Master's Thesis

Profinite Homotopy Theory

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Chapter 1

Introduction

Profinite homotopy theory originated from the desire to apply methods of algebraic topology to algebraic geometry, in particular to assign invariants from algebraic topology to schemes. It was believed that by using such invariants, in particular a good cohomology theory for schemes, the Weil conjectures could be proved. In SGA1 ([Gro+71]), Grothendieck defined the étale fundamental group of a connected and locally Noetherian scheme using finite étale coverings. Étale cohomology was developed by Grothendieck and M. Artin, and was published in [Art62] and SGA4 ([GAV72]).

Michael Artin and Barry Mazur refined the theory of étale cohomology by associating a "pro-homotopy type" to a connected and locally Noetherian scheme in their book *Etale homotopy* [AM69], called the étale homotopy type of the scheme. A pro-homotopy type is a pro-object in the homotopy category of simplicial sets $\mathrm{Ho}(\mathbf{S})$. A pro-object in a category is defined as a functor from a cofiltered category to this category. There exists a good notion of morphism between pro-objects, and they in particular form a category.

For these pro-homotopy types, there exists a good notion of the fundamental group and of cohomology, and these turn out to be equal to the étale fundamental group (after applying a certain profinite completion) and étale cohomology of the scheme, respectively. Pro-homotopy types share many properties with the usual homotopy types of spaces, and allow for the application of methods from algebraic topology to algebraic geometry.

The study of the étale fundamental group, étale cohomology and these étale homotopy types has found many applications. An important application of étale cohomology is the proof of the last Weil conjecture by Deligne [Del74]. Remarkable applications of étale homotopy theory can be found in the proofs of the Adams conjecture by Friedlander [Fri73] and Sullivan [Sul74].

The motivation for this thesis came from a recent application of ètale homotopy theory, namely the proof by Geoffroy Horel that the group of homotopy automorphisms of the profinite completion of the little 2-discs operad is isomorphic to the profinite Grothendieck-Teichmüller group [Hor17]. The goal of this thesis is to gain a better understanding of certain parts of this proof. We will provide a brief sketch of Horel's proof, to give some context and motivation for the work done in this thesis. Those readers who are not familiar with operads or the Grothendieck-Teichmüller group should not worry. Although Horel's article [Hor17] is about profinite completion of operads and the

Grothendieck-Teichmüller group, both operads and the Grothendieck-Teichmüller group do not appear anywhere in this thesis, except in the following sketch of Horel's proof.

In [Hor17], Horel first defines the profinite completion of an operad over simplicial sets, and of an operad over groupoids. These profinite completions land in the category of so-called "weak operads" over "profinite spaces" and "profinite groupoids", respectively. For both of these categories of "weak operads", there exists a model structure in a natural way. In particular, it is possible to talk about the group of homotopy automorphisms of these (weak) operads. The profinite completion functor for groupoids over simplicial sets is applied to the little 2-discs operad. The little 2-discs operad is weakly equivalent to the operad $B(\mathcal{P}a\mathcal{B})$, where $\mathcal{P}a\mathcal{B}$ is the operad of "parenthesized braids", an operad over groupoids. Here B is the functor which associates to a groupoid A its nerve $BA \in \mathbf{S}$, applied levelwise to the operad $\mathcal{P}a\mathcal{B}$. There is a profinite completion functor both for operads over simplicial sets and for operads over groupoids. It is shown that for the operad $\mathcal{P}a\mathcal{B}$, the objects $B(\widehat{\mathcal{P}a\mathcal{B}})$ and $\widehat{B(\mathcal{P}a\mathcal{B})}$ are weakly equivalent, where $\widehat{(\cdot)}$ denotes the profinite completion functor. In particular, to study the homotopy automorphisms of the profinite completion of $B(\mathcal{P}a\mathcal{B})$, one can also first apply the profinite completion functor (for operads over groupoids) to $\mathcal{P}a\mathcal{B}$, and then apply the nerve functor B. The homotopy automorphism group of $B(\widehat{\mathcal{P}a\mathcal{B}})$ is equal to that of $\widehat{\mathcal{P}a\mathcal{B}}$. The homotopy automorphism group of the latter operad can be computed, and turns out to be isomorphic to the profinite Grothendieck-Teichmüller group.

In [AM69], Artin and Mazur construct a profinite completion functor, which associates a pro-object in $\mathrm{Ho}(\mathbf{S})_{fin}$ to a pro-object in $\mathrm{Ho}(\mathbf{S})$. Here $\mathrm{Ho}(\mathbf{S})_{fin}$ denotes the full subcategory of $\mathrm{Ho}(\mathbf{S})$ of simplicial sets whose homotopy groups are finite. Such a profinite completion functor is often used in applications of étale homotopy theory. However, it is usually desirable to work in a model category when doing homotopy theory, instead of the homotopy category itself, as homotopy categories often behave badly. Working in the pro-category of a homotopy category is therefore undesirable. In Geoffroy Horel's proof, to define the profinite completion of an operad, it is necessary to have a profinite completion functor on the level of simplicial sets. More precisely, a model category is needed in which one can study "profinite spaces", meaning that its homotopy category should, in a certain sense, be the category of profinite homotopy types. Furthermore, there should be a profinite completion functor, which should be a functor from simplicial sets to this model category. Such a model category, together with a profinite completion functor, is constructed in [Qui08]. To define the profinite completion of an operad over groupoids, a good profinite completion functor for groupoids is needed, which maps to a model category whose objects are "profinite groupoids". Such a model category is constructed by Horel himself in [Hor17, §4].

The aim of this thesis is to understand these "profinite spaces" and "profinite groupoids" and the model categories in which they live, as defined by Quick in [Qui08] and by Horel in [Hor17, §4]. We show that the fibrantly generated model structures, defined by Quick and Horel, indeed exist, filling in some gaps in their proofs and correcting a few minor mistakes.

1.1 Overview

This thesis consists of three chapters, aside from this introduction, and an appendix.

Chapter 2 covers the category $\mathrm{Pro}(\mathbf{C})$ of pro-objects in $\mathbf{C}$. We start by defining pro-objects and morphisms between objects, and then some of their properties. Many of the properties of pro-categories that we deduce in this chapter are used throughout chapters 3 and 4.

In chapter 3, we define and study profinite groupoids. In particular, weak equivalences between profinite groupoids are defined and studied extensively. At the end of this chapter, we construct a fibrantly generated model structure on the category of profinite groupoids.

Chapter 4 is about profinite spaces, which we also define there. We construct their fundamental groupoid, a profinite groupoid in the sense of chapter 3, and cohomology with local coefficients. We subsequently use these to define weak equivalences of profinite spaces, and prove that they are the weak equivalences for a certain fibrantly generated model structure. We end this chapter by developing a theory of coverings for profinite spaces.

This is followed by an appendix on fibrantly generated model categories, the dual of the more common cofibrantly generated model categories.

1.2 Preliminaries

Some familiarity with category theory is required; the reader should at least be comfortable with heavy use of (co)limits and adjunctions. Some basic knowledge of model categories is also required, chapter 7 and some parts of chapter 8 of [Hir03] should suffice. Fibrantly generated model categories, which play an important role in chapters 3 and 4, are treated in the appendix. Familiarity with simplicial sets is needed as well, at least the first three chapters of [Lam68], although some experience with cohomology of simplicial sets and the Dold-Kan equivalence might also be useful.

Chapter 2

Pro-categories

This chapter is devoted to defining and studying pro-categories and related notions. It should be seen as an introduction to pro-categories, and it covers many definitions and results which are needed for the study of profinite groupoids and profinite spaces presented in chapters 3 and 4, respectively.

In the first section, we define the category of pro-objects in a given category, which we will call the pro-category of this given category. We then study some basic properties of these pro-categories. In particular some results on functors between pro-categories and the existence of (co)limits in pro-categories are proved. In the second section, we study a few concrete examples of pro-categories, namely the categories of profinite sets, profinite groups and profinite G-sets. We show that there exist point-set topological models for these objects. More specifically, we show that one can view profinite sets as Stone spaces, profinite groups as topological groups whose underlying space is a Stone space, and profinite G-sets as Stone spaces with a continuous G-action. In section 3, we construct a so-called profinite completion functor, and study it in the context of profinite sets and profinite groups. The last section is devoted to categories of the form $\mathrm{Pro}(C)^I$. We show that, under some assumptions on the categories I and $\mathbf{C}$, this category is equivalent to the pro-category $\mathrm{Pro}(\mathbf{D})$ for a certain full subcategory $\mathbf{D}$ of $\mathbf{C}^I$. This will prove useful in chapter 4, when we study simplicial profinite sets, or as they are called there, profinite spaces.

The main sources for the material in this chapter are [Isa01] and [GAV72]. In section 2.2 some results are taken from [RZ10], although most of the results are the author's own work. The inspiration for the main result shown in section 2.4, Theorem 2.82, came from the proof of [BHH17, Proposition 7.4.1].

2.1 The category of pro-objects

We begin this section by defining pro-objects in a category $\mathbf{C}$ and the appropriate notion of morphism between pro-objects, to obtain a category $\mathrm{Pro}(\mathbf{C})$ of pro-objects. We then consider some basic examples of pro-categories and of their dual notion, ind-categories. After these examples, we study the behaviour of cofiltered limits in pro-categories and prove a universal property characterizing pro-categories. The last part of this section is devoted to the study of Pro($\mathbf{C}$) when $\mathbf{C}$ is essentially small and has finite limits, as the pro-category has many useful and desirable properties in this case.

2.1.1 Pro-objects

Recall that a *directed set* is a preorder $(I, \leq)$ such that $I \neq \emptyset$ and for any $i, j \in I$, there exists a $k \in I$ such that $i, j \leq k$. Dually, a *codirected set* $(I, \leq)$ is a nonempty preorder such that for any $i, j \in I$, there exists a $k \in I$ such that $k \leq i, j$. For a preorder $(I, \leq)$, we denote by $(I^{op}, \leq^{op})$ the pre-order such that $i \leq^{op} j$ if and only if $j \leq i$ in I. Note that I is directed if and only if I^{op} is codirected. If the preorder is also a partial order, then we call it a *(co)directed poset*. A directed poset I is *cofinite* if the set $\{j \in I \mid j \leq i\}$ is finite for every $i \in I$, and dually a codirected poset is *cofinite* if I^{op} is cofinite. (Co)directed sets can be used to define a special kind of (co)limit. We can view I as a category, where the objects are the elements of I and there is a unique arrow $i \to j$ if and only if $i \leq j$.

Definition 2.1. Let I be a directed set and $\mathbf{C}$ a category. A diagram of the form $I \to \mathbf{C}$ is called an *inductive diagram*. Dually, a diagram of the form $I \to \mathbf{C}$ with I codirected is called a *projective diagram*. A limit of a projective diagram is called a *projective limit* and a colimit of an inductive diagram is called an *inductive limit*. $\diamondsuit$

Notable examples of projective and inductive limits are the ones over $I = \mathbb{N}^{op}$ and $I = \mathbb{N}$, respectively, which are sometimes called sequential limits. It is possible to generalize the notion of a directed set to categories which have multiple arrows between objects.

Definition 2.2. A nonempty category I is said to be *cofiltered* if

(i) For any objects $i, j \in I$, there exists a $k \in I$ such that there are arrows $k \to i$ and $k \to j$.

(ii) If we are given two morphisms $\alpha, \beta \colon i \to j$ in I, then for some k there exists a morphism $\gamma \colon k \to i$ such that $\alpha\gamma = \beta\gamma$.

Dually, a category I is said to be *filtered* if I^{op} is cofiltered. If I is (co)filtered and $I \to \mathbf{C}$ is a diagram, then we call this a *(co)filtered diagram*. A limit over a cofiltered category is called a *cofiltered limit*, and dually a colimit over a filtered diagram is a *filtered colimit*. $\diamondsuit$

Remark 2.3. It is easy to see that if I is a codirected set, then as a category it is cofiltered (note that (ii) is trivially satisfied, since any two arrows $i \to j$ are equal). In fact, codirected sets are precisely those cofiltered categories with at most one arrow between any two objects. $\diamondsuit$

Remark 2.4. If I and J are cofiltered categories, then the product category $I \times J$ is also cofiltered. In fact, any arbitrary product of cofiltered categories is again cofiltered. If I and J are (co)directed sets, then the product category $I \times J$ also has at most one arrow between any two objects, hence it is also a (co)directed set. More specifically, the order on $I \times J$ is defined by $(i, j) \leq (i', j')$ if and only if $i \leq i'$ and $j \leq j'$. $\diamondsuit$

We will see later that cofiltered limits can always be given as projective limits, i.e. limits over a codirected set. The reason to work with cofiltered limits is that in some contexts they occur more naturally than projective limits.

A well-known fact on filtered colimits is that they commute with finite limits in **Set**, the category of sets.

Theorem 2.5. *Let I be a filtered category, J a finite category and $D\colon I \times J \to \mathbf{Set}$ be a diagram. Then*

$$\operatorname*{colim}_i \lim_j D(i,j) = \lim_j \operatorname*{colim}_i D(i,j).$$

For a proof, see Theorem IX.2.1 in [Mac88].

Definition 2.6. Let $\mathbf{C}$ be a category. A *pro-object* in $\mathbf{C}$ is a diagram $I \to \mathbf{C}$ where I is cofiltered. Dually, an *ind-object* in $\mathbf{C}$ is a diagram $I \to \mathbf{C}$ where I is filtered. ♢

Remark 2.7. Note that a pro-object is by definition the same as a cofiltered diagram, and that an ind-object is a filtered diagram. However, as we will treat them differently than diagrams, we define them seperately. In particular, morphisms between pro-objects and ind-objects will not be defined as natural transformations, although this is the usual way of defining a morphism between diagrams. ♢

We will often write $\{A_i\}_{i\in I}$ or $\{A_i\}$ for a diagram in a category $\mathbf{C}$ indexed by I. We will also use this notation for pro-objects, implicitly assuming that I is cofiltered.

Note that pro stands for projective and ind for inductive in the above definition. A pro-object can be seen as a formal projective limit. The reason for working with pro-objects is that in computing the actual limit, one can lose certain information. For example, we will see in Example 2.13 that a pro-object in the category **Set** contains more structure than the actual limit in **Set**. There turns out to be a natural way to define maps between pro-objects (or, dually, ind-objects).

Definition 2.8. For a category $\mathbf{C}$, define the category $\mathrm{Pro}(\mathbf{C})$ with as objects all pro-objects in $\mathbf{C}$. For two pro-objects $X : I \to \mathbf{C}$ and $Y : J \to \mathbf{C}$, define

$$\mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}(X,Y) = \lim_j \operatorname*{colim}_i \mathrm{Hom}_{\mathbf{C}}(X(i), Y(j)),$$

where the (co)limits are taken in **Set**. One can dually define the category $\mathrm{Ind}(\mathbf{C})$ of ind-objects by setting

$$\mathrm{Hom}_{\mathrm{Ind}(\mathbf{C})}(X,Y) = \lim_i \operatorname*{colim}_j \mathrm{Hom}_{\mathbf{C}}(X(i), Y(j)).$$ ♢

We usually call $\mathrm{Pro}(\mathbf{C})$ the *pro-category* of $\mathbf{C}$, and $\mathrm{Ind}(\mathbf{C})$ the *ind-category* of $\mathbf{C}$. One can deduce from the above definition that $(\mathrm{Ind}(\mathbf{C}^{op}))^{op} = \mathrm{Pro}(\mathbf{C})$. We can associate to any object $C \in \mathbf{C}$ the diagram $* \to \mathbf{C}$ with value C, where $*$ is the category with one object and one arrow. This defines a functor $\iota\colon \mathbf{C} \to \mathrm{Pro}(\mathbf{C})$ and $\iota\colon \mathbf{C} \to \mathrm{Ind}(\mathbf{C})$. Sometimes we will abusively write C for $\iota(C)$. One can easily deduce that both functors ι are fully faithful, hence we can view $\mathbf{C}$ as a full subcategory of $\mathrm{Pro}(\mathbf{C})$ and $\mathrm{Ind}(\mathbf{C})$ respectively. We call the objects in the image of ι *representables*. In Proposition 2.36, we will see that the representable presheaves in $(\mathbf{Set}^{\mathbf{C}})^{op}$ and $\mathbf{Set}^{\mathbf{C}^{op}}$ correspond to the representables in $\mathrm{Pro}(\mathbf{C})$ and $\mathrm{Ind}(\mathbf{C})$, respectively, thus justifying their name.

Remark 2.9. The constructions of Ind($\mathbf{C}$) (resp. Pro($\mathbf{C}$)) can be seen as "freely adjoining filtered colimits (resp. cofiltered limits)" to $\mathbf{C}$. In the case that $\mathbf{C}$ already has these (co)limits, this does not mean that Ind($\mathbf{C}$) (resp. Pro($\mathbf{C}$)) is again the category $\mathbf{C}$ (compare this to $\mathbf{Set}^{\mathbf{C}^{op}}$, the free cocompletion of $\mathbf{C}$). In the case that $\mathbf{C}$ has all small filtered colimits, then $\iota\colon \mathbf{C} \to \mathrm{Ind}(\mathbf{C})$ is right adjoint to $|\cdot|\colon \mathrm{Ind}(\mathbf{C}) \to \mathbf{C}$, where the functor $|\cdot|$ computes the actual colimit of an ind-object in $\mathbf{C}$. Indeed, one can easily verify that there is a natural bijection $\mathrm{Hom}_{\mathbf{C}}(\mathrm{colim}_i X(i), Y) \simeq \mathrm{Hom}_{\mathrm{Ind}(\mathbf{C})}(X, \iota Y)$ for any ind-object $X\colon I \to \mathbf{C}$ and any object $Y \in \mathbf{C}$. In the case that $\mathbf{C}$ has all small cofiltered limits, then $\iota\colon \mathbf{C} \to \mathrm{Pro}(\mathbf{C})$ is left adjoint to $|\cdot|\colon \mathrm{Pro}(\mathbf{C}) \to \mathbf{C}$, where $|\{X_i\}_i| = \lim_i X_i$ for any pro-object $\{X_i\}_{i\in I}$. This can be proved in a similar way, or by noting that $\mathrm{Pro}(\mathbf{C}) = \mathrm{Ind}(\mathbf{C}^{op})^{op}$. $\diamondsuit$

The above definition of the Hom-sets of Pro($\mathbf{C}$) are somewhat abstract, and it may also be unclear how to compose morphisms. We will therefore unravel their definition. There are easy descriptions for filtered colimits and cofiltered limits in $\mathbf{Set}$. If we are given a filtered diagram $D\colon I \to \mathbf{Set}$, then

$$\operatorname*{colim}_i D(i) = \left(\coprod_i D(i)\right) \Big/ \sim .$$

For $x \in D(i)$ and $y \in D(j)$, we define $x \sim y$ if and only for some k there exist $\alpha\colon i \to k$, $\beta\colon j \to k$ such that $D(\alpha)(x) = D(\beta)(y)$ in $D(k)$. If we are given a cofiltered diagram $E\colon J \to \mathbf{Set}$, then

$$\lim_i E(i) = \left\{ (x_i) \in \prod_i E(i) \;\middle|\; E(\alpha)(x_i) = x_j \text{ for all } \alpha\colon i \to j \text{ in } J \right\}. \tag{2.1}$$

Looking at the above definition of the Hom-sets of Pro($\mathbf{C}$), we see that a morphism $f\colon X \to Y$ is a J-indexed tuple $([f_j])_j$, where $[f_j]$ is an equivalence class of maps $X(i) \to Y(j)$. Here i may vary, and the representative f_j is a map $X(i_j) \to Y(j)$ for some $i_j \in I$. Using this it is not hard to prove the following proposition, which can also be taken as a definition. It is taken from [EH76]. The details are left to the reader.

Proposition 2.10. *Let X, Y be pro-objects of $\mathbf{C}$ indexed by the cofiltering categories I and J, respectively. A morphism $f\colon X \to Y$ in* Pro($\mathbf{C}$) *can be represented by a map $\theta\colon \mathrm{Ob}(J) \to \mathrm{Ob}(I)$ (which need not be order preserving) and morphisms $f_j\colon X(\theta(j)) \to Y(j)$ for every $j \in J$, such that for every $\alpha\colon j \to j'$ in J, there exist $\beta\colon i \to \theta(j)$ and $\gamma\colon i \to \theta(j')$ for which the diagram*

$$\begin{tikzcd} X(i) \arrow[r, "X(\beta)"] \arrow[dr, "X(\gamma)"'] & X(\theta(j)) \arrow[r, "f_j"] & Y(j) \arrow[d, "Y(\alpha)"] \\ & X(\theta(j')) \arrow[r, "f_{j'}"] & Y(j') \end{tikzcd}$$

commutes. (θ, f_j) *and* (θ', f'_j) *represent the same morphism in* Pro($\mathbf{C}$) *precisely if for every* $j \in J$ *there exist* $\alpha\colon i \to \theta(j)$ *and* $\beta\colon i \to \theta'(j)$ *such that*

$$\begin{tikzcd} X(i) \arrow[r, "X(\alpha)"] \arrow[d, "X(\beta)"'] & X(\theta(j)) \arrow[d, "f_j"] \\ X(\theta'(j)) \arrow[r, "f'_j"'] & Y(j) \end{tikzcd}$$

commutes. ■

Remark 2.11. Note that if we are given two pro-objects $X, Y\colon I \to \mathbf{C}$ indexed by the same cofiltered category I, then a morphism $X \to Y$ is not the same as a natural transformation from X to Y. Any natural transformation induces a morphism $X \to Y$ in Pro($\mathbf{C}$), as can easily be seen from the above proposition. However not all morphisms come from a natural transformation and different natural transformations need not induce different morphisms of pro-objects, as can be seen in the next example. ♢

Example 2.12. Consider the two cofiltered diagrams X, Y in **Set** given by $\{0,1\} \leftarrow \{0,1\}$, where the arrow is the identity map for X, and the constant map 0 for Y. There exist four natural transformations from Y to X, and four morphisms $Y \to X$ in Pro(**Set**) (which correspond to the four maps $\{0,1\} \to \{0,1\}$). However, one can verify that only two of these morphisms are induced by a natural transformation from Y to X. We conclude that not all morphisms come from natural transformation, and that different natural transformations can induce the same morphism in Pro(**Set**). ♢

Example 2.13. This example illustrates that a pro-object contains more information that its actual limit. Consider the $\mathbb{N}$-indexed diagram

$$\mathbb{N}_{\geq 1} \xleftarrow{\cdot 2} \mathbb{N}_{\geq 1} \xleftarrow{\cdot 2} \mathbb{N}_{\geq 1} \xleftarrow{\cdot 2} \cdots$$

in **Set**. The limit of this diagram in **Set** is empty, however there exist many non-equal morphisms in Pro(**Set**) with this diagram as their domain or codomain. The reader is advised to try his or her hand at finding such morphisms using Proposition 2.10. ♢

Before we begin studying the structure of the category Pro($\mathbf{C}$), we will consider some examples of ind- and pro-categories.

Example 2.14. Let **FinSet** be the category of finite sets. Then Ind(**FinSet**) $\simeq$ **Set**, the category of sets. ♢

Example 2.15. Let **FinVect** be the category of finite dimensional vector spaces. Then Ind(**FinVect**) $\simeq$ **Vect**. Similarly, the category of groups **Grp** is the ind-category of the category of finitely generated groups, and the same holds for the category of abelian groups **Ab**. ♢

Example 2.16. There is also a nice description for pro-objects in **FinSet**. Denote the category Pro(**FinSet**) by $\widehat{\mathbf{Set}}$. If we are given a cofiltered diagram $D \colon I \to \mathbf{FinSet}$ of finite sets, then its limit in **Set** can be computed as in (2.1). If we give all these finite sets the discrete topology and $\lim_i D(i)$ the subspace topology of the product topology on $\prod D(i)$, then morphisms in $\widehat{\mathbf{Set}}$ can be shown to correspond to continuous maps between these actual limits. This means that $\widehat{\mathbf{Set}}$ is a full subcategory of the category **Top** of topological spaces, with all cofiltered limits of discrete finite spaces as objects. One can show that a space is a cofiltered limit of finite discrete spaces precisely if it is a *Stone space*, i.e. a totally disconnected compact Hausdorff space. We therefore have an equivalence of categories $\widehat{\mathbf{Set}} \simeq \mathbf{Stone}$. This will be proved in detail in section 2.2. ♢

Example 2.17. Using Stone duality, there exists a particularly slick proof for the above equivalence of categories. As in the above examples, if one uses the ind-construction on the category of finite boolean algebras **FinBool**, then one obtains the category of boolean algebras **Bool**. Stone duality states that $\mathbf{Bool}^{op} \simeq \mathbf{Stone}$ (and this equivalence restricts to an equivalence $\mathbf{FinBool}^{op} \simeq \mathbf{FinSet}$), hence we have

$$\mathrm{Pro}(\mathbf{FinSet}) \simeq \mathrm{Ind}(\mathbf{FinSet}^{op})^{op} \simeq \mathrm{Ind}(\mathbf{FinBool})^{op} \simeq \mathbf{Bool}^{op} \simeq \mathbf{Stone}. \qquad \diamondsuit$$

Example 2.18. Another example that one often encounters is that of a *profinite group*, i.e. a pro-object in the category of finite groups **FinGrp**. Denote the category of profinite groups by $\widehat{\mathbf{Grp}}$. The construction of a cofiltered limit as in (2.1) also works for groups. If we give these finite groups the discrete topology, then the cofiltered limit will be a topological group that is also a Stone space. One can show that the category of profinite groups is equivalent to the category of topological groups that are Stone spaces. We will also discuss this proof in section 2.2. ♢

Example 2.19. The two categories that we will mainly study in this thesis are Pro(**FinG**) and $\mathrm{Pro}(\mathbf{S}_{cofin})$. Here **FinG** is the category of finite groupoids, and $\mathbf{S}_{cofin}$ the category of simplicial finite sets which are k-coskeletal for some k. We will see that Pro(**FinG**) is a full subcategory of the category of topological groupoids, and that $\mathrm{Pro}(\mathbf{S}_{cofin})$ is the category of simplicial objects in $\widehat{\mathbf{Set}}$, or equivalently in **Stone**. ♢

2.1.2 Reindexing pro-objects and level representations

In many situations we need to replace a pro-object by an isomorphic one that is indexed in a more convenient way. For example, in certain situations it is convenient if a pro-object is indexed by a codirected poset instead of an arbitrary cofiltered category. The following proposition, which states that the limit of a cofiltered diagram is this diagram itself in Pro(**C**), is particularly important when constructing such reindexed pro-objects.

Proposition 2.20. *Let $X \colon I \to \mathbf{C}$ be a pro-object in a category* **C** *and let $\iota \colon \mathbf{C} \to \mathrm{Pro}(\mathbf{C})$ be the inclusion. Then $\lim_i \iota(X(i)) = X$ in* Pro(**C**).

Proof. One can easily show that X has the universal property of the limit $\lim_i \iota(X(i))$. Assume we are given a pro-object Y and natural morphisms $\mu_i \colon Y \to \iota(X(i))$. Then it is a straightforward application of Proposition 2.10 to see that this precisely defines a unique morphism $Y \to X$. ■

The above proposition is particularly useful since it implies that if we are given an initial functor $F\colon I \to J$ between cofiltered categories, and a pro-object $Y\colon J \to \mathbf{C}$, then YF and Y are isomorphic pro-objects (by uniqueness of limits). Recall that a functor $F\colon I \to J$ is called *initial* if for every $j \in J$, the comma category $(F \downarrow j)$ is nonempty and connected, and that we have the following theorem (for a proof, see Theorem IX.3.1 of [Mac88]).

Theorem 2.21. *If $F\colon I \to J$ is initial and $Y\colon J \to \mathbf{C}$ is a functor such that $\lim_i YF(i)$ exists, then $\lim_j F(j)$ exists and these limits are equal.*

The comma category $(F \downarrow j)$ is nonempty if there exists an $i \in I$ such that there is an arrow $F(i) \to j$. It is connected if for any two such arrows $F(i) \to j$ and $F(i') \to j$ there exists a commutative diagram of the form

$$\begin{tikzcd}
F(i) \arrow[r, "F(\alpha_1)"] \arrow[d] & F(i_2) \arrow[d] & F(i_3) \arrow[l, "F(\alpha_2)"'] \arrow[r] \arrow[d] & \cdots & F(i_n) \arrow[l] \arrow[d] & F(i') \arrow[l, "F(\alpha_n)"'] \arrow[d] \\
j \arrow[r, equal] & j \arrow[r, equal] & j \arrow[r, equal] & \cdots \arrow[r, equal] & j \arrow[r, equal] & j
\end{tikzcd}$$

in J. If we are working with cofiltered categories and F is full, i.e. surjective on Hom-sets, then this last condition of connectedness can be dropped. In particular, an initial functor between codirected sets I and J is simply an order-preserving map f such that for any $j \in J$, there exists an $i \in I$ with $f(i) \le j$.

By uniqueness of limits, we obtain the following result.

Corollary 2.22. *Let $\{A_i\}_{i\in I}$ be a pro-object in a category $\mathbf{C}$, and let $f\colon J \to I$ be an initial functor of cofiltered categories. Then $\{A_{f(j)}\}_{j\in J} \cong \{A_i\}_{i\in I}$ in $\mathrm{Pro}(\mathbf{C})$.* ■

The following construction can be found in [GAV72, Exposé 1, 8.1.6]. For a given cofiltered category I, let $M(I)$ be the set of all finite diagrams in I with an initial point, i.e. all functors $D\colon K \to I$ where K is a finite category with an initial object. If $x \in K$ is the initial object of K, then we call $D(x)$ the initial point of D. We say that $D' \le D$ if D' is a subdiagram of D. If $D\colon K \to I$ and $D'\colon K' \to I$, this means that K' is a subcategory of K and that $D|_{K'} = D$. It is immediate that this defines a poset. It is in fact codirected. To see this, let D and D' be diagrams with domain K and K' respectively. Define the category K'' by taking the disjoint union of K and K' and adjoining an initial object x. Let y and y' be the initial objects of K and K' respectively, pick $i \in I$ with arrows $i \to D(y)$ and $i \to D'(y')$. Then we can extend D and D' to diagram $D''\colon K'' \to I$ with $D''(x) = i$, such that $D, D' \le D''$. We conclude that $M(I)$ is codirected. Since all diagrams are finite, it is a cofinite codirected poset. Hence we have the following theorem.

Theorem 2.23. *The functor $t\colon M(I)^{op} \to I$ which sends a diagram D to its initial point, is initial. As a corollary the category $\mathrm{Pro}(\mathbf{C})$ is equivalent to the full subcategory of pro-objects indexed by a cofinite codirected poset.*

Proof. If we are given two diagrams $D' \le D$, then there is a canonical arrow from the initial point of D to the inital point of D', making t indeed into a functor. Using what is stated above, the proof is straightforward and left to the reader. ■

From hereon we assume that objects of Pro($\mathbf{C}$) are indexed by a codirected poset, unless stated otherwise. We still talk about cofiltered limits sometimes, when we want to stress that a certain statement not only holds for projective limits, but in practice we only work with projective diagrams indexed by a codirected poset.

In order to compute (finite) limits and colimits in pro-categories, it is useful to have so-called level representations of diagrams in pro-categories.

Definition 2.24. Let $X = \{X_i\}_{i \in I}$ and $Y = \{Y_j\}_{j \in J}$ be pro-objects in a category $\mathbf{C}$, and let $f\colon X \to Y$ be a morphism of pro-objects. A *representation* of f is a map $g\colon X_i \to Y_j$ for some $i \in I$ and $j \in J$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X_i & \xrightarrow{g} & Y_j \end{array}$$

commutes. More generally, if $D\colon K \to \text{Pro}(\mathbf{C})$ is a diagram, we say that $D'\colon K \to \mathbf{C}$ *represents* D if every map in D' is a representation of the corresponding map in the diagram D. More precisely, for any arrow α in K, we ask that $D'(\alpha)$ is a representation of $D(\alpha)$. $\diamondsuit$

Definition 2.25. Let $f\colon X \to Y$ be a map of pro-objects in a category $\mathbf{C}$. A *level representation* of f consists of two cofiltered diagrams $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ indexed by the same category, natural maps $f_i\colon X_i \to Y_i$ and isomorphisms $X \cong \{X_i\}$, $Y \cong \{Y_i\}$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \{X_i\} & \xrightarrow{\{f_i\}} & \{Y_i\} \end{array}$$

commutes. More generally, if $D\colon K \to \text{Pro}(\mathbf{C})$ is a diagram, then a *level representation* of D consists of a diagram $\widetilde{D}\colon I \times K \to \mathbf{C}; (i,k) \mapsto \widetilde{D}_i(k)$ with I a cofiltered category and isomorphisms $D(k) \cong \{\widetilde{D}_i(k)\}_{i \in I}$, such that for every arrow $\alpha\colon k \to k'$ in K, the maps $\{\widetilde{D}_i(\alpha)\}_{i \in I}\colon \{\widetilde{D}_i(k,i)\}_{i \in I} \to \{\widetilde{D}_i(k',i)\}_{i \in I}$ are a level representation for $D(\alpha)\colon D(k) \to D(k')$. $\diamondsuit$

The above definition of a level representation of a diagram may seem somewhat cryptic. In the definition of a level representation of a map f of pro-objects, we replace the domain and codomain of the map f by pro-objects that are indexed by the same index category, in such a way that f can be represented by a natural transformation between these pro-objects. What is actually meant in the definition of a level representation of a diagram is that we pick level representations of all the maps at the same time, all indexed by the same index category, and such that everything commutes levelwise. In fact, this can be seen as a pro-object in the diagram category $\mathbf{C}^K$. When trying to construct a level representation of a diagram in a pro-category, we are actually trying to associate a pro-object in $\mathbf{C}^K$ to it. This viewpoint will be studied more thoroughly in section 2.4.

Level representations of maps always exist, but level representations of arbitrary diagrams in Pro($\mathbf{C}$) do not always exist. However, under certain assumptions on the index category of the diagram and the category $\mathbf{C}$, they do exist. Here we consider the case where the index category is a finite loopless category, and we will discuss some other cases in section 2.4. The following is based on appendix 3 of [AM69].

Definition 2.26. Let K be a category. We say that K is *loopless* if for any chain of composable non-identity arrows $\alpha_1, \ldots, \alpha_n$, the domain and codomain of the composition $\alpha_1\alpha_2 \ldots \alpha_n$ are distinct. We say that K is *finite* if it has finitely many arrows. $\diamondsuit$

Note that a category K is loopless precisely if the only cycles in its underlying directed graph are the identity arrows.

Proposition 2.27. *Let $D\colon K \to$ Pro($\mathbf{C}$) be a diagram, where K is finite and loopless, and $\mathbf{C}$ is any category. Then D has a level representation $\widetilde{D}$. Moreover, this level representation $\widetilde{D}$ can be chosen in such a way that, for any k, if $D(k) = \{X_{i_k}\}_{i \in I_k}$, then for every $i \in I$, where I indexes the level representation, there exists an $i_k \in I_k$ such that $\widetilde{D}_i(k) = X_{i_k}$.*

The second part of this proposition might look somewhat technical. It says that $\widetilde{D}_j$ represents D in the sense of Definition 2.24. The idea is that, when constructing the level representation $\widetilde{D}$ of D, we do not change the objects of $\mathbf{C}$ that we are working with, but only change the way in which they are indexed. This is useful, for example, when one of the pro-objects $X = D(k)$ in the diagram D is indexed as $X = \{X_j\}_{j \in J}$ where the X_j have certain convenient properties. The pro-object $\{\widetilde{D}_i(k)\}$ corresponding to X in this level representation is then still a cofiltered diagram consisting of objects having these convenient properties. Note that since for every cofiltered category I' there is a codirected poset I with an initial functor $I \to I'$, we can assume that every cofiltered diagram is indexed by a codirected poset. In particular, level representations can always be indexed by a codirected poset.

Proof. We proceed by induction on the number of objects of the index category. If K consists of one object, then the proposition is trivial. Assume the proposition holds for index categories with n objects, and let K be a finite loopless category with $n+1$ objects. Let $D\colon K \to$ Pro($\mathbf{C}$) be a diagram. Since K is finite and loopless, there exists an object $k_0 \in K$ such that no arrow in K has k_0 as target. Write K' for the category obtained by removing k_0 and all arrows coming out of k_0. Then the diagram $D' := D|_{K'}$ has a level representation by induction, as its index category has n objects. Let $\{\widetilde{D}'_j\colon K' \to \mathbf{C}\}_{j \in I}$ be the level representation of D', indexed by a codirected poset J. Furthermore, write $X = D(k_0)$, and assume $X = \{X_i\}_{i \in I}$, indexed by a codirected poset. Define the set L to consist of triples (i, j, D_l) where $i \in I$, $j \in J$, and where $D_l\colon K \to \mathbf{C}$ is a diagram such that $D_l|_{K'} = \widetilde{D}'_j$, for which $D_l(k_0) = X_i$ and such that D_l represents D, i.e. for any $\alpha\colon k_0 \to k$ in K, the square

$$\begin{tikzcd} D(k_0) \arrow[r, "D(\alpha)"] \arrow[d] & D(k) \arrow[d] \\ D_l(k_0) \arrow[r, "D_l(\alpha)"] & D_l(k) \end{tikzcd}$$

commutes. Note that we only have to check this for maps out of k_0, as we already know that $\widetilde{D}'_j$ represents D' by the induction hypothesis. We say that $(i, j, D_l) \leq (i', j', D_{l'})$ if $i \leq i'$, $j \leq j'$ and if the induced map $D_l \to D_{l'}$ is a morphism of diagrams (meaning it is natural). This makes L into a poset.

Now note that we can define a diagram $\widetilde{D} \colon L \times K \to \mathbf{C}$ by mapping $((i, j, D_l), k)$ to $D_l(k)$. We will show that $\widetilde{D}$ is a level representation of D. As the second statement of this proposition clearly holds for the diagram $\widetilde{D}$, this completes the proof. Showing that $\widetilde{D}$ is a level representation amounts to showing that L is codirected and that the projections $p_I \colon L \to I$, $p_J \colon L \to J$ onto the first and second coordinate, respectively, are initial. The fact that these maps are initial will provide us with the isomorphisms $D(k) \cong \{\widetilde{D}'_j(k)\}_{j \in J} \cong \{\widetilde{D}_l(k)\}_{l \in L}$ for $k \in K'$ and $D(k_0) \cong \{X_i\}_{i \in I} \cong \{\widetilde{D}_l(k_0)\}_{l \in L}$ that we need. Proving these three properties of L, which we do in the rest of this proof, is not very hard, but rather technical. It might be more instructive for the reader to convince themself that these properties are true, and then skip the rest of this proof.

We first show that $L \to J$ is initial. Let $j \in J$ be given. For every $\alpha \colon k_0 \to k$ in K, there exists for some i a map $X_i \to \widetilde{D}'_j(k)$ that represents $D(\alpha) \colon D(k_0) \to D(k)$, by definition of morphisms between pro-objects. Since I is codirected and K finite, for some i there exists such a map $X_i \to \widetilde{D}'_j(k)$ for every α coming out of k_0. By Proposition 2.10, and the fact that K is finite, we can choose $i' \leq i$ such that the maps $X_{i'} \to \widetilde{D}'_j(k)$ fit in a diagram $D_l \colon K \to \mathbf{C}$ satisfying $D_l|_{K'} = \widetilde{D}'_j$ (i.e. such that everything that should commute, does so). Then $(i', j, D_l) \in L$, and $p_J(i', j, D_l) = j$, so $p_J \colon L \to J$ is surjective, hence initial.

To observe that $L \to I$ is indeed initial, let $i \in I$ be given. By the above proof that $L \to J$ is initial, we see that L is nonempty, so let (i', j, D_l) be in L. Pick $i'' \in I$ such that $i'' \leq i'$ and $i'' \leq i$. Then, by precomposing the maps $X_{i'} \to \widetilde{D}_j(k)$ with $X_{i''} \to X_{i'}$, we obtain a diagram $D_{l'} \colon K \to \mathbf{C}$ such that $(i'', j, D_{l'})$ is an element of L. As $p_I(i'', j, D_{l'}) \leq i$, we see that p_I is initial.

To see that L is codirected, let (i_0, j_0, D_{l_0}) and (i_1, j_1, D_{l_1}) be elements of L. Pick j_2 such that $j_2 \leq j_1$ and $j_2 \leq j_0$. By the above proof that $L \to J$ is surjective, we know that there exists an element (i_2, j_2, D_{l_2}) in L. Now let $\alpha \colon k_0 \to k$ be given, and let $D_{l_2}(\alpha) \colon X_{i_2} \to D_{l_2}(k) = \widetilde{D}_{j_2}(k)$ and $D_{l_0}(\alpha) \colon X_{i_0} \to D_{l_0}(k) = \widetilde{D}_{j_0}(k)$ be the maps representing $D(\alpha) \colon D(k_0) \to D(k)$. As they represent the same map, we know that for some $i_3 \leq i_2$, the diagram

$$\begin{tikzcd}
X_{i_3} \arrow[r] \arrow[dr] & X_{i_2} \arrow[r, "D_{l_2}(\alpha)"] & \widetilde{D}_{j_2}(k) \arrow[d] \\
 & X_{i_0} \arrow[r, "D_{l_0}(\alpha)"] & \widetilde{D}_{j_0}(k)
\end{tikzcd}$$

commutes. Since I is codirected, we can pick i_3 such that this holds for all arrows α in K coming out of k_0, and we can pick i_3 such that this holds for the diagram D_{l_1} as well, and not just D_{l_0}. Let D_{l_3} be the diagram obtained by precomposing the arrows $X_{i_2} \to D_{l_2}(k)$ with $X_{i_3} \to X_{i_2}$. We then see that (i_3, j_2, D_{l_3}) is an element of L and that $(i_3, j_2, D_{l_3}) \leq (i_1, j_1, D_{l_1})$ and $(i_3, j_2, D_{l_3}) \leq (i_0, j_0, D_{l_0})$. We conclude that L is codirected. ∎

Corollary 2.28. *For any category* $\mathbf{C}$*, any map in* $\mathrm{Pro}(\mathbf{C})$ *has a level representation.*

Proof. Apply the above proposition with $* \to *$ as index category K. ∎

2.1.3 Cofiltered limits in pro-categories

Using the fact that we only need to consider cofinite codirected posets, we will prove that $\mathrm{Pro}(\mathbf{C})$ is complete with respect to cofiltered limits. The idea is to find a level representation $\widetilde{D}$ of a diagram $D\colon I \to \mathrm{Pro}(\mathbf{C})$ where I is a cofinite codirected poset. This level representation $\widetilde{D}$ will be a diagram $\widetilde{D}\colon I \times J \to \mathbf{C}$ for a cofinite codirected poset J. As the product $I \times J$ is again a cofiltered category, this diagram is an object of $\mathrm{Pro}(\mathbf{C})$. We will show that this object is the limit $\lim_i D(i)$ of D in $\mathrm{Pro}(\mathbf{C})$. The following proof is taken from [Isa01, Theorem 3.3].

Theorem 2.29. *Let* I *be a cofinite codirected poset and* $D\colon I \to \mathrm{Pro}(\mathbf{C})$ *a diagram of pro-objects. Then there exists a level representation* $\widetilde{D}\colon I \times J \to \mathbf{C}$ *of* D *with* J *a cofinite codirected poset.*

Proof. Let, for each $a \in I$, the pro-object $D(a)$ be given by $D(a) = X^a = \{X^a_j\}_{j \in J^a}$, where J^a is a cofinite codirected poset. Now let J be an arbitrary cofinite codirected set with cardinality greater than or equal to the cardinalities of each of J^a. For example, we can take J to be the collection of finite subsets of some set X, where the cardinality of X is greater than $\sqcup_a J^a$, ordered by reverse inclusion. Also pick arbitrary surjections $h^a\colon J \to J^a$ (which only need to be surjections of sets, they need not be order-preserving).

For every $a \in I$, we will construct a new pro-object $\widetilde{X}^a$ isomorphic to X^a, by constructing an order-preserving map $f^a\colon J \to J^a$ which is initial. The pro-object $\widetilde{X}^a$ is then the composition of f^a with $X^a\colon J^a \to \mathbf{C}$. More explicitly, $\widetilde{X}^a = \{X^a_{f^a(j)}\}_{j \in J}$. The level representation $\widetilde{D}$ of D will be given by $\widetilde{D}_j(a) = \widetilde{X}^a_j = X^a_{f^a(j)}$. We will construct these maps f^a by induction on a, which is possible since J is cofinite codirected.

Let $a \in I$ be given, and assume that $f^b\colon J \to J^b$ has already been constructed for all $b > a$. We define f^a inductively, utilizing the fact that J is cofinite codirected. Let $j \in J$ and assume $f^a(k)$ has already been defined for $k > j$. We will choose a sufficiently small $f^a(j)$, such that all of the following holds:

(i) Choose $f^a(j)$ such that $f^a(j) \leq f^a(k)$ for all $k > j$. This will imply that $f^a\colon J \to J^a$ is order-preserving.

(ii) Choose $f^a(j)$ such that $f^a(j) \leq h^a(k)$ for all $k > j$. This will imply that $f^a\colon J \to J^a$ is initial.

(iii) Choose a sufficiently small $f^a(j)$ such that for all $b > a$ in I, there are maps $X^a_{f^a(j)} \to$

$X^b_{f^b(j)}$ representing $X^a \to X^b$ and satisfying that

$$\begin{tikzcd} X^a_{f^a(j)} \arrow[r] \arrow[d] & X^b_{f^b(j)} \arrow[d] \\ X^a_{f^a(k)} \arrow[r] & X^b_{f^b(k)} \end{tikzcd}$$

commutes for all $k > j$. This will guarantee that the resulting morphism $\widetilde{X}^a \to \widetilde{X}^b$ is a level representation of $X^a \to X^b$.

(iv) Choose $f^a(j)$ sufficiently small such that for all $c > b > a$ in I, the diagram

$$\begin{tikzcd} X^a_{f^a(j)} \arrow[r] \arrow[dr] & X^b_{f^b(j)} \arrow[d] \\ & X^c_{f^c(j)} \end{tikzcd}$$

commutes. This will guarantee that, for $j \in J$ fixed, the diagram $I \to \mathbf{C}$, given by $a \mapsto \widetilde{X}^a_{f^a(j)}$, is indeed a diagram. Hence, it guarantees that $\widetilde{D}$ is a level representation of D.

Note that in properties (iii) and (iv), there are only finitely many $b > a$ and $c > b > a$. For each of these b and c, we can find a $j' \in J^a$ which satisfies these properties. Then, by codirectedness of J^a, there always exists a small enough j' satisfying these properties for all b and c at the same time. Then, since we only stated a finite number of properties above, we can always find a j' that is small enough such that it satisfies all four of them. We then define $f^a(j)$ to be such a j'.

Since the maps $f^a \colon J \to J^a$ are initial, we see that the induced morphisms $\{X^a_{j'}\}_{j' \in J^a} \to \{X^a_{f^a(j)}\}_{j \in J}$ are an isomorphism for every $a \in I$. By construction $\widetilde{D}$ is a level representation of D. ■

Since we have $\{\widetilde{D}_j(i)\}_{j \in J} = \lim_{j \in J} \widetilde{D}_j(i)$, we see that

$$\lim_{i \in I} D(i) = \lim_{i \in I} \{\widetilde{D}_j(i)\}_{j \in J} = \lim_{i \in I} \left(\lim_{j \in J} \widetilde{D}_j(i) \right) = \lim_{(i,j) \in I \times J} \widetilde{D}_j(i).$$

Note that we write $\widetilde{D}_j(i)$ for the object in $\mathrm{Pro}(\mathbf{C})$ which corresponds to $\widetilde{D}_j(i)$ under the inclusion $\mathbf{C} \to \mathrm{Pro}(\mathbf{C})$. Since the right-hand side is equal to the pro-object $\{\widetilde{D}_j(i)\}$, we see that $\lim_i D(i)$ exists in $\mathrm{Pro}(\mathbf{C})$. We have proved the following.

Theorem 2.30. *The category* $\mathrm{Pro}(\mathbf{C})$ *is complete with respect to cofiltered limits.* ■

The construction of $\mathrm{Pro}(\mathbf{C})$ can be seen as "freely adjoining all cofiltered limits to $\mathbf{C}$". It is therefore not surprising that it can be characterized by the following universal property.

Proposition 2.31. *For any category $\mathbf{C}$ and any category $\mathbf{D}$ that is complete with respect to cofiltered limits, there exists a natural equivalence between functors $\mathbf{C} \to \mathbf{D}$ and functors $\mathrm{Pro}(\mathbf{C}) \to \mathbf{D}$ that preserve cofiltered limits.*

Proof. Let a functor $F\colon \mathbf{C} \to \mathbf{D}$ be given. We will construct a functor $\widetilde{F}\colon \mathrm{Pro}(\mathbf{C}) \to \mathbf{D}$ making the diagram

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "F"] \arrow[d, "\iota"'] & \mathbf{D} \\
\mathrm{Pro}(\mathbf{C}) \arrow[ur, "\widetilde{F}"'] &
\end{tikzcd}$$

commute. For any pro-object $X = X_{ii\in I}$ in $\mathbf{C}$, set $\widetilde{F}(X) = \lim_{i\in I} F(X_i)$. This defines a functor $\widetilde{F}\colon \mathrm{Pro}(\mathbf{C}) \to \mathbf{D}$ satisfying $F = \widetilde{F}\iota$. It preserves cofiltered limits of representables by construction. By an argument similar to the proof of Theorem 2.30, i.e. by writing a cofiltered limit in $\mathrm{Pro}(\mathbf{C})$ as a cofiltered limit of representables, we see that it preserves all cofiltered limits.

To obtain a functor $F\colon \mathbf{C} \to \mathbf{D}$ from a functor $\widetilde{F}\colon \mathrm{Pro}(\mathbf{C}) \to \mathbf{D}$, of course define $F = \widetilde{F}\iota$.

If we are given another functor $\widetilde{F}'$ also extending F, then we obtain a natural isomorphism $\widetilde{F}' \to \widetilde{F}$ from the fact that limits are unique up to a canonical isomorphism. This shows that we have a natural equivalence between functors $\mathbf{C} \to \mathbf{D}$ and functors $\mathrm{Pro}(\mathbf{C}) \to \mathbf{D}$ preserving cofiltered limits. ∎

Corollary 2.32. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor. Then F induces a cofiltered limit-preserving functor $\widetilde{F}\colon \mathrm{Pro}(\mathbf{C}) \to \mathrm{Pro}(\mathbf{D})$. Furthermore, a fully faithful functor induces a fully faithful one, adjoints induce adjoints, and equivalences induce equivalences.*

Proof. We define $\widetilde{F}(\{C_i\}_i) = \{F(C_i)\}_i$. Note that $\widetilde{F}\colon \mathrm{Pro}(\mathbf{C}) \to \mathrm{Pro}(\mathbf{D})$ is the functor corresponding to $\iota \circ F\colon \mathbf{C} \to \mathrm{Pro}(\mathbf{D})$ in the universal property of $\iota\colon \mathbf{C} \to \mathrm{Pro}(\mathbf{C})$ proved above. In particular, it preserves cofiltered limits. One can easily see that $\widetilde{\mathrm{id}}_{\mathbf{C}} = \mathrm{id}_{\mathrm{Pro}(\mathbf{C})}$ and $\widetilde{F}\widetilde{G} = \widetilde{FG}$.

For the case where $F\colon \mathbf{C} \to \mathbf{D}$ is an equivalence with inverse G, note that the natural isomorphisms $GF \simeq \mathrm{id}_{\mathbf{C}}$ and $FG \simeq \mathrm{id}_{\mathbf{D}}$ induce natural isomorphisms $\widetilde{G}\widetilde{F} \simeq \mathrm{id}_{\mathrm{Pro}(\mathbf{C})}$ and $\widetilde{F}\widetilde{G} \simeq \mathrm{id}_{\mathrm{Pro}(\mathbf{D})}$. One can also see that adjoints are sent to adjoints by a similar argument, noting that the unit and counit of $F \dashv G$ induce a unit and counit for $\widetilde{F} \dashv \widetilde{G}$. This can also be seen directly. If we assume $F \dashv G$, then

$$\begin{aligned}
\mathrm{Hom}(\widetilde{F}\{C_i\}, \{D_j\}) &= \lim_j \operatorname*{colim}_i \mathrm{Hom}(FC_i, D_j) \cong \lim_j \operatorname*{colim}_i \mathrm{Hom}(C_i, GD_j) \\
&= \mathrm{Hom}(\{C_i\}, \widetilde{G}\{D_j\}),
\end{aligned}$$

so $\widetilde{F} \dashv \widetilde{G}$.

