} < \#\hat{X}(\mathbb{Z}/p^m\mathbb{Z})$. Consider for $n \in \mathbb{N}$ the following sequences

$$a_n := \sum_{\substack{\boldsymbol{x} \in \mathbb{Z}^{n+1} \\ F_1(\boldsymbol{x})=0 \\ n(\boldsymbol{x})=n}} w(\boldsymbol{x}/P), \quad \text{where } n(\boldsymbol{x}) = \prod_{\substack{p \mid \mathcal{P} \\ [\boldsymbol{x}]_{p^m} \in \hat{\Omega}^c_{p^m}}} p. \tag{5.3}$$

If $\boldsymbol{x} \in \mathbb{Z}^{n+1}$ with $F_1(\boldsymbol{x}) = 0$ and $[\boldsymbol{x}]_{p^m} \in \hat{\Omega}_{p^m}$ for all p, then $n(\boldsymbol{x}) = 1$ and thus then the term $w(\boldsymbol{x}/P)$ appears in a_1. Hence

$$\sum_{\substack{\boldsymbol{x} \in \mathbb{Z}^{n+1}, F_1(\boldsymbol{x})=0 \\ [\boldsymbol{x}]_{p^m} \in \hat{\Omega}_{p^m} \text{ for all p}}} w(\boldsymbol{x}/P) \leq \sum_{\substack{n \in \mathbb{N} \\ (n,\mathcal{P})=1}} a_n. \tag{5.4}$$

On the other hand if $\boldsymbol{x} \in \mathbb{Z}^{n+1}$ with $F_1(\boldsymbol{x}) = 0$ and $(n(\boldsymbol{x}), \mathcal{P}) = 1$, then $[\boldsymbol{x}]_{p^m} \in \Omega_{p^m}$ for all $p|\mathcal{P}$ and for all primes p for which $\omega_p = 0$. We can not say that $[\boldsymbol{x}]_{p^m} \in \Omega_{p^m}$ for all primes p, so the inequality (5.4) is not necessarily an equality.

We conclude that to find a found for $N(P, \Omega)$, it suffices to find a bound for $\sum_{\substack{n \in \mathbb{N} \\ (n,\mathcal{P})=1}} a_n$, with a_n as in (5.3).

6 Sieving on Quadrics

Thanks to the previous section we know how we can write the problem for finding an upper bound for $N(P,\Omega)$ as a sieve problem. In this section we apply Selberg's sieve as desribed in section 3 to find this upper bound.

Before we apply Selberg's sieve we need a formula for $\mathcal{A}_d$ as in (3.2). If we assume that $F_1 \in \mathbb{R}[x_0,\dots,x_n]$ is a smooth quadratic, a formula for A_d is already known which is stated in Theorem 6.1. After that, in Theorem 6.2, the upper bound for $N(P,\Omega)$ is given.

We use the same notation as in the previous section; let a_n and $n(\boldsymbol{x})$ be as in (5.3), and let $\mathcal{P}$ denote the product of distinct primes $p<\xi$ for which $\omega_p>0$. Let $d|\mathcal{P}$. We would like to write $\mathcal{A}_d$ as $g(d)X+r_d$, where $X>0$ and $g(d)$ a multiplicative function with $0<g(p)<1$ for every $p|\mathcal{P}$. We have

$$\begin{aligned}\mathcal{A}_d &= \sum_{d|n} a_n \\ &= \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^{n+1}, F_1(\boldsymbol{x})=0\\ d|n(\boldsymbol{x})}} \omega(\boldsymbol{x}/P) \qquad (6.1)\end{aligned}$$

We have that $d\,|\,n(\boldsymbol{x})$ if and only if $[\boldsymbol{x}]_{p^m}\in\hat{\Omega}^c_{p^m}$ for all $p\,|\,d$, for any $\boldsymbol{x}$ appearing in the definition of a_n. Hence (6.1) becomes

$$\mathcal{A}_d = \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^{n+1},\, F_1(\boldsymbol{x})=0\\ [\boldsymbol{x}]_m\in\Omega^c_{d^m}}} \omega(\boldsymbol{x}/P). \qquad (6.2)$$

We define

$$\hat{N}(P,\Omega_M) := \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^{n+1},\, F_1(\boldsymbol{x})=0\\ [\boldsymbol{x}]_M\in\Omega_M}} \omega(\boldsymbol{x}/P).$$

Theorem 6.1. *(Browning and Loughran [8, Theorem 4.1]). Assume that $n\geq 5$ and that $\nabla F_1(\boldsymbol{x})\gg 1$ for all $\boldsymbol{x}\in supp(w)$. Assume that M is coprime to $2\Delta_{F_1}$ and let Ω_M be as in (4.1). Then*

$$\hat{N}(P,\Omega_M) = \sigma_\infty(w)P^{n-1}\prod_{p\nmid M}\sigma_p\prod_{p^m||M}\frac{\#\Omega_{p^m}}{p^{mn}} + O_{\epsilon,F_1,\omega}(P^{(n+1)/2+\epsilon}M^{(n+1)/2+\epsilon}),\ \forall\epsilon>0,$$

where $\sigma_\infty(w)$ is the weighted real density associated to F_1 and ω as defined in [12, Th. 3] and where σ_p is the p-adic density associated to F_1 and defined as

$$\sigma_p = \lim_{k\to\infty} p^{-nk}\#\{\boldsymbol{x}\in(\mathbb{Z}/p^k\mathbb{Z})^{n+1} : p\nmid \boldsymbol{x},\ F_1(\boldsymbol{x})\equiv 0 \ mod\ p^k\}$$

for each prime p.

If we choose $M=d^m$ and $\Omega_M=\hat{\Omega}_{d^m}$ in the previous theorem, we derive from (6.2) that

$$\mathcal{A}_d = \hat{N}(P,\hat{\Omega}^c_{d^m}) = g(d)P^{n-1}\sigma_\infty(\omega)\prod_p\sigma_p + O_{\epsilon,X}(d^{m(n+1)/2+\epsilon/4}P^{(n+1)/2+\epsilon}), \qquad (6.3)$$

where

$$g(d) = \prod_{p|d} \Big(1 - \frac{\#\hat{\Omega}_{p^m}}{\#\hat{X}(\mathbb{Z}/p^m\mathbb{Z})}\Big). \tag{6.4}$$

In deriving the formula $g(d)$ we used that for all $p|d$ we have $\#\hat{X}(\mathbb{Z}/p^m\mathbb{Z}) = \#\hat{X}(\mathbb{Z}/p\mathbb{Z}) \cdot p^{(m-1)n}$ and $\sigma_p = \hat{X}(\mathbb{Z}/p\mathbb{Z}) \cdot p^{-n}$, which follows from a quantitative version of Hensel's lemma [8, Lemma 2.1]. We indeed have

$$\begin{aligned} g(d) \prod_{p|M} \sigma_p &= \prod_{p|M} \Big(1 - \frac{\#\hat{\Omega}_{p^m}}{\#\hat{X}(\mathbb{Z}/p^m\mathbb{Z})}\Big)\sigma_p \\ &= \prod_{p|M} \frac{\#\hat{\Omega}^c_{p^m}}{\#\hat{X}(\mathbb{Z}/p^m\mathbb{Z})} \cdot \sigma_p \\ &= \prod_{p|M} \frac{\#\hat{\Omega}^c_{p^m}}{\#\hat{X}(\mathbb{Z}/p\mathbb{Z}) \cdot p^{(m-1)n}} \cdot \frac{\hat{X}(\mathbb{Z}/p\mathbb{Z})}{p^n} \\ &= \prod_{p^m||M} \frac{\#\hat{\Omega}^c_{p^m}}{p^{mn}}. \end{aligned}$$

Since we have now a formula for $\mathcal{A}_d$ we can plug this in in the Selberg sieve [11, p. 93] and find the following theorem.

Theorem 6.2. *(Browning and Loughran [8, Theorem 1.7]). Assume that $X \subset \mathbb{P}^n$ is a smooth quadratic of dimension at least 3 over $\mathbb{Q}$. Let $m \in \mathbb{N}$ and let $\Omega_{p^m} \subset X(\mathbb{Z}/p^m\ Z)$ for each prime p. Further, let ω_p be as in (5.2). Assume that*

$$0 \leq \omega_p < 1 \quad \textit{for all } p.$$

Then, for any $\xi > 1$ and any $\epsilon > 0$, we have

$$N(P, \Omega) \ll_{\epsilon,X} \frac{P^{n-1}}{J(\xi)} + \xi^{m(n+1)+2+\epsilon} P^{(n+1)/2+\epsilon},$$

where $J(\xi) = \sum_{k<\xi} \mu^2(k) \prod_{p|k} \left(\frac{\omega_p}{1-\omega_p}\right)$.

Proof. First we use (6.3). It is clear from the definition of $g(d)$ in (6.4) that $g(d)$ is multiplicative. Now apply Selberg's sieve as in Theorem 3.1 and find that

$$\sum_{(n,\mathcal{P})=1} a_n \ll_{\epsilon,X} \frac{P^{n-1}}{J(\xi)} + \sum_{d \leq \xi^2} \tau_3(d) d^{m(n+1)/2+\epsilon/4} P^{(n+1)/2+\epsilon},$$

where $J(\xi) = \sum_{k<\xi} \mu^2(k) \prod_{p|k} \left(\frac{\omega_p}{1-\omega_p}\right)$. Taking the trivial bound $\tau_3(d) \ll d^{\epsilon/4}$ and summing over $d \leq \xi^2$, we get that

$$\sum_{(n,\mathcal{P})=1} a_n \ll_{\epsilon,X} \frac{P^{n-1}}{J(\xi)} + \xi^{m(n+1)+2+\epsilon} P^{(n+1)/2+\epsilon}.$$

□

7 The Sieve Problem for a System of Quadratics

One can wonder if we can also find a formula for $N(P,\Omega)$ if $R>1$. The answer is that this is possible. We will give a proof in the following sections. In this section we formulate the problem in to a sieve problem. In the next section we prove a analogue for Theorem 6.1 for the case when $R>1$. In section 9 we are ready to prove a formula for $N(P,\Omega)$ if $R>1$.

Recall from (4.2) that

$$N(P,\Omega)\le \#\{\boldsymbol{x}\in\mathbb{Z}^{n+1}: |\boldsymbol{x}|\le P,\ F_1(\boldsymbol{x})=0,\dots,F_R(\boldsymbol{x})=0,[\boldsymbol{x}]_{p^m}\in\hat{\Omega}_{p^m}\text{ for all }p\}$$

Let $\mathcal{B}$ be a box in $\mathbb{R}^{n+1}$, contained in the box $[-1,1]^{n+1}$ and having sides of length 1 which are parallel to the coordinate axes. We are interested in a upper bound for

$$\tilde{N}(P,\Omega)=\#\{\boldsymbol{x}\in\mathbb{Z}^{n+1}:\boldsymbol{x}\in P\mathcal{B},\boldsymbol{F}(x)=0,\boldsymbol{x}\in\Omega_{p^m}\text{ for all }p\},\tag{7.1}$$

where $\boldsymbol{F}$ is the vector $(F_1,F_2,\dots,F_R)^T$ and F_i are smooth quadratics. If we compare this with the notations we see that if we choose $\mathcal{B}=[-\frac{1}{2},\frac{1}{2}]^{n+1}$, we have

$$N(P,\Omega)\le\tilde{N}(2P,\hat{\Omega}),$$

where $N(P,\hat{\Omega})$ is as in section 4. We use this notation because later it will be more usefull. We have

$$\tilde{N}(P,\Omega)=\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^{n+1}\cap P\mathcal{B}\\ \boldsymbol{F}(x)=0\\ \boldsymbol{x}\in\Omega}}1.$$

If we compare this with the sieve problem in section 5 it looks similar, only now we don't have the weight function ω. This ω was needed in section 5 because then there is known an asymptotic for $N(P,\Omega_M)$, which uses this weight function ω. For our approach we don't need this weighted function.

Let $P,\xi\ge 1$ and let $\mathcal{P}$ denote the product over distinct primes $p<\xi$ for which $\#\Omega_{p^m}<\#X(\mathbb{Z}/p^m\mathbb{Z})$. Consider for $n\in\mathbb{N}$ the following sequences

$$a_n:=\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^{n+1}\cap P\mathcal{B}\\ \boldsymbol{F}(\boldsymbol{x})=0\\ n(\boldsymbol{x})=n}}1,\qquad\text{where } n(\boldsymbol{x})=\prod_{\substack{p|\mathcal{P}\\ [\boldsymbol{x}]_{p^m}\in\Omega^c_{p^m}}}p.$$

If $\boldsymbol{x}\in\mathbb{Z}^{n+1}\cap P\mathcal{B}$ with $\boldsymbol{F}(\boldsymbol{x})=0$ and $[\boldsymbol{x}]_{p^m}\in\Omega_{p^m}$ for all p, then $n(\boldsymbol{x})=1$ and thus this $\boldsymbol{x}$ is counted in a_1. Hence

$$\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^{n+1}\cap P\mathcal{B}\\ \boldsymbol{F}(x)=0\\ \boldsymbol{x}\in\Omega}}1\le\sum_{\substack{n\in\mathbb{N}\\ (n,\mathcal{P})=1}}a_n.\tag{7.2}$$

On the other hand if $\boldsymbol{x}\in\mathbb{Z}^{n+1}\cap P\mathcal{B}$ with $\boldsymbol{F}(\boldsymbol{x})=0$ and $(n(\boldsymbol{x}),\mathcal{P})=1$, then $[\boldsymbol{x}]_{p^m}\in\Omega_{p^m}$ for all $p|\mathcal{P}$ and for all primes p for which $\#\Omega_{p^m}=\#\hat{X}(\mathbb{Z}/p^m\mathbb{Z})$. We can not say that $[\boldsymbol{x}]_{p^m}\in\Omega_{p^m}$

for all primes p, so the inequality (7.2) is not necessarily an equality. So it is enough to find an upper bound for $\sum_{\substack{n\in\mathbb{N} \\ (n,\mathcal{P})=1}} a_n$.

Now we have formulated our problem into a sieve problem, Selberg's sieve comes in the picture. Before we can use this sieve, we have to find a formula for $\mathcal{A}_d(x)$. We have

$$\begin{aligned}\mathcal{A}_d &= \sum_{d|n} a_n \\ &= \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^{n+1}\cap P\mathcal{B} \\ \boldsymbol{F}(\boldsymbol{x})=0 \\ d|n(\boldsymbol{x})}} 1 \end{aligned} \tag{7.3}$$

We have that $d \,|\, n(\boldsymbol{x})$ if and only if $[\boldsymbol{x}]_{p^m} \in \Omega^c_{p^m}$ for all $p \,|\, d$, for any $\boldsymbol{x}$ appearing in the definition of a_n. Hence (7.3) becomes

$$\mathcal{A}_d = \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^{n+1}\cap P\mathcal{B} \\ \boldsymbol{F}(\boldsymbol{x})=0 \\ [\boldsymbol{x}]_m\in\Omega^c_{d^m}}} 1.$$

So if we have an asymptotic formula for $\mathcal{A}_d$, which we will do in the next section, then by applying Selberg's sieve we are able to find an upper bound for $\tilde{N}(P,\Omega)$, which will we do in section 9.

8 Systems of Quadratics

In this section we find a asymptotic formula for $\tilde{N}(P,\Omega_M)$ (defined below). This formula is stated in Theorem 1.4, which will be the main theorem of this section and which will help us (in the next section) to find a formula for $\mathcal{A}_d$ (previous section).

First we will fix the notation and state Theorem 1.4. In section 8.2 we will use exponential sums to rewrite $\tilde{N}(P,\Omega_M)$. After that an auxiliary inequality will be discussed. In section 8.4 the circle method will be used to prove some upper bounds, which will be put together in section 8.5 to prove Theorem 1.4.

8.1 Notation and Main Theorem

Definition 8.1. *Let f_i be polynomials in n variables. Define*

$$\tilde{N}(P,\Omega_M) = \#\{\boldsymbol{x} \in \mathbb{Z}^n : \boldsymbol{x} \in P\mathcal{B}, \boldsymbol{f}(x) = 0, [\boldsymbol{x}]_M \in \Omega_M\},$$

where $\boldsymbol{f} = (f_1, \ldots, f_R)^T$.

We are interested in an asymptotic formula for $\tilde{N}(p,\Omega_M)$.

From now $f_1, f_2, \ldots, f_R$ are polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ of degree $d = 2$. We will often write d instead of just 2, so that it will be clear how the degree effects $\tilde{N}(p,\Omega_M)$. We write $f_i^{[d]}$ (or $f_i^{[2]}$) for the degree $d = 2$ part of f_i and we write $\boldsymbol{f}$ for the vector $(f_1, f_2, \ldots, f_R)^T$ in $(\mathbb{R}[x])^R$.

We follow a paper by Myerson [20]. In that paper he proves, under some conditions, a asymptotic formula for $\tilde{N}(P,\Omega_M)$ if $\Omega_{p^m} = X(\mathbb{Z}/p^m\mathbb{Z})$. We will use this proof to find a asymptotic formula for $\tilde{N}(P,\Omega_M)$ if Ω_{p^m} is not necessarily equal to $X(\mathbb{Z}/p^m\mathbb{Z})$. Our goal will be to prove Theorem 1.4.

Definition 8.2. *Let $h(\boldsymbol{x})$ be any polynomial of degree $d \geq 2$ with real coefficients in n variables $x_1, \ldots, x_n$. For $i = 1, \ldots, n$ we define*

$$m_i^{(h)}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}) = \sum_{j_1,\ldots,j_{d-1}=1}^{n} x_{j_1}^{(1)} \cdots x_{j_{d-1}}^{(d-1)} \frac{\partial^d h(\boldsymbol{x})}{\partial x_{j_1} \cdots \partial x_{j_{d-1}} \partial x_i},$$

where we write $\boldsymbol{x}^{(j)}$ for a vector of n variables $(x_1^{(j)}, \ldots, x_n^{(j)})^T$. This gives an n-tuple

$$\boldsymbol{m}^{(h)}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}) \in \mathbb{R}[\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}]^n.$$

For each $B \geq 1$ we put $N_h^{\mathrm{aux}}(B)$ for the number of $(d-1)$-tuples of integer n-vectors $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)}$ with

$$|\boldsymbol{x}^{(1)}|, \ldots, |\boldsymbol{x}^{(d-1)}| \leq B,$$

$$|\boldsymbol{m}^{(h)}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(d-1)})| < |h^{[d]}|_\infty B^{d-2},$$

where we let $|h^{[d]}|_\infty = \frac{1}{d!} \max_{\boldsymbol{j} \in \{1,\ldots,n\}^d} |\frac{\partial^d h(\boldsymbol{x})}{\partial x_{j_1} \cdots \partial x_{j_d}}|$.

