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Zeta Functions of Multilinear Varieties

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1 Introduction

One of the most famous zeta functions is the Hasse-Weil zeta function. It played an important rol in the development of algebraic geometry in the twentieth century ([**Musta**]). Such a zeta function is defined for a certain algebraic variety. Algebraic varieties are one of the central objects of study in algebraic geometry and they are defined as a set of solutions of a system of polynomial equations. The Hasse-Weil zeta function captures all the information conveyed by a certain sequence of numbers in a power series. As we will see in **Section 3** these numbers denote the cardinality of a certain algebraic variety. These Hasse-Weil zeta functions appear to be rational. The proof of the rationality can be found in ([**Kob84**]). In this thesis we will compute the Hasse-Weil zeta function of multilinear hypersurfaces, which are algebraic varieties defined by the zeroes of a multilinear polynomial.

We will start with some field theory including theory about finite fields, since we will work with polynomials with coefficients in a finite field. After that, we will continue with defining the hypersurfaces and their Hasse-Weil zeta functions in **Section 3**. This extensive list of definitions and theorems makes this thesis almost self-contained. As soon as we have this knowledge we will start computing zeta functions of multilinear hypersurfaces.

2 Preliminaries

In this chapter we will provide some theory about fields. We will first start with the basic definition of a field and some properties. Furthermore we will give the definition of a finite field.

2.1 Field Theory

The formal definition of a field is given by:

Definition 2.1 (Field)**.** *(**[How06]**) A field is a set $\mathbb{F}$ with two operations called addition '+' and multiplication '$\cdot$', which satisfies the following axioms for $a, b, c \in \mathbb{F}$:*

A1. Associativity of addition and multiplication: $a+(b+c) = (a+b)+c$ and $a\cdot(b\cdot c) = (a\cdot b)\cdot c$.

A2. Commutativity of addition and multiplication: $a + b = b + a$ and $a \cdot b = b \cdot a$.

A3. Additive and multiplicative identity: there exist two different elements 0 and 1 in $\mathbb{F}$ such that $a + 0 = a$ and $a \cdot 1 = a$.

A4. Additive inverses: for every $a \neq 0$ in $\mathbb{F}$, there exists an element in $\mathbb{F}$, denoted $-a$, called the additive inverse of a, such that $a + (-a) = 0$.

A5. Distributivity of multiplication over addition: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

A6. Multiplicative inverses: for every $a \neq 0$ in $\mathbb{F}$, there exists an element in $\mathbb{F}$, denoted a^{-1} or $\frac{1}{a}$, called the multiplicative inverse of a, such that $a \cdot a^{-1} = 1$.

Definition 2.2 (Prime field)**.** *Let $\mathbb{F}$ be a field. The prime field of $\mathbb{F}$ is the intersection of all subfields of $\mathbb{F}$.*

Theorem 2.3. *Let $\mathbb{F}$ be a field. The prime field of $\mathbb{F}$ is either isomorphic to $\mathbb{Q}$, or to $\mathbb{Z}/p\mathbb{Z}$ for some prime p.*

Definition 2.4 (The Characteristic)**.** *Let $\mathbb{F}$ be a field. Suppose its prime field is $\mathbb{Z}/p\mathbb{Z}$. Then we say that the characteristic of $\mathbb{F}$ is p. When the prime field is $\mathbb{Q}$, we say that the characteristic is 0.*

Definition 2.5 (Irreducible Polynomial)**.** *(**[How06]**) Let $\mathbb{F}$ be a field. A non-constant polynomial with coefficients in $\mathbb{F}$ is irreducible over $\mathbb{F}$ if it cannot be written as a product of two non-constant polynomials with coefficients in $\mathbb{F}$. Otherwise, the polynomial will be called reducible.*

2.2 Finite Fields

In this section we shall be looking at finite fields. This will be the type of fields we will work with in the rest of this paper.

Definition 2.6 (Finite field)**.** *(**[Beu18]**) A field is called a finite field if it contains a finite number of elements.*

Definition 2.7. *(**[Beu18]**) Let $\mathbb{F}$ be a finite field and let p be its characteristic. We define multiplication between an element of $\mathbb{Z}/p\mathbb{Z}$ by an element of $\mathbb{F}$ as follows*

$$\mathbb{Z}/p\mathbb{Z} \times \mathbb{F} \to \mathbb{F}, (k, x) \mapsto k \cdot x,$$

where we choose an integer representative for k and where we use repeated addition.

Theorem 2.8 (Existence and Unicity). *([**Beu18**]) a) Let $\mathbb{F}$ be a finite field, then $\mathbb{F}$ has p^n elements for some $n \in \mathbb{N}$, where p is the characteristic of $\mathbb{F}$. b) For every $n \in \mathbb{N}$ and prime p there exists precisely one field (up to isomorphism) with $q = p^n$ elements, denoted by $\mathbb{F}_q$. Furthermore, there are no other finite fields.*

Proof. a) Let $\mathbb{F}$ be a finite field. Let $\mathbb{K}$ be the prime field of $\mathbb{F}$. It follows immediately from the definition of the prime field that $\mathbb{K}$ is also finite. This implies that the characteristic of $\mathbb{F}$ is p for some prime p. Since $\mathbb{K}$ is a subfield of $\mathbb{F}$, $\mathbb{F}$ is called a finite field extension of $\mathbb{K}$. Then $\mathbb{F}$ can be considered as a finite dimensional $\mathbb{K}$-vectorspace. Suppose that $\dim_{\mathbb{K}}(\mathbb{F}) = n$ for some $n \in \mathbb{N}$, then there exists a basis $\{x_1, ..., x_n\}$ of the $\mathbb{K}$-vectorspace $\mathbb{F}$. This means that the elements of $\mathbb{F}$ can be written uniquely as

$$\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n \text{ for } \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}.$$

Remark that $|\mathbb{K}| = p$, so there are exactly p^n distinct elements in $\mathbb{F}$.

b) We first prove existence. Let $n \in \mathbb{N}$ and $q = p^n$ for some prime p. Let $\mathbb{F}$ be the splitting field of the polynomial $X^q - X \in \mathbb{F}_p[X]$. We define

$$\mathbb{H} := \{x \in \mathbb{F} | x^q - x = 0\},$$

which is the set of zeroes in $\mathbb{F}$ of the polynomial $X^q - X$. We want to prove that $\mathbb{H}$ is a subfield of $\mathbb{F}$. It is clear that $1 \in \mathbb{H}$. Let $\alpha, \beta \in \mathbb{H}$. Since the characteristic is p, we know that $p \cdot \alpha = 0$. Hence the additive inverse of α is given by $(-\alpha) = (p-1) \cdot \alpha$. Remark that $(\alpha^{-1})^p = (\alpha^p)^{-1} = \alpha^{-1}$. In other words, the multiplicative inverse of α is also contained in $\mathbb{H}$. So it suffices to show that $\mathbb{H}$ is closed under addition and multiplication. By using the Binomial Theorem we find

$$(\alpha + \beta)^q = \sum_{k=0}^{q} \binom{q}{k} \alpha^{q-k} \beta^k = \alpha^q + \beta^q = \alpha + \beta,$$

since $\binom{q}{k}$ is 0 mod p whenever $0 < k < p$. Hence $\mathbb{H}$ is indeed closed under addition. Remark that $(\alpha\beta)^q = \alpha^q \beta^q = \alpha\beta$, so $\mathbb{H}$ is also closed under multiplication. We have now shown that $\mathbb{H}$ is a subfield of $\mathbb{F}$. It follows now immediately from the definition of the splitting field that $\mathbb{H} = \mathbb{F}$. The fact that $X^q - X$ has q distinct roots tells us that $\mathbb{F}$ is a field with q elements.

To show uniqueness take any prime power $q = p^n$ and let $\mathbb{F}$ be a field with q elements. Since $\mathbb{F}$ is a field we know that $|\mathbb{F}^*| = q - 1$, where $\mathbb{F}^*$ is the unit group of $\mathbb{F}$. Hence $x^{q-1} = 1$ for all $x \in \mathbb{F}^*$. This implies that $x^q - x = 0$ for all $x \in \mathbb{F}$. In other words, $\mathbb{F}$ is the splitting field of the polynomial $X^q - X$. Since the splitting field of a polynomial is uniquely determined (up to isomorphism) we can conclude that the same holds for $\mathbb{F}$. □

Remark 2.9. *We have seen in the above that the field $\mathbb{F}_{p^n}$ is isomorphic to the splittingfield of the polynomial $X^{p^n} - X \in \mathbb{F}_p[X]$ for every $n \in \mathbb{N}$ and prime p.*

Theorem 2.10. *([**And17**]) For p a prime and $n, m \in \mathbb{N}$, we have that $\mathbb{F}_{p^m}$ is a subfield of $\mathbb{F}_{p^n}$ if and only if $m|n$.*

Proof. Suppose that $\mathbb{F}_{p^m} \subset \mathbb{F}_{p^n}$. We can consider $\mathbb{F}_{p^n}$ as a $\mathbb{F}_{p^m}$-vector space. Since we are working with finite fields the dimension of $\mathbb{F}_{p^n}$ as a $\mathbb{F}_{p^m}$-vector space is a number $d \in \mathbb{N}$. This implies that $p^n = |\mathbb{F}_{p^n}| = |\mathbb{F}_{p^m}|^d = p^{md}$. We conclude that $m|n$.