If F is fully faithful, note that

$$\begin{aligned}
\mathrm{Hom}(\widetilde{F}\{C_i\}, \widetilde{F}\{C'_j\}) &= \lim_j \operatorname*{colim}_i \mathrm{Hom}(FC_i, FC'_j) \cong \lim_j \operatorname*{colim}_i \mathrm{Hom}(C_i, C'_j) \\
&= \mathrm{Hom}(\{C_i\}, \{C'_j\})
\end{aligned}$$

so $\widetilde{F}$ is fully faithful. ■

As an application of the above proposition, consider the functor $\mathrm{Hom}_{\mathbf{C}}(-,C)\colon \mathbf{C} \to \mathbf{Set}^{op}$ for any category $\mathbf{C}$ and any object $C \in \mathbf{C}$. Its essentially unique extension, in the sense of Proposition 2.31, is the functor $\mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}(-,C)\colon \mathrm{Pro}(\mathbf{C}) \to \mathbf{Set}^{op}$ which takes cofiltered limits to cofiltered limits in $\mathbf{Set}^{op}$, which are filtered colimits in $\mathbf{Set}$. We therefore see that $\mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}(\lim_i X_i, Y) = \mathrm{colim}_i \mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}(X_i, Y)$ if Y is a pro-object in the image of $\mathbf{C} \hookrightarrow \mathrm{Pro}(\mathbf{C})$. Such objects can be thought of as "small" in a certain sense.

Definition 2.33. Let X be an object in a category $\mathbf{C}$ with all filtered colimits. X is *compact* if, for any filtered colimit $\mathrm{colim}_i Y_i$ in $\mathbf{C}$, the canonical map

$$\operatorname*{colim}_i \mathrm{Hom}_{\mathbf{C}}(X, Y_i) \to \mathrm{Hom}_{\mathbf{C}}(X, \operatorname*{colim}_i Y_i)$$

is a bijection. Dually, if $\mathbf{C}$ has all cofiltered limits, then we say that X is *cocompact* if, for any cofiltered limits $\lim_i Y_i$ in $\mathbf{C}$, the canonical map

$$\operatorname*{colim}_i \mathrm{Hom}_{\mathbf{C}}(Y_i, X) \to \mathrm{Hom}_{\mathbf{C}}(\lim_i Y_i, X)$$

is a bijection. ♢

Example 2.34. In $\mathbf{Set}$, the compact objects are finite sets. In fact, in many categories those objects which are "finitely presented" in some sense are the compact objects. If we consider the category of topological spaces, and restrict our attention to cofiltered limits of "sufficiently nice" inclusions (for example inclusions of sub-CW-complexes) instead of all maps, then compact topological spaces are compact in the above sense. In the case of sub-CW-complexes, this follows from the fact that compact subsets of CW-complexes are contained in a finite subcomplex. ♢

Remark 2.35. By what we saw above, representables in ind-categories are compact, and representables in pro-categories are cocompact. In general, these need not be all the compact or cocompact objects of an ind-category or pro-category. However, one can show that if a pro-object is isomorphic to one where all arrows are epimorphisms (which is often the case), then it is cocompact if and only if it is isomorphic to a pro-object in the image of $\iota\colon \mathbf{C} \to \mathrm{Pro}(\mathbf{C})$. ♢

Recall that $\mathbf{Set}^{\mathbf{C}^{op}}$ is the free cocompletion of $\mathbf{C}$. By duality of limits and colimits, we see that $(\mathbf{Set}^{\mathbf{C}})^{op}$ is the free completion of $\mathbf{C}$, where the inclusion $\mathbf{C} \to (\mathbf{Set}^{\mathbf{C}})^{op}$ comes from the Yoneda embedding $\mathbf{C}^{op} \to \mathbf{Set}^{\mathbf{C}}$. This means that there is a natural equivalence between functors $\mathbf{C} \to \mathbf{D}$ and functors $(\mathbf{Set}^{\mathbf{C}})^{op} \to \mathbf{D}$ if $\mathbf{D}$ is complete. From this one can deduce that $\mathrm{Pro}(\mathbf{C})$ is contained as a full subcategory of $(\mathbf{Set}^{\mathbf{C}})^{op}$.

Proposition 2.36. $\mathrm{Pro}(\mathbf{C})$ *is equivalent to the full subcategory of* $(\mathbf{Set}^{\mathbf{C}})^{op}$ *whose objects are all cofiltered limits of representables.*

Proof. The inclusion $\mathrm{Pro}(\mathbf{C}) \to (\mathbf{Set}^{\mathbf{C}})^{op}$ is obtained by mapping a cofiltered diagram $X\colon I \to \mathbf{C}$ to the corresponding limit of representables in $(\mathbf{Set}^{\mathbf{C}})^{op}$. To see that this is an equivalence onto the full subcategory of cofiltered limits of representables, one can show that this full subcategory also has the universal property of Proposition 2.31. One can also give a direct proof that $\mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}(X,Y) \cong \mathrm{Hom}_{(\mathbf{Set}^{\mathbf{C}})^{op}}(\lim_i y(X_i), \lim_j y(Y_j))$, where $X = \{X_i\}$, $Y = \{Y_j\}$ and y is the (co)Yoneda embedding $\mathbf{C} \to (\mathbf{Set}^{\mathbf{C}})^{op}$. ■

2.1.4 Pro-categories of categories with finite limits

We will now study Pro($\mathbf{C}$) for (small) categories $\mathbf{C}$ with finite limits. We will show that Pro($\mathbf{C}$) is both complete and cocomplete in this case. We also provide a nice description of Pro$(\mathbf{C})^{op}$ as a full subcategory of $\mathbf{Set}^{\mathbf{C}}$, which will be used to define profinite completion in section 2.3.

We can prove the following regarding limits in pro-categories.

Proposition 2.37. *Let* $\mathbf{C}$ *be a category.*

(i) If $\mathbf{C}$ *has finite products, then* Pro($\mathbf{C}$) *also has finite products. Furthermore, the product of* $X = \{X_i\}$ *and* $Y = \{Y_i\}$ *is given by*

$$X \times Y = \{X_i \times Y_j\}_{(i,j)\in I\times J}.$$

(ii) If $\mathbf{C}$ *has pullbacks, then* Pro($\mathbf{C}$) *also has pullbacks.*

(iii) If $\mathbf{C}$ *has all finite limits, then* Pro($\mathbf{C}$) *is complete.*

Proof. The first statement is left to the reader as an exercise.

For the second statement, let the pullback diagram $X \to Z \leftarrow Y$ be given. Then there exists a level representation of this diagram by Proposition 2.27, i.e. an isomorphic diagram $\{X_i\} \to \{Z_i\} \leftarrow \{Y_i\}$ where all objects are indexed by the same index set I, and where the two morphisms come from natural transformations. For such a level representation, it is straightforward to verify that $X \times_Z Y = \{X_i \times_{Z_i} Y_i\}_{i\in I}$. In particular all pullbacks exist in Pro($\mathbf{C}$).

For the third statement, note that Pro($\mathbf{C}$) has finite products and pullbacks by the first two statements. It also has all cofiltered limits by Theorem 2.30. Since arbitrary products can be written as cofiltered limits of finite products (see Theorem IX.1.1 of [Mac88]), and any category with arbitrary products and pullbacks is complete, we conclude that Pro($\mathbf{C}$) is complete. ∎

From the above proof, we also learn the following.

Proposition 2.38. *Let* $\mathbf{C}$ *be a category with finite limits, and let* $\mathbf{D}$ *be a category with all limits. Then the equivalence between functors* $\mathbf{C} \to \mathbf{D}$ *and the functors* Pro$(\mathbf{C}) \to \mathbf{D}$ *which preserve cofiltered limits, restricts to an equivalence between functors* $\mathbf{C} \to \mathbf{D}$ *that preserve finite limits and functors* Pro$(\mathbf{C}) \to \mathbf{D}$ *that preserve all limits.*

Proof. Recall from Proposition 2.31 that the equivalence is given by associating to $F\colon \mathbf{C} \to \mathbf{D}$ the functor $\widetilde{F}\colon \text{Pro}(\mathbf{C}) \to \mathbf{D}$ given by $\widetilde{F}(\{C_i\}) = \lim_i F(C_i)$. Conversely, to a functor $\widetilde{F}\colon \text{Pro}(\mathbf{C}) \to \mathbf{D}$ we associate the functor $\widetilde{F} \circ \iota\colon \mathbf{C} \to \mathbf{D}$, where $\iota\colon \mathbf{C} \to \text{Pro}(\mathbf{C})$ is the inclusion. We see from the construction of pullbacks and products in the above proof that ι preserves finite limits. In particular, if $\widetilde{F}$ preserves all limits, then $\widetilde{F} \circ \iota$ preserves all finite ones.

Conversely, we need to show that if F preserves finite limits, then $\widetilde{F}$ preserves all limits. We first show that $\widetilde{F}$ preserves finite limits. It is sufficient to show that $\widetilde{F}$

preserves pullbacks and finite products. This follows by choosing level representations of the diagrams, noting that finite limits are computed levelwise, and that cofiltered limits commute with finite limits. For example, if $\{X_i\} \to \{Z_i\} \leftarrow \{Y_i\}$ is a level representation for a pullback diagram, then we see that

$$\begin{aligned}\widetilde{F}(\{X_i \times_{Z_i} Y_i\}) &= \lim_i F(X_i \times_{Z_i} Y_i) = \lim_i F(X_i) \times_{F(Z_i)} F(Y_i) \\ &= \lim_i F(X_i) \times_{\lim_i F(Z_i)} \lim_i F(Y_i) = \widetilde{F}(\{X_i\}) \times_{F(\{Z_i\})} F(\{Y_i\}).\end{aligned}$$

Now note that $\widetilde{F}$ also preserves cofiltered limits. We can deduce from this that $\widetilde{F}$ preserves all products, since arbitrary products can be written as cofiltered limits of finite products in a canonical way (see Theorem IX.1.1 of [Mac88]). We conclude that $\widetilde{F}$ preserves all limits, since it preserves finite limits and arbitrary products. ■

For small categories $\mathbf{C}$ with finite limits, the following proposition gives a nice characterization of $\mathrm{Pro}(\mathbf{C})^{op}$ as a full subcategory of $\mathbf{Set}^{\mathbf{C}}$, also found in [GAV72, Théorème I.8.3.3].

Proposition 2.39. *If $\mathbf{C}$ is small and has all finite limits, then a functor $\mathbf{C} \to \mathbf{Set}$ is a filtered colimit of representable functors if and only if it preserves finite limits. In particular it uniquely extends to a representable functor $\mathrm{Pro}(\mathbf{C}) \to \mathbf{Set}$.*

Proof. Note that representable functors $\mathbf{C} \to \mathbf{Set}$ preserve limits, in particular finite limits. By Theorem 2.5 finite limits commute with filtered colimits in $\mathbf{Set}$, hence a filtered colimit of representable functors must preserve finite limits. For the converse we will make use of the fact that any $F\colon \mathbf{C} \to \mathbf{Set}$ is a colimit of representable functors. Recall the basic fact[1] that F is the colimit of the diagram

$$\mathrm{el}(F)^{op} \xrightarrow{U} \mathbf{C}^{op} \xrightarrow{Y} \mathbf{Set}^{\mathbf{C}}.$$

Here $\mathrm{el}(F)$ is the category of elements of F, whose objects are pairs (C, x) with $C \in \mathbf{C}$ and $x \in F(C)$, and whose arrows $(C, x) \to (C', x')$ are all arrows $f\colon C \to C'$ for which $F(f)(x) = x'$. The functor U is the forgetful functor mapping (C, x) to C, and Y is the Yoneda embedding. To see that F is in fact a filtered colimit of representables, we will show that $\mathrm{el}(F)^{op}$ is a filtered category, or equivalently that $\mathrm{el}(F)$ is cofiltered. Assume we are given (C, x) and (C', x') in $\mathrm{el}(F)$. Since $\mathbf{C}$ has all finite limits and F preserves these, we see that there is a unique element $y \in F(C \times C') \cong F(C) \times F(C')$ such that $F(p_1)(y) = x$ and $F(p_2)(y) = x'$. Here p_1 and p_2 are the projections of $C \times C'$ onto C and C' respectively. In particular there are arrows $(C \times C', y) \to (C, x)$ and $(C \times C', y) \to (C', x')$, so property (i) of Definition 2.2 holds. For property (ii) the proof is similar. If we are given two arrows $(C, x) \to (C', x')$, then these come from two arrows $f, g\colon C \to C'$. By constructing their equalizer in $\mathbf{C}$ and using the fact that F preserves equalizers, property (ii) follows.

[1] This fact, or actually its dual, usually goes under the name that "every presheaf is a colimit of representables".

For the last statement, note that

$$F = \operatorname*{colim}_{(C,x)\in \mathrm{el}(F)^{op}} YU(C,x) = \operatorname*{colim}_{\mathrm{el}(F)^{op}} \mathrm{Hom}_{\mathbf{C}}(C,-) \cong \mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}\left(\lim_{\mathrm{el}(F)} C, -\right),$$

so F is represented by $\{C\}_{(C,x)\in\mathrm{el}(F)} = \lim_{\mathrm{el}(F)} C$ in $\mathrm{Pro}(\mathbf{C})$. This pro-object is unique up to canonical isomorphism. To see this, let $\{C_i\}$ and $\{D_j\}$ be pro-objects of $\mathbf{C}$, both representing a functor that extends F. Then we have natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}(\{C_i\},\{A_k\}) \cong \lim_k \mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}(\{C_i\}, A_k) \cong \lim_k F(A_k),$$

and similarly $\mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}(\{D_i\},\{A_k\}) \cong \lim_k F(A_k)$. This means that $\{C_i\}$ and $\{D_i\}$ represent the same functor $\mathrm{Pro}(\mathbf{C}) \to \mathbf{Set}$, so by Yoneda's lemma they are canonically isomorphic. ■

As we saw above, $\mathrm{Pro}(\mathbf{C})^{op}$ is the full subcategory of $\mathbf{Set}^{\mathbf{C}}$ whose objects are filtered colimits of representables. By the above theorem, this is equivalent to saying that $\mathrm{Pro}(\mathbf{C})^{op}$ is the full subcategory of $\mathbf{Set}^{\mathbf{C}}$ with as objects the functors $\mathbf{C} \to \mathbf{Set}$ which preserve all finite limits. This allows for a simple proof of the cocompleteness of $\mathrm{Pro}(\mathbf{C})$. Given two categories $\mathbf{C}$ and $\mathbf{D}$ with finite limits, we denote the category of functors $\mathbf{C} \to \mathbf{D}$ that preserve finite limits by $\mathrm{Fun}_L(\mathbf{C},\mathbf{D})$.

Theorem 2.40. *Let $\mathbf{C}$ be a small category with finite limits. Then $\mathrm{Pro}(\mathbf{C})$ is complete and cocomplete.*

Proof. We already proved completeness in Proposition 2.37. Cocompleteness of $\mathrm{Pro}(\mathbf{C})$ is equivalent to completeness of $\mathrm{Pro}(\mathbf{C})^{op}$. Identify $\mathrm{Pro}(\mathbf{C})^{op}$ with the full subcategory $\mathrm{Fun}_L(\mathbf{C},\mathbf{D})$ of $\mathbf{Set}^{\mathbf{C}}$. As limits commute with finite limits, and limits in $\mathbf{Set}^{\mathbf{C}}$ are computed pointwise, we see that for a diagram $I \to \mathrm{Fun}_L(\mathbf{C},\mathbf{Set})$, its limit in $\mathbf{Set}^{\mathbf{C}}$ is again in $\mathrm{Fun}_L(\mathbf{C},\mathbf{Set})$. This in particular implies that $\mathrm{Fun}_L(\mathbf{C},\mathbf{Set})$ is complete. ■

One can also use the identification of $\mathrm{Pro}(\mathbf{C})^{op}$ with $\mathrm{Fun}_L(\mathbf{C},\mathbf{Set})$ to show that $\mathrm{Pro}(\mathbf{C})^I \simeq \mathrm{Pro}(\mathbf{C}^I)$ for any finite diagram I if $\mathbf{C}$ is small and has all finite limits, as is done in [Mey80, §4]. In section 2.4, we will use this result to study pro-categories of diagram categories where the index category I has finite Hom-sets, but its set of objects may be infinite. We will prove that $\widehat{\mathbf{S}} = \mathbf{s}(\widehat{\mathbf{Set}})$ is equivalent to $\mathrm{Pro}(\mathbf{S}_{cofin})$ for a certain full subcategory $\mathbf{S}_{cofin}$ of $\mathbf{sFinSet}$.

Corollary 2.41. *If $\mathbf{C}$ is small and has finite limits, then cofiltered limits and finite colimits commute in $\mathrm{Pro}(\mathbf{C})$.*

Proof. This is dual to saying that filtered colimits and finite limits commute in $\mathrm{Pro}(\mathbf{C})^{op}$. Note that filtered colimits and finite limits commute in $\mathbf{Set}$ by Theorem 2.5, so this also holds in $\mathbf{Set}^{\mathbf{C}}$. The inclusion $\mathrm{Pro}(\mathbf{C})^{op} \to \mathbf{Set}^{\mathbf{C}}$ preserves filtered colimits by construction, and it preserves all limits by the above proof. It therefore follows that cofiltered limits and finite limits also commute in $\mathrm{Pro}(\mathbf{C})^{op}$. ■

Remark 2.42. The above theorem in fact holds for any category $\mathbf{C}$, see Theorem 6.1 of [Isa01]. The above proof will not work for any category $\mathbf{C}$, however, since it is necessary that the inclusion $\mathrm{Pro}(\mathbf{C})^{op} \to \mathbf{Set}^{\mathbf{C}}$ preserves finite limits, for which we use Proposition 2.39. ♢

2.2 Profinite sets and Stone spaces

In this section we will prove some useful results on projective diagrams of finite sets. We will also give a direct proof that $\widehat{\mathbf{Set}}$ is equivalent to **Stone**, without using Stone duality, and similarly show that $\widehat{\mathbf{Grp}}$ is equivalent to **StoneGrp**, the category of group objects in **Stone**. We will then prove a similar result for profinite G-sets, and deduce some of their basic properties which we need in the subsequent chapters.

2.2.1 Cofiltered limits of finite sets

The following theorem is an essential ingredient in many proofs involving projective limits of finite sets. The proof is taken from [RZ10], Proposition 1.1.4.

Theorem 2.43. *A cofiltered limit of finite nonempty sets is nonempty.*

Proof. As we have already seen, any cofiltered diagram can be replaced by a projective diagram indexed by a codirected poset. So let $\{X_i\}_{i\in I}$ be a projective diagram of nonempty finite sets, with maps $p^i_j\colon X_i \to X_j$ for $i \leq j$. Give all sets X_i the discrete topology. Then $\lim_i X_i$ is a compact space by Tychonoff's theorem. For each $j \in I$, let $Y_j \subseteq \prod_i X_i$ be the set consisting of all $(x_i)_{i\in I}$ such that if $j \leq k$, then $p^j_k(x_j) = x_k$. Then clearly $\lim_i X_i = \bigcap_i Y_i$. Using the axiom of choice, we see that Y_j is nonempty for every j (since products of nonempty sets are nonempty). One can also easily verify that it is closed. Now assume $\bigcap_i Y_i = \emptyset$. By compactness, this implies that $Y_{i_1} \cap \ldots \cap Y_{i_n} = \emptyset$ for some $i_1, \ldots, i_n \in I$. Pick $j \leq i_1, \ldots, i_n$. Then $Y_j \subseteq Y_{i_1} \cap \ldots \cap Y_{i_n}$ by construction, which is a contradiction since $Y_j \neq \emptyset$. We conclude that $\lim_i X_i = \bigcap_i Y_i \neq \emptyset$. ■

Remark 2.44. The above proof more generally works for any projective limit of nonempty compact Hausdorff spaces. ♢

The Mittag-Leffler condition is a condition on a projective diagram, which is often considered in the context of abelian groups. Its interest comes from the fact that it ensures that the derived functor $\lim^1$ vanishes. We will consider the condition for diagrams of sets and show that it holds for all diagrams of finite sets.

Definition 2.45. Let I be a codirected set and let $\{X_i\}_{i\in I}$ be a projective diagram of sets with maps $p^j_i\colon X_j \to X_i$. We say that $\{X_i\}_{i\in I}$ satisfies the *Mittag-Leffler condition* if for every $i \in I$ there exists a $j \leq i$ such that

$$\bigcap_{k\leq i} p^k_i(X_k) = p^j_i(X_j). \qquad ♢$$

Remark 2.46. The Mittag-Leffler condition can be defined in any category with a suitable notion of "image" and "intersection of images". However, doing this here would needlessly complicate matters. ♢

Proposition 2.47. *Any projective diagram of finite sets $\{X_i\}$ satisfies the Mittag-Leffler condition. Furthermore,* $\operatorname{im}(p_i) = \bigcap_{k\leq i} p^k_i(X_k)$ *for each $i \in I$, where $p_i\colon \lim_i X_i \to X_i$ is the projection map. In particular* $\operatorname{im}(p_i) = p^j_i(X_k)$ *for some $j \leq i$.*

Proof. Let $\{X_i\}_{i\in I}$ be a diagram of finite sets with maps $p_i^j\colon X_j \to X_i$, and let $i \in I$ be given. Let $\{x_1, \dots, x_n\} = X_i \setminus \bigcap_{k\leq i} p_i^k(X_k)$. Then there are $j_1, \dots, j_n \leq i$ such that $x_m \notin p_i^{j_m}(X_{j_m})$ for each $1 \leq m \leq n$. Let $j \leq j_1, \dots, j_n$. Then $x_m \notin p_i^j(X_k)$ for each m, hence $p_i^j(X_i) = \bigcap_{k\leq i} p_i^k(X_i)$.

For the second part of the proposition, note that $p_i(\lim_i X_i) \subseteq \bigcap_{k\leq i} p_i^k(X_k)$ trivially. For the other direction, let $x \in \bigcap_{k\leq i} p_i^k(X_k)$. Note that the inclusion of $I_{\leq i} = \{j \in I \mid j \leq i\}$ in I is initial, so $\lim_{j\leq i} X_j = \lim_{j\in I} X_j$. Now define $Y_j = (p_i^j)^{-1}(\{x\})$ for each $j \leq i$. Then the Y_j are all nonempty, and the maps $p_j^k|_{Y_k}$ make $\{Y_j\}_{j\leq i}$ into a projective diagram. By Theorem 2.43, $\lim_{j\leq i} Y_j$ is nonempty, and we also have $\lim_{j\leq i} Y_j \subseteq \lim_{j\leq i} X_j = \lim_{j\in I} X_j$. An element $y \in \lim_{j\leq i} Y_j$ corresponds to an element $x' \in \lim_i X_i$ with $p_i(x') = x$ by construction, hence $x \in p_i(\lim_i X_i)$. ■

Remark 2.48. Since the image of a group homomorphism is simply the set theoretic image, we see that the above proposition also holds for diagrams of finite groups. ♢

Corollary 2.49. *A projective diagram of finite sets or finite groups is isomorphic to a projective diagram where all maps are surjective, indexed by the same codirected set.*

Proof. Let $\{X_i\}$ be a projective diagram of sets or groups, with maps $p_i^j\colon X_j \to X_i$ for $j \leq i$. Then set $X_i' := \bigcap_{k\leq i} p_i^k(X_k)$. The maps p_i^j can be restricted to maps $X_j' \to X_i'$, making $\{X_i\}$ into a projective diagram indexed by the same codirected set. Using the Mittag-Leffler condition, one can show that the maps $X_j' \to X_i'$ are surjective and that the levelwise inclusion of $\{X_i'\}$ into $\{X_i\}$ is an isomorphism of pro-objects. ■

2.2.2 Profinite sets and profinite groups

We will now prove that $\widehat{\mathbf{Set}} \simeq \mathbf{Stone}$. The following proposition is useful when trying to recognize if a category is a certain pro-category. Recall from Definition 2.33 that an object X in a category $\mathbf{C}$ that has cofiltered limits is called cocompact if $\operatorname{Hom}(\lim_i Y_i, X) = \operatorname{colim}_i \operatorname{Hom}(Y_i, X)$ for any cofiltered limit $\lim_i Y_i$.

Proposition 2.50. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor with $\mathbf{D}$ cofiltered complete, and assume that*

(i) F is fully faithful,

(ii) every $D \in \mathbf{D}$ is the limit of a cofiltered diagram of the form $I \longrightarrow \mathbf{C} \xrightarrow{F} \mathbf{D}$, and

(iii) $F(C)$ is cocompact in $\mathbf{D}$ for every $C \in \mathbf{C}$.

Then the induced functor $\widetilde{F}\colon \operatorname{Pro}(\mathbf{C}) \to \mathbf{D}$ is an equivalence of categories.

Proof. To see that $\widetilde{F}$ is essentially surjective, note that every $D \in \mathbf{D}$ can be written as a cofiltered limit $\lim_i F(X_i)$, hence $D \cong \widetilde{F}(\{X_i\})$.

Recall that for a pro-object $\{X_i\}$, $\widetilde{F}$ is defined by $\widetilde{F}(\{X_i\}) = \lim_i F(X_i)$. Considering that every object of the form $F(C)$ is cocompact, we see that

$$\begin{aligned}\mathrm{Hom}_{\mathbf{D}}(\widetilde{F}(\{X_i\}), \widetilde{F}(\{Y_j\})) &\cong \mathrm{Hom}_{\mathbf{D}}(\lim_i F(X_i), \lim_j F(Y_i)) \cong \lim_j \mathrm{Hom}_{\mathbf{D}}(\lim_i F(X_i), F(Y_i)) \\ &\cong \lim_j \operatorname*{colim}_i \mathrm{Hom}_{\mathbf{D}}(F(X_i), F(Y_i)) \cong \lim_j \operatorname*{colim}_i \mathrm{Hom}_{\mathbf{C}}(X_i, Y_i),\end{aligned}$$

so $\widetilde{F}$ is fully faithful. ■

Remark 2.51. In the above proof, we only need that $F(C)$ is cocompact in $\mathbf{D}$ with respect to projective limits of objects of the form $F(C_i')$. In particular, to apply the above lemma, we only need to prove that $\mathrm{colim}_i \mathrm{Hom}_{\mathbf{D}}(F(C_i'), F(C)) = \mathrm{Hom}_{\mathbf{D}}(\lim_i F(C_i'), F(C))$ for projective diagrams $\{C_i'\}$ in $\mathbf{C}$. ♢

From Example 2.16, we already know that a projective limit $\lim_i X_i$ of finite discrete spaces is a Stone space. This induces a functor $F\colon \widehat{\mathbf{Set}} \to \mathbf{Stone}$, which computes the limit of a pro-object of finite sets as topological space, where all finite sets are given the discrete topology.

Theorem 2.52. *The functor $F\colon \widehat{\mathbf{Set}} \to \mathbf{Stone}$ is an equivalence of categories.*

Proof. We need to verify that the assumption of Proposition 2.50 holds for the inclusion $\mathbf{FinSet} \to \mathbf{Stone}$. The proof that a projective limit of Stone spaces is again a Stone space (which implies that $\mathbf{Stone}$ is cofiltered complete) is left to the reader.

(i) It is clear that $\mathbf{FinSet} \to \mathbf{Stone}$ is full and faithful.

(ii) We need to show that any Stone space X is a projective limit of finite sets. Define $\mathcal{R}(X)$ to be the set of all equivalence relations on X such that each equivalence class is open in X, ordered by inclusion. By compactness, for each $R \in \mathcal{R}(X)$, X/R is finite and discrete. The inclusion of an equivalence relation $R \subseteq R'$ induces a map $X/R \to X/R'$. The finite sets X/R therefore define a projective diagram in $\mathbf{FinSet}$. By the universal property of limits, the quotient maps $X \to X/R$ induce a continuous map $f\colon X \to \lim_{R\in\mathcal{R}(X)} X/R$. Since all spaces are compact Hausdorff, this map is a homeomorphism if and only if it is bijective.

For injectivity, let $x \neq y$ be elements of X. Then there is a clopen $U \subseteq X$ such that $x \in U$ and $y \notin U$. Let R be the equivalence relation on X with equivalence classes U and $X \setminus U$. Then $R \in \mathcal{R}(X)$, and we have $q_R = p_R \circ f$, where $q_R\colon X \to X/R$ is the quotient and $p_R\colon \lim_{R'\in\mathcal{R}(X)} X/R' \to X/R$ the projection. Since $q_R(x) \neq q_R(y)$, we see that $f(x) \neq f(y)$, so injectivity follows.

For surjectivity, let $x \in \lim X/R$. Let $p_R(x)$ denote the equivalence class corresponding to x in X/R. Then $p_R(x)$ is a closed subset of X, and $f(x') = x$ if and only if $x' \in \bigcap_{R\in\mathcal{R}(X)} p_R(x)$. We therefore need to prove that $\bigcap_{R\in\mathcal{R}(X)} p_R(x) \neq \emptyset$. Assume that this intersection is empty. Then, by compactness, there are $R_1, \dots, R_n \in \mathcal{R}(X)$ such that $p_{R_1}(x) \cap \dots \cap p_{R_n}(x) = \emptyset$. Choose an $R \in \mathcal{R}(X)$ such that $R \subseteq R_1, \dots, R_n$. Then $p_R(x) \subseteq p_{R_1}(x) \cap \dots \cap p_{R_n}(x)$, yet $p_R(x) \neq \emptyset$ by definition. We conclude that f is surjective.

(iii) To see that finite discrete sets are cocompact in **Stone**, note that we only need to show that the canonical map $\operatorname{colim}_i \operatorname{Hom}(X_i, Y) \to \operatorname{Hom}(\lim_i X_i, Y)$ is a bijection for $\{X_i\}$ a projective diagram of finite sets and Y a finite set. Write $X = \lim X_i$, let $p_i \colon X \to X_i$ be the projection map and denote the maps $X_j \to X_i$ for $j \leq i$ by p_i^j.

We will first prove surjectivity. Assume we are given a continuous map $f \colon X \to Y$. Note that, by definition of the product topology, an open $U \subseteq X$ contains a subset of the form $(\prod_i Z_i) \cap X$, where $Z_i \subseteq X_i$ for all i and $Z_i = X_i$ for all but finitely many i. Since I is codirected and we are intersecting $\prod_i Z_i$ by X, we can assume that $Z_i = X_i$ for all but one i. Now let $y \in Y$ be given, then $f^{-1}(y)$ contains a subset of the form $\prod_i Z_i \cap X$ with $Z_i = X_i$ for all i except some $i_y \in I$. Since Y is finite, we can pick $j \in I$ such that $j \leq i_y$ for all y. Then there is a map $f_j \colon p_j(X) \to Y$ such that $f_j p_j = f$. Now use Proposition 2.47 to pick a $k \leq j$ such that $\operatorname{im}(p_j) = \operatorname{im}(p_j^k)$ and define $f_k \colon X_k \to Y$ by $f_k = f_j p_j^k$. Then $f_k p_k = f$.

For injectivity, assume we are given $f \colon X_i \to Y$ and $f' \colon X_j \to Y$ such that $f p_i = f' p_j$. We need to show that $f p_i^l = f' p_j^l$ for some $l \in I$, as this means that f and f' represent the same element in $\operatorname{colim}_i \operatorname{Hom}(X_i, Y)$. Since there is a k such that $k \leq i$ and $k \leq j$, we may assume without loss of generality that $i = j = k$. Now denote by X_k' the set $\{x \in X_k \mid f(x) \neq f'(x)\}$. There can be no $y \in X$ such that $p_k(y) \in X_k'$, since then $f(p_k(y)) = f'(p_k(y))$ by definition of f and f'. Pick $l \leq k$ such that $\operatorname{im}(p_k^l) = \operatorname{im}(p_k)$. Then, as f and f' are obviously equal on $\operatorname{im}(p_k)$, we see that $f p_k^l = f' p_k^l$, hence f and f' represent the same element in $\operatorname{colim}_i \operatorname{Hom}(X_i, Y)$. ■

The above proof can be modified to obtain the well-known similar result for profinite groups. We have a similar functor $F \colon \widehat{\mathbf{Grp}} \to \mathbf{StoneGrp}$ coming from the inclusion **FinGrp** $\to$ **StoneGrp**. One problem occurs when translating the above proof to the context of profinite groups. We will, for now, call a topological group that is compact Hausdorff and totally disconnected a *Stone group*. To show that any Stone group is a projective limit of finite discrete groups, we need to consider equivalence relations on a Stone group G for which the quotient is a finite discrete group. One easily sees that such equivalence relations correspond to open normal subgroups. We will therefore need to show that the canonical map

$$f \colon G \to \lim_{\substack{N \trianglelefteq G \\ N \text{ open}}} G/N$$

is an isomorphism if we want to prove that any Stone group is a projective limit of finite discrete groups. Surjectivity follows in the same way as above, but injectivity is slightly more complex, since Stone groups have fewer quotients than Stone spaces. Injectivity essentially follows from the following lemma, which states that there exist "enough" open normal subgroups.

Lemma 2.53. *Let G be a Stone group. Then*

$$\bigcap_{\substack{N \trianglelefteq G \\ N \text{ open}}} N = \{1\}.$$

For a proof, see [RZ10], Theorem 2.1.3.

Theorem 2.54. *The functor* $F\colon \widehat{\mathbf{Grp}} \to \mathbf{StoneGrp}$ *is an equivalence of categories.*

Proof. We again check the assumption of Proposition 2.50. To see that **StoneGrp** is cofiltered complete, one can simply show that a projective limit of Stone groups is again a Stone group.

(i) **FinGrp** $\to$ **StoneGrp** is clearly fully faithful.

(ii) We need to show that the canonical map $f\colon G \to \lim_N G/N$ is an isomorphism. It is surjective by the exact same proof as above. For injectivity, assume that $g \in \ker(f)$. Then by definition $g \in N$ for each open normal subgroup N of G, so by the above theorem $g = 1$, the identity element, hence f is injective. We conclude that f is a bijection, hence an isomorphism of Stone groups.

(iii) For cocompactness of finite discrete groups, let $\{G_i\}$ be a projective diagram of finite groups and H a finite group. The proof that $\operatorname{colim}_i \operatorname{Hom}(G_i, H) \to \operatorname{Hom}(\lim_i G_i, H)$ is an isomorphism is exactly the same as in the above proof for finite sets and Stone spaces. The only problem is that it is not immediately clear that, given a continuous map $f\colon \lim_i G_i \to H$, the obtained map $f_k\colon G_k \to H$ is a group homomorphism. Considering that k was chosen such that $\operatorname{im}(p_j^k) = \operatorname{im}(p_j)$ in the proof of Theorem 2.50, this is however not difficult to verify, and is therefore left to the reader. ■

One easily sees that the above equivalence restricts to an equivalence $\widehat{\mathbf{Ab}} \simeq \mathbf{StoneAb}$, where $\widehat{\mathbf{Ab}}$ denotes $\operatorname{Pro}(\mathbf{FinAb})$.

Corollary 2.55. *The functor* $F\colon \widehat{\mathbf{Ab}} \to \mathbf{StoneAb}$ *is an equivalence of categories.* ■

2.2.3 Profinite G-sets and profinite G-modules

In chapters 3 and 4, we will encounter continuous actions by profinite groups. In particular, we will need some results on profinite G-sets. By a *profinite G-set* we mean a profinite set X together with an action of G on X such that the map $G \times X \to X$ is continuous. Similarly, by a *profinite G-module*, we mean a profinite abelian group A together with an action of G on A such that the map $G \times A \to A$ is continuous. We will consider left-G-actions below, but of course everything proved will also hold for right-G-actions.

We denote the category of profinite G-sets by $\widehat{\mathbf{Set}}_G$ and the category of profinite G-modules by $\widehat{\mathbf{Ab}}_G$. By a finite G-set (resp. G-module) we mean a profinite G-set (resp. G-module) such that the underlying set is finite. The corresponding categories are denoted by $\mathbf{FinSet}_G$ and $\mathbf{FinAb}_G$. Then $\widehat{\mathbf{Set}}_G \simeq \operatorname{Pro}(\mathbf{FinSet}_G)$ and $\widehat{\mathbf{Ab}}_G \simeq \operatorname{Pro}(\mathbf{FinAb}_G)$.

Theorem 2.56. *For any profinite group G, there is an equivalence of categories* $\widehat{\mathbf{Set}}_G \simeq \operatorname{Pro}(\mathbf{FinSet}_G)$.

Proof. The proof follows the same steps as the proof of Theorem 2.54. Parts (i) and (iii) can both be translated directly to the context of profinite G-sets. Part (ii), showing that every profinite G-set is a projective limit of finite G-sets, requires some further work. We will first look at the case where G is a finite group.

If G is a finite group, then a profinite G-set X is simply a profinite set together with a group homomorphism $G \to \mathrm{Aut}(X)$. Since G is discrete, the corresponding map $G \times X \to X$ will always be continuous, so we can drop this requirement. If we view G as a category (or groupoid, in fact) with one object, then a homomorphism $G \to \mathrm{Aut}(X)$ is the same as a functor $G \to \widehat{\mathbf{Set}}$ which maps the object of G to X. We therefore see that $\widehat{\mathbf{Set}}_G$ is just the functor category $\mathrm{Pro}(\mathbf{FinSet})^G$. It is proved in [Mey80, §4], stated as Proposition 2.75 of this thesis, that for a finite category I and an (essentially) small category $\mathbf{C}$ with finite limits, there is an equivalence $\mathrm{Pro}(\mathbf{C}^I) \to \mathrm{Pro}(\mathbf{C})^I$. Applying this to the case $\mathbf{C} = \mathbf{FinSet}$ and $I = G$, we obtain the desired equivalence $\mathrm{Pro}(\mathbf{FinSet}_G) \simeq \widehat{\mathbf{Set}}_G$. In particular, any profinite G-set is a projective limit of finite G-sets when G is a finite group.

Now assume that G is profinite. As stated above, we only need to show that a profinite G-set is a projective limit of finite G-sets. Let X be a profinite G-set. We will show that the canonical map $X \to \lim_R X/R$, where R ranges over all G-equivariant equivalence relations on X for which X/R is finite and discrete, is an isomorphism. By a G-equivariant equivalence relation on X, we mean that xRy if and only if $(g \cdot x)R(g \cdot y)$, which is the necessary condition to give X/R an induced G-action. This canonical map is surjective by the same argument as in Theorem 2.52. To see that it is injective, let $x, y \in X$ be two distinct points. We need to show that there exists a map of G-sets $f \colon X \to Y$ such that Y is a finite G-set and $f(x) \neq f(y)$. The collection of $g \in G$ satisfying $g \cdot x = y$ is a closed (possibly empty) subset of G, not containing the unit 1. In the proof of [RZ10], Theorem 2.1.3, it is shown that for a profinite group G, any open neighborhood of 1 contains an open normal subgroup H. In particular, there exists an open normal subgroup H such that no $h \in H$ satisfies $h \cdot x = y$. Then x and y remain distinct in the quotient X/H. The action of G on X/H factors through the finite group G/H, i.e. X/H is a profinite G/H-set. Then, since G/H is finite, we can write X/H as a projective limit of finite G/H-sets as we saw above. In particular there is a map $X/H \to Y$ for some finite G/H-module Y, such that x and y are mapped to different elements in Y. Composing the action of G/H on Y with the quotient map $G \to G/H$ gives Y the structure of a G-set, and the composition $X \to X/H \to Y$ is the desired map of G-sets. ■

The following theorem follows by combining Lemma 5.1.1(b) and Proposition 5.3.6(c) of [RZ10], to which we refer the reader for a proof.

Theorem 2.57. *For any profinite group G, there is an equivalence of categories $\widehat{\mathbf{Ab}}_G \simeq \mathrm{Pro}(\mathbf{FinAb}_G)$.*

This theorem in particular implies that for any profinite G-module M, the canonical map $M \to \lim_N M/N$, where N ranges over all open G-submodules of M, is an isomorphism.

We will now prove some results on profinite G-sets that are used in chapters 3 and 4.

Lemma 2.58. *Let S be a profinite G-set for a profinite group G. Then the quotient S/G, computed in* **Top***, is a Stone space.*

Proof. We view S as a Stone space, on which G acts continuously. We will first show that S/G is Hausdorff. This is equivalent to showing that the diagonal $\Delta_{S/G} \subseteq S/G \times S/G$

is closed. Note that the quotient map $\pi\colon S \to S/G$ is open, since for any $U \subseteq S$ we have $\pi^{-1}(\pi(U)) = \bigcup_{g\in G} g(U)$, which is open. The product of the quotient map $S \times S \to S/G \times S/G$ is therefore also open. The inverse image of $\Delta_{S/G}$ is given by

$$R := \{(s, g \cdot s) \mid s \in S \text{ and } g \in G\}.$$

This is the image of $S \times G \to S \times S$, $(s, g) \mapsto (s, g \cdot s)$. Since $S \times G$ is compact, we see that R is closed, hence $(S \times S) \setminus R$ is open. Now note that the image of $(S \times S) \setminus R$ under $S \times S \to S/G \times S/G$ is open, and that it is precisely the complement of $\Delta_{S/G}$. We conclude that $\Delta_{S/G}$ is closed, hence that S/G is Hausdorff.

To see that S/G is a Stone space, we also need to show that it is totally disconnected. So let $[s], [t] \in S/G$ be given. Pick an open neighborhood U of $[s]$ in S/G which does not contain $[t]$. Then $\pi^{-1}(U)$ is an open subset in S/G which is disjoint from the closed subset $[t]$. We can write $\pi^{-1}(U)$ as a union of clopens. Since finitely many of these clopens cover $[s]$, we obtain a clopen subset C of S which contains $[s]$, and which is disjoint from $[t]$. Now note that π is an open map because of what we saw above, and that it is a closed map since S and S/G are compact Hausdorff. We therefore see that $\pi(C)$ is a clopen neighborhood of $[s]$ in S/G, and that it does not contain $[t]$. We conclude that any two points in S/G can be seperated by clopens, and hence that S/G is a Stone space. ■

Lemma 2.59. *Let G be a profinite group and S a profinite G-set. Then $\lim_N S/N \cong S$, where N ranges over all open normal subgroups of G.*

Proof. We will view S as a Stone space. The quotient S/N is a Stone space by the above lemma. S/N inherits an action from G, defined by $g \cdot [s] = [g \cdot s]$, where $[s]$ denotes an equivalence class in S/N. This is well-defined since N is a normal subgroup of G. We therefore also have a G-action on $\lim_N S/N$. There is a canonical map $\phi\colon S \to \lim_N S/N$ given by $s \mapsto ([s]_N)$, where $[s]_N$ denotes the element of S/N represented by s. This map is clearly G-equivariant. To see that it is a homeomorphism, we only need to show that it is a bijection, as a continuous bijection between Stone spaces is a homeomorphism.

For injectivity of ϕ, let $s, t \in S$ and assume $\phi(s) = \phi(t)$. Define the subset $H_N \subseteq N$ by

$$H_N = \{h \in N \mid h \cdot s = t\}$$

for every N. Then H_N is nonempty since $\phi(s) = \phi(t)$. It is also closed since it is the inverse image of t under the map $g \mapsto g \cdot s$. In particular, H_N is compact Hausdorff for every N and hence $\lim_N H_N$ is nonempty. But this is just $\bigcap_N H_N$. As $H_N \subseteq N$ for every N, we see that $\bigcap_N H_N \subseteq \bigcap_N N = \{1\}$, so $1 \in H_N$ for every N. By definition of H_N, this implies that $s = 1 \cdot s = t$, so ϕ is injective.

For surjectivity, let $([s_N]_N) \in \lim_N S/N$. Then $[s_N]_N$ is a closed subset of S for every N, and $\phi(s) = ([s_N]_N)$ precisely if $s \in \bigcap_N [s_N]_N = \lim_N [s_N]_N$ for every N. It follows by the same argument as above that the right-hand side is nonempty, hence that such an s exists. We conclude that ϕ is surjective, hence an isormorphism. ■

We will also need to study free profinite G-sets. It turns out that any free continuous G-action on a profinite set is trivial if G is a profinite group.

Lemma 2.60. *Let S be a Stone space together with a free G-action for some finite group G, and let $T = S/G$. Then $S \cong G \times T$ by a G-equivariant homeomorphism.*

Proof. Denote the map $S \to T$ by p. Let $x \in T$. Then $p^{-1}(x)$ is discrete, hence finite, say $p^{-1}(x) = \{y_1, \dots, y_n\}$. Since any compact Hausdorff space is normal, and S has a basis of clopens, there exists a clopen U such that $U \cap p^{-1}(x) = \{y_1\}$. Then for any $g \in G$, $g(U)$ is a clopen such that $g(U) \cap p^{-1}(x) = \{y_i\}$ for some $i > 1$. Choose a clopen $V \subseteq U$ around y_1 such that $V \cap g(U) = \emptyset$ for any $g \in G \setminus \{e\}$. Then $g(V) \cap V = \emptyset$ since $V \subseteq U$. Let $W = p(V) \subseteq T$. Then W is clopen, as $p^{-1}(W) = \bigcup_{g \in G} g(V)$ is clopen in S. We also see that $G \times W \cong p^{-1}(W)$. This isomorphism is given by first noting that $p\colon V \to W$ is a homeomorphism, and then mapping $(g, v) \in G \times V$ to $g \cdot v \in g(V) \subseteq p^{-1}(W)$.

We have now seen that there is a cover of clopens of T such that $S \to T$ is trivial over each of these clopens. By compactness, we can pick a finite cover of such clopens, and by intersecting these, we can assume that they are also disjoint. In particular $S \to T$ is a disjoint union of trivial G-bundles, hence it is a trivial G-bundle. ■

Proposition 2.61. *Let S be a Stone space together with a free, continuous G-action for some profinite group G, and let $T = S/G$. Then $S \cong G \times T$ by a G-equivariant homeomorphism.*

The proof of this lemma is very similar to that of Proposition 1 of [Ser97, Chapter 1]. This proposition states that any continuous, surjective homomorphism between profinite groups has a continuous section (which is usually not a homomorphism). In fact, it is easily seen that this proposition is a special case of the lemma we are about to prove.

Proof. The statement that $S \cong G \times T$ is equivalent to saying that a continuous section $T \to S$ of the quotient map exists. Indeed, if $s\colon T \to S$ is such a section, then the map $G \times T \to S$, $(g, t) \mapsto g \cdot s(t)$ is a G-equivariant homeomorphism. We will show that such a section s exists. Define the set $\mathcal{P}$ to consist of pairs (H, s), where $H \leq G$ is a closed subgroup, and where $s\colon T \to S/H$ is a section of the quotient map $S/H \to T$. We give $\mathcal{P}$ an ordering by saying that $(H, s) \leq (H', s')$ if $H' \subseteq H$ and

$$\begin{tikzcd} S/H' \arrow[rr] & & S/H \\ & T \arrow[ul, "s'"] \arrow[ur, "s"'] & \end{tikzcd}$$

commutes, where $S/H' \to S/H$ is the quotient map.

The set $\mathcal{P}$ is nonempty since it contains the element (G, id_T). Assume we are given a totally ordered subset $\{(H_i, s_i)\}$ of $\mathcal{P}$. We will see that this subset has an upper bound.

Let $H = \bigcap_i H_i$. Then the canonical map $\phi\colon S/H \to \lim_i S/H_i$ is a homeomorphism. To see this, we need to show that it is injective and surjective. The proof is similar to that of Lemma 2.59. For surjectivity, let $([x_i]_{H_i})_i \in \lim_i S/H_i$ be given, where $[x_i]_{H_i}$ is the class in S/H_i containing x_i. Then $[x_i]_{H_i}$ is a closed subset of S for every i, hence $\bigcap_i [x_i]_{H_i} = \lim_i [x_i]_{H_i} \neq \emptyset$. Any x in this intersection satisfies $\phi([x]_H) = ([x_i]_{H_i})_i$. For injectivity, let $\phi([x]_H) = \phi([y]_H)$. Define $L_i \subseteq H_i$ by $g \in L_i$ if and only if $g \cdot x = y$. Then

L_i is nonempty and closed for every i, hence $\bigcap_i L_i = \lim_i L_i$ is nonempty. As $\bigcap_i L_i \subseteq H$, we conclude that $[x]_H = [y]_H$.

The sections $s_i \colon T \to S/H_i$ induce a section $s \colon T \to \lim_i S/H_i \cong S/H$ by the universal property of the limit. We see that (H, s) is an upper bound of $\{(H_i, s_i)\}$. By Zorn's lemma, there is a maximal element $(H, s) \in \mathcal{P}$. We will show that $H = \{1\}$, completing the proof. Assume $H \neq \{1\}$. Then there is some $h \in H$ such that $h \neq 1$. By Lemma 2.53, there is an open normal subgroup N of G such that $h \notin N$. Then $N \cap H$ is also an open normal subgroup of H, hence $H/(N \cap H)$ is a finite group. Now consider

$$\begin{tikzcd} S/(N \cap H) \arrow[r] & S/H \arrow[r] \arrow[l, dotted, bend right, "s'"'] & T. \arrow[l, bend right, "s"'] \end{tikzcd}$$

We see that $H/(N \cap H)$ acts freely on $S/(N \cap H)$ and that its quotient is S/H, hence by Lemma 2.60 there exists a section $s' \colon S/H \to S/(N \cap H)$. Then $(N \cap H, s's)$ is an element of $\mathcal{P}$, and $(N, s) < (N \cap H, s's)$, contradicting the maximality of (H, s). We therefore conclude that $H = \{1\}$ and that $S \cong G \times T$ by a G-equivariant homeomorphism. ■

2.3 Profinite completion

We will encounter the following situation many times. We are given a category $\mathbf{C}$ and a subcategory $\mathbf{FinC}$, whose objects are the objects of $\mathbf{C}$ that are finite in a certain sense. To any object $C \in \mathbf{C}$ we want to associate some object $\widehat{C} \in \mathrm{Pro}(\mathbf{FinC})$ in a natural way. We denote $\mathrm{Pro}(\mathbf{FinC})$ by $\widehat{\mathbf{C}}$ for brevity. More specifically, we want a natural bijection $\mathrm{Hom}_{\mathbf{C}}(C, D) \cong \mathrm{Hom}_{\widehat{\mathbf{C}}}(\widehat{C}, D)$ for any $D \in \mathbf{FinC}$. We first give a general construction of such a profinite completion functor, under some mild conditions on the categories $\mathbf{FinC}$ and $\mathbf{C}$. We then study the profinite completion functor in two specific cases, namely profinite sets and profinite groups.

2.3.1 General construction of the profinite completion functor

The main ingredient in constructing a *profinite completion* functor is Proposition 2.39, which states that pro-objects correspond to $\mathbf{Set}$-valued functors that preserve finite limits.

Theorem 2.62. *Let $\mathbf{C}$ be a complete category and $\mathbf{D}$ a small, full subcategory which has finite limits, such that the inclusion $\mathbf{D} \to \mathbf{C}$ preserves finite limits. Then there exists a functor $\widehat{(\cdot)} \colon \mathbf{C} \to \mathrm{Pro}(\mathbf{D})$ and a morphism $i \colon C \to \widehat{C}$ in $\mathrm{Pro}(\mathbf{C})$ for every $C \in \mathbf{C}$, such that for every morphism $f \colon C \to D$ in $\mathbf{C}$ with $D \in \mathbf{D}$, there is a unique $\overline{f} \colon \widehat{C} \to D$ such that the diagram*

$$\begin{tikzcd} C \arrow[r, "i"] \arrow[dr, "f"'] & \widehat{C} \arrow[d, "\overline{f}"] \\ & D \end{tikzcd}$$

commutes in $\mathrm{Pro}(\mathbf{C})$. In particular $\widehat{(\cdot)}$ is left adjoint to $|\cdot| \colon \mathrm{Pro}(\mathbf{D}) \to \mathbf{C}$, the functor mapping a pro-object to its limit in $\mathbf{C}$.