Let us recall Theorem 1.4.

Theorem 1.4. *Let $f_i \in \mathbb{Z}[x_1, \ldots, x_n]$ be quadratic forms with integer coefficients and $n \geq 2$ and $\dim(\tilde{X}^*) \leq n-1$, where $\tilde{X}^*$ is defined as in §8.4.1. Let $N_h^{aux}(B)$ as in Definition 8.2. Suppose that the f_i are linearly independent and that*

$$N^{aux}_{\boldsymbol{\beta}\cdot\boldsymbol{f}}(B) \leq C_0 B^{(d-1)n-2^d\mathscr{C}}$$

for some $C_0 \geq 1, \mathscr{C} > dR$ and all $\boldsymbol{\beta} \in \mathbb{R}^R$ and $B \geq 1$, where $\boldsymbol{\beta}\cdot\boldsymbol{f} = \beta_1 f_1 + \ldots + \beta_R f_R$. Then for all $P \geq 1$ we have

$$\tilde{N}(P, \Omega_M) = \sum_{[\boldsymbol{v}]_M \in \Omega_M} (\mathfrak{J}\mathfrak{S}(\boldsymbol{v}; M) P^{n-dR} M^{-n} + O(P^{n-dR-\delta} M^{-n+\delta_2})),$$

where the implied constant depends at most on $C_0, \mathscr{C}$ and the f_i, and δ and δ_2 are positive constants depending at most on $\mathscr{C}, d$ and R. Here $\mathfrak{J}$ and $\mathfrak{S}$ are as in Lemma 8.11 and Lemma 8.10.

If $\Omega_{p^m} = X(\mathbb{Z}/p^m\mathbb{Z})$, the asymptotic formula that Myerson [20] found, under the same conditions as Theorem 1.4, is

$$\tilde{N}(P, X(\mathbb{Z}/p^m\mathbb{Z})) = \mathfrak{J}\mathfrak{S} P^{n-dR} + O(P^{n-dR-\delta}), \tag{8.1}$$

where the error term may depend on $\boldsymbol{f}$. One can wonder if we can use this formula directly in the case when $\Omega_{p^m} \neq X(\mathbb{Z}/p^m\mathbb{Z})$ in the following way. Let $[\boldsymbol{v}]_M \in \Omega_M$ and $\boldsymbol{G}_{M,\boldsymbol{v}}(\boldsymbol{x}) = \boldsymbol{F}(M\boldsymbol{x}+\boldsymbol{v})$. Now we view M and $\boldsymbol{v}$ as coefficients of $\boldsymbol{G}_{M,\boldsymbol{v}}$. We first compute $\tilde{N}(P, X(\mathbb{Z}/p^m\mathbb{Z}))$ for $\boldsymbol{G}$ with Myersons formula (8.1) and then sum over all $[\boldsymbol{v}]_M \in \Omega$, finding that

$$\tilde{N}(P, \Omega_{p^m}) = \sum_{[\boldsymbol{v}]_M \in \Omega_M} \Big(\mathfrak{J}\mathfrak{S} P^{n-dR} + O(P^{n-dR-\delta})\Big). \tag{8.2}$$

The problem with this is that if we use Myerson's formula for $\boldsymbol{G}_{M,\boldsymbol{v}}$ the error term may depend on M and $\boldsymbol{v}$. Hence the error term in (8.2) may depend on $\boldsymbol{v}$ and M.

8.2 The Exponential Sum

Definition 8.3. *For each $[\boldsymbol{v}]_M \in \Omega_M$, $\boldsymbol{\alpha} \in \mathbb{R}^R$ and $P \geq 1$ we define the exponential sum*

$$S(\boldsymbol{\alpha}; P; \boldsymbol{v}; M) = \sum_{\substack{\boldsymbol{y} \in \mathbb{Z}^n \\ M\boldsymbol{y}+\boldsymbol{v} \in P\mathcal{B}}} e(\boldsymbol{\alpha}\cdot\boldsymbol{f}(M\boldsymbol{y}+\boldsymbol{v})),$$

where $\cdot$ denotes the standard inproduct and $e(x) = e^{2\pi i x}$.

Lemma 8.4. *If the f_i have integer coefficients, then we have*

$$\tilde{N}(P, \Omega_M) = \sum_{[\boldsymbol{v}]_M \in \Omega_M} \int_{[0,1]^R} S(\boldsymbol{\alpha}; P; \boldsymbol{v}; M)\, d\boldsymbol{\alpha}.$$

Proof. First we note that $\tilde{N}(P,\Omega_M) = \sum_{[\boldsymbol{v}]_M\in\Omega_M}\tilde{N}(P,[\boldsymbol{v}]_M)$, so it is sufficient to prove that for every $[\boldsymbol{v}]_M \in \Omega_M$ we have $\tilde{N}(P,[\boldsymbol{v}]_M) = \int_{[0,1]^R} S(\boldsymbol{\alpha};P;\boldsymbol{v};M)\,\mathrm{d}\boldsymbol{\alpha}$. Let $[\boldsymbol{v}]_M \in \Omega_M$. If $\boldsymbol{x} \in P\mathcal{B}\cap\mathbb{Z}^n$ with $\boldsymbol{f}(\boldsymbol{x}) = 0$ and $\boldsymbol{x} = \boldsymbol{v} \bmod M$, then there is an $\boldsymbol{y} \in \mathbb{Z}^n$ such that $\boldsymbol{x} = M\boldsymbol{y} + \boldsymbol{v}$. Hence

$$\int_{[0,1]^R} e(\boldsymbol{\alpha}\cdot\boldsymbol{f}(M\boldsymbol{y}+\boldsymbol{v}))\,\mathrm{d}\boldsymbol{\alpha} = \int_{[0,1]^R} e(\boldsymbol{\alpha}\cdot 0)\,\mathrm{d}\boldsymbol{\alpha} = \int_{[0,1]^R} 1\ \mathrm{d}\boldsymbol{\alpha} = 1.$$

Conversely if $f_i(\boldsymbol{x}) \neq 0$ and $\boldsymbol{x} = M\boldsymbol{y}+\boldsymbol{v}$ for some $\boldsymbol{y}\in\mathbb{Z}^n$, then

$$\int_{[0,1]} e(\boldsymbol{\alpha}_i\cdot f_i(M\boldsymbol{y}+\boldsymbol{v}))\,\mathrm{d}\boldsymbol{\alpha}_i = \frac{1}{f_i(\boldsymbol{x})}e(\boldsymbol{\alpha}_i\cdot f_i(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{\alpha}\Big|_0^1 = \frac{1}{f_i(\boldsymbol{x})}(1-1) = 0.$$

Hence if $\boldsymbol{f}(\boldsymbol{x}) \neq 0$ and $\boldsymbol{x} = M\boldsymbol{y}+\boldsymbol{v}$ for some $\boldsymbol{y}\in\mathbb{Z}^n$, then

$$\int_{[0,1]^R} e(\boldsymbol{\alpha}\cdot\boldsymbol{f}(M\boldsymbol{y}+\boldsymbol{v}))\,\mathrm{d}\boldsymbol{\alpha} = 0,$$

which completes the proof. □

8.3 The Auxiliary Inequality

Definition 8.5. *Let $h, \boldsymbol{m}^{(h)}(\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(d-1)})$ be as in Definition 8.2. Given $B \geq 1$ and $\delta > 0$, we let $U_h(B,\delta)$ be the number of $(d-1)$-tuples of integer n-vectors $\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(d-1)}$ such that*

$$|\boldsymbol{x}^{(1)}|,\ldots,|\boldsymbol{x}^{(d-1)}| \leq B \qquad \min_{\boldsymbol{y}\in\mathbb{Z}^n}|\boldsymbol{y}-\boldsymbol{m}^{(h)}(\boldsymbol{x}^{(1)},\ldots,\boldsymbol{x}^{(d-1)})| < \delta.$$

Note that U_h only depends on the degree d part of h. For this reason we have

$$U_{h(M\boldsymbol{x}+\boldsymbol{v})}(B,\delta) = U_{M^d h(\boldsymbol{x})}(B,\delta)$$

for every $[\boldsymbol{v}]_M \in \Omega_M$.

Lemma 8.6. *Let $U_h(B,\delta)$ be as in Definition 8.5. Let $[\boldsymbol{v}]_M \in \Omega_M$. For all $\epsilon > 0$, $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{R}^R$ and $\theta \in (0,1]$, we have*

$$\min\left\{\left|\frac{S(\boldsymbol{\alpha};P;\boldsymbol{v};M)}{(P/M)^{n+\epsilon}}\right|,\left|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta};P;\boldsymbol{v};M)}{(P/M)^{n+\epsilon}}\right|\right\}^{2^d} \ll_{d,n,\epsilon} \frac{U_{\boldsymbol{\beta}\cdot M^d\boldsymbol{f}}((P/M)^\theta,(P/M)^{(d-1)\theta-d})}{(P/M)^{(d-1)\theta n}} \tag{8.3}$$

where the implied constants depend only on d, n and ϵ.

This lemma shows us that if the right hand side is small then you get a saving for one of the exponential sums of the left hand side.

Proof. To prove (8.3) it is sufficient to prove

$$\left|\frac{S(\boldsymbol{\alpha};P;\boldsymbol{v};M)S(\boldsymbol{\alpha}+\boldsymbol{\beta};P;\boldsymbol{v};M)}{(P/M)^{2(n+\epsilon)}}\right|^{2^{d-1}} \ll_{d,n,\epsilon} \frac{U_{\boldsymbol{\beta}\cdot M^d\boldsymbol{f}}((P/M)^\theta,(P/M)^{(d-1)\theta-d})}{(P/M)^{(d-1)\theta n}}. \tag{8.4}$$

Let us first eliminate $\boldsymbol{\alpha}$ in the exponential sums in the left hand side of (8.4). We use the fact that if $M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B}$ and $M(\boldsymbol{x}+\boldsymbol{z})+\boldsymbol{v}\in P\mathcal{B}$, then $|\boldsymbol{z}|\leq P/M$.

$$\begin{aligned}
&S(\boldsymbol{\alpha}+\boldsymbol{\beta};P;\boldsymbol{v};M)\overline{S(\boldsymbol{\alpha};P;\boldsymbol{v};M)}\\
&\qquad= \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B}}}\ \sum_{\substack{\boldsymbol{z}\in\mathbb{Z}^n\\ M(\boldsymbol{x}+\boldsymbol{z})+\boldsymbol{v}\in P\mathcal{B}}} e((\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\boldsymbol{f}(M\boldsymbol{x}+\boldsymbol{v})-\boldsymbol{\alpha}\cdot\boldsymbol{f}(M(\boldsymbol{x}+\boldsymbol{z})+\boldsymbol{v}))\\
&\qquad\leq \sum_{\substack{\boldsymbol{z}\in\mathbb{Z}^n\\ |\boldsymbol{z}|\leq P/M}}\Big|\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B}}} e((\boldsymbol{\alpha}+\boldsymbol{\beta})\cdot\boldsymbol{f}(M\boldsymbol{x}+\boldsymbol{v})-\boldsymbol{\alpha}\cdot\boldsymbol{f}(M(\boldsymbol{x}+\boldsymbol{z})+\boldsymbol{v}))\Big|\\
&\qquad\leq \sum_{\substack{\boldsymbol{z}\in\mathbb{Z}^n\\ |\boldsymbol{z}|\leq P/M}}\Big|\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B}}} e(\boldsymbol{\beta}\cdot M^d\boldsymbol{f}^{[d]}(\boldsymbol{x})+g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z},M,\boldsymbol{v}}(\boldsymbol{x}))\Big|
\end{aligned}$$

for some real polynomial $g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z},M,\boldsymbol{v}}(\boldsymbol{x})$ of degree at most $d-1$ in $\boldsymbol{x}$. Now first apply Lemma 12.2 and then Lemma 12.1 to deduce that

$$\begin{aligned}
&|S(\boldsymbol{\alpha}+\boldsymbol{\beta};P;\boldsymbol{v};M)\overline{S(\boldsymbol{\alpha};P;\boldsymbol{v};M)}|^{2^{d-1}}\\
&\quad\leq\Big(\sum_{\substack{\boldsymbol{z}\in\mathbb{Z}^n\\ |\boldsymbol{z}|\leq P/M}}\Big|\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B}}} e(\boldsymbol{\beta}\cdot M^d\boldsymbol{f}^{[d]}(\boldsymbol{x})+g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z},M,\boldsymbol{v}}(\boldsymbol{x}))\Big|\Big)^{2^{d-1}}\\
&\quad\leq (2P/M+1)^{n(2^{d-1}-1)}\Big|\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B}}} e(\boldsymbol{\beta}\cdot M^d\boldsymbol{f}^{[d]}(\boldsymbol{x})+g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z},M,\boldsymbol{v}}(\boldsymbol{x}))\Big|^{2^{d-1}}\\
&\quad\ll_{d,n} (2P/M)^{2^{d-1}n}\Big|\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B}}} e(\boldsymbol{\beta}\cdot M^d\boldsymbol{f}^{[d]}(\boldsymbol{x})+g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z},M,\boldsymbol{v}}(\boldsymbol{x}))\Big|^{2^{d-1}}. \qquad (8.5)
\end{aligned}$$

Thanks to Myerson [20, Proof of Lemma 3.1] we have the following bound

$$\Big(\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ \boldsymbol{x}\leq P\mathcal{B}}} e(\boldsymbol{\alpha}\cdot\boldsymbol{h}(x))\Big)^{2^{d-1}} \ll_{d,n,\epsilon} P^{2^{d-1}n-(d-1)n\theta+\epsilon}U_{\boldsymbol{\alpha}\cdot\boldsymbol{h}}(P^{\theta},P^{(d-1)\theta-d}), \qquad (8.6)$$

for some $\boldsymbol{h}\in(\mathbb{Z}[x])^R$ where all h_i are of degree $d\geq 2$. The implied constant in (8.6) does not depend on $\boldsymbol{h}$. The inner sum in (8.5) has the same form as the left hand side of (8.6), with $(P\mathcal{B}-\boldsymbol{v})/M$ in place of $P\mathcal{B}$ and with $\boldsymbol{\beta}\cdot M^d\boldsymbol{f}^{[d]}(\boldsymbol{x})+g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z},M}(\boldsymbol{x})$ in place of $\boldsymbol{\alpha}\cdot\boldsymbol{h}$ as underlying polynomial. So applying (8.6) to (8.5) gives

$$\begin{aligned}
&\Big|\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B}}} e(\boldsymbol{\beta}\cdot M^d\boldsymbol{f}^{[d]}(\boldsymbol{x})+g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z},M,\boldsymbol{v}}(\boldsymbol{x}))\Big|^{2^{d-1}}\\
&\ll_{d,n,\epsilon}\Big(\frac{P}{M}\Big)^{2^{d-1}n-(d-1)n\theta+\epsilon}U_{\boldsymbol{\beta}\cdot M^d\boldsymbol{f}^{[d]}(\boldsymbol{x})+g_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{z},M}(\boldsymbol{x})}\Big(\Big(\frac{P}{M}\Big)^{\theta},\Big(\frac{P}{M}\Big)^{(d-1)\theta-d}\Big) \qquad (8.7)\\
&=\Big(\frac{P}{M}\Big)^{2^{d-1}n-(d-1)n\theta+\epsilon}U_{\boldsymbol{\beta}\cdot M^d\boldsymbol{f}(\boldsymbol{x})}\Big(\Big(\frac{P}{M}\Big)^{\theta},\Big(\frac{P}{M}\Big)^{(d-1)\theta-d}\Big),
\end{aligned}$$

as U_h depends only on the degree part of h.

Putting it all together gives

$$|S(\boldsymbol{\alpha}+\boldsymbol{\beta};P;\boldsymbol{v};M)\overline{S(\boldsymbol{\alpha};P;\boldsymbol{v};M)}|^{2^{d-1}} \ll_{d,n,\epsilon} \left(\frac{P}{M}\right)^{2^{d-1}n}\left(\frac{P}{M}\right)^{2^{d-1}n-(d-1)n\theta+\epsilon} U_{\boldsymbol{\beta}\cdot M^d\boldsymbol{f}(\boldsymbol{x})}\Big(\Big(\frac{P}{M}\Big)^{\theta},\Big(\frac{P}{M}\Big)^{(d-1)\theta-d}\Big),$$

which proves (8.4). We can see now why it is so important that the implied constant of (8.6) does not depend on $\boldsymbol{h}$, namely then the implied constant of (8.7) does not depend on $M^d\boldsymbol{f}$, in particular not on M. And this will give us the statement of the lemma, where the implied constant does not depend on M. In the end we are interested in a bound for $S(\alpha;P;\boldsymbol{v};M)$ which does not depend on M. □

Proposition 8.7. *Let $N_h^{\mathrm{aux}}(B)$, $|f|_\infty$ be as in Definition 8.2. Suppose that we are given $C_0 \geq 1$ and $\mathscr{C} > 0$ such that for all $\boldsymbol{\beta} \in \mathbb{R}^R$ and $B \geq 1$ we have*

$$N_{\boldsymbol{\beta}\cdot\boldsymbol{f}}^{\mathrm{aux}}(B) \leq C_0 B^{(d-1)n-2^d\mathscr{C}}. \tag{8.8}$$

Let $\kappa > \mu > 0$ such that for all $\boldsymbol{\beta} \in \mathbb{R}^R$ we have

$$\mu|\boldsymbol{\beta}| \leq |\boldsymbol{\beta}\cdot\boldsymbol{f}^{[d]}|_\infty \leq \kappa|\boldsymbol{\beta}|, \tag{8.9}$$

Let $\epsilon > 0$. Then there exists a constant $C \geq 1$, depending on C_0, d, n, μ, κ and ϵ, such that the bound

$$\min\Big\{\Big|\frac{S(\boldsymbol{\alpha};P;\boldsymbol{v};M)}{(P/M)^{n+\epsilon}}\Big|,\Big|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta};P;\boldsymbol{v};M)}{(P/M)^{n+\epsilon}}\Big|\Big\} \leq C\max\{P^{-d}|\boldsymbol{\beta}|^{-1}, |\boldsymbol{\beta}|^{\frac{1}{d-1}}M^{\frac{d}{d-1}}\}^{\mathscr{C}} \tag{8.10}$$

holds for all $P/M \geq 1$ and all $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{R}^R$.