Suppose that $m|n$, then there exists a $d \in \mathbb{N}$ such that $n = dm$. Let $x \in \mathbb{F}_{p^m}$, then according to **Remark 2.9** we have $x^{p^m} = x$. Since $n = dm$ we find

$$p^n - 1 = p^{dm} - 1 = (p^m)^d - 1 = (p^m - 1)((p^m)^{d-1} + ... + p^m + 1).$$

This implies that $p^m - 1$ divides $p^n - 1$. In an analogous way we find that $x^{p^m - 1} - 1 \big| x^{p^n - 1} - 1$. It now follows that $x^{p^m} - x \big| x^{p^n} - x$, so $x^{p^n} = x$ and $x \in \mathbb{F}_{p^n}$. $\square$

3 Hypersurfaces and their Zeta Functions

3.1 Homogeneous Polynomials

Let $\mathbb{F}$ be a field and $n \in \mathbb{N}$. Let $p(X_0, ..., X_n) \in \mathbb{F}[X_0, ..., X_n]$ be a polynomial of degree d, then we call p a **homogeneous** polynomial if it is a linear combination of monomials of the same total degree d.

Example 3.1. *The polynomial* $p(X_0, X_1, X_2, X_3) = X_0^4 - X_1^2 X_3^2 + 2X_0X_1X_2X_3$ *is a homogeneous polynomial in* $\mathbb{Z}[X_0, X_1, X_2, X_3]$ *of degree* 4.

It is possible to make a homogeneous polynomial of degree d out of a non-homogeneous polynomial $p(X_1, ..., X_n) \in \mathbb{F}[X_1, ..., X_n]$ of degree d. This can be done by adding a new variable and by defining the homogeneous polynomial in the following way ([**Kob84**])

$$\widetilde{p}(X_0, ..., X_n) := X_0^d p(X_1/X_0, ..., X_n/X_0).$$

This polynomial is called the **homogeneous completion** of $p(X_1, ..., X_n)$.

Lemma 3.2. *The homogeneous completion of a non-homogeneous polynomial* $p(X_1, ..., X_n)$ *in* $\mathbb{F}[X_1, ..., X_n]$ *of degree* d *is indeed a homogeneous polynomial of degree* d.

Proof. Remark that $p(X_1, ..., X_n)$ is a polynomial, so we can write it as a linear combination of m monomials for some $m \in \mathbb{N}$. In other words,

$$p(X_1, ..., X_n) = \sum_{i=1}^{m} a_i \prod_{j=1}^{n} X_j^{b_{i,j}}, \tag{1}$$

for some $a_i \in \mathbb{F}$ and $b_{i,j} \in \mathbb{N}$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ such that $\sum_{j=1}^{n} b_{i,j} \leqslant d$. By rewriting the definition of the homogeneous completion of $p(X_1, ..., X_n)$ using (1), we find

$$X_0^d p(X_1/X_0, ..., X_n/X_0) = (X_0)^d \sum_{i=1}^{m} a_i \prod_{j=1}^{n} \left(\frac{X_j}{X_0}\right)^{b_{i,j}} = \sum_{i=1}^{m} a_i X_0^{d - \sum_{j=1}^{n} b_{i,j}} \prod_{j=1}^{n} X_j^{b_{i,j}}.$$

It follows immediately that this is a homogeneous polynomial of degree d, since we have $\sum_{j=1}^{n} b_{i,j} \leqslant d$. □

Example 3.3. *The polynomial given in* ***Example 3.1*** *is the homogeneous completion of the polynomial* $q(X_1, X_2, X_3) = 1 - X_1^2 X_3^2 + 2X_1X_2X_3$.

3.2 Hypersurfaces

Let $\mathbb{F}$ be a field and $n \in \mathbb{N}$. Let $\mathbb{F}^n$ be the set consisting of the ordered n-tuples $(x_1, ..., x_n)$ with $x_i \in \mathbb{F}$ for $1 \leqslant i \leqslant n$. Let $p(X_1, ..., X_n) \in \mathbb{F}[X_1, ..., X_n]$ be a polynomial.

Definition 3.4. *([**Kob84**]) The affine hypersurface defined by the polynomial* p *in* $\mathbb{F}^n$ *is given by*

$$H_p(\mathbb{F}^n) := \{(x_1, ..., x_n) \in \mathbb{F}^n \mid p(x_1, ..., x_n) = 0\}.$$

Definition 3.5. *([**Kob84**]) We define the* n*-dimensional projective space over the field* $\mathbb{F}$, $\mathbb{P}^n(\mathbb{F})$, *as the quotiëntspace of* $\mathbb{F}^{n+1}\backslash\{0\}$ *with the equivalence relation given by*

$$(x_0, ..., x_n) \sim (x_0', ..., x_n') \Leftrightarrow \exists \lambda \in \mathbb{F}^* \text{ such that } x_i = \lambda x_i' \text{ for } i = 0, ..., n.$$

In other words, $\mathbb{P}^n(\mathbb{F}) := (\mathbb{F}^{n+1}\backslash\{0\})/\sim$.

Definition 3.6. *([**Kob84**]) Let $\widetilde{p}(X_0, ..., X_n) \in \mathbb{F}[X_0, ..., X_n]$ be a homogeneous polynomial. The projective hypersurface defined by $\widetilde{p}$ in $\mathbb{P}^n(\mathbb{F})$ is given by*

$$\widetilde{H}_{\widetilde{p}}(\mathbb{P}^n(\mathbb{F})) := \{[x_0 : ... : x_n] \in \mathbb{P}^n(\mathbb{F}) \mid \widetilde{p}(x_0, ..., x_n) = 0\}.$$

Remark that it makes sense to talk about the set of equivalence classes of $(n+1)$-tuples of $\mathbb{P}^n(\mathbb{F})$ at which $\widetilde{p}$ vanishes. For $\lambda \in \mathbb{F}^*$ we namely have that $\widetilde{p}(\lambda x_0, ..., \lambda x_n) = 0$ if $\widetilde{p}(x_0, ..., x_n) = 0$, since $\widetilde{p}$ is homogeneous. Let $\mathbb{K}$ be a field containing $\mathbb{F}$. Then we can also define the set

$$H_p(\mathbb{K}^n) := \{(x_1, ..., x_n) \in \mathbb{K}^n \mid p(x_1, ..., x_n) = 0\},$$

where $p(X_1, ..., X_n)$ is a polynomial with coefficients in $\mathbb{F}$. If $\widetilde{p}(X_1, ..., X_n)$ is a homogeneous polynomial, we can similarly define the set

$$\widetilde{H}_{\widetilde{p}}(\mathbb{P}^n(\mathbb{K})) := \{[x_0 : ... : x_n] \in \mathbb{P}^n(\mathbb{K}) \mid \widetilde{p}(x_0, ..., x_0) = 0\}.$$

3.3 Zeta function

Let $\mathbb{F} = \mathbb{F}_q$ be a finite field and $\mathbb{K} = \mathbb{F}_{q^s}$ be a finite field extension for some $s \in \mathbb{N}$. Let $p(X_1, ..., X_n) \in \mathbb{F}[X_1, ..., X_n]$ be a polynomial and $\widetilde{p}(X_0, ..., X_n) \in \mathbb{F}[X_0, ..., X_n]$ be a homogeneous polynomial. We define the sequences $(N_s)_{s\in\mathbb{N}}$ and $(\widetilde{N}_s)_{s\in\mathbb{N}}$, where

$$N_s := \left|H_p(\mathbb{F}^n_{q^s})\right|, \ \widetilde{N}_s := \left|\widetilde{H}_{\widetilde{p}}(\mathbb{P}^n(\mathbb{F}_{q^s}))\right|.$$

Definition 3.7 (Hasse-Weil Zeta function)**.** *([**Musta**],2.3.2) The Hasse-Weil zeta function of the affine hypersurface $H_p(\mathbb{F}^n_q)$ is defined as*

$$Z(H_p(\mathbb{F}^n_q); T) := \exp\left(\sum_{s=1}^{\infty} \frac{N_s T^s}{s}\right).$$

The Hasse-Weil zeta function of the projective hypersurface is defined in a similar way where $H_p(\mathbb{F}^n_q)$ and N_s are respectively replaced by $\widetilde{H}_{\widetilde{p}}(\mathbb{P}^n(\mathbb{F}_q))$ and $\widetilde{N}_s$.