Proof. Under the assumptions of this theorem, the functor $\mathrm{Hom}_{\mathbf{C}}(C,-)\colon \mathbf{D} \to \mathbf{Set}$ preserves all finite limits, hence it is representable by a unique pro-object by Proposition 2.39. We define this pro-object to be $\widehat{C}$. We now have a natural bijection $\mathrm{Hom}_{\mathbf{C}}(C,D) \cong \mathrm{Hom}_{\mathrm{Pro}(\mathbf{D})}(\widehat{C},D)$ by construction. The pro-object $\widehat{C}$ can be described concretely as the diagram $\mathrm{el}(\mathrm{Hom}_{\mathbf{C}}(C,-)) \to \mathbf{D}$. Note that an object in $\mathrm{el}(\mathrm{Hom}_{\mathbf{C}}(C,-))$ is a pair (D,f) with $D \in \mathbf{D}$ and $f \in \mathrm{Hom}_{\mathbf{C}}(C,D)$. These maps $f\colon C \to D$ together define a natural transformation, hence a morphism $i\colon C \to \widehat{C}$ in $\mathrm{Pro}(\mathbf{C})$. One now easily sees that the bijection $\mathrm{Hom}_{\mathrm{Pro}(\mathbf{D})}(\widehat{C},D) \cong \mathrm{Hom}_{\mathbf{C}}(C,D)$ is given by $\overline{f} \mapsto \overline{f} \circ i$, proving the first claim of the theorem.

If we are given a morphism $g\colon C \to C'$ in $\mathbf{C}$, then we obtain an induced functor $g^*\colon \mathrm{el}(\mathrm{Hom}_{\mathbf{C}}(C',-)) \to \mathrm{el}(\mathrm{Hom}_{\mathbf{C}}(C,-))$ which maps a pair (D,f) to (D,fg). Precomposing this with the diagram $\mathrm{el}(\mathrm{Hom}_{\mathbf{C}}(C,-)) \to \mathbf{D}$, which is by definition equal to $\widehat{C}$, we obtain an arrow $\widehat{g}\colon \widehat{C} \to \widehat{C'}$ in $\mathrm{Pro}(\mathbf{D})$, so the profinite completion $\widehat{(\cdot)}$ is indeed a functor $\mathbf{C} \to \mathrm{Pro}(\mathbf{D})$.

For the second claim, let $\{D_i\}$ be a pro-object in $\mathbf{D}$. Then

$$\mathrm{Hom}_{\mathbf{C}}(C, \lim_i D_i) \cong \lim_i \mathrm{Hom}_{\mathbf{C}}(C,D_i) \cong \lim_i \mathrm{Hom}_{\mathrm{Pro}(\mathbf{D})}(\widehat{C},D_i) \cong \mathrm{Hom}_{\mathrm{Pro}(\mathbf{D})}(\widehat{C},\{D_i\}). \quad \blacksquare$$

Remark 2.63. Let $\mathbf{C}$ and $\mathbf{D}$ be as above. Suppose we are given an object $C \in \mathbf{C}$ and a set I_0 consisting of pairs (D,f), where $D \in \mathbf{D}$ and $f\colon C \to D$ (i.e. I_0 is a subset of the set of objects of $\mathrm{el}(\mathrm{Hom}_{\mathbf{C}}(C,-))$). Furthermore, assume that any $f'\colon C \to D'$, where $D' \in \mathbf{D}$, factors as $C \xrightarrow{f} D \xrightarrow{g} D'$ for some $(D,f) \in I_0$ and some $g\colon D \to D'$. Then the inclusion $I \hookrightarrow \mathrm{el}(\mathrm{Hom}_{\mathbf{C}}(C,-))$, where I is the full subcategory that has I_0 as its set of objects, is an initial functor. For example in $\mathbf{Set}$, any map $X \to Y$ factors as $X \to X/R \to Y$ for some equivalence relation R on X. This means that for the profinite completion of a set X, we only need to consider finite quotients X/R of X, instead of all maps $X \to Y$ for all finite sets Y. $\diamondsuit$

Remark 2.64. A natural choice for the set I_0 in the above remark would be the set of all pairs $\{(D,f)\}$ where f is epi. In many examples, the inclusion $I \hookrightarrow \mathrm{el}(\mathrm{Hom}_{\mathbf{C}}(C,-))$ is indeed an initial functor, where I is the full subcategory whose set of objects is I_0. As a morphism $g\colon (D,f) \to (D',f')$ is precisely a morphism $D \to D'$ satisfying $gf = f'$, only one such morphism can exist. In particular, the subcategory I of $\mathrm{el}(\mathrm{Hom}_{\mathbf{C}}(C,-))$ is a codirected set, instead of just a cofiltered category. If we furthermore pick precisely one representative for each isomorphism class of I, we obtain a codirected poset. $\diamondsuit$

The cases of interest to us are $\mathbf{Set}$, $\mathbf{Grp}$, the category of groupoids $\mathbf{G}$ and the category of simplicial sets $\mathbf{S}$. The full subcategories that we will consider are $\mathbf{FinSet}$, $\mathbf{FinGrp}$, the category of finite groupoids $\mathbf{FinG}$ and the category of cofinite simplicial sets $\mathbf{S}_{cofin}$. A finite groupoid is a groupoid whose set of arrows is finite, which automatically implies that its set of objects is finite as well. A cofinite simplicial set is a simplicial finite set which is k-coskeletal for some k. Theorem 2.62 gives us profinite completion functors for all four of these cases. Note that, strictly speaking, these subcategories are not small; however we can always restrict our attention to an equivalent small subcategory.

The rest of this section will be devoted to the study of the profinite completion of sets and groups. We will go into the profinite completion of groupoids and simplicial sets in chapters 3 and 4, respectively.

2.3.2 Profinite completion of sets and groups

As mentioned above, we only need to consider quotients of X when constructing the profinite completion of a set X. We will state this as a definition.

Definition 2.65. Let X be a set, and let $\mathcal{R}(X)$ be the set of equivalence classes on X for which X/R is finite. Then $\mathcal{R}(X)$, ordered by reverse inclusion, is a codirected set. Define the *profinite completion* $\widehat{X}$ as the pro-object $\lim_{R\in\mathcal{R}(X)} X/R$. $\diamondsuit$

Remark 2.66. The reader familiar with profinite groups might recognize this construction. For the profinite completion of a group, instead of using the collection of equivalence relations $\mathcal{R}(X)$, one uses the collection of all normal subgroups with a finite index. $\diamondsuit$

There is another description of the profinite completion of a set which might be more familiar. Recall that $\widehat{\mathbf{Set}} \simeq \mathbf{Stone}$. A construction having the same universal property as $\widehat{(\cdot)}$, but with respect to all compact Hausdorff spaces, is the Stone-Čech compactification (applied to a set viewed as a discrete space). To see that $\widehat{X}$ is indeed the Stone-Čech compactification of X, we need to prove the following lemma.

Lemma 2.67. *Let X be a discrete space. Then the Stone-Čech compactification βX is totally disconnected.*

Proof. Let $i\colon X \to \beta X$ be the canonical map. We will first show that the image of X is dense in βX. Let $Y \subseteq \beta X$ be the closure of $i(X)$. Then Y is compact Hausdorff since it is closed in βX. By the universal property of βX, we obtain a map $f\colon \beta X \to Y$ such that $f \circ i = i$. We also see by the universal property that $\iota_Y \circ f = \mathrm{id}_{\beta X}$, where $\iota_Y\colon Y \hookrightarrow \beta X$. This implies that ι_Y is surjective, hence $Y = \beta X$.

To prove that βX is totally disconnected, note that we can associate a clopen $\widetilde{U} \subseteq \beta X$ to any subset $U \subseteq X$ in the following way. Let $\chi_U\colon X \to \{0,1\}$ be the characteristic function of U, i.e. $\chi_U(x) = 1$ if and only if $x \in U$. Then χ_U extends to a unique continuous map $\widetilde{\chi}_U\colon \beta X \to \{0,1\}$, let $\widetilde{U} = \widetilde{\chi}_U^{-1}(1) \subseteq \beta X$. To see that βX is totally disconnected, it is enough to show that for any $a, b \in \beta X$ there exists a clopen set $V \subseteq \beta X$ containing a but not b. To find such a set, choose disjoint opens U_a and U_b containing a and b respectively. Using that $i(X)$ is dense in βX, one now easily sees that $\widetilde{i^{-1}(U_a)}$ is a clopen set containing a but not containing b. ∎

Corollary 2.68. *The profinite completion $\widehat{X}$ of a set X is the Stone-Čech compactification βX of the discrete space X.*

Proof. Using the universal properties of $\widehat{X}$ and βX, we obtain continuous maps $\widehat{X} \to \beta X$ and $\beta X \to \widehat{X}$ which must be inverse to each other. ∎

The profinite completion functor for sets has the very unfortunate property that it does not preserve products. It is interesting to remark that this problem is precisely the reason why Horel needs to consider "weak operads" instead of operads when defining the profinite completion of an operad in [Hor17].

Proposition 2.69. *The profinite completion functor for sets does not preserve products.*

Proof. We will use that the profinite completion is the Stone-Čech compactification. For a discrete set X, the Stone-Čech compactification βX consists of all ultrafilters on X, given the Stone topology. If profinite completion preserves products, then the canonical map $\beta(X \times Y) \to \beta X \times \beta Y$ must be an isomorphism. We will show that this is not the case for $X = Y = \mathbb{N}$.

The map $\phi\colon \beta(\mathbb{N} \times \mathbb{N}) \to \beta\mathbb{N} \times \beta\mathbb{N}$ maps an ultrafilter $\mathcal{U}$ on $\mathbb{N} \times \mathbb{N}$ to the ultrafilters $\mathcal{V}_1$ and $\mathcal{V}_2$, where $\mathcal{V}_i = \{V \subseteq \mathbb{N} \mid p_i^{-1}(V) \in \mathcal{U}\}$, where p_i is the projection map $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ on the i-th coordinate. This map ϕ fails to be injective. If we are given a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, then consider the set $\mathcal{S} = \{U \times V \mid U, V \in \mathcal{U}\}$. Any ultrafilter $\mathcal{V}$ with $\mathcal{S} \subseteq \mathcal{V}$ satisfies $\phi(\mathcal{V}) = (\mathcal{U}, \mathcal{U})$. Let $N_1 = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m < n\}$ and let $N_2 = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m > n\}$. Define $\mathcal{F}_1$ and $\mathcal{F}_2$ to be the smallest filters containing $\mathcal{S} \cup \{N_1\}$ and $\mathcal{S} \cup \{N_2\}$ respectively. These both satisfy $\emptyset \notin \mathcal{F}_i$, hence there exist ultrafilters $\mathcal{V}_1$ and $\mathcal{V}_2$ containing $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. Since $N_i \in \mathcal{V}_i$ and $N_1 \cap N_2 = \emptyset$, we see that $\mathcal{V}_1 \neq \mathcal{V}_2$. However, $\phi(\mathcal{V}_1) = \phi(\mathcal{V}_2) = (\mathcal{U}, \mathcal{U})$, so we conclude that ϕ is not injective. ∎

As a fun application of the profinite completion functor, we will show that $\widehat{\mathbf{Set}}$ is not cartesion closed. Recall that a category is called *cartesian closed* if it has binary products and if for each object X, the functor $(-) \times X$ has a right adjoint. In particular, if $\widehat{\mathbf{Set}}$ is cartesian closed, then $(-) \times X$ preserves colimits for every profinite set X. Note that as the profinite completion functor is left adjoint, it preserves colimits. This can sometimes be used to compute certain colimits in pro-categories. Two examples of such computations are given in the following proof. It will follow from these computations that $(-) \times \widehat{\mathbb{N}}$ does not preserve colimits.

Proposition 2.70. *The category of profinite sets* $\widehat{\mathbf{Set}}$ *is not cartesian closed.*

Proof. Recall that $\widehat{\mathbf{Set}} \simeq \mathbf{Stone}$, so we may work with Stone spaces instead of profinite sets. We will show that $(-) \times \widehat{\mathbb{N}}$ does not preserve colimits, where $\widehat{\mathbb{N}}$ is the profinite completion of $\mathbb{N}$. Consider the coproduct $\sqcup_{\mathbb{N}}\{*\}$ over a countably infinite number of copies of the one-point space in **Stone**. If we compute this coproduct in **Set**, then we obtain $\mathbb{N}$. Since profinite completion preserves coproducts, this implies that $\sqcup_{\mathbb{N}}\{*\} = \widehat{\mathbb{N}}$ in **Stone**. In particular, $(\sqcup_{\mathbb{N}}\{*\}) \times \widehat{\mathbb{N}} \cong \widehat{\mathbb{N}} \times \widehat{\mathbb{N}}$.

In order to compute $\sqcup_{\mathbb{N}}(\{*\} \times \widehat{\mathbb{N}})$, note that the inclusion $\mathbf{FinSet} \hookrightarrow \mathbf{Top}$, viewing finite sets as finite discrete spaces, satisfies all the assumptions of Theorem 2.62. In particular, there is a profinite completion functor $\widehat{(\cdot)}\colon \mathbf{Top} \to \mathbf{Stone}$ left adjoint to the inclusion $\mathbf{Stone} \hookrightarrow \mathbf{Top}$. Note that this notation is unambiguous, as the profinite completion functor for sets agrees with the above one on discrete topological spaces. The coproduct $\sqcup_{\mathbb{N}}(\{*\} \times \widehat{\mathbb{N}})$ is equal to $\mathbb{N} \times \widehat{\mathbb{N}}$ when computed in **Top**, so this coproduct is equal to $\widehat{\mathbb{N} \times \widehat{\mathbb{N}}}$ when computed in **Stone**. We have the following canonical isomorphisms for every Stone space S, considering that $\widehat{(\cdot)}$ is left adjoint to the inclusion **Stone** $\hookrightarrow$ **Top**:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Stone}}(\widehat{\mathbb{N} \times \widehat{\mathbb{N}}}, S) &\cong \mathrm{Hom}_{\mathbf{Top}}(\mathbb{N} \times \widehat{\mathbb{N}}, S) \cong \sqcup_{\mathbb{N}} \mathrm{Hom}_{\mathbf{Top}}(\widehat{\mathbb{N}}, S) \cong \sqcup_{\mathbb{N}} \mathrm{Hom}_{\mathbf{Top}}(\mathbb{N}, S) \\ &\cong \mathrm{Hom}_{\mathbf{Top}}(\mathbb{N} \times \mathbb{N}, S) \cong \mathrm{Hom}_{\mathbf{Stone}}(\widehat{\mathbb{N} \times \mathbb{N}}, S).\end{aligned}$$

By Yoneda's lemmma, we see that $\widehat{\mathbb{N} \times \widehat{\mathbb{N}}} \cong \widehat{\mathbb{N} \times \mathbb{N}}$. In the above proof, we saw that $\widehat{\mathbb{N} \times \mathbb{N}} \not\cong \widehat{\mathbb{N}} \times \widehat{\mathbb{N}}$, so we conclude that $(-) \times \widehat{\mathbb{N}}$ does not preserve coproducts. In particular, $\widehat{\mathbf{Set}}$ is not cartesian closed. ∎

Contrary to the case of sets, the profinite completion of groups does preserve products. Note that, similarly to the case of sets, the profinite completion of a group G can also be defined as

$$\widehat{G} = \lim_{N} G/N,$$

or, more formally, as the diagram $\{G/N\}_N$. Here N ranges over all normal subgroups of G such that G/N is finite. One easily verifies that

$$\widehat{G} \times \widehat{H} = \lim_{N_G} G/N_G \times \lim_{N_H} H/N_H = \lim_{N_G, N_H} (G \times H)/(N_G \times N_H),$$

so $\widehat{G} \times \widehat{H}$ is the limit of a subdiagram of the diagram used to define $\widehat{G \times H}$.

Proposition 2.71. *The profinite completion functor for groups preserves products.*

Proof. Let G, H be groups. As stated above, the profinite group $\widehat{G} \times \widehat{H}$ is the limit of the diagram $\{(G \times H)/(N_G \times N_H)\}_{N_G, N_H}$, which is a subdiagram of $\{(G \times H)/N\}_N$. The limit of the latter diagram is by definition $\widehat{G \times H}$, so the proof that $\widehat{G} \times \widehat{H} \cong \widehat{G \times H}$ amounts to proving that this inclusion is initial. As we work with diagrams of codirected posets, this comes down to proving that for any cofinite $N \trianglelefteq G \times H$, there are cofinite $N_G \trianglelefteq G$ and $N_H \trianglelefteq H$ such that $N_G \times N_H \subseteq N$. Let ι_G and ι_H be the inclusions of G, H in $G \times H$ respectively, and let $q \colon G \times H \to (G \times H)/N$ be the quotient map. Define $N_G = \ker(q \circ i_G)$ and $N_H = \ker(q \circ i_H)$. Since G/N_G and H/N_H are isomorphic to subgroups of $(G \times H)/N$, they are finite. It is also clear that $N_G \times N_H \subseteq N$. We conclude that $\widehat{G} \times \widehat{H} \cong \widehat{G \times H}$. ∎

2.4 Pro-objects in presheaf categories

In this section we will study the relation between the categories $\mathrm{Pro}(\mathbf{C}^I)$ and $\mathrm{Pro}(\mathbf{C})^I$, that is, the relation between diagrams of pro-objects and pro-objects of diagrams. We will apply the obtained results to the case where $I = \Delta^{op}$, the opposite simplex category, to gain a better understanding of simplicial objects in pro-categories.

2.4.1 Diagrams in pro-categories versus pro-objects in diagram categories

Note that there is an obvious functor $\mathrm{Pro}(\mathbf{C}^I) \to \mathrm{Pro}(\mathbf{C})^I$ for any pair of categories $\mathbf{C}$ and I. If we are given a pro-object in $\mathrm{Pro}(\mathbf{C}^I)$, say a cofiltered diagram $X\colon J \to \mathbf{C}^I$, then we obtain a diagram $I \to \mathbf{C}^J$. Since J is cofiltered, we also obtain a diagram $I \to \mathrm{Pro}(\mathbf{C})$, i.e. an object in $\mathrm{Pro}(\mathbf{C})^I$. The reader is invited to check that this defines a functor, in particular that morphisms in $\mathrm{Pro}(\mathbf{C}^I)$ induce morphisms in $\mathrm{Pro}(\mathbf{C})^I$. This functor can also be obtained by noting that $\mathrm{Pro}(\mathbf{C})^I$ has all cofiltered limits, hence that the obvious map $\mathbf{C}^I \to \mathrm{Pro}(\mathbf{C})^I$ extends to a functor $\mathrm{Pro}(\mathbf{C}^I) \to \mathrm{Pro}(\mathbf{C})^I$ by Proposition 2.31. In the rest of this section this functor will be denoted by L. It is straightforward to verify that a diagram $D\colon I \to \mathrm{Pro}(\mathbf{C})$ has a level representation in the sense of Definition 2.25 precisely if D is in the essential image of $L\colon \mathrm{Pro}(\mathbf{C}^I) \to \mathrm{Pro}(\mathbf{C})^I$.

This functor may in general fail to be fully faithful and to be essentially surjective. The following example shows that fully faithfulness can fail. For an example in which L is not essentially surjective, see [Mey80, §5].

Example 2.72. Let X be any simplicial finite set that is not n-coskeletal for any n (this notion is defined in Example 2.78). We can view X as an object of $\mathrm{Pro}(\mathbf{FinSet}^{\Delta^{op}})$ via the inclusion $\mathbf{FinSet}^{\Delta^{op}} \to \mathrm{Pro}(\mathbf{FinSet}^{\Delta^{op}})$, and similarly as an object of $\mathrm{Pro}(\mathbf{FinSet})^{\Delta^{op}} = \widehat{\mathbf{S}}$. Under these identifications we have $LX = X$. We can also define the pro-object $X' := \{\operatorname{cosk}_n X\}_{n\in\mathbb{N}^{op}}$ in $\mathrm{Pro}(\mathbf{FinSet}^{\Delta^{op}})$. This comes with a map $f\colon X \to \{\operatorname{cosk}_n X\}$, coming from the canonical maps $X \to \operatorname{cosk}_n X$ for every n. Using the above definition of L, we see that $(LX')_i = \{(\operatorname{cosk}_n X)_i\}_{n\in\mathbb{N}^{op}} = \lim_{n\in\mathbb{N}^{op}}(\operatorname{cosk}_n X)_i$ for every $i \in \mathbb{N}$. Since $(\operatorname{cosk}_n X)_i = X_i$ for all $n > i$, we see that $(LX')_i \cong X_i$ for all $i \geq 0$, and that $L(f)\colon LX \to LX'$ is an isomorphism in $\mathrm{Pro}(\mathbf{FinSet})^{\Delta^{op}}$. However, f is not an isomorphism. If f were an isomorphism, then for some n there must exist a morphism $g\colon \operatorname{cosk}_n(X) \to X$ such that $gf = \mathrm{id}_X$. However, one can deduce from this that $fg = \mathrm{id}_{\operatorname{cosk}_n(X)}$ must hold as well, hence that $X \cong \operatorname{cosk}_n(X)$ is n-coskeletal. We conclude that $\mathrm{Pro}(\mathbf{FinSet}^{\Delta^{op}}) \to \widehat{\mathbf{S}}$ is not fully faithful, as it does not preserve isomorphisms.

Note that, however, $\mathrm{Pro}(\mathbf{FinSet}^{\Delta^{op}}) \to \widehat{\mathbf{S}}$ is essentially surjective, which we will see in Corollary 2.83. ♢

In the above example, fully faithfulness of L fails because Δ^{op} is an infinite category. Indeed, if we consider finite diagrams, then we have the following proposition.

Proposition 2.73. *Let $\mathbf{C}$ be any category, and let I be a finite category. Then the functor $L\colon \mathrm{Pro}(\mathbf{C}^I) \to \mathrm{Pro}(\mathbf{C})^I$ is fully faithful.*

Proof. Note that for two diagrams $A, B \in \mathbf{C}^I$, we have that

$$\mathrm{Hom}_{\mathbf{C}^I}(A,B) = \mathrm{eq}\left(\prod_{i\in\mathrm{Ob}(I)} \mathrm{Hom}_{\mathbf{C}}(A(i),B(i)) \rightrightarrows \prod_{\alpha\in\mathrm{Ar}(I)} \mathrm{Hom}_{\mathbf{C}}(A(s(\alpha)),B(t(\alpha)))\right).$$

Here s and t denote the source and target, and the two maps in the equalizer are given by mapping the tuple $(f_i)_{i\in\mathrm{Ob}(i)}$ to the tuples $(f_{t(\alpha)} \circ A(\alpha))_{\alpha\in\mathrm{Ar}(I)}$ and $(B(\alpha) \circ f_{s(\alpha)})_{\alpha\in\mathrm{Ar}(I)}$.

Now let $X, Y \in \mathrm{Pro}(\mathbf{C}^I)$ be given, say $X\colon J \to \mathbf{C}^I$ and $Y\colon K \to \mathbf{C}^I$ with J and K cofiltered. Then

$$\mathrm{Hom}_{\mathrm{Pro}(\mathbf{C}^I)}(X,Y) = \lim_k \operatorname*{colim}_j \mathrm{eq}\left(\prod_{i\in\mathrm{Ob}(I)} \mathrm{Hom}_{\mathbf{C}}(X(j)(i), Y(k)(i)) \right.$$
$$\left. \rightrightarrows \prod_{\alpha\in\mathrm{Ar}(I)} \mathrm{Hom}_{\mathbf{C}}(X(j)(s(\alpha)), Y(k)(t(\alpha)))\right).$$

Since I is finite, we see that both the products occuring in the equalizer are finite. Furthermore, an equalizer is of course also a finite limit. Since limits commute with each other and filtered colimits commute with finite limits by Theorem 2.5, we can put the limit and colimit over K and J inside the products. Noting that $\lim_k \mathrm{colim}_j \mathrm{Hom}_{\mathbf{C}}(X(j)(i), Y(k)(i'))$ is by definition equal to $\mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}((LX)(i), (LY)(i'))$ for all $i, i' \in \mathrm{Ob}(I)$, we see that

$$\mathrm{Hom}_{\mathrm{Pro}(\mathbf{C}^I)}(X,Y) \cong \mathrm{eq}\left(\prod_{i\in\mathrm{Ob}(I)} \mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}((LX)(i), (LY)(i)) \right.$$
$$\left. \rightrightarrows \prod_{\alpha\in\mathrm{Ar}(I)} \mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}((LX)(s(\alpha)), (LY)(t(\alpha)))\right).$$

The right-hand side of this expression is, as we saw at the beginning of this proof, equal to $\mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})^I}(LX, LY)$. We therefore see that $\mathrm{Hom}_{\mathrm{Pro}(\mathbf{C}^I)}(X,Y) \cong \mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})^I}(LX, LY)$ and conclude that $L\colon \mathrm{Pro}(\mathbf{C}^I) \to \mathrm{Pro}(\mathbf{C})^I$ is fully faithful. ■

In the above proof, it is essential that the filtered colimits commutes with the products indexed by $\mathrm{Ob}(I)$ and $\mathrm{Ar}(I)$. We should therefore not hope for the same result if I is infinite.

A level representation of $X \in \mathrm{Pro}(\mathbf{C})^I$ is the same as an isomorphism from X to an object in the image of $L\colon \mathrm{Pro}(\mathbf{C}^I) \to \mathrm{Pro}(\mathbf{C})^I$. Saying that this functor is essentially surjective is equivalent to saying that every I-indexed diagram has a level representation. We have already seen two such cases, namely when I is a finite loopless category (Proposition 2.27) and when I is a cofinite codirected poset (Theorem 2.29).

We saw in the above proposition that $\mathrm{Pro}(\mathbf{C}^I) \to \mathrm{Pro}(\mathbf{C})^I$ is fully faithful if I is finite. In many of the cases that we will consider, it is even an equivalence. By combining Proposition 2.27 with the above example, we obtain the following result.

Proposition 2.74. *Let* $\mathbf{C}$ *be a category and let* I *be a finite loopless category. Then* $\mathrm{Pro}(\mathbf{C}^I) \to \mathrm{Pro}(\mathbf{C})^I$ *is an equivalence of categories.* ■

If $\mathbf{C}$ has finite limits, then we may in fact consider any finite index category I.

Proposition 2.75. *Let* $\mathbf{C}$ *be a small category with finite limits, and let* I *be a finite category. Then* $\mathrm{Pro}(\mathbf{C}^I) \to \mathrm{Pro}(\mathbf{C})^I$ *is an equivalence of categories.*

This is proved in [Mey80, §4].

The rest of this section is devoted to the case where the index category I has finite Hom-sets. The above proposition is not quite true in this case, as we have already seen in Example 2.72. However, we will prove a very similar result.

2.4.2 When the index category has finite Hom-sets

In [BHH17, Proposition 7.4.1], Barnea, Harpaz and Horel prove that $\mathrm{Pro}(\mathbf{S}_{cofin}) \simeq \widehat{\mathbf{S}}$, where $\mathbf{S}_{cofin}$ is the category of "cofinite spaces", i.e. coskeletal simplicial finite sets, and where $\mathbf{s}(\widehat{\mathbf{Set}}) = \widehat{\mathbf{S}}$. We will prove a generalization of this proposition. This proposition is useful as it gives us many cocompact objects for free, which are needed for constructing a fibrantly generated model structure using the cosmall object argument (see Appendix A). Furthermore, the universal property of pro-categories allows us to extend many known functors to the profinite context. For profinite spaces and profinite groupoids, this is illustrated in section 4.2. Here we obtain a profinite version of the usual nerve and fundamental groupoid, satisfying many useful properties, and without having to do any hard work.

We will first need some lemmas. In the following, a category which has finite Hom-sets is called a *locally finite category*.

Using Kan extensions, we can prove the following.

Lemma 2.76. *Let $F\colon J \to I$ be a functor, I a small and locally finite category, I a finite category, and $\mathbf{C}$ a category with finite limits. Then the induced $F^*\colon \mathbf{C}^I \to \mathbf{C}^J$ has a right adjoint F_*. Furthermore, if F is fully faithful, then F_* is fully faithful.*

Proof. We apply Theorem X.3.1 of [Mac88]. Here it is shown that if, for $X \in \mathbf{C}^J$, we define

$$F_*(X)(i) = \varprojlim(X \circ Q\colon (i \downarrow F) \to J \to \mathbf{C})),$$

then $F_*(X)$ defines a functor $I \to \mathbf{C}$ and that F_* is right adjoint of F^* (assuming that all these limits exist). The category $(i \downarrow F)$ is the comma category, whose objects are pairs (f, j), where $f\colon i \to F(j)$. An arrow $(f, j) \to (f', j')$ is an arrow $g\colon j \to j'$ such that the diagram

$$\begin{tikzcd} & i \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ F(j) \arrow[rr, "F(g)"] & & F(j') \end{tikzcd}$$

commutes. The functor $Q\colon (i \downarrow F) \to J$ is the "forgetful functor" which maps an object (f, j) to j, and an arrow $g\colon (f, j) \to (f', j')$ to $g\colon j \to j'$. Since J is finite and I locally finite, we see that $(i \downarrow F)$ is finite, hence the above limit exists. We conclude by Theorem X.3.1 of [Mac88] that F^* has a right adjoint F_*.

For the second statement, note that since F_* is right adjoint to F^*, it is fully faithful precisely if $F^*F_* \simeq \mathrm{id}_{\mathbf{C}^J}$. If F is fully faithful, then for any $j \in J$, we see that $(\mathrm{id}_{F(j)}, F(j))$ is an initial object in $(F(j) \downarrow F)$, hence the limit in the above definition of $F_*(X)(F(j))$ is canonically isomorphic to $X(j)$. We therefore see that $(F^*F_*(X))(j) = F_*(X)(F(j)) \cong$

$X(j)$ for all objects $j \in J$. Using the definition of $F_*(X)$ on arrows (see the proof of [Mac88, Theorem X.3.1]), we see that the isomorphisms are natural and conclude that $F^*F_*(X) = X$. ■

Remark 2.77. If, in the above lemma, $\mathbf{C}$ has all finite colimits, then the dual statement holds, namely that F^* has a left adjoint. Since most categories that we consider have both finite limits and colimits, the functor F^* usually has both a left and a right adjoint. ♢

Example 2.78. A notable example of the above lemma is the n-coskeleton of a simplicial set. Given the simplex category Δ, we denote the full subcategory on the linear orders $[0], \ldots, [n]$ by $\Delta_{\leq n}$. The inclusion $\Delta_{\leq n} \to \Delta$ induces a functor $\tau_n \colon \mathbf{C}^{\Delta^{op}} \to \mathbf{C}^{\Delta^{op}_{\leq n}}$ for any category $\mathbf{C}$. This functor is called the n-th truncation functor, since it cuts off all simplices above dimension n. If $\mathbf{C}$ has finite colimits, then the left adjoint of τ_n is usually denoted sk_n and called the (simplicial) n-skeleton. If $\mathbf{C}$ has finite limits, then the right adjoint of τ_n is denoted cosk_n and called the n-coskeleton. We call a simplicial object $X \in \mathbf{C}^{\Delta^{op}}$ n-skeletal if it is in the image of sk_n, and n-coskeletal if it is in the image of cosk_n. We denote by $\mathbf{S}_{cofin}$ the full subcategory of **sFinSet** whose objects are n-coskeletal for some n. We return to this example in more detail in section 4.1. ♢

The above example inspires the following definition.

Definition 2.79. Assume we are given a small category I, a full subcategory J and a category $\mathbf{C}$. Denote by $\tau_J \colon \mathbf{C}^I \to \mathbf{C}^J$ the functor associated to the inclusion $J \to I$. Denote its right adjoint by cosk_J and its left adjoint by sk_J, if they exist. Define $(\mathbf{C}^I)_{cofin}$ to be the full subcategory of $\mathbf{C}^I$ whose objects are in the image of cosk_J for a finite full subcategory J of I. ♢

One can verify that $\mathbf{S}_{cofin}$ as defined in the above example agrees with $(\mathbf{FinSet}^{\Delta^{op}})_{cofin}$ as defined in the above definition. If we are given $X \in \mathbf{C}^I$, we will abusively write $\mathrm{cosk}_J(X)$ and $\mathrm{sk}_J(X)$ for $\mathrm{cosk}_J(\tau_J(X))$ and $\mathrm{sk}_J(\tau_J(X))$, respectively.

Lemma 2.80. *Let I be a locally finite small category and let $\mathbf{C}$ be a category with finite limits. Then for any $X \in \mathbf{C}^I$ we have*

$$X \cong \lim_{J \subseteq I} \mathrm{cosk}_J X,$$

where J ranges over all finite full subcategories of I, and this limit is a projective limit.

Proof. To see that the limit is projective, note that the collection of finite full subcategories of I is directed. If we are given two finite full subcategories $J' \subseteq J$, then note that $\tau_{J'} \mathrm{cosk}_J(X) = \tau_{J'}(X)$, using that $\tau_J \mathrm{cosk}_J(X) = \tau_J(X)$ and that $\tau_{J'}$ factors through τ_J. We therefore see that

$$\mathrm{Hom}(\tau_{J'}(X), \tau_{J'}(X)) = \mathrm{Hom}(\tau_{J'} \mathrm{cosk}_J(X), \tau_{J'}(X)) \cong \mathrm{Hom}(\mathrm{cosk}_J X, \mathrm{cosk}_{J'} X),$$

so the identity morphism $\tau_{J'}(X) \to \tau_{J'}(X)$ induces a morphism $\mathrm{cosk}_J X \to \mathrm{cosk}_{J'} X$, which is the unique morphism satisfying that $(\mathrm{cosk}_J X)(C) \to (\mathrm{cosk}_{J'} X)(C)$ is the identity morphism for every $C \in J'$. These morphisms make $\{\mathrm{cosk}_J X\}_J$ into a projective diagram.

To see that X is indeed the limit of this diagram, note that limits are computed pointwise in $\mathbf{C}^I$, meaning that $(\lim_J \operatorname{cosk}_J X)(C) = \lim_J(\operatorname{cosk}_J X)(C)$. As $(\operatorname{cosk}_J X)(C) = X(C)$ for any full subcategory J containing I, we see that $\lim_J(\operatorname{cosk}_J X)(C) = X(C)$ and conclude that $\lim_J(\operatorname{cosk}_J X) = X$ ■

Remark 2.81. In the above example, we do not need to consider all finite full subcategories J of I. It is of course enough to consider a (directed) collection of finite full subcategories whose union is I. For Δ^{op}, we can in particular consider the filtration $\Delta^{op}_{\leq 0} \subseteq \Delta^{op}_{\leq 1} \subseteq \Delta^{op}_{\leq 2} \subseteq \ldots$, and see that

$$X = \lim_{n \in \mathbb{N}} \operatorname{cosk}_n X. \qquad \diamondsuit$$

Theorem 2.82. *If I is a locally finite small category and $\mathbf{C}$ is small and has finite limits, then*

$$\operatorname{Pro}((\mathbf{C}^I)_{cofin}) \simeq \operatorname{Pro}(\mathbf{C})^I.$$

Proof. By Proposition 2.50, we need to show that $(\mathbf{C}^I)_{cofin} \to \operatorname{Pro}(\mathbf{C})^I$ is fully faithful, that all objects in its image are cocompact in $\operatorname{Pro}(\mathbf{C})^I$, and that every object of $\operatorname{Pro}(\mathbf{C})^I$ is a cofiltered limit of objects in $(\mathbf{C}^I)_{cofin}$.

For the first property, note that $\iota\colon \mathbf{C} \to \operatorname{Pro}(\mathbf{C})$ is fully faithful, hence $\mathbf{C}^I \to \operatorname{Pro}(\mathbf{C})^I$ is fully faithful. Since $(\mathbf{C}^I)_{cofin}$ is a full subcategory of $\mathbf{C}^I$, we conclude that $(\mathbf{C}^I)_{cofin} \to \operatorname{Pro}(\mathbf{C})^I$ is fully faithful.

For the second property, let $X \in (\mathbf{C}^I)_{cofin}$. Then $X = \operatorname{cosk}_J(X')$ for some finite full subcategory J of I and some $X' \in \mathbf{C}^J$. We now have

$$\operatorname{Hom}(\lim_i Y_i, X) = \operatorname{Hom}(\lim_i Y_i, \operatorname{cosk}_J(X')) = \operatorname{Hom}(\lim_i \tau_J(Y_i), X').$$

Here we use that $\tau_J(\lim_i Y_i) = \lim_i \tau_J(Y_i)$, which follows since limits in functor categories are computed levelwise. Now note that $\operatorname{Pro}(\mathbf{C})^J \simeq \operatorname{Pro}(\mathbf{C}^J)$ and hence that X' is cocompact in $\operatorname{Pro}(\mathbf{C})^J$, which follows from Proposition 2.75. We conclude from this that

$$\operatorname{Hom}(\lim_i \tau_J(Y_i), X') = \operatorname*{colim}_i \operatorname{Hom}(\tau_J(Y_i), X') = \operatorname*{colim}_i \operatorname{Hom}(Y_i, X),$$

hence X is cocompact.

Lastly, we need to show that any $X \in \operatorname{Pro}(\mathbf{C})^I$ is a cofiltered limit of objects in $(\mathbf{C}^I)_{cofin}$. We see by Lemma 2.80 that $X = \lim_{J \subseteq I}(\operatorname{cosk}_J X)$, with J ranging over all finite full subcategories of I. Since a cofiltered limit of cofiltered limits is again a cofiltered limit (see Theorem 2.29), it suffices to show that $\operatorname{cosk}_J X$ is a cofiltered limit of objects in $(\mathbf{C}^I)_{cofin}$. Since $\operatorname{Pro}(\mathbf{C})^J \simeq \operatorname{Pro}(\mathbf{C}^J)$, we see that $\tau_J(X)$ is a cofiltered limit $\lim_i X_i$ of objects of $\mathbf{C}^J$. Since cosk_J is a right adjoint, it preserves limits, hence $\operatorname{cosk}_J(X) = \lim_i(\operatorname{cosk}_J X_i)$. We conclude that any $X \in \operatorname{Pro}(\mathbf{C})^I$ is a cofiltered limit of objects of $(\mathbf{C}^I)_{cofin}$, and hence that $\operatorname{Pro}((\mathbf{C}^I)_{cofin}) \simeq \operatorname{Pro}(\mathbf{C})^I$. ■

Noting that we have the following commutative triangle

$$\begin{tikzcd}
\operatorname{Pro}((\mathbf{C}^I)_{cofin}) \arrow[rr, "\simeq"] \arrow[dr, hook] & & \operatorname{Pro}(\mathbf{C})^I \\
& \operatorname{Pro}(\mathbf{C}^I), \arrow[ur, "L"'] &
\end{tikzcd}$$

we conclude the following.

Corollary 2.83. *If I is a locally finite small category and $\mathbf{C}$ has finite limits, then*

$$\mathrm{Pro}(\mathbf{C}^I) \to \mathrm{Pro}(\mathbf{C})^I$$

is essentially surjective. ■

By Remark 2.81, we see that we do not need to consider all full subcategories of Δ^{op} for the above theorem to hold. In particular, we can use the filtration $\Delta^{op}_{\leq 0} \subseteq \Delta^{op}_{\leq 1} \subseteq \ldots \subseteq \Delta^{op}$.

Corollary 2.84. *The inclusion $\mathbf{S}_{cofin} \to \widehat{\mathbf{S}}$ induces an equivalence $\mathrm{Pro}(\mathbf{S}_{cofin}) \simeq \widehat{\mathbf{S}}$.* ■

Remark 2.85. There is of course also a dual statement. Assume that the same hypotheses hold as in Theorem 2.82, except that $\mathbf{C}$ needs to have all finite colimits instead of finite limits. Then

$$\mathrm{Ind}((\mathbf{C}^I)_{skfin}) \simeq \mathrm{Ind}(\mathbf{C})^I,$$

where $(\mathbf{C}^I)_{skfin}$ is the full subcategory of $\mathbf{C}^I$ spanned by all objects which are in the image of sk_J for some finite full subcategory $J \subseteq I$. This for example implies that the ind-category of the category of simplicial sets with finitely many non-degenerate simplices is the category of all simplicial sets. ♢

Chapter 3

Profinite groupoids

In this chapter we will study profinite groupoids, which are defined as pro-objects in the category of finite groupoids. The goal is to put a fibrantly generated model structure on $\widehat{\mathbf{G}}$, the category of profinite groupoids. The main source for the material of this chapter is Horel's paper [Hor17], which in particular contains the construction of the fibrantly generated model structure. We will fill in some gaps of Horel's construction, and correct a few minor mistakes. The main difference between Horel's approach and the approach presented here is that we extensively study weak equivalences between profinite groupoids before we prove the existence of the fibrantly generated model structure. We also discuss a different way of looking at profinite groupoids than as pro-objects in the category of finite groupoids, namely as certain topological groupoids. This viewpoint allows for a characterization of all connected profinite groupoids.

In the first section of this chapter, we recall some basic notions about groupoids and define profinite groupoids. In the second section we compare profinite groupoids to topological groupoids, proving that the category of profinite groupoids is a full subcategory of the category of topological groupoids. We also provide a simple description of all connected profinite groupoids. The third section is devoted to the profinite completion functor for profinite groupoids. We obtain a concrete description of the profinite completion functor, and show that it preserves products of profinite groupoids with finitely many objects. In the fourth section, weak equivalences are defined and studied. We show that there are three equivalent definitions of weak equivalences of profinite groupoids, and prove that weak equivalences of connected profinite groupoids are in fact homotopy equivalences. In the last section, we prove the existence of the fibrantly generated model structure of [Hor17].

The main source for the material in this chapter is [Hor17, §4]. Section 3.2 of this chapter, where we compare profinite groupoids to topological groupoids, is the author's own work. The proof in section 3.3 that the profinite completion functor preserves products of groupoids with finitely many objects, is the author's own work, as the proof given in [Hor17] was based on the wrong assumption that the set-theoretical image of a morphism of groupoids is again a groupoid. Most of the proofs and construction in section 3.4 on weak equivalences are work by the author himself. The proof of the model structure in section 3.5 is based on Horel's proof [Hor17, Theorem 4.12], where the main differences are due to our different approach to weak equivalences in section 3.4.

3.1 Some basic facts on (profinite) groupoids

We start this section by recalling some basic constructions of groupoids, introducing notation that we will use along the way. We then recall the construction of the nerve of a groupoid, which relates groupoids to simplicial sets. We end this section by introducing profinite groupoids, the main object of study in this chapter.

3.1.1 Basic constructions with groupoids

Recall that a *groupoid* is a small category in which all morphisms are invertible. We will call the morphisms in a groupoid *arrows* to distinguish them from morphisms between groupoids. Morphisms between groupoids are functors. We denote the category of groupoids by $\mathbf{G}$. For a groupoid A and two objects x, y of A, we denote the set of arrows from x to y by $A(x, y)$, and the arrows form x to x by $A(x)$. Note that $A(x)$ is a group. A groupoid A consists of a set of *objects* $\mathrm{Ob}(A)$, a set of *arrows* $\mathrm{Ar}(A)$, a *source* map s, a *target* map t, an *identity* or *unit* map e, an *inverse* map ι and a *multiplication* or *composition* map m, pictured as

$$\begin{tikzcd} \mathrm{Ar}(A) \times_{\mathrm{Ob}(A)} \mathrm{Ar}(A) \arrow[r, "m"] & \mathrm{Ar}(A) \arrow[loop above, "\iota"] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & \mathrm{Ob}(A). \arrow[l, "e"'] \end{tikzcd}$$

These maps of course satisfy all of the relations that make A into a groupoid, i.e. a category where every map has an inverse.

In the context of homotopy theory groupoids arise naturally, the most notable example being the fundamental groupoid of a space or a simplicial set. These can be seen as a generalization of the fundamental group. We will first consider some important examples of groupoids.

Example 3.1. Let X be a topological space. Define the *fundamental groupoid* $\Pi_1(X)$ with as objects the points of X. The arrows $x \to y$ are given by the homotopy classes of paths from x to y, relative to the beginning and end-point. The composition of two arrows is given by concatenating paths. One can easily check that this indeed defines a groupoid. The fundamental groupoid can be seen as a generalization of the fundamental group. In particular, for any $x \in X$, the group $\pi_1(X, x)$ is the group of automorphisms of x in $\Pi_1(X)$. $\diamondsuit$

Example 3.2. There is also an analogue of the above construction for simplicial sets X. The *fundamental groupoid* $\Pi_1(X)$ has X_0 as its set of objects. Given $x, y \in X_0$, define precisely one arrow $x \to y$ for every $u \in X_1$ with $d_1(u) = x$ and $d_0(u) = y$. This defines a directed graph with set of edges X_0. Let $\Pi_1'(X)$ be the free category on this graph, modulo the relation $u \circ v = w$ if there is a $z \in X_2$ with $d_0(z) = u$, $d_1(z) = w$ and $d_2(z) = v$.

Define $\Pi_1(X)$ to be the groupoid completion of this category, i.e. the groupoid obtained by adding inverses for each arrow in $\Pi_1'(X)$.

For a more concrete description, let $X_1^{-1} = \{u^{-1} \mid u \in X_1\}$ be the set of formal inverses of the 1-simplices of X. Define $d_0(u^{-1}) = d_1(u)$ and $d_1(u^{-1}) = d_0(u)$, since we view inverses as paths in the reverse direction. Then an arrow $x \to y$ is an equivalence class of finite sequences $v_1 \dots v_n$ of elements of $X_1 \sqcup X_1^{-1}$ which satisfy $d_1(v_1) = x$, $d_0(v_i) = d_1(v_{i+1})$ and $d_0(v_n) = y$. Here the equivalence relation between such sequences is defined in the obvious way (one can replace uv by w if there is a $z \in X_2$ with $d_0(z) = u$, $d_1(z) = w$ and $d_2(z) = v$, etc.), and composition of arrows is defined by concatenating such finite sequences. ♢

Example 3.3. Let S be a set with a left action by a group G. Denote by $S /\!\!/ G$ the translation groupoid. Its set of objects is S, and the arrows $s \to t$ are the $g \in G$ for which $g \cdot s = t$. For the one element set $*$ and a trivial G-action, obtain a groupoid $* /\!\!/ G$ with one object whose arrows are precisely those of G. The assignment $G \mapsto * /\!\!/ G$ embeds **Grp** in **G** as a full subcategory. We will abbreviate $* /\!\!/ G$ by G_*. ♢

Example 3.4. Given a set S, the discrete groupoid $\mathrm{Disc}(S)$ has S as its set of objects, and the only arrows are the identities for each object. The codiscrete groupoid $\mathrm{Codisc}(S)$ also has S as its set of objects, but instead there is exactly one arrow $s \to t$ for every $s, t \in S$. ♢

Example 3.5. Let X be a set with an equivalence relation R. We then define the groupoid $X /\!\!/ R$ with set of objects X, and with one arrow $x \to y$ precisely if xRy. The discrete and codiscrete groupoid are two special cases of groupoids of the form $X /\!\!/ R$, namely the case where any object is only equivalent to itself, and the case where any two objects are equivalent. ♢

Example 3.6. We write $I[n]$ for the groupoid $\mathrm{Codisc}(\{0, 1, \dots, n\})$. We will later see that $I[1]$ serves as a unit interval. Note that there are morphisms $\delta_i \colon I[n-1] \to I[n]$ and $\sigma_i \colon I[n+1] \to I[n]$ for $0 \le i \le n$ which, on objects, are defined by

$$\delta_i(m) = \begin{cases} m & \text{if } m < i \\ m+1 & m \ge i \end{cases}; \qquad \sigma_i(m) = \begin{cases} m & \text{if } m \le i \\ m+1 & m - 1 > i. \end{cases}$$

The groupoids $I[n]$ together with these morphisms define a cosimplicial groupoid I. ♢

Example 3.7. If we are given two groupoids A and B, then the product $A \times B$ is simply the product of A and B as categories. We define B^A to be the groupoid whose objects are morphisms $A \to B$ and whose arrows are natural transformations. Since all arrows in a groupoid are isomorphisms, one sees that a natural transformation between two morphisms $A \to B$ is in fact a natural isomorphism, hence B^A is again a groupoid. The functor $A \times (-)$ is left adjoint to $(-)^A$. ♢

Example 3.8. Given a set S and a group G, denote by $G[S]$ the groupoid with S as its set of objects, and whose arrows $s \to t$ are the elements of G, for any $s, t \in S$. The composition $h \circ g$ of $g \colon s \to t$ and $h \colon t \to u$ is then defined as $hg \colon s \to u$. Note that $G[S]$ is canonically isomorphic to the groupoid $\mathrm{Codisc}(S) \times G_*$. ♢

Denote the functors $\mathbf{G} \to \mathbf{Set}$ sending a groupoid to its underlying set of objects or arrows by Ob and Ar respectively. Note that the functor Ob is represented by $I[0]$, and Ar by $I[1]$. A groupoid is called *connected* if for any objects x and y, there exists an arrow $x \to y$, and a connected component is a maximal connected subgroupoid. The functor $\pi_0 \colon \mathbf{G} \to \mathbf{Set}$ maps a groupoid to its set of connected components. More concretely, $\pi_0 A$ is defined as the coequalizer of the source and target map $\mathrm{Ar}(A) \rightrightarrows \mathrm{Ob}(A)$. Together with the functors Disc and Codisc, we have some easy-to-prove adjunctions:

$$\pi_0 \dashv \mathrm{Disc} \dashv \mathrm{Ob} \dashv \mathrm{Codisc}\,.$$

If we are given a connected groupoid A with set of objects S and whose automorphism group at some point $x \in S$ is G, then there is a non-canonical isomorphism $A \cong G[S]$. Such an isomorphism can be obtained by picking, for each $y \in S \setminus \{x\}$, some arrow $\alpha_y \colon x \to y$, and mapping an arrow $h \colon y \to y'$ in A to $\alpha_{y'}^{-1} h \alpha_y \colon y \to y'$ in $G[S]$.

Given a groupoid A, we will work with the groupoid $A^{I[1]}$ when defining the notion of homotopy for morphisms of groupoids. We can view objects of $A^{I[1]}$ as arrows in A, and an arrow k between two arrows α, β of A consists of two arrows k_0, k_1 such that $k_1 \alpha = \beta k_0$. Since k_1 is uniquely determined as $\beta k_0 \alpha^{-1}$, we can simply view an arrow k between two arrows α, β of A as an arrow between the sources $s(\alpha)$ and $s(\beta)$. Denote by ev_0, ev_1 the maps which send an arrow α of A to its source $s(\alpha)$ or target $t(\alpha)$ respectively. An arrow k in $A^{I[1]}$ between two objects α, β can be seen as an arrow $k \colon s(\alpha) \to s(\beta)$ in A, and we define $ev_0(k) = k$, $ev_1(k) = \beta k \alpha^{-1}$.

The category of groupoids is complete and cocomplete. We have the following proposition on the computation of limits in $\mathbf{G}$, which reduces it to a computation in $\mathbf{Set}$. The proof is left to the reader.

Proposition 3.9. *Let $\{G_i\}$ be a diagram of groupoids. Then $\lim_i G_i$ has $\lim_i \mathrm{Ob}(G_i)$ as its set of objects, and $\lim_i \mathrm{Ar}(G_i)$ as its set of arrows. The source, target, inverse, identity and composition maps are induced by the corresponding maps in G_i.* ∎

3.1.2 The nerve of a groupoid

One can view groupoids as spaces which only contain information in dimensions 0 and 1. This idea is made precise by the construction of the nerve of a groupoid. The nerve of a groupoid will be a Kan complex X in which every horn $\Lambda^n_k \to X$ (with $n > 1$) has a unique filler. This uniqueness implies, in a sense, that X contains no information in dimensions greater than 1. In particular, such a Kan complex is 2-coskeletal, meaning that any map $\partial\Delta^n \to X$ extends uniquely to a map $\Delta^n \to X$ when $n > 2$. For the definition of the nerve of a groupoid, recall the construction of the cosimplicial groupoid I from Example 3.6.