Proof. Since (8.8) has to hold for every $\boldsymbol{\beta}$, we replace $N_{\boldsymbol{\beta}\cdot\boldsymbol{f}}^{aux}((P/M)^\theta)$ by $N_{\boldsymbol{\beta}\cdot M^d\boldsymbol{f}}^{aux}((P/M)^\theta)$. Let us first suppose that for some $\theta > 0$ we have

$$N_{\boldsymbol{\beta}\cdot M^d\boldsymbol{f}}^{aux}((P/M)^\theta) < U_{\boldsymbol{\beta}\cdot M^d\boldsymbol{f}}((P/M)^\theta,(P/M)^{(d-1)\theta-d}). \tag{8.11}$$

Then there must be a $(d-1)$-tuple of vectors $\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)} \in \mathbb{Z}^n$ which is included in the count $U_{\boldsymbol{\beta}\cdot M^d\boldsymbol{f}}((P/M)^\theta,(P/M)^{(d-1)\theta-d})$ but not in $N_{\boldsymbol{\beta}\cdot M^d\boldsymbol{f}}^{aux}((P/M)^\theta)$.

Since the $(d-1)$-tuple $(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})$ is counted by $U_{\boldsymbol{\beta}\cdot M^d\boldsymbol{f}}((P/M)^\theta,(P/M)^{(d-1)\theta-d})$, the inequality $|\boldsymbol{x}^{(i)}| \leq (P/M)^\theta$ holds for each $i = 1,\dots,d-1$, and we have the bound

$$|\boldsymbol{y}-\boldsymbol{m}^{(\boldsymbol{\beta}\cdot M^d\boldsymbol{f})}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})| < (P/M)^{(d-1)\theta-d}, \tag{8.12}$$

for some $\boldsymbol{y} \in \mathbb{Z}^n$. Since the $(d-1)$-tuple $(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})$ is not counted by $N_{\boldsymbol{\beta}\cdot M^d\boldsymbol{f}}^{aux}((P/M)^\theta)$, we must also have

$$|\boldsymbol{y}-\boldsymbol{m}^{(\boldsymbol{\beta}\cdot M^d\boldsymbol{f})}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})| \geq |\boldsymbol{\beta}\cdot M^d\boldsymbol{f}^{[d]}|_\infty(P/M)^{(d-2)\theta}. \tag{8.13}$$

we use (8.12) and (8.13) to relate $(P/M)^\theta$ and $|\boldsymbol{\beta}|$. It follows from (8.12) that either

$$|\boldsymbol{m}^{(\boldsymbol{\beta}\cdot M^d\boldsymbol{f})}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})| < (P/M)^{(d-1)\theta-d}, \tag{8.14}$$

or

$$|\boldsymbol{m}^{(\boldsymbol{\beta}\cdot M^d\boldsymbol{f})}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})| \geq \frac{1}{2}, \tag{8.15}$$

When (8.14) holds, then (8.13) implies

$$|\boldsymbol{\beta}\cdot M^d\boldsymbol{f}^{[d]}|_\infty < \frac{(P/M)^{(d-1)\theta-d}}{(P/M)^{(d-2)\theta}} = (P/M)^{\theta-d}. \tag{8.16}$$

When on the other hand (8.15) holds, then the bound $|\boldsymbol{x}^{(i)}| \leq (P/M)^\theta$ implies

$$|\boldsymbol{m}^{(\boldsymbol{\beta}\cdot M^d\boldsymbol{f})}(\boldsymbol{x}^{(1)},\dots,\boldsymbol{x}^{(d-1)})| \ll |\boldsymbol{\beta}\cdot M^d\boldsymbol{f}^{[d]}|_\infty (P/M)^{(d-1)\theta},$$

and it follows by (8.15) that

$$|\boldsymbol{\beta}\cdot M^d\boldsymbol{f}^{[d]}|_\infty \gg (P/M)^{-(d-1)\theta}. \tag{8.17}$$

Either (8.16) or (8.17) holds. So by rearranging and applying (8.9) we infer

$$(P/M)^{-\theta} \ll_{\mu,\kappa} \max\{P^{-d}|\boldsymbol{\beta}|^{-1}, |\boldsymbol{\beta}|^{\frac{1}{d-1}}M^{\frac{d}{d-1}}\}. \tag{8.18}$$

We have shown that (8.11) implies (8.18).

Lemma 8.6 shows that for $\theta\in(0,1]$ we have

$$U_{\boldsymbol{\beta}\cdot M^d\boldsymbol{f}}((P/M)^\theta,(P/M)^{(d-1)\theta-d}) \gg_\epsilon (P/M)^{(d-1)\theta n}\min\left\{\left|\frac{S(\boldsymbol{\alpha};P;\boldsymbol{v};M)}{(P/M)^{n+\epsilon}}\right|,\left|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta};P;\boldsymbol{v};M)}{(P/M)^{n+\epsilon}}\right|\right\}^{2^d}.$$

We also have the assumption (8.8):

$$N^{\mathrm{aux}}_{\boldsymbol{\beta}\cdot M^d\boldsymbol{f}}((P/M)^\theta) \leq C_0(P/M)^{\theta((d-1)n-2^d\mathscr{C})}.$$

This shows that (8.11) holds provided that $\theta\in(0,1]$ and that

$$(P/M)^{\theta((d-1)n-2^d\mathscr{C})} \leq C_1^{-1}(P/M)^{(d-1)\theta n}\min\left\{\left|\frac{S(\boldsymbol{\alpha};P;\boldsymbol{v};M)}{(P/M)^{n+\epsilon}}\right|,\left|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta};P;\boldsymbol{v};M)}{(P/M)^{n+\epsilon}}\right|\right\}^{2^d} \tag{8.19}$$

for some $C_1\geq 1$ depending only on C_0,d,n and ϵ. Define θ by

$$(P/M)^\theta = C_1^{1/2^d\mathscr{C}}\min\left\{\left|\frac{S(\boldsymbol{\alpha};P;\boldsymbol{v};M)}{(P/M)^{n+\epsilon}}\right|,\left|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta};P;\boldsymbol{v};M)}{(P/M)^{n+\epsilon}}\right|\right\}^{-1/\mathscr{C}}, \tag{8.20}$$

so that the inequality (8.19) holds. □

We consider three cases.

The first case is when $\theta \le 0$ holds. We can rule this out. If $\theta \le 0$ then (8.20) gives

$$\min\Big\{\Big|\frac{S(\boldsymbol{\alpha};P;\boldsymbol{v};M)}{(P/M)^{n+\epsilon}}\Big|,\Big|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta};P;\boldsymbol{v};M)}{(P/M)^{n+\epsilon}}\Big|\Big\} \ge C_1^{-1/2^d}. \tag{8.21}$$

To prove (8.10), we can assume without loss of generality that $P/M \gg_\epsilon 1$ holds. But then (8.21) is false, since $|S(\boldsymbol{\alpha};P;\boldsymbol{v};M)| \le (P/M+1)^n$ by Definition 8.3.

The second case is when $0 < \theta \le 1$ holds. We saw above that in this case (8.11) and hence (8.18) holds. Now (8.10) follows from (8.18) by substituting the value of θ from (8.20) and choosing C to satisfy the bound $C \gg_{\mu,\kappa} C_1^{1/2^d}$.

The third and last case is when $\theta > 1$. In this case we have by (8.20) that

$$\min\Big\{\Big|\frac{S(\boldsymbol{\alpha};P;\boldsymbol{v};M)}{(P/M)^{n+\epsilon}}\Big|,\Big|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta};P;\boldsymbol{v};M)}{(P/M)^{n+\epsilon}}\Big|\Big\} < C_1^{1/2^d}(P/M)^{-\mathscr{C}}. \tag{8.22}$$

Now for any $t > 0$ we have $\max\{(P/M)^{-d}t^{-1}, t^{\frac{1}{d-1}}\} \ge (P/M)^{-1}$, and hence

$$\max\{P^d|\boldsymbol{\beta}|^{-1}, |\boldsymbol{\beta}|^{\frac{1}{d-1}}M^{\frac{d}{d-1}}\}^{\mathscr{C}} \ge (P/M)^{-\mathscr{C}}.$$

So (8.10) follows from (8.22) on choosing C such that $C \ge C_1^{1/2^d}$ holds.

8.4 Circle Method

In this section the circle method will be used to prove some upper bounds. Hardy and Littlewood developed what is nowadays called Hardy-Littlewood circle method. Initially, this method involved a contour integral over the unit circle, which explains the word circle. However, in the modern formulation, exponential sums take over the role of the contour.

8.4.1 Notation for the Circle Method

We split the domain $[0,1]^R$ into two regions. Let $\Delta \in (0,1)$, we define the *major arcs*

$$\mathfrak{M}_{P,d,\delta} = \bigcup_{\substack{q\in\mathbb{N}\\ q\le M^{dR}P^\Delta}} \bigcup_{\substack{0\le a_1,\dots,a_R\le q\\ (a_1,\dots,a_R,q)=1}} \{\boldsymbol{\alpha}\in[0,1)^R : |\boldsymbol{\alpha}-\frac{\boldsymbol{a}}{q}| \le M^{dR}P^{\Delta-d}\},$$

and the *minor arcs*

$$\mathfrak{m}_{P,d,\delta} = [0,1]^R \backslash \mathfrak{M}_{P,d,\delta}.$$

For each $[\boldsymbol{v}]_M \in \Omega_M$, $q\in\mathbb{N}$ and $\boldsymbol{a}\in\mathbb{Z}^R$ we set

$$S_q(\boldsymbol{a};\boldsymbol{v};M) = q^{-n}\sum_{\boldsymbol{y}\in\{1,\dots,q\}^n} e(\frac{\boldsymbol{a}}{q}\cdot\boldsymbol{f}(M\boldsymbol{y}+\boldsymbol{v})).$$

$$\mathfrak{S}(P;\boldsymbol{v};M) = \sum_{q\le M^{dR}P^\Delta}\sum_{\substack{\boldsymbol{a}\in\{1,\dots,q\}^R\\ (a_1,\dots,a_R,q)=1}} S_q(\boldsymbol{a};\boldsymbol{v};M).$$

For each $\boldsymbol{\gamma} \in \mathbb{R}^R$, set

$$S_\infty(\boldsymbol{\gamma}) = \int_{\mathcal{B}} e(\boldsymbol{\gamma} \cdot \boldsymbol{f}^{[d]}(\boldsymbol{t}))\mathrm{d}\boldsymbol{t},$$

and let

$$\mathfrak{J}(P; M) = \int_{\substack{\boldsymbol{\alpha}\in\mathbb{R}^R \\ |\boldsymbol{\alpha}|\leq M^{dR}P^{\Delta-d}}} \Big(\frac{P}{M}\Big)^n S_\infty(P^d\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha}.$$

Further we set

$$\delta_0 = \frac{n - \dim(\tilde{X}^*)}{(d-1)2^{d-1}R},$$

where $\tilde{X}^*$ is the Birch singular locus defined as the variety consisting of all points $\boldsymbol{x} \in \mathbb{C}^n$ for which

$$rk\Big(\frac{\partial f_i^{[d]}}{\partial x_j}(\boldsymbol{x})\Big)_{i,j} < R.$$

We will always assume that $\dim(\tilde{X}^*) \leq n - 1$.

8.4.2 The Minor Arcs

Lemma 8.8. *Suppose that the polynomials f_i have integer coefficients and $\dim(\tilde{X}^*) \leq n-1$. Let $\Delta, \mathfrak{m}_{P,d,\delta}$ and δ_0 be as in §8.4.1 and let $\epsilon > 0$. Further let $S(\boldsymbol{\alpha}; P; \boldsymbol{v}; M)$ be as in Definition 8.3. Then we have*

$$\sup_{\boldsymbol{\alpha}\in\mathfrak{m}_{P,d,\delta}} |S(\boldsymbol{\alpha}; P; \boldsymbol{v}; M)| \ll_\epsilon (P/M)^{n-\Delta\delta_0+\epsilon}$$

where the implicit constant depends only on $\boldsymbol{f}, d, n, R$ and ϵ. The constant δ_0 satisfies $\delta_0 \geq \frac{1}{(d-1)2^{d-}R}$.

Proof. Let $0 < \Delta' < 1$. We will use [13, Lemma 2.13], which says that one of the following holds:

(i) $|\sum_{\boldsymbol{y}\in\mathbb{Z}^n\cap(P/M)\mathcal{B}} e(\boldsymbol{\alpha} \cdot \boldsymbol{f}(\boldsymbol{y}))| \ll_R (P/M)^{n-\Delta'\delta_0+\epsilon}$

(ii) there is a rational approximation $\boldsymbol{a}/q$ to $\boldsymbol{\alpha}$ with $\boldsymbol{a} \in \mathbb{Z}^R_{\geq 0}$ and $q \in \mathbb{N}$ satisfying

$$\begin{aligned}
&(\boldsymbol{a}, q) = 1 \\
&|\boldsymbol{\alpha} - \frac{\boldsymbol{a}}{q}| \leq \tilde{C}^{(R-1)}(P/M)^{\Delta'-d} \\
&1 \leq q \leq \tilde{C}^R(P/M)^{\Delta'},
\end{aligned}$$

where $\tilde{C}$ denotes the maximum of the absolute values of the coefficients of $\boldsymbol{f}$. Let K be any constant. By choosing Δ' such that $(P/M)^{\Delta'} = (P/M)^\Delta K$, we deduce a variant of [13, Lemma 2.13]; one of the following holds:

(i) $|\sum_{\boldsymbol{y}\in\mathbb{Z}^n\cap(P/M)\mathcal{B}} e(\boldsymbol{\alpha} \cdot \boldsymbol{f}(\boldsymbol{y}))| \ll_{\boldsymbol{f},d,n,R,K} (P/M)^{n-\Delta\delta_0+\epsilon}$

(ii) there is a rational approximation $\boldsymbol{a}/q$ to $\boldsymbol{\alpha}$ with $\boldsymbol{a} \in \mathbb{Z}^R_{\geq 0}$ and $q \in \mathbb{N}$ satisfying

$$\begin{aligned}
&(\boldsymbol{a}, q) = 1\\
&|\boldsymbol{\alpha} - \frac{\boldsymbol{a}}{q}| \leq K\tilde{C}^{(R-1)}(P/M)^{\Delta-d}\\
&1 \leq q \leq K\tilde{C}^R(P/M)^{\Delta}.
\end{aligned}$$

Let $\boldsymbol{\alpha} \in \mathfrak{m}_{P,d,\delta}$. By the definition of the minor arcs we have that for all $\boldsymbol{a} \in \mathbb{Z}^R_{\geq 0}$ and $q \in \mathbb{N}$ with $(\boldsymbol{a}, q) = 1$ one of the following holds

$$|q\boldsymbol{\alpha} - \boldsymbol{a}| > M^{dR}P^{\Delta-d}q \qquad \text{or} \qquad q > M^{dR}P^{\Delta}.$$

We have the inequalities

$$\begin{aligned}
M^{dR}P^{\Delta-d}q &\geq M^{dR}P^{\Delta-d}\\
&\geq M^{dR}M^{\Delta-d}(P/M)^{\Delta-d}\\
&\geq M^{d(R-1)}(P/M)^{\Delta-d}
\end{aligned}$$

and

$$\begin{aligned}
M^{dR}P^{\Delta} &\geq M^{dR}(P/M)^{\Delta}M^{\Delta}\\
&\geq M^{dR}(P/M)^{\Delta}.
\end{aligned}$$

So we have for all $\boldsymbol{a} \in \mathbb{Z}^R_{\geq 0}$ and $q \in \mathbb{N}$ with $(\boldsymbol{a}, q) = 1$ that

$$|q\boldsymbol{\alpha} - \boldsymbol{a}| > M^{d(R-1)}(P/M)^{\Delta-d} \qquad \text{or} \qquad q > M^{dR}(P/M)^{\Delta}.$$

Now choose $K = \min\{\tilde{C}^{1-R}, \tilde{C}^{-R}\}$, where $\tilde{C}$ denotes the maximum of the absolute values of the coefficients of $\boldsymbol{f}(\boldsymbol{x})$. Then we have for all $\boldsymbol{a} \in \mathbb{Z}^R_{\geq 0}$ and $q \in \mathbb{N}$ with $(\boldsymbol{a}, q) = 1$ that

$$|q\boldsymbol{\alpha} - \boldsymbol{a}| > K(\tilde{C}M)^{d(R-1)}P^{\Delta-d} \qquad \text{or} \qquad q > (\tilde{C}M)^{dR}P^{\Delta}.$$

Now we will apply the variant of [13, Lemma 2.13] on the polynomial $\boldsymbol{f}(M\boldsymbol{x} + \boldsymbol{v})$. We note that if we replace $\boldsymbol{f}(\boldsymbol{x})$ by $\boldsymbol{f}(M\boldsymbol{x} + \boldsymbol{v})$ then $\tilde{C}$ changes in $M^d\tilde{C}$. Hence the variant of [13, Lemma 2.13] gives

$$|\sum_{\boldsymbol{y} \in \mathbb{Z}^n \cap (P/M)\mathcal{B}} e(\boldsymbol{\alpha} \cdot \boldsymbol{f}(M\boldsymbol{y} + \boldsymbol{v}))| \ll_{\boldsymbol{f},d,n,R,K} P^{n-\Delta\delta_0+\epsilon},$$

which proves the first statement. The second statement from the lemma follows directly from the definition of δ_0 and the assumption that $\dim(\tilde{V}^*) \leq n-1$. □