3.4 Zeta Function of a Multilinear Hypersurface in $\mathbb{P}^1(\mathbb{F}_q)^n$

We shall be interested in multilinear hypersurfaces and their zeta functions. We can generalise the definition of the Hasse-Weil zeta function to hypersurfaces in $\mathbb{P}^1(\mathbb{F}_q)^n$. The multiprojective space over the field $\mathbb{F}_q$, $\mathbb{P}^1(\mathbb{F}_q)^n$, is defined as the cartesian product of 1-dimensional projective spaces, i.e. $\mathbb{P}^1(\mathbb{F}_q)^n := \mathbb{P}^1(\mathbb{F}_q) \overset{n-times}{\times ... \times} \mathbb{P}^1(\mathbb{F}_q)$ for $n \in \mathbb{N}$. Let $p(X_1, ..., X_n) \in \mathbb{F}_q[X_1, ..., X_n]$ be a multilinear polynomial of degree n, i.e. the polynomial contains a term consisting of a product of all the different variables. We define the homogeneous completion of $p(X_1, ..., X_n)$ as follows

$$\widetilde{p}(X_1, Y_1, ..., X_n, Y_n) := \left(\prod_{i=1}^{n} Y_i\right) p(X_1/Y_1, ..., X_n/Y_n).$$

Notice that this is indeed a homogeneous polynomial by recalling that $p(X_1, ..., X_n)$ is a multilinear polynomial. We can now take a look at the following hypersurface

$$\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^n) := \{([x_1 : y_1], ..., [x_n : y_n]) \in \mathbb{P}^1(\mathbb{F}_q)^n | \widetilde{p}(x_1, y_1, ..., x_n, y_n) = 0\}.$$

We can again conclude from the fact that $\widetilde{p}$ is homogeneous that for every $\lambda_1, ..., \lambda_n \in \mathbb{F}^*_q$ we have that $\widetilde{p}(\lambda_1 x_1, \lambda_1 y_1, ..., \lambda_n x_n, \lambda_n y_n) = 0$ if $\widetilde{p}(x_1, y_1, ..., x_n, y_n) = 0$. So it makes sense to consider the above hypersurface. Similarly, we can define the set

$$\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_{q^s})^n) := \{([x_1 : y_1], ..., [x_n : y_n]) \in \mathbb{P}^1(\mathbb{F}_{q^s})^n | \widetilde{p}(x_1, y_1, ..., x_n, y_n) = 0\},$$

for $s \in \mathbb{N}$. We can now define the Hasse-Weil zeta function of the multiprojective hypersurface $\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^n)$ in the following way

$$Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^n);T) := \exp\left(\sum_{s=1}^{\infty} \frac{\widetilde{N}_s T^s}{s}\right),$$

where $\widetilde{N}_s := \left|\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_{q^s})^n)\right|$ for $s \in \mathbb{N}$.

4 Zeta Function of the Affine Hypersurface in $\mathbb{F}_q^2$

Let $p(X_1, X_2) \in \mathbb{F}_q[X_1, X_2]$ be a multilinear irreducible polynomial of degree 2, i.e.

$$p(X_1, X_2) = aX_1X_2 + bX_1 + cX_2 + d$$

for some $a, b, c, d \in \mathbb{F}_q$ and $a \neq 0$. In this section we shall determine the zeta function of the affine hypersurface in $\mathbb{F}_q^2$ defined by $p(X_1, X_2)$. The following two lemmas will help us to determine this zeta function.

Lemma 4.1. *If the polynomial $p(X_1, X_2) = aX_1X_2 + bX_1 + cX_2 + d$ is reducible, then there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{F}_q$ such that $p(X_1, X_2) = (\alpha_1X_1 + \beta_1)(\alpha_2X_2 + \beta_2)$.*

Proof. Suppose that $p(X_1, X_2)$ is reducible, then there exist polynomials $q(X_1, X_2), r(X_1, X_2)$ in $\mathbb{F}_q[X_1, X_2]$ of degree 1 such that $p(X_1, X_2) = q(X_1, X_2)r(X_1, X_2)$. In other words,

$$\begin{aligned} p(X_1, X_2) &= (a_1X_1 + b_1X_2 + c_1)(a_2X_1 + b_2X_2 + c_2) \\ &= a_1a_2X_1^2 + b_1b_2X_2^2 + (a_1b_2 + a_2b_1)X_1X_2 + (a_1c_2 + a_2c_1)X_1 + (b_1c_2 + b_2c_1)X_2 + c_1c_2. \end{aligned}$$

for some $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{F}_q$. It follows from the above equations that $a_1a_2 = 0$, $b_1b_2 = 0$ and $a_1b_2 + a_2b_1 \neq 0$. This implies that $a_1 = b_2 = 0$ or $a_2 = b_1 = 0$, since a field has no zero divisors. In both cases we see that we get a factorization of the desired form. □

Lemma 4.2. *The multilinear polynomial $p(X_1, X_2) = aX_1X_2 + bX_1 + cX_2 + d$ of degree 2 is reducible if and only if $d = ca^{-1}b$.*

Proof. "$\Longrightarrow$" Suppose that $p(X_1, X_2)$ is reducible, then by using **Lemma 4.1** we find that

$$p(X_1, X_2) = (\alpha_1X_1 + \beta_1)(\alpha_2X_2 + \beta_2) = (\alpha_1\alpha_2X_1X_2 + \alpha_1\beta_2X_1 + \beta_1\alpha_2X_2 + \beta_1\beta_2).$$

This implies that $\alpha_1\alpha_2 = a$, $\alpha_1\beta_2 = b$, $\alpha_2\beta_1 = c$ and $d = \beta_1\beta_2$. Since $a \neq 0$ we know that α_1 and α_2 have an inverse. Using this we find

$$d = \beta_1\beta_2 = \alpha_2^{-1}cb\alpha_1^{-1} = ca^{-1}b.$$

"$\Longleftarrow$" Assume that $d = ca^{-1}b$, then it follows from the following calculations that $p(X_1, X_2)$ is indeed reducible.

$$p(X_1, X_2) = aX_1X_2 + bX_1 + cX_2 + ca^{-1}b = (X_1 + ca^{-1})(aX_2 + b).$$

□

By using the lemmas above we can determine the zeta function of the affine hypersurface $H_p(\mathbb{F}_q^2) := \{(x_1, x_2) \in \mathbb{F}_q^2 |\ p(x_1, x_2) = 0\}$, or equivalently we can prove the following theorem.

Theorem 4.3. *The Hasse-Weil zeta function of the affine hypersurface $H_p(\mathbb{F}_q^2)$ is given by*

$$Z(H_p(\mathbb{F}_q^2); T) = \frac{1 - T}{1 - qT}.$$

Proof. By looking at the definition of the Hasse-Weil zeta function of the affine hypersurface in **Section 3.3** we see that we need to determine the number $N_s = \left|H_p(\mathbb{F}_{q^s}^2)\right|$ for all $s \in \mathbb{N}$. We first determine N_1. Remark that we can rewrite the equation $p(x_1, x_2) = 0$ in the following way

$$p(x_1, x_2) = 0 \Leftrightarrow x_1(ax_2 + b) + cx_2 + d = 0 \Leftrightarrow x_1 = -\frac{cx_2 + d}{ax_2 + b} \text{ if } x_2 \neq -a^{-1}b.$$

In other words, for all $x_2 \in \mathbb{F}_q \backslash \{-a^{-1}b\}$ we can find x_1 such that $p(x_1, x_2) = 0$. Hence we can find $q-1$ solutions of $p(x_1, x_2) = 0$ in $\mathbb{F}_q^2$. Suppose that $x_2 = -a^{-1}b$, then

$$p(x_1, -a^{-1}b) = x_1(-aa^{-1}b + b) + (-ca^{-1}b) + d = 0 \Leftrightarrow d = ca^{-1}b.$$

By using **Lemma 4.2** and the above result we see that there are no solutions with $x_2 = -a^{-1}b$, since $p(X_1, X_2)$ is irreducible by assumption. This means that $N_1 = \left|H_p(\mathbb{F}_q^2)\right| = q - 1$. It follows from an analogous argument that $N_s = q^s - 1$ for $s \in \mathbb{N}$. So the zeta function of the affine hypersurface $H_p(\mathbb{F}_q^2)$ is given by

$$Z(H_p(\mathbb{F}_q^2); T) = \exp\left(\sum_{s=1}^{\infty} \frac{(q^s-1)T^s}{s}\right) = \exp(-\log(1-qT))\exp(\log(1-T)) = \frac{1-T}{1-qT}.$$

□

5 Zeta Function of the Hypersurface in $\mathbb{P}^1(\mathbb{F}_q)^2$

In this chapter we shall determine the zeta function of a multilinear hypersurface in $\mathbb{P}^1(\mathbb{F}_q^2)$, where we recall the theory discussed in **Section 3.4**. Let $p(X_1, X_2) \in \mathbb{F}_q[X_1, X_2]$ be the multilinear polynomial of degree 2, i.e.

$$p(X_1, X_2) = aX_1X_2 + bX_1 + cX_2 + d$$

for some $a, b, c, d \in \mathbb{F}_q$ and $a \neq 0$. The homogeneous completion of $p(X_1, X_2)$ is defined as follows

$$\widetilde{p}(X_1, Y_1, X_2, Y_2) := Y_1Y_2p(X_1/Y_1, X_2/Y_2) = aX_1X_2 + bX_1Y_2 + cX_2Y_1 + dY_1Y_2.$$

We can now determine the zeta function of the following hypersurface

$$\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^2) := \{([x_1 : y_1], [x_2 : y_2]) \in \mathbb{P}^1(\mathbb{F}_q)^2 | \widetilde{p}(x_1, y_1, x_2, y_2) = 0\}. \tag{2}$$

We will consider the case in which $p(X_1, X_2)$ is irreducible and the case in which $p(X_1, X_2)$ is reducible.

5.1 Zeta Function of the Irreducible Hypersurface in $\mathbb{P}^1(\mathbb{F}_q)^2$

We will start with proving the following theorem about the zeta function of the irreducible hypersurface $\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^2)$, which means that $p(X_1, X_2)$ is irreducible.