Definition 3.10. Define the nerve functor $B \colon \mathbf{G} \to \mathbf{S}$ as follows. For any groupoid A, the nerve BA is given by

$$(BA)_n = \mathrm{Hom}_{\mathbf{G}}(I[n], A), \qquad d_i = \delta_i^* \quad \text{and} \quad s_i = \sigma_i^*$$

where $\delta_i\colon I[n-1] \to I[n]$ and $\sigma_i\colon I[n+1] \to I[n]$ are as in Example 3.6. If we are given a morphism of groupoids $f\colon A \to C$, define $Bf\colon BA \to BC$ by $(Bf)_n(g) = fg$ for any $g \in (BA)_n = \mathrm{Hom}_{\mathbf{G}}(I[n], A)$ and any $n \geq 0$. ♢

Note that a morphism of groupoids $\phi\colon I[n] \to A$ is precisely a sequence composible of arrows

$$x_0 \xrightarrow{\alpha_1^0} x_1 \xrightarrow{\alpha_2^1} \cdots \longrightarrow x_{n-1} \xrightarrow{\alpha_n^{n-1}} x_n.$$

In particular, we can view $(BA)_n$ as $\mathrm{Ar}\,A \times_{\mathrm{Ob}\,A} \ldots \times_{\mathrm{Ob}\,A} \mathrm{Ar}\,A$. We will write this as $(\alpha_1^0, \ldots, \alpha_n^{n-1})$ in A. Here α_{i+1}^i is the image of the unique arrow $i \to i+1$ in $I[n]$ under ϕ. The face and degeneracy maps are then given by

$$\begin{aligned}
d_0(\alpha_1^0, \ldots, \alpha_n^{n-1}) &= (\alpha_2^1, \ldots, \alpha_n^{n-1}) \\
d_n(\alpha_1^0, \ldots, \alpha_n^{n-1}) &= (\alpha_1^0, \ldots, \alpha_{n-1}^{n-2}) \\
d_i(\alpha_1^0, \ldots, \alpha_n^{n-1}) &= (\alpha_1^0, \ldots, \alpha_{i+1}^i \alpha_i^{i-1}, \ldots, \alpha_n^{n-1}) \qquad \text{for } 0 < i < n \\
s_i(\alpha_1^0, \ldots, \alpha_n^{n-1}) &= (\alpha_1^0, \ldots, \alpha_i^{i-1}, \mathrm{id}_{x_i}, \alpha_{i+1}^i, \ldots, \alpha_n^{n-1}).
\end{aligned}$$

Remark 3.11. One can define the nerve of any category C. If one replaces $I[n]$ by the category with one unique arrow $i \to j$ for every $i \leq j$, and no arrow $i \to j$ if $j < i$, then one obtains a cosimplicial object in **Cat**, the category of small categories. Using this cosimplicial object, one can generalize the above definition to a functor $\mathbf{Cat} \to \mathbf{S}$ which agrees with $B\colon \mathbf{G} \to \mathbf{S}$ on groupoids. ♢

To see that any horn $\Lambda_k^n \to BA$ with $n > 1$ has a unique filler, first note that this is clear for $n = 2$ using that any morphism can be inverted. Now assume we are given an n-horn $f\colon \Lambda_k^n \to BA$ with $n > 2$. Denote by $[i, j]$ with $i \leq j$ the 2-simplex of Λ_k^n connecting the i-th and j-th vertex. Since $k > 2$, this 2-simplex exists for all $0 \leq i \leq j \leq n$. Define $\alpha_{i+1}^i = f([i, i+1])$. Then $\alpha := (\alpha_1^0, \ldots, \alpha_n^{n-1})$ is an n-simplex of BA which fills the horn $f\colon \Lambda_k^n \to BA$. Uniqueness follows trivially from this. To see that the faces of α indeed agree with those of this horn, one can use (by induction) that $(n-1)$-horns have unique fillers.

If X is a simplicial set where horns have unique fillers, then it must be 2-coskeletal. To see this, let a simplicial map $f\colon \partial\Delta^n \to X$ with $n > 2$ be given. If we now remove the face $d_0(\Delta^n)$, then we obtain a horn, which has a unique filler $\tilde{f}\colon \Delta^n \to X$. To see that $\tilde{f}$ agrees with f on $d_0(\Delta^n)$, remove the face $d_0 d_0(\Delta^n)$ and use that this horn has a unique filler, which must be equal to both $f(d_0(\Delta^n))$ and $\tilde{f}(d_0(\Delta^n))$.

One can show that $\Pi_1(BA) = A$ for any groupoid A, and that there is a natural bijection $\mathrm{Hom}_{\mathbf{G}}(\Pi_1 X, A) \to \mathrm{Hom}_{\mathbf{S}}(X, BA)$. We therefore have the following theorem.

Theorem 3.12. *The functor $B\colon \mathbf{G} \to \mathbf{S}$ is full and faithful and is right adjoint to Π_1.* ■

3.1.3 Profinite groupoids

We say that a groupoid A is finite if $\mathrm{Ar}(A)$ is finite, and denote the category of finite groupoids by **FinG**. As limits in **FinG** can be computed by considering the sets of arrows and computing the corresponding limit in **Set**, we see that a finite limit of finite groupoids is again finite. We denote the category Pro(**FinG**) of profinite groupoids by $\widehat{\mathbf{G}}$. Most of the constructions from the first part of this section also work for profinite groupoids; in particular one can define the profinite groupoids G_*, $\mathrm{Disc}(S)$, $\mathrm{Codisc}(S)$ and $G[S]$ for any profinite group G and profinite set S. For example, if $S = \{S_i\}_{i\in I}$ and $G = \{G_j\}_{j\in J}$, then one can define $G[S] = \lim_{(i,j)\in I\times J} G_j[S_i]$. One can also define the translation groupoid $S /\!\!/ G$ for a profinite G-set S, by noting that any profinite G-set is a projective limit of finite G-sets, and that the action of a profinite group G on a finite set always factors through a finite quotient of G.

For a profinite groupoid $A = \{A_i\}_{i\in I}$, we define the profinite sets, or Stone spaces, $\mathrm{Ob}(A)$ and $\mathrm{Ar}(A)$ by $\{\mathrm{Ob}(A_i)\}_{i\in i}$ and $\{\mathrm{Ar}(A_i)\}_{i\in i}$, respectively. The source, target, unit, inverse and multiplication map of the groupoids A_i induce source, target, unit, inverse and multiplication maps between $\mathrm{Ar}(A) \times_{\mathrm{Ob}(A)} \mathrm{Ar}(A)$, $\mathrm{Ar}(A)$ and $\mathrm{Ob}(A)$.

All functors in the sequence of adjunctions $\pi_0 \dashv \mathrm{Disc} \dashv \mathrm{Ob} \dashv \mathrm{Codisc}$ restrict to adjunctions between **FinG** and **FinSet**. By Corollary 2.32, we obtain an induced sequence of adjunctions $\pi_0 \dashv \mathrm{Disc} \dashv \mathrm{Ob} \dashv \mathrm{Codisc}$ between $\widehat{\mathbf{G}}$ and $\widehat{\mathbf{Set}}$. The adjunction $\Pi_1 \dashv B$ can similarly be extended to an adjunction between profinite spaces and profinite groupoids. This will be done in section 4.2 of chapter 4, once we have developed some theory on profinite spaces.

We will shortly discuss connectedness of profinite groupoids. Note that there are two ways of defining connectedness. One could define a profinite groupoid A to be connected if $\pi_0(A) = \{*\}$, or if $\mathrm{Ob}(A)$ is nonempty and for every $x, y \in \mathrm{Ob}(A)$ there exists an $\alpha \in \mathrm{Ar}(A)$ with source x and target y. Note that for the second definition, we identify the profinite sets $\mathrm{Ob}(A)$ and $\mathrm{Ar}(A)$ with Stone spaces (or with their limit in **Set**), so that we are able to talk about elements of $\mathrm{Ob}(A)$ and $\mathrm{Ar}(A)$. Luckily, the two above-mentioned notions of connectedness turn out to be equivalent.

Proposition 3.13. *Let $x, y \in \mathrm{Ob}(A)$. Then $\mathrm{Ob}(A) \to \pi_0(X)$ maps x and y to the same connected component precisely if there is an $\alpha \in \mathrm{Ar}(A)$ with source x and target y. In particular, A is connected precisely if $\mathrm{Ob}(A)$ is nonempty and for any $x, y \in \mathrm{Ob}(A)$, there exists an $\alpha \in \mathrm{Ar}(A)$ with source x and target y.*

Proof. Let $A = \{A_i\}_{i\in I}$ be a profinite groupoid, and let $(x_i)_i, (y_i)_i \in \lim_i \mathrm{Ob}(A_i) = \mathrm{Ob}(A)$ be given. Let $(\alpha_i) \in \lim_i \mathrm{Ar}(A_i) = \mathrm{Ar}(A)$ be an arrow with source $(x_i)_i$ and target $(y_i)_i$. Then $s(\alpha_i) = x_i$ and $t(\alpha_i) = y_i$ by definition, so x_i and y_i get mapped to the same object in $\pi_0 A_i$ for every i, hence they get mapped to the same connected component by $\mathrm{Ob}(A) \to \pi_0 A = \lim_i \pi_0 A_i$.

For the converse, assume that $(x_i)_i$ and $(y_i)_i$ get mapped to the same connected component by $A \to \pi_0 A$. Then x_i and y_i get mapped to the same connected component by $A_i \to \pi_0 A_i$ for every i. In particular, the set $A_i(x_i, y_i)$ is nonempty for every i. This set is also finite, and we see that for $j \leq i$, the map $A_j \to A_i$ restricts to a map $A_j(x_j, y_j) \to A_i(x_i, y_i)$. Therefore $\{A_i(x_i, y_i)\}_{i\in I}$ is a projective diagram of finite nonempty sets. By Theorem 2.43, the limit $\lim_i A_i(x_i, y_i) = A((x_i)_i, (y_i)_i)$ is nonempty. Any $(\alpha_i)_i \in \lim_i A_i(x_i, y_i)$ has source $(x_i)_i$ and target $(y_i)_i$. ■

Recall that for profinite sets and profinite groups, we can replace a pro-object by one where all maps are surjective. For profinite groupoids, this is not possible since the set-theoretic image of a morphism $f\colon A \to B$ between groupoids is not always a groupoid. By the set-theoretic image, we mean $f(\mathrm{Ar}(A)) \subseteq \mathrm{Ar}(B)$. However, there is still a good notion of image for maps between groupoids. Namely, we define $\mathrm{im}(f)$, the *image of* f, to be the smallest subgroupoid of B containing all arrows of the form $f(\alpha)$ with $\alpha \in \mathrm{Ar}(A)$. One can show that such a groupoid indeed exists, and that $\mathrm{im}(f)$ consists precisely of those arrows β of B that can be written as a composition $f(\alpha_1)f(\alpha_2)\dots f(\alpha_n)$ for certain $\alpha_1, \dots, \alpha_n \in \mathrm{Ar}(A)$. Note that although $f(\alpha_1), \dots, f(\alpha_n)$ can be composed in B, this does not have to hold for $\alpha_1, \dots, \alpha_n$ in A.

One can also show that f is epi if and only if $\mathrm{im}(f) = B$. It is easy to see that $\mathrm{im}(f) = B$ implies that f is epi; we leave this to reader to verify. The proof that $\mathrm{im}(f) = B$ holds if f is epi is more involved, and somewhat similar to the proof that epimorphisms of groups are surjective. As we do not ever need this result, we will not prove it here.

Using the notion of image defined above, we can prove the following proposition. Note that the proof uses Corollary 3.19 from the next section. However, the author found it more natural to state it in this section.

Proposition 3.14. *Let A be a profinite groupoid. Then there exists a projective diagram $\{A_i\}$ with A_i finite groupoids and with $A \cong \{A_i\}$, such that for each $j \leq i$, the map $A_j \to A_i$ is epi, and such that the projection $A \to A_i$ is epi for every i.*

Proof. Let $A = \{A'_i\}_{i \in I}$ be a profinite groupoid. Let $p_i\colon A \to A'_i$ be the projections, and denote the maps $A'_i \to A'_j$ for $i \leq j$ by p^i_j. Define $A_i = \mathrm{im}(p_i) \subseteq A'_i$, where $\mathrm{im}(p_i)$ denotes the smallest subgroupoid of A'_i containing $p_i(\mathrm{Ar}(A))$. Note that $p^j_i(\mathrm{im}(p_j)) \subseteq \mathrm{im}(p_i)$, i.e. $p^j_i(A_j) \subseteq A_i$, so the maps $p^j_i\colon A'_j \to A'_i$ restrict to maps $A_j \to A_i$. In particular $\{A_i\}_{i\in I}$ is a projective diagram of finite groupoids. The inclusions $A_i \hookrightarrow A'_i$ induce a map of profinite groupoids $\{A_i\}_{i\in I} \to \{A'_i\}_{i\in I}$. The corresponding map on arrows $\lim_i \mathrm{Ar}(A_i) \to \lim_i \mathrm{Ar}(A'_i) = \mathrm{Ar}(A)$ is clearly injective. It is also surjective. Indeed, for $(a_i)_i \in \lim_i \mathrm{Ar}(A'_i) = \mathrm{Ar}(A)$, we see that $a_j = p_j((a_i)_i) \in p_j(\mathrm{Ar}(A)) \subseteq A'_j$ for every j, hence $(a_i)_i \in \lim_i \mathrm{Ar}(A_i)$. Since a continuous bijective map of Stone spaces is a homeomorphism, we conclude by Corollary 3.19 that $\{A_i\} \cong A$.

Since $\mathrm{im}(p_j) = A_j$, we see that $p_j\colon A \to A_j$ is epi. Since $p_i = p^j_i p_j$, we conclude that $p^j_i\colon A_j \to A_i$ is epi as well. ∎

3.2 Profinite groupoids as topological groupoids

Recall from Example 2.16 that profinite sets can be seen as certain topological spaces, and that similarly $\widehat{\mathbf{Grp}}$ is a full subcategory of the category of topological groups. More specifically, $\widehat{\mathbf{Set}} \simeq \mathbf{Stone}$ and $\widehat{\mathbf{Grp}} \simeq \mathbf{StoneGrp}$, where **StoneGrp** is the category of group objects in the category of Stone spaces. In this section we will show that $\widehat{\mathbf{G}}$ similarly is a full subcategory of the category of groupoids internal to the category of Stone spaces, which in turn is a full subcategory of the category of topological groupoids **TopG**. We start by defining topological groupoids.

Recall that a groupoid A consists of two sets $\mathrm{Ob}(A)$ and $\mathrm{Ar}(A)$ and several maps

$$\begin{tikzcd} \mathrm{Ar}(A) \times_{\mathrm{Ob}(A)} \mathrm{Ar}(A) \arrow[r, "m"] & \mathrm{Ar}(A) \arrow[loop above, "\iota"] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & \mathrm{Ob}(A) \arrow[l, "e" description] \end{tikzcd} \tag{3.1}$$

which satisfy certain relations that can be expressed through commutative diagrams. Here m is the composition or multiplication map, ι maps an arrow to its inverse, s and t map an arrow to its source and target respectively, and e maps an object to its identity arrow. By $\mathrm{Ar}(A) \times_{\mathrm{Ob}(A)} \mathrm{Ar}(A)$ we mean the pullback

$$\begin{tikzcd} \mathrm{Ar}(A) \times_{\mathrm{Ob}(A)} \mathrm{Ar}(A) \arrow[r] \arrow[d] & \mathrm{Ar}(A) \arrow[d, "s"] \\ \mathrm{Ar}(A) \arrow[r, "t"'] & \mathrm{Ob}(A). \end{tikzcd}$$

This construction can be generalized to any category **C** having pullbacks, in particular to **Top** and **Stone**.

Definition 3.15. A *topological groupoid* consists of two topological spaces $\mathrm{Ar}(A)$ and $\mathrm{Ob}(A)$ and continuous maps m, ι, s, e and t as in (3.1). These morphisms must satisfy the usual relations defining a groupoid. A morphism between two topological groupoids A and B consists of continuous maps $\mathrm{Ar}(A) \to \mathrm{Ar}(B)$ and $\mathrm{Ob}(A) \to \mathrm{Ob}(B)$ which commute with all the maps of (3.1). A *Stone groupoid* is a topological groupoid A such that $\mathrm{Ob}(A)$ and $\mathrm{Ar}(A)$ are Stone spaces. The category of Stone groupoids **StoneG** is defined as a full subcategory of the category of topological groupoids **TopG**. $\diamondsuit$

Note that pullbacks of Stone spaces (in **Top**) are again Stone spaces. In particular, the pullback $\mathrm{Ar}(A) \times_{\mathrm{Ob}(A)} \mathrm{Ar}(A)$ is the same when computed in **Stone** as when computed in **Top**, so the above definition of a Stone groupoid indeed makes sense. Since e is a section of s, we can regard $\mathrm{Ob}(A)$ as a subspace of $\mathrm{Ar}(A)$. Many of the constructions of section 3.1 also exists for topological groupoids and Stone groupoids. When speaking about a connected (topological or Stone) groupoid, we will mean that the underlying groupoid itself is connected. This has nothing to do with the connectedness of the spaces $\mathrm{Ob}(A)$ and $\mathrm{Ar}(A)$.

Example 3.16. Let $\Pi_1 S^1$ be the fundamental groupoid of the topological space S^1. We can give $\Pi_1 S^1$ the structure of a topological groupoid. Assume we are given $[\gamma] \in \mathrm{Ar}(\Pi_1 S^1)$, represented by $\gamma\colon I \to S^1$. Let $U_0, U_1 \subseteq S^1$ be open with $\gamma(0) \in U_0$ and $\gamma(1) \in U_1$. Define $V_\gamma(U_0, U_1) \subseteq \mathrm{Ar}(A)$ by

$$V_\gamma(U_0, U_1) = \{[\alpha_0 \cdot \gamma \cdot \alpha_1] \mid \alpha_i\colon I \to U_i, \alpha_0(1) = \gamma(0), \alpha_1(0) = \gamma_1\}$$

The subsets of the form $V_\gamma(U_0, U_1)$ generate a topology on $\mathrm{Ar}(\Pi_1 S^1)$, giving $\Pi_1 S^1$ the structure of a topological groupoid. For any $x \in S^1$, the subspace of $\mathrm{Ar}(\Pi_1 S^1)$ of paths starting at x is the universal cover of S^1. $\diamondsuit$

If we are given a profinite groupoid $A = \{A_i\}_{i\in I}$, then we can associate a topological groupoid TA to it, by viewing A_i as topological groupoids with a discrete set of arrows, and letting $TA = \lim_i A_i$. In this way we obtain a functor from $\widehat{\mathbf{G}}$ to **StoneG**. We denote this functor $\widehat{\mathbf{G}} \to \mathbf{StoneG}$ by T. If we are given a profinite groupoid $A = \{A_i\}$, then the associated topological groupoid TA satisfies $\mathrm{Ar}(TA) = \lim_i \mathrm{Ar}(A_i)$ and $\mathrm{Ob}(TA) = \lim_i \mathrm{Ob}(A_i)$. In fact, any limit in **TopG** is computed by $\mathrm{Ar}(\lim_i A_i) = \lim_i \mathrm{Ar}(A_i)$ and $\mathrm{Ob}(\lim_i A_i) = \lim_i \mathrm{Ob}(A_i)$. As in the proof that $\widehat{\mathbf{Set}} \simeq \mathbf{Stone}$, Theorem 2.52, we will use lemma Proposition 2.50 to show that $\widehat{\mathbf{G}}$ is the full subcategory of **StoneG** of those Stone groupoids that can be written as a cofiltered limit of finite discrete groupoids. It is immediately apparent that this subcategory has all cofiltered limits, that $\mathbf{FinG} \to \mathbf{StoneG}$ is fully faithful, and that any object is a cofiltered limit of objects in the image of $\mathbf{FinG} \to \mathbf{StoneG}$. We therefore only have to show that finite groupoids are cocompact in this full subcategory.

Proposition 3.17. *The category* $\widehat{\mathbf{G}}$ *is equivalent to a full subcategory of* **StoneG**. *Both the inclusions* $\widehat{\mathbf{G}} \to \mathbf{StoneG}$ *and* $\widehat{\mathbf{G}} \to \mathbf{TopG}$ *have a left adjoint* $\widehat{(\cdot)}$.

Proof. Let $\mathbf{D} \subseteq \mathbf{StoneG}$ denote the full subcategory of Stone groupoids that are a cofiltered limit of finite groupoids. As stated above, we only have to check the cocompactness assumption of Proposition 2.50. Since any object in $\mathbf{D}$ is a cofiltered limit of finite discrete groupoids, we only have to check cocompactness with respect to projective limits of finite discrete groupoids.

Let $\{A_i\}$ be a projective diagram of finite groupoids, and let B be a finite groupoid. Denote $A = \lim_i A_i$, write p_i for the projection map $A \to A_i$, and denote the maps $A_i \to A_j$ for $i \leq j$ by p^i_j. We need to show that the canonical map $\mathrm{colim}_i \mathrm{Hom}(A_i, B) \to \mathrm{Hom}(\lim_i A_i, B)$ is a bijection. For injectivity, note that a morphism $f\colon \lim_i A_i \to B$ is fully determined by the induced map of profinite sets $\lim_i \mathrm{Ar}(A_i) \to \mathrm{Ar}(B)$. Injectivity therefore follows from the fact that the canonical map $\mathrm{colim}_i \mathrm{Hom}_{\mathbf{Stone}}(\mathrm{Ar}(A_i), \mathrm{Ar}(B)) \to \mathrm{Hom}_{\mathbf{Stone}}(\lim_i \mathrm{Ar}(A_i), \mathrm{Ar}(B))$ is injective (see Theorem 2.52).

For surjectivity, let $f\colon \lim_i A_i \to B$ be given, again noting that it is fully determined by $f'\colon \lim_i \mathrm{Ar}(A_i) \to \mathrm{Ar}(B)$. By Theorem 2.52, there is some i and some map $f_i\colon \mathrm{Ar}(A_i) \to \mathrm{Ar}(B)$ such that $f' = f_i p_i$. This map f_i does not in general correspond to a morphism of groupoids. To obtain a morphism of groupoids, note that $\mathrm{Ar}(A) \times_{\mathrm{Ob}(A)} \mathrm{Ar}(A) = \lim_i(\mathrm{Ar}(A_i) \times_{\mathrm{Ob}(A_i)} \mathrm{Ar}(A_i))$. Let $q_i\colon \mathrm{Ar}(A) \times_{\mathrm{Ob}(A)} \mathrm{Ar}(A) \to \mathrm{Ar}(A_i) \times_{\mathrm{Ob}(A_i)} \mathrm{Ar}(A_i)$ be the projection and q^i_j the map $\mathrm{Ar}(A_i) \times_{\mathrm{Ob}(A_i)} \mathrm{Ar}(A_i) \to \mathrm{Ar}(A_j) \times_{\mathrm{Ob}(A_j)} \mathrm{Ar}(A_j)$ for $i \leq j$. There is a $k \leq i$ such that $\mathrm{im}(q_i) = \mathrm{im}(q^k_i)$ by Proposition 2.47. The map $f' p^k_i\colon \mathrm{Ar}(A_k) \to \mathrm{Ar}(B)$ induces a map of groupoids $f_k\colon A_k \to B$ satisfying $f_k p_k = f$; we leave it to the reader to verify this.

The existence of the left adjoints follows directly from Theorem 2.62, which asserts that there exist profinite completion functors $\mathbf{StoneG} \to \widehat{\mathbf{G}}$ and $\mathbf{TopG} \to \widehat{\mathbf{G}}$. ■

Remark 3.18. The above proof could also have been given in an easier, but less direct, way. Namely, note that a Stone groupoid is in particular a diagram $F\colon I \to \mathbf{Stone} \simeq \widehat{\mathbf{Set}}$

where I is given by

$$\begin{tikzcd} A_2 \arrow[r, "m"] & A_1 \arrow[loop above, "\iota"] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & A_0, \arrow[l, "e" description] \end{tikzcd}$$

where the arrows satisfy the usual relations of a groupoid. Such a diagram F is a Stone groupoid precisely if $F(A_2) = F(A_1) \times_{F(A_0)} F(A_1)$. In particular, we see that **StoneG** is a full subcategory of $\mathbf{Stone}^I$. We already know that $\mathbf{Stone}^I \simeq \mathrm{Pro}(\mathbf{FinSet}^I)$, since $\mathbf{Stone} \simeq \mathrm{Pro}(\mathbf{FinSet})$, I is finite and **FinSet** is essentially small and has finite limits. If we now view **FinG** as a full subcategory of $\mathbf{FinSet}^I$, then we can view $\widehat{\mathbf{G}}$ as a full subcategory of $\mathrm{Pro}(\mathbf{FinSet}^I)$. By inspecting the equivalence $\mathrm{Pro}(\mathbf{FinSet}^I) \to \mathbf{Stone}^I$, we see that this full subcategory gets mapped to a full subcategory of $\mathbf{StoneG} \subseteq \mathbf{Stone}^I$, which also proves the above proposition. ♢

Corollary 3.19. *Let $f\colon A \to B$ be a morphism of profinite groupoids such that the induced map $\mathrm{Ar}(A) \to \mathrm{Ar}(B)$ is a bijection. Then f is an isomorphism.*

Proof. As $\widehat{\mathbf{G}}$ is a full subcategory of **StoneG**, we can view f as a morphism between Stone groupoids. Then $\mathrm{Ar}(A) \to \mathrm{Ar}(B)$ is a continuous bijection, hence a homeomorphism. The inverse map $\mathrm{Ar}(B) \to \mathrm{Ar}(A)$ defines a map of groupoids $g\colon B \to A$, and is continuous, hence it is a morphism of Stone groupoids. We conclude that f has an inverse g, hence that it is an isomorphism. ■

3.2.1 Characterizing profinite groupoids among Stone groupoids

When seeing Proposition 3.17, one might wonder which Stone groupoids are a cofiltered limit of finite ones. The examples of profinite groups, profinite G-sets and profinite G-modules from section 2.2 might suggest that the fully faithful functor $\widehat{\mathbf{G}} \to \mathbf{StoneG}$ is, in fact, an equivalence of categories. This is, however, not the case, as Example 3.20 below will illustrate. The author has not been able to give a precise characterization of profinite groupoids among Stone groupoids, but some partial results are presented here.

In section 3.1, we defined $\pi_0\colon \widehat{\mathbf{G}} \to \widehat{\mathbf{Set}}$ by extending the usual $\pi_0\colon \mathbf{FinG} \to \mathbf{FinSet}$ to a functor that preserves cofiltered limits. This means that, for a profinite groupoid $A = \{A_i\}_{i\in I}$, $\pi_0 A$ is defined as $\{\pi_0 A_i\}_{i\in I}$. According to Proposition 3.13, for any two objects $x, y \in \mathrm{Ob}(A)$, there is an arrow $x \to y$ in $\mathrm{Ar}(A)$ precisely if x and y are mapped to the same element of $\pi_0 A$, viewing $\mathrm{Ob}(A)$, $\mathrm{Ar}(A)$ and $\pi_0 A$ as Stone spaces. As a surjective map between Stone spaces is always a quotient map, we see that $\pi_0 A$ is a quotient of $\mathrm{Ob}(A)$. Denote the equivalence relation corresponding to this quotient by $\sim$, so $\sim$ is defined by $x \sim y$ if and only if there exists an $\alpha \in \mathrm{Ar}(A)$ with source x and target y. Note that this implies that the quotient $\pi_0(A) = \mathrm{Ob}(A)/\sim$, when computed in **Top**, is always a Stone space.

For a Stone groupoid B, we can define the same equivalence relation, meaning that for $x, y \in \mathrm{Ob}(B)$, we have $x \sim y$ if and only if there exists an arrow $x \to y$. Denote the $\mathrm{Ob}(B)/\sim$, computed in **Top**, by $\pi_0^{\mathbf{T}}(B)$. If B is a profinite groupoid, then by what we just discussed, $\pi_0^{\mathbf{T}}(B)$ is a Stone space. Hence a necessary condition for B to be a profinite groupoid, is that $\pi_0^{\mathbf{T}}(B)$ is a Stone space. However, as the next example illustrates, this is not always the case.

Example 3.20. Let $C = \{0,1\}^{\mathbb{N}}$ be the space of infinite sequences of 0's and 1's, with the product topology. As this is a product of finite discrete spaces, it is a Stone space. Recall that Cantor constructed an embedding $C \to [0,1]$, where $[0,1]$ is the unit interval, by mapping a sequence $(a_n)_{n\in\mathbb{N}}$ to the infinite sum

$$\sum_{n=0}^{\infty} \frac{2a_n}{3^{n+1}}.$$

The image of this embedding, usually called the Cantor set, can also be constructed by removing the middle 1/3rd of the unit interval, then removing the middle 1/3rd of the remaining two (closed) intervals, and continuing this procedure indefinitely. Points in the Cantor set can be described by stating in which of the two intervals (left or right) the point lies, at every stage of this construction. If we write 0 for left, and right to 1, then the above embedding $C \to [0,1]$ precisely maps a sequence to the point in the Cantor set that it describes.

We can consider a variation of this map, which is surjective instead of injective. Define $f\colon C \to [0,1]$ by

$$f((a_n)_{n\in\mathbb{N}}) = \sum_{n=0}^{\infty} \frac{2a_n}{2^{n+1}}.$$

If we are given a point $r \in [0,1]$, then it is either in the left half $[0,1/2]$ or right half $[1/2,1]$ of $[0,1]$ (possibly in both). It is then, again, either in the left half or right half of this interval, and so on. This way, we can associate a (not unique) sequence $(a_n)_{n\in\mathbb{N}}$ of 0's and 1's to r, where 0 means left and 1 means right. Then $f((a_n)_{n\in\mathbb{N}}) = r$, so f is surjective. Define $R \subseteq C \times C$ by

$$R = \{(x,y) \in C \times C \mid f(x) = f(y)\}.$$

Then R is closed in C, hence a Stone space, and $C/R \cong [0,1]$ in **Top**, as a surjection between compact Hausdorff spaces is always a quotient map.

Define a Stone groupoid A by $\mathrm{Ob}(A) = C$ and $\mathrm{Ar}(A) = R$. The projections $p_1, p_2\colon R \to C$ are the source and target map. This defines a Stone groupoid in the obvious way: for two arrows $(z,w),(x,y) \in \mathrm{Ar}(A) = R$, their composition $(z,w)(x,y)$ exists if and only if $z = y$, and in this case it is given by (x,w). We see that $\pi_0^{\mathbf{T}}(A) \cong [0,1]$, so A is not a profinite groupoid. $\diamondsuit$

We call a Stone groupoid A *connected* if $\pi_0^{\mathbf{T}}(A) = \{*\}$. We will show that any connected Stone groupoid is a projective limit of finite ones. This means that the functor $T\colon \widehat{\mathbf{G}} \to \mathbf{StoneG}$ becomes an equivalence when we restrict connected profinite groupoids and connected Stone groupoids. We will in fact prove something stronger, namely that any connected Stone groupoid is of the form $G[S]$, where G is a profinite group, and S a profinite set.

Proposition 3.21. *If A is a connected Stone groupoid, then A is of the form $G[S]$ for a profinite group G and a profinite set S. In particular, A is a projective limit of finite groupoids.*

Proof. Pick $a \in \mathrm{Ob}(A)$. Then $t^{-1}(a)$ is a closed subset of $\mathrm{Ar}(A)$, consisting precisely of all arrows whose target is a. Define $G := A(a)$. Then $A(a)$ acts on $t^{-1}(a)$ by defining $\alpha \cdot \gamma = \alpha\gamma$ for any $\alpha \in A(a)$ and $\gamma \in t^{-1}(a)$. This action is continuous since the multiplication of the Stone groupoid is continuous, and it is clearly a free action. We therefore obtain an isomorphism $\phi\colon t^{-1}(a) \to (t^{-1}(a)/G) \times G$ by Proposition 2.61. Note that the map $s\colon t^{-1}(a) \to \mathrm{Ob}(A)$ is a quotient map, and that $s(\beta) = s(\gamma)$ if and only if $\beta = \alpha\gamma$ for some $\alpha \in G$. We therefore obtain a canonical isomorphism $t^{-1}(a)/G \cong \mathrm{Ob}(A)$. By translating if necessary, we can assume without loss of generality that $\phi(\mathrm{id}_a) = (a, \mathrm{id}_a) \in \mathrm{Ob}(A) \times A$. Define the continuous map $q\colon \mathrm{Ob}(A) \to \mathrm{Ar}(A)$ by $q(a') = \phi^{-1}(a', \mathrm{id}_a)$. We then see that $t(q(a')) = a$ and $s(q(a')) = a'$ for every $a' \in \mathrm{Ob}(A)$.

Now define a map of Stone groupoids $f\colon A \to G[\mathrm{Ob}(A)]$ by $f = \mathrm{id}_{\mathrm{Ob}(A)}$ on objects, and by $f(\alpha) = q(t(\alpha))\alpha q(s(\alpha))^{-1}$. This map is continuous since the maps q, s and t are continuous, and since the inverse map and multiplication map of A are continuous. It is easily verified that f is indeed a map of groupoids, and that it is in fact a bijection on arrows. Since continuous bijections between Stone spaces are homeomorphisms, we conclude that f is an isomorphism of Stone groupoids.

To see that A is a projective limit of finite groupoids, let $G = \lim_{i \in I} G_i$ and $\mathrm{Ob}(A) = \lim_{j \in J} S_j$, with G_i finite groups and S_j finite sets. Then

$$A \cong G[\mathrm{Ob}(A)] \cong \lim_{(i,j) \in I \times J} G_i[S_j]. \qquad \blacksquare$$

In particular, we have the following useful corollary. However, note that the isomorphism $A \cong G[S]$ is not canonical.

Corollary 3.22. *Let A be a connected profinite groupoid. Then $A \cong G[S]$ for some profinite group G and profinite set S.* $\blacksquare$

We have the following (partial) characterization of profinite groupoids among Stone groupoids.

Corollary 3.23. *Let A be a Stone groupoid. If $\pi_0^{\mathbf{T}}(A)$ is finite, then A is a profinite groupoid. If A is a profinite groupoid, then $\pi_0^{\mathbf{T}}(A)$ must be a Stone space.* $\blacksquare$

3.3 Profinite completion

In this section we show that the profinite completion functor for groupoids can be described in terms of the profinite completion functor for groups and sets and the coproduct of profinite groupoids. Unfortunately, the coproduct of infinitely many profinite groupoids is not easy to describe, but we do obtain a fairly concrete description of the profinite completion of a groupoid with finitely many connected components. We then deduce from this that the profinite completion functor behaves well with respect to products if we work with groupoids that have a finite set of objects.

The following proposition is a direct consequence of Theorem 2.62.

Proposition 3.24. *There exists a profinite completion functor* $\widehat{(\cdot)}\colon \mathbf{G} \to \widehat{\mathbf{G}}$*, left adjoint to the functor* $|\cdot|\colon \widehat{\mathbf{G}} \to \mathbf{G}$ *that sends a profinite groupoid* $\{A_i\}_{i\in I}$ *to its limit* $\lim_i A_i$ *in* $\mathbf{G}$*.* ■

Recall that the profinite completion functor of sets does not preserve products, yet the profinite completion functor of groups does preserve products. One easily sees that the profinite completion of the groupoid $\mathrm{Codisc}(S)$ is $\mathrm{Codisc}(\widehat{S})$ for any set S, in particular for $\mathbb{N} \times \mathbb{N}$. However, we see that

$$\mathrm{Codisc}(\widehat{\mathbb{N} \times \mathbb{N}}) \not\cong \mathrm{Codisc}(\widehat{\mathbb{N}} \times \widehat{\mathbb{N}}) \cong \widehat{\mathrm{Codisc}(\mathbb{N})} \times \widehat{\mathrm{Codisc}(\mathbb{N})},$$

so profinite completion of groupoids in general does not preserve products. However, it does preserve products of groupoids with finitely many objects, as we will prove below.

Recall that, for a morphism $f\colon A \to B$ of groupoids, we defined $\mathrm{im}(f)$ to be the smallest subgroupoid of B containing all arrows of the form $f(\alpha)$ with $\alpha \in \mathrm{Ar}(A)$. If $B = \mathrm{im}(f)$, then f is epi. We in particular see that any morphism of groupoids $f\colon A \to B$ factors as an epi followed by a mono $A \to \mathrm{im}(f) \to B$. If B is finite, then $\mathrm{im}(f)$ is also finite. By Remark 2.63, we see the following.

Proposition 3.25. *Let A be a groupoid. Define I to be the category with objects (B, f), where B is a finite groupoid and $f\colon A \to B$ is epi, and whose arrows $(B, f) \to (B', f')$ are morphisms $g\colon B \to B'$ such that $gf = f'$. Then I is cofiltered (in fact it is a codirected poset) and the forgetful functor $I \to$* **FinG** *defines a pro-object canonically isomorphic to* $\widehat{A}$*.* ■

Remark 3.26. One can in fact show that any morphism of groupoids $A \to B$ with B finite, factors as $A \to C \to B$ with $A \to C$ surjective on arrows, and C finite. As a corollary, we only need to consider morphisms $A \to B$ that are surjective on arrows when constructing $\widehat{A}$. As we will not need this stronger version of the above proposition, we leave the proof of this to the reader. ♢

Lemma 3.27. *Let G be a group and S a finite set. Then $\widehat{G[S]} \cong \widehat{G}[S]$.*

Proof. We will show that any morphism $G[S] \to A$ with A finite, factors through the map $q\colon G[S] \to G/N[S]$ for some normal subgroup N of G with G/N finite, where q is induced by the quotient map $G \to G/N$. By Remark 2.63, this implies $\widehat{A} = \{G/N[S]\}_N$, where N ranges over all normal subgroups of G with G/N finite, which is by definition equal to $\widehat{G}[S]$.

To see that this indeed holds, let $f\colon G[S] \to A$ with A finite be given. Since we only need to consider epimorphisms by the above proposition, we see that A must be connected. Without loss of generality $A = G'[S']$ for some finite group G' and some finite set S'. Pick $x \in S$. Let f_x the homomorphism $G \to G'$ induced by f at x, and let $N = \ker(f_x)$. Note that N does not depend on the choice of $x \in S$. Now f factors through $G \to G/N[S]$. To see this, define $\widetilde{f}\colon G/N[S] \to G'[S']$ by mapping the arrow $[g]\colon y \to z$, for some $y, z \in S$, to the arrow $f(g)$ from $f(y)$ to $f(z)$, where we see g as an arrow $y \to z$ in $G[S]$. It is left to the reader to verify that this is well-defined. ■

Remark 3.28. By slightly modifying the above proof, one can in fact show that for any group G and any set S one has $\widehat{G[S]} \cong \widehat{G}[\widehat{S}]$. Noting that any groupoid A is of the form $\bigsqcup_i G_i[S_i]$ for certain groups G_i and sets S_i, we see that $\widehat{A} \cong \bigsqcup_i \widehat{G}_i[\widehat{S}_i]$. Here we use that profinite completion is left adjoint, hence that it preserves coproducts. However, if the coproduct $\bigsqcup_i \widehat{G}_i[\widehat{S}_i]$ is infinite, then it is not computed by taking "disjoint unions", as the result will not be a profinite groupoid (infinite disjoint unions of compact spaces are not compact). This coproduct can be computed by first computing it in the category of topological groupoids **TopG**, where it is just a disjoint union, and then applying the profinite completion functor $\mathbf{TopG} \to \widehat{\mathbf{G}}$, which exists by Theorem 2.62. $\diamondsuit$

Proposition 3.29. *Let A and B be groupoids with finitely many objects. Then $\widehat{A \times B} \cong \widehat{A} \times \widehat{B}$.*

Proof. Pick isomorphisms $A \cong \sqcup_i G_i[S_i]$ and $B \cong \sqcup_j H_j[T_j]$. Then $A \times B \cong \sqcup_{i,j} G_i \times H_j[S_i \times S_j]$. Since profinite completion is left adjoint, it preserves coproducts. Since profinite completion of groups preserves products, we see that

$$\widehat{A \times B} \cong \sqcup_{i,j}(\widehat{G_i} \times \widehat{H_j})[S_i \times S_j] \cong \sqcup_i \widehat{G_i}[S_i] \times \sqcup_j \widehat{G_j}[S_j] \cong \widehat{A} \times \widehat{B}. \qquad \blacksquare$$

3.4 Weak equivalences of profinite groupoids

In the next section we will construct a model structure on $\widehat{\mathbf{G}}$ which in some sense corresponds to the canonical model structure on $\mathbf{G}$. This section is devoted to defining and studying the weak equivalences in $\widehat{\mathbf{G}}$. We will look at their behaviour with respect to cofiltered limits, show that they are precisely the fully faithful and essentially surjective morphisms of profinite groupoids, that they have a level representation by weak equivalences, and we will show that they agree with homotopy equivalences between connected profinite groupoids.

First recall the well-known *canonical model structure* on $\mathbf{G}$.

Proposition 3.30. *There exists a model structure on $\mathbf{G}$ such that*

(i) a morphism $f\colon A \to B$ is a weak equivalence precisely if it is an equivalence in the categorical sense, i.e. f is fully faithful and essentially surjective,

(ii) a morphism $f\colon A \to B$ is a cofibration if it is injective on objects, and

(iii) a morphism $f\colon A \to B$ is a fibration if it has the right lifting property with respect to the inclusion $\delta_1\colon I[0] \to I[1]$, meaning that for any commutative square of the form

$$\begin{tikzcd} I[0] \arrow[r] \arrow[d, hook, "\delta_1"'] & A \arrow[d, "f"] \\ I[1] \arrow[r] \arrow[ur, dotted] & B, \end{tikzcd}$$

there exists a diagonal (the dotted arrow) making both triangles commute.

In this model structure all objects are fibrant and cofibrant. ■

Note that a natural isomorphism between morphisms of groupoids $A \to B$ can be seen as morphism $H\colon A \to B^{I[1]}$ such that $ev_0 H = f$ and $ev_1 H = g$. This is a straightforward check and left to the reader. We will call f and g *homotopic* if such a morphism H exists, and we call f a *homotopy equivalence* if an inverse up to homotopy exists. Denote by $\pi\mathbf{G}$ the category whose objects are those of $\mathbf{G}$, and whose morphisms are homotopy classes of morphisms in $\mathbf{G}$. The homotopy category $\mathrm{Ho}(\mathbf{G})$ corresponding to the canonical model structure on $\mathbf{G}$ is equivalent to $\pi\mathbf{G}$. The reason for this is that every object of $\mathbf{G}$ is fibrant and cofibrant and that $B^{I[1]}$ is a path object for every groupoid B.

3.4.1 The definition of a weak equivalence

For a profinite groupoid $A = \{A_i\}$, note that $A \times I[1] = \{A_i \times I[1]\}$ defines the product with $I[1]$ in $\widehat{\mathbf{G}}$. If we define $A^{I[1]} = \{A_i^{I[1]}\}$, then $(-) \times I[1]$ is left adjoint to $(-)^{I[1]}$. The evaluation morphisms $\mathrm{ev}_0, \mathrm{ev}_1\colon A_i^{I[1]} \to A_i$ induce morphisms $\mathrm{ev}_0, \mathrm{ev}_1\colon A^{I[1]} \to A$. The naive approach to doing homotopy theory in $\widehat{\mathbf{G}}$ would be to define two morphisms of profinite groupoids $f, g\colon A \to B$ to be homotopic if there exists some $H\colon A \to B^{I[1]}$ with $\mathrm{ev}_0 H = f$ and $\mathrm{ev}_1 H = g$, and define the homotopy category $\pi\widehat{\mathbf{G}}$ by identifying homotopic maps.

Definition 3.31. We say that two morphisms $f, g\colon A \to B$ in $\mathbf{G}$ (resp. $\widehat{\mathbf{G}}$) are *homotopic* if there exists some $H\colon A \to B^{I[1]}$ with $\mathrm{ev}_0 H = f$ and $\mathrm{ev}_1 H = g$. This defines a congruence on the category $\mathbf{G}$ (resp. $\widehat{\mathbf{G}}$), and we denote the category obtained by identifying all homotopic morphisms by $\pi\mathbf{G}$ (resp. $\pi\widehat{\mathbf{G}}$). ♢

Note that, in the above definition, $\mathrm{Hom}_{\pi\mathbf{G}}(A, B)$ is by definition the coequalizer of the pair $(\mathrm{ev}_0)_*, (\mathrm{ev}_1)_*\colon \mathrm{Hom}_{\mathbf{G}}(A, B^{I[1]}) \to \mathrm{Hom}_{\mathbf{G}}(A, B)$, and similar for $\mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(A, B)$. However, when defining the homotopy category of $\widehat{\mathbf{G}}$ in this way, we run into some problems. For example, if we are given two profinite groupoids $\{A_i\}$, $\{B_i\}$ indexed by the same set, and a levelwise weak equivalence $f_i\colon A_i \to B_i$ (i.e. a natural transformation consisting of weak equivalences), then the induced morphism $f\colon \{A_i\} \to \{B_i\}$ will not always be a homotopy equivalence of profinite groupoids. An example of such a morphism will be given in Example 3.50, in the last part of this section.

We therefore need a "weaker" notion of weak equivalence. Using the Yoneda lemma, we see that a morphism $f\colon A \to B$ of groupoids (resp. profinite groupoids) is a homotopy equivalence if and only if the induced map $f^*\colon \mathrm{Hom}_{\pi\mathbf{G}}(B, C) \to \mathrm{Hom}_{\pi\mathbf{G}}(A, C)$ (resp. $f^*\colon \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(B, C) \to \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(A, C)$) is an isomorphism for every groupoid (resp. profinite groupoid) C. The following notion of weak equivalence turns out to be the right one in $\widehat{\mathbf{G}}$.

Definition 3.32. Let $f\colon A \to B$ be a morphism in $\widehat{\mathbf{G}}$. We say that f is a weak equivalence if for every finite groupoid C, the induced map

$$f^*\colon \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(B, C) \to \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(A, C)$$

is a bijection. ♢

To check if $f\colon A \to B$ is a weak equivalence, we only need to check the cases $C = \mathrm{Disc}(S)$ and $C = G_*$ for finite sets S and finite groups G.

Proposition 3.33. *Let $f\colon A \to B$ be a morphism of profinite groupoids, and assume that for any finite set S and any finite group G, the maps*

$$f^*\colon \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(B, \mathrm{Disc}\, S) \to \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(A, \mathrm{Disc}\, S)$$
$$f^*\colon \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(B, G_*) \to \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(A, G_*)$$

are bijections. Then f is a weak equivalence.

Proof. We call f a C-equivalence if the map

$$f^*\colon \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(B, C) \to \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(A, C)$$

is an isomorphism. We want to show that f is a C-equivalence for all finite groupoids C, so let C be any finite groupoid. Then C is homotopy equivalent to a finite groupoid of the form $G_1 \sqcup \ldots \sqcup G_n$ where $G_1, \ldots, G_n$ are finite groupoids with one object. A homotopy equivalence between finite groupoids is an isomorphism in $\pi\widehat{\mathbf{G}}$, so f is a C-equivalence if and only if f is a $G_1 \sqcup \ldots \sqcup G_n$-equivalence. Now note that $\mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(D, E \times F) \cong \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(D, E) \times \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(D, F)$, for all (pro)finite groupoids D, E and F, so f is an $E \times F$-equivalence if F is both an E-equivalence and an F-equivalence. In particular, f is by assumption a $G_1 \times \ldots \times G_n \times \{1, \ldots, n\}$-equivalence, where $\{1, \ldots, n\}$ denotes the discrete groupoid with n objects. We see that $G_1 \sqcup \ldots \sqcup G_n$ is a retract of $G_1 \times \ldots \times G_n \times \{1, \ldots, n\}$, where the inclusion is given by mapping $g \in G_i$ to $(0, \ldots, 0, g, 0 \ldots, 0, i)$, and where the retraction is given by mapping $(g_1, \ldots, g_n, i)$ to $g_i \in G_i$. We leave it to the reader to verify that if f is a D-equivalence, and E is a retract of D, then f is also an E-equivalence. This completes the proof. ■

Remark 3.34. Horel's approach in [Hor17] to defining weak equivalences is slightly different. He defines the cohomology $H^0(-; S)$ and $H^1(-; G)$, and defines f to be a weak equivalence if it induces a bijection on $H^0(-; S)$ and $H^1(-; G)$ for any finite set S and finite group G. It is then proved that $H^0(-; S) \cong \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(-, \mathrm{Disc}(S))$ and $H^1(-; G) \cong \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(-, G_*)$, and later that a weak equivalence induces bijections $\mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(B, C) \to \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(A, C)$ for all finite groupoids C. This definition of a weak equivalence is convenient when comparing the model structure on $\widehat{\mathbf{G}}$ to the model structure on $\widehat{\mathbf{S}}$, which we will define in chapter 4, but the above definition seems more natural when working in a pro-category. ♢

3.4.2 Basic properties of weak equivalences

The following proposition is immediate from Yoneda's lemma.

Proposition 3.35. *Let $f\colon A \to B$ be a morphism of finite groupoids. Then f is a weak equivalence in $\mathbf{G}$ if and only if f is a weak equivalence in $\widehat{\mathbf{G}}$.* ■

The universal property of the profinite completion functor implies that it preserves weak equivalences.

Proposition 3.36. *Let $f\colon A \to B$ be a weak equivalence in $\mathbf{G}$. Then $\widehat{f}\colon \widehat{A} \to \widehat{B}$ is a weak equivalence in $\widehat{\mathbf{G}}$.*

Proof. Note that there is a natural bijection $\mathrm{Hom}_{\mathbf{G}}(A, C) \cong \mathrm{Hom}_{\widehat{\mathbf{G}}}(\widehat{A}, C)$ for any finite groupoid C. Thus there is also a natural bijection

$$\begin{aligned}\mathrm{Hom}_{\pi\mathbf{G}}(A, C) &= \mathrm{coeq}(\mathrm{Hom}_{\mathbf{G}}(A, C^{I[1]}) \rightrightarrows \mathrm{Hom}_{\mathbf{G}}(A, C)) \\ &\cong \mathrm{coeq}(\mathrm{Hom}_{\widehat{\mathbf{G}}}(\widehat{A}, C^{I[1]}) \rightrightarrows \mathrm{Hom}_{\widehat{\mathbf{G}}}(\widehat{A}, C)) = \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(\widehat{A}, C).\end{aligned}$$

Since $f^*\colon \mathrm{Hom}_{\pi\mathbf{G}}(B, C) \to \mathrm{Hom}_{\pi\mathbf{G}}(A, C)$ is a bijection, we conclude by the above natural bijection that $f^*\colon \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(\widehat{B}, C) \to \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(\widehat{A}, C)$ is so as well. ■

By noting that a groupoid with finitely many connected components is weakly equivalent to a groupoid of the form $\coprod_{i=1}^n (G_i)_*$, we obtain the following.

Corollary 3.37. *Let A be a groupoid with finitely many connected components. Then $\widehat{A}$ is weakly equivalent to a profinite groupoid of the form $\coprod_{i=1}^n (\widehat{G}_i)_*$.* ■

As stated above, a levelwise weak equivalence $\{f_i\}$ between two profinite groups is generally not a homotopy equivalence in $\widehat{\mathbf{G}}$, i.e. an isomorphism in $\pi\widehat{\mathbf{G}}$. This in particular implies that $\mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(A, B)$ is in general not equal to $\lim_j \mathrm{colim}_i \mathrm{Hom}_{\pi\mathbf{G}}(A_i, B_j)$ for profinite groupoids $A = \{A_i\}$ and $B = \{B_j\}$. If this were to hold, then the homotopy inverses g_i of f_i would induce a homotopy inverse of $\{f_i\}$. However, it does hold if either $\{A_i\}$ or $\{B_j\}$ is a finite groupoid.