8.4.3 The Major Arcs

Lemma 8.9. *Suppose that the polynomials f_i have integer coefficients. Let $[\boldsymbol{v}]_M \in \Omega_M$ with $0 < |\boldsymbol{v}| \leq M$. Let $\Delta, \mathfrak{M}_{P,d,\delta}, S_\infty(\boldsymbol{\gamma}; \boldsymbol{v}; M), S_q(\boldsymbol{a}; \boldsymbol{v}; M), \mathfrak{G}(P)$ and $\mathfrak{J}(P)$ be as above. Then for all $\boldsymbol{a} \in \mathbb{Z}^{\mathbb{R}}$ and all $q \in \mathbb{N}$ such that $q \leq P/M$, we have*

$$S(\frac{\boldsymbol{a}}{q} + \boldsymbol{\alpha}; P; \boldsymbol{v}; M) = S_q(\boldsymbol{a}; \boldsymbol{v}; M)\Big(\frac{P}{M}\Big)^n S_\infty(P^d\boldsymbol{\alpha}) + O\Big(q\Big(\frac{P}{M}\Big)^{n-1}(1 + P^d|\boldsymbol{\alpha}|)\Big) \quad (8.23)$$

and it follows that

$$\int_{\mathfrak{M}_{P,d,\delta}} S(\boldsymbol{\alpha};P;\boldsymbol{v};M)\,d\boldsymbol{\alpha} = \mathfrak{S}(P;\boldsymbol{v};M)\mathfrak{J}(P;M) + O\Big(M^{1-n+3dR+2dR^2}P^{n-1+\Delta(2R+3)-dR}\Big) \tag{8.24}$$

We remark that in the case when $\boldsymbol{a}=\mathbf{0}$ and $q=1$ equation (8.23) gives

$$S(\boldsymbol{\alpha};P;\boldsymbol{v};M) = \Big(\frac{P}{M}\Big)^n S_\infty(P^d\boldsymbol{\alpha}) + O\Big(\Big(\frac{P}{M}\Big)^{n-1}(1+P^d|\boldsymbol{\alpha}|)\Big).$$

Proof. First observe that

$$\begin{aligned} S(\frac{\boldsymbol{a}}{q}+\boldsymbol{\alpha};P;\boldsymbol{v};M) &= \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B}}} e(\frac{\boldsymbol{a}}{q}\cdot\boldsymbol{f}(M\boldsymbol{x}+\boldsymbol{v}))e(\boldsymbol{\alpha}\cdot\boldsymbol{f}(M\boldsymbol{x}+\boldsymbol{v})) \\ &= \sum_{1\le y_1,\dots,y_m\le q} e(\frac{\boldsymbol{a}}{q}\cdot\boldsymbol{f}(M\boldsymbol{y}+\boldsymbol{v})) \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B}\\ \boldsymbol{x}\equiv\boldsymbol{y} \bmod q}} e(\boldsymbol{\alpha}\cdot\boldsymbol{f}(M\boldsymbol{x}+\boldsymbol{v})) \end{aligned} \tag{8.25}$$

since if $\boldsymbol{y}\equiv\boldsymbol{x} \bmod q$, then $\boldsymbol{f}(M\boldsymbol{y}+\boldsymbol{v})\equiv\boldsymbol{f}(M\boldsymbol{x}+\boldsymbol{v}) \bmod q$ and there is a $\boldsymbol{b}\in\mathbb{Z}^n$ such that $\boldsymbol{f}(M\boldsymbol{y}+\boldsymbol{v})=\boldsymbol{f}(M\boldsymbol{x}+\boldsymbol{v})+q\boldsymbol{b}$; hence

$$e(\frac{\boldsymbol{a}}{q}\cdot(\boldsymbol{f}(M\boldsymbol{y}+\boldsymbol{v})) = e(\frac{\boldsymbol{a}}{q}\cdot(\boldsymbol{f}(M\boldsymbol{x}+\boldsymbol{v})+q\boldsymbol{b}))) = e(\frac{\boldsymbol{a}}{q}\cdot\boldsymbol{f}(M\boldsymbol{x}+\boldsymbol{v})).$$

Observe that $\boldsymbol{\alpha}\cdot\boldsymbol{f}(M\boldsymbol{x}+\boldsymbol{v}) = \boldsymbol{\alpha}\cdot M^d\boldsymbol{f}^{[d]}(\boldsymbol{x}) + O(M^d|\boldsymbol{x}|^{d-1}|\boldsymbol{\alpha}|)$. With Lemma 12.3 we see that

$$e(\boldsymbol{\alpha}\cdot\boldsymbol{f}(M\boldsymbol{x}+\boldsymbol{v})) = e(\boldsymbol{\alpha}\cdot M^d\boldsymbol{f}^{[d]}(\boldsymbol{x})) + O(M^d|\boldsymbol{x}|^{d-1}|\boldsymbol{\alpha}|). \tag{8.26}$$

By substituting (8.26) into (8.25) we get

$$\begin{aligned} &S(\frac{\boldsymbol{a}}{q}+\boldsymbol{\alpha};P;\boldsymbol{v};M) \\ &= \sum_{1\le y_1,\dots,y_m\le q} e(\frac{\boldsymbol{a}}{q}\cdot\boldsymbol{f}(M\boldsymbol{y}+\boldsymbol{v})) \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B}\\ \boldsymbol{x}\equiv\boldsymbol{y} \bmod q}} \Big(e(\boldsymbol{\alpha}\cdot M^d\boldsymbol{f}^{[d]}(\boldsymbol{x})) + O(M^d|\boldsymbol{x}|^{d-1}|\boldsymbol{\alpha}|)\Big) \\ &= \sum_{1\le y_1,\dots,y_m\le q} e(\frac{\boldsymbol{a}}{q}\cdot\boldsymbol{f}(M\boldsymbol{y}+\boldsymbol{v})) \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B}\\ \boldsymbol{x}\equiv\boldsymbol{y} \bmod q}} e(\boldsymbol{\alpha}\cdot M^d\boldsymbol{f}^{[d]}(\boldsymbol{x})) \\ &+ O\Big(q^nM^d\Big(\frac{P}{M}+1\Big)^{d-1}|\boldsymbol{\alpha}|\Big). \\ &= \sum_{1\le y_1,\dots,y_m\le q} e(\frac{\boldsymbol{a}}{q}\cdot\boldsymbol{f}(M\boldsymbol{y}+\boldsymbol{v})) \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B}\\ \boldsymbol{x}\equiv\boldsymbol{y} \bmod q}} e(\boldsymbol{\alpha}\cdot M^d\boldsymbol{f}^{[d]}(\boldsymbol{x})) \\ &+ O\Big(M^d\Big(\frac{P}{M}\Big)^{n+d-1}|\boldsymbol{\alpha}|\Big). \end{aligned} \tag{8.27}$$

If ψ is any differentiable complex-valued function on $\mathbb{R}^n$ we have

$$\psi(\boldsymbol{x}) = q^{-n} \int_{\substack{\boldsymbol{u}\in\mathbb{R}^n \\ |\boldsymbol{u}|\le q/2}} \psi(\boldsymbol{x}+\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} + O_n(q \max_{\substack{\boldsymbol{u}\in\mathbb{R}^n \\ |\boldsymbol{u}|\le q/2}} |\nabla_{\boldsymbol{u}}\psi(\boldsymbol{x}+\boldsymbol{u})|).$$

Setting $\psi(\boldsymbol{x}) = e(\boldsymbol{\alpha}\cdot M^d \boldsymbol{f}^{[d]}(\boldsymbol{x}))$ gives

$$\begin{aligned}
&\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n \\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B} \\ \boldsymbol{x}\equiv\boldsymbol{y} \bmod q}} e(\boldsymbol{\alpha}\cdot M^d \boldsymbol{f}^{[d]}(\boldsymbol{x})) \\
&= q^{-n} \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n \\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B} \\ \boldsymbol{x}\equiv\boldsymbol{y} \bmod q}} \int_{\substack{\boldsymbol{u}\in\mathbb{R}^n \\ |\boldsymbol{u}|\le q/2}} e(\boldsymbol{\alpha}\cdot M^d \boldsymbol{f}^{[d]}(\boldsymbol{x}+\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u} \\
&+ \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n \\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B} \\ \boldsymbol{x}\equiv\boldsymbol{y} \bmod q}} O_n(q \max_{\substack{\boldsymbol{u}\in\mathbb{R}^n \\ |\boldsymbol{u}|\le q/2}} |\nabla_{\boldsymbol{u}} e(\boldsymbol{\alpha}\cdot M^d \boldsymbol{f}^{[d]}(\boldsymbol{x}+\boldsymbol{u}))|) \\
&= q^{-n} \int_{\substack{\boldsymbol{u}\in\mathbb{R}^n \\ M\boldsymbol{u}+\boldsymbol{v}\in P\mathcal{B}}} e(\boldsymbol{\alpha}\cdot M^d \boldsymbol{f}^{[d]}(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u} + q^{-n} O_n\Big(q\Big(\frac{P}{M}+q\Big)^{n-1}\Big) \\
&+ \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n \\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B} \\ \boldsymbol{x}\equiv\boldsymbol{y} \bmod q}} O_n\Big(q \max_{\substack{\boldsymbol{u}\in\mathbb{R}^n \\ |\boldsymbol{u}|\le q/2}} M^d d(\boldsymbol{x}+\boldsymbol{u})^{d-1}|\boldsymbol{\alpha}|\Big) \\
&= q^{-n} \int_{\substack{\boldsymbol{u}\in\mathbb{R}^n \\ M\boldsymbol{u}+\boldsymbol{v}\in P\mathcal{B}}} e(\boldsymbol{\alpha}\cdot M^d \boldsymbol{f}^{[d]}(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u} \\
&+ O_{n,d}\Big(\Big(\frac{P}{qM}\Big)^n qM^d\Big(\frac{P}{M}+1+\frac{q}{2}\Big)^{d-1}|\boldsymbol{\alpha}| + q^{1-n}\Big(\frac{P}{M}+q\Big)^{n-1}\Big) \\
&= q^{-n} \int_{\substack{\boldsymbol{u}\in\mathbb{R}^n \\ M\boldsymbol{u}+\boldsymbol{v}\in P\mathcal{B}}} e(\boldsymbol{\alpha}\cdot M^d \boldsymbol{f}^{[d]}(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u} \\
&+ O_{n,d}\Big(q^{1-n}M^d\Big(\frac{P}{M}\Big)^{n+d-1}|\boldsymbol{\alpha}| + q^{1-n}\Big(\frac{P}{M}\Big)^{n-1}\Big),
\end{aligned} \tag{8.28}$$

where the error term $q(\frac{P}{M}+q)^{n-1}$ occurs since it can happen that $\sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n \\ M\boldsymbol{x}+\boldsymbol{v}\in P\mathcal{B} \\ \boldsymbol{x}\equiv\boldsymbol{y} \bmod q}} \int_{\substack{\boldsymbol{u}\in\mathbb{R}^n \\ |\boldsymbol{u}|\le q/2}}$ integrates over a larger box than only $(P\mathcal{B}+\boldsymbol{v})/M$. In the worst case all sides of $(P\mathcal{B}+\boldsymbol{v})/M$ are increased by length $q/2$. This extension increases the value of the integral over $(P\mathcal{B}+\boldsymbol{v})/M$ at most with $n\cdot 2\frac{q}{2}(\frac{P}{M}+q)^{n-1}$.

Substituting (8.28) into (8.27) and using $S_q(\boldsymbol{a};\boldsymbol{v};M) = O(1)$ shows

$$\begin{aligned}
&S(\frac{\boldsymbol{a}}{q}+\boldsymbol{\alpha};P;\boldsymbol{v};M) \\
&= S_q(\boldsymbol{a};\boldsymbol{v};M)\Big(\int_{\substack{\boldsymbol{u}\in\mathbb{R}^n \\ M\boldsymbol{u}+\boldsymbol{v}\in P\mathcal{B}}} e(\boldsymbol{\alpha}\cdot M^d \boldsymbol{f}^{[d]}(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u} + O\Big(qM^d\Big(\frac{P}{M}\Big)^{n+d-1}|\boldsymbol{\alpha}| + q\Big(\frac{P}{M}\Big)^{n-1}\Big)\Big) \\
&+ O\Big(M^d\Big(\frac{P}{M}\Big)^{n+d-1}|\boldsymbol{\alpha}|\Big). \\
&= S_q(\boldsymbol{a};\boldsymbol{v};M)\int_{\substack{\boldsymbol{u}\in\mathbb{R}^n \\ M\boldsymbol{u}+\boldsymbol{v}\in P\mathcal{B}}} e(\boldsymbol{\alpha}\cdot M^d \boldsymbol{f}^{[d]}(\boldsymbol{u}))\,\mathrm{d}\boldsymbol{u} + O\Big(qM^d\Big(\frac{P}{M}\Big)^{n+d-1}|\boldsymbol{\alpha}| + q\Big(\frac{P}{M}\Big)^{n-1}\Big)
\end{aligned}$$

Setting $M\boldsymbol{u}+\boldsymbol{v}=P\boldsymbol{t}$ and using the definition of $S_\infty(\boldsymbol{\gamma})$ from §8.4.1 we conclude

$$\begin{aligned}
&S(\frac{\boldsymbol{a}}{q}+\boldsymbol{\alpha};P;\boldsymbol{v};M)\\
&=S_q(\boldsymbol{a};\boldsymbol{v};M)\Big(\frac{P}{M}\Big)^n\int_{\boldsymbol{t}\in\mathcal{B}}e(\boldsymbol{\alpha}\cdot P^d\boldsymbol{f}^{[d]}(\boldsymbol{t}))\,\mathrm{d}\boldsymbol{t}+O\Big(q\Big(\frac{P}{M}\Big)^{n-1}(1+P^d|\boldsymbol{\alpha}|)\Big)\\
&=S_q(\boldsymbol{a};\boldsymbol{v};M)\Big(\frac{P}{M}\Big)^n S_\infty(P^d\boldsymbol{\alpha})+O\Big(q\Big(\frac{P}{M}\Big)^{n-1}(1+P^d|\boldsymbol{\alpha}|)\Big),
\end{aligned}$$

which proves (8.23). We will now use equation (8.23) and the definition of $\mathfrak{M}_{P,d,\delta}$ (§8.4.1) to prove (8.24).

$$\begin{aligned}
&\int_{\mathfrak{M}_{P,d,\delta}}S(\boldsymbol{\alpha};P;\boldsymbol{v};M)\,\mathrm{d}\boldsymbol{\alpha}\\
&\qquad=\sum_{\substack{q\in\mathbb{N}\\ q\le M^{dR}P^{\Delta}}}\sum_{\substack{0\le a_1,\dots,a_R\le q\\ (a_1,\dots,a_R,q)=1}}\int_{|\boldsymbol{\alpha}-\frac{\boldsymbol{a}}{q}|\le M^{dR}P^{\Delta-d}}S(\boldsymbol{\alpha};P;\boldsymbol{v};M)\,\mathrm{d}\boldsymbol{\alpha}\\
&\qquad=\sum_{\substack{q\in\mathbb{N}\\ q\le M^{dR}P^{\Delta}}}\sum_{\substack{0\le a_1,\dots,a_R\le q\\ (a_1,\dots,a_R,q)=1}}\int_{|\boldsymbol{\beta}|\le M^{dR}P^{\Delta-d}}S(\boldsymbol{\beta}+\frac{\boldsymbol{a}}{q};P;\boldsymbol{v};M)\,\mathrm{d}\boldsymbol{\beta}\\
&\qquad=\sum_{\substack{q\in\mathbb{N}\\ q\le M^{dR}P^{\Delta}}}\sum_{\substack{0\le a_1,\dots,a_R\le q\\ (a_1,\dots,a_R,q)=1}}\int_{|\boldsymbol{\beta}|\le M^{dR}P^{\Delta-d}}S_q(\boldsymbol{a};\boldsymbol{v};M)\Big(\frac{P}{M}\Big)^n S_\infty(P^d\boldsymbol{\beta})\,\mathrm{d}\boldsymbol{\beta}\\
&\qquad+O\Big(\sum_{\substack{q\in\mathbb{N}\\ q\le M^{dR}P^{\Delta}}}\sum_{\substack{0\le a_1,\dots,a_R\le q\\ (a_1,\dots,a_R,q)=1}}\int_{|\boldsymbol{\beta}|\le M^{dR}P^{\Delta-d}}q\Big(\frac{P}{M}\Big)^{n-1}(1+P^d|\boldsymbol{\beta}|)\,\mathrm{d}\boldsymbol{\beta}\Big)\\
&\qquad=\mathfrak{S}(P;\boldsymbol{v};M)\mathfrak{J}(P;M)\\
&\qquad+O\Big(\sum_{\substack{q\in\mathbb{N}\\ q\le M^{dR}P^{\Delta}}}\sum_{\substack{0\le a_1,\dots,a_R\le q\\ (a_1,\dots,a_R,q)=1}}\int_{|\boldsymbol{\beta}|\le M^{dR}P^{\Delta-d}}q\Big(\frac{P}{M}\Big)^{n-1}(1+P^d|\boldsymbol{\beta}|)\,\mathrm{d}\boldsymbol{\beta}\Big).
\end{aligned}\tag{8.29}$$