Theorem 5.1. *The Hasse-Weil zeta function of the irreducible hypersurface $\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^2)$ is given by*

$$Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^2); T) = \frac{1}{(1-T)(1-qT)}.$$

Proof. Assume that the polynomial $p(X_1, X_2)$ is irreducible. If we want to determine the zeta function of the hypersurface given at (2) we have to find the number $\widetilde{N}_s := \left|\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right|$ for $s \in \mathbb{N}$. We will take a look at the number $\widetilde{N}_s$ by rewriting the equation $\widetilde{p}(x_1, y_1, x_2, y_2) = 0$ in the following way

$$\widetilde{p}(x_1, y_1, x_2, y_2) = x_1(ax_2 + by_2) + y_1(cx_2 + dy_2) = 0. \tag{3}$$

We choose $[x_2 : y_2] \in \mathbb{P}^1(\mathbb{F}_{q^s})$ and by using the above equation we find the following cases.

1. $ax_2 + by_2$ and $cx_2 + dy_2$ are not both equal to zero.
2. $ax_2 + by_2 = cx_2 + dy_2 = 0$.

We first consider the situation in which $ax_2 + by_2$ and $cx_2 + dy_2$ are not both equal to zero. We have to take a look at three different cases.

- If $ax_2 + by_2 = 0$ and $cx_2 + dy_2 \neq 0$, then it follows from (3) that $y_1 = 0$ and $x_1 = \xi$ for some $\xi \in \mathbb{F}_{q^s}^*$.
- If $ax_2 + by_2 \neq 0$ and $cx_2 + dy_2 = 0$, then (3) implies that $x_1 = 0$ and $y_1 \in \mathbb{F}_{q^s}^*$.
- If $ax_2 + by_2 \neq 0 \neq cx_2 + dy_2$, then $x_1 \neq 0 \neq y_1$ and $\frac{x_1}{y_1} = \frac{cx_2+dy_2}{ax_2+by_2}$.

We conclude from the above that for every $[x_2 : y_2] \in \mathbb{P}^1(\mathbb{F}_{q^s})$ satisfying the first situation there is exactly one $[x_1 : y_1] \in \mathbb{P}^1(\mathbb{F}_{q^s})$ such that $\widetilde{p}(x_1, y_1, x_2, y_2) = 0$.

Remark that we can rewrite the second case, in which $ax_2 + by_2 = cx_2 + dy_2 = 0$, in the following way

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since we pick $[x_2 : y_2] \in \mathbb{P}^1(\mathbb{F}_{q^s})$ the above can only happen if

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0 \Leftrightarrow ad - bc = 0 \Leftrightarrow d = ca^{-1}b.$$

By using **Lemma 4.2** and the fact that $p(X_1, X_2)$ is irreducible we see that this cannot occur.

It follows from the results found above that there are $q^s + 1$ solutions of $\widetilde{p}(x_1, y_1, x_2, y_2) = 0$ in $\mathbb{P}^1(\mathbb{F}_{q^s})^2$, since $[x_2 : y_2]$ can be chosen arbitrarily. The zeta function of the hypersurface defined at (2) is now given by

$$\begin{aligned} Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^2); T) &= \exp\left(\sum_{s=1}^{\infty} \frac{(q^s + 1)T^s}{s} \right) = \exp(-\log(1 - qT)) \exp(-\log(1 - T)) \\ &= \frac{1}{(1 - T)(1 - qT)}. \end{aligned}$$

□

5.2 Zeta Function of the Reducible Hypersurface in $\mathbb{P}^1(\mathbb{F}_q)^2$

We will now take a look at the zeta function of the reducible hypersurface $\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^2)$.

Theorem 5.2. *The Hasse-Weil zeta function of the reducible hypersurface $\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^2)$ is given by*

$$Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^2); T) = \frac{1}{(1 - T)(1 - qT)^2}.$$

Proof. Suppose that $p(X_1, X_2)$ is reducible, then it follows from **Lemma 4.2** that $d = ca^{-1}b$. This implies that

$$\widetilde{p}(X_1, Y_1, X_2, Y_2) = aX_1X_2 + bX_1Y_2 + cY_1X_2 + ca^{-1}bY_1Y_2 = (X_1 + ca^{-1}Y_1)(aX_2 + bY_2). \quad (4)$$

Since we want to find the zeta function of the hypersurface given at (2) we have to determine the number $\widetilde{N}_s := \left|\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right|$ for $s \in \mathbb{N}$. We define the following polynomials

$$\begin{aligned} A(X_1, Y_1, X_2, Y_2) &:= X_1 + ca^{-1}Y_1 \\ B(X_1, Y_1, X_2, Y_2) &:= aX_2 + bY_2, \end{aligned}$$

and remark that $\widetilde{p}(X_1, Y_1, X_2, Y_2) = A(X_1, Y_1, X_2, Y_2)B(X_1, Y_1, X_2, Y_2)$. We define the following sets for $s \in \mathbb{N}$

$$\begin{aligned} \mathcal{O}_A(\mathbb{P}^1(\mathbb{F}_{q^s})^2) &:= \{([x_1 : y_1], [x_2 : y_2]) \in \mathbb{P}^1(\mathbb{F}_{q^s})^2 | A(x_1, y_1, x_2, y_2) = 0\} \\ \mathcal{O}_B(\mathbb{P}^1(\mathbb{F}_{q^s})^2) &:= \{([x_1 : y_1], [x_2 : y_2]) \in \mathbb{P}^1(\mathbb{F}_{q^s})^2 | B(x_1, y_1, x_2, y_2) = 0\}. \end{aligned}$$

We can now conclude from the above that

$$\widetilde{N}_s = \left|\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right| = \left|\mathcal{O}_A(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right| + \left|\mathcal{O}_B(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right| - \left|\mathcal{O}_A(\mathbb{P}^1(\mathbb{F}_{q^s})^2) \cap \mathcal{O}_B(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right|. \quad (5)$$

We will first determine the number $\left|\mathcal{O}_A(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right|$. Remark that this is equivalent to finding the number of solutions of $A(x_1, y_1, x_2, y_2) = 0$ in $\mathbb{P}^1(\mathbb{F}_{q^s})^2$. Rewriting this statement into the equation $x_1 = -ca^{-1}y_1$ implies that $[x_1 : y_1] = [-ca^{-1} : 1]$. So the solutions of the equation $A(x_1, y_1, x_2, y_2) = 0$ are given by $([-ca^{-1} : 1], [x_2 : y_2]) \in \mathbb{P}^1(\mathbb{F}_{q^s})^2$ for every $[x_2 : y_2] \in \mathbb{P}^1(\mathbb{F}_{q^s})$. So there are $q^s + 1$ solutions in $\mathbb{P}^1(\mathbb{F}_{q^s})^2$ of the equation $A(x_1, y_1, x_2, y_2) = 0$, since $[x_2 : y_2]$ can be chosen arbitrarily and $\left|\mathbb{P}^1(\mathbb{F}_{q^s})\right| = q^s + 1$. In other words $\left|\mathcal{O}_A(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right| = q^s + 1$.

Similarly, we can determine the number $\left|\mathcal{O}_B(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right|$ by counting the number of solutions of $B(x_1, y_1, x_2, y_2) = 0$, or equivalently $ax_2 = -by_2$, in $\mathbb{P}^1(\mathbb{F}_{q^s})^2$. This implies that the solutions of $B(x_1, y_1, x_2, y_2) = 0$ in $\mathbb{P}^1(\mathbb{F}_{q^s})^2$ are given by the elements $([x_1 : y_1], [-a^{-1}b : 1])$ with $[x_1 : y_1] \in \mathbb{P}^1(\mathbb{F}_{q^s})$. We can now conclude from the above that $\left|\mathcal{O}_B(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right| = q^s + 1$. Also, notice by looking at the explicit solutions found above that

$$\left|\mathcal{O}_A(\mathbb{P}^1(\mathbb{F}_{q^s})^2) \cap \mathcal{O}_B(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right| = \left|\{([-ca^{-1} : 1], [-a^{-1}b : 1])\}\right| = 1.$$

By using (5) and the above results we find that $\widetilde{N}_s = 2q^s + 1$. The zeta function of the hypersurface given at (2) will now be given by

$$\begin{aligned} Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^2); T) &= \exp\left(\sum_{s=1}^{\infty} \frac{(2q^s + 1)T^s}{s}\right) = \exp(-2\log(1 - qT))\exp(-\log(1 - T)) \\ &= \frac{1}{(1 - T)(1 - qT)^2}. \end{aligned}$$

$\square$

6 Zeta Function of the Hypersurface in $\mathbb{P}^1(\mathbb{F}_q)^3$

In this chapter we shall determine the zeta function of a hypersurface in $\mathbb{P}^1(\mathbb{F}_q)^3$ defined by a multilinear polynomial of degree 3.
Let $p(X_1, X_2, X_3) \in \mathbb{F}_q[X_1, X_2, X_3]$ be a multilinear polynomial of degree 3, i.e.

$$p(X_1, X_2, X_3) = aX_1X_2X_3 + bX_1X_2 + cX_2X_3 + dX_1X_3 + eX_1 + fX_2 + gX_3 + h.$$

for some $a, b, c, d, e, f, g, h \in \mathbb{F}_q$ and $a \neq 0$. We define the polynomials

$$\begin{array}{rcl} A(X_2, Y_2, X_3, Y_3) & := & aX_2X_3 + bX_2Y_3 + dY_2X_3 + eY_2Y_3 \\ B(X_2, Y_2, X_3, Y_3) & := & cX_2X_3 + fX_2Y_3 + gY_2X_3 + hY_2Y_3 \end{array}$$

and remark that the homogeneous completion of $p(X_1, X_2, X_3)$ is given by

$$\begin{array}{rcl} \widetilde{p}(X_1, Y_1, X_2, Y_2, X_3, Y_3) & := & Y_1Y_2Y_3 p(X_1/Y_1, X_2/Y_2, X_3/Y_3) \\ & = & X_1 A(X_2, Y_2, X_3, Y_3) + Y_1 B(X_2, Y_2, X_3, Y_3). \end{array}$$

We will now determine the zeta function of the hypersurface given by

$$\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3) := \{([x_1 : y_1], [x_2 : y_2], [x_3 : y_3]) \in \mathbb{P}^1(\mathbb{F}_q)^3 | \widetilde{p}(x_1, y_1, x_2, y_2, x_3, y_3) = 0\}. \quad (6)$$

We will distinguish between the case that $p(X_1, X_2, X_3)$ is irreducible and the case that it is reducible.