Proposition 3.38. *Let $A = \{A_i\}_{i\in I}$ be a profinite groupoid, let B be a profinite groupoid with finitely many objects and let C be a finite groupoid. Then there are natural isomorphisms*

$$\mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(A, C) \cong \operatorname*{colim}_i \mathrm{Hom}_{\pi\mathbf{G}}(A_i, C)$$

and

$$\mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(B, A) \cong \lim_i \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(B, A_i).$$

Proof. The first isomorphism follows since coequalizers commute with colimits, and since $\mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(A, C)$ is the coequalizer of $\mathrm{Hom}_{\widehat{\mathbf{G}}}(A, C^{I[1]}) \rightrightarrows \mathrm{Hom}_{\widehat{\mathbf{G}}}(A, C)$.

The second isomorphism requires more work to prove. Denote the maps $A_j \to A_i$ for $j \leq i$ by p_i^j. Let $f\colon B \to A$ be a morphism of profinite groupoids. Then f consists of maps $f_i\colon B \to A_i$ satisfying $f_i p_i^j = f_j$, so we obtain an element in $\lim_i \mathrm{Hom}_{\pi\mathbf{G}}(B, A_i)$. To see that this defines a well-defined map $\phi\colon \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(B, A) \to \lim_i \mathrm{Hom}_{\pi\mathbf{G}}(B, A_i)$, assume $f \simeq g$. Let $H\colon B \to A^{I[1]}$ be a homotopy. Then H consists of homotopies $H_i\colon B \to A_i^{I[1]}$ from f_i to g_i by definition, hence $(f_i)_i$ and $(g_i)_i$ represent the same element in $\lim_i \mathrm{Hom}_{\pi\mathbf{G}}(B, A_i)$.

To see that ϕ is injective and surjective, observe that since B has finitely many objects, for any $f_i\colon B \to A_i$ there can only exist finitely many homotopies $H_i\colon B \to A_i^{I[1]}$ satisfying $\mathrm{ev}_0 H_i = f_i$. Indeed, assume we are given such a homotopy H_i, and let $\alpha\colon x \to y$ be an arrow in B. Then $H_i(\alpha)$ is a natural transformation between $H_i(x)$ and $H_i(y)$, i.e. a pair of arrows β, β' in A_i satisfying $H_i(y)\beta = \beta' H_i(x)$. Since $\mathrm{ev}_0 H_i = f_i$, we see that $\beta = \mathrm{ev}_0 H(\alpha) = f(\alpha)$, and $\beta' = H_i(y)\beta H_i(x)^{-1}$, so H_i is fully determined by some map $\mathrm{Ob}(B) \to A_i^{I[1]}$. Since both $\mathrm{Ob}(B)$ and $A_i^{I[1]}$ are finite, we conclude that only finitely many homotopies H_i with $\mathrm{ev}_0 H_i = f_i$ can exist.

For surjectivity, let $([f_i])_i \in \lim_i \mathrm{Hom}_{\pi\mathbf{G}}(B, A_i)$ be given, where $[f_i]$ is the homotopy class of the map $f_i\colon B \to A_i$. Then by the above observation, $[f_i]$ is a finite set for every i. We also see that the maps $p_i^j\colon A_j \to A_i$ restrict to maps $[f_j] \to [f_i]$, so $\{[f_i]\}_{i\in I}$ is a projective diagram of finite, nonempty sets. In particular its limit is nonempty by Theorem 2.43, so let $f \in \lim_i [f_i]$. Then f is by definition a morphism $f\colon B \to A$ satisfying $\phi([f]) = ([f_i])_i$, so ϕ is surjective.

For injectivity, let $\phi([f]) = \phi([g])$, with $f = (f_i)_i$ and $g = (g_i)_i$. Let X_i be the set of all homotopies $H_i\colon B \to A_i^{I[1]}$ satisfying $\mathrm{ev}_0 H_i = f_i$ and $\mathrm{ev}_1 H_i = g_i$. By the above observation each X_i is a finite nonempty set. The maps $p_i^j\colon A_j \to A_i$ induce maps $X_j \to X_i$, so by Theorem 2.5, there is some $H \in \lim_i X_i$. Then H is by construction a homotopy $B \to A^{I[1]}$ satisfying $\mathrm{ev}_0 H = f$ and $\mathrm{ev}_1 H = g$, hence $[f] = [g]$. We conclude that ϕ is a bijection. ∎

Corollary 3.39. *Let A, B be profinite groupoids with finitely many objects, and let $f\colon A \to B$ be a weak equivalence. Then f is a homotopy equivalence.*

Proof. By the second isomorphism of Proposition 3.38, we see that if $f^*\colon \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(B, C) \to \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(A, C)$ is an isomorphism for all finite groupoids C, then it is so for every profinite groupoid. By Yoneda's lemma f is a homotopy equivalence. ∎

Proposition 3.40. *Weak equivalences in $\widehat{\mathbf{G}}$ are stable under cofiltered limits. In particular, if $A = \{A_i\}$ and $B = \{B_i\}$ are profinite groupoids with the same index category, and if $f\colon A \to B$ is levelwise a weak equivalence, meaning f is represented by weak equivalences $f_i\colon A_i \to B_i$ in $\mathbf{G}$, then f is a weak equivalence in $\widehat{\mathbf{G}}$.*

Proof. The contravariant functor $\mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(-, C)$ preserves cofiltered limits of representables (or, more precisely, maps them to the corresponding filtered colimit) by the above proposition. By a proof similar to Proposition 2.31, it preserves all cofiltered limits. Now let $F, G\colon I \to \widehat{\mathbf{G}}$ be two cofiltered diagrams, let $\{f_i\}\colon F \to G$ be a natural transformation that is levelwise a weak equivalence, and let $f\colon \lim_i F(i) \to \lim_i G(i)$ denote the corresponding morphism between the limits. Then

$$f^*\colon \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(\lim_i G(i), C) \to \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(\lim_i F(i), C)$$

is the colimit of the maps

$$f_i^*\colon \mathrm{Hom}_{\pi\mathbf{G}}(G(i), C) \to \mathrm{Hom}_{\pi\mathbf{G}}(F(i), C).$$

Since all the f_i^* are isomorphisms, we see that f^* is an isomorphism, and conclude that f is a weak equivalence. ∎

3.4.3 Weak equivalences are fully faithful and essentially surjective

In this part we will show that weak equivalences are fully faithful and essentially surjective. A morphism $f\colon A \to B$ between profinite groupoids is *essentially surjective* or *fully faithful* if the morphism $|f|\colon |A| \to |B|$ between the underlying groupoids is so. We start by showing that a weak equivalence of profinite groupoids induces an isomorphism on the set of connected components, which in particular implies that it is essentially surjective. Note that the set of connected components is a profinite set, and not just a set.

Proposition 3.41. *Let $f\colon A \to B$ be a morphism of profinite groupoids. Then the induced map $\pi_0(f)\colon \pi_0(A) \to \pi_0(B)$ is an isomorphism of profinite sets if and only if*

$$f^*\colon \operatorname{Hom}_{\pi\widehat{\mathbf{G}}}(B, \operatorname{Disc}(S)) \to \operatorname{Hom}_{\pi\widehat{\mathbf{G}}}(A, \operatorname{Disc}(S))$$

is an isomorphism for all finite sets S. In particular, if f is a weak equivalence of profinite groupoids, then $\pi_0(f)\colon \pi_0(A) \to \pi_0(B)$ is an isomorphism of profinite sets.

Proof. Note that, for any finite set S and any profinite groupoid A,

$$\operatorname{Hom}_{\pi\widehat{\mathbf{G}}}(A, \operatorname{Disc}(S)) = \operatorname{Hom}_{\widehat{\mathbf{G}}}(A, \operatorname{Disc}(S)),$$

which follows from $\operatorname{Disc}(S)^{I[1]} = \operatorname{Disc}(S)$. Also note that $\pi_0 \dashv \operatorname{Disc}$. We therefore see that $f^*\colon \operatorname{Hom}_{\pi\widehat{\mathbf{G}}}(B, \operatorname{Disc}(S)) \to \operatorname{Hom}_{\pi\widehat{\mathbf{G}}}(A, \operatorname{Disc}(S))$ is an isomorphism if and only if $\pi_0(f)^*\colon \operatorname{Hom}_{\widehat{\mathbf{Set}}}(\pi_0(B), S) \to \operatorname{Hom}_{\widehat{\mathbf{Set}}}(\pi_0(A), S)$ is so. But if the latter holds for all finite sets S, then it also holds for any profinite set S, hence by Yoneda's lemma $\pi_0(f)\colon \pi_0(A) \to \pi_0(B)$ is an isomorphism of profinite sets. ∎

As one would expect, a weak equivalence of profinite groupoids also induces weak equivalences on each connected component. However, this is not that straightforward, as a profinite groupoid is generally not a coproduct of connected profinite groupoids.

Proposition 3.42. *Let $f\colon A \to B$ be a weak equivalence of profinite groupoids. Then f is also a weak equivalence on each connected component.*

Proof. Let $A = \{A_i\}_{i\in I}$ and $B = \{B_j\}_{j\in J}$, and let $p_i\colon A \to A_i$, $q_j\colon B \to B_j$ be the projection maps. Pick a point $x = (x_i)_i \in \operatorname{Ob}(A)$ and denote the connected component of x by A_x. By Proposition 3.13, A_x consists of all arrows α in A such that there are arrows from x to both the source and target of α. Let $A_{i,x}$ denote the connected component containing $p_i(x)$ in A_i. One easily verifies that $A_x = \{A_{i,x}\}$. Define B_x to be the connected component containing $f(x)$ and define $B_{j,x} \subseteq B_j$ similarly to $A_{i,x}$. Note that f can be represented by maps $f_j\colon A_{\theta(j)} \to B_j$, where θ is a map $J \to I$. Since $f_j(A_{\theta(j),x}) \subseteq B_{j,x}$, we see that f restricts to a map $f'\colon A_x \to B_x$ represented by restrictions $f'_j\colon A_{\theta(j),x} \to B_{j,x}$ of f_j. We will show that f' is a weak equivalence.

For surjectivity, let $u'\colon A_x \to C$ be given with C a finite groupoid. Then u' is represented by some map $A_{i,x} \to C$. We can extend this to a map $A_i \to C$ by mapping all other connected components of A_i to $\mathrm{id}_{u'(x)} \in \operatorname{Ar}(C)$, hence we obtain a map $u\colon A \to C$

such that $u|_{A_x} = u'$. Since f is a weak equivalence, there is some $v\colon B \to C$ such that $vf \simeq u$. This homotopy restricts to a homotopy $v'f' \simeq u'$, where $v' = v|_{B_x}$, so we see that $(f')^*\colon \operatorname{Hom}_{\pi\widehat{\mathbf{G}}}(B_x, C) \to \operatorname{Hom}_{\pi\widehat{\mathbf{G}}}(A_x, C)$ is surjective.

For injectivity, let $u', v'\colon B_x \to C$ be given and assume $u'f' \simeq v'f'$. We then have a commutative diagram of the form

$$\begin{tikzcd} A_x \arrow[r, "H'"] \arrow[d, "f'"'] & C^{I[1]} \arrow[d, "(\mathrm{ev}_0\,,\mathrm{ev}_1)"] \\ B_x \arrow[r, "(u',v')"'] & C^2 \end{tikzcd}$$

so there are i and j such that the diagram

$$\begin{tikzcd} A_{i,x} \arrow[r, "H'_i"] \arrow[d, "f'_j"'] & C^{I[1]} \arrow[d, "(\mathrm{ev}_0\,,\mathrm{ev}_1)"] \\ B_{j,x} \arrow[r, "(u'_j,v'_j)"'] & C^2 \end{tikzcd}$$

commutes, where u'_j, v'_j and H'_i represent u', v' and H'. We can extend u'_j, v'_j and H'_i to maps $(u_j, v_j)\colon B_j \to C^2$ and $H_i\colon A_i \to C^{I[1]}$ such that the diagram still commutes. These represent morphisms $u, v\colon B \to C$ together with a homotopy $uf \simeq vf$. Since f is a weak equivalence, u and v are homotopic as well. Restricting this homotopy to B_x gives a homotopy $u' \simeq v'$, since u and v extend u' and v' by construction. We conclude that $(f')^*\colon \operatorname{Hom}_{\pi\widehat{\mathbf{G}}}(B_x, C) \to \operatorname{Hom}_{\pi\widehat{\mathbf{G}}}(A_x, C)$ is a bijection. ■

To prove that a weak equivalence is fully faithful, we need the following lemma on profinite groups.

Lemma 3.43. *Let $f\colon G \to H$ be a homomorphism of profinite groups and assume that for any finite group K, the map*

$$f^*\colon \operatorname{Hom}_{\pi\widehat{\mathbf{G}}}(H_*, K_*) \to \operatorname{Hom}_{\pi\widehat{\mathbf{G}}}(G_*, K_*)$$

is a bijection. Then f is an isomorphism.

Proof. By the second isomorphism of Proposition 3.38, we see that

$$f^*\colon \operatorname{Hom}_{\pi\widehat{\mathbf{G}}}(H_*, K_*) \to \operatorname{Hom}_{\pi\widehat{\mathbf{G}}}(G_*, K_*)$$

is a bijection for any profinite group K. By the Yoneda lemma (applied to the full subcategory of $\pi\widehat{\mathbf{G}}$ of profinite groupoids with one object), we see that $f\colon G_* \to H_*$ is a homotopy equivalence. Let $g\colon H_* \to G_*$ be the homotopy inverse. Note that a homotopy equivalence between maps of profinite groups is just an inner automorphism of the codomain of these maps. In particular there are automorphisms $\phi\colon G \to G$ and $\psi\colon H \to H$ such that $\phi g f = \mathrm{id}_G$ and $f g \psi = \mathrm{id}_H$, hence f has both a left and right inverse. We conclude that f is an isomorphism of profinite groups. ■

Theorem 3.44. *Let $f\colon A \to B$ be a weak equivalence. Then f is essentially surjective and fully faithful.*

Proof. The fact that f is essentially surjective follows from Proposition 3.13 and Proposition 3.41. To see that f is fully faithful, note that a morphism $g\colon C \to D$ between connected (profinite) groupoids is fully faithful if and only if there is some $x \in \mathrm{Ob}(C)$ such that $g\colon C(x) \to D(g(x))$ is an isomorphism. Here $C(x)$ and $D(g(x))$ denote the (profinite) groups of arrows which all have source and target x and $g(x)$, respectively. Since f is a weak equivalence on each connected component by Proposition 3.42, we may assume that A and B are connected profinite groupoids. Now let $x = (x_i)_i \in \mathrm{Ob}(A)$ be given. Then the inclusion $A(x) \to A$ is represented by the levelwise inclusion $A_i(x_i) \to A_i$. Since this is levelwise a weak equivalence, we see that $A(x) \to A$ is a weak equivalence. We similarly see that $B(f(x)) \to B$ is a weak equivalence. We now have a commutative square

$$\begin{tikzcd} A(x) \arrow[r, hook] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ B(f(x)) \arrow[r, hook] & B \end{tikzcd}$$

where f' is the restriction of f to $A(x)$. Since three of these maps are weak equivalences, we see that f' is a weak equivalence as well. By Lemma 3.43, f' is an isomorphism. We conclude that f is fully faithful. ■

3.4.4 Level representations of weak equivalences

To show that an essentially surjective and fully faithful morphism of profinite groupoids is indeed a weak equivalence, we will need to consider a certain construction, where we change the (profinite) set of objects of a groupoid. This will not only allow us to show that fully faithful and essentially surjective maps are weak equivalences, but also that they have level representations by weak equivalences.

First, assume we are given a groupoid A (not profinite), and let $f\colon S \to \mathrm{Ob}(A)$ be any map of sets. We define a groupoid A_S by $\mathrm{Ob}(A_S) = S$ and by letting $\mathrm{Ar}(A_S)$ be the pullback

$$\begin{tikzcd} \mathrm{Ar}(A_S) \arrow[r] \arrow[d] & \mathrm{Ar}(A) \arrow[d, "{(s,t)}"] \\ S \times S \arrow[r, "{(f,f)}"] & \mathrm{Ob}(A) \times \mathrm{Ob}(A), \end{tikzcd}$$

where s and t are the source and target map, respectively. The map $\mathrm{Ar}(A_S) \to S \times S$ is then the source and target map. Note that an arrow of $\mathrm{Ar}(A_S)$ consists of two elements $x, y \in S$, and an arrow $\alpha\colon f(x) \to f(y)$ in A. The omposition of arrows in A_S is defined by composing them in A. It is clear that this defines a groupoid with set of objects S. The construction also gives a canonical map $A_S \to A$ which is fully faithful. If the map $S \to \mathrm{Ob}(A)$ hits each component of A at least once (i.e. $S \to \pi_0(A)$ is surjective), then $A_S \to A$ is furthermore essentially surjective, hence a weak equivalence.

To generalize this construction to profinite groupoids, remark that for any (profinite) groupoid A there is a canonical morphism $A \to \mathrm{Codisc}(\mathrm{Ob}(A))$, the unit of the adjunction $\mathrm{Ob} \dashv \mathrm{Codisc}$. Concretely, this morphism is given by mapping an arrow $\alpha\colon x \to y$ to the unique arrow $x \to y$ in $\mathrm{Codisc}(\mathrm{Ob}(A))$. If A is a groupoid, S a set and $f\colon S \to \mathrm{Ob}(A)$ any map, then it is a straightforward verification that A_S, as defined above, can also be given as the pullback

$$\begin{tikzcd} A_S \arrow[r] \arrow[d] & A \arrow[d] \\ \mathrm{Codisc}(S) \arrow[r, "f"] & \mathrm{Codisc}(\mathrm{Ob}(A)). \end{tikzcd}$$

This construction can also be carried out in $\widehat{\mathbf{G}}$.

Definition 3.45. Let A be a profinite groupoid and let $f\colon S \to \mathrm{Ob}(A)$ be a map of profinite sets. We define A_S, the *pullback of A along f*, by the pullback

$$\begin{tikzcd} A_S \arrow[r] \arrow[d] & A \arrow[d] \\ \mathrm{Codisc}(S) \arrow[r, "f"] & \mathrm{Codisc}(\mathrm{Ob}(A)) \end{tikzcd}$$

in $\widehat{\mathbf{G}}$. ♢

Note that the functor $\mathrm{Ar}\colon \widehat{\mathbf{G}} \to \widehat{\mathbf{Set}}$ preserves limits, as it is corepresented by $I[1]$ (i.e. $\mathrm{Hom}_{\widehat{\mathbf{G}}}(I[1], A) \cong \mathrm{Ar}(A)$ for all profinite groupoids A). Applying Ar to the pullback square defining A_S above, we see that

$$\begin{tikzcd} \mathrm{Ar}(A_S) \arrow[r] \arrow[d] & \mathrm{Ar}(A) \arrow[d, "(s,t)"] \\ S \times S \arrow[r, "(f,f)"] & \mathrm{Ob}(A) \times \mathrm{Ob}(A) \end{tikzcd}$$

is a pullback square in $\widehat{\mathbf{Set}}$. In particular the above definition of A_S is the correct generalization to $\widehat{\mathbf{G}}$.

Lemma 3.46. *Let A be a profinite groupoid and $S \to \mathrm{Ob}(A)$ be a map of profinite sets such that $S \to \pi_0(A)$ is surjective. Then there is a level representation of $A_S \to A$ by weak equivalences. If $S \to \mathrm{Ob}(A)$ is injective, then the level representation can be chosen in such a way that every map is injective on objects.*

Proof. Note that giving a map $f\colon S \to \mathrm{Ob}(A)$, with A a profinite groupoid, is equivalent to giving a morphism of profinite groupoids $f\colon \mathrm{Disc}(S) \to A$. Pick a level representation $f_i\colon B_i \to A_i$ of f. We may assume by Proposition 3.14 that the diagrams $\{B_i\}$ and $\{A_i\}$ are indexed by epimorphisms, and that the morphisms $A \to A_i$, $\mathrm{Disc}(S) \to B_i$ are epi for every i. Since $\mathrm{Disc}(S)$ is discrete, this implies that B_i is discrete for every i, hence $B_i = \mathrm{Disc}(S_i)$ with $\lim_i S_i = S$. Since epimorphisms induce surjections on the (profinite) sets of objects, we see that $\mathrm{Ob}(A) \to \mathrm{Ob}(A_i)$ is surjective for every i, and in particular that $\pi_0 A \to \pi_0 A_i$ is surjective for every i.

As pullbacks can be computed levelwise, we see that $A_S = \lim_i (A_i)_{S_i}$, where $(A_i)_{S_i}$ is the pullback of A_i along $f_i\colon S_i \to \mathrm{Ob}(A_i)$, and that $A_S \to A$ has a level representation given by $(A_i)_{S_i} \to A_i$. Now note that $S \to \pi_0(A) \to \pi_0(A_i)$ is surjective, and that

$$\begin{array}{ccc} S & \longrightarrow & \pi_0(A) \\ \downarrow & & \downarrow \\ S_i & \longrightarrow & \pi_0(A_i) \end{array}$$

commutes, hence $S_i \to \pi_0(A_i)$ is surjective. By the discussion above, $(A_i)_{S_i} \to A_i$ is an essentially surjective and fully faithful morphism of finite groupoids, i.e. a weak equivalence, for every i.

For the last part of the lemma, assume that $S \to \mathrm{Ob}(A)$ is an injection. Let the maps $f_i\colon S_i \to \mathrm{Ob}(A_i)$ be as above. Define $S_i' = S_i/{\sim_i}$, where $x \sim_i y$ if and only if $f_i(x) = f_i(y)$. The quotient maps $S_i \to S_i'$ induce a map $\lim_i S_i \to \lim_i S_i'$. One easily shows that $\lim_i S_i' \cong \mathrm{im}(f) \subseteq \mathrm{Ob}(A)$, and hence that the canonical map $\lim_i S_i \to \lim_i S_i'$ is a bijection by injectivity of f, and therefore an isomorphism. This implies that in the above proof, we can replace S_i by S_i' for every i. As $S_i' \to \mathrm{Ob}(A)$ is injective by construction, we see that the maps $(A_i)_{S_i} \to A_i$ are all injective on objects, so all the maps in the level representation are injective on objects. ■

The above lemma can be used to show that any fully faithful and essentially surjective morphism in $\widehat{\mathbf{G}}$ has a level representation by weak equivalences.

Proposition 3.47. *Let $f\colon A \to B$ be an essentially surjective and fully faithful morphism between profinite groupoids. Then there exists a level representation $\{f_i\}\colon \{A_i\} \to \{B_i\}$ of f such that $f_i\colon A_i \to B_i$ is essentially surjective and fully faithful for every i. If f is injective on objects, then the level representation can be chosen such that f_i is injective on objects for every i.*

Proof. We start with the observation that $A \cong B_{\mathrm{Ob}(A)}$, i.e. that A is the pullback of B along $\mathrm{Ob}(A) \to \mathrm{Ob}(B)$. To see that this is indeed the case, note that $f\colon A \to B$ and $A \to \mathrm{Codisc}(\mathrm{Ob}(A))$ induce $\widetilde{f}\colon A \to B_{\mathrm{Ob}(A)}$ by the universal property of the pullback. This map is by construction the identity on objects, and it is fully faithful since f is so. We therefore see that $\mathrm{Ar}(A) \to \mathrm{Ar}(B_{\mathrm{Ob}(A)})$ is an isomorphism, so $\widetilde{f}$ is an isomorphism of profinite groupoids by Corollary 3.19.

By the lemma proved above and the fact that f is essentially surjective, we see that $B_{\mathrm{Ob}(A)} \to B$ has a level representation by weak equivalences, i.e. there are projective diagrams of finite groupoids $\{A_i\}$ and $\{B_i\}$ and natural weak equivalences $f_i\colon A_i \to B_i$ and isomorphims $A \cong B_{\mathrm{Ob}(A)} \cong \{A_i\}$, $B \cong \{B_i\}$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\cong} & & \downarrow{\cong} \\ \{A_i\} & \xrightarrow{\{f_i\}} & \{B_i\} \end{array}$$

commutes.

It follows from the last part of Lemma 3.46 that we can choose the level representation such that all maps f_i are injective on objects. ■

We arrive at the following characterization of weak equivalences.

Theorem 3.48. *Let $f\colon A \to B$ be a morphism between profinite groupoids. Then the following are equivalent:*

(i) f is a weak equivalence;

(ii) f is essentially surjective and fully faithful;

(iii) f has a level representation by weak equivalences between finite groupoids.

Proof. This follows by combining Proposition 3.40, Theorem 3.44 and Proposition 3.47. ■

3.4.5 Comparing weak equivalences to homotopy equivalences

As promised earlier, we start with an example of a morphism $f\colon A \to B$ that has a level representation by weak equivalence, but that is not a homotopy equivalence. To give this example, we need a surjection between Stone spaces that has no section.

Lemma 3.49. *There exists a continuous surjection between Stone spaces that has no continuous section.*

Proof. By Stone duality, this is equivalent to giving a monomorphism between Boolean algebras that has no retraction. We will give such a monomorphism. $\mathcal{P}(\mathbb{N})$, the power set of the natural numbers, has the structure of a Boolean algebra. Here objects are ordered by inclusion, and the meet and join of X and Y are given by $X \cap Y$ and $X \cup Y$, respectively. It has a Boolean subalgebra $\mathcal{P}_f(\mathbb{N})$ defined by

$$\mathcal{P}_f(\mathbb{N}) = \{X \subseteq \mathbb{N} \mid X \text{ is finite or } \mathbb{N} \setminus X \text{ is finite}\}.$$

The inclusion $\mathcal{P}_f(\mathbb{N}) \hookrightarrow \mathcal{P}(\mathbb{N})$ is a monomorphism since it is injective. To see that it has no retraction, assume such a retraction $r\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}_f(\mathbb{N})$ exists. Then r must be order preserving, and $r(X) = X$ for all $X \subseteq \mathbb{N}$ that are either finite or cofinite. If $X \subseteq \mathbb{N}$ is arbitrary, then for any $x \in X$, we have $\{x\} = r(\{x\}) \subseteq r(X)$, hence $X \subseteq r(X)$. Now note that $r(2\mathbb{N}) \cup r(2\mathbb{N}+1) = r(\mathbb{N}) = \mathbb{N}$ and $r(2\mathbb{N}) \cap r(2\mathbb{N}+1) = r(\emptyset) = \emptyset$, so one of $r(2\mathbb{N})$ and $r(2\mathbb{N}+1)$ must be finite and the other cofinite. However, as $2\mathbb{N} \subseteq r(2\mathbb{N})$ and $2\mathbb{N}+1 \subseteq r(2\mathbb{N}+1)$, this is not possible. We conclude that no retraction exists. The map that corresponds to the inclusion $\mathcal{P}_f(\mathbb{N}) \hookrightarrow \mathcal{P}(\mathbb{N})$ under Stone duality is a surjection that has no section. ■

Example 3.50. Let $f\colon S \to T$ be a surjection between Stone spaces that has no section. Define the groupoid B by $B = \mathrm{Disc}(T)$, and define A as the pullback of B along f, i.e. A is defined by the pullback

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d] & \mathrm{Disc}(T) \arrow[d] \\ \mathrm{Codisc}(S) \arrow[r, "f"] & \mathrm{Codisc}(T). \end{tikzcd}$$

We see that for any $x, y \in \mathrm{Ob}(A) = S$, there is precisely one arrow $x \to y$ if $f(x) = f(y)$, and there is no arrow $x \to y$ otherwise. It follows directly from this that $\pi_0 A = T$, and that g induces an isomorphism $\pi_0 A \to \pi_0 B$. Since g is also fully faithful, we conclude that g is a weak equivalence. Now assume that $h\colon B \to A$ is a homotopy inverse. As $B = \mathrm{Disc}(T)$, we see that h is just a map $T \to \mathrm{Ob}(A) = S$. The fact that h is a homotopy inverse implies that $gh(x)$ and x are in the same path component of B for every $x \in \mathrm{Ob}(B) = T$. Since $B = \mathrm{Disc}(T)$, this implies $gh(x) = fh(x) = x$ for every $x \in T$. This contradicts the fact that f has no continuous section, so we conclude that g cannot be a homotopy equivalence. $\diamondsuit$

The above example illustrates that for finding a homotopy inverse, certain continuous sections have to exist. However, we saw in Proposition 2.61 that for a free G-action on a Stone space S, with G profinite, the quotient map always has a section. In Corollary 3.22, we used this to prove that any connected profinite groupoid is of the form $G[S]$ for a profinite group G and a profinite set S. This can be used to show that for any connected profinite groupoid A and any $x \in \mathrm{Ob}(A)$, the inclusion $A(x) \hookrightarrow A$ is a homotopy equivalence. It will follow from this that any weak equivalence between connected profinite groupoids is a homotopy equivalence.

Lemma 3.51. *Let A be a profinite groupoid and $x \in \mathrm{Ob}(A)$ any point. Then $A(x) \hookrightarrow A$ is a homotopy equivalence, with a retract $A \to A(x)$ as homotopy inverse.*

Proof. Write $G = A(x)$. By Corollary 3.22, we can assume without loss of generality that $A = G[S]$, where S is a profinite set, and that $i\colon G_* \to G[S]$ is the inclusion at a given $s_0 \in S$. We define a retract $r\colon G[S] \to G_*$ by mapping an arrow $g\colon t \to t'$ to the corresponding element g of G. Then clearly $ri = \mathrm{id}_{G_*}$. To see that $ir \simeq \mathrm{id}_{G[S]}$, we need to construct a homotopy $H\colon G[S] \to (G[S])^{I[1]}$. Note that $\mathrm{Ob}((G[S])^{I[1]}) = \mathrm{Ar}(G[S])$. On objects, we define H by $H(t) = (e\colon t \to s_0)$, where e is the identity element of G. On arrows, H is defined by

$$(g\colon t \to t') \quad \mapsto \quad \begin{tikzcd} t \arrow[r, "g"] \arrow[d, "e"] & t' \arrow[d, "e"] \\ s_0 \arrow[r, "g"] & s_0. \end{tikzcd}$$

It follows immediately from the definition of H that $\mathrm{ev}_0 H = \mathrm{id}_{G[S]}$ and that $\mathrm{ev}_1 H = ir$. We conclude that i is a homotopy equivalence. ∎

Theorem 3.52. *Let $f\colon A \to B$ be a weak equivalence of connected profinite groupoids. Then f is a homotopy equivalence.*

Proof. Pick $x \in \mathrm{Ob}(A)$. Since f is a weak equivalence, it is fully faithful, so it induces an isomorphism $A(x) \to B(f(x))$. In particular, we obtain a commutative diagram of the form

$$\begin{tikzcd} & A \arrow[dd, "f"] \\ A(x) \arrow[ur, hook] \arrow[dr, hook] & \\ & B. \end{tikzcd}$$

Since two of the arrows are homotopy equivalences by Lemma 3.51, we conclude that $f\colon A \to B$ is a homotopy equivalence as well. ■

3.5 A fibrantly generated model structure on $\widehat{\mathbf{G}}$

We will now construct a fibrantly generated model structure on $\widehat{\mathbf{G}}$ using the dual of Theorem 11.3.1 in [Hir03], which is Theorem A.18 in this thesis. We will follow the approach of Horel in [Hor17], Theorem 4.12. The proof is (a slight modification) of Horel's proof given there. Let $\mathcal{S}$ be a set containing at least one representative for each isomorphism class of finite sets. Let $\mathcal{G}$ be the set of all groups whose underlying set is in $\mathcal{S}$. Define P to be the set containing all morphisms of the form

(i) $G /\!\!/ G \to G_*$, where $(k\colon g \to h) \mapsto k$;

(ii) $(\mathrm{ev}_0, \mathrm{ev}_1)\colon (G_*)^{I[1]} \to (G_*)^2$;

(iii) $G_* \to *$;

(iv) $\mathrm{Disc}(S) \to *$; and

(v) $\mathrm{Disc}(\{0\}) \to \mathrm{Disc}(\{0,1\})$

for all $S \in \mathcal{S}$ and $G \in \mathcal{G}$. We can also view the map $G /\!\!/ G \to G_*$ as $\mathrm{Codisc}(G) \to G_*$, where $(g \to h) \mapsto hg^{-1}$, and where $\mathrm{Codisc}(G)$ is the codiscrete groupoid on the underlying set of G.

Define Q to be the set containing all morphisms of the form $\mathrm{Codisc}(S) \to *$, with $S \in \mathcal{S}$ nonempty.

Before proving that these sets are the generating (trivial) fibrations for a model structure on $\widehat{\mathbf{G}}$, we will show that the following lemmas hold.

Lemma 3.53. *For any finite groupoid C, the maps $C \to *$ and $(\mathrm{ev}_0, \mathrm{ev}_1)\colon C^{I[1]} \to C^2$ are P-fibrations.*

Proof. The main ingredient in this proof is that the P-fibrations are closed under finite coproducts. Note that the coproduct of two profinite groupoids $\{A_i\}$ and $\{B_j\}$ is just $\{A_i \sqcup B_j\}$. If $\{0,1\}$ is the discrete groupoid with two elements, then

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & \{0\} \arrow[d] \\ A \sqcup B \arrow[r] & \{0,1\}, \end{tikzcd}$$

is a pullback square, so the inclusion $A \to A \sqcup B$ is a P-fibration. Here the bottom map is defined by sending A to 0 and B to 1.

Now assume we are given two P-fibrations $f\colon A \to B$ and $g\colon C \to D$ with A and C nonempty. Then pick $a_0 \in \mathrm{Ob}(A)$ and $c_0 \in \mathrm{Ob}(C)$, and define $A \sqcup C \hookrightarrow A \times C \times \{0,1\}$ by including A as $A \times \{c_0\} \times \{0\}$ and C as $C \times \{a_0\} \times \{1\}$, and define $A \times C \times \{0,1\} \to A \sqcup C$ by projecting $A \times C \times \{0\}$ on A and $A \times C \times \{1\}$ on C. Define similar maps for $B \sqcup D$, with $f(a_0)$ and $g(c_0)$ as basepoints. Then the diagram

$$\begin{tikzcd} A \sqcup C \arrow[r] \arrow[d] \arrow[rr, bend left, "\mathrm{id}"] & A \times C \times \{0,1\} \arrow[r] \arrow[d] & A \sqcup C \arrow[d] \\ B \sqcup D \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & B \times D \times \{0,1\} \arrow[r] & B \sqcup D \end{tikzcd}$$

commutes, hence $A \sqcup C \to B \sqcup D$ is a retract of $A \times C \times \{0,1\} \to B \times D \times \{0,1\}$. Since products of P-fibrations are again P-fibrations, we see that $A \sqcup C \to B \sqcup D$ is a P-fibration.

Now if C is a finite connected groupoid, then $C \cong G[S] \cong G_* \times \mathrm{Codisc}(S)$. Since $G_* \to *$ and $\mathrm{Codisc}(S) \to *$ are P-fibrations, we see that $C \to *$ is so as well. If C is not connected, then C is a finite disjoint union $\sqcup_i G_i[S_i]$. By the above, $\sqcup_i G_i[S_i] \to \sqcup_i *$ is a P-fibration. Since $\sqcup_i * \to *$ is also a P-fibration, we conclude that $C \to *$ is a P-fibration for every finite groupoid C.

For $C^{I[1]} \to C^2$, again assume C is connected, say $C = G[S]$. Then

$$\begin{tikzcd} G[S]^{I[1]} \arrow[r] \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"'] & G_*^{I[1]} \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ G[S]^2 \arrow[r] & G_*^2 \end{tikzcd}$$

is a pullback square, where the horizontal maps are given by forgetting about S, i.e. by mapping an arrow $g\colon x \to y$ to the arrow $g\colon * \to *$ in G_*. We therefore see that $C^{I[1]} \to C^2$ is a P-fibration. Now assume we are given A, B such that $A^{I[1]} \to A^2$ and $B^{I[1]} \to B^2$ are P-fibrations. Then $(A \sqcup B)^{I[1]} \cong A^{I[1]} \sqcup B^{I[1]}$ and $(A \sqcup B)^2 \cong A^2 \sqcup B^2 \sqcup (A \times B) \sqcup (B \times A)$. Under these isomorphisms, the map $(A \sqcup B)^{I[1]} \to (A \sqcup B)^2$ corresponds to the map

$$A^{I[1]} \sqcup B^{I[1]} \to A^2 \sqcup B^2 \to A^2 \sqcup B^2 \sqcup (A \times B) \sqcup (B \times A).$$

Both of these maps are P-fibrations by the above discussion on coproducts of P-fibrations. We now inductively see that $C^{I[1]} \to C^2$ is a P-fibration for every finite groupoid C. ∎

Lemma 3.54. *Let $f\colon A \to B$ is Q-projective if and only if f is injective on objects.*

Proof. f is Q-projective precisely if $f^*\colon \mathrm{Hom}_{\widehat{\mathbf{G}}}(B, \mathrm{Codisc}(S)) \to \mathrm{Hom}_{\widehat{\mathbf{G}}}(A, \mathrm{Codisc}(S))$ is a surjection for all finite sets S. By the adjunction $\mathrm{Ob} \dashv \mathrm{Codisc}$, this is equivalent to $f^*\colon \mathrm{Hom}_{\widehat{\mathbf{Set}}}(\mathrm{Ob}(B), S) \to \mathrm{Hom}_{\widehat{\mathbf{Set}}}(\mathrm{Ob}(A), S)$ being a surjection for every finite set S. Using that for any $x, y \in \mathrm{Ob}(A)$, there is a continuous map $h\colon \mathrm{Ob}(A) \to \{0, 1\}$ for which $h(x) = 0$ and $h(y) = 1$, we see that f is injective. For the converse, note that if a map $X \to Y$ between Stone spaces is injective, then any map $X \to S$ for any finite nonempty set S extends to a map $Y \to S$, so the result follows again by the adjunction $\mathrm{Ob} \dashv \mathrm{Codisc}$. ∎

Lemma 3.55. *Let $f\colon A \to B$ be a weak equivalence of profinite groupoids, let C and D be finite groupoids, and let a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & C \arrow[d, "g"] \\ B \arrow[r, "v"] & D \end{tikzcd}$$

be given. Then there exists a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "u"] & A' \arrow[r] \arrow[d, "f'"] & C \arrow[d, "g"] \\ B \arrow[r] \arrow[rr, bend right, "v"'] & B' \arrow[r] & D \end{tikzcd}$$

where A' and B' are finite groupoids and where $f'\colon A' \to B'$ is a weak equivalence. If f is injective on objects, then f' can be chosen to be injective on objects as well.

Proof. Pick a level representation $f_i\colon A_i \to B_i$ of f by weak equivalences, which exists by Theorem 3.48. It follows from Proposition 2.10 that there exist an i and maps $A_i \to C$, $B_i \to D$ such that

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "u"] & A_i \arrow[r] \arrow[d, "f_i"] & C \arrow[d, "g"] \\ B \arrow[r] \arrow[rr, bend right, "v"'] & B_i \arrow[r] & D \end{tikzcd}$$

commutes. Note that by Proposition 3.47, we may assume f_i is injective on objects if f is so. ∎

Corollary 3.56. *Let $f\colon C \to D$ be a map between finite groupoids and assume f has the right-lifting property with respect to all weak equivalences between finite groupoids that are injective on objects. Then f has the right-lifting property with respect to all Q-projective weak equivalences between profinite groupoids.*

Proof. This follows directly by applying Lemma 3.54 and the previous lemma. ∎

Theorem 3.57. *The category $\widehat{\mathbf{G}}$ has a model structure for which P is a set of generating fibrations, Q a set of generating trivial fibrations, and where the weak equivalences are as in Definition* 3.32. *In this model structure, the cofibrations are the maps that are injective on objects.*

Proof. We will check all the conditions of Theorem A.18. The class of weak equivalences is clearly closed under retracts and satisfies the two out of three property.

1. Since all codomains of morphisms in P and Q are finite groupoids, they are cosmall by Example A.4. In particular P and Q permit the cosmall object argument.

2. To show that Q-fibrations are weak equivalences, we only have to show this for Q-cocell complexes, since the weak equivalences are closed under retracts. Since weak equivalences are stable under cofiltered limits (so in particular under transfinite precomposition) by Proposition 3.40, we only have to prove that pullbacks of maps in Q are weak equivalences. Let $A = \{A_i\}$ be a profinite groupoid, and let $\mathrm{Codisc}(S) \to *$ be in Q. The pullback of this map along $A \to *$ is the product $A \times \mathrm{Codisc}(S) \to A$. This map is induced by the levelwise projection $\{A_i \times \mathrm{Codisc}(S)\} \to \{A_i\}$. Since these projections are weak equivalences of groupoids, we see by Proposition 3.40 that $A \times \mathrm{Codisc}(S) \to A$ is a weak equivalence.

   To see that Q-fibrations are P-fibrations, we only need to show that maps in Q are P-fibrations. Let S be a nonempty finite set. Let G be any group with underlying set S. Then $G /\!\!/ G \cong \mathrm{Codisc}(S)$. Since $G /\!\!/ G \to G_*$ and $G_* \to *$ are in P, we see that their composition $G /\!\!/ G \to *$ is a P-fibration.

3. A P-projective map has the left-lifting property against all P-fibrations, so it is in particular Q-projective.

   To see that P-projective maps are weak equivalences, let $f\colon A \to B$ be P-projective. Then f has the left-lifting property against the maps $C \to *$ and $(\mathrm{ev}_0, \mathrm{ev}_1)\colon C^{I[1]} \to C^2$ for any finite groupoid C by Lemma 3.53. The left-lifting property against $C \to *$ implies that $f^*\colon \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(B, C) \to \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(A, C)$ is surjective. For injectivity, let $g, h\colon B \to C$ be given and assume $gf \simeq hf$, i.e. there exists a map $H\colon A \to C^{I[1]}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "H"] \arrow[d, "f"'] & C^{I[1]} \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ B \arrow[r, "(g,h)"'] \arrow[ur, dotted, "\widehat{H}"] & C^2 \end{tikzcd}$$

   commutes. Since f is P-projective, there exists a lift $\widehat{H}\colon B \to C^{I[1]}$ which is the desired homotopy from g to h. We conclude that f is a weak equivalence.

4. We will show that any map that is Q-projective and a weak equivalence is also P-projective. This amounts to showing that any map in P has the right-lifting property with respect to Q-projective weak equivalences. By Corollary 3.56, we only need to prove that maps in P have the right-lifting property with respect to weak equivalences $f\colon A \to B$ between finite groupoids that are injective on objects. Such a map f is a trivial cofibration in the canonical model structure on $\mathbf{G}$. Since all maps in P are fibrations in the canonical model structure on $\mathbf{G}$, we conclude that f is P-projective by Proposition 3.30.

It follows from Lemma 3.54 that the cofibrations are indeed the maps that are injective on objects. ■

Since the cofibrations are the maps that are injective on objects, we see that all profinite groupoids are cofibrant in this model structure. Recall that at the end of section 3.4, we showed that a weak equivalence between connected profinite groupoids is a homotopy equivalence. This might suggest that connected profinite groupoids are always fibrant. This is not quite true; however, we have the following characterization of the fibrant objects among the connected profinite groupoids.

Proposition 3.58. *In the model structure of Theorem* 3.57*, any profinite groupoid is cofibrant. Furthermore, a connected profinite groupoid A is fibrant precisely if* $\mathrm{Ob}(A)$ *is an injective object in* $\widehat{\mathbf{Set}}$.

Proof. We directly see from Theorem 3.57 that any profinite groupoid is cofibrant.

For the second part of this proposition, we first show that profinite groupoids with one object are fibrant. Let G be a profinite group, and let $G_* \to RG_*$ be a fibrant replacement. Then this map is a weak equivalence. In the proof of Lemma 3.51, we saw that such a map has a retraction. In particular G_* is a retract of RG_*, hence fibrant.

Let B be any connected profinite groupoid. By Corollary 3.22, we see that $B \cong G[S]$ for a profinite group G and for $S = \mathrm{Ob}(B)$. Since $G[S] = G_* \times \mathrm{Codisc}(S)$, we see that $G[S]$ is fibrant precisely if G_* and $\mathrm{Codisc}(S)$ are so. We already saw that G_* is fibrant. To see when $\mathrm{Codisc}(S)$ is fibrant, note that $\mathrm{Codisc}(S) \to *$ is a weak equivalence, so $\mathrm{Codisc}(S)$ is fibrant precisely if it has the right-lifting property with respect to all morphisms $C \to D$ of profinite groupoids that are injective on objects. By the adjunction $\mathrm{Ob} \dashv \mathrm{Codisc}$, we see that $\mathrm{Codisc}(S) \to *$ has the right-lifting property with such morphisms of profinite groupoids, precisely if for any injective map $X \to Y$ of Stone spaces and any map $X \to S$, an extension exists as in the following diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & S \\ Y. \arrow[ur, dotted] & \end{tikzcd}$$

Since a map between Stone spaces is a monomorphism precisely if it is injective, we see that this is equivalent to S being an injective object in $\widehat{\mathbf{Set}}$. ■

Recall that the profinite completion functor $\widehat{(\cdot)}\colon \mathbf{G} \to \widehat{\mathbf{G}}$ is left adjoint to the functor $|\cdot|\colon \widehat{\mathbf{G}} \to \mathbf{G}$ which maps a profinite groupoid $\{A_i\}$ to its limit $\lim_i A_i$, computed in $\mathbf{G}$. This adjunction is a Quillen pair when considering the above model structure on $\widehat{\mathbf{G}}$ and the canonical model structure on $\mathbf{G}$.

Proposition 3.59. *The functors $\widehat{(\cdot)}$ and $|\cdot|$ form a Quillen pair. Furthermore, they both preserve weak equivalences.*

Proof. By [Hir03, Proposition 8.5.3], to see that $(\widehat{(\cdot)}, |\cdot|)$ is a Quillen pair it is sufficient to show that $|\cdot|$ preserves (trivial) fibrations. As the model structure on $\widehat{\mathbf{G}}$ is fibrantly generated, we only have to check this for the generating (trivial) fibrations. As all morphisms in P are morphisms between finite groupoids, and they are fibrations in the canonical model structure on $\mathbf{G}$, it follows that $|\cdot|$ preserves fibrations. Since $\mathrm{Codisc}(S) \to *$ is clearly a weak equivalence in $\mathbf{G}$ for any finite set S, we see that $|\cdot|$ preserves trivial fibrations as well. Recall from Proposition 3.36 that $\widehat{(\cdot)}$ preserves weak equivalences. To see that $|\cdot|$ preserves weak equivalences, note that weak equivalences $f\colon A \to B$ in $\widehat{\mathbf{G}}$ are fully faithful and essentially surjective by Theorem 3.44, which by definition means that $|f|\colon |A| \to |B|$ is essentially surjective and fully faithful. In particular $|\cdot|$ preserves weak equivalences. ∎

Chapter 4

Profinite spaces

The goal of this chapter is to put a fibrantly generated model structure on $\widehat{\mathbf{S}}$, the category of profinite spaces. This is the model structure defined by Quick in [Qui08], which was later corrected in [Qui11a][1]. We will take a slightly different approach than Quick, also filling in some of the gaps in his proof, using the theory on profinite groupoids developed in chapter 3. The main difference is that we will use a different construction of the profinite fundamental group(oid), and that we will give a more precise and rigorous definition of cohomology with local coefficients. Our definition of a weak equivalence will also be slightly different, but we will come back to this difference in section 4.5, when we have constructed the model structure on $\widehat{\mathbf{S}}$.

In the first section of this chapter, we will recall some basic notions about simplicial sets, and define some basic notion for profinite spaces, which will be used throughout the rest of the chapter. In the second section, we define what connectedness means for a profinite space (in particular we define π_0), and we define the fundamental groupoid. The third section is used to develop the theory of principal G-bundles of profinite spaces. The fourth section is then devoted to cohomology with local coefficients, which we will use to define our weak equivalences. In the fifth section we will give a detailed proof of the existence of the model structure of [Qui08]. The last section is about covering spaces and the fundamental groupoid as defined by Quick in [Qui08], which we will compare to the fundamental groupoid as defined in section 4.2. We will show that they are naturally isomorphic whenever we are able to make Quick's construction work.

The main sources for the material in this chapter are [Qui08] and [Qui11a]. The proof in section 4.3 that profinite principal G-bundles are classified by the profinite space BG, is the author's own work. The treatment of cohomology with local coefficients in section 4.4 is based on sections 2.2 of [Qui08] and VI.4 of [GJ09], although most proofs are the author's own work. The construction of the model structure in section 4.5 is taken from [Qui08] and [Qui11b], with a slightly different approach to weak equivalences. The work on profinite coverings in section 4.6 is the author's own work, where [Gro+71, Exposé V

[1] There is an updated version of this paper, since the set of generating fibrations was still chosen too small. This version can be found on G. Quick's homepage `https://folk.ntnu.no/gereonq/` as the set of generating fibrations was too small. The fibrations, cofibrations and weak equivalences of the model structure are defined in Definition 2.9 of this updated version, and in Theorem 2.10 it is proved that this defines a (fibrantly generated) model structure.

§4] and [Len97] were used as sources for the material on Galois categories.

4.1 Some basic facts on simplicial (profinite) sets

Define Δ to be the category with objects $[n] = \{0 < 1 < \ldots < n\}$, and with order preserving maps as morphisms. Recall that a simplicial set is a functor $\Delta^{op} \to \mathbf{Set}$. We will usually write $\mathbf{S}$ for the category of simplicial sets, although we will sometimes write $\mathbf{Set}^{\Delta^{op}}$ if we want to stress the fact that simplicial sets are functors, or $\mathbf{sSet}$ if we want to stress that they are simplicial objects in the category $\mathbf{Set}$. We will denote the image of $[n]$ in $\mathbf{S}$ under the Yoneda embedding by Δ^n. There is a natural isomorphism $\mathrm{Hom}_{\mathbf{S}}(\Delta^n, X) \cong X_n$ by the Yoneda lemma. We can of course define simplicial objects in any category $\mathbf{C}$ as functors $\Delta^{op} \to \mathbf{C}$. If we define morphisms of simplicial objects in $\mathbf{C}$ to be natural transformations, then we obtain the category of simplicial objects in $\mathbf{C}$, which we will denote by $\mathbf{sC}$ or $\mathbf{C}^{\Delta^{op}}$. The category we are particularly interested in is $\mathbf{s}(\widehat{\mathbf{Set}})$, the category of simplicial profinite sets. We will denote this category by $\widehat{\mathbf{S}}$, and we will call its objects *profinite spaces*. Since Δ^n is a simplicial finite set, we see that Δ^n is an object of $\widehat{\mathbf{S}}$ for every $n \geq 0$. In particular, we see that the natural bijection $\mathrm{Hom}_{\widehat{\mathbf{S}}}(\Delta^n, X) \cong X_n$ holds for profinite spaces as well.

4.1.1 Skeletal and coskeletal simplicial sets

We first recall the notions of skeletal and coskeletal simplicial sets. For $n \geq 0$, we define $\Delta_{\leq n}$ be the full subcategory of Δ on the objects $[0], \ldots, [n]$. We define the *n-truncation* functor $\tau_n \colon \mathbf{Set}^{\Delta^{op}} \to \mathbf{Set}^{\Delta^{op}_{\leq n}}$ by precomposing a functor $X \colon \Delta^{op} \to \mathbf{Set}$ with the inclusion $\Delta^{op}_{\leq n} \to \Delta^{op}$. The truncation functor τ_n has both a left and right adjoint, denoted sk_n and cosk_n respectively.

Definition 4.1. Let X be a simplicial set. Then X is called

(i) *n-skeletal* if $X = \mathrm{sk}_n Y$ for some $Y \in \mathbf{Set}^{\Delta^{op}_{\leq n}}$;

(ii) *n-coskeletal* if $X = \mathrm{cosk}_n Y$ for some $Y \in \mathbf{Set}^{\Delta^{op}_{\leq n}}$;

(iii) *skeletal* if X is n-skeletal for some $n \geq 0$; and

(iv) *coskeletal* if X is n-coskeletal for some $n \geq 0$. ♢

Remark 4.2. In Example 2.78, we showed that sk_n and cosk_n indeed exist. The proof given there, i.e. the proof of Lemma 2.76, in fact applies for any category with finite limits and colimits, not just $\mathbf{Set}$. This in particular implies that $\mathrm{sk}_n X$ and $\mathrm{cosk}_n X$ are simplicial finite sets if X is so, since $\mathbf{FinSet}$ has finite (co)limits and sk_n and cosk_n are left and right adjoint to the n-truncation $\mathbf{FinSet}^{\Delta^{op}} \to \mathbf{FinSet}^{\Delta^{op}_{\leq n}}$. The proof also implies that sk_n and cosk_n are fully faithful. ♢

Remark 4.3. We will usually write sk_n and cosk_n for the functors $\mathrm{sk}_n \circ \tau_n$ and $\mathrm{cosk}_n \circ \tau_n$ as well. In this notation, we easily see that $\mathrm{sk}_n \dashv \mathrm{cosk}_n$. ♢

There is a nice description for n-(co)skeletal simplicial sets.