We have

$$
\begin{aligned}
O\Big(\sum_{\substack{q\in\mathbb{N}\\ q\le M^{dR}P^{\Delta}}}\ \sum_{\substack{0\le a_1,\dots,a_R\le q\\ (a_1,\dots,a_R,q)=1}} \int_{|\boldsymbol{\beta}|\le M^{dR}P^{\Delta-d}} q\Big(\frac{P}{M}\Big)^{n-1}(1+P^d|\boldsymbol{\beta}|)\,\mathrm{d}\boldsymbol{\beta}\Big) \\
&= O\Big(\sum_{\substack{q\in\mathbb{N}\\ q\le M^{dR}P^{\Delta}}} \int_{|\boldsymbol{\beta}|\le M^{dR}P^{\Delta-d}} q^{R+1}\Big(\frac{P}{M}\Big)^{n-1}(1+P^d|\boldsymbol{\beta}|)\,\mathrm{d}\boldsymbol{\beta}\Big) \\
&= O\Big(\sum_{\substack{q\in\mathbb{N}\\ q\le M^{dR}P^{\Delta}}} \int_{|\boldsymbol{\beta}|\le M^{dR}P^{\Delta-d}} q^{R+1}\Big(\frac{P}{M}\Big)^{n-1}(1+M^{dR}P^{\Delta})\,\mathrm{d}\boldsymbol{\beta}\Big) \\
&= O\Big(\sum_{\substack{q\in\mathbb{N}\\ q\le M^{dR}P^{\Delta}}} q^{R+1}\Big(\frac{P}{M}\Big)^{n-1} M^{dR+dR^2}P^{\Delta+(\Delta-d)R}\Big) \\
&= O\Big((M^{dR}P^{\Delta}+1)^{R+2}M^{1-n+dR+dR^2}P^{n-1+\Delta+(\Delta-d)R}\Big) \\
&= O\Big(M^{1-n+3dR+2dR^2}P^{n-1+\Delta(2R+3)-dR}\Big). \qquad (8.30)
\end{aligned}
$$

Here we used Lemma 12.4 in the second last step. Combining (8.29) and (8.30) completes the proof. □

Lemma 8.10. *Let $[\boldsymbol{v}]_M \in \Omega_M$ and let $\mathcal{B}$ be the box $[0.1]^R$ and let $S_q(a)$ be as in §8.4.1. Suppose the polynomials f_i have integer coefficients and $\dim(\tilde{V}^*) \le n-1$. Let $\epsilon \ge 0$ and $C \ge 1$ such that the bound (8.10) in Lemma 8.7 holds. Then:*

(i) There is $\epsilon' > 0$ such that $\epsilon' = O_{\mathscr{C}}(\epsilon)$ and

$$\min\{|S_q(\boldsymbol{a};\boldsymbol{v};M)|, |S_{q'}(\boldsymbol{a}';\boldsymbol{v};M)|\} \ll_C (q'+q)^{\epsilon} M^{\frac{d\mathscr{C}}{d-1}} \Big|\frac{\boldsymbol{a}}{q} - \frac{\boldsymbol{a}}{q'}\Big|^{\frac{\mathscr{C}-\epsilon'}{d-1}}, \qquad (8.31)$$

for all $\boldsymbol{a},\boldsymbol{a}' \in \{1,\dots,q\}^R$ such that $\frac{\boldsymbol{a}}{q} \neq \frac{\boldsymbol{a}}{q'}$

(ii) If $\mathscr{C} > \epsilon'$, then for all $t > 0$ and $q_0 \in \mathbb{N}$ we have

$$\#\{\tfrac{\boldsymbol{a}}{q} \in \mathbb{Q}^R \cap [0,1)^R :\ q \le q_0, |S_q(\boldsymbol{a};\boldsymbol{v};M)| \ge t\} \ll_C M^{-dR}(q_0^{-\epsilon}t)^{-\frac{(d-1)R}{\mathscr{C}-\epsilon'}},$$

where $\frac{\boldsymbol{a}}{q}$ are in lowest terms.

(iii) Let δ_0 as in §8.4.1 and let $\epsilon'' > 0$. For all $q \in \mathbb{N}$ and all $\boldsymbol{a} \in \mathbb{Z}^R$ such that $(a_1,\dots,a_r,q) = 1$, we have

$$|S_q(\boldsymbol{a};\boldsymbol{v};M)| \ll_{\epsilon''} M^{dR\delta_0} q^{-\delta_0+\epsilon''}.$$

(iv) Let Δ and $\mathfrak{S}(P)$ as in §8.4.1. Suppose that ϵ is sufficiently small in terms of $\mathscr{C}$, d and R. Provided the inequality $\mathscr{C} > (d-1)R$ holds, we have

$$\mathfrak{S}(P;\boldsymbol{v};M) - \mathfrak{S}(\boldsymbol{v};M) \ll_{C,\mathscr{C}} P^{-\Delta\delta_1}M^{dR(\delta_1-1)} \qquad (8.32)$$

for some $\mathfrak{S}(\boldsymbol{v}; M) \in \mathbb{C}$ *and some* $\delta_1 > 0$ *depending at most on* $\mathscr{C}$, d *and* R. *We have* $\mathfrak{S}(\boldsymbol{v}; M) \geq 0$ *and*

$$\mathfrak{S}(\boldsymbol{v}; M) = \prod_p \lim_{k\to\infty} \frac{1}{p^{k(n-R)}} \#\{\boldsymbol{b} \in \{1, 2, \ldots, , p^k\}^n : \\ f_1(M\boldsymbol{b} + \boldsymbol{v}) \equiv 0, \ldots, f_R(M\boldsymbol{b} + \boldsymbol{v}) \equiv 0 \text{ mod } p^k\} \tag{8.33}$$

where the product is over the primes p and converges absolutely.

Proof. Proof of part (i). Let P/M be a parameter, to be chosen later. Then (8.10) gives

$$\min\Big\{\Big|\frac{S(\boldsymbol{\alpha}; P; \boldsymbol{v}; M)}{(P/M)^{n+\epsilon}}\Big|, \Big|\frac{S(\boldsymbol{\alpha}+\boldsymbol{\beta}; P; \boldsymbol{v}; M)}{(P/M)^{n+\epsilon}}\Big|\Big\} \leq C \max\{P^{-d}|\boldsymbol{\beta}|^{-1}, |\boldsymbol{\beta}|^{\frac{1}{d-1}} M^{\frac{d}{d-1}}\}^{\mathscr{C}}. \tag{8.34}$$

Since $\mathcal{B} = [0.1]^n$ the equality $S_\infty(\boldsymbol{0}) = 1$ holds, ans so (8.23) implies that

$$\frac{S(\frac{\boldsymbol{a}}{q}; P; \boldsymbol{v}; M)}{(P/M)^n} = S_q(\boldsymbol{a}; \boldsymbol{v}; M) + O(q(P/M)^{-1}), \tag{8.35}$$

$$\frac{S(\frac{\boldsymbol{a}'}{q'}; P; \boldsymbol{v}; M)}{(P/M)^n} = S_q(\boldsymbol{a}'; \boldsymbol{v}; M) + O(q'(P/M)^{-1}). \tag{8.36}$$

Together (8.34), (8.35) and (8.36) yield

$$\min\{S_q(\boldsymbol{a}; \boldsymbol{v}; M), S_{q'}(\boldsymbol{a}'; P; \boldsymbol{v}; M)\} \leq C(P/M)^\epsilon P^{-d\mathscr{C}} |\tfrac{\boldsymbol{a}'}{q'} - \tfrac{\boldsymbol{a}}{q}|^{-\mathscr{C}} \\ + C(P/M)^\epsilon |\tfrac{\boldsymbol{a}'}{q'} - \tfrac{\boldsymbol{a}}{q}|^{\frac{\mathscr{C}}{d-1}} M^{\frac{\mathscr{C}d}{d-1}} + O((q'+q)(P/M)^{-1})$$

Observe that for P sufficiently large the term $C(P/M)^\epsilon |\frac{\boldsymbol{a}'}{q'} - \frac{\boldsymbol{a}}{q}| M^{\frac{\mathscr{C}d}{d-1}}$ dominates the right-hand side. We claim that this is the case for

$$P/M = (q'+q)|\tfrac{\boldsymbol{a}'}{q'} - \tfrac{\boldsymbol{a}}{q}|^{-\frac{1+\mathscr{C}}{d-1}}.$$

Indeed, since $|\frac{\boldsymbol{a}'}{q'} - \frac{\boldsymbol{a}}{q}| \leq 1$ it follows that

$$\min\{S_q(\boldsymbol{a}; \boldsymbol{v}; M), S_{q'}(\boldsymbol{a}'; P; \boldsymbol{v}; M)\} \leq C(P/M)^\epsilon (q'+q)^{-d\mathscr{C}} M^{-d\mathscr{C}} |\tfrac{\boldsymbol{a}'}{q'} - \tfrac{\boldsymbol{a}}{q}|^{\frac{\mathscr{C}+\mathscr{C}^2 d}{d-1}} \\ + C(P/M)^\epsilon |\tfrac{\boldsymbol{a}'}{q'} - \tfrac{\boldsymbol{a}}{q}|^{\frac{\mathscr{C}}{d-1}} M^{\frac{\mathscr{C}d}{d-1}} + O(|\tfrac{\boldsymbol{a}'}{q'} - \tfrac{\boldsymbol{a}}{q}|^{\frac{1+\mathscr{C}}{d-1}}) \\ \ll_C (P/M)^\epsilon M^{\frac{\mathscr{C}d}{d-1}} |\tfrac{\boldsymbol{a}'}{q'} - \tfrac{\boldsymbol{a}}{q}|^{\frac{\mathscr{C}}{d-1}}.$$

By again substituting our choice for P/M we see

$$\min\{S_q(\boldsymbol{a}; \boldsymbol{v}; M), S_{q'}(\boldsymbol{a}'; P; \boldsymbol{v}; M)\} \ll_C ((q'+q)|\tfrac{\boldsymbol{a}'}{q'} - \tfrac{\boldsymbol{a}}{q}|^{-\frac{1+\mathscr{C}}{d-1}})^\epsilon M^{\frac{\mathscr{C}d}{d-1}} |\tfrac{\boldsymbol{a}'}{q'} - \tfrac{\boldsymbol{a}}{q}|^{\frac{\mathscr{C}}{d-1}} \\ \ll_C (q'+q)^\epsilon M^{\frac{\mathscr{C}d}{d-1}} |\tfrac{\boldsymbol{a}'}{q'} - \tfrac{\boldsymbol{a}}{q}|^{\frac{\mathscr{C}}{d-1} - \frac{1+\mathscr{C}}{d-1}\epsilon}.$$

By choosing $\epsilon' = \frac{1+\mathscr{C}}{d-1}\epsilon = O_{\mathscr{C}}(\epsilon)$, statement (i) follows.

Proof of part (ii). If $\epsilon' < \mathscr{C}$ is small, then by part (i), the points in the set

$$\{\tfrac{\boldsymbol{a}}{q} \in \mathbb{Q}^R \cap [0,1)^R : q \leq q_0, |S_q(\boldsymbol{a};\boldsymbol{v};M)| \geq t\}$$

are seperated by caps of size

$$\begin{aligned} |\tfrac{\boldsymbol{a}'}{q'} - \tfrac{\boldsymbol{a}}{q}| &\gg_C (tq_0^{-\epsilon} M^{\frac{d\mathscr{C}}{d-1}})^{\frac{d-1}{\mathscr{C}-\epsilon'}} \\ &\gg_C (tq_0^{-\epsilon})^{\frac{d-1}{\mathscr{C}-\epsilon'}} M^{\frac{d\mathscr{C}}{\mathscr{C}-\epsilon'}} \\ &\gg_C (tq_0^{-\epsilon})^{\frac{d-1}{\mathscr{C}-\epsilon'}} M^d. \end{aligned}$$

At most $O_C((M^{-dR}(q_0^{-\epsilon}t)^{-\frac{(d-1)R}{\mathscr{C}-\epsilon'}})$ such points fit in the box $[0,1)^R$, proving the claim.

Proof of part (iii). The bound follows from [13, Lemma 2.14].

Proof of part (iv). With $\boldsymbol{a}/q$ we mean $(\frac{a_1}{q},\ldots,\frac{a_R}{q})$ where $\boldsymbol{a} \in \mathbb{Z}^R$, $q \in \mathbb{N}$ and $(a_1,\ldots,a_R,q) = 1$. It is sufficient to show that

$$s(Q) := \sum_{\substack{\boldsymbol{a}/q \in [0,1)^R \\ Q < q \leq 2Q}} |S_q(\boldsymbol{a};\boldsymbol{v};M)| \ll_{C,\mathscr{C}} M^{-dR}(QM^{-2dR})^{-\delta_1}, \tag{8.37}$$

for all $Q \geq 1$ and some δ_1 depending only on $\mathscr{C}, d$ and R, since if (8.37), then

$$\begin{aligned} \Big|\mathfrak{S}(P;\boldsymbol{v};M) - \sum_{\boldsymbol{a}/q\in[0,1)^R} S_q(\boldsymbol{a};\boldsymbol{v};M)\Big| &\leq \sum_{\substack{\boldsymbol{a}/q\in[0,1)^R \\ q > P^{\Delta}M^{dR}}} |S_q(\boldsymbol{a};\boldsymbol{v};M)| \\ &= \sum_{\substack{Q=2^kP^{\Delta}M^{dR} \\ k=0,1,\ldots}} s(Q) \\ &\ll_{C,\mathscr{C}} \sum_{k=0}^{\infty} M^{-dR}(2^kP^{\Delta}M^{dR}M^{-2dR})^{-\delta_1} \\ &\ll_{C,\mathscr{C}} M^{-dR}P^{-\Delta\delta_1}M^{dR\delta_1} \\ &\ll_{C,\mathscr{C}} P^{-\Delta\delta_1}M^{dR(\delta_1-1)} \end{aligned}$$

which proves (8.32) with $\mathfrak{S}(\boldsymbol{v};M) = \sum_{\boldsymbol{a}/q\in[0,1)^R} S_q(\boldsymbol{a};\boldsymbol{v};M)$, where this sum is absolutely converges since

$$\begin{aligned} \sum_{\boldsymbol{a}/q\in[0,1)^R} S_q(\boldsymbol{a};\boldsymbol{v};M) &= \sum_{\substack{\boldsymbol{a}/q\in[0,1)^R \\ q=1}} S_q(\boldsymbol{a};\boldsymbol{v};M) + \sum_{\substack{Q=2^k \\ k=0,1,\ldots}} s(Q) \\ &\ll_{C,\mathscr{C}} 1 + \sum_{k=0,1,\ldots} M^{-dR}2^{-\delta_1 k}M^{2dR\delta_1} \\ &\ll_{C,\mathscr{C}} 1 + M^{dR(2\delta_1-1)}. \end{aligned}$$

Then (8.33) follows as in §7 of Birch [5], and $\mathfrak{S}(v;M) \geq 0$ follows from Myerson [20, Theorem 1.3].

We will now prove (8.37). Let $l \in \mathbb{Z}$. We have

$$\begin{aligned} s(Q) = &\sum_{\substack{\boldsymbol{a}/q \in [0,1)^R \\ |S_q(\boldsymbol{a};\boldsymbol{v};M)| \geq 2^{-l} \\ Q<q\leq 2Q}} |S_q(\boldsymbol{a};\boldsymbol{v};M)| + \sum_{i=l}^{\infty} \sum_{\substack{\boldsymbol{a}/q \in [0,1)^R \\ 2^{-i} > |S_q(\boldsymbol{a};\boldsymbol{v};M)| \geq 2^{-i-1} \\ Q<q\leq 2Q}} |S_q(\boldsymbol{a};\boldsymbol{v};M)| \\ &\leq \#\{\frac{\boldsymbol{a}}{q} \in \mathbb{Q}^R \cap [0,1)^R : q \leq 2Q, |S_q(\boldsymbol{a};\boldsymbol{v};M)| \geq 2^{-l}\} \cdot \sup_{q>Q} |S_q(\boldsymbol{a};\boldsymbol{v};M)| \\ &+ \sum_{i=l}^{\infty} \#\{\frac{\boldsymbol{a}}{q} \in \mathbb{Q}^R \cap [0,1)^R : q \leq 2Q, |S_q(\boldsymbol{a};\boldsymbol{v};M)| \geq 2^{-i-1}\} \cdot 2^{-i}. \end{aligned} \tag{8.38}$$

Now parts (ii) and (iii) show that

$$\#\{\frac{\boldsymbol{a}}{q} \in \mathbb{Q}^R \cap [0,1)^R : q \leq 2Q, |S_q(\boldsymbol{a};\boldsymbol{v};M)| \geq t\} \ll_C (Q^{\epsilon} t)^{-\frac{(d-1)R}{\mathscr{C}-\epsilon'}} M^{-dR}$$

and that

$$\sup_{q>Q} |S_q(\boldsymbol{a};\boldsymbol{v};M)| \ll Q^{-\delta_0/2} M^{dR\delta_0}.$$

Substituting these bounds into (8.38) gives

$$s(Q) \ll M^{-dR} Q^{O_{\mathscr{C}}(\epsilon)-\delta_0/2} M^{dR\delta_0} 2^{l\frac{(d-1)R}{\mathscr{C}-\epsilon'}} + M^{-dR} Q^{O_{\mathscr{C}}(\epsilon)} \sum_{i=l}^{\infty} 2^{(i+1)\frac{(d-1)R}{\mathscr{C}-\epsilon}-i}.$$

We have $\mathscr{C} > (d-1)R$ and we have assumed that ϵ' is small in terms of $\mathscr{C}, d$ and R, so we may assume that the bound $\mathscr{C} > (d-1)R + \epsilon'$ holds. So we may sum the geometric progression to find that

$$s(Q) \ll_{C,\mathscr{C}} M^{-dR} Q^{O_{\mathscr{C}}(\epsilon)} 2^{l\frac{(d-1)R}{\mathscr{C}-\epsilon'}} (Q^{-\delta_0/2} M^{d\delta_0 R} + 2^{-l}).$$

Picking $l = \lfloor \log_2(Q^{\delta_0/2} M^{-dR\delta_0}) \rfloor$ shows that

$$s(Q) \ll_{C,\mathscr{C}} M^{-dR} Q^{O_{\mathscr{C}}(\epsilon)} (QM^{-2dR})^{\frac{\delta_0}{2}\left(\frac{(d-1)R}{(\mathscr{C}-\epsilon)}-1\right)}.$$

We have $\delta_0 \geq \frac{1}{(d-1)2^{d-1}R}$, by Lemma 8.8. As ϵ is small in terms of $\mathscr{C}, d$ and R it follows that $s(Q) \ll_{C,\mathscr{C}} M^{-dR} Q^{-\delta_1} M^{2dR\delta_1}$ for some $\delta_1 > 0$ depending only on $\mathscr{C}, d$ and R. This proves (8.38). □

Lemma 8.11. *Let $C, \mathscr{C}$ and ϵ be as in Lemma 8.7.*

(i) For all $\gamma \in \mathbb{R}^R$ we have

$$S_{\infty}(\gamma) \ll_C |\gamma|^{-\mathscr{C}+\epsilon'}, \tag{8.39}$$

for some $\epsilon' = O_{\mathscr{C}}(\epsilon)$.