6.1 Zeta Function of the Irreducible Hypersurface in $\mathbb{P}^1(\mathbb{F}_q)^3$

We will first determine the zeta function of the irreducible hypersurface in $\mathbb{P}^1(\mathbb{F}_q)^3$. We start by defining the number $D := (ah + bg - ce - df)^2 - 4(ag - cd)(bh - ef)$. This number will show up in a natural way during the proof of **Theorem 6.2**. The following lemma will be useful by determining this zeta function.

Lemma 6.1. *If $ag - cd = bh - ef = ah + bg - ce - df = 0$, then $p(X_1, X_2, X_3)$ is reducible.*

Proof. Suppose that $ag - cd = bh - ef = ah + bg - ce - df = 0$, then

$$\det \begin{pmatrix} a & c \\ d & g \end{pmatrix} = 0 = \det \begin{pmatrix} b & e \\ f & h \end{pmatrix}.$$

This implies that there exist $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ such that

$$\begin{array}{rcl} (a, c) & = & \lambda(\alpha, \beta), \\ (d, g) & = & \mu(\alpha, \beta), \\ (b, e) & = & \rho(\gamma, \delta), \\ (f, h) & = & \sigma(\gamma, \delta) \end{array} \quad (7)$$

for some $\lambda, \mu, \rho, \sigma \in \mathbb{F}_q$. Substituting these results in the equation $ah + bg - ce - df = 0$ hands us the following

$$0 = ah + bg - ce - df = \lambda\alpha\sigma\delta + \rho\gamma\mu\beta - \lambda\beta\rho\delta - \mu\alpha\sigma\gamma = (\lambda\sigma - \mu\rho)(\alpha\delta - \beta\gamma).$$

This implies that $\lambda\sigma - \mu\rho = 0$ or $\alpha\delta - \beta\gamma = 0$. If $\lambda\sigma - \mu\rho = 0$, then there exists $\xi \in \mathbb{F}_q$ such that $(\lambda, \mu) = \xi(\rho, \sigma)$. Using this and the results at (7) we find that

$$(a, b, c, d, e, f, g, h) = (\alpha\xi\rho, \rho\gamma, \beta\xi\rho, \alpha\xi\sigma, \rho\delta, \sigma\gamma, \beta\xi\sigma, \sigma\delta).$$

Now notice the following to conclude that $p(X_1, X_2, X_3)$ is reducible

$$(\rho X_1 + \sigma)(\alpha\xi X_2 X_3 + \gamma X_2 + \beta\xi X_3 + \delta) = p(X_1, X_2, X_3).$$

If $\alpha\delta - \beta\gamma = 0$, then there exists $\omega \in \mathbb{F}_q$ such that $(\alpha, \beta) = \omega(\gamma, \delta)$. Combining this with the results found in (7) hands us the following

$$(a, b, c, d, e, f, g, h) = (\lambda\omega\gamma, \rho\gamma, \lambda\omega\delta, \mu\omega\gamma, \rho\delta, \sigma\gamma, \mu\omega\delta, \sigma\delta).$$

This implies that $p(X_1, X_2, X_3)$ is reducible, since

$$(\gamma X_2 + \delta)(\lambda\omega X_1 X_3 + \rho X_1 + \mu\omega X_3 + \sigma) = p(X_1, X_2, X_3).$$

□

Theorem 6.2. *The Hasse-Weil zeta function of the irreducible hypersurface $\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3)$ is given in the following five cases:*

1. *If $ag - cd \neq 0$ and $D = 0$,*

$$Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3); T) = \frac{1}{(1-T)(1-qT)^3(1-q^2T)}.$$

2. *If $ag - cd \neq 0$, $D \neq 0$ and D a square in $\mathbb{F}_q$,*

$$Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3); T) = \frac{1}{(1-T)(1-qT)^4(1-q^2T)}.$$

3. *If $ag - cd \neq 0$, $D \neq 0$ and D not a square in $\mathbb{F}_q$,*

$$Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3); T) = \frac{1}{(1-T)(1+qT)(1-qT)^3(1-q^2T)}.$$

4. *If $ag - cd = 0$ and $D = 0$,*

$$Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3); T) = \frac{1}{(1-T)(1-qT)^3(1-q^2T)}.$$

5. *If $ag - cd = 0$ and $D \neq 0$,*

$$Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3); T) = \frac{1}{(1-T)(1-qT)^4(1-q^2T)}.$$

Proof. Assume that $p(X_1, X_2, X_3)$ is irreducible. According to the definition of the zeta function we need to find the number $\widetilde{N}_s := \left|\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_{q^s})^3)\right|$ for $s \in \mathbb{N}$. This means that we have to determine how many solutions the equation

$$\widetilde{p}(x_1, y_1, x_2, y_2, x_3, y_3) = x_1 A(x_2, y_2, x_3, y_3) + y_1 B(x_2, y_2, x_3, y_3) = 0 \tag{8}$$

has in $\mathbb{P}^1(\mathbb{F}_{q^s})^3$. We choose $([x_2 : y_2], [x_3 : y_3]) \in \mathbb{P}^1(\mathbb{F}_{q^s})^2$ and by using (8) we find the following cases:

1. $A(x_2, y_2, x_3, y_3)$ and $B(x_2, y_2, x_3, y_3)$ are not both equal to zero.

2. $A(x_2, y_2, x_3, y_3) = B(x_2, y_2, x_3, y_3) = 0$, or equivalently

$$\begin{pmatrix} A(x_2, y_2, x_3, y_3) \\ B(x_2, y_2, x_3, y_3) \end{pmatrix} = \begin{pmatrix} ax_3 + by_3 & dx_3 + ey_3 \\ cx_3 + fy_3 & gx_3 + hy_3 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Notice that in the first case we get a unique $[x_1 : y_1] \in \mathbb{P}^1(\mathbb{F}_{q^s})$ from the equation given at (8). Also, remark that the second case can only happen in $\mathbb{P}^1(\mathbb{F}_{q^s})^2$ if

$$\det \begin{pmatrix} ax_3 + by_3 & dx_3 + ey_3 \\ cx_3 + fy_3 & gx_3 + hy_3 \end{pmatrix} = (ag - cd)x_3^2 + (ah + bg - ce - df)x_3y_3 + (bh - ef)y_3^2 = 0 \quad (9)$$

We define the following set consisting of the elements $([x_2 : y_2], [x_3 : y_3]) \in \mathbb{P}^1(\mathbb{F}_{q^s})^2$ belonging to the second case

$$\mathcal{O}(\mathbb{P}^1(\mathbb{F}_{q^s})^2) := \{([x_2 : y_2], [x_3 : y_3]) \in \mathbb{P}^1(\mathbb{F}_{q^s})^2 | A(x_2, y_2, x_3, y_3) = B(x_2, y_2, x_3, y_3) = 0\}.$$

We can now conclude from the above that

$$\widetilde{N}_s = (q^s + 1)^2 - \left|\mathcal{O}(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right| + \left|\mathcal{O}(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right|(q^s + 1) = (q^s + 1)^2 + q^s\left|\mathcal{O}(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right|, \quad (10)$$

since for every $([x_2 : y_2], [x_3 : y_3]) \in \mathbb{P}^1(\mathbb{F}_{q^s})^2$ with $A(x_2, y_2, x_3, y_3) = B(x_2, y_2, x_3, y_3) = 0$ we see that $\widetilde{p}(x_1, y_1, x_2, y_2, x_3, y_3) = 0$ for every $[x_1 : y_1] \in \mathbb{P}^1(\mathbb{F}_{q^s})$. Consequently, we need to determine $\left|\mathcal{O}(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right|$ if we want to find the zeta function of (6). We will use the following lemma to determine this number.

Lemma 6.3. *Since $p(X_1, X_2, X_3)$ is an irreducible polynomial we have*

$$\begin{pmatrix} ax_3 + by_3 & dx_3 + ey_3 \\ cx_3 + fy_3 & gx_3 + hy_3 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for every $[x_3 : y_3] \in \mathbb{P}^1(\mathbb{F}_{q^s})$.