Proposition 4.4. *A simplicial set X is n-skeletal if and only if all its simplices above degree n are degenerate, i.e. if X has dimension $\leq n$. A simplicial set X is n-coskeletal if and only if for any map $\partial\Delta^k \to X$ with $k > n$, there exists a unique extension $\Delta^k \to X$.*

Proof. The statement about n-skeletal simplicial sets is immediate from the fact that sk_n is left adjoint to τ_n.

For the statement about n-coskeletal sets, note that we have natural isomorphisms

$$(\operatorname{cosk}_n X)_k \cong \operatorname{Hom}_{\mathbf{S}}(\Delta^k, \operatorname{cosk}_n X) \cong \operatorname{Hom}_{\mathbf{S}}(\operatorname{sk}_n \Delta^k, X).$$

We also have the natural isomorphism $X_k \cong \operatorname{Hom}_{\mathbf{S}}(\Delta^k, X)$. Noting that X is n-coskeletal precisely if $X = \operatorname{cosk}_n X$, we see that X is coskeletal if and only if the map $\operatorname{Hom}_{\mathbf{S}}(\Delta^k, X) \to \operatorname{Hom}_{\mathbf{S}}(\operatorname{sk}_n \Delta^k, X)$ coming from the inclusion $\operatorname{sk}_n \Delta^k \to \Delta^k$ is an isomorphism. This can be rephrased as saying that any map $\operatorname{sk}_n \Delta^k \to X$ extends uniquely to a map $\Delta^k \to X$. It follows inductively that this is the case precisely if for any $k > n$, all maps $\partial\Delta^k \to X$ extend uniquely to maps $\Delta^k \to X$. ∎

The skeletal simplicial finite sets can, in a sense, be seen as the "finitely generated" simplicial sets, as they contain finitely many non-degenerate simplices. If $\mathbf{S}_{fin}$ denotes the full subcategory of $\mathbf{S}$ with as objects all skeletal simplicial finite sets, then one can prove that $\operatorname{Ind}(\mathbf{S}_{fin}) \simeq \mathbf{S}$. Compare this to Example 2.14 and Example 2.15, where we saw that a similar statement holds for finite sets, finite dimensional vector spaces and finitely generated groups. In section 2.4, we proved that a statement similar to (and dual to) $\operatorname{Ind}(\mathbf{S}_{fin}) \simeq \mathbf{S}$ holds in general for categories of functors $\mathbf{D}^{\mathbf{C}}$, under some assumptions on the categories $\mathbf{D}$ and $\mathbf{C}$. In this chapter, Corollary 2.84, which is Proposition 7.4.1 of [BHH17], will play an important role, so we state it as a seperate theorem here. Define $\mathbf{S}_{cofin}$ to be the category of all coskeletal simplicial finite sets, viewed as full subcategory of $\mathbf{S}$. Note that $\mathbf{S}_{cofin}$ is also a full subcategory of $\widehat{\mathbf{S}} = \widehat{\mathbf{Set}}^{\Delta^{op}}$ in the obvious way.

Theorem 4.5. *The inclusion $\mathbf{S}_{cofin} \to \widehat{\mathbf{Set}}^{\Delta^{op}}$ induces an equivalence $\operatorname{Pro}(\mathbf{S}_{cofin}) \to \widehat{\mathbf{Set}}^{\Delta^{op}}$.* ∎

In light of this equivalence, we will write $\widehat{\mathbf{S}}$ for the categories $\operatorname{Pro}(\mathbf{S}_{cofin})$, $\widehat{\mathbf{Set}}^{\Delta^{op}}$ and $\mathbf{Stone}^{\Delta^{op}}$. Whenever we are in a situation where it matters in which of these categories we work, it should be clear which category is meant from the context. The usefulness of the above theorem lies in the fact that it allows us to use both the abstract theory of pro-categories, as developed in chapter 2, and point-set topological arguments, when studying profinite spaces. For example, when defining and studying the profinite fundamental groupoid in section 4.2, we will mostly use abstract arguments, while section 4.3 on principal G-bundles will have a more point-set topological flavour. The above theorem also provides us with a lot of cosmall objects in $\widehat{\mathbf{S}}$, namely all coskeletal simplicial finite sets, which will be useful since we're constructing a fibrantly generated model structure on $\widehat{\mathbf{S}}$.

4.1.2 Simplicial homotopy

Throughout this chapter, we will use the notion of simplicial homotopy many times.

Definition 4.6. Let $f, g\colon X \to Y$ be maps of simplicial sets or profinite spaces. We say that $H\colon X \times \Delta^1 \to Y$ is a *simplicial homotopy* from f to g if $Hi_0 = f$ and $Hi_1 = g$, where i_0 and i_1 are the inclusions $X \to X \times \Delta^1$ at the starting point and endpoint, respectively. If there is a simplicial homotopy from f to g, we say that f is *simplicially homotopic* to g. ♢

Remark 4.7. Note that simplicial homotopy in general does not give an equivalence relation on $\mathrm{Hom}(X, Y)$ for X, Y simplicial sets of profinite spaces. It is therefore common to consider the equivalence relation generated by simplicial homotopy, and call two maps simplicially homotopic if they are equivalent under this equivalence relation. In this chapter, however, we will only need to consider homotopic maps with a given simplicial homotopy, and so this problem will not play any role. ♢

Remark 4.8. Note that the above definition makes sense for both simplicial sets and profinite spaces. Δ^1 is a simplicial finite set, so in particular a simplicial profinite set, i.e. an object of $\widehat{\mathbf{S}}$. We can therefore define the product $X \times \Delta^1$ for any $X \in \widehat{\mathbf{S}}$. ♢

Note that in the case of profinite groupoids, because of the adjunction $(-) \times I[1] \dashv (-)^{I[1]}$ we can see homotopies as maps $A \to B^{I[1]}$. A similar statement holds for simplicial sets and profinite spaces. We first note that $\mathbf{S}$ (like any presheaf category) is cartesian closed, meaning that for any $X \in \mathbf{S}$, the functor $(-) \times X$ has a right adjoint $(-)^X$. Explicitly, Y^X is given by $(Y^X)_n = \mathrm{Hom}_{\mathbf{S}}(X \times \Delta^n, Y)$. For any $\alpha\colon [n] \to [m]$, the Yoneda embedding provides us with a map $\alpha_*\colon \Delta^n \to \Delta^m$, which in turn induces a map $\alpha^*\colon (Y^X)_m \to (Y^X)_n$, giving Y^X the structure of a simplicial set. We in particular get from this that $(-) \times \Delta^1$ has a left adjoint $(-)^{\Delta^1}$, and hence that we can view a simplicial homotopy from f to g as a map $H\colon X \to Y^{\Delta^1}$ satisfying $\mathrm{ev}_0 H = f$ and $\mathrm{ev}_1 H = g$.

The category of profinite spaces $\widehat{\mathbf{S}}$ is not cartesian closed, since $\mathrm{Hom}_{\widehat{\mathbf{S}}}(X, Y)$ is not a profinite set in any natural way. The proof in Proposition 2.70 that $\widehat{\mathbf{Set}}$ is not cartesian closed can be translated directly to $\mathbf{s}(\widehat{\mathbf{Set}})$. However, if X is a simplicial finite set, then there does exist a natural way to give $\mathrm{Hom}_{\widehat{\mathbf{S}}}(X, Y)$ the structure of a profinite set. To see this, write $Y = \lim_i Y_i$ with $Y_i \in \mathbf{S}_{cofin}$ for all i. Then $\mathrm{Hom}_{\widehat{\mathbf{S}}}(X, Y) = \lim_i \mathrm{Hom}_{\widehat{\mathbf{S}}}(X, Y_i)$. Since Y_i is coskeletal for some n, we see that

$$\mathrm{Hom}_{\widehat{\mathbf{S}}}(X, Y_i) = \mathrm{Hom}_{\mathbf{sFinSet}}(X, Y_i) = \mathrm{Hom}_{\mathbf{FinSet}^{\Delta^{op}_{\leq n}}}(\tau_n X, \tau_n Y_i)$$

for some n. Since the right-hand side is clearly a finite set, we see that $\mathrm{Hom}_{\widehat{\mathbf{S}}}(X, Y)$ is a projective limit of finite sets, hence a profinite set. In particular, $\mathrm{Hom}_{\widehat{\mathbf{S}}}(X \times \Delta^n, Y)$ is a profinite set for every n if X is a finite simplicial set, so we can define $Y^X \in \widehat{\mathbf{S}}$ by $(Y^X)_n = \mathrm{Hom}_{\widehat{\mathbf{S}}}(X \times \Delta^n, Y)$. Note that $(Y^X)_n = \lim_i (Y_i^X)_n$ levelwise, so $Y^X = \lim_i Y_i^X$. We therefore see that $(-)^X$ is just the extension of the functor $(-)^X\colon \mathbf{S}_{cofin} \to \widehat{\mathbf{S}}$ in the sense of Proposition 2.31.

Proposition 4.9. *Let X be a simplicial finite set. Then $(-) \times X\colon \widehat{\mathbf{S}} \to \widehat{\mathbf{S}}$ is right adjoint to $(-)^X$.*

Proof. We let $(-)^X$ be as in the discussion above. Then it is the unique extension of $(-)^X : \mathbf{S}_{cofin} \to \widehat{\mathbf{S}}$ which preserves projective limits. Since products commute with limits, we see that $(-) \times X$ also preserves projective limits. Now let $Y, Z \in \widehat{\mathbf{S}}$ be given and write $Y = \lim_i Y_i$, $Z = \lim_j Z_j$ with Y_i and Z_j coskeletal and finite for all i, j. We see that

$$\begin{aligned}\operatorname{Hom}_{\widehat{\mathbf{S}}}(Y \times X, Z) &\cong \lim_j \operatorname{Hom}_{\widehat{\mathbf{S}}}(\lim_i (Y_i \times X), Z_j) \cong \lim_j \operatorname*{colim}_i \operatorname{Hom}_{\widehat{\mathbf{S}}}(Y_i \times X, Z_j)\\ &\cong \lim_j \operatorname*{colim}_i \operatorname{Hom}_{\widehat{\mathbf{S}}}(Y_i, Z_j^X),\end{aligned}$$

where for the last equality we used that morphisms between simplicial finite sets in $\widehat{\mathbf{S}}$ are just morphisms in **sFinSet**, and we know that $(-) \times X$ is left adjoint to $(-)^X$ there.

On the other hand, we also have

$$\operatorname{Hom}_{\widehat{\mathbf{S}}}(Y, Z^X) \cong \operatorname{Hom}_{\widehat{\mathbf{S}}}(\lim_i Y_i, (\lim_j Z_j)^X) \cong \lim_j \operatorname{Hom}_{\widehat{\mathbf{S}}}(\lim_i Y_i, Z_j^X).$$

If Z_j^X happens to be coskeletal and finite, then the right-hand side of this equation will be equal to $\lim_j \operatorname{colim}_i \operatorname{Hom}_{\widehat{\mathbf{S}}}(Y_i, Z_j^X)$, proving that $(-) \times X$ is left adjoint to $(-)^X$.

This indeed turns out to be the case. Assume Z_j is n-coskeletal. We already saw that $(Z_j^X)_k$ is finite for every k in the discussion above, since

$$(Z_j^X)_k = \operatorname{Hom}_{\mathbf{FinSet}^{\Delta^{op}_{\leq n}}}(\tau_n(X \times \Delta^k), \tau_n Z_j).$$

To see that Z_j^X is n-coskeletal, first note that τ_n preserves products. We therefore see that $\tau_n((-) \times X) \cong \tau_n(-) \times X$ as functors. They have right adjoints, $\operatorname{cosk}_n(-)^X$ and $\operatorname{cosk}_n((-)^X)$, respectively, so by uniqueness of adjoints we conclude that there is a natural isomorphism $\operatorname{cosk}_n(-)^X \cong \operatorname{cosk}_n((-)^X)$. Since $Z_i = \operatorname{cosk}_n W$ for some $W \in \mathbf{FinSet}^{\Delta^{op}_{\leq n}}$, we conclude that $Z_i^X \cong \operatorname{cosk}_n(W^X)$. So Z_i^X is indeed coskeletal, and we conclude that $(-) \times X$ is left adjoint to $(-)^X$. ∎

From the above proposition we deduce that we can view a simplicial homotopy between maps of profinite spaces $f, g\colon X \to Y$ as a map $H\colon X \to Y^{\Delta^1}$ satisfying $\operatorname{ev}_0 H = f$ and $\operatorname{ev}_1 H = g$, where $\operatorname{ev}_0, \operatorname{ev}_1\colon Y^{\Delta^1} \to Y$ are obtained by composing the inclusions $i_0, i_1\colon Y^{\Delta^1} \to Y^{\Delta^1} \times \Delta^1$ with the evaluation map $\operatorname{ev}\colon Y^{\Delta^1} \times \Delta^1 \to Y$.

4.1.3 Profinite completion

In the last part of this section we will investigate the profinite completion functor. Note that $\mathbf{S}_{cofin}$ has finite limits. Indeed, let $\{X_i\}$ be a finite diagram in $\mathbf{S}_{cofin}$. Choose n_i such that X_i is n_i-coskeletal for every i. If we pick n such that $n \geq n_i$ for all i, then every X_i is n-coskeletal. We now see that $\lim_i X_i = \operatorname{cosk}_n(\lim_i \tau_n(X_i))$, using that $\operatorname{cosk}_n \tau_n(X_i) = X_i$, that cosk_n preserves limits, and that $\mathbf{FinSet}^{\Delta^{op}_{\leq n}}$ has all finite limits. In particularc $\widehat{\mathbf{S}}$ has all limits and colimits by Theorem 2.40[2] and there is a profinite

[2]Note that this in fact follows immediately when we view $\widehat{\mathbf{S}}$ as $\widehat{\mathbf{Set}}^{\Delta^{op}}$, since we can compute limits and colimits pointwise.

completion functor $\widehat{(\cdot)}\colon \mathbf{S} \to \widehat{\mathbf{S}}$ by Theorem 2.62. By the same theorem, this profinite completion functor is left adjoint to the functor $|\cdot|\colon \widehat{\mathbf{S}} \to \mathbf{S}$ which sends a profinite space to its underlying simplicial set. When we view $\widehat{\mathbf{S}}$ as $\mathbf{Stone}^{\Delta^{op}}$, then $|\cdot|$ is levelwise just the forgetful functor $\mathbf{Stone} \to \mathbf{Set}$, whose left adjoint is the Stone-Čech compactification $\beta\colon \mathbf{Set} \to \mathbf{Stone}$. This implies that the left adjoint of $|\cdot|$ is levelwise given by the Stone-Čech compactification, so by uniqueness of adjoints, we see that the profinite completion functor is levelwise the Stone-Čech compactification.

Proposition 4.10. *The profinite completion functor $\widehat{(\cdot)}\colon \mathbf{S} \to \widehat{\mathbf{S}}$ is levelwise given by the Stone-Čech compactification, when viewing $\widehat{\mathbf{S}}$ as $\mathbf{Stone}^{\Delta^{op}}$.* ■

As the Stone-Čech compactification does not preserve products (see Proposition 2.69), we immediately see that the profinite completion functor $\widehat{(\cdot)}\colon \mathbf{S} \to \widehat{\mathbf{S}}$ does not preserve products.

Corollary 4.11. *Profinite completion of simplicial sets does not preserve products.* ■

4.2 Connectedness and the fundamental groupoid

In the first part of this section, we discuss the definition of $\pi_0 X$, the profinite set of path components of a profinite space X. In the second part, we define the fundamental groupoid $\Pi_1 X$ of a profinite space X, which will be a profinite groupoid.

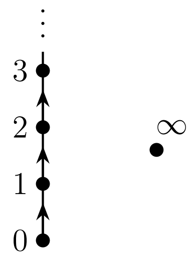
4.2.1 Connected profinite spaces

Given a profinite space $X \in \mathbf{s}(\widehat{\mathbf{Set}})$, a first guess might be to say that X is connected if its underlying simplicial set is connected. A natural definition for π_0 would then be the set of connected components of the underlying simplicial set. By "underlying simplicial set", we mean the simplicial set we obtain by forgetting about the topology on each Stone space X_n. To (hopefully) give π_0 the topology of a Stone space, we could define π_0 as a quotient of X_0 (more precisely, as the coequalizer of $d_0, d_1\colon X_1 \rightrightarrows X_0$ in $\mathbf{Top}$). The next example illustrates that this is the wrong approach though.

Example 4.12. Let I^1 be the simplicial unit interval, i.e. $I^1 = \Delta[1]$. We define I^2 by attaching two copies of I^1 to each other, the endpoint of the first to the starting point of the second (i.e. $d_0(I^1)$ gets glued to $d_1(I^1)$ of the second copy). Repeating this construction, we define I^{n+1} by attaching a copy of I^1 to I^n. For every $n > 1$, we define a map $I^n \to I^{n-1}$ by mapping $I^{n-1} \subseteq I^n$ to itself, and by mapping the attached copy of I^1 to a degeneracy in I^{n-1}. For clarity, the map $I^4 \to I^3$ is drawn below.

$$
\begin{tikzcd}
 & \bullet\, 4 \arrow[dl, mapsto] \\
3\, \bullet & \bullet\, 3 \arrow[l, mapsto] \arrow[u] \\
2\, \bullet \arrow[u] & \bullet\, 2 \arrow[l, mapsto] \arrow[u] \\
1\, \bullet \arrow[u] & \bullet\, 1 \arrow[l, mapsto] \arrow[u] \\
0\, \bullet \arrow[u] & \bullet\, 0 \arrow[l, mapsto] \arrow[u]
\end{tikzcd}
$$

If we now consider the limit I^∞ of the diagram $\{I^n\}_{n\in\mathbb{N}}$ in $\mathbf{s}(\widehat{\mathbf{Set}})$, we obtain a profinite space whose underlying simplicial set looks like the following



It can be seen as an infinitely long line with an endpoint at infinity. The space of 0-simplices I_0^∞ is the limit $\lim_{n\in\mathbb{N}} I_0^n = \lim_{n\in\mathbb{N}}[n]$, where $[n] = \{0,\ldots,n\}$ and where the map $[n] \to [n-1]$ is the identity on $[n-1]$ and maps n to $n-1$. We see that this limit is the Stone space $\mathbb{N}\cup\{\infty\}$, where $U \subseteq \mathbb{N}\cup\{\infty\}$ is open precisely if $U \subseteq \mathbb{N}$ or if $\infty \in \mathbb{N}$ and the complement of U is finite. We see that $\lim_n I^n$ is not connected in the sense discussed above, even though I^n is connected for every n. We see that $\pi_0(I^\infty)$ contains two elements, yet $\pi_0(I^n)$ contains only one element for $n \in \mathbb{N}$. When working with pro-categories, it is natural to ask of our functors that they preserve projective limits, so this suggests that $\pi_0(I^\infty)$ should only contain one element. We also see that, if $\pi_0(I^\infty)$ is considered as a quotient of I_0^∞, then it is not Hausdorff, hence not a Stone space.

Recall that if a nonempty simplicial set (or a topological space) X is not connected, then there always exists nonconstant map $X \to \{0,1\}$. We however see that any map $I^\infty \to \{0,1\}$ must be constant due to the topology on I_0^∞. Indeed, if $f\colon I^\infty \to \{0,1\}$, then $f(\infty) = f(x)$ for all but finitely many $x \in I_0^\infty$. Since all $x \in I_0^\infty \setminus \{\infty\}$ are connected by a sequence of 1-simplices, we see that this implies that $f(\infty) = f(x)$ for all $x \in I_0^\infty$, and hence that f is constant. This suggests that a good notion of connectedness should imply that I^∞ is connected.

A third reason why this notion of connectedness is not the right one, is that the connected components of a profinite space need not be profinite spaces. In the above example, we see that the connected component which is an infinite line, is not a profinite space, as it is infinite and has a discrete topology. This is problematic as we will sometimes need to prove statements by checking them for each connected component, which will require these components to be profinite spaces. $\diamondsuit$

The above example suggests we need another notion of connectedness, and another definition of π_0 for profinite spaces. Note that we can view a set S as a discrete simplicial set, by identifying S with the constant functor $\Delta^{op} \to \mathbf{Set}$ equal to S. If X is a simplicial set, the map $X \to \pi_0(X)$ which maps a simplex of X to the connected component it is contained in, is universal. This explicitly means that if we are given a map $X \to S$ for some discrete simplicial set S, then there is a unique map $\pi_0(X) \to S$ making the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & S \\ \pi_0(X) \arrow[ur] & \end{tikzcd}$$

commute. Note that this is equivalent to saying that $\pi_0(X)$ is the coequalizer of $X_1 \rightrightarrows X_0$.

We can use this approach to define $\pi_0(X)$ for a profinite space X, by defining $X \to \pi_0(X)$ as the arrow having the universal property expressed in the above diagram for all profinite sets S (where we view profinite sets as objects in $\mathbf{s}(\widehat{\mathbf{Set}})$ having only degenerate simplices above dimension 0). This is equivalent to defining $\pi_0(X)$ as the coequalizer of $d_0, d_1 \colon X_1 \rightrightarrows X_0$ in $\widehat{\mathbf{Set}}$. This coequalizer always exists, as $\widehat{\mathbf{Set}}$ is cocomplete. Note that we can also compute the coequalizer in **Stone**, of course. One should be careful however, as colimits in **Stone** are in general not the same as colimits in **Top**.

Definition 4.13. Let X be a profinite space. Define its profinite set of connected components $\pi_0(X)$ as the coequalizer of $d_0, d_1 \colon X_1 \rightrightarrows X_0$ in $\widehat{\mathbf{Set}}$. We say that X is *connected* if $\pi_0(X) = \{*\}$. If the underlying simplicial set of X is connected as well, then we call X *strongly connected.* ♢

Proposition 4.14. π_0 *preserves cofiltered limits.*

Proof. Note that $\widehat{\mathbf{Set}}$, finite colimits commute with cofiltered limits by Corollary 2.41. Since π_0 is a coequalizer in $\widehat{\mathbf{Set}}$, and cofiltered limits in $\widehat{\mathbf{S}} = \mathbf{s}(\widehat{\mathbf{Set}})$ are computed levelwise, we see that π_0 commutes with cofiltered limits. ■

The distinction between these two notions of connectedness can be important when considering covering spaces. In [Qui08], Quick uses covering spaces to define the profinite fundamental group and the profinite fundamental groupoid of a profinite space X. It is stated that for a (connected) profinite space X, the category of finite covering spaces forms a Galois category in the sense of [Gro+71]. However, using Quick's notion of a covering space, it turns out that axioms (G3) and (G6) of the definition of a Galois category in [Gro+71] need not be satisfied if the profinite space is not strongly connected. This is for example the case for the profinite space I^∞ of the above example. We will show this in Example 4.68 of section 4.6. We will fix this problem in this section by giving a slightly different definition of a covering space, which works for all connected profinite spaces. For now we will take a different approach to defining the fundamental group of a profinite space, based on Horel's definition of the profinite fundamental groupoid in [Hor17, p. 33]. Many desirable properties are easier to deduce using this definition, and we will show in section 4.6 that both approaches are equivalent for connected profinite spaces.

4.2.2 The fundamental groupoid of a profinite space

For Horel's definition of the fundamental groupoid, we need to view $\widehat{\mathbf{S}}$ as $\mathrm{Pro}(\mathbf{S}_{cofin})$. We write Π_1^c for the "classical" fundamental groupoid of a simplical set, which is a functor $\mathbf{S} \to \mathbf{G}$. Recall its definition from Example 3.2.

Definition 4.15. Define $\Pi_1 \colon \widehat{\mathbf{S}} \to \widehat{\mathbf{G}}$ by $\Pi_1(\{X_i\}) = \lim_i \widehat{\Pi_1^c X_i}$, where $\widehat{(\cdot)} \colon \mathbf{G} \to \widehat{\mathbf{G}}$ is the profinite completion functor, and where $\{X_i\}$ is a pro-object in $\mathbf{S}_{cofin}$. ♢

Remark 4.16. We need to consider the profinite completion of $\Pi_1^c X_i$ in the above definition, since $\Pi_1^c X_i$ for $X_i \in \mathbf{S}_{cofin}$ does not have to be a finite groupoid[3]. For example $\Pi_1^c(\Delta^1/\partial\Delta^1) \cong \mathbb{Z}_*$. $\diamondsuit$

Remark 4.17. Recall that the nerve of a groupoid is always 2-coskeletal. Since the nerve of finite groupoid is clearly finite, we obtain a fully faithful functor $\mathbf{FinG} \to \mathbf{S}_{cofin}$, hence a fully faithful functor $B\colon \widehat{\mathbf{G}} \to \widehat{\mathbf{S}}$ by Corollary 2.32. $\diamondsuit$

We now list some basic properties of $\Pi_1\colon \widehat{\mathbf{S}} \to \widehat{\mathbf{G}}$.

Proposition 4.18.

(i) $\Pi_1\colon \widehat{\mathbf{S}} \rightleftarrows \widehat{\mathbf{G}} : B$ *is an adjunction, and* $\Pi_1 BA = A$ *for any profinite groupoid* A.

(ii) Π_1 *preserves projective limits.*

(iii) $\Pi_1(\widehat{X}) = \widehat{\Pi_1^c X}$ *for any simplicial set* X.

(iv) $\pi_0\Pi_1 X = \pi_0(X)$ *for any profinite space* X.

(v) $\mathrm{Ob}(\Pi_1 X) = X_0$ *for any profinite space* X.

Proof.

(i) Let $X = \{X_i\}$ be a profine space and $A = \{A_j\}$ a profinite groupoid. We see that

$$\begin{aligned}\mathrm{Hom}_{\widehat{\mathbf{G}}}(\Pi_1 X, A) &= \mathrm{Hom}_{\widehat{\mathbf{G}}}(\lim_i \widehat{\Pi_1^c X_i}, \{A_j\}) \cong \lim_j \operatorname*{colim}_i \mathrm{Hom}_{\widehat{\mathbf{G}}}(\widehat{\Pi_1^c X_i}, A_j)\\ &\cong \lim_j \operatorname*{colim}_i \mathrm{Hom}_{\mathbf{G}}(\Pi_1^c X_i, A_j) \cong \lim_j \operatorname*{colim}_i \mathrm{Hom}_{\mathbf{S}_{cofin}}(X_i, BA_j)\\ &\cong \mathrm{Hom}_{\widehat{\mathbf{S}}}(X, BA),\end{aligned}$$

where all isomorphisms are natural, hence $\Pi_1 \dashv B$. Since B is fully faithful, we see that $\Pi_1 BA = A$ for any profinite groupoid A.

(ii) This follows from Proposition 2.31.

(iii) Let $|\cdot|_{\mathbf{G}}\colon \widehat{\mathbf{G}} \to \mathbf{G}$ be the functor that maps a profinite groupoid to its limit in $\mathbf{G}$, and let $|\cdot|_{\mathbf{S}}\colon \widehat{\mathbf{S}} \to \mathbf{S}$ map a profinite space to its limit in $\mathbf{S}$. Both of these functors are right adjoint to a profinite completion functor. We therefore see that the functor $\widehat{(\cdot)} \circ \Pi_1^c$ is left adjoint to $B \circ |\cdot|_{\mathbf{G}}$, and $\Pi_1 \circ \widehat{(\cdot)}$ is left adjoint to $|\cdot|_{\mathbf{S}} \circ B$. As B preserves limits, we see that $B \circ |\cdot|_{\mathbf{G}}$ and $|\cdot|_{\mathbf{S}} \circ B$ are naturally isomorphic. By uniqueness of adjoints, we see that there is a canonical isomorphism $\Pi_1(\widehat{X}) \cong \widehat{\Pi_1^c X}$.

(iv) Note that for finite simplicial sets X, $\pi_0(X) = \pi_0\Pi_1 X$. Since both π_0 and $\pi_0\Pi_1$ are functors $\widehat{\mathbf{S}} \to \widehat{\mathbf{Set}}$ that preserve cofiltered limits, and they agree on $\mathbf{S}_{cofin}$, we see that they are naturally isomorphic by Proposition 2.31.

[3]It is mistakenly assumed that $\Pi_1^c X$ is a finite groupoid for any $X \in \mathbf{S}_{cofin}$ in [Hor17], in between Remark 5.6 and Proposition 5.7

(v) Left to the reader. ■

One can similarly define the fundamental group $\pi_1(X, x)$ by writing X, with basepoint x, as a cofiltered limit $\{X_i\}$ of simplicial finite sets with basepoints x_i. Then $\pi_1(X, x)$ is defined as $\lim_i \widehat{\pi_1(X_i, x_i)}$. This is clearly equivalent to the following definition.

Definition 4.19. Let X be a profinite space and $x \in X_0$. Then define $\pi_1(X, x) := (\Pi_1 X)(x)$, the automorphism group of $\Pi_1 X$ at $x \in \mathrm{Ob}(\Pi_1 X)$. ♢

4.3 Principal G-bundles and $H^1(X; G)$

In this section we will study principal G-bundles over profinite spaces, for profinite groups G. We will prove a profinite analogue of the statement that BG classifies principal G-bundles. This will allow us to view the set of principal G-bundles over a profinite space X as $H^1(X; G)$, the first cohomology class of X with coefficients in a (not necessarily abelian!) profinite group G.

First recall the following definition.

Definition 4.20. Let G be a group and X, E simplicial sets. An *action* of G on E is a group homomorphism $G \to \mathrm{Aut}(E)$. A simplicial set E together with an action of a group G is called a G-space. The quotient E/G is the simplicial set defined by $(E/G)_n = E_n/G$ for each $n \in \mathbb{N}$. A map $p\colon E \to X$ is called a *principal G-bundle* if G acts freely on E, p is G-invariant and p induces an isomorphism $E/G \to X$. ♢

Example 4.21. For G a group, we can construct the simplicial set BG as in Definition 3.10, where we view G as a groupoid with one object. Then $(BG)_n = G^n$. For an object $(g_1, \ldots, g_n)$ of $(BG)_n$, we have

$$\begin{aligned}
d_0(g_1, \ldots, g_n) &= (g_2, \ldots, g_n) \\
d_n(g_1, \ldots, g_n) &= (g_1, \ldots, g_{n-1}) \\
d_i(g_1, \ldots, g_n) &= (g_1, \ldots, g_{i-1}, g_{i+1}g_i, g_{i+2}, \ldots g_n) \qquad \text{for } 0 < i < n \\
s_i(g_1, \ldots, g_n) &= (g_1, \ldots, g_i, e, g_{i+1}, \ldots, g_n).
\end{aligned}$$

Define the simplicial set EG by $(EG)_n = G^{n+1}$, and define the face maps and degeneracy maps by

$$\begin{aligned}
d_i(g_0, \ldots, g_n) &= (g_0, \ldots, g_{i-1}, g_{i+1}g_i, g_{i+2}, \ldots, g_n) \qquad \text{for } 0 \le i < n \\
d_n(g_0, \ldots, g_n) &= (g_0, \ldots, g_{n-1}) \\
s_i(g_0, \ldots, g_n) &= (g_0, \ldots, g_{i-1}, g_i, g_i, g_{i+1}, \ldots, g_n).
\end{aligned}$$

Define the map $p\colon EG \to BG$ by $p(g_0, \ldots, g_n) = (g_1, \ldots, g_n)$. There is an action of G on EG given by $h \cdot (g_0, \ldots, g_n) = (g_0 h^{-1}, \ldots, g_n)$. The reader is invited to check that $p\colon EG \to BG$ is a principal G-bundle. This bundle is called the *universal G-bundle*, and BG the *classifying space of G*, names which are justified by the following theorem. Note that the above constructions also work for profinite groups G, giving $(BG)_n$ and $(EG)_n$

the product topology. Furthermore note that $EG = B(G /\!\!/ G)$, and that $p\colon EG \to BG$ comes from the map $G /\!\!/ G \to G_*$ which maps an arrow $g \to h$ in $G /\!\!/ G$ to hg^{-1}. The reader should verify that EG is 0-coskeletal. ♢

The following well-known theorem states that principal G-bundles are classified by homotopy classes of maps to BG.

Theorem 4.22 ([GJ09, Theorem V.3.9]). *The space BG classifies principal G-bundles. More specifically, for any simplicial set X, there is a bijection between homotopy classes of maps $X \to BG$ and isomorphism classes of principal G-bundles $E \to X$. This bijection is given by sending the class of a map $f\colon X \to BG$ to the pullback $f^*(EG) \to X$ of $EG \to BG$.*

In the rest of this section will prove a profinite version of the above theorem.

4.3.1 BG classifies profinite G-bundles

We of course start by defining profinite principal G-bundles.

Definition 4.23. Let X be a profinite space and G a profinite group. An *action* of G on X is a levelwise continuous map $G \times X \to X$ satisfying the usual axioms of a group action. A profinite space together with a group action of a profinite group G is called a *(profinite) G-space.* ♢

Definition 4.24. Let $p\colon E \to X$ be a map of profinite spaces and let G be a profinite group acting continuously on E. Then p is called a *(profinite) principal G-bundle* if p is G-invariant and induces an isomorphism $E/G \to X$. A morphism of principal G-bundles over X is a G-equivariant map $f\colon E \to E'$ of profinite spaces. ♢

Example 4.25. The above example of the principal G-bundle $EG \to BG$ can be generalized to the context of profinite G-bundles. Note that if G is a profinite group, then G^n is a Stone space for every natural number n, so we can view EG and BG as objects of $\widehat{\mathbf{S}}$. We again see that $EG = B(G /\!\!/ G)$, where $G /\!\!/ G$ is a profinite groupoid. ♢

Example 4.26. This example is similar to the above one, and can be seen as a generalization. If we are given a profinite groupoid A and an object $x \in A$, we can form the profinite groupoid $A \downarrow x$ whose objects are arrows $x' \to x$ in A with target x, and whose arrows are commutative diagrams of the form

$$\begin{tikzcd} x'' \arrow[rr] \arrow[dr] & & x' \arrow[dl] \\ & x. & \end{tikzcd}$$

To see that we can define a profinite groupoid this way, let $A = \{A_i\}$ and let $x_i \in \mathrm{Ob}(A_i)$ correspond to x for every i. If we then set $A \downarrow x = \{A_i \downarrow x_i\}$, we see that $A \downarrow x$ is a profinite groupoid satisfying $\mathrm{Ob}(A \downarrow x) = \{\alpha \in \mathrm{Ar}(A) \mid t(\alpha) = x\}$ and $\mathrm{Ar}(A \downarrow x) =$

$\mathrm{Ob}(A \downarrow x) \times_{\mathrm{Ob}(A)} \mathrm{Ar}(A)$. The forgetful functor $A \downarrow x \to A$ is then a morphism of profinite groupoids.

Note that the profinite group $A(x)$ acts on $A \downarrow x$. On objects, this action is given by $\alpha \cdot \beta = \alpha\beta$, where $\alpha \in A(x)$ and where $\beta\colon x' \to x$ is an object of $A \downarrow x$. If A is a connected profinite groupoid, then this action makes $B(A \downarrow x) \to BA$ into a principal $A(x)$-bundle. ♢

To deduce the main result about profinite principal G-bundles, we need a few lemmas.

Lemma 4.27. *Let $f\colon E \to E'$ be a morphism of principal G-bundles over X. Then f is an isomorphism.*

Proof. Let $p\colon E \to X$ and $p'\colon E' \to X$ be the quotient maps. Since $f_n\colon E_n \to E'_n$ is a map of Stone spaces for each $n \geq 0$, we just need to show that f_n is a bijection for every n. For injectivity, assume $f_n(y) = f_n(z)$ for $y, z \in E_n$. Then $p(y) = p(z)$, since $p = p'f$. We can therefore pick $g \in G$ such that $g \cdot y = z$. Then $f_n(z) = f_n(g \cdot y) = g \cdot f_n(y) = g \cdot f_n(z)$. Since G acts freely on E_n, we see that $g = e$, hence $z = e \cdot y = y$.

For surjectivity, let $y' \in E'_n$. Choose $y \in E_n$ such that $p'(f(y)) = p(y) = p'(y')$. Such a y always exists since the fibers of $E \to X$ are nonempty Then there is some $g \in G$ such that $g \cdot f(y) = y'$. Since f is G-equivariant, we see that $f(g \cdot y) = y'$, so f is surjective. ■

Lemma 4.28. *Let X be a profinite G-space for some profinite group G. There is a 1-1 correspondence between continuous G-equivariant maps $X_0 \to G$, with G acting on itself from the left, and G-equivariant maps of profinite spaces $X \to EG$.*

Proof. Note that G acts on $(EG)_0 = G$ by $g \cdot h = hg^{-1}$. As a profinite G-set, this is isomorphic to G acting on itself from the left, under the isomorphism $G \to G$, $h \mapsto h^{-1}$. So a G-equivariant map $X_0 \to G$ is the same as a G-equivariant map $X_0 \to (EG)_0$. Noting that EG is 0-coskeletal, this is the same as a G-equivariant map $X \to EG$. ■

Lemma 4.29. *Let $p\colon E \to X$ be a (profinite) principal G-bundle. Then for any $n > 0$ and any $0 \leq i \leq n$, the square*

$$\begin{array}{ccc} E_n & \xrightarrow{d_i} & E_{n-1} \\ \big\downarrow{\scriptstyle p_n} & & \big\downarrow{\scriptstyle p_{n-1}} \\ X_n & \xrightarrow{d_i} & X_{n-1} \end{array}$$

is a pullback.

Proof. Let $n > 0$ and $0 \leq i \leq n$ be given. To see that the above square is a pullback, consider the continuous map $E_n \to X_n \times_{X_{n-1}} E_{n-1}$ given by $y \mapsto (p_n(y), d_i(y))$, arising from the universal property of the pullback. This is a map of principal G-bundles over X_n, so it is an isomorphism by Lemma 4.27. ■

Proposition 4.30. *Let $f, g\colon X \to X'$ be homotopic maps of profinite spaces and let $E \to X'$ be a principal G-bundle. Then f^*E and g^*E are isomorphic as G-bundles over X' by a canonical isomorphism.*

Proof. Let $h\colon X \times \Delta^1 \to X'$ be a homotopy from f to g, and let ι_0, ι_1 be the inclusion $X \to X \times \Delta^1$ at the vertices 0 and 1 of Δ^1. Then $h\iota_0 = f$ and $h\iota_1 = g$, so f^*E and g^*E are the pullback of h^*E along ι_0 and ι_1 respectively. We therefore are reduced to proving that for any profinite space X and any bundle $E \to X \times \Delta^1$, the bundles ι_0^*E and ι_1^*E are isomorphic. We will show that $\iota_0^*E \times \Delta^1$ is isomorphic to E. By a similar argument $\iota_1^*E \times \Delta^1 \cong E$, and the result will follow.

To see that $\iota_0^*E \times \Delta^1 \cong E$, note that by Lemma V.3.4 of [GJ09], there exists such an isomorphism ϕ which agrees with the inclusion $\iota_0^*E \hookrightarrow E$ on $\iota_0^*E \times \{0\}$ if we forget the topology on $\iota_0^*E \times \Delta^1$, E and G. We will show that such an isomorphism is continuous. From this proof it will also follow that ϕ is the unique isomorphism $\iota_0^*E \times \Delta^1 \cong E$ which extends the inclusion $\iota_0^*E \hookrightarrow E$.

Note that we can write $(\iota_0^*E \times \Delta^1)_1$ as a disjoint union $\sqcup_{\alpha \in (\Delta^1)_1} \iota_0^*E_1$. On the component corresponding to $s_0(0) \in (\Delta^1)_1$, we see that ϕ_1 is just the inclusion of $(\iota_0^*E)_1$ into E_1, hence continuous. Write D for the component corresponding to the nondegenerate 1-simplex 01 connecting the vertices 0 and 1. Then $d_1\phi_1 = \phi_0 d_1$, where ϕ_0 is the inclusion of $(\iota_0^*E)_0$ into E_0, so in particular ϕ_0 is continuous. We therefore see that $\phi_1|_D$ must be the map arising from the pullback

$$\begin{tikzcd}
D \arrow[drr, bend left, "\phi_0 d_1"] \arrow[dr, dotted, "\phi_1|_D"] \arrow[ddr, bend right] & & \\
& E_1 \arrow[r, "d_1"] \arrow[d] & E_0 \arrow[d] \\
& (X \times \Delta^1)_1 \arrow[r, "d_1"'] & (X \times \Delta^1)_0.
\end{tikzcd}$$

In particular, $\phi_1|_D$ is continuous. By similar arguments one can show that ϕ is continuous on $(\iota_0^*E)_0 \times \{1\}$ and on the third component of $(\iota_0^*E \times \Delta^1)_1$, the one corresponding to $s_0(1) \in (\Delta^1)_1$. This means that ϕ_0 and ϕ_1 are continuous. Now note that ϕ_n can be obtained from ϕ_{n-1} via the pullback

$$\begin{tikzcd}
(\iota_0^*E \times \Delta^1)_n \arrow[drr, bend left, "\phi_{n-1} d_0"] \arrow[dr, dotted, "\phi_n"] \arrow[ddr, bend right] & & \\
& E_n \arrow[r, "d_0"] \arrow[d] & E_{n-1} \arrow[d] \\
& (X \times \Delta^1)_n \arrow[r, "d_0"'] & (X \times \Delta^1)_{n-1},
\end{tikzcd}$$

so we conclude by induction that ϕ_n is continuous for every n. ■

Theorem 4.31. *Let X be a profinite space and G a profinite group. Let $PB_G(X)$ denote the set of isomorphism classes of principal G-bundles over X. Then the map*

$$\phi\colon [X, BG] \to PB_G(X); \quad f \mapsto f^*EG$$

is a bijection.

Proof. This map is well-defined by Proposition 4.30. For surjectivity of ϕ, let $p\colon E \to X$ be a principal G-bundle. Since E_0 is a Stone space and G acts freely on E_0, we see by Proposition 2.61 that E_0 is isomorphic to $G \times X_0$ as a G-space, so in particular there exists a G-equivariant map $E_0 \to G$. By Lemma 4.28, there is G-equivariant map $f'\colon E \to EG$ extending this map. This map factors through the quotients as a map $f\colon X \to BG$. We claim that $E \cong f^*EG$. By the universal property of the pullback, there exists a map $g\colon E \to f^*EG$ making the diagram

$$\begin{tikzcd}
E \arrow[dr, dotted, "g"] \arrow[drr, bend left, "f'"] \arrow[ddr, bend right, "p"'] & & \\
& f^*EG \arrow[r] \arrow[d] & EG \arrow[d] \\
& X \arrow[r, "f"'] & BG
\end{tikzcd}$$

commute. This is in particular a map of G-bundles, hence an isomorphism by Lemma 4.27.

For injectivity of ϕ, let $f, g\colon X \to BG$ and assume that $f^*EG \cong g^*EG$. We can identify both f^*EG and g^*EG with the same vector bundle $p\colon E \to X$. Then there exist commutative squares

$$\begin{tikzcd}
E \arrow[r, "f'"] \arrow[d, "p"] & EG \arrow[d] \\
X \arrow[r, "f"] & BG
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "g'"] \arrow[d, "p"] & EG \arrow[d] \\
X \arrow[r, "g"] & BG,
\end{tikzcd}$$

where f' and g' are G-equivariant. If we can construct a G-equivariant homotopy from f' to g', then we will obtain an induced homotopy $f \simeq g$. To do this, note that $E \times \Delta^1$ is naturally a G-space, where we let G act trivially on Δ^1. A G-equivariant homotopy $E \times \Delta^1 \to EG$ is then the same as a G-equivariant map $(E \times \Delta^1)_0 \to (EG_0)$. Since $(E \times \Delta^1)_0 = E_0 \times \{0, 1\}$, we can define such a G-equivariant map h_0 by $h_0(y, 0) = f'_0(y)$ and $h_0(y, 1) = g'_0(y)$. By Lemma 4.28, we obtain a G-equivariant homotopy $h\colon E \times \Delta^1 \to EG$ with $h(\cdot, 0) = f'$ and $h(\cdot, 1) = g'$, and therefore a homotopy $f \simeq g$. ∎

Remark 4.32. The above theorem can in fact be strengthened a bit. One can give $[X, BG]$ and $PB_G(X)$ both the structure of a groupoid. For $[X, BG]$ the objects of this groupoid are morphisms $f\colon X \to BG$, and the arrows of this groupoid are simplicial homotopies. Two homotopies can be composed using the fact that horns in BG have unique fillers. For $PB_G(X)$, the objects are principal G-bundles and the arrows are morphisms of principal G-bundles, which are always isomorphisms. The above map $\phi\colon [X, BG] \to PB_G(X)$ can be interpreted as a morphism of groupoids. We constructed an isomorphism $f^*EG \to g^*EG$ out of a homotopy H from f to g in Proposition 4.30. We constructed this isomorphism by considering a certain lift $\iota_0^*E \times \Delta^1 \to H^*EG$, where $H\colon X \times \Delta^1 \to BG$ is the homotopy. It can be shown that this lift is unique, and therefore that there is a canonical isomorphism $f^*EG \to g^*EG$ corresponding to H. One can also show that the composition of two homotopies corresponds to the composition of the induced isomorphisms, so ϕ is indeed a morphism of groupoids. The above theorem then states that ϕ induces a bijection on the connected components of these groupoids. However, more can be shown, namely that ϕ is also fully faithful, and hence that ϕ is an equivalence of groupoids. To see this, one needs to show that for any $f\colon X \to BG$, ϕ induces a bijection between the homotopies $f \simeq f$ and automorphisms of f^*EG. By a careful analysis of how, in the proof of Theorem 4.31, one obtains an isomorphism from a homotopy, and how one obtains a homotopy from an isomorphism, this indeed follows. We leave this to the reader to verify, as we will not explicitly need it. $\diamondsuit$

Recall that for simplicial sets, there is a natural bijection between $H^1(-;A)$ and $[-, BA]$, where A is any abelian group.

Definition 4.33. Let X be a profinite space and G a profinite group. Define the first cohomology group of X with coefficients in G by

$$H^1(X;G) := [X, BG]. \qquad \diamondsuit$$

Remark 4.34. In general, $H^1(X;G)$ does not have the structure of a group, so we should view $H^1(-;G)$ as a contravariant functor $\widehat{\mathbf{S}} \to \mathbf{Set}$. Note that since cohomology is contravariant, cofiltered diagrams are mapped to filtered diagrams. This is why the codomain of $H^1(-;G)$ is $\mathbf{Set} = \mathrm{Ind}(\mathbf{FinSet})$ instead of $\widehat{\mathbf{Set}}$. $\diamondsuit$

Proposition 4.35. *Let A be a profinite groupoid. Then $[X, BA] = \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(\Pi_1 X, A)$.*

Proof. Note that there is a 1-1 correspondence between $\mathrm{Hom}_{\widehat{\mathbf{S}}}(X, BA)$ and $\mathrm{Hom}_{\widehat{\mathbf{G}}}(\Pi_1 X, A)$. To see that this 1-1 correspondence takes homotopic maps to homotopic maps, it is enough to note that $(\mathrm{ev}_0, \mathrm{ev}_1)\colon (BA)^{\Delta^1} \to (BA)^2$ is equal to $B(\mathrm{ev}_0, \mathrm{ev}_1)\colon B(A^{I[1]}) \to B(A^2)$. This is left as an exercise to the reader. ■

Corollary 4.36. *Let $X = \lim_i X_i$ be a cofiltered limit of profinite spaces, and let G be a finite group. Then $H^1(X;G) \cong \mathrm{colim}_i H^1(X_i;G)$.*

Proof. If we combine the above proposition and Proposition 3.38, then we see that

$$H^1(X;G) \cong \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(\Pi_1 X, G_*) \cong \operatorname*{colim}_i \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(\Pi_1 X_i, G_*) \cong \operatorname*{colim}_i H^1(X_i;G),$$

using that G is finite. ■

Corollary 4.37. *Let $f\colon X \to Y$ be a map of profinite spaces. Then $\Pi_1 f\colon \Pi_1 X \to \Pi_1 Y$ is a weak equivalence if and only if $\pi_0(f)$ and $f^*\colon H^1(Y;G) \to H^1(X;G)$ are isomorphisms for all finite groups G.*

Proof. By the above proposition, $H^1(Y;G) \to H^1(X;G)$ is an isomorphism precisely if $\mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(\Pi_1 Y, G_*) \to \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(\Pi_1 X, G_*)$ is so. Furthermore, the fact that $\pi_0(f)$ is an isomorphism is equivalent to $\pi_0\Pi_1 f$ being an isomorphism, which is equivalent to the map $\mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(\Pi_1 Y, \mathrm{Disc}\, S) \to \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(\Pi_1 X, \mathrm{Disc}\, S)$ being an isomorphism for all finite sets S. Combining these two things and Proposition 3.33 implies the desired result. ■

Corollary 4.38. *Let X be a connected profinite space, let $x \in X_0$ and let G be a profinite group. Then there is a 1-1 correspondence between isomorphism classes of principal G-bundles and continuous homomorphisms $\pi_1(X, x) \to G$ up to inner automorphism, i.e.*

$$PB_G(X) \cong H^1(X; G) \cong \mathrm{Hom}_{\widehat{\mathbf{Grp}}}(\pi_1(X, x), G)/G,$$

where G acts on $\mathrm{Hom}_{\widehat{\mathbf{Grp}}}(\pi_1(X, x), G)$ by conjugation.

Proof. Note that $PB_G(X) \cong [X, BG]$. By Proposition 4.35 and the fact that the inclusion $\pi_1(X, x) \to \Pi_1 X$ is a homotopy equivalence by Theorem 3.52, we obtain isomorphisms

$$[X, BG] \cong \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(\Pi_1 X, G_*) \cong \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(\pi_1(X, x)_*, G_*).$$

The result now follows by noting that two maps $f, g \colon \pi_1(X, x)_* \to G_*$ are homotopic precisely if there is an inner automorphism $h \colon G \to G$ such that $fh = g$. ■

4.4 Cohomology with local coefficients

Recall that instead of defining (co)homology with coefficients in a fixed abelian group, it is also possible to define (co)homology with coefficients in a so-called *local system of abelian groups*. The advantage of defining (co)homology with local coefficients, is that it takes into account the action of the fundamental group of the space. In this section, we will define cohomology with local coefficients for profinite spaces. We follow Quick's approach in [Qui08], making some slight modifications and filling in some gaps. His approach is based on the alternative characterization of cohomology with local coefficients as given in [GJ09].

Remark 4.39. The approach to cohomology with local coefficients for profinite spaces presented in this section, is actually not the right one. The problem lies in the definition of a local coefficient system on a profinite groupoid. This definition does not really take the profinite structure of the groupoid into account. If we view a profinite groupoid A as a topological groupoid (see section 3.2), then the definition of a local coefficient system given below only takes into account the topology present on $A(x, y)$ for all $x, y \in \mathrm{Ob}(A)$, but not the topology present on $\mathrm{Ob}(A)$ or $\mathrm{Ar}(A)$. Especially if $\pi_0(A)$ is not finite, this may lead to strange behaviour of cohomology with local coefficients. As an example, see part (i) of Lemma 4.48. This statement involves a product, but a statement involving cofiltered limits and filtered colimits should actually be expected. However, in the rest of this thesis, we only need to consider finite local coefficient systems on finite groupoids. These are, of course, simply functors to the category of finite abelian groups **FinAb**, since both finite abelian groups and finite groupoids have discrete topologies. In particular, there is no extra structure present that has to be taken into account in the definition of a finite local coefficient system on a finite groupoid. We do obtain the right definition of cohomology with local coefficients in this case, so we will not encounter any real problems in the rest of this thesis by using the definition presented in this section.