(ii) If the conclusion of part (i) holds and $\mathscr{C} - \epsilon' > R$, then there exists a complex number $\mathfrak{I} \in \mathbb{C}$ such that for all $P \geq 1$ we have

$$\frac{M^n}{P^{n-dR}} \mathfrak{I}(P;M) - \mathfrak{I} \ll_{\mathscr{C},C,\epsilon'} P^{-\Delta(\mathscr{C}-\epsilon'-R)} M^{-dR(\mathscr{C}-\epsilon'-R)}. \tag{8.40}$$

Furthermore we have $\mathfrak{J} \geq 0$ and

$$\mathfrak{J} = \lim_{P\to\infty} \frac{1}{P^{n-dR}} \lambda\{\boldsymbol{t} \in \mathbb{R}^n : \frac{1}{P}\boldsymbol{t} \in \mathcal{B}, |f_1^{[d]}(\boldsymbol{t})| \leq \frac{1}{2}, \ldots, |f_R^{[d]}(\boldsymbol{t})| \leq \frac{1}{2}\}, \tag{8.41}$$

where $\lambda\{\cdot\}$ denotes the Lebesgue measure.

Proof of part (i). For this part we set $M = 1$ and we will use the bound (8.10) and (8.23) to find a bound for $S_\infty(\boldsymbol{\gamma})$. First, for all $\boldsymbol{\beta} \in \mathbb{R}^R$ we have $|S(\boldsymbol{\beta}; P; \boldsymbol{v}; M)| \leq S(\boldsymbol{0}; P; \boldsymbol{v}; M)$, from Definition 8.3. Taking $\boldsymbol{\alpha} = 0, \boldsymbol{\beta} = P^{-d}\boldsymbol{\gamma}$ in (8.10) shows that

$$|S(P^{-d}\boldsymbol{\gamma}; P; \boldsymbol{v}; M)| \leq CP^{n+\epsilon} \max\{|\boldsymbol{\gamma}|^{-1}, P^{-\frac{d}{d-1}}|\boldsymbol{\gamma}|^{\frac{1}{d-1}}\}^{\mathscr{C}}.$$

If we take $\boldsymbol{a} = 0, q = 1, \boldsymbol{\alpha} = P^{-d}\boldsymbol{\gamma}$ in (8.23) we have

$$S(P^{-d}\boldsymbol{\gamma}; P; \boldsymbol{v}; M) = P^n S_\infty(\boldsymbol{\gamma}) + O(P^{n-1}(1 + |\boldsymbol{\gamma}|))$$

Combining these equality's gives

$$S_\infty(\boldsymbol{\gamma}) \ll_C P^\epsilon \max\{|\boldsymbol{\gamma}|^{-1}, P^{-\frac{d}{d-1}}|\boldsymbol{\gamma}|^{\frac{1}{d-1}}\}^{\mathscr{C}} + P^{-1} + P^{-1}|\boldsymbol{\gamma}| \tag{8.42}$$

If we have $|\boldsymbol{\gamma}| \leq 1$, then we choose $P = 1$. We then have

$$S_\infty(\boldsymbol{\gamma}) \ll_C \max\{|\boldsymbol{\gamma}|^{-1}, |\boldsymbol{\gamma}|^{\frac{1}{d-1}}\}^{\mathscr{C}} + 1 + |\boldsymbol{\gamma}| \ll |\boldsymbol{\gamma}|^{-\mathscr{C}}$$

Hence (i) follows. Otherwise we put $P = |\boldsymbol{\gamma}|^{1+\mathscr{C}}$. We then have

$$\begin{aligned} S_\infty(\boldsymbol{\gamma}) &\ll_C P^\epsilon \max\{|\boldsymbol{\gamma}|^{-1}, P^{-\frac{d}{d-1}}|\boldsymbol{\gamma}|^{\frac{1}{d-1}}\}^{\mathscr{C}} + |\boldsymbol{\gamma}|^{-1-\mathscr{C}} + |\boldsymbol{\gamma}|^{-\mathscr{C}} \\ &\leq 2|\boldsymbol{\gamma}|^{-\mathscr{C}+(1+\mathscr{C})\epsilon}. \end{aligned}$$

If we set $\epsilon' = (1 + \mathscr{C})\epsilon = O_{\mathscr{C}}(\epsilon)$, then (ii) follows.

Proof of part (ii). By the definition of $\mathfrak{J}(P; M)$ in §8.4.1 we have

$$\mathfrak{J}(P; M) = \int_{|\boldsymbol{\gamma}| \leq P^\Delta M^{dR}} P^{n-dR} M^{-n} S_\infty(\boldsymbol{\gamma})\, \mathrm{d}\boldsymbol{\gamma}.$$

If the inequality $\mathscr{C} - \epsilon' > R$ holds, then by (8.39) we have

$$\begin{aligned} \tfrac{M^n}{P^{n-dR}} \mathfrak{J}(P; M) - \mathfrak{J} &= \int_{|\boldsymbol{\gamma}| > P^\Delta M^{dR}} S_\infty(\boldsymbol{\gamma})\, \mathrm{d}\boldsymbol{\gamma} \\ &\ll_{\mathscr{C}, C, \epsilon'} P^{-\Delta(\mathscr{C}-\epsilon'-R)} M^{-dR(\mathscr{C}-\epsilon'-R)}, \end{aligned}$$

where the integrals converge absolutely. This proves (8.40) with

$$\mathfrak{J} = \int_{\boldsymbol{\gamma} \in \mathbb{R}^R} S_\infty(\boldsymbol{\gamma})\, \mathrm{d}\boldsymbol{\gamma}.$$

Now (8.41) follows from Myerson [20, Lemma 2.6], where again it is important to note that our definition of S_∞ coincides with the definition of S_∞ in Myerson. Lastely, $\mathfrak{J} \geq 0$ follows from Myerson [20, Theorem 1.3]. □

8.5 Proof of Theorem 1.4

In this section we deduce Theorem 1.4. For that we need two technical lemmas. The proof of the first one can be found in Myerson [20, Lemma 2.1]. The second lemma is a variant of Myerson [20, Lemma 2.2], which we will prove below. After that, in §8.5.2 we prove the main theorem of this section, Theorem 1.4.

8.5.1 Some Technical Lemmas

We show that the bound (8.10) implies an upper bound for the integral of the function $S(\boldsymbol{\alpha}; P\boldsymbol{v}; M)$ over any bounded measurable set. Müller [19] and Bentkus and Götze [3],[4] previously used similar ideas to treat quadratic forms with real coefficients. This was a main motivation in the work of Myerson [20].

Lemma 8.12. *(Myerson [20, Lemma 2.1]).*

Let $r_1 : (0, \infty) \to (0, \infty)$ be a strictly decreasing bijection, and let $r_2 : (0, \infty) \to (0, \infty)$ be a strictly increasing bijection. Write r_1^{-1} and r_2^{-1} for the inverses of these maps. Let $w > 0$ and let E_0 be a hypercube in $\mathbb{R}^R$ whose sides are of length w and parallel to the coordinate axes. Let E be a measurable subset of E_0 and let $\varphi : E \to [0, \infty)$ be a measurable function. Suppose that for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}$ such that $\boldsymbol{\alpha} \in E$ and $\boldsymbol{\alpha} + \boldsymbol{\beta} \in E$, we have

$$\min\{\varphi(\boldsymbol{\alpha}), \varphi(\boldsymbol{\alpha} + \boldsymbol{\beta})\} \leq \max\left\{r_1^{-1}(|\boldsymbol{\beta}|), r_2^{-1}(|\boldsymbol{\beta}|)\right\}. \tag{8.43}$$

Then, for any integers k and l with $k < l$, we have

$$\int_E \varphi(\boldsymbol{\alpha}) \, \mathrm{d}\,\boldsymbol{\alpha} \ll_R w^R 2^k + \sum_{i=k}^{l-1} 2^i \left(\frac{v r_1(2^i)}{\min\{r_2(2^i), w\}}\right)^R + \left(\frac{w r_1(2^l)}{\min\{r_2(2^l), w\}}\right)^R \sup_{\boldsymbol{\alpha} \in E} \varphi(\boldsymbol{\alpha}), \tag{8.44}$$

where the implied constant depends only on R.

We see that (8.43) looks a lot like (8.10). We will see below that if we choose

$$\varphi(\boldsymbol{\alpha}) = \frac{|S(\boldsymbol{\alpha}; P; v; M)|}{C(P/M)^{n+\epsilon}}, \qquad r_1(t) = t^{-1/\mathscr{C}} P^{-d}, \qquad r_2(t) = t^{\frac{d-1}{\mathscr{C}}} M^{-d},$$

that then (8.43) and (8.10) becomes identical. This will enable us to apply Lemma 8.12 to bound the integral $\int_{\mathfrak{m}_{P,d,\delta}} S(\boldsymbol{\alpha}; P; v; M)$, where $\mathfrak{m}_{P,d,\delta}$ is a set of minor arcs on which $S(\boldsymbol{\alpha}; P; v; M)$ is somehwat small.

Lemma 8.13. *(Myerson [20, Lemma 2.2]). Let T be a complex valued measurable function on $\mathbb{R}^R$. Let E_0 be a hypercube in $\mathbb{R}^R$ whose sides are length w and parallel to the coordinate axes, and let E be a measurable subset of E_0. Suppose that the inequality*

$$\min\left\{\left|\frac{T(\boldsymbol{\alpha})}{(P/M)^n}\right|, \left|\frac{T(\boldsymbol{\alpha}+\boldsymbol{\beta})}{(P/M)^n}\right|\right\} \leq \max\{P^{-d}|\boldsymbol{\beta}|^{-1}, |\boldsymbol{\beta}|^{\frac{1}{d-1}} M^{\frac{d}{d-1}}\}^{\mathscr{C}} \tag{8.45}$$

holds for some $P \geq 1$ and $\mathscr{C} \geq 0$ and all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^R$. Suppose that $\mathscr{C} > dR$ and that

$$\sup_{\boldsymbol{\alpha} \in E} |T(\boldsymbol{\alpha})| \leq (P/M)^{n-\delta} \tag{8.46}$$

for some $\delta \geq 0$. Then we have

$$\int_E \frac{|T(\boldsymbol{\alpha})|}{(P/M)^n}\, d\boldsymbol{\alpha} \\ \ll_{\mathscr{C},d,R} w^R (P/M)^{-dR-\delta(1-dR/\mathscr{C})} + P^{-dR}(P/M)^{-\delta(1-R/\mathscr{C})}. \tag{8.47}$$

Corollary 8.13.1. *If in addition in Theorem 8.13 we have $w = 1$, then (8.47) becomes*

$$\int_E \frac{|T(\boldsymbol{\alpha})|}{(P/M)^n}\, d\boldsymbol{\alpha} \ll_{\mathscr{C},d,R} (P/M)^{-dR-\delta(1-dR/\mathscr{C})}$$

Proof. We have

$$M^{dR}(P/M)^{\delta dR/\mathscr{C}} \geq (P/M)^{\delta R/\mathscr{C}}.$$

Multiplying both sides with $P^{-dR}(P/M)^{-\delta}$ gives

$$(P/M)^{-dR}(P/M)^{-\delta(1-dR/\mathscr{C})} \geq P^{-dR}(P/M)^{-\delta(1-R/\mathscr{C})},$$

which proves the claim. □

Proof of Lemma 8.13. We want to apply Lemma 8.12, in particular we want to choose $T(\boldsymbol{\alpha})$, r_1 and r_2 in such a way that the bound (8.43) follows from (8.45). It is an obvious choice to take $\varphi(\boldsymbol{\alpha}) = \frac{|T(\boldsymbol{\alpha})|}{(P/M)^n}$. We now only have to choose r_1 and r_2 in a smart way. We want that

$$r_1^{-1}(|\boldsymbol{\beta}|) = (P^{-d}|\boldsymbol{\beta}|^{-1})^{\mathscr{C}} \tag{8.48}$$

$$r_2^{-1}(|\boldsymbol{\beta}|) = (M^{\frac{d}{d-1}}|\boldsymbol{\beta}|^{\frac{1}{d-1}})^{\mathscr{C}}. \tag{8.49}$$

For deducing a formula for r_1 we set $t = (P^{-d}|\boldsymbol{\beta}|^{-1})^{\mathscr{C}}$. Rewriting gives $|\boldsymbol{\beta}| = P^{-d}t^{-1/\mathscr{C}}$. Then (8.48) is equivalent to

$$r_1(t) = |\boldsymbol{\beta}| = P^{-d}t^{-1/\mathscr{C}}$$

For deducing a formula for r_2 we set $t = (M^{\frac{d}{d-1}}|\boldsymbol{\beta}|^{\frac{1}{d-1}})^{\mathscr{C}}$. Rewriting gives $|\boldsymbol{\beta}| = M^{-d}t^{(d-1)/\mathscr{C}}$. Then (8.49) is equivalent to

$$r_2(t) = |\boldsymbol{\beta}| = M^{-d}t^{(d-1)/\mathscr{C}}.$$

Hence we choose

$$\varphi(\boldsymbol{\alpha}) = \frac{|T(\boldsymbol{\alpha})|}{(P/M)^n}, \qquad r_1(t) = t^{-1/\mathscr{C}}P^{-d}, \qquad r_2(t) = t^{\frac{d-1}{\mathscr{C}}}M^{-d}, \tag{8.50}$$

noting that the bound (8.43) then follows from (8.45). Clearly r_1 is a strictly decreasing bijection and r_2 is a strictly increasing bijection. It remains to choose the parameters k and

l from (8.44). We will choose these so that the right-hand side of (8.44) is dominated by the sum $\sum_{i=k}^{l-1}$, rather than either of the other two terms. More precisely, take

$$k = \left\lfloor \log_2(P/M)^{-\mathscr{C}} \right\rfloor, \qquad l = \left\lceil \log_2(P/M)^{-\delta} \right\rceil$$

Observing that

$$\frac{1}{2}(P/M)^{-\mathscr{C}} < 2^k \leq (P/M)^{-\mathscr{C}}, \qquad (P/M)^{-\delta} \leq 2^l < 2(P/M)^{-\delta}. \tag{8.51}$$

We may assume that $\mathscr{C} > \delta$, for otherwise we have the bound $\int_E T(\boldsymbol{\alpha})\, \mathrm{d}\,\boldsymbol{\alpha} \leq w^R(P/M)^{n-\delta}$ which follows from (8.46), is stronger than the bound (8.47). We then have $k < l$ and so we can apply Lemma 8.12. Substituting in our choises (8.50) for the parameters yields

$$\begin{aligned} \int_E \frac{|T(\boldsymbol{\alpha})|}{(P/M)^n}\, \mathrm{d}\boldsymbol{\alpha} \ll_R w^R 2^k + \sum_{i=k}^{l-1} 2^i \left(\frac{vP^{-d}2^{-i/\mathscr{C}}}{\min\{M^{-d}2^{(d-1)i/\mathscr{C}}, w\}} \right)^R \\ + \left(\frac{wP^{-d}2^{-l/\mathscr{C}}}{\min\{M^{-d}2^{(d-1)l/\mathscr{C}}, w\}} \right)^R \sup\nolimits_{\boldsymbol{\alpha}\in E} \varphi(\boldsymbol{\alpha}), \end{aligned} \tag{8.52}$$

By (8.46) and (8.51) we have $\sup_{\boldsymbol{\alpha}\in E} \frac{|T(\boldsymbol{\alpha})|}{(P/M)^n} \leq 2^l$, and so we may extend the sum in (8.52) from $\sum_{i=k}^{l-1}$ to $\sum_{i=k}^{l}$ to obtain

$$\int_E \frac{|T(\boldsymbol{\alpha})|}{(P/M)^n}\, \mathrm{d}\boldsymbol{\alpha} \ll_R w^R 2^k + \sum_{i=k}^{l} 2^i \left(\frac{vP^{-d}2^{-i/\mathscr{C}}}{\min\{M^{-d}2^{(d-1)i/\mathscr{C}}, w\}} \right)^R$$

Since

$$\begin{aligned} \frac{P^{-d}2^{-i/\mathscr{C}}}{\min\{M^{-d}2^{(d-1)i/\mathscr{C}}, w\}} &\leq P^{-d}2^{-di/\mathscr{C}}M^d + w^{-1}P^{-d}2^{-i/\mathscr{C}} \\ &\leq (P/M)^{-d}2^{-di/\mathscr{C}} + w^{-1}P^{-d}2^{-i/\mathscr{C}} \end{aligned}$$

we deduce that

$$\int_E \frac{|T(\boldsymbol{\alpha})|}{(P/M)^n}\, \mathrm{d}\boldsymbol{\alpha} \ll_R w^R 2^k + \sum_{i=k}^{l} w^R (P/M)^{-dR} 2^{i(1-dR/\mathscr{C})} + \sum_{i=k}^{l} P^{-dR} 2^{i(1-R/\mathscr{C})}. \tag{8.53}$$

Note that

$$\sum_{i=k}^{l} 2^{i(1-dR/\mathscr{C})} \ll_{\mathscr{C},d,R} 2^{l(1-dR/\mathscr{C})}$$

and

$$\sum_{i=k}^{l} 2^{i(1-R/\mathscr{C})} \ll_{\mathscr{C},d,R} 2^{l(1-R/\mathscr{C})}.$$

Recall from (8.51) that we have $2^l \leq 2(P/M)^{-\delta}$. It follows that

$$\sum_{i=k}^{l} 2^{i(1-dR/\mathscr{C})} \ll_{\mathscr{C},d,R} (P/M)^{-\delta(1-dR/\mathscr{C})}$$

and
$$\sum_{i=k}^{l} 2^{i(1-R/\mathscr{C})} \ll_{\mathscr{C},d,R} (P/M)^{-\delta(1-R/\mathscr{C})}.$$
Substituting these bounds in (8.53) gives
$$\int_E \frac{|T(\boldsymbol{\alpha})|}{(P/M)^n}\,\mathrm{d}\boldsymbol{\alpha} \\ \ll_{\mathscr{C},d,R} w^R 2^k + w^R (P/M)^{-dR}(P/M)^{-\delta(1-dR/\mathscr{C})} + P^{-dR}(P/M)^{-\delta(1-R/\mathscr{C})}.$$
Lastly we have the bound $2^k \leq (P/M)^{-\mathscr{C}}$ from (8.51) and we have $-\mathscr{C} < -dR$, hence
$$\int_E \frac{|T(\boldsymbol{\alpha})|}{(P/M)^n}\,\mathrm{d}\boldsymbol{\alpha} \\ \ll_{\mathscr{C},d,R} w^R (P/M)^{-\mathscr{C}} + w^R (P/M)^{-dR-\delta(1-dR/\mathscr{C})} + P^{-dR}(P/M)^{-\delta(1-R/\mathscr{C})} \\ \ll_{\mathscr{C},d,R} w^R (P/M)^{-dR-\delta(1-dR/\mathscr{C})} + P^{-dR}(P/M)^{-\delta(1-R/\mathscr{C})},$$
which completes the proof. □

8.5.2 Proof of Theorem 1.4

Theorem 1.4. *Let $f_i \in \mathbb{Z}[x_1, \ldots, x_n]$ be quadratic forms with $n \geq 2$ and integer coefficients and $\dim(\tilde{X}^*) \leq n-1$. Suppose*
$$N^{aux}_{\boldsymbol{\beta}\cdot\boldsymbol{F}}(B) \leq C_0 B^{(d-1)n-2^d\mathscr{C}}$$
for some $C_0 \geq 1, \mathscr{C} > dR$ and all $\boldsymbol{\beta} \in \mathbb{R}^R$ and $B \geq 1$. For all $P \geq M$ we have
$$\tilde{N}(P, \Omega_M) = \sum_{[\boldsymbol{v}]_M \in \Omega_M} (\mathfrak{J}\mathfrak{S}(\boldsymbol{v}; M) P^{n-dR} M^{-n} + O(P^{n-dR-\delta} M^{-n+\delta_2})),$$
where the implied constant depends at most on $C_0, \mathscr{C}$ and the f_i, and δ and δ_2 are positive constants depending at most on $\mathscr{C}, d$ and R.