Proof. Suppose that there exists $[x_3 : y_3] \in \mathbb{P}^1(\mathbb{F}_{q^s})$ such that

$$\begin{pmatrix} ax_3 + by_3 & dx_3 + ey_3 \\ cx_3 + fy_3 & gx_3 + hy_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then we see that $ax_3 + by_3 = dx_3 + ey_3 = 0$, or equivalently

$$\begin{pmatrix} a & b \\ d & e \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $[x_3 : y_3] \neq [0 : 0]$ it follows from the above that

$$\det \begin{pmatrix} a & b \\ d & e \end{pmatrix} = 0.$$

This implies that there exist $\alpha, \beta \in \mathbb{F}_q$ such that $\lambda(\alpha, \beta) = (a, b)$ and $\mu(\alpha, \beta) = (d, e)$ for some $\lambda, \mu \in \mathbb{F}_q$. We can find in an analogical way that there exist $\alpha, \beta \in \mathbb{F}_q$ such that

$$\begin{aligned} \gamma(\alpha, \beta) &= (a, b) \text{ for some } \gamma \in \mathbb{F}_q, \\ \delta(\alpha, \beta) &= (d, e) \text{ for some } \delta \in \mathbb{F}_q, \\ \varepsilon(\alpha, \beta) &= (c, f) \text{ for some } \varepsilon \in \mathbb{F}_q, \\ \zeta(\alpha, \beta) &= (g, h) \text{ for some } \zeta \in \mathbb{F}_q. \end{aligned}$$

Now notice that the following is true to conclude that $p(X_1, X_2, X_3)$ is reducible

$$(\alpha X_3 + \beta)(\gamma X_1 X_2 + \delta X_1 + \varepsilon X_2 + \zeta) = p(X_1, X_2, X_3).$$

□

The above lemma tells us that for every $[x_3 : y_3] \in \mathbb{P}^1(\mathbb{F}_{q^s})$ satisfying (9) there is exactly one $[x_2 : y_2] \in \mathbb{P}^1(\mathbb{F}_{q^s})$ such that $([x_2 : y_2], [x_3 : y_3]) \in \mathcal{O}(\mathbb{P}^1(\mathbb{F}_{q^s})^2)$. So the number $\left|\mathcal{O}(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right|$ is equal to the number of solutions of (9) in $\mathbb{P}^1(\mathbb{F}_{q^s})$. Computing these solutions will cause the following possible cases:

1. $ag - cd \neq 0$.
2. $ag - cd = 0$.

We will first take a look at the case in which $ag - cd \neq 0$. Notice that it follows from (9) that there are no solutions in $\mathbb{P}^1(\mathbb{F}_{q^s})$ with $y_3 = 0$, since $ag - cd \neq 0$ by assumption. So we may assume that $y_3 \neq 0$. Recall that $D := (ah + bg - ce - df)^2 - 4(ag - cd)(bh - ef)$. We will consider the case that $D = 0$ and $D \neq 0$ in $\mathbb{F}_q$. When $D \neq 0$ we will need to distinguish between the case that D is a square in $\mathbb{F}_q$ and the case that D is not a square in $\mathbb{F}_q$.

1. $D = 0$.
   Remark that $D = 0$ if and only if $(ah+bg-ce-df)^2 = 4(ag-cd)(bh-ef)$, since $y_3 \neq 0$. It follows from (9) that

$$\frac{x_3}{y_3} = \frac{-(ah + bg - ce - df)}{2(ag - cd)}.$$

   So there is exactly one solution of (9) in $\mathbb{P}^1(\mathbb{F}_{q^s})$. Now recall that the number $\left|\mathcal{O}(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right|$ is equal to the number of solutions of (9) in $\mathbb{P}^1(\mathbb{F}_{q^s})$. By using this and (10) we find that $\widetilde{N}_s = (q^s + 1)^2 + q^s = q^{2s} + 3q^s + 1$. The zeta function of (6) will now be given by

$$\begin{aligned} Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3); T) &= \exp\left(\sum_{s=1}^{\infty} \frac{(q^{2s} + 3q^s + 1)T^s}{s}\right) \\ &= \exp(-\log(1 - q^2T)) \exp(-3\log(1 - qT)) \exp(-\log(1 - T)) \\ &= \frac{1}{(1 - T)(1 - qT)^3(1 - q^2T)}. \end{aligned}$$

2. $D \neq 0$

- D is a square in $\mathbb{F}_q$. In this case it follows from (9) that

$$x_3 = \frac{-(ah + bg - ce - df) \pm y_3\sqrt{D}}{2(ag - cd)} \in \mathbb{F}_{q^s}.$$

  So there are exactly two solutions of (9) in $\mathbb{P}^1(\mathbb{F}_{q^s})$. This implies that $\left|\mathcal{O}(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right| = 2$ and thus it follows from (5) that $\widetilde{N}_s = (q^s+1)^2+2q^s = q^{2s}+4q^s+1$. So the zeta function of the hypersurface given at (6) is given by

$$\begin{aligned} Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3); T) &= \exp\left(\sum_{s=1}^{\infty} \frac{(q^{2s} + 4q^s + 1)T^s}{s}\right) \\ &= \exp(-\log(1 - q^2T)) \exp(-4\log(1 - qT)) \exp(-\log(1 - T)) \\ &= \frac{1}{(1 - T)(1 - qT)^4(1 - q^2T)}. \end{aligned}$$

- D is not a square in $\mathbb{F}_q$. Consider the field $\mathbb{F}_q(\sqrt{D}) := \{\alpha + \beta\sqrt{D} | \alpha, \beta \in \mathbb{F}_q\}$. Notice that this field has exactly q^2 elements. It follows from **Theorem 2.8** that this field has to be isomorphic to $\mathbb{F}_{q^2}$, so D is a square in $\mathbb{F}_{q^2}$. We can also conclude that D is not a square in $\mathbb{F}_{q^{2m+1}}$ for $m \in \mathbb{N}$ and that D is a square in $\mathbb{F}_{q^{2n}}$ for $n \in \mathbb{N}$ by using **Theorem 2.10**. This implies that there are no solutions of (9) in $\mathbb{P}^1(\mathbb{F}_{q^{2m+1}})$ for $m \in \mathbb{N}$. As we have seen in the previous case there will be two solutions of (9) in $\mathbb{P}^1(\mathbb{F}_{q^{2n}})$ for $n \in \mathbb{N}$. It now follows from (5) that

$$\begin{aligned}\widetilde{N}_s &= \begin{cases}(q^s+1)^2 = q^{2s}+2q^s+1 & \text{if } s \text{ is odd.}\\ (q^s+1)^2+2q^s = q^{2s}+4q^s+1 & \text{if } s \text{ is even.}\end{cases}\\ &= q^{2s}+3q^s+1+(-q)^s\end{aligned}$$

The zeta function of (6) is now given by

$$\begin{aligned}Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3);T) &= \exp\left(\sum_{n=1}^{\infty}\frac{(q^{2s}+3q^s+1+(-q)^s)T^s}{s}\right)\\ &= \frac{1}{(1-q^2T)(1-qT)^3(1+qT)(1-T)}.\end{aligned}$$

We will now compute the zeta function of (6) in the case that $ag-cd=0$. We can now rewrite (9) in the following way

$$(ah+bg-ce-df)x_3y_3+(bh-ef)y_3^2=0. \tag{11}$$

Also, notice that $D=(ah+bg-ce-df)^2$ since $ag-cd=0$. We will now consider the following possibilities:

- $D=0$.
  This implies that $ah+bg-ce-df=0$. By using **Lemma 6.1** we can conclude that $bh-ef\neq 0$, since $p(X_1,X_2,X_3)$ is irreducible by assumption. This means that the only solution of (11) in $\mathbb{P}^1(\mathbb{F}_{q^s})$ is $[1:0]$. Consequently, $\widetilde{N}_s=(q^s+1)^2+q^s=q^{2s}+3q^s+1$ and the zeta function will thus be given by

$$\begin{aligned}Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3);T) &= \exp\left(\sum_{s=1}^{\infty}\frac{(q^{2s}+3q^s+1)T^s}{s}\right)\\ &= \exp(-\log(1-q^2T))\exp(-3\log(1-qT))\exp(-\log(1-T))\\ &= \frac{1}{(1-T)(1-qT)^3(1-q^2T)}.\end{aligned}$$

- $D\neq 0$.
  This means that $ah+bg-ce-df\neq 0$. We will need to take a look at the case in which $bh-ef=0$ and the case in which $bh-ef\neq 0$. If $bh-ef=0$, the only solutions of (11) in $\mathbb{P}^1(\mathbb{F}_{q^s})$ are $[1:0]$ and $[0:1]$. If $bh-ef\neq 0$, there will also be two solutions of (11) in $\mathbb{P}^1(\mathbb{F}_{q^s})$, since $\frac{y_3}{x_3}=\frac{-(ah+bg-ce-df)\pm(ah+bg-ce-df)}{2(bh-ef)}$. So we conclude in the same way as before that $\widetilde{N}_s=(q^s+1)^2+2q^s=q^{2s}+4q^s+1$. The corresponding zeta function will now be given by

$$\begin{aligned}Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3);T)) &= \exp\left(\sum_{s=1}^{\infty}\frac{(q^{2s}+4q^s+1)T^s}{s}\right)\\ &= \exp(-\log(1-q^2T))\exp(-4\log(1-qT))\exp(-\log(1-T))\\ &= \frac{1}{(1-q^2T)(1-qT)^4(1-T)}.\end{aligned}$$

□

6.2 Zeta Function of the Reducible Hypersurface in $\mathbb{P}^1(\mathbb{F}_q)^3$

Suppose that $p(X_1, X_2, X_3)$ is reducible, then there exist $r(X_1, X_2, X_3), s(X_1, X_2, X_3) \in \mathbb{F}_q[X]$ such that $p(X_1, X_2, X_3) = r(X_1, X_2, X_3)s(X_1, X_2, X_3)$. Remark that $p(X_1, X_2, X_3)$ is multilinear, so the two factors can not depend on the same variable. This means that we have the following three possibilities:

$$\begin{aligned} p(X_1, X_2, X_3) &= (\alpha X_1 + \beta)(\gamma X_2 X_3 + \delta X_2 + \varepsilon X_3 + \zeta), \\ p(X_1, X_2, X_3) &= (\alpha X_2 + \beta)(\gamma X_1 X_3 + \delta X_1 + \varepsilon X_3 + \zeta), \\ p(X_1, X_2, X_3) &= (\alpha X_3 + \beta)(\gamma X_1 X_2 + \delta X_1 + \varepsilon X_2 + \zeta) \end{aligned} \tag{12}$$

for some $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{F}_q$. Remark that $\alpha\gamma = a \neq 0$, so $\alpha \neq 0 \neq \gamma$. Assume that $r(X_1, X_2, X_3)$ is the factor of degree 1 and $s(X_1, X_2, X_3)$ is the factor of degree 2.