The author has found several ways of defining profinite local coefficient systems which fix the above issue, but unfortunately he did not have enough time to work this out. This section will hopefully be rewritten somewhere in the near future, using the correct definition of a profinite local coefficient system. ♢

4.4.1 Profinite local coefficient systems

Definition 4.40. Let A be a profinite groupoid. A *(profinite) local coefficient system* $\mathcal{M}$ on A is a functor $\mathcal{M}\colon A \to \widehat{\mathbf{Ab}}$ such that, for any $x, y \in \mathrm{Ob}(A)$, the map

$$A(x,y) \times \mathcal{M}(x) \to \mathcal{M}(y); (\alpha, m) \mapsto \mathcal{M}(\alpha)(m)$$

is continuous. Here $\widehat{\mathbf{Ab}}$ is the category of profinite abelian groups and $A(x,y)$ is the profinite set of arrows $x \to y$ in A. By a functor $A \to \widehat{\mathbf{Ab}}$, we mean a functor $|A| \to \widehat{\mathbf{Ab}}$, where $|A|$ is the underlying groupoid of the profinite groupoid A. $\diamondsuit$

Definition 4.41. Let X be a profinite space. A *(profinite) local coefficient system* on X is a profinite local coefficient system on $\Pi_1 X$. $\diamondsuit$

Remark 4.42. Local coefficient systems are a generalization of G-modules for (profinite) groups G. Indeed, viewing G as a (profinite) groupoid with one object, we see that a (profinite) local coefficient system on G is the same as a G-module. $\diamondsuit$

Example 4.43. As an example, consider the space $X = \Delta^1/\partial\Delta^1$. We see that $\Pi_1 X = \widehat{\mathbb{Z}}$, so local coefficient systems on X are $\widehat{\mathbb{Z}}$-modules (where we view $\widehat{\mathbb{Z}}$ as a group, not as a ring!). An example of a $\widehat{\mathbb{Z}}$-module could be any finite abelian group A with a distinguished automorphism $\sigma\colon A \to A$. To see this, note that a $\widehat{\mathbb{Z}}$-module structure on A is just a continuous map $\widehat{\mathbb{Z}} \to \mathrm{Aut}(A)$, hence, by the universal property of $\widehat{\mathbb{Z}}$, just a map $\mathbb{Z} \to \mathrm{Aut}(A)$. Maps $\mathbb{Z} \to \mathrm{Aut}(A)$, however, simply correspond to elements of $\mathrm{Aut}(A)$. In particular, a finite local coefficient system on X is a finite abelian group A with a distinguished automorphism. $\diamondsuit$

Recall the construction of $A \downarrow x$ of example Example 4.26, together with the $A(x)$-bundle map $A \downarrow x \to A$. Any arrow $x \to y$ in A induces a morphism $A \downarrow x \to A \downarrow y$ by composition, which commutes with the maps $A \downarrow x \to A$, so we obtain a functor $A \to \widehat{\mathbf{G}}/A$. Here $\widehat{\mathbf{G}}/A$ is the category whose objects are morphisms with codomain A, and whose morphisms are commutative triangles, similarly to the definition of $A \downarrow x$. Composing with the functor $B\colon \widehat{\mathbf{G}} \to \widehat{\mathbf{S}}$ defines a functor $A \to \widehat{\mathbf{S}}/BA$. Define, for any $f\colon X \to BA$ in $\widehat{\mathbf{S}}/BA$, the functor $\widetilde{X}\colon A \to \widehat{\mathbf{S}}$ by the pullback

$$\begin{tikzcd} \widetilde{X}(a) \arrow[r] \arrow[d] & B(A \downarrow a) \arrow[d] \\ X \arrow[r, "f"] & BA \end{tikzcd}$$

for any $a \in \mathrm{Ob}(A)$. Since any arrow $a \to a'$ in A induces a map $B(A \downarrow a) \to B(A \downarrow a')$, we also obtain a map $\widetilde{X}(a) \to \widetilde{X}(a')$ from the universal property of the pullback. $\widetilde{X}$ is called the *covering system* of f. Any commutative triangle

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & BA & \end{tikzcd}$$

induces a natural transformation $\widetilde{X} \to \widetilde{Y}$, also following from the universal property of the pullback. Note that given a covering system $\widetilde{X}$, we obtain for any n the functor $\widetilde{X}_n \colon A \to \widehat{\mathbf{Set}}$, mapping $a \in \mathrm{Ob}(A)$ to the profinite set of n-simplices of $\widetilde{X}(a)$.

Definition 4.44. Let A be a profinite groupoid, $X \to BA$ a map of profinite spaces, and let $\mathcal{M}$ be a local coefficient system on A. Define the cochain complex $\overline{C}^*_A(X;\mathcal{M})$ by

$$\overline{C}^n_A(X;\mathcal{M}) = \{\text{all natural transformations } \widetilde{X}_n \to \mathcal{M}\},$$

where we view $\mathcal{M}$ as a functor landing in $\widehat{\mathbf{Set}}$. We define the cochain complex $C^*_A(X;\mathcal{M})$ to consists of those natural transformations that are zero on all degenerate simplices of X. Note that the maps $d_i \colon \widetilde{X}_{n+1} \to \widetilde{X}_n$ induce maps $d_i^* \colon \overline{C}^n_A(X;\mathcal{M}) \to \overline{C}^{n+1}_A(X;\mathcal{M})$. Define the differential by

$$d^n \colon C^n_A(X;\mathcal{M}) \to C^{n+1}_A(X;\mathcal{M}); \quad d^n = \sum_{i=0}^{n+1}(-1)^i d_i^*.$$

If $A = \Pi_1 X$ and $X \to B\Pi_1 X$ is the unit of the adjunction $\Pi_1 \dashv B$, then write $C^*(X;\mathcal{M}) := C^*_{\Pi_1 X}(X;\mathcal{M})$. We usually write d for d^n. $\diamondsuit$

Note that in the above definition, the simplicial identities for d_i and s_j ensure that $d(\mu)$ vanishes on all degenerate simplices if $\mu \colon \widetilde{X}_n \to \mathcal{M}$ does so. The cochain complex $C^*_A(X;\mathcal{M})$ can be seen as the normalized version of $\overline{C}^*_A(X;\mathcal{M})$. Note that we can also view $C^n_A(X;\mathcal{M})$ as the collection of natural transformation $N(\widetilde{X}_n) \to \mathcal{M}$, where by $N(\widetilde{X}_n)$ we denote the non-degenerate simplices of $\widetilde{X}_n$.

Definition 4.45. Let A, $\mathcal{M}$ and $X \to BA$ be as above. Define the *cohomology of X with local coefficients in $\mathcal{M}$*, denoted $H^*_A(X;\mathcal{M})$, to be the cohomology of the chain complex $C^*_A(X;\mathcal{M})$. As above, if $A = \Pi_1 X$, we write $H^*(X;\mathcal{M}) := H^*_A(X;\mathcal{M})$. $\diamondsuit$

It is not hard to see that $C^*_A(-;\mathcal{M})$ is a contravariant functor $\widehat{\mathbf{S}}/BA \to \mathbf{coCh}_{\geq 0}$, by using the above observation that any map $X \to Y$ over BA induces a map of covering systems $\widetilde{X} \to \widetilde{Y}$. In particular, $H^*_A(-;\mathcal{M})$ is a contravariant functor.

We should also investigate what happens if we are given a map of profinite groupoids $f \colon A \to C$. Let such a map be given and let $\mathcal{M}$ be a local coefficient system on C. Precomposition with f gives a local coefficient system $f^*\mathcal{M}$ on A. Now assume we are given a map of profinite spaces $X \to BA$. Composition with $Bf \colon BA \to BC$ gives us a map $X \to BC$. Denote the corresponding covering systems by $\widetilde{X}_A$ and $\widetilde{X}_C$. The map $f \colon A \to C$ induces maps $A \downarrow a \to C \downarrow f(a)$, so the universal property of the pullback gives us natural maps $\widetilde{X}_A(a) \to \widetilde{X}_C(f(a))$. Precomposition with these maps induces a map of chain complexes $C^*_C(X;\mathcal{M}) \to C^*_A(X; f^*\mathcal{M})$ and in particular a map $f^* \colon H^*_C(X;\mathcal{M}) \to H^*_A(X; f^*\mathcal{M})$ in cohomology.

Now let $f \colon X \to Y$ be profinite spaces, and $\mathcal{M}$ a local coefficient system on Y. If we now combine the above two cases, with $A = \Pi_1 X$ and $C = \Pi_1 Y$, then we obtain a chain map $C^*(Y;\mathcal{M}) \to C^*(X; f^*\mathcal{M})$ and hence a map $f^* \colon H^*(Y;\mathcal{M}) \to H^*(X; f^*\mathcal{M})$ in cohomology. Here $f^*\mathcal{M}$ is the local coefficient system on X obtained by precomposing $\mathcal{M} \colon \Pi_1 Y \to \widehat{\mathbf{Ab}}$ with $\Pi_1 f \colon \Pi_1 X \to \Pi_1 Y$. We first look at the behaviour of H^*_A with respect to cofiltered limits.

Proposition 4.46. *Let A be a finite groupoid, $\mathcal{M}$ a finite local coefficient system on A and let $X = \lim_{i \in I} X_i$ be a cofiltered limit of profinite spaces. Furthermore let a map $X \to BA$ be given. Then $H^*_A(X; \mathcal{M}) = \operatorname{colim}_{i \in I^{op}} H^*_A(X_i; \mathcal{M})$.*

Proof. Without loss of generality, assume the index category I is a codirected poset. The statement that $H^*_A(X; \mathcal{M}) = \operatorname{colim}_{i \in I^{op}} H^*_A(X_i; \mathcal{M})$ is a slight abuse of notation, as we do not have maps $X_i \to BA$. However, since BA is a coskeletal simplicial finite set, we see that $X \to BA$ factors through X_i for some $i \in I$, so we can define $\operatorname{colim}_{j \in J^{op}} H^*_A(X_j; \mathcal{M})$, where $J = \{j \in I \mid j \leq i\}$. Since the inclusion $J \hookrightarrow I$ is initial, we see that $\lim_{j \in J} X_j = X$.

Note that $C^n_A(X; \mathcal{M})$ consists of natural transformations $\widetilde{X}_n \to \mathcal{M}$, where $\widetilde{X}(a) = X \times_{BA} B(A \downarrow a)$. Since pullbacks and limits commute, we see that $\widetilde{X}(a) = \lim_j (X_j \times_{BA} B(A \downarrow a))$. Write $\widetilde{X}_j$ for the covering system of $X_j \to BA$, i.e. $\widetilde{X}_j(a) = (X_j \times_{BA} B(A \downarrow a))$. Since $\mathcal{M}(a)$ is finite for every a, and A is finite, we see that any natural transformation $\widetilde{X}_n \to \mathcal{M}$ factors through $\widetilde{(X_j)}_n$ for some $j \in J$. Since $\operatorname{Hom}_{\widehat{\mathbf{Set}}}(\widetilde{X}_n(a), \mathcal{M}(a)) \cong \operatorname{colim}_j \operatorname{Hom}_{\widehat{\mathbf{Set}}}((\widetilde{X}_j)_n(a), \mathcal{M}(a))$ holds for any $a \in \operatorname{Ob}(A)$, we see that $C^n_A(X; \mathcal{M}) = \operatorname{colim}_j C^n_A(X_j; \mathcal{M})$. Since taking filtered colimits is an exact functor, we conclude that $\operatorname{colim}_j H^*_A(X_j; \mathcal{M}) = H^*(X; \mathcal{M})$. ■

Corollary 4.47. *Let $\{f_i \colon X_i \to Y_i\}_{i \in I}$ be natural maps of profinite sets, indexed by a cofiltered category. Let A be a finite groupoid, $\mathcal{M}$ a finite local coefficient system on A and let natural maps $Y_i \to BA$ be given. Assume that $f_i^* \colon H^n_A(Y_i; \mathcal{M}) \to H^n_A(X_i; \mathcal{M})$ is an isomorphism for every i. Then $f^* \colon H^n_A(Y; \mathcal{M}) \to H^n_A(X; \mathcal{M})$ is an isomorphism.*

Proof. By the above proposition, $f^* \colon H^n_A(Y; \mathcal{M}) \to H^n_A(X; \mathcal{M})$ is a filtered colimit of the maps $f_i^* \colon H^n_A(Y_i; \mathcal{M}) \to H^n_A(X_i; \mathcal{M})$, which are all isomorphisms. ■

4.4.2 Homotopy invariance

The maps defined above, rather unsurprisingly, turn out to be invariant under homotopy. We first need the following lemmas.

Lemma 4.48. *Let X be a profinite space, A a profinite groupoid, $f \colon X \to BA$ a map of profinite spaces and $\mathcal{M}$ a local coefficient system on A. Then the following properties hold:*

(i) For $A_i \in \pi_0 A$ a connected component of A, write X_i for $f^{-1}(BA_i)$ and $\mathcal{M}_i$ for the restriction of $\mathcal{M}$ to A_i. Then there is a canonical isomorphism $\prod_{A_i \in \pi_0 A} H^n_{A_i}(X_i; \mathcal{M}_i) \to H^n_A(X; \mathcal{M})$ for every $n \geq 0$.

(ii) If A is connected, then for any $a \in \operatorname{Ob}(A)$, the chain complex $C^*_A(X; \mathcal{M})$ is naturally isomorphic to the chain complex of $A(a)$-equivariant maps $\widetilde{X}_n(a) \to \mathcal{M}(a)$. Here $A(a)$ denotes the profinite groups of arrows in A whose source and target is a.

Proof. For property (i), write $\widetilde{X}$ for the covering system corresponding to $X \to BA$ and $\widetilde{X}_i$ for the covering system corresponding to $X_i \to BA_i$. It is straightforward to see that a natural transformation $\mu \colon \widetilde{X}_n \to \mathcal{M}$ is the same as a tuple of natural transformations

$\mu_i\colon (\widetilde{X}_i)_n \to \mathcal{M}_i$ indexed by $A_i \in \pi_0(A)$. We therefore obtain a canonical isomorphism $\prod_{A_i \in \pi_0 A} C^n_{A_i}(X_i;\mathcal{M}_i) \to C^n_A(X;\mathcal{M})$ which induces the desired isomorphism on cohomology.

To prove (ii), let A be a connected profinite groupoid and let $a \in \mathrm{Ob}(A)$. Note that a natural transformation $\widetilde{X}_n \to \mathcal{M}$ is uniquely determined by the $A(a)$-equivariant map $\widetilde{X}_n(a) \to \mathcal{M}(a)$, and that any such $A(a)$-equivariant map can be uniquely extended to a natural transformation $\widetilde{X}_n \to \mathcal{M}$. This gives the desired isomorphism of chain complexes. ■

Lemma 4.49. *Let A be a profinite groupoid, let X be a profinite space, let $h\colon X \times \Delta^1 \to BA$ be a homotopy from f to g and let $\widetilde{X}^f$, $\widetilde{X}^g$ and $\widetilde{X}^h$ be the corresponding covering systems. Then there are (canonical) isomorphisms $\widetilde{X}^f \times \Delta^1 \cong \widetilde{X}^h \cong \widetilde{X}^g \times \Delta^1$ over $X \times I$, such that the restrictions $\widetilde{X}^f \times \{0\} \to \widetilde{X}^h$ and $\widetilde{X}^g \times \{1\} \to \widetilde{X}^h$ are the inclusions of $\widetilde{X}^f$ and $\widetilde{X}^g$ in $\widetilde{X}^h$.*

Proof. Note that we can restrict to connected components of A. Indeed, let $a \in \mathrm{Ob}(A)$ be given and let A' be the connected component containing a. Then $A \downarrow a \cong A' \downarrow a$, since any arrow with target a is contained in A'. For a homotopy $h\colon X \times \Delta^1 \to BA$, we see that $h^{-1}(B(A')) = X' \times \Delta^1$ for some $X' \subseteq X$, and

$$\widetilde{X}^h(a) = X \times_{BA} B(A \downarrow a) \cong X' \times_{B(A')} B(A' \downarrow a) = (\widetilde{X}')^h(a).$$

Therefore, when constructing the isomorphisms $\widetilde{X}^f(a) \times \Delta^1 \cong \widetilde{X}^h(a) \cong \widetilde{X}^g(a) \times \Delta^1$ for all $a \in A'$, we can restrict our attention to the connected profinite groupoid $A' \subseteq A$ and the profinite subspace $X' \subseteq X$. So from now on, assume A is connected.

Since A is connected, we see that the map $B(A \downarrow a) \to BA$ is a principal $A(a)$-bundle for every $a \in \mathrm{Ob}(A)$. We will only construct the isomorphism $\widetilde{X}^f \times \Delta^1 \cong \widetilde{X}^h$, the isomorphism for g is contructed similarly. By the proof of Proposition 4.30, we see that there is a unique isomorphism ϕ_a

$$\begin{tikzcd}
\widetilde{X}^f(a) \arrow[r, hook] \arrow[d, "\iota_0"'] & \widetilde{X}^h(a) \arrow[d] \\
\widetilde{X}^f(a) \times \Delta^1 \arrow[r] \arrow[ur, dotted, "\phi_a"'] & X \times \Delta^1
\end{tikzcd}$$

of $A(a)$-bundles, for every $a \in \mathrm{Ob}(A)$, such that the restriction $\widetilde{X}^f(a) \times \{0\} \to \widetilde{X}^h(a)$ agrees with the inclusion $\widetilde{X}^f(a) \hookrightarrow \widetilde{X}^h(a)$.

We are left with proving naturality for these isomorphisms $\widetilde{X}^f(a) \times \Delta^1 \to \widetilde{X}^h(a)$. This follows from their uniqueness. Assume we are given an arrow $\alpha\colon a' \to a$. Then we get induced isomorphisms $\alpha_*\colon \widetilde{X}^f(a') \to \widetilde{X}^f(a)$ and $\alpha_*\colon \widetilde{X}^h(a') \to \widetilde{X}^h(a)$. We need to show that

$$\begin{tikzcd}
\widetilde{X}^f(a') \times \Delta^1 \arrow[r, "\phi_{a'}"] \arrow[d, "\alpha_* \times \mathrm{id}_{\Delta^1}"'] & \widetilde{X}^h(a') \arrow[d, "\alpha_*"] \\
\widetilde{X}^f(a) \times \Delta^1 \arrow[r, "\phi_a"'] & \widetilde{X}^h(a)
\end{tikzcd}$$

commutes. The arrow α also induces an isomorphism $A(a') \to A(a)$ by conjugation, which allows us to see the maps α_* as maps of principal $A(a)$-bundles. The uniqueness of the isomorphism ϕ_a now implies naturality. Namely, by the uniqueness of ϕ_a, we see that $\phi_a = \alpha_* \circ \phi_{a'} \circ (\alpha_* \times \mathrm{id}_{\Delta^1})^{-1}$, hence $\phi_a \circ (\alpha_* \times \mathrm{id}_{\Delta^1}) = \alpha_* \circ \phi_{a'}$. ∎

Proposition 4.50. *Let $\mathcal{M}$ be a local coefficient system on a profinite groupoid C.*

(i) If $f, g\colon X \to Y$ are homotopic maps of profinite spaces and $Y \to BC$ is any map, then $f^*, g^*\colon H_C(Y;\mathcal{M}) \to H_C(X;\mathcal{M})$ are equal.

(ii) If $f\colon A \to C$ is weak equivalence of profinite groupoids, and $X \to BA$ is any map of profinite spaces, then $f^*\colon C^*_C(X;\mathcal{M}) \to C^*_A(X; f^*\mathcal{M})$ is an isomorphism.

(iii) If $f\colon X \to Y$ is a homotopy equivalence of profinite spaces and $\mathcal{N}$ is a local coefficient system on Y, then $f^*\colon H^*(Y;\mathcal{N}) \to H^*(X; f^*\mathcal{N})$ is an isomorphism.

Proof. (i) Let h be a homotopy from f to g. Denote the map $Y \to BC$ by ϕ. The maps $\phi f, \phi g\colon X \to BC$ are then homotopic by the homotopy $\phi \circ h$. Using Lemma 4.49, we see that there are natural isomorphisms $\widetilde{X}^h \cong \widetilde{X}^f \times \Delta^1 \cong \widetilde{X}^g \times \Delta^1$, and that this restricts to an isomorphism $\widetilde{X}^g \to \widetilde{X}^f$ over X, by restricting the isomorphism to $\widetilde{X}^g \times \{1\}$. We therefore identify $\widetilde{X}^g$ with $\widetilde{X}^f$. We have a commutative diagram of the form

$$\begin{tikzcd}
\widetilde{X}^f(c) \times \Delta^1 \arrow[d, "\cong"] \arrow[dr, "\widetilde{h}_c"] & & \\
\widetilde{X}^h(c) \arrow[r] \arrow[d] & \widetilde{Y}(c) \arrow[r] \arrow[d] & B(C \downarrow c) \arrow[d] \\
X \times I \arrow[r, "h"] & Y \arrow[r, "\phi"] & BC,
\end{tikzcd}$$

where the map $\widetilde{h}_c$ is natural in c. We will omit the subscript c from the notation. The map $\widetilde{h}\iota_0$ is the map $\widetilde{X}^f(c) \to \widetilde{Y}(c)$ obtained from the pullback along f. The map $\widetilde{h}\iota_1$, under the identification $\widetilde{X}^f \cong \widetilde{X}^g$ made above, is the map $\widetilde{X}^f(c) \to Y(c)$ induced by g. We therefore need to show that the chain maps $C^*_C(Y;\mathcal{M}) \to C^*_C(X;\mathcal{M})$ corresponding to $\widetilde{h}\iota_0$ and $\widetilde{h}\iota_1$ induce the same map in cohomology. By a standard argument, we see that the homotopy $\widetilde{h}$ induces a chain homotopy between these two chain maps, so this is indeed the case. For an example of such an argument, dualize the proof of [Lam68, Satz V.1.4], where homotopy invariance of homology is proved.

(ii) Let $f\colon A \to C$ be a weak equivalence. Then f is fully faithful and essentially surjective. In particular, it induces an isomorphism $\pi_0(A) \to \pi_0(C)$, so by Lemma 4.48, we can reduce to the case where A and C are connected. One can show in this case that for any $a \in \mathrm{Ob}(A)$, the diagram

$$\begin{tikzcd}
A \downarrow a \arrow[r] \arrow[d] & C \downarrow f(a) \arrow[d] \\
A \arrow[r, "f"] & C
\end{tikzcd}$$

is a pullback square. If we write $\widetilde{X}^A$ and $\widetilde{X}^C$ for the covering systems corresponding to $X \to BA$ and $X \to BA \to BC$ respectively, then we see that the left and right square of the diagram

$$\begin{tikzcd} \widetilde{X}^A(a) \arrow[r] \arrow[d] & B(A \downarrow a) \arrow[r] \arrow[d] & B(C \downarrow f(a)) \arrow[d] \\ X \arrow[r] & BA \arrow[r, "Bf"] & BC \end{tikzcd}$$

are pullbacks (B preserves pullbacks since it is left adjoint), hence the whole square is a pullback as well. This implies that the map $\widetilde{X}^A(a) \to \widetilde{X}^C(f(a))$ induced by f is an isomorphism. By part (ii) of Lemma 4.48, we can view $C^n_C(X; \mathcal{M})$ as the $C(f(a))$-equivariant maps $\widetilde{X}^C_n(f(a)) \to \mathcal{M}(f(a))$, and $C^n_A(X; \mathcal{M})$ as the $A(a)$-equivariant maps $\widetilde{X}^A_n(a) \to \mathcal{M}(f(a))$. As f induces isomorphisms $X^A(a) \to X^C(f(a))$ and $A(a) \to C(f(a))$, we see that the chain map $f^*\colon C^*_C(X; \mathcal{M}) \to C^*_A(X; f^*\mathcal{M})$ is indeed an isormorphism.

(iii) Note that Π_1 preserves homotopies, since Π_1 preserves products and $\Pi_1\Delta^1 = I[1]$. In particular $\Pi_1 f$ is a homotopy equivalence, hence a weak equivalence. The result now follows from the fact that $H^*(Y; \mathcal{N}) \to H^*(X; g^*\mathcal{N})$ factors as $H^*_{\Pi_1 Y}(Y; \mathcal{N}) \to H^*_{\Pi_1 Y}(X; \mathcal{N}) \to H^*_{\Pi_1 X}(X; g^*\mathcal{N})$. The map on the left is an isomorphism by part (i), and the map on the right by part (ii) with $A = \Pi_1 X$ and $C = \Pi_1 Y$. ■

It is well known that the Eilenberg-MacLane spaces $K(A, n)$ represent cohomology with coefficients in some abelian group A. One way to prove this is by using the Dold-Kan equivalence, as is done in [Lam68, §VIII.3]. We will construct profinite spaces similar to $K(A, n)$ which represent cohomology with local coefficients for profinite spaces X.

4.4.3 Representing cohomology with local coefficients

We first recall the situation for ordinary cohomology and the construction of $K(A, n)$ using the Dold-Kan equivalence. We will try to find a suitable translation to the situation of profinite spaces and local coefficients. The Dold-Kan equivalence is an equivalence $M\colon \mathbf{sAb} \rightleftarrows \mathbf{Ch}_{\geq 0} : D$. Here M computes the so-called Moore complex of a simplicial abelian group, and D recovers the simplicial abelian group from its Moore complex. The functor M furthermore satisfies $\pi_n(A_\bullet) = H_n(M(A_\bullet))$ for any simplicial abelian group $A_\bullet$. The functor D is defined by

$$(DC_*)_n = \bigoplus_{\substack{[n] \to [k] \\ surjective}} C_k,$$

where the sum ranges over all possible order preserving surjections $[n] \to [k]$, for any k. For more information on this functor, and a proof of the Dold-Kan equivalence, see [Wei94, §8.4].

Note that there is also an adjunction $\mathbb{Z}[-]\colon \mathbf{sSet} \leftrightarrows \mathbf{sAb} : U$, where U is the forgetful functor and $\mathbb{Z}[-]$ is levelwise the free abelian group on a simplicial set X, i.e. $\mathbb{Z}[X]_n =$

$\mathbb{Z}[X_n]$. Combining this with Dold-Kan equivalence, we obtain an adjunction $C_* \colon \mathbf{sSet} \leftrightarrows \mathbf{Ch}_{\geq 0} : D$, where C_* is just the (normalized) chain complex of X. Both the functors C_* and D preserve homotopies, hence there is a natural bijection between homotopy classes of maps $[C_*(X), A_*] \cong [X, D(A_*)]$ for every simplicial set X and chain complex A (see [Lam68, Satz 2.3][4]). Define, for an abelian group A and $n \geq 0$, two chain complexes $k_*(A,n)$ and $l_*(A,n)$ by

$$k_i(A,n) = \begin{cases} A & \text{if} \quad i = n \\ 0 & \text{if} \quad i \neq n \end{cases}; \qquad l_i(A,n) = \begin{cases} A & \text{if} \quad i = n, n+1 \\ 0 & \text{if} \quad i \neq n \end{cases}$$

where the differential $l_{n+1}(A,n) \to l_n(A,n)$ is given by id_A. Now define the simplicial sets $K(A,n) := D(k_*(A,n))$ and $L(A,n) := D(l_*(A,n))$. Note that there is an obvious map $l_*(A,n) \to k_*(A,n+1)$, and hence a map $L(A,n) \to K(A,n+1)$. Using the properties of C_* and D discussed above, one can deduce the following proposition. We write $Z^n(X;A) = \ker(C^n(X;A) \to C^{n+1}(X;A))$ and $B^n(X;A) = \mathrm{im}(C^{n-1}(X;A) \to C^n(X;A))$ for the cocycles and coboundaries, respectively.

Proposition 4.51. *Let X, Y be a simplicial set, $f \colon X \to Y$ a map, A an abelian group and $n \geq 0$.*

(i) There are natural bijections

$$\mathrm{Hom}_{\mathbf{sSet}}(X, K(A,n)) \cong Z^n(X;A) \text{ and } \mathrm{Hom}_{\mathbf{sSet}}(X, L(A,n)) \cong C^n(X;A).$$

(ii) There is a natural bijection $[X, K(A,n)] \cong H^n(X;A)$.

(iii) f has the left-lifting property (llp) with respect to $L(A,n) \to *$ if and only if $f^* \colon C^n(Y;A) \to C^n(X;A)$ is surjective. This is in particular the case if f is a monomorphism.

(iv) f has the llp with respect to $K(A,n) \to *$ if and only if $f^* \colon Z^n(Y;A) \to Z^n(X;A)$ is surjective, which in particular implies that $f^* \colon H^n(Y;A) \to H^n(X;A)$ is surjective. If f is a monomorphism, then the converse also holds, i.e. $f^* \colon Z^n(Y;A) \to Z^n(X;A)$ is surjective if $f^* \colon H^n(Y;A) \to H^n(X;A)$ is surjective.

(v) If f has the llp with respect to $L(A,n) \to K(A,n+1)$, then $(f^*)^{-1}(B^{n+1}(X;A)) \subseteq B^{n+1}(Y;A)$, or, equivalently, $f^* \colon H^{n+1}(Y;A) \to H^{n+1}(X;A)$ is injective. If f is a monomorphism and $f^* \colon H^{n+1}(Y;A) \to H^{n+1}(X;A)$ is an isomorphism, then the converse holds as well.

The author is aware that the proof below is not the most efficient one, however it can be directly translated to the case of cohomology of profinite spaces with local coefficients, as we will see later.

[4] Although a relative version of the statement is proved there, the proof of the absolute case is identical.

Proof. Statements (i) and (ii) follow directly by applying the adjunction $C_* \dashv D$ and noting that maps $C_*(X) \to k_*(A,n)$ and $C_*(X) \to l_*(A,n)$ are just cocycles and cochains, see for example [Lam68, Satz VIII.3.6].

For the proof of statement (iii), note that by the adjunction $C_* \dashv D$, a lift in the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & L(A,n) \\ Y \arrow[ur, dotted] & \end{tikzcd}$$

is equivalent to a lift in the diagram

$$\begin{tikzcd} C_*(X) \arrow[r, "\sigma"] \arrow[d, "f_*"] & l_*(A,n) \\ C_*(Y). \arrow[ur, dotted] & \end{tikzcd}$$

Since chain maps $C_*(X) \to l_*(A,n)$ correspond to cochains $C_n(X) \to A$, and similarly for $C_*(Y)$, we see that a lift exists if and only if there is some $\tau \in C^n(X)$ such that $f^*(\tau) = \sigma$. We conclude that the llp with respect to $L(A,n) \to *$ is equivalent to $C^n(Y;A) \to C^n(X;A)$ being surjective. For the case that f is a monomorphism, note that we can view $C^n(X;A)$ as all possible maps $N(X_n) \to A$, where $N(X_n)$ denotes the set of non-degenerate n-simplices of X. Since f induces an injection $N(X_n) \to N(Y_n)$, we see that any map $N(X_n) \to A$ extends to a map $N(Y_n) \to A$, hence $f^*\colon C^n(Y;A) \to C^n(X;A)$ is surjective.

Statement (iv) is similar. Again using the adjunction $C_* \dashv D$ and noting that maps $C_*(X) \to k_*(A,n)$ are cocycles, we see that the llp with respect to $K(A,n) \to *$ is equivalent to $Z^n(Y;A) \to Z^n(X;A)$ being surjective. Now assume f is a monomorphism and $f^*\colon H^n(Y;A) \to H^n(X;A)$ is surjective. Let $\sigma \in Z^n(X;A)$ be given. Then by assumption $\sigma = f^*(\tau) + d(\nu)$ for some $\tau \in Z^n(X;A)$ and some $\nu \in C^{n+1}(Y;A)$. By part (iii), we can pick $\nu' \in C^{n+1}(X;A)$ such that $f^*(\nu') = \nu$. Then $\tau + d(\nu') \in Z^n(X;A)$ and $f^*(\tau + d(\nu')) = \sigma$, so indeed $f^*\colon Z^n(Y;A) \to Z^n(X;A)$ is surjective.

For statement (v), by similar arguments as above, the llp with respect to $L(A,n) \to K(A,n+1)$ is equivalent to the statement that for any $\sigma \in C^n(X;A)$ and $\tau \in Z^n(Y;A)$ such that $f^*(\tau) = d(\sigma)$, there exists a $\nu \in C^n(Y;A)$ such that $d(\nu) = \tau$ and $f^*(\nu) = \sigma$. This can be neatly phrased by stating that the map

$$C^n(Y;A) \to C^n(X;A) \times_{Z^{n+1}(X;A)} Z^{n+1}(Y;A)$$

is surjective. This in particular implies that $(f^*)^{-1}(B^{n+1}(X;A)) \subseteq B^{n+1}(Y;A)$, i.e., that $f^*\colon H^{n+1}(Y;A) \to H^{n+1}(X;A)$ is injective.

For the latter part of statement (v), assume that f is a monomorphism and that $f^*\colon H^{n+1}(Y;A) \to H^{n+1}(X;A)$ is an isomorphism. Then in particular $f^*\colon Z^n(Y;A) \to Z^n(X;A)$ is surjective by (iv). Assume that we are given some $\sigma \in C^n(X;A)$ and $\tau \in Z^n(Y;A)$ such that $f^*(\tau) = d(\sigma)$. From the injectivity of $f^*\colon H^{n+1}(Y;A) \to H^{n+1}(X;A)$ we see that $\tau = d(\tau')$ for some τ', and from the surjectivity of $f^*\colon C^n(Y;A) \to C^n(X;A)$, we see that $f^*(\sigma') = \sigma$ for some σ'. Then $d(f^*(\sigma' - \tau')) = d(f^*(\sigma')) - f^*(d(\tau')) = 0$, so by surjectivity of $f^*\colon Z^{n+1}(Y;A) \to Z^{n+1}(X;A)$, there is some ν' such that $f^*(\nu') = f^*(\sigma' - \tau')$ and $d(\nu') = 0$. Now set $\nu = \nu' + \tau'$. Then $d(\nu) = d(\nu') + d(\tau') = \tau$ and $f^*(\nu) = f^*(\nu') + f^*(\tau') = f^*(\sigma') = \sigma$, so $\nu \in C^n(Y;A)$ defines the desired lift. ■

We will now consider the case of profinite spaces and cohomology with local coefficients over a profinite group. Note that we are working with profinite groups here, and not groupoids. This will however be enough to prove the existence of the model structure in section 4.5. Let a profinite group G be given, and let M be a profinite left-G-module, i.e. a profinite abelian group with a continuous left-G-action. Recall that for an abelian group A, the simplicial abelian groups $L(A,n)$ and $K(A,n)$ are levelwise just direct sums of copies of A. In particular, we can define simplicial profinite abelian groups $L(M,n)$ and $K(M,n)$ by endowing these simplicial abelian groups with the product topology in every degree. Noting that, for every $l \in \mathbb{N}$, there is a G-action on $\bigoplus_{i=1}^{l} M$ by letting G act on every copy of M simultaneously, we obtain a left action of G on $L(M,n)$ and $K(M,n)$.

Recall that a map of simplicial sets $f\colon X \to L(M,n)$ corresponds to the n-cochain $N(X_n) \to M$ given by $x \mapsto f(x) \in M = L(M,n)$, where $N(X_n)$ is the set of non-degenerate n-simplices of X. We therefore immediately see that for a profinite space X with a continuous G-action, the G-equivariant maps $X \to L(M,n)$ correspond to G-equivariant maps $N(X_n) \to M$. Similarly the G-equivariant maps $f\colon X \to K(M,n)$ correspond to G-equivariant maps $\sigma\colon N(X_n) \to M$ such that $d\sigma = \sum_{i=0}^{n}(-1)^i \sigma \circ d_i = 0$. If we are given a map of profinite spaces $X \to BG$, then the covering system $\widetilde{X}$ of this map consists of one space with a left-G-action. For the single object $* \in \mathrm{Ob}(G_*)$, we see that $B(G_* \downarrow *) = EG$, so the covering system of $X \to BG$ is simply the space $EG \times_{BG} X$ with its natural left-G-action.

Remark 4.52. Note that in the above discussion, a more formal approach is possible as well. The categorie $\widehat{\mathbf{Ab}}_G$ of profinite G-modules can be shown to be abelian. In particular, this implies that there is a Dold-Kan equivalence $\mathbf{s}(\widehat{\mathbf{Ab}}_G) \simeq \mathbf{Ch}_{\geq 0}(\widehat{\mathbf{Ab}}_G)$ which preserves homotopies. The forgetful functor $U\colon \widehat{\mathbf{Ab}}_G \to \widehat{\mathbf{Set}}_G$ has a left adjoint $F_G\colon \widehat{\mathbf{Set}} \to \widehat{\mathbf{Ab}}_G$, hence the forgetful functor $U\colon \mathbf{s}(\widehat{\mathbf{Ab}}_G) \to \mathbf{s}(\widehat{\mathbf{Set}}_G)$ has a left adjoint as well. One can show that this left adjoint also preserves simplicial homotopies. Composing this forgetful functor with the equivalence $\mathbf{s}(\widehat{\mathbf{Ab}}_G) \to \mathbf{Ch}_{\geq 0}(\widehat{\mathbf{Ab}}_G)$ defines a homotopy-preserving functor $C_*^G\colon \mathbf{s}(\widehat{\mathbf{Set}}_G) \to \mathbf{Ch}_{\geq 0}(\widehat{\mathbf{Ab}}_G)$ left adjoint to $D\colon \mathbf{Ch}_{\geq 0}(\widehat{\mathbf{Ab}}_G) \to \mathbf{s}(\widehat{\mathbf{Set}}_G)$. It can be shown that $C_G^*(X;M)$ as defined above is the dual of the complex $C_*^G(X)$. The above discussion on ordinary cohomology of simplicial sets therefore translates well to the context of profinite sets and cohomology with local coefficients. ◊

The following proposition relates $\widehat{\mathbf{S}}_G$, the category of profinite spaces with a continuous G-action, to $\widehat{\mathbf{S}}/BG$, the category of profinite spaces over BG.

Proposition 4.53. *For any profinite group G, there is an adjunction $EG \times_{BG} (-) \dashv EG \times_G (-)$, i.e. there is a natural bijection*

$$\mathrm{Hom}_{\widehat{\mathbf{S}}_G}(EG \times_{BG} X, Y) \cong \mathrm{Hom}_{\widehat{\mathbf{S}}/BG}(X, EG \times_G Y)$$

for every $X \in \widehat{\mathbf{S}}/BG$ *and* $Y \in \widehat{\mathbf{S}}_G$*. Here* $EG \times_G Y$ *is called the* Borel construction *or the* homotopy orbit space*. These functors also preserve homotopies, in the sense that* G*-equivariant homotopies are sent to homotopies over* BG *and vice versa.*

Proof. The functor $EG \times_{BG} (-) \colon \widehat{\mathbf{S}}/BG \to \widehat{\mathbf{S}}_G$ is defined by taking the pullback. Recall that $(EG)_n = G^{n+1}$, and that G acts from the left on EG by $h \cdot (g_0, \ldots, g_n) = (g_0 h^{-1}, \ldots, g_n)$. The G-action on $EG \times_{BG} X$ is simply defined to be the action of G on EG.

For defining the functor $EG \times_G (-) \colon \widehat{\mathbf{S}}_G \to \widehat{\mathbf{S}}/BG$, recall that if G acts from the right on the set Z, and form the left on W, then one defines

$$Z \times_G W := \frac{Z \times W}{\{(zg, w) \sim (z, gw)\}}.$$

We define $EG \times_G Y$ by $(EG \times_G Y)_n = EG_n \times_G Y_n$ degreewise, with the obvious boundary and degeneracy maps. Elements of $EG \times_G Y$ will be denoted as $[g_0, \ldots, g_n, y]$, where $[g_0, \ldots, g_n, y] = [g_0 h^{-1}, \ldots, g_n, hy]$ for all $h \in G$. Note that as the map $p \colon EG \to BG$ is G-invariant, it induces a map $EG \times_G Y \to BG$, explicitly given by $[g_0, \ldots, g_n, y] \mapsto (g_1, \ldots, g_n)$ in degree n. We also obtain, for every n, a map $q_n \colon (EG \times_G Y)_n \to Y_n$ given by $q_n([g_0, \ldots, g_n, y]) = g_0 y$. This map is not a map in $\widehat{\mathbf{S}}/BG$ however, as it generally does not commute with the boundary maps.

Now assume we are given a map of profinite spaces $f \colon X \to BG$, a profinite G-space Y and a G-equivariant morphism $\phi \colon EG \times_{BG} X \to Y$. Then we can define a morphism $\widetilde{\phi} \colon X \to EG \times_G Y$ by $\widetilde{\phi}(x) = [e, f(x), \phi(e, f(x), x)]$, noting that $f(x) \in G^n$ for $x \in X_n$. It directly follows that $\widetilde{\phi}$ is a morphism in $\widehat{\mathbf{S}}/BG$. For the converse, assume we are given a morphism $\psi \colon X \to EG \times_G Y$. Then define a morphism $\widetilde{\psi} \colon EG \times_B GX \to Y$ by $\widetilde{\psi}(g_0, \ldots, g_n, x) = g_0^{-1} \cdot q(\psi(x))$. One can show that this indeed defines the desired natural bijection

$$\mathrm{Hom}_{\widehat{\mathbf{S}}_G}(EG \times_{BG} X, Y) \cong \mathrm{Hom}_{\widehat{\mathbf{S}}/BG}(X, EG \times_G Y),$$

the details of which are left to the reader.

To see that these functors preserve homotopies, note that a G-equivariant homotopy is just a G-equivariant map $Y \times \Delta^1 \to Z$ for profinite G-spaces Y and Z, where G acts trivially on Δ^1. Similarly, a homotopy over BG is just a map $X \times \Delta^1 \to W$ with X, W spaces over BG, for which the square

$$\begin{tikzcd} X \times \Delta^1 \arrow[r] \arrow[d, "p_X"] & W \arrow[d] \\ X \arrow[r] & BG \end{tikzcd}$$

commutes. Since $EG \times_{BG} (X \times \Delta^1) = (EG \times_{BG} X) \times \Delta^1$ and $EG \times_G (Y \times \Delta^1) = (EG \times_G Y) \times \Delta^1$ in these two cases, we see that both functors preserve homotopies. ∎

Now note that $\mathrm{Hom}_{\widehat{\mathbf{S}}_G}(EG \times_{BG} X, L(M, n))$ corresponds to the G-equivariant continuous maps $N((EG \times_{BG} X)_n) \to M$ as we saw above, which are by definition the elements of $C^n_G(X;M)$, and that similarly $\mathrm{Hom}_{\widehat{\mathbf{S}}_G}(EG \times_{BG} X, K(M,n)) \cong Z^n_G(X;M)$. We therefore define $L^G(M,n) := EG \times_G L(M,n)$ and $K^G(M,n) := EG \times_G K(M,n)$. Note that $L^G(M,n)$ and $K^G(M,n)$ should be considered as profinite spaces over BG, i.e. they naturally come with a map to BG. We can now translate Proposition 4.51 to the context of profinite spaces and cohomology with local coefficients.

Proposition 4.54. *Let G be a profinite group, M a G-module and $n \geq 0$. For statements (i) and (ii), let $X \to BG$ be an object of $\widehat{\mathbf{S}}/BG$, and for statements (iii)-(v), let $f\colon X \to Y$ be a morphism in $\widehat{\mathbf{S}}$.*

(i) There are natural bijections

$$\mathrm{Hom}_{\widehat{\mathbf{S}}/BG}(X, K^G(M,n)) \cong Z^n_G(X;M) \quad \text{and} \quad \mathrm{Hom}_{\widehat{\mathbf{S}}/BG}(X, L^G(M,n)) \cong C^n_G(X;M).$$

(ii) There is a natural bijection $[X, K^G(M,n)] \cong H^n_G(X;M)$. Here $[X, K^G(M,n)]$ consists of the homotopy classes of maps $X \to K^G(M,n)$ over BG, where the homotopies are also homotopies over BG.

(iii) f has the llp with respect to $L^G(M,n) \to BG$ in $\widehat{\mathbf{S}}$ if and only if $f^*\colon C^n_G(Y;M) \to C^n_G(X;M)$ is surjective for any map $Y \to BG$. This is in particular the case if f is a monomorphism in $\widehat{\mathbf{S}}$.

(iv) f has the llp with respect to $K^G(M,n) \to BG$ in $\widehat{\mathbf{S}}$ if and only if $f^*\colon Z^n_G(Y;M) \to Z^n_G(X;M)$ is surjective for any map $Y \to BG$, which in particular implies that $f^*\colon H^n_G(Y;M) \to H^n_G(X;M)$ is surjective. If f is a monomorphism, then the converse also holds, i.e. $f^*\colon Z^n_G(Y;M) \to Z^n_G(X;M)$ is surjective if $f^*\colon H^n_G(Y;M) \to H^n_G(X;M)$ is surjective.

(v) If f has the left-lifting property with respect to $L^G(M,n) \to K^G(M,n+1)$, then $(f^*)^{-1}(B^{n+1}_G(X;M)) \subseteq B^{n+1}_G(Y;M)$, or, equivalently, the map $f^*\colon H^{n+1}_G(Y;M) \to H^{n+1}_G(X;M)$ is injective for any map $Y \to BG$. If f is a monomorphism and $f^*\colon H^{n+1}_G(Y;M) \to H^{n+1}_G(X;M)$ is an isomorphism for any map $Y \to BG$, then the converse holds as well.

Proof. For (i), note that we have $\mathrm{Hom}_{\widehat{\mathbf{S}}/BG}(X, K^G(M,n)) \cong \mathrm{Hom}_{\widehat{\mathbf{S}}_G}(EG \times_{BG} X, K(M,n))$. We already saw in the discussion below the proof of Proposition 4.51 that the latter is naturally isomorphic to $Z^n_G(X;M)$. The proof for $L^G(M,n)$ is similar.

For (ii), note that we already have the bijection

$$\mathrm{Hom}_{\widehat{\mathbf{S}}/BG}(X, K^G(M,n)) \cong Z^n_G(X;M)$$

from part (i). A map $f\colon X \to K^G(M,n)$ is sent to a G-equivariant map $\widetilde{f}\colon (EG \times_{BG} X)_n \to M$. By Proposition 4.53, we see that a homotopy between maps $f, g\colon X \to K^G(M,n)$ over BG corresponds to a G-equivariant homotopy $h\colon (EG \times_{BG} X) \times \Delta^1 \to K(M,n)$. We have seen in Proposition 4.51 that a homotopy $h\colon (EG \times_{BG} X) \times \Delta^1 \to K(M,n)$ corresponds to a map $\widetilde{h}\colon (EG \times_{BG} X)_{n-1} \to M$ satisfying $d\widetilde{h} = \widetilde{f} - \widetilde{g}$. Under this correspondence, G-equivariant homotopies are mapped to G-equivariant maps

$\widetilde{h}\colon (EG \times_{BG} X)_{n-1} \to M$, a fact which we leave to the reader to check. In particular, $[\widetilde{f}] = [\widetilde{g}]$ in $H^n_G(X; M)$ if and only if $f \simeq g$ as maps over BG.

For statement (iii), assume we are given a square of the form

$$\begin{tikzcd} X \arrow[r, "\widetilde{\sigma}"] \arrow[d, "f"] & L^G(M,n) \arrow[d] \\ Y \arrow[r, "g"] & BG. \end{tikzcd}$$

In particular, we have the maps $g\colon Y \to BG$ and $gf\colon X \to BG$, so we can view Y and X as spaces over BG. Under the natural bijection of part (i), $\widetilde{\sigma}$ corresponds to a $\sigma \in C^n_G(X; M)$, and we see that a lift exists if and only if $f^*(\tau) = \sigma$ for some $\tau \in C^n_G(Y; M)$. We conclude that f has the llp with respect to $L^G(M, n) \to BG$ if and only if $f^*\colon C^n_G(Y; M) \to C^n_G(X; M)$ is surjective for any $Y \to BG$.

Now assume $f\colon X \to Y$ is a monomorphism. We need to show that $f^*\colon C^n_G(Y; M) \to C^n_G(X; M)$ is surjective. This means that for any continuous G-equivariant $\sigma\colon N((EG \times_{BG} X)_n) \to M$, there should be a continuous G-equivariant map $\tau\colon N((EG \times_{BG} Y)_n) \to M$ for which $\sigma = \tau \circ f$. Since f is a monomorphism, it induces an injection $N((EG \times_{BG} X)_n) \to N((EG \times_{BG} Y)_n)$. Since G acts freely on EG_n, it acts freely on $N((EG \times_{BG} Y)_n)$. By the lemma below, the desired τ exists.

The proofs of statements (iv) and (v) are analogous to the proofs in Proposition 4.51. ∎

Lemma 4.55. *Let G be a profinite group and M a profinite G-module. Then for any injective G-equivariant map $X \to Y$ of free profinite G-sets, and any continuous G-equivariant map $f\colon X \to M$, there exists an extension $Y \to M$ making*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & M \\ Y \arrow[ur, dotted] & \end{tikzcd}$$

commute.

The following proof is a modification of the proof of [Qui08, Lemma 2.7] to the case of profinite G-modules and profinite G-sets.

Proof. Let a diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & M \\ Y & \end{tikzcd}$$

be given, with X, Y free profinite G-sets. Let $\mathcal{P}$ be the set of pairs (S, s) with $S \subseteq M$ a closed sub-G-module, and $s\colon Y \to M/S$ an extension of $X \to M \to M/S$. We order $\mathcal{P}$ by $(S, s) \leq (S', s')$ if and only if $S' \subseteq S$ and $qs' = s$, where $q\colon M/S' \to M/S$ is the quotient map. This makes $\mathcal{P}$ into a poset, which is nonempty since $(M, 0) \in \mathcal{P}$, where 0 denotes the trivial map. We want to apply Zorn's lemma, so assume we are given a chain $\{(S_i, s_i)\}_{i\in I}$ in $\mathcal{P}$. Define $T = \cap_{i\in I} S_i$. Then T is a closed submodule of M, and $M/T = \lim_i M/S_i$. Let $t\colon Y \to T/M$ be the map $\lim_i s_i$. Then $(T, t) \in \mathcal{P}$, and (T, t) is an upper bound for $\{(S_i, s_i)\}_{i\in I}$. By Zorn's lemma, there exists a maximal element $(S, s) \in \mathcal{P}$. We will show that $S = \{0\}$, which concludes the proof.

Suppose $S \neq \{0\}$. Then pick an open submodule $U \subseteq M$ such that $S \cap U \neq S$. Such a submodule must exist since $M = \lim_V M/V$ by Theorem 2.57, where the limit ranges over all open submodules V of M. Then $U \cap S$ is an open submodule of S, hence $S/(U \cap S)$ is finite. There is a homeomorphism of Stone spaces $M/(U \cap S) \cong M/S \times S/(U \cap S)$ such that the quotient map $M/(U \cap S) \to M/S$ corresponds to the projection onto the first coordinate. To see this, note that by Proposition 1 of [Ser97, Chapter I], or Exercise 2.2.3 of [RZ10], there exists a continuous section $\sigma\colon M/S \to M/(U \cap S)$ of the quotient map $M/(U \cap S) \to M/S$. (Note that since U is an open in M, it has finite index, hence it is also closed.) The map

$$M/S \times S/(U \cap S) \to M/(U \cap S); \qquad ([m], [n]) \mapsto (\sigma([m]) + [n])$$

then defines a continuous bijection, hence a homeomorphism. The problem of finding an extension $Y \to M/(U \cap S)$ is therefore reduced to finding an extension $Y \to M/S$ and $Y \to S/(U \cap S)$. For the first extension we take s, for the second extension we use the following lemma, noting that $S/(U \cap S)$ is finite. Denote the constructed extension by $s'\colon Y \to M/S$. Then $(S \cap U, s') > (S, s)$, contradicting the maximality of (S, s). We therefore conclude that $S = \{0\}$. ∎

Lemma 4.56. *For a profinite group G, let $X \to Y$ be an injective map of profinite G-sets and let $f\colon X \to Z$ be a map of profinite G-sets with Z finite. Assume furthermore that G acts freely on X and Y. Then f can be extended to a map $Y \to Z$.*

Proof. Let $X \hookrightarrow Y$ be a monomorphism in $\widehat{\mathbf{Set}}_G$, i.e. a continuous, injective G-equivariant map. We can just view X as a closed subspace of Y for which $g \cdot x \in X$ for any $x \in X$ and $g \in G$. By Proposition 2.61, there is a G-equivariant homeomorphism $Y \cong G \times T$ for some Stone space T, where G acts trivially on T. As X is a closed subspace of Y, we see that this homeomorphism restricts to a homeomorphism $X \cong G \times S$ for some closed subspace S of T.