Proof of Theorem 1.4. Let $(P/M) \geq 1$ and $\Delta = \frac{1}{4R+6}$. By Lemma 8.4 we have
$$\tilde{N}(P, \Omega_M) = \sum_{\boldsymbol{v} \in \Omega_M} \Big(\int_{\mathfrak{m}_{P,d,\delta}} S(\boldsymbol{\alpha}; P; v; M)\,\mathrm{d}\boldsymbol{\alpha} + \int_{\mathfrak{M}_{P,d,\delta}} S(\boldsymbol{\alpha}; P; v; M)\,\mathrm{d}\boldsymbol{\alpha} \Big),$$
where $\mathfrak{m}_{P,d,\delta}$ and $\mathfrak{M}_{P,d,\delta}$ are defined as in §8.4.1. We apply Lemma 8.13 with
$$T(\boldsymbol{\alpha}) = C^{-1}\Big(\frac{P}{M}\Big)^{-\epsilon} S(\boldsymbol{\alpha}; P; \boldsymbol{v}; M), \quad E_0 = [0,1]^R, \quad E = \mathfrak{m}_{P,d,\delta}, \quad \delta = \Delta\delta_0.$$
Lemma 8.7 then gives
$$\min\Big\{\Big|\frac{T(\boldsymbol{\alpha})}{(P/M)^n}\Big|, \Big|\frac{T(\boldsymbol{\alpha}+\boldsymbol{\beta})}{(P/M)^n}\Big|\Big\} \leq \max\{P^{-d}|\boldsymbol{\beta}|^{-1}, |\boldsymbol{\beta}|^{\frac{1}{d-1}} M^{\frac{d}{d-1}}\}^{\mathscr{C}}$$

Hence with these choices (8.45) holds. Lemma 8.8 shows that $\sup_{\boldsymbol{\alpha}\in\mathfrak{m}_{P,d,\delta}} CT(\boldsymbol{\alpha}) \ll (P/M)^{n-\delta}$, and after increasing C if necessary this gives us (8.46). This verifies the hypotheses of Lemma 8.13. Now Corollary 8.13.1 gives

$$\int_{\mathfrak{m}_{P,d,\delta}} S(\boldsymbol{\alpha}; P; \boldsymbol{v}; M)\,\mathrm{d}\boldsymbol{\alpha} \ll_{C,\mathscr{C}} (P/M)^{n-dR-\Delta\delta_0(1-\frac{dR}{\mathscr{C}})+\epsilon}. \tag{8.54}$$

For the major arcs, since $\Delta = \frac{1}{4R+6}$ we have by Lemma 8.9 that

$$\begin{aligned}\int_{\mathfrak{M}_{P,d,\delta}} S(\boldsymbol{\alpha}; P; \boldsymbol{v}; M)\,d\boldsymbol{\alpha} = \mathfrak{S}(P;\boldsymbol{v};M)\mathfrak{J}(P;M)&\\ + O\Big(P^{n-dR-\frac{1}{2}}M^{1-n+3dR+2dR^2}\Big)&,\end{aligned} \tag{8.55}$$

where $\mathfrak{S}(P;\boldsymbol{v};M)$ and $\mathfrak{J}(P;M)$ are defined as in §8.4.1.

Since $\mathscr{C} > dR$ holds, $\dim(\tilde{V}^*) \le n-1$, and ϵ is small in terms of $\mathscr{C}, d$ and R, both of Lemmas 8.10 and 8.11 apply. We have $\mathfrak{J} = O(1)$, $\mathfrak{S}(\boldsymbol{v};\mathfrak{M}) = O(M^{dR(2\delta_1-1)})$. Now (8.32) from Lemma 8.10 and (8.40) from Lemma 8.11 shows that

$$\begin{aligned}&P^{dR-n}M^n\mathfrak{S}(P;\boldsymbol{v};M)\mathfrak{J}(P;M)\\ &= (\mathfrak{J} + O_{\mathscr{C},C}(P^{-\Delta(\mathscr{C}-\epsilon'-R)}M^{-dR(\mathscr{C}-\epsilon'-R)}))(\mathfrak{S}(\boldsymbol{v};M) + O_{\mathscr{C},C}(P^{-\Delta\delta_1}M^{dR(\delta_1-1)}))\\ &= \mathfrak{J}\mathfrak{S}(\boldsymbol{v};M) + O_{\mathscr{C},C}(P^{-\Delta\delta_1}M^{dR(\delta_1-1)}) + O_{\mathscr{C},C}(P^{-\Delta(\mathscr{C}-\epsilon'-R)}M^{-dR(\mathscr{C}-\epsilon'-R)}M^{dR(2\delta_1-1)})\\ &= \mathfrak{J}\mathfrak{S}(\boldsymbol{v};M) + O_{\mathscr{C},C}(P^{-\Delta\delta_1}M^{dR(\delta_1-1)} + P^{-\Delta(\mathscr{C}-R)/2}M^{dR(2\delta_1-1+(R-\mathscr{C})/2)}),\end{aligned} \tag{8.56}$$

where $\delta_1 > 0$ depends at most on $\mathscr{C}, d$ and R. Combining (8.54), (8.55) and (8.56) gives

$$\begin{aligned}\tilde{N}(P,[\boldsymbol{v}]_M) = \mathfrak{J}\mathfrak{S}(\boldsymbol{v};M)P^{n-dR}M^{-n} + O_{\mathscr{C},C}(&+P^{n-dR-\Delta\delta_1}M^{-n+dR(\delta_1-1)}\\ &+P^{n-dR-\Delta(\mathscr{C}-R)/2}M^{-n+dR(2\delta_1-1+(R-\mathscr{C})/2)}\\ &+P^{n-dR-\frac{1}{2}}M^{-n+1+3dR+2dR^2}\\ &+P^{n-dR-\Delta\delta_0(1-\frac{dR}{\mathscr{C}})}M^{-n+dR+\Delta\delta_0(1-\frac{dR}{\mathscr{C}})+\epsilon})\end{aligned}$$

Let

$$\delta = \min\{\Delta\delta_1, \quad \Delta(\mathscr{C}-R)/2, \quad \frac{1}{2}, \quad \Delta\delta_0(1-\frac{dR}{\mathscr{C}})\}$$

and

$$\delta_2 = \max\{R(\delta_1-1), \quad dR(2\delta_1-1+(R-\mathscr{C})/2), \quad 1+3dR+2dR^2, \quad dR+\Delta\delta_0(1-\frac{dR}{\mathscr{C}})+\epsilon\}.$$

Then

$$\tilde{N}(P,[\boldsymbol{v}]_M) = \mathfrak{J}\mathfrak{S}(\boldsymbol{v};M)P^{n-dR}M^{-n} + O_{\mathscr{C},C}(P^{n-dR-\delta}M^{-n+\delta_2}),$$

which proves the theorem. □

9 Sieving on System of Quadrics: Proof of Theorem 1.3

Now we have a bound for $\tilde{N}(P, \Omega_M)$, we can use this to plug this into Selberg's sieve to find an upper bound for $N(P, \Omega)$. We use the same notation as in section 7 and 8. So let $F_1, \ldots, F_R \in \mathbb{Z}[x_0, \ldots, x_n]$ be quadratics with $n \geq 2$, X the corresponding projective variety and a_n and $\mathcal{A}_d$ as in section 7.

Definition 9.1. *Let $\sigma_{\mathbb{R}}$ be the element of $\{0, \ldots, n\}$ defined by*

$$\sigma_{\mathbb{R}} = 1 + \max_{\boldsymbol{\beta} \in \mathbb{R}^R \setminus \{\mathbf{0}\}} \dim \operatorname{Sing} V(\boldsymbol{\beta} \cdot \boldsymbol{F}),$$

and $V(\boldsymbol{\beta} \cdot \boldsymbol{F})$ is the hypersurface cut out in $\mathbb{P}^n_{\mathbb{R}}$ by $\beta_1 F_1 + \ldots + \beta_R F_R = 0$.

Theorem 9.2. *Assume $\dim X = n - R$, $\dim(\tilde{X}^*) \leq n - 1$ ($\tilde{X}^*$ defined in §8.4.1) and $n + 1 - \sigma_{\mathbb{R}} > 8R$. Let $\mathfrak{J}$ be as in Theorem 1.4 and*

$$\sigma_p = \lim_{k \to \infty} \frac{1}{p^{k(n+1-R)}} \#\{\boldsymbol{b} \in \{1, 2, \ldots, p^k\}^{n+1} : \boldsymbol{F}(\boldsymbol{b}) \equiv 0 \bmod p^k\}.$$

Then

$$\mathcal{A}_d = \prod_{p | d^m} \Big(1 - \frac{\#\hat{\Omega}_{p^m}}{X(\#\mathbb{Z}/p^m\mathbb{Z})}\Big) P^{n+1-2R} \mathfrak{J} \prod_p \sigma_p + O(P^{n+1-2R-\delta} d^{m(-n-1+\delta_2)} \#\hat{\Omega}^c_{d^m})$$

Proof. We let $\mathscr{C} = \frac{n+1-\sigma_{\mathbb{R}}}{4}$. Then

$$N^{\text{aux}}_{\boldsymbol{\beta} \cdot \boldsymbol{F}}(B) \leq C_0 B^{n+1-4\mathscr{C}}$$

for some $C_0 \geq 1$ and all $\boldsymbol{\beta} \in \mathbb{R}^R$ and $B \geq 1$ will follow from the proof of Theorem 1.2 in [20]. Theorem 1.4 now gives

$$\mathcal{A}_d = \sum_{d|n} a_n = \sum_{[\boldsymbol{v}]_{d^m} \in \hat{\Omega}^c_{d^m}} (\mathfrak{J}\mathfrak{S}(\boldsymbol{v}; d^m) P^{n+1-2R} d^{-m(n+1)} + O(P^{n+1-2R-\delta} d^{m(-(n+1)+\delta_2)})).$$

To simplify the notation let $M = d^m$. Define

$$\sigma_p(\boldsymbol{v}, M) = \lim_{k \to \infty} \frac{1}{p^{k(n+1-R)}} \#\{\boldsymbol{b} \in \{1, 2, \ldots, p^k\}^{n+1} : \boldsymbol{F}(M\boldsymbol{b} + \boldsymbol{v}) \equiv 0 \bmod p^k\}.$$

Note $\sigma_p(0, 1) = \sigma_p$. Let

$$g(d) = \sum_{\boldsymbol{v} \in \hat{\Omega}^c_M} \prod_{p | M} \frac{\sigma_p(\boldsymbol{v}, M)}{\sigma_p} M^{-n-1}.$$

and

$$X = P^{n+1-2R} \mathfrak{J} \prod_p \sigma_p$$

For $p \nmid M$ we have $\sigma_p(\boldsymbol{v}, M) = \sigma_p$, since $\boldsymbol{b} \in (\mathbb{Z}\backslash p^k\mathbb{Z})^{n+1}$ runs over the same elements as $M\boldsymbol{b} + \boldsymbol{v} \in (\mathbb{Z}\backslash p^k\mathbb{Z})^{n+1}$. Moreover by lemma 8.10 we have $\mathfrak{S}(\boldsymbol{v}; M) = \prod_p \sigma_p(\boldsymbol{v}, M)$. Hence

$$\begin{aligned} g(d)X &= \sum_{\boldsymbol{v} \in \hat{\Omega}^c_M} \prod_{p|d} \frac{\sigma_p(\boldsymbol{v}, M)}{\sigma_p} M^{-n-1} P^{n+1-2R} \mathfrak{J} \prod_p \sigma_p \\ &= \sum_{\boldsymbol{v} \in \hat{\Omega}^c_M} \prod_p \sigma_p(\boldsymbol{v}, M) M^{-n-1} P^{n+1-2R} \mathfrak{J} \\ &= \sum_{\boldsymbol{v} \in \hat{\Omega}^c_M} \mathfrak{S}(\boldsymbol{v}; M) M^{-n-1} P^{n+1-2R} \mathfrak{J}. \end{aligned}$$

So it suffices to prove that $g(d) = \prod_{p|M} \left(1 - \frac{\#\hat{\Omega}_{p^m}}{X(\#\mathbb{Z}/p^m\mathbb{Z})}\right)$. We will use that for all $p|d$ we have $\#X(\mathbb{Z}/p^m\mathbb{Z}) = \#X(\mathbb{Z}/p\mathbb{Z}) \cdot p^{(m-1)(n+1-R)}$ and $\sigma_p = X(\mathbb{Z}/p\mathbb{Z}) \cdot p^{-(n+1-R)}$ and $\sum_{[v]_M \in \hat{\Omega}^c_M} \sigma_p(v, M) = p^{mR} \#\hat{\Omega}^c_{p^m}$, which follow from a quantitative version of Hensel's lemma [8, Lemma 2.1]. We have

$$\begin{aligned} g(d) &= \sum_{\boldsymbol{v} \in \hat{\Omega}^c_M} \prod_{p|M} \frac{\sigma_p(\boldsymbol{v}, M)}{\sigma_p} p^{-(n+1)m} \\ &= \prod_{p|M} \frac{\sum_{\boldsymbol{v} \in \hat{\Omega}^c_M} \sigma_p(\boldsymbol{v}, M)}{\#X(\mathbb{Z}/p\mathbb{Z})} p^{-(n+1)m+(n+1-R)} \\ &= \prod_{p|M} \frac{\#\hat{\Omega}^c_{p^m}}{\#X(\mathbb{Z}/p\mathbb{Z})} p^{-(n+1)m+(n+1-R)+mR} \\ &= \prod_{p|M} \frac{\#\hat{\Omega}^c_{p^m}}{\#X(\mathbb{Z}/p^m\mathbb{Z})} p^{-(m-1)(n+1-R)} \\ &= \prod_{p|M} \left(1 - \frac{\#\hat{\Omega}_{p^m}}{X(\#\mathbb{Z}/p^m\mathbb{Z})}\right). \end{aligned}$$

□

Recall the definition of the density function from (1.3)

$$\omega_p = 1 - \frac{\#\hat{\Omega}_{p^m}}{\#\hat{X}(\mathbb{Z}/p^m\mathbb{Z})}.$$

Also, recall Theorem 1.3.

Theorem 1.3. *Assume that $X \subset \mathbb{P}^n(\mathbb{Q})$ is a smooth variety defined by the quadratics $F_1, \ldots, F_R$ with integer coefficients, $\dim X = n - R$, $\dim(\tilde{X}^*) \leq n - 1$ and $n + 1 - \sigma_{\mathbb{R}} > 8R$. Let $m \in \mathbb{N}$ and let $\Omega_{p^m} \subset X(\mathbb{Z}/p^m\mathbb{Z})$ for each prime p. Assume that*

$$0 \leq \omega_p < 1.$$

Then for every $\xi \geq 1$ and any $\epsilon > 0$, we have

$$N(P,\Omega) \ll_{X,\epsilon} \frac{P^{n+1-2R}}{J(\xi)} + P^{n+1-2R-\delta}\xi^{2m\delta_2+2+\epsilon},$$

where

$$J(\xi) = \sum_{k<\xi} \mu^2(k) \prod_{p|k} \left(\frac{\omega_p}{1-\omega_p} \right).$$

Proof. Recall from section 7 that

$$N(P,\Omega) \leq \tilde{N}(2P,\hat{\Omega}) \leq \sum_{(n,\mathcal{P})=1} a_n.$$

The main term follows from combining Theorem 9.2 and applying Selberg's sieve as in Theorem 3.1. For the error term we have

$$\begin{aligned} |R(\mathcal{A},\Lambda^2)| &\leq \sum_{d<\xi^2} \tau_3(d)|r_d(\mathcal{A})| \\ &\leq \sum_{d<\xi^2} \tau_3(d) P^{n+1-2R-\delta} d^{m(-n+\delta_2)} \#\hat{\Omega}^c_{d^m} \\ &\leq \sum_{d<\xi^2} \tau_3(d) P^{n+1-2R-\delta} d^{m\delta_2} \end{aligned}$$

Taking the trivial bound $\tau_3(d) \ll d^{\epsilon/2}$ and summing over $d \leq \xi^2$, we see

$$|R(\mathcal{A},\Lambda^2)| \ll_{X,\epsilon} P^{n+1-2R-\delta}\xi^{2m\delta_2+2+\epsilon},$$

which completes the proof.