Theorem 6.4. *The Hasse-Weil zeta function of the reducible hypersurface $\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3)$ is given in the following two cases:*

1. *If $s(X_1, X_2, X_3)$ is irreducible, then*

$$Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3); T) = \frac{1}{(1-T)(1-qT)^3(1-q^2T)^2}.$$

2. *If $s(X_1, X_2, X_3)$ is reducible, then*

$$Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3); T) = \frac{1}{(1-T)(1-qT)^3(1-q^2T)^3}.$$

Proof. Since we want to find the zeta function of (6), we need to determine $\widetilde{N}_s := \left|\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_{q^s})^3)\right|$ for $s \in \mathbb{N}$. Since the above situations (12) are quite similar, we will only compute the zeta function in the case that $p(X_1, X_2, X_3)$ is reducible in X_3. We define the following polynomials

$$\begin{aligned} g(X_1, Y_1, X_2, Y_2, X_3, Y_3) &:= (\alpha X_3 + \beta Y_3) \\ h(X_1, Y_1, X_2, Y_2, X_3, Y_3) &:= (\gamma X_1 X_2 + \delta X_1 Y_2 + \varepsilon Y_1 X_2 + \zeta Y_1 Y_2), \end{aligned}$$

and notice that the homogeneous completion of $p(X_1, X_2, X_3)$ is as follows

$$\begin{aligned} \widetilde{p}(X_1, Y_1, X_2, Y_2, X_3, Y_3) &:= Y_1 Y_2 Y_3 p(X_1/Y_1, X_2/Y_2, X_3/Y_3) \\ &= (\alpha X_3 + \beta Y_3)(\gamma X_1 X_2 + \delta X_1 Y_2 + \varepsilon Y_1 X_2 + \zeta Y_1 Y_2) \\ &= g(X_1, Y_1, X_2, Y_2, X_3, Y_3) h(X_1, Y_1, X_2, Y_2, X_3, Y_3). \end{aligned}$$

We define the following sets of zeroes for $s \in \mathbb{N}$

$$\begin{aligned} \mathcal{O}_g(\mathbb{P}^1(\mathbb{F}_{q^s})^3) &:= \{([x_1 : y_1], [x_2 : y_2], [x_3 : y_3]) \in \mathbb{P}^1(\mathbb{F}_{q^s})^3 | g(x_1, y_1, x_2, y_2, x_3, y_3) = 0\} \\ \mathcal{O}_h(\mathbb{P}^1(\mathbb{F}_{q^s})^3) &:= \{([x_1 : y_1], [x_2 : y_2], [x_3 : y_3]) \in \mathbb{P}^1(\mathbb{F}_{q^s})^3 | h(x_1, y_1, x_2, y_2, x_3, y_3) = 0\}. \end{aligned}$$

We now remark from the above that

$$\widetilde{N}_s = \left|\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_{q^s})^3)\right| = \left|\mathcal{O}_g(\mathbb{P}^1(\mathbb{F}_{q^s})^3)\right| + \left|\mathcal{O}_h(\mathbb{P}^1(\mathbb{F}_{q^s})^3)\right| - \left|\mathcal{O}_g(\mathbb{P}^1(\mathbb{F}_{q^s})^3) \cap \mathcal{O}_h(\mathbb{P}^1(\mathbb{F}_{q^s})^3)\right|. \tag{13}$$

We will first determine the number of solutions of $g(x_1, y_1, x_2, y_2, x_3, y_3) = 0$ in $\mathbb{P}^1(\mathbb{F}_{q^s})^3$. We can rewrite this statement as $x_3 = -\alpha^{-1}\beta y_3$. Consequently, we find that $[x_3 : y_3] = [-\alpha^{-1}\beta : 1]$. So the solutions will be given by $([x_1 : y_1], [x_2 : y_2], [-\alpha^{-1}\beta : 1]) \in \mathbb{P}^1(\mathbb{F}_{q^s})^3$ for every element $[x_1 : y_1], [x_2 : y_2] \in \mathbb{P}^1(\mathbb{F}_{q^s})$. It follows that $\left|\mathcal{O}_g(\mathbb{P}^1(\mathbb{F}_{q^s})^3)\right| = (q^s + 1)^2$.

Notice that we have to distinguish between the case in which $s(X_1, X_2, X_3)$ is irreducible and the case in which it is reducible if we want to determine the number of solutions of $h(X_1, Y_1, X_2, Y_2, X_3, Y_3)$ in $\mathbb{P}^1(\mathbb{F}_{q^s})^3$, as we have seen in **Section 5**.

1. Suppose that $s(X_1, X_2, X_3)$ is irreducible. We will now determine the number of solutions of $h(x_1, y_1, x_2, y_2, x_3, y_3) = 0$ in $\mathbb{P}^1(\mathbb{F}_{q^s})^3$. Since $\gamma \neq 0$ we can use **Theorem 5.1** and the results found in the proof of this theorem. Since $[x_3 : y_3]$ can be chosen arbitrarily in $\mathbb{P}^1(\mathbb{F}_{q^s})^3$ we can directly conclude that $\left|\mathcal{O}_h(\mathbb{P}^1(\mathbb{F}_{q^s})^3)\right| = (q^s+1)^2$. By looking at the solutions of the equation $g(x_1, y_1, x_2, y_2, x_3, y_3) = 0 = h(x_1, y_1, x_2, y_2, x_3, y_3)$ we immediately notice that the number $\left|\mathcal{O}_g(\mathbb{P}^1(\mathbb{F}_{q^s})^3) \cap \mathcal{O}_h(\mathbb{P}^1(\mathbb{F}_{q^s})^3)\right|$ is given by

$$\left|\{([x_1 : y_1], [x_2 : y_2]) \in \mathbb{P}^1(\mathbb{F}_{q^s})^2 | h(x_1, y_1, x_2, y_2, -\alpha^{-1}\beta, 1) = 0\}\right| = q^s + 1.$$

We can now conclude that $\widetilde{N}_s = (q^s+1)^2 + (q^s+1)^2 - (q^s+1) = 2q^{2s} + 3q^s + 1$ by using (13) and the results found above. The zeta function of the hypersurface given at (6) will thus be given by

$$\begin{aligned}
Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3); T) &= \exp\left(\sum_{s=1}^{\infty} \frac{(2q^{2s} + 3q^s + 1)T^s}{s}\right) \\
&= \exp(-2\log(1 - q^2T)) \exp(-3\log(1 - qT)) \exp(-\log(1 - T)) \\
&= \frac{1}{(1 - T)(1 - qT)^3(1 - q^2T)^2}.
\end{aligned}$$

2. Assume that $s(X_1, X_2, X_3)$ is reducible. Since $\gamma \neq 0$ we can use the results in **Theorem 5.2** to find the number of solutions of $h(x_1, y_1, x_2, y_2, x_3, y_3) = 0$ in $\mathbb{P}^1(\mathbb{F}_{q^s})^3$. It immediately follows that $\left|\mathcal{O}_h(\mathbb{P}^1(\mathbb{F}_{q^s})^3)\right| = (q^s+1)(2q^s+1)$. We notice in a similar way as in the previous case that $\left|\mathcal{O}_g(\mathbb{P}^1(\mathbb{F}_{q^s})^3) \cap \mathcal{O}_h(\mathbb{P}^1(\mathbb{F}_{q^s})^3)\right| = 2q^s + 1$. We have now found the requirements to conclude from (13) that

$$\widetilde{N}_s = (q^s+1)^2 + (q^s+1)(2q^s+1) - (2q^s+1) = 3q^{2s} + 3q^s + 1.$$

The above hands us the following zeta function corresponding to the hypersurface given at (6)

$$\begin{aligned}
Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^3); T) &= \exp\left(\sum_{s=1}^{\infty} \frac{(3q^{2s} + 3q^s + 1)T^s}{s}\right) \\
&= \exp(-3\log(1 - q^2T)) \exp(-3\log(1 - qT)) \exp(-\log(1 - T)) \\
&= \frac{1}{(1 - T)(1 - qT)^3(1 - q^2T)^3}.
\end{aligned}$$