□

10 Thin Sets: Proof of Theorem 1.2

An application of the asymptotic formula for $\tilde{N}(P,\Omega)$ is found when we try to count points in a thin subset. Our aim in this section is to prove Theorem 1.2. First we give the definition of thin sets from [22, §3.1].

Definition 10.1. *Let X be an integral variety over a field F. A type I thin subset is a set of the form $Z(F) \subset X(F)$, where Z is a closed subvariety with $Z \neq X$. A type II thin subset is a set of the form $\pi(Y(F))$, where $\pi : Y \to X$ is a generically finite dominant morphism with $\deg \pi \geq 2$ and Y geometrically integral. A thin subset is a subset contained in a finite union of thin subsets of type I and II.*

An example of a thin set of type I is the zero set of $x^2 - y^2$ in $\mathbb{P}^1(\mathbb{Q})$. An example of a thin set of type II is the set of all squares $\mathbb{Q}$, i.e. the image of $\mathbb{Q}$ of the map from $\mathbb{Q}$ to $\mathbb{Q}$ defined as $x \to x^2$.

Let us recall Theorem 1.2.

Theorem 1.2. *Let $X(\mathbb{Q}) \subset \mathbb{P}^n(\mathbb{Q})$ be a smooth variety defined by the quadratics $F_1, \ldots, F_R$ with integer coefficients. Suppose $\dim X = n - R$, $\dim \tilde{X}^* \leq n - 1$ and $n + 1 - \sigma_{\mathbb{R}} > 8R$. Let $\Upsilon \subset X(\mathbb{Q})$ be a thin set. Then there exists $\theta_n > 0$ such that*

$$\#\{x \in \Upsilon : |x| \leq P\} \ll_{\Upsilon,X} P^{n+1-2R-\theta_n}.$$

There is a similar result for only one quadratic, proved by Browning and Loughran [8, Th. 1.8]. The proof of Theorem 1.2 is based on that one. We will see below that $\theta_n < \delta/(2m\delta_2+3)$ is admissible. To prove Theorem 1.2, we require information on thin sets modulo p.

Lemma 10.2. *Let $X \to \operatorname{Spec} \mathbb{Z}$ be a smooth integral finite type scheme of relative dimension n and $\Upsilon \subset X(\mathbb{Z})$ be thin in $X(\mathbb{Q})$.*

- *If Υ has type I then $\#(\Upsilon \bmod p) \ll_\Upsilon p^{n-1}$*
- *If Υ has type II, then there exists a finite Galois extension $\mathbb{Q}_\Upsilon/\mathbb{Q}$ and a constant $c_\Upsilon \in (0,1)$ such that for all primes p which split completely in $\mathbb{Q}_\Upsilon$ we have $\#(\Upsilon \bmod p) \leq c_\Upsilon p^n + O_\Upsilon(p^{n-1/2})$.*

Proof. The first part follows from applying the Lang–Weil estimates [15] to each component of the closure of Υ. The second part is [22, Theorem 3.6.2]. □

Proof of Theorem 1.2. To prove Theorem 1.2, it suffices to consider thin sets of type I and II. We begin with type II. By Lemma 10.2 there is a set of primes $\mathcal{P}$ of positive natural density θ and a constant $c \in (0,1)$, such that for each $p \in \mathcal{P}$ we have

$$\#(\Upsilon \bmod p) \leq cp^{n-1} + O_\Upsilon(p^{n-3/2}).$$

Taking $m = 1$ in ω_p for each $p \in \mathcal{P}$ gives

$$\omega_p = 1 - \frac{\#(\Upsilon \bmod p)}{X(\mathbb{Z}/p\mathbb{Z})} \geq 1 - \frac{\#(\Upsilon \bmod p)}{p^{n-1}}$$
$$\geq 1 - \frac{cp^{n-1} + O_\Upsilon(p^{n-3/2})}{p^{n-1}} \geq 1 - c + O_\Upsilon(p^{-1/2})$$

It follows that there exists $\eta < (1-c)/c$ such that

$$\frac{\omega_p}{1-\omega_p} \geq \eta$$

for large enough $p \in \mathcal{P}$. Let $\mathcal{P}^o$ denote the set of such $p \in \mathcal{P}$. An application of [8, Lemma 3.11] now yields

$$J(\xi) \geq \sum_{\substack{a \leq \xi \\ p|a \Rightarrow p \in \mathcal{P}^o}} \mu^2(a)\eta^{\omega(a)} \gg_{\Upsilon,X} \xi(\log \xi)^{\eta\delta-1} \gg_{\epsilon,\Upsilon,X} \xi^{1-\epsilon},$$

for any $\epsilon > 0$ and where we denote $\omega(a)$ for the number of primes dividing a. Now it follows from Theorem 1.3 that

$$\#\{x \in \Upsilon : |x| \leq P\} \ll_{\Upsilon,X,\epsilon} \xi^{\epsilon-1}P^{n+1-2R} + P^{n+1-2R-\delta}\xi^{2m\delta_2+2+\epsilon}.$$

Balancing the terms by choosing $\xi = P^{\theta_n}$ with $\theta_n = \delta/(2m\delta_2 + 3)$, gives the statement for thin sets of type II.

For thin sets of type I, we let $Z \subset X$ be a Zariski closed subset with $Z \neq X$. For any prime p, Lemma 10.2 implies that $Z(\mathbb{F}_p) \leq cp^{n-2}$, for some $c = c(Z) > 0$. Then $\omega_p \geq 1 - cp^{-1}$ ant it follows that

$$\frac{\omega_p}{1-\omega_p} \geq \frac{1-cp^{-1}}{cp^{-1}} = \frac{p}{c} - 1.$$

A further application of [8, Lemma 3.11] now implies that $J(\xi) \gg_{\epsilon,X} \xi^{2-\epsilon}$ for all $\epsilon > 0$. We complete the proof by the same argument as above. □

11 Almost primes

Let B be a fixed finite set of primes. An application of Theorem 9.2 is found when we try to count the following set

$$\{\boldsymbol{x} \in \mathbb{Z}^n | \boldsymbol{x} \in P\mathcal{B}, \boldsymbol{F}(\boldsymbol{x}) = 0, x_1 \cdots x_n \in P_r(B)\},$$

where $P_r(B)$ is the set of positive integers with at most r prime divisors outside B. If

$$a_n = \sum_{\substack{\boldsymbol{x}\in\mathbb{Z}^n\cap P\mathcal{B} \\ \boldsymbol{F}(\boldsymbol{x})=0 \\ n=x_1\cdots x_n}} 1 \tag{11.1}$$

then

$$\sum_{n\in P_r(B)} a_n = \{\boldsymbol{x} \in \mathbb{Z}^n | \boldsymbol{x} \in P\mathcal{B}, \boldsymbol{F}(\boldsymbol{x}) = 0, x_1 \cdots x_n \in P_r(B)\}.$$

In general, to estimate $\sum_{\omega(n)\le r} a_n$ we need to know how $\mathcal{A}$ is distributed, i.e. for each square-free d for which $(d, B) = 1$ we need a formula for

$$\mathcal{A}_d = \sum_{n=0(\bmod d)} a_n = \{\boldsymbol{x} \in \mathbb{Z}^n | \boldsymbol{x} \in P\mathcal{B}, \boldsymbol{F}(\boldsymbol{x}) = 0, d | x_1 \cdots x_n\}.$$

Just as in section 3 we write

$$|\mathcal{A}_d| = \omega_d X - R_d,$$

where X is an approximation to $|\mathcal{A}_1| = |\mathcal{A}|$ and ω_d is a non-negative multiplicative function satisfying

$$\begin{cases} \omega_1 = 1; \\ 0 \le \omega_p < 1, & \text{if } p \notin B \\ \omega_p = 0, & \text{if } p \in B. \end{cases} \tag{11.2}$$

Suppose for fixed (independent of z, z_1) constants $\kappa > 1$ and $A \ge 2$ we have

$$\prod_{z_1\le p<z} (1 - \omega_p)^{-1} \le \Big(\frac{\log z}{\log z_1}\Big)^{\kappa} \Big(1 + \frac{A}{\log z_1}\Big), \qquad \text{for } 2 \le z_1 < z. \tag{11.3}$$

The term $\omega_d X$ s considered as an approximation to $|\mathcal{A}_d|$, and therefore we suppose that the errors R_d are small on average, i.e. for some constants τ with $0 < \tau < 0$, $A_1 \ge 1$, and $A_2 \ge 2$,

$$\sum_{\substack{d<X^\tau \log^{-A_1} X \\ (d,B)=1}} \mu^2(d) 4^{\nu(d)} |R_d| \le A_2 \frac{X}{\log^{\kappa+1} X}, \tag{11.4}$$

where ν denotes the number of prime factors of d. Before we can state the two essential theorems for the problem we discuss , we introduce another constant, μ, by

$$\max_{a_n\in\mathcal{A}} a_n \le X^{\tau\mu}.$$

Note that in our sieving problem we have $a_l = 0$ if $l > P^n$.

To give an lower bound for $\sum_{\omega(n)\le r} a_n$, we need two theorems from Diamond and Halberstam [10]. In the first one we introduce the number β_κ, which we will use in the second one.

Theorem 11.1. *(Diamond and Halberstam [10, Theorem 0]). Let $\kappa > 1$ be given, and let $\sigma_\kappa(u)$ be the continuous of the differential-difference problem*

$$\begin{cases} \mu^{-\kappa}\sigma(u) = A_\kappa^{-1}, & \text{for } 0 < u \le 2,\ A_\kappa = (2e^\gamma)^\kappa\Gamma(\kappa+1), \\ (u^{-\kappa}\sigma(u))' = -\kappa u^{-\kappa-1}\sigma(u-2), & \text{for } 2 < u; \end{cases}$$

here γ denotes the Euler constant. Then there exist two number α_κ and β_κ satisfying

$$\alpha_\kappa \ge \beta_\kappa \ge 2$$

such that the simultaneous differential-difference system

$$\begin{cases} F(u) = 1/\sigma_\kappa(u) & \text{for } 0 < u \le \alpha_\kappa, \\ f(u) = 0 & \text{for } 0 < u \le \beta_\kappa, \\ (u^\kappa F(u))' = \kappa u^{\kappa-1} f(u-1) & \text{for } u > \alpha_\kappa, \\ (u^\kappa f(u))' = \kappa u^{\kappa-1} F(u-1) & \text{for } u > \beta_\kappa \end{cases}$$

has continuous solutions $F_\kappa(u)$ and $f_\kappa(u)$ with the properties that

$$F_\kappa(u) = 1 + O(e^{-u}), \quad f_\kappa(u) = 1 + O(e^{-u}),$$

and that $F_\kappa(u)$ and $f_\kappa(u)$ respectively, decreases and increases monotonically towards 1 as $u \to \infty$.

Theorem 11.2. *(Diamond and Halberstam [10, Theorem 1]). Let $\mathcal{A}$ and B described as above, in particular (11.2), (11.3) and (11.4) hold. For any two real numbers u and v satisfying*

$$\frac{1}{\tau} < u \le v, \quad \beta_\kappa < \tau\nu$$

we have

$$\sum_{\omega(n)\le r} a_n \gg X \prod_{p<X^{1/v}} (1-\omega_p)$$

provided that

$$r > \tau\mu u - 1 + \frac{\kappa}{f_\kappa(\tau v)} \int_1^{v/u} F_\kappa(\tau v - s)(1 - \frac{u}{v}s)\frac{ds}{s}.$$

Theorem 1.6. *Let $\boldsymbol{F}$ is a system of R linear independent quatrics in n variables and integer coefficients. Assume $\dim(\tilde{X}^*) \le n-1$ and $n+1-\sigma_\mathbb{R} > 8R$. Let $\mathfrak{J}, \sigma_p$ be as in Theorem 9.2,*

$$\Omega_p = \{\boldsymbol{x} \in (\mathbb{Z}/p\mathbb{Z})^n : \boldsymbol{F}(\boldsymbol{x}) \equiv 0, x_1\cdots x_n \equiv 0 \ mod\ p\}$$

and $\hat{\Omega}_p$ be as in Section 2. Let B be the set of primes for which $\#\Omega_p = 0$. Let a_n described as in (11.1) and assume (11.2), (11.3) and (11.4) hold. For any two reals numbers u and v satisfying

$$\frac{1}{\tau} < u \leq v, \quad \beta_\kappa < \tau\mu$$

we have

$$\sum_{\omega(n)\leq r} a_n \gg P^{n-2R}\mathfrak{J}\prod_p \sigma_p \prod_{p<(P^{n-2R}\mathfrak{J}\prod_p \sigma_p)^{1/v}} \frac{\#\hat{\Omega}_p}{\hat{X}(\mathbb{Z}/p\mathbb{Z})},$$

provided that

$$r > \tau\mu u - 1 + \frac{\kappa}{f_\kappa(\tau v)}\int_1^{v/u} F_\kappa(\tau v - s)(1 - \frac{u}{v}s)\frac{ds}{s}.$$

Proof. Combing the previous theorem with Theorem 9.2 with $m = 1$ gives the result. □

Corollary 11.2.1. *If in addition to Theorem 1.6 we have*

$$P^{n-2R} \prod_{p<(P^{n-2R}\mathfrak{J}\prod_p \sigma_p)^{1/v}} \frac{\#\hat{\Omega}_p}{\hat{X}(\mathbb{Z}/p\mathbb{Z})} \gg P^\epsilon \tag{11.5}$$

for some $\epsilon > 0$, we know that there are infinitely many solution for $\boldsymbol{F}(\boldsymbol{x}) = 0$ where $\boldsymbol{x}$ is r-almost prime.

12 Appendix

We will often use the following lemmas.

Lemma 12.1. *Let $k \in \mathbb{N}$. If $f(x) \ll g(x)$, then*

$$(f(x) + g(x))^k \ll_k g(x)^k.$$

Proof. Let $x_0, C > 0$ such that $|f(x)| \leq Cg(x)$ for all $x > x_0$. Expanding $(f(x)+g(x))^k$ gives

$$(f(x) + g(x))^k = \sum_{i=1}^{k} \binom{k}{i} g(x)^i f(x)^{k-i} \leq \sum_{i=1}^{k} \binom{k}{i} C^{k-i} g(x)^k,$$

for $x > x_0$, which proves the lemma. □

Lemma 12.2. *Let $d \in \mathbb{N}$ and $\mathcal{I}$ a finite subset of $\mathbb{N}$. Suppose for all $i \in \mathcal{I}$ we have $\lambda_i \in \mathbb{C}$, then*

$$\Big|\sum_{i\in\mathcal{I}} \lambda_i\Big|^d \leq (\#\mathcal{I})^{d-1} \sum_{i\in\mathcal{I}} |\lambda_i|^d.$$

Proof. We have

$$\Big|\sum_{i\in\mathcal{I}} \lambda_i\Big|^d \leq \Big(\sum_{i\in\mathcal{I}} |\lambda_i|\Big)^d.$$

After expanding the latter one we can bound every term $|\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_d}|$ by $(\max_{1\leq j\leq d}\{|\lambda_{i_j}|\})^d$. Let $k \in \mathcal{I}$, then there are at most $(\#\mathcal{I})^{d-1}$ terms of the form $|\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{d-1}}\lambda_k|$ for which $\max_{1\leq j\leq d-1}\{|\lambda_{i_j}|, |\lambda_k|\} = |\lambda_k|$. Hence

$$\Big(\sum_{i\in\mathcal{I}} |\lambda_i|\Big)^d \leq (\#\mathcal{I})^{d-1} \sum_{i\in\mathcal{I}} |\lambda_i|^d.$$

□

Lemma 12.3. *If $h \in \mathbb{R}$, then for every $x \in \mathbb{R}$ we have*

$$|e(x) - e(x+h)| \ll |h|.$$

Proof. First we note that $|1 - e(h)| = |e(x)| \cdot |1 - e(h)| = |e(x) - e(x+h)|$, so we only need to show that $|1 - e(h)| \ll |h|$. If $|h| \geq \frac{1}{4\pi}$, then $|1 - e(h)| \leq 2 \leq 8\pi|h|$. If $|h| < \frac{1}{4\pi}$ consider the power series of $1 - e(h)$;

$$\begin{aligned}
|1 - e(h)| &= \Big|\sum_{n=1}^{\infty} \frac{(2\pi i h)^n}{n!}\Big| \\
&\leq \sum_{n=1}^{\infty} (2\pi|h|)^n \\
&\leq (2\pi|h|) \sum_{n=0}^{\infty} \Big(\frac{2\pi}{4\pi}\Big)^n \ll |h|.
\end{aligned}$$

□

Lemma 12.4. *Let $m \in \mathbb{N}$ and $Q \in \mathbb{R}_{>0}$, then*

$$\sum_{\substack{x \in \mathbb{N} \\ x < Q}} x^m = O(Q^{m+1})$$

Proof. We have

$$\sum_{\substack{x \in \mathbb{N} \\ x < Q}} x^m \leq \int_0^Q x^m = (Q)^{m+1}.$$

□

Remark 12.5. *One can easily modify the proof and prove a similar statement for $m \in \mathbb{Q}$. Also, something stronger holds: $Q^{m+1} \sim \sum_{\substack{x \in \mathbb{N} \\ x < Q}} x^m$ as $Q \to \infty$, i.e. $\lim_{Q \to \infty} \frac{Q^{m+1}}{\sum_{x<Q} x^m} = 1$.*

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