□

7 Zeta Function of the Hypersurface in $\mathbb{P}^1(\mathbb{F}_q)^4$

In this chapter we will take a look at the zeta function of a hypersurface in $\mathbb{P}^1(\mathbb{F}_q)^4$ defined by a multilinear polynomial of degree 4.
Let $p(X_1, X_2, X_3, X_4) \in \mathbb{F}_q[X_1, X_2, X_3, X_4]$ be a multilinear polynomial of degree 4. This means that there exist multilinear polynomials $A(X_2, X_3, X_4), B(X_2, X_3, X_4) \in \mathbb{F}_q[X_2, X_3, X_4]$ such that

$$p(X_1, X_2, X_3, X_4) = X_1 A(X_2, X_3, X_4) + B(X_2, X_3, X_4).$$

We define the homogeneous completions of $A(X_2, X_3, X_4)$ and $B(X_2, X_3, X_4)$ as follow

$$\begin{aligned} \widetilde{A}(X_2, Y_2, X_3, Y_3, X_4, Y_4) &:= Y_2 Y_3 Y_4 A(X_2/Y_2, X_3/Y_3, X_4/Y_4) \\ \widetilde{B}(X_2, Y_2, X_3, Y_3, X_4, Y_4) &:= Y_2 Y_3 Y_4 B(X_2/Y_2, X_3/Y_3, X_4/Y_4), \end{aligned}$$

and remark that these are again multilinear polynomials. Consequently, the homogeneous completion of $p(X_1, X_2, X_3, X_4)$ is defined as follows

$$\begin{aligned} \widetilde{p}(X_1, Y_1, ..., X_4, Y_4) &:= Y_1 Y_2 Y_3 Y_4 p(X_1/Y_1, ..., X_4/Y_4) \\ &= X_1 \widetilde{A}(X_2, Y_2, X_3, Y_3, X_4, Y_4) + Y_1 \widetilde{B}(X_2, Y_2, X_3, Y_3, X_4, Y_4). \end{aligned} \tag{14}$$

We will now determine the zeta function of the hypersurface given by

$$\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^4) := \{([x_1 : y_1], ..., [x_4 : y_4]) \in \mathbb{P}^1(\mathbb{F}_q)^4 | \widetilde{p}(x_1, y_1, ..., x_4, y_4) = 0\}. \tag{15}$$

This means that we have to determine the number $\widetilde{N}_s := \left|\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^4)\right|$ for $s \in \mathbb{N}$. We choose $([x_2 : y_2], [x_3 : y_3], [x_4 : y_4]) \in \mathbb{P}^1(\mathbb{F}_q)^3$ and by rewriting the equation $\widetilde{p}(x_1, y_1, ..., x_4, y_4) = 0$ as follows

$$x_1 \widetilde{A}(x_2, y_2, x_3, y_3, x_4, y_4) + y_1 \widetilde{B}(x_2, y_2, x_3, y_3, x_4, y_4) = 0, \tag{16}$$

we find the following possibilities:

1. $\widetilde{A}(x_2, y_2, x_3, y_3, x_4, y_4)$ and $\widetilde{B}(x_2, y_2, x_3, y_3, x_4, y_4)$ are not both equal to zero. In this situation there follows a unique $[x_1 : y_1] \in \mathbb{P}^1(\mathbb{F}_{q^s})$ from (16).

2. $\widetilde{A}(x_2, y_2, x_3, y_3, x_4, y_4) = 0 = \widetilde{B}(x_2, y_2, x_3, y_3, x_4, y_4)$. Remark that

$$\begin{aligned} \widetilde{A}(X_2, Y_2, X_3, Y_3, X_4, Y_4) &= X_2 \alpha(X_3, Y_3, X_4, Y_4) + Y_2 \beta(X_3, Y_3, X_4, Y_4) \\ \widetilde{B}(X_2, Y_2, X_3, Y_3, X_4, Y_4) &= X_2 \gamma(X_3, Y_3, X_4, Y_4) + Y_2 \delta(X_3, Y_3, X_4, Y_4), \end{aligned}$$

for some polynomials $\alpha(X_3, Y_3, X_4, Y_4), \beta(X_3, Y_3, X_4, Y_4), \gamma(X_3, Y_3, X_4, Y_4), \delta(X_3, Y_3, X_4, Y_4)$ in $\mathbb{F}_q[X_3, Y_3, X_4, Y_4]$. This means that we can rewrite this situation in the following way

$$\begin{pmatrix} \alpha(x_3, y_3, x_4, y_4) & \beta(x_3, y_3, x_4, y_4) \\ \gamma(x_3, y_3, x_4, y_4) & \delta(x_3, y_3, x_4, y_4) \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{17}$$

Notice that this can only happen in $\mathbb{P}^1(\mathbb{F}_{q^s})^3$ if

$$\det \begin{pmatrix} \alpha(x_3, y_3, x_4, y_4) & \beta(x_3, y_3, x_4, y_4) \\ \gamma(x_3, y_3, x_4, y_4) & \delta(x_3, y_3, x_4, y_4) \end{pmatrix} = 0 \Leftrightarrow (\alpha\delta - \beta\gamma)(x_3, y_3, x_4, y_4) = 0. \tag{18}$$

We define the following set of zeroes

$$\mathcal{O}_{\alpha\delta-\beta\gamma}(\mathbb{P}^1(\mathbb{F}_{q^s})^2) := \{[x_3 : y_3], [x_4 : y_4] \in \mathbb{P}^1(\mathbb{F}_{q^s})^2 | (\alpha\delta - \beta\gamma)(x_3, y_3, x_4, y_4) = 0\}.$$

From now on we will assume that the 2×2-matrix given at (17) is not equal to the zero matrix for every $([x_3 : y_3], [x_4 : y_4]) \in \mathbb{P}^1(\mathbb{F}_{q^s})^2$. This implies that for every $([x_3 : y_3], [x_4 : y_4]) \in \mathbb{P}^1(\mathbb{F}_{q^s})^2$ that satisfies (18) there exists exactly one $[x_2 : y_2] \in \mathbb{P}^1(\mathbb{F}_{q^s})$ such that (17) holds. We define $\xi_s := \left|\mathcal{O}_{\alpha\delta-\beta\gamma}(\mathbb{P}^1(\mathbb{F}_{q^s})^2)\right|$ for $s \in \mathbb{N}$, and conclude from the above that

$$\widetilde{N}_s = \left|\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^4)\right| = (q^s+1)^3 + (q^s+1)\xi_s - \xi_s = (q^s+1)^3 + q^s\xi_s = q^{3s} + 3q^{2s} + 3q^s + 1 + q^s\xi_s$$

The zeta function of the hypersurface definded at (15) is given by

$$\begin{aligned} Z(\widetilde{H}_{\widetilde{p}}(\mathbb{P}^1(\mathbb{F}_q)^4); T) &= \exp\left(\sum_{s=1}^{\infty} \frac{(q^{3s} + 3q^{2s} + 3q^s + 1 + q^s\xi_s)T^s}{s}\right) \\ &= \frac{1}{(1-q^3T)(1-q^2T)^3(1-qT)^3(1-T)} Z(\mathcal{O}_{\alpha\delta-\beta\gamma}(\mathbb{P}^1(\mathbb{F}_{q^s})^2); qT), \end{aligned}$$

where $Z(\mathcal{O}_{\alpha\delta-\beta\gamma}(\mathbb{P}^1(\mathbb{F}_{q^s})^2); T)$ denotes the zeta function of $\mathcal{O}_{\alpha\delta-\beta\gamma}(\mathbb{P}^1(\mathbb{F}_{q^s})^2)$.

8 Conclusion

In this thesis we computed the Hasse-Weil zeta functions of certain multilinear hypersurfaces. We started with hypersurfaces in $\mathbb{P}^1(\mathbb{F}_q)^2$ defined by multilinear polynomials of degree 2 in $\mathbb{F}_q[X_1, X_2]$. We have found two different zeta functions, one belonging to the irreducible hypersurfaces, and one belonging to the reducible hypersurfaces. We continued computing the zeta functions of hypersurfaces in $\mathbb{P}^1(\mathbb{F}_q)^3$ defined by multilinear polynomials of degree 3 in $\mathbb{F}_q[X_1, X_2, X_3]$. As we have seen in **Section 6** the zeta functions of these hypersurfaces depend on the coefficients of the polynomials defining the hypersurfaces.

After all, we computed the zeta functions for some hypersurfaces in $\mathbb{P}^1(\mathbb{F}_q)^4$ defined by multilinear polynomials of degree 4 in $\mathbb{F}_q[X_1, X_2, X_3, X_4]$. During computing this zeta function we made an assumption and we eventually found a zeta function depending of another zeta function. Further investigation is needed to find out what the zeta function is if the hypersurface is defined by a polynomial that does not satisfy the assumption.

References

[Beu18] F. Beukers (2018), *Rings and Galois Theory*, Utrecht University, Department of Mathematics,

[Kob84] N. Koblitz (1984), *p-adic Numbers, p-adic Analysis, and Zeta-Functions*, second edition. New York: Springer Verlag, p 109-111.

[Musta] M. Mustaţă, *Zeta functions in algebraic geometry*, lecture notes available on the web. Retrieved from `http://www-personal.umich.edu/~mmustata/zeta_book.pdf`, visited at 19-10-2018.

[How06] John M. Howie (2006), *Fields and Galois Theory*, first edition, London: Springer Verlag.

[And17] Andrew Sutherland (2017), *18.783 Elliptic Curves Lecture 3*, 15-2-2017. Retrieved from `https://math.mit.edu/classes/18.783/2017/LectureNotes3.pdf`, visited at 12-10-2018.