There is a 1-1 correspondence between continuous G-equivariant maps $X \to Z$ and continuous maps $S \to Z$, and similarly for Y. To see this, let $h\colon S \to Z$ be given. Then we can associate to h the map $G \times S \to Z$ which maps (g, x) to $g \cdot h(x)$. This reduces the problem to showing that for an injection $S \hookrightarrow T$ of Stone spaces, any continuous map $S \to Z$ with Z finite, nonempty and discrete can be extended to a continuous map $T \to Z$, which is indeed possible. ∎

4.5 A fibrantly generated model structure on $\widehat{\mathbf{S}}$

In this section we will prove the existence of a fibrantly generated model structure on $\widehat{\mathbf{S}}$, as in Theorem 2.10 of Quick's paper [Qui11a]. We first repeat the definition of weak equivalences, cofibrations and fibrations as given in Definition 2.9 of Quick's paper, with a slightly different definition of a weak equivalence.

Definition 4.57. A morphism $f\colon X \to Y$ in $\widehat{\mathbf{S}}$ is called

(i) a *weak equivalence* if the induced map $\Pi_1 f\colon \Pi_1 X \to \Pi_1 Y$ is a weak equivalence of profinite groupoids, and $f^*\colon H^n_G(Y;M) \to H^n_G(X;M)$ is an isomorphism for every finite group G, every finite G-module M, every morphism $Y \to BG$ and every $n \geq 0$;

(ii) a *cofibration* if f is a (levelwise) monomorphism; and

(iii) a *fibration* if it has the right-lifting property with respect to every cofibrations that is also a weak equivalence. $\diamondsuit$

4.5.1 Weak equivalences

Before proving the existence of the fibrantly generated model structure on $\widehat{\mathbf{S}}$, we will study the notion of weak equivalence defined above. Like in the case of profinite groupoids, the weak equivalences in $\widehat{\mathbf{S}}$ are stable under cofiltered limits.

Proposition 4.58. *Weak equivalences in $\widehat{\mathbf{S}}$ are stable under cofiltered limits. More precisely, if $\{f_i\colon X_i \to Y_i\}_{i\in I}$ is a diagram of morphisms of profinite sets, indexed by a cofiltered category I, and f_i is a weak equivalence of profinite spaces for every i, then the limit $f\colon \lim_i X_i \to \lim_i Y_i$ is a weak equivalence.*

Proof. Write $X = \lim_i X_i$ and $Y = \lim_i Y_i$. Without loss of generality I is a codirected poset. Since Π_1 preserves cofiltered limits, we see that $\Pi_1 f\colon \Pi_1 X \to \Pi_1 Y$ is a weak equivalence. Now let a finite group G, a finite G-module M and a map $Y \to BG$ be given. Then $Y \to BG$ factors through Y_i for some i. Let J consist of those $j \in I$ that satisfy $j \leq i$. Then the inclusion $J \hookrightarrow I$ is initial, and $f_j^*\colon H^*(Y_j;M) \to H^*(X_j;M)$ is an isomorphism for every $j \in J$. By Proposition 4.46, we see that $f^*\colon H^*(Y;M) \to H^*(X;M)$ is the colimit of the isomorphisms $f_j^*\colon H^*(Y_j;M) \to H^*(X_j;M)$, hence an isomorphism. ∎

In [Qui11a], Quick defines weak equivalences as maps $f\colon X \to Y$ which induce isomorphisms $\pi_0(X) \to \pi_0(Y)$, $\pi_1(X,x) \to \pi_1(Y,f(x))$ for every basepoint $x \in X_0$, and $H^n(Y;\mathcal{M}) \to H^n(X;f^*\mathcal{M})$ for every local coefficient system of finite abelian groups $\mathcal{M}$ on Y. Since weak equivalences between profinite groupoids are the same as essentially surjective and fully faithful morphisms, we see that this is the same as asking that $\Pi_1 f\colon \Pi_1 X \to \Pi_1 Y$ is an isomorphism of profinite groupoids, and that $f^*\colon H^n(Y;\mathcal{M}) \to H^n(X;f^*\mathcal{M})$ is an isomorphism for every local coefficient system of finite abelian groups $\mathcal{M}$ on Y. We will show that this notion of weak equivalence agrees with the one that we define above for maps between connected profinite spaces. This proof does not work for profinite spaces that are not connected, and the proof of uses some ideas from the theory of (profinite) covering spaces, which we develop in section 4.6. For these reasons we use the notion of weak equivalence given in Definition 4.57, instead of the definition of a weak equivalence given in [Qui11a].

For a map $f\colon X \to Y$ of profinite spaces, $\Pi_1 f\colon \Pi_1 X \to \Pi_1 Y$ is a weak equivalence precisely if $\pi_0(X) \to \pi_0(Y)$ is an isomorphism of profinite sets and $\pi_1(X,x) \to \pi_1(Y, f(x))$ is an isomorphisms for any basepoint $x \in X_0$. We are therefore left with proving the following proposition, if we want to show that both notions of weak equivalence agree for connected profinite spaces. For the second part of the proof, the reader has to be familiar with the covering space theory which we develop in section 4.6. The proof is given at this stage, since it feels more natural to already include this proposition in this section.

Proposition 4.59. *Let $f\colon X \to Y$ be a morphism of connected profinite spaces, and assume that $\Pi_1 f\colon \Pi_1 X \to \Pi_1 Y$ is a weak equivalence. Then $f^*\colon H^n_G(Y;M) \to H^n_G(X;M)$ is an isomorphism for every finite group G, every finite G-module M, every morphism $Y \to BG$ and every $n \geq 0$, precisely if $f^*\colon H^n(Y;\mathcal{N}) \to H^n(X; f^*\mathcal{N})$ is an isomorphism for every $n \geq 0$ and every local coefficient system of finite abelian groups $\mathcal{N}$ on Y.*

Proof. For the first direction, let a finite local coefficient system $\mathcal{N}$ on Y be given and assume that $f^*\colon H^*_G(Y;M) \to H^*_G(X;M)$ is an isomorphism for every finite group G, every finite G-module M and every morphism $Y \to BG$. The morphism $\Pi_1 f\colon \Pi_1 X \to \Pi_1 Y$ is a weak equivalence, so by property (ii) of Proposition 4.50, we see that it induces an isomorphism $C^*_{\Pi_1 Y}(X;\mathcal{N}) \to C^*(X; f^*\mathcal{N})$ of chain complexes. So $f^*\colon H^*(Y;\mathcal{N}) \to H^*(X; f^*\mathcal{N})$ is an isomorphism if $f^*\colon H^*_{\Pi_1 Y}(Y;\mathcal{N}) \to H^*_{\Pi_1 Y}(X;\mathcal{N})$ is so.

Now pick $y \in Y$. Note that the inclusion $\pi_1(Y,y) \to \Pi_1 Y$ is a homotopy equivalence, which has a retract $\Pi_1 Y \to \pi_1(Y,y)$ as homotopy inverse, by Lemma 3.51. By property (ii) of Proposition 4.50, the following diagram commutes

$$\begin{tikzcd}
C^*(Y,\mathcal{N}) \arrow[r, "\cong"] \arrow[d, "f^*"] & C^*(Y,(ir)^*\mathcal{N}) \arrow[r, "\cong"] \arrow[d, "f^*"] & C^*_{\pi_1(Y,y)}(Y, i^*\mathcal{N}) \arrow[d, "f^*"] \\
C^*_{\Pi_1 Y}(X,\mathcal{N}) \arrow[r, "\cong"] & C^*_{\Pi_1 Y}(X,(ir)^*\mathcal{N}) \arrow[r, "\cong"] & C^*_{\pi_1(Y,y)}(X, i^*\mathcal{N}),
\end{tikzcd}$$

where all horizontal maps are isomorphisms. We are therefore left with proving that $H^n_{\pi_1(Y,y)}(Y, i^*\mathcal{N}) \to H^n_{\pi_1(Y,y)}(X, i^*\mathcal{N})$ is an isomorphism for every $n \geq 0$. Here $i^*\mathcal{N}$ is a finite local coefficient system on $\pi_1(Y,y)$, or equivalently, a finite $\pi_1(Y,y)$-module, for which we will write N.

Since N is finite, the action of $\pi_1(Y,y)$ on N factors through a finite quotient $G := \pi_1(Y,y)/U$ of $\pi_1(Y,y)$. Note that $C^n_{\pi_1(Y,y)}(Y,N)$ is the collection of $\pi_1(Y,y)$-equivariant maps $Y_n \times_{B(\pi_1(Y,y))_n} E(\pi_1(Y,y))_n \to N$. Since the action of $\Pi_1(Y,y)$ on N factors through G, we see that these are just the G-equivariant maps $Y'_n \to N$, where we define Y' to be the quotient $(Y \times_{B(\pi_1(Y,y))} E(\pi_1(Y,y)))/U$. The profinite space Y' is canonically isomorphic to $Y \times_{BG} EG$ over Y, so we see that we can just consider G-equivariant maps $Y_n \times_{BG_n} EG_n \to N$. However, this is the definition of $C^n_G(Y,N)$, so the quotient map $\pi_1(Y,y) \to G$ induces an isomorphism $C^n_{\pi_1(Y,y)}(Y,N) \to C^n_G(Y,N)$. Of course, this argument works for any profinite space with a map to $B\pi_1(Y,y)$, in particular we also obtain an isomorphism $C^n_{\pi_1(Y,y)}(X,N) \to C^n_G(X,N)$.

By assumption, the map $f^*\colon H^*_G(Y,N) \to H^*_G(X,N)$ is an isomorphism. We therefore see that $f^*\colon H^*(Y;\mathcal{N}) \to H^*(X; f^*\mathcal{N})$ must be an isomorphism as well.

For the converse, assume that $f^*\colon H^*(Y;\mathcal{N}) \to H^*(X;f^*\mathcal{N})$ is an isomorphism for every local coefficient system of finite abelian groups $\mathcal{N}$ on Y. Let a finite group G, a morphism $Y \to BG$ and a finite G-module M be given. The morphism $Y \to BG$ factors as $Y \xrightarrow{\eta} B\Pi_1 Y \xrightarrow{B\phi} BG$ for a certain morphism of profinite groupoids $\phi\colon \Pi_1 Y \to G$, where η is the unit of the adjunction $\Pi_1 \dashv B$. We will show that $\phi^*\colon C^*_G(Y;M) \to C^*(Y;\phi^* M)$ is an isomorphism of chain complexes. The same argument will show that we also have an isomorphism of chain complexes $C^*_G(X;M) \to C^*(X;(\phi\circ\Pi_1 f)^* M)$, since $X \to BG$ factors as

$$Y \xrightarrow{\quad\eta\quad} B\Pi_1 Y \xrightarrow{B(\phi\circ\Pi_1 f)} BG.$$

To see that $\phi^*\colon : C^*_G(Y;M) \to C^*(Y;\phi^*M)$ is an isomorphism, note that by property (ii) of Lemma 4.48 we can view $C^n(Y;\phi^*M)$ as the $\pi_1(Y,y)$-equivariant maps $(Y \times_{B\Pi_1 Y} B(\Pi_1 Y \downarrow y))_n \to M$ for an arbitrary fixed basepoint $y \in Y_0$. Here the action of $\pi_1(Y,y)$ on M comes from the map $\phi\colon \Pi_1 Y \to G$, restricted to $\pi_1(Y,y)$. We will denote this restriction to $\pi_1(Y,y)$ by ϕ as well. By Proposition 4.99, we see that $Y \times_{B\Pi_1 Y} B(\Pi_1 Y \downarrow y)$ is a universal cover of Y with a distinguished basepoint (the point corresponding the the identity arrow $y \to y$ in $\Pi_1 Y \downarrow y$), so we will denote it by $\widetilde{Y}$ from now on, and its basepoint by $\widetilde{y}$. In particular, $\pi_1(Y,y)$ is the automorphism group of $\widetilde{Y}$ as object over Y, and it acts freely and transitively on each fiber. The map ϕ induces a canonical map $B(\Pi_1 Y \downarrow y) \to EG$ which maps the basepoint of $B(\Pi_1 Y \downarrow y)$ (the identity arrow $y \to y$ of $\Pi_1 Y$) to the basepoint 1 of EG, where 1 is the unit of $G = (EG)_0$. By the universal property of pullbacks, we obtain an induced map $\widetilde{\phi}\colon \widetilde{Y} \to Y \times_{BG} EG$, and the map $\phi^*\colon : C^*_G(Y;M) \to C^*(Y;\phi^*M)$ is given by mapping a G-equivariant map $\sigma\colon (Y \times_{BG} EG)_n \to M$ to the $\pi_1(Y,y)$-equivariant map $\sigma\circ\widetilde{\phi}\colon \widetilde{Y} \to M$, where we use that we can view $C^*(Y,\phi^*M)$ as the $\pi_1(Y,y)$-equivariant maps $\widetilde{Y} \to M$.

Since $\widetilde{Y}$ is a universal cover of Y, we see that $\widetilde{\phi}(\widetilde{Y}) \subseteq Y \times_{BG} EG$ is a (connected) cover of Y by Proposition 4.86. If we identify the fiber above y with $\pi_1(Y,y)$, we see that $\widetilde{\phi}$, when restricted to the fiber above y, is precisely the map $\phi\colon \pi_1(Y,y) \to G$. In particular, we see that that $H := \mathrm{im}(\phi\colon \pi_1(Y,y) \to G)$ acts on $\widetilde{\phi}(\widetilde{Y})$. If $h \in H$, then the map $\widetilde{Y} \to Y \times_{BG} EG$ given by $z \mapsto h\cdot\widetilde{\phi}(z)$ maps $\widetilde{y}$ to $h \in (EG)_0$, which is contained in $\widetilde{\phi}(\widetilde{Y})$. Since $\widetilde{Y}$ is connected, we see that $\widetilde{Y}$ gets mapped to the connected component $\widetilde{\phi}(\widetilde{Y})$ of $Y \times_{BG} EG$. We see that $h \in H$ precisely if $h(\widetilde{\phi}(\widetilde{Y})) = \widetilde{\phi}(\widetilde{Y})$ for every $h \in H$.

We will now show that restricting a map $(Y \times_{BG} EG)_n \to M$ to a map $\widetilde{\phi}(\widetilde{Y})_n \to M$ gives a 1-1 correspondence between G-equivariant maps $(Y \times_{BG} EG)_n \to M$ and H-equivariant maps $\widetilde{\phi}(\widetilde{Y})_n \to M$. We then show that H-equivariant maps $\widetilde{\phi}(\widetilde{Y})_n \to M$ correspond to $\pi_1(Y,y)$-equivariant maps $\widetilde{Y} \to M$. We denote the G-equivariant maps $(Y \times_{BG} EG)_n \to M$ by $\mathrm{Hom}_G((Y \times_{BG} EG)_n, M)$, and similarly use the notation $\mathrm{Hom}_H(\widetilde{\phi}(\widetilde{Y})_n, M)$ and $\mathrm{Hom}_{\pi_1(Y,y)}(\widetilde{Y}_n, M)$. For the first statement, let a G-equivariant map $\sigma\colon (Y \times_{BG} EG)_n \to M$ be given. Since H is a subgroup of G, it is clear that restricting to $\widetilde{\phi}(\widetilde{Y})_n$ gives an H-equivariant map $\sigma|_{\widetilde{\phi}(\widetilde{Y})_n}\colon \widetilde{\phi}(\widetilde{Y})_n \to M$. To see that the restriction to $\widetilde{\phi}(\widetilde{Y})_n$ induces an a bijection $\mathrm{Hom}_G((Y \times_{BG} EG)_n, M) \to \mathrm{Hom}_H(\widetilde{\phi}(\widetilde{Y})_n, M)$, note that the image of $\widetilde{\phi}\colon \widetilde{Y} \to Y \times_{BG} EG$ is a nonempty covering of Y. In particular, for any $y' \in Y_n$, there is a $\widetilde{y}' \in \widetilde{Y}$ such that $\widetilde{\phi}(\widetilde{y}')$ lies in the fiber above y'. Since G acts transitively and freely on each fiber, and $\widetilde{\phi}(\widetilde{Y})_n$ contains at least one point of every fiber over Y, we see that an H-equivariant map $\sigma' \colon \widetilde{\phi}(\widetilde{Y})_n \to M$ extends uniquely to a G-equivariant map $\sigma \colon (Y \times_{BG} EG)_n \to M$. This proves that $\mathrm{Hom}_G((Y \times_{BG} EG)_n, M) \to \mathrm{Hom}_H(\widetilde{\phi}(\widetilde{Y})_n, M)$ is a bijection.

To see that the map $\phi^* \colon \mathrm{Hom}_H(\widetilde{\phi}(\widetilde{Y})_n, M) \to \mathrm{Hom}_{\pi_1(Y,y)}(\widetilde{Y}_n, M)$, given by precomposition with ϕ, is a bijection, note that injectivity follows from the fact that ϕ is levelwise a surjection. To see that ϕ^* is a surjection, let $\sigma \colon \widetilde{Y}_n \to M$ be a $\pi_1(Y,y)$-equivariant map. We need to show that it factors through $\widetilde{\phi}(\widetilde{Y})$, so let $z, w \in \widetilde{Y}_n$ be given, and assume $\widetilde{\phi}(z) = \widetilde{\phi}(w)$. Then z and w lie in the same fiber over Y, so there is an $\alpha \in \pi_1(Y,y)$ such that $\alpha \cdot z = w$. Then $\widetilde{\phi}(w)\widetilde{\phi}(\alpha \cdot z) = \phi(\alpha) \cdot \widetilde{\phi}(z) = \phi(\alpha) \cdot \widetilde{\phi}(w)$, so $\phi(\alpha) = 1$, since H acts freely on $\widetilde{\phi}(\widetilde{Y})$. We now see that $\sigma(w) = \sigma(\alpha \cdot z) = \phi(\alpha) \cdot \sigma(z) = \sigma(z)$, so indeed σ factors through $\widetilde{\phi}(\widetilde{Y})$. It is clear from the definition of the $\pi_1(Y,y)$-module structure on M that it factors as an H-equivariant map, so we conclude that $\phi^* \colon \mathrm{Hom}_H(\widetilde{\phi}(\widetilde{Y})_n, M) \to \mathrm{Hom}_{\pi_1(Y,y)}(\widetilde{Y}_n, M)$ is a bijection. Combining this with the restriction map $\mathrm{Hom}_G((Y \times_{BG} EG)_n, M) \to \mathrm{Hom}_H(\widetilde{\phi}(\widetilde{Y})_n, M)$, we see that $\phi^* \colon C^n_G(Y; M) \to C^n(Y; \phi^* M)$ is an isomorphism for every n. In particular $\phi^* \colon H^*_G(Y; M) \to H^*(Y; \phi^* M)$ is an isomorphism. By the similar argument for X, we see that $\phi^* \colon H^*_G(Y; M) \to H^*(Y; (\phi \circ \Pi_1 f)^* M)$ is an isomorphism as well. Note that the local coefficient system $(\phi \circ \Pi_1 f)^* M$ is by definition equal to $f^* \phi^* M$. We now have the commutative diagram

$$\begin{array}{ccc} H^*_G(Y; M) & \xrightarrow{f^*} & H^*_G(X; M) \\ {\scriptstyle\cong}\big\downarrow{\scriptstyle\phi^*} & & {\scriptstyle\cong}\big\downarrow{\scriptstyle\phi^*} \\ H^*(Y; \phi^* M) & \xrightarrow[\cong]{f^*} & H^*(X; f^* \phi^* M), \end{array}$$

where the bottom map is an isomorphism by assumption, noting that $\phi^* M$ is a finite local coefficient system on Y. We conclude that the map $H_G(Y; M) \to H_G(X; M)$ is an isomorphism. ■

As the proof of the two out of three property requires some work, we state it as a seperate lemma.

Lemma 4.60. *The class of weak equivalences is closed under retracts and satisfies the two out of three property.*

Proof. The proof for retracts is straightforward and left to the reader. For the two out of three property, there is only one difficult case. Namely, showing that if $f \colon X \to Y$ and $g \colon Y \to Z$ are morphisms such that gf and g are weak equivalences, then f is so as well. We have already seen that if $\Pi_1 g$ and $\Pi_1(gf)$ are weak equivalences, then this also holds for $\Pi_1 f$. So what remains, is proving that $f^* \colon H^n_G(Y; M) \to H^n_G(X; M)$ is an isomorphism for all maps $s \colon Y \to BG$ and all finite G-modules M. The problem is that such a map s does not need to factor through some map $Z \to BG$, so in principle we do not have a commutative diagram of the form

$$\begin{tikzcd} H^n_G(X;M) \arrow[rr, "(gf)^*", "\cong"'] \arrow[dr, "f^*"'] & & H^n_G(Z;M) \\ & H^n_G(Y;M), \arrow[ur, "\cong", "g^*"'] & \end{tikzcd}$$

since $H^n_G(Z;M)$ is not defined. However, the map $s\colon Y \to BG$ does factor through $Z \to BG$ up to homotopy, which will be enough to prove the desired statement.

Since $\Pi_1 g$ is a weak equivalence, by definition the induced map $\mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(\Pi_1 Z, G_*) \to \mathrm{Hom}_{\pi\widehat{\mathbf{G}}}(\Pi_1 Y, G_*)$ is a bijection. By Proposition 4.35, $g^*\colon [Z, BG] \to [Y, BG]$ is a bijection, so there is a map $t\colon Z \to BG$ such that $tg \simeq s$. Now note that the chain complex $C^*_G(Y;M)$ is given, in degree n, by the continuous G-equivariant maps $(Y \times_{BG} EG)_n = (s^*EG)_n \to M$ which are zero on degenerate simplices. By Proposition 4.30, s^*EG and $(tg)^*EG$ are isomorphic. This means that we can replace the map $s\colon Y \to BG$ by $tg\colon Y \to BG$ in the definition of $C^*_G(Y;M)$ and $H^*_G(Y;M)$. Since sf and tgf are homotopic by the same homotopy (precomposed with f), we can also replace the map $sf\colon X \to BG$ by $tgf\colon X \to BG$ in the definition of $H^*_G(X;M)$. Since g and gf are weak equivalences, we get a commutative diagram as above, and hence $f^*\colon H^n_G(Y;M) \to H^n_G(X;M)$ is an isomorphism for all $n \geq 0$. ■

4.5.2 The model structure

We define two (countable) sets of morphisms P and Q in $\widehat{\mathbf{S}}$, which will be the generating fibrations and generating trivial fibrations respectively. Let $\mathcal{S}$ be a countable set containing at least one representative for every isomorphism class of finite sets, let $\mathcal{G}$ be the set of all finite groups whose underlying set is in $\mathcal{S}$, and for every $G \in \mathcal{G}$, let $\mathcal{L}_G$ be the set of all finite G-modules whose underlying set is in $\mathcal{S}$. P consists of the following maps

(i) $\{0\} \hookrightarrow \{0,1\}$;

(ii) $S \to *$;

(iii) $EG \to BG$;

(iv) $BG \to *$;

(v) $(\mathrm{ev}_0, \mathrm{ev}_1)\colon (BG)^{\Delta[1]} \to BG \times BG$;

(vi) $L^G(M,n) \to K^G(M,n)$; and

(vii) $K^G(M,n) \to BG$

for every finite set $S \in \mathcal{S}$, every $G \in \mathcal{G}$, every $M \in \mathcal{L}_G$ and every $n \geq 0$. Here S denotes the discrete zero-dimensional simplicial set with S as it set of 0-simplices.

We define Q to be the set of morphisms

(i) $EG \to *$; and

(ii) $L^G(M,n) \to BG$

for every $G \in \mathcal{G}$, every $M \in \mathcal{L}_G$ and every $n \geq 0$.

If Q is the set of generating trivial fibrations, and the cofibrations are the monomorphisms, then the monomorphisms should precisely be the maps having the left-lifting property with respect to all maps in Q.

Lemma 4.61. *The Q-projective maps are precisely the monomorphisms in $\widehat{\mathbf{S}}$.*

Proof. Note that a map in $\widehat{\mathbf{Set}}^{\Delta^{op}}$ is a monomorphism precisely if it is levelwise a monomorphism. We will show that a morphism of profinite spaces $f\colon X \to Y$ is Q-projective if and only if $f_n\colon X_n \to Y_n$ is a monomorphism for every n. To see that a monomorphism has the left-lifting property with respect to $EG \to *$ for every finite group G, note that since EG is 0-skeletal and $(EG)_0 = G$, this follows since finite sets are injective objects in $\widehat{\mathbf{Set}}$. By Proposition 4.54, property (iii), we see that f is in $\mathrm{llp}(Q)$ if f_n is a monomorphism for every n. For the converse, consider the case where G is the trivial group and $M = \mathbb{Z}/2$. Then the map $L^G(M,n) \to BG$ is just the map $L(\mathbb{Z}/2, n) \to *$, so the left-lifting property with respect to this map is equivalent to $f^*\colon C^n(Y;\mathbb{Z}/2) \to C^n(X;\mathbb{Z}/2)$ being surjective. Note that $C^n(X;\mathbb{Z}/2)$ is just the collection of all maps $N(X_n) \to \mathbb{Z}/2$, where $N(X_n)$ are the non-degenerate simplices of X_n, and the similar statement holds for $C^n(Y;\mathbb{Z}/2)$. By an argument similar to the proof of Lemma 3.54, this implies that f restricts to an injection $N(X_n) \to N(Y_n)$ (note that one also has to prove that $f_n(N(X_n)) \subseteq N(Y_n)$). One can show that f is levelwise a monomorphism if and only if it restricts to a levelwise monomorphism on non-degenerate simplices, so we conclude that f is a monomorphism. ∎

In order to use the cosmall objects argument, we need that the codomains of all maps in P and Q are cosmall. Since $\widehat{\mathbf{S}} = \mathrm{Pro}(\mathbf{S}_{cofin})$, this follows if they are (finite and) k-coskeletal for some k. As this is not directly clear for $K^G(M,n)$, we prove this as a lemma.

Lemma 4.62. *For any finite group G and finite G-module M, the simplicial set $K^G(M,n)$ is coskeletal.*

Proof. We will show that $K^G(M,n)$ is $(n+1)$-coskeletal. First note that $K(M,n)$ is $(n+1)$-coskeletal. To see this, recall that $K(M,n)$ is a minimal Kan complex (meaning that $K(M,n) \to *$ is a minimal fibration), and that its homotopy groups vanish above degree n. Assume we are given a map $\partial\Delta^k \to K(M,n)$, with $k > n+1$. Noting that $\pi_{k-1}(K(M,n))$ vanishes, we can use arguments similar to the proof of Theorem I.7.10 of [GJ09] to conclude that this map extends to a map $\Delta^k \to K(M,n)$. The vanishing of $\pi_k(K(M,n))$ then guarantees that this extension is unique up to homotopy (rel. $\partial\Delta^k$). Since $K(M,n)$ is minimal, this implies that this extension is unique.

To complete the proof, we will show the general statement that if $p\colon X \to Z$ is a principal G-bundle, Y a G-set and if Z and Y are both n-coskeletal, then $X \times_G Y$ is n-coskeletal. This implies the lemma, since $K^G(M,n) = EG \times_G K(M,n)$. The definition of $X \times_G Y$ is the same as the definition of $EG \times_G Y$ as given in Proposition 4.53, where we note that X has a right-G-action given by $x \cdot g = g^{-1}x$.

Assume $W\colon \partial\Delta^k \to X \times_G Y$ is given with $k > n$. The map $p\colon X \to Z$ is G-invariant, so it induces a map $p'\colon X \times_G Y \to Z$. Then the map $p'W\colon \partial\Delta^k \to Z$ uniquely extends to a k-simplex $z \in Z_k$. Let $x \in X_k$ be a k-simplex such that $p(x) = z$. Now write $W = ([x_0, y_0], \dots, [x_n, y_n])$, where $[x_i, y_i] \in (X \times_G Y)_{k-1}$ is the i-th face of W. We can pick x_i and y_i such that $d_i x = x_i$ for every i, since G acts transitively on X, and y_i is uniquely determined by the choice of x_i since G acts freely on X. We see that for any $i < j$,

$$[d_i x_j, d_i y_j] = d_i([x_j, y_j]) = d_{j-1}([x_i, y_i]) = [d_{j-1}x_i, d_{j-1}y_i].$$

Since $d_i x_j = d_{j-1}x_i$ for all $i < j$, and since G acts freely on X, we see that $d_i y_j = d_{j-1}y_i$ for all $i < j$. This implies that $(y_0, \dots, y_n)$ determines a map $\partial\Delta^k \to Y$. Let $y \in Y_k$ be the unique k-simplex that extends this map to a map $y\colon \Delta^k \to Y$. Then $[x, y] \in (X \times_G Y)_k$ is the unique k-simplex extending $W\colon \partial\Delta^k \to X \times_G Y$. ∎

Theorem 4.63. *The category $\widehat{\mathbf{S}}$ has a fibrantly generated model structure with the weak equivalences, cofibrations and fibrations as in Definition 4.57. The set P is a set of generating fibrations, and Q a set of generating trivial fibrations for this model structure. Any object is cofibrant in this model structure.*

Proof. We will check all the conditions of Theorem A.18. The class of weak equivalences is closed under retracts and satisfies the two out of three property by Lemma 4.60.

1. To see that P and Q permit the cosmall object argument, we will show that the domains of all the maps in P and Q are a coskeletal simplicial finite set. The result then follows since $\widehat{\mathbf{S}} = \mathrm{Pro}(\mathbf{S}_{cofin})$. First note that a finite discrete set of points is 1-coskeletal. The fact that BG is coskeletal has been proved in section 3.1, when we defined the nerve of a groupoid. Products of coskeletal simplicial sets are again coskeletal since cosk_n is right adjoint for every n, hence preserves limits. By Lemma 4.62, $K^G(M, n)$ is coskeletal.

2. Since any map in Q is a composition of maps in P, we see that $Q \subseteq \mathrm{fib}(P)$, hence $\mathrm{fib}(Q) \subseteq \mathrm{fib}(P)$. To see that any Q-fibration is a weak equivalence, note that the Q-projective maps are precisely the monomorphisms in $\widehat{\mathbf{S}}$, so $\mathrm{fib}(Q)$ consists precisely of those maps that have the right-lifting property with respect to all monomorphisms. Now let $f\colon X \to Y$ be a Q-fibration. Then there exists a lift g in the following diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, dotted, "g"] & B \end{tikzcd}$$

hence there is a $g\colon B \to A$ satisfying $fg = \mathrm{id}_B$. We also see that the following diagram commutes

$$\begin{tikzcd} A \sqcup A \arrow[r, "\mathrm{id}_A \sqcup gf"] \arrow[d, "\iota_0 \sqcup \iota_1"'] & A \arrow[d, "f"] \\ A \times \Delta^1 \arrow[r, "fp_A"'] \arrow[ur, dotted, "h"] & B, \end{tikzcd}$$

so there exists a lift $h\colon A \times \Delta^1 \to A$ which is a homotopy from gf to id_B. We conclude that f is a simplicial homotopy equivalence. Since Π_1 preserves homotopies, we in particular see that $\Pi_1 f$ is a weak equivalence. We proved in part (i) of Proposition 4.50 that a homotopy equivalence induces isomorphisms $H^n_G(Y; M) \to H^n_G(X; M)$, so we conclude that f is a weak equivalence.

3. Let $f\colon X \to Y$ be a P-projective map. Since $\mathrm{fib}(Q) \subseteq \mathrm{fib}(P)$, we see that f must be Q-projective as well. In particular, it is a monomorphism.

   To see that $\Pi_1 f$ is a weak equivalence, note that by the adjunction $\Pi_1 \dashv B$, $\Pi_1 f$ has the left-lifting property with respect to $\mathrm{Disc}(\{0\}) \to \mathrm{Disc}(\{0,1\})$, $\mathrm{Disc}(S) \to *$, $G /\!\!/ G \to G$, $G \to *$ and $G^{I[1]} \to G \times G$. For the maps $G /\!\!/ G \to G$ and $G^{I[1]} \to G \times G$, note that $EG = B(G /\!\!/ G)$ and that the map $(\mathrm{ev}_0, \mathrm{ev}_1)\colon (BG)^{\Delta[1]} \to BG \times BG$ is the map $B(\mathrm{ev}_0, \mathrm{ev}_1)\colon B(G^{I[1]}) \to B(G \times G)$. By part 3 of the proof of Theorem 3.57, we see that this implies that $\Pi_1 f$ is a weak equivalence of profinite groupoids.

   Now let a map $Y \to BG$ and a finite G-module M be given, with G a finite group. By properties (iv) and (v) of Proposition 4.54 and the llp with respect to $L^G(M,n) \to K^G(M,n)$ and $K^G(M,n) \to BG$, we see that $f^*\colon H^n_G(Y; M) \to H^n_G(X; M)$ is an isomorphism for every $n \geq 0$. We conclude that f is a weak equivalence.

4. For the last step we will show that any Q-projective map that is a weak equivalence, is P-projective. Note that a Q-projective weak equivalence is just a monomorphism that is also a weak equivalence. Let $f\colon X \to Y$ be such a map. To see that $\Pi_1 f$ is a weak equivalence of groupoids, we again use the adjunction $\Pi_1 \dashv B$. By this adjunction, f has the llp with respect to the maps (i) - (v) of P precisely if $\Pi_1 f$ has the llp with respect to $\mathrm{Disc}(\{0\}) \to \mathrm{Disc}(\{0,1\})$, $\mathrm{Disc}(S) \to *$, $G /\!\!/ G \to G$, $G \to *$ and $G^{I[1]} \to G \times G$. By part 4 of the proof of Theorem 3.57, this is indeed the case if $\Pi_1 f$ is injective on objects and a weak equivalence.

   It therefore remains to show that f has the llp with respect to $L^G(M,n) \to K^G(M,n)$ and $K^G(M,n) \to BG$. This follows from parts (iv) and (v) of Proposition 4.54, the injectivity of f and the fact that $f^*\colon H^n_G(Y; M) \to H^n_G(X; M)$ is an isomorphism for all finite groups G, all finite G-modules M and all morphisms $Y \to BG$.

As the cofibrations are the levelwise injective maps, we see that every object of $\widehat{\mathbf{S}}$ is cofibrant. ∎

4.5.3 Relation to other model categories

We have seen a couple of adjunctions between the categories $\widehat{\mathbf{S}}$, $\widehat{\mathbf{G}}$, $\mathbf{S}$ and $\mathbf{G}$, which are expressed in the following commutative diagram:

$$\begin{tikzcd}
\mathbf{S} \arrow[r, bend left, "\Pi_1"] \arrow[r, phantom, "\perp"] \arrow[d, bend right, "\widehat{(\cdot)}"'] \arrow[d, phantom, "\dashv"] & \mathbf{G} \arrow[l, bend left, "B"] \arrow[d, bend right, "\widehat{(\cdot)}"'] \arrow[d, phantom, "\dashv"] \\
\widehat{\mathbf{S}} \arrow[r, bend left, "\Pi_1"] \arrow[r, phantom, "\perp"] \arrow[u, bend right, "|\cdot|"'] & \widehat{\mathbf{G}}. \arrow[l, bend left, "B"] \arrow[u, bend right, "|\cdot|"']
\end{tikzcd} \tag{4.1}$$

If we give $\mathbf{S}$ the Kan-Quillen model structure, $\mathbf{G}$ the canonical model structure, $\widehat{\mathbf{G}}$ the model structure of section 3.5 and $\widehat{\mathbf{S}}$ the model structure defined above, then all these adjunctions are Quillen pairs.

Proposition 4.64. *All four adjunctions of diagram* (4.1) *are Quillen pairs. Furthermore, each of these functors preserve weak equivalences, except for* $|\cdot|\colon \widehat{\mathbf{S}} \to \mathbf{S}$.

Proof. The adjunction $\widehat{(\cdot)} \dashv |\cdot|$ between $\mathbf{G}$ and $\widehat{\mathbf{G}}$ has already been treated in Proposition 3.59.

Now consider the adjunction $\widehat{(\cdot)} \dashv |\cdot|$ between $\mathbf{S}$ and $\widehat{\mathbf{S}}$. Note that the generating fibrations in $\widehat{\mathbf{S}}$ are all Kan fibrations between simplicial finite sets, hence the functor $|\cdot|$ maps them to fibrations in $\mathbf{S}$. We also see that $EG \to *$ and $L^G(M,n) \to BG$ are trivial fibrations in $\mathbf{S}$, as they are Kan fibrations with contractible fibers. We therefore see that $|\cdot|$ preserves fibrations and trivial fibrations, hence $(\widehat{(\cdot)}, |\cdot|)$ is a Quillen pair. We see by [Hir03, Proposition 8.5.7] that $|\cdot|$ preserves weak equivalences between fibrant objects, and that $\widehat{(\cdot)}$ preserves weak equivalences between cofibrant objects. As every object in $\mathbf{S}$ is cofibrant, we conclude that $\widehat{(\cdot)}$ preserves weak equivalences.

The functor $\Pi_1\colon \widehat{\mathbf{S}} \to \widehat{\mathbf{G}}$ preserves cofibrations. This follows since cofibrations in $\widehat{\mathbf{S}}$ are levelwise injective maps, cofibrations in $\widehat{\mathbf{G}}$ are maps that are injective on objects, and $\mathrm{Ob}(\Pi_1 X) = X_0$. We also see that Π_1 preserves weak equivalences by definition of the weak equivalences in $\widehat{\mathbf{S}}$. In particular, Π_1 preserves trivial cofibrations, hence (Π_1, B) is a Quillen pair. To see that $B\colon \widehat{\mathbf{G}} \to \widehat{\mathbf{S}}$ preserves weak equivalences, let $f\colon A \to C$ be a weak equivalence in $\widehat{\mathbf{G}}$. By Theorem 3.48, there is a level representations $\{f_i\colon A_i \to C_i\}$ of f by weak equivalences. A weak equivalence between finite groupoids is a homotopy equivalence, and B preserves homotopies, so $Bf_i\colon BA_i \to BC_i$ is a (simplicial) homotopy equivalence between profinite spaces, in particular a weak equivalence. By Proposition 4.58, $Bf\colon BA \to BC$ is a weak equivalence.

Lastly, we look at the adjunction $\Pi_1 \dashv B$ between $\mathbf{S}$ and $\mathbf{G}$. The proof that Π_1 preserves (trivial) cofibrations and weak equivalences is identical to the one given above. To see that B preserves weak equivalences, note that weak equivalences are homotopy equivalences in $\mathbf{G}$, and that B preserves homotopy equivalences. ■

4.6 Coverings of profinite spaces

In this section, we develop a theory of covering spaces for profinite spaces. We begin by correcting Quick's definition of a finite covering given in [Qui08]. We then prove that the finite coverings of a connected profinite space indeed form a Galois category, providing us with an alternative construction of the fundamental group. Subsequently, we deduce some basic properties and define profinite coverings of profinite spaces. These notions are then used to provide an alternative construction of the fundamental groupoid of a connected profinite space X. It is shown that this fundamental groupoid is naturally isomorphic to the one defined in section 4.2. In particular, the profinite fundamental group constructed using finite coverings agrees with the one defined in section 4.2.

4.6.1 Finite coverings

It was mentioned in section 4.2 that for a connected profinite space X, the category of finite covering spaces as defined by Quick in [Qui08] does not always satisfy the axioms of a Galois category given in [Gro+71, Exposé V §4]. Quick defines a map $Y \to X$ of profinite spaces to be a covering map if for any solid arrow diagram of the form

$$\begin{tikzcd} \Delta^1 \arrow[r] \arrow[d] & Y \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted, "\exists!"] & X \end{tikzcd}$$

there exists a unique lift $\Delta^n \to Y$. A covering $p\colon Y \to X$ is called finite if for every $x \in X_0$, the profinite set $p^{-1}(x) \subseteq Y_0$ is finite. It is then claimed that, fixing some $x \in X_0$, the category of finite coverings of X, as a full subcategory of $\widehat{\mathbf{S}}/X$, together with the fiber functor F_x which maps a covering space $p\colon Y \to X$ to the fiber $p^{-1}(x)$, is a Galois category.

However, we mentioned in section 4.2 that axioms (G3) and (G6) of a Galois category are not always satisfied when using this definition, even for a connected base space X. As promised there, we will illustrate this with an example, and then offer a slightly different definition of a covering space, which fixes this issue. We first recall the definition of a Galois category as originally given in [Gro+71, Exposé V §4]. Our treatment of Galois categories is based on the one given in chapter 3 of [Len97].

Definition 4.65. Let $\mathbf{C}$ be an essentially small category, and $F\colon \mathbf{C} \to \mathbf{FinSet}$ a functor. We call $\mathbf{C}$ a *Galois category* with *fundamental functor* F if the following is satisfied.

(G1) $\mathbf{C}$ has a terminal object and pullbacks (or, equivalently, all finite limits).

(G2) $\mathbf{C}$ has finite coproducts (including an initial object) and quotients by finite group actions.

(G3) Any morphism u in $\mathbf{C}$ can be factored as $u'u''$ where u' is a monomorphism and u'' an epimorphism. Furthermore, any monomorphism $u\colon X \to Y$ is an isomorphism with a direct summand of Y.

(G4) F preserves finite limits.

(G5) F preserves finite coproducts, epimorphisms and quotients by finite group actions.

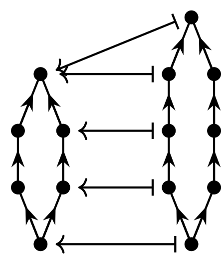
(G6) For any morphism u in $\mathbf{C}$, if $F(u)$ is an isomorphism, then u is an isomorphism. $\diamondsuit$

Remark 4.66. If we call a category $\mathbf{C}$ a Galois category, then we implicitly assume that a fundamental functor $F\colon \mathbf{C} \to \mathbf{FinSet}$ is also given. $\diamondsuit$

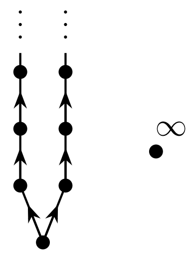
Remark 4.67. Note that in axiom (G3) we dropped the strictness from the assumption that u'' is a strict epimorphism. According to the text at the beginning of chapter 3 of [Len97], this strictness is not needed to prove that a Galois category is equivalent to the category $\mathbf{FinSet}_G$ for some some profinite group G. In particular, the above axioms (G1) - (G6) already imply that any epimorphism is strict, as this is true in $\mathbf{FinSet}_G$. $\diamondsuit$

Before giving the right definition of a finite covering, we consider an example of a connected space X for which its category of finite coverings does not satisfy (G3) and (G6). The profinite space I^∞ of Example 4.12 can be used for this, but we will give a slightly different example. Namely, we consider an example of a profinite space with a nontrivial fundamental group, which therefore should entail an interesting theory of covering spaces.

Example 4.68. Let C^1 be the space obtained from two copies of Δ^1, where the beginning and endpoints are glued to each other. More generally, we define C^n to be the space obtained from two copies of I^n (the spaces of Example 4.12), where the beginning and endpoints are glued to each other. The maps $I^{n+1} \to I^n$ then induce maps $C^{n+1} \to C^n$ for all $n \geq 1$. Below is a picture of the profinite spaces and map $C^4 \to C^3$.



We now define $C^\infty = \lim_n C^n$. Then C^∞ can be pictured as the following space, which resembles some sort of infinitely long circle.



We can now consider the inclusion $\Delta^1 \hookrightarrow C^\infty$ at the point ∞, and the identity map $C^\infty \to C^\infty$. Both are covering spaces according to the definition given above. One of the things that axiom (G3) states is that a monomorphism is an isomorphism onto a direct summand. The map $\iota_\infty \colon \Delta^1 \hookrightarrow C^\infty$, seen as a map between covering spaces, is a monomorphism, yet its image, the point ∞, is not a direct summand of Y, since $C^\infty \setminus \{\infty\}$ is not a profinite space (the topology on $(C^\infty)_0 \setminus \{\infty\}$ is not compact). To see that (G6) is not satisfied, we consider the functor F which maps a covering space $p \colon Y \to X$ to the fiber $p^{-1}(\infty)$. We see that $F(\iota_\infty)$ is an isomorphism, yet ι_∞ is not an isomorphism of covering spaces. ♢

To fix this problem, we should exclude covering spaces such as $\Delta^1 \hookrightarrow C^\infty$. This can be done by asking that, for a covering $Y \to X$, the map $Y_0 \to X_0$ is a covering as well (i.e. a locally trivial map with discrete fibers). The fact that the map $\Delta^1 \hookrightarrow C^\infty$ is then excluded as a covering is left as an exercise for the reader. Note that since Y_0 and X_0 are Stone spaces, the fibers of this covering must all be finite (as they are discrete and a closed subset of Y_0), so we can only use this to define finite covering spaces.

Definition 4.69. A map $p \colon Y \to X$ of profinite spaces is called a *finite covering (map) of X* if, for any solid arrow diagram of the form

$$\begin{tikzcd} \Delta^1 \arrow[r] \arrow[d] & Y \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted, "\exists!"] & X, \end{tikzcd} \tag{4.2}$$

there exists a unique lift $\Delta^n \to Y$, and the map $p_0 \colon Y_0 \to X_0$ is a finite covering of topological spaces. The category of finite coverings over X is defined as the full subcategory of $\widehat{\mathbf{S}}/X$ whose objects are finite covering maps of X, and will be denoted R_f/X. ♢

Note that the map $\Delta^1 \to \Delta^n$ in the above diagram can be the inclusion of Δ^1 as any vertex of Δ^n, not necessarily the zeroth one.

Pullbacks of finite coverings are easily seen to be finite coverings as well, so any map $f \colon X \to X'$ induces a functor $R_f/X' \to R_f/X$ by pullback. The following lemma gives an equivalent description of a covering space. Compare it to Lemma 4.29.

Lemma 4.70. *Let $p \colon Y \to X$ be a map such that $p_0 \colon Y_0 \to X_0$ is a finite covering of topological spaces. Then p is a finite covering map if and only if*

$$\begin{tikzcd} Y_n \arrow[r, "d_i"] \arrow[d, "p_n"] & Y_{n-1} \arrow[d, "p_{n-1}"] \\ X_n \arrow[r, "d_i"] & X_{n-1} \end{tikzcd} \tag{4.3}$$

is a pullback square for any $n > 0$ and any $0 \leq i \leq n$. In particular $p_n \colon Y_n \to X_n$ is a covering map for any $n \geq 0$.

Proof. Let $n > 0$ and $0 \leq i \leq n$ be given. Note that we have the maps $p_n \colon Y_n \to X_n$ and $d_i \colon Y_n \to Y_{n-1}$ which induce a map $Y_n \to X_n \times_{X_{n-1}} Y_{n-1}$ by the universal property of the pullback. The above square is a pullback precisely if this map is a bijection, as continuous bijections of Stone spaces are homeomorphisms. This amounts to showing that for any $y \in Y_{n-1}$ and $x \in X_n$ satisfying $p(y) = d_i(x)$, there is a unique $y' \in Y_n$ satisfying $p(y') = x$ and $d_i(y') = y$. This is equivalent to showing that for any diagram of the form

$$\begin{tikzcd} \Delta^{n-1} \arrow[r] \arrow[d, hook, "\delta^i"] & Y \arrow[d, "p"] \\ \Delta^n \arrow[r] & X, \end{tikzcd}$$

there is a unique lift, where δ^i is the inclusion of Δ^{n-1} as the i-th face of Δ^n. It follows inductively that this holds for all $n > 0$ and $0 \leq i \leq n$ if and only if p is a covering map. ■

Corollary 4.71. *Let G be a finite group. Then any principal G-bundle $E \to X$ is a finite covering.*

Proof. By Lemma 2.60, the map $E_0 \to X_0$ is a finite covering of topological spaces. By Lemma 4.29 and the above lemma, we see that $E \to X$ is a finite covering of X. ■

4.6.2 Verification of the axioms of a Galois category

For $x \in X_0$, we define a functor $F_x \colon R_f/X \to \mathbf{FinSet}$ by sending a cover $p \colon Y \to X$ to $p^{-1}(x)$. We will show that if X is connected (and not necessarily strongly connected), and $x \in X_0$ is any vertex, then R_f/X is a Galois category with fundamental functor F_x. The verification of axioms (G1) - (G6) follows by combining some ideas of the covering theory of topological spaces, and some of the covering theory of simplicial sets. For a reminder on the theory of covering spaces for simplicial sets, we refer the reader to section III.3 of [Lam68]. We require the following point-set topological lemma, which is Lemma 3.8 of [Len97]. We direct the reader to those notes for the proof.

Lemma 4.72. *Let X, Y, Z be topological spaces, $p \colon Y \to X$ and $q \colon Z \to X$ finite coverings, $f \colon Y \to Z$ a continuous map with $p = qf$ and let $x \in X$. Then there exists an open neighborhood U of x in X such that f, g and h are "trivial above U", i.e., such that there exist finite discrete sets D and E, homeomorphisms $\alpha \colon p^{-1}(U) \to U \times D$ and $\beta \colon q^{-1}(U) \to U \times E$ and a map $\phi \colon D \to E$ such that the diagram*

$$\begin{tikzcd} p^{-1}(U) \arrow[rrr, "f"] \arrow[dr, "\alpha"] \arrow[dd, "p"] & & & q^{-1}(U) \arrow[dl, "\beta"'] \arrow[dd, "q"] \\ & U \times D \arrow[r, "\mathrm{id}_U \times \phi"] \arrow[dl, "p_U"] & U \times E \arrow[dr, "p_U"] & \\ U \arrow[rrr, "\mathrm{id}_U"] & & & U \end{tikzcd}$$

is commutative. In particular, the map f is also a finite covering.

The last sentence is actually not in Lemma 3.8 of [Len97], but it is a direct consequence, since the fibers of f are clearly finite, and it trivializes over each copy of U in $q^{-1}(U)$. If we are given profinite spaces X, Y, Z, finite coverings $p\colon Y \to X$ and $q\colon Z \to X$ and a map of profinite spaces $f\colon Y \to Z$ satisfying $p = qf$, then it is easy to see that the lifting property of Definition 4.69 is satisfied by f as well. Since the above lemma shows that $f_0\colon Y_0 \to Z_0$ is a finite covering of topological spaces, we see that f is a finite covering of profinite spaces.

Corollary 4.73. *If $f\colon Y \to Z$ is a morphism in R_f/X, then $f\colon Y \to Z$ is a finite covering of Z.* ■

We are now ready to show that R_f/X is indeed a Galois category.

Theorem 4.74. *For any connected profinite space X and any $x \in X_0$, the category R_f/X together with the fiber functor $F_x\colon R_f/X \to$ **FinSet** is a Galois category.*

Proof. (G1) The terminal object of R_f/X is $\mathrm{id}_X\colon X \to X$. Now assume we are given covering spaces Y_1, Y_2, Y_3 of X and fiber preserving maps $Y_1 \to Y_3$, $Y_2 \to Y_3$. The pullback $Y_1 \times_{Y_3} Y_2$ in $\widehat{\mathbf{S}}$ naturally comes with the map $Y_1 \times_{Y_3} Y_2 \to Y_1 \to X$, and $Y_1 \times_{Y_3} Y_2$ together with this map is the pullback in $\widehat{\mathbf{S}}/X$ as well. Hence, if we can show that $Y_1 \times_{Y_3} Y_2$ is an object of R_f/X, then we are done. To see that this is indeed the case, note that $Y_2 \to Y_3$ is a finite covering by Corollary 4.73, hence the pullback $Y_1 \times_{Y_3} Y_2 \to Y_1$ is a finite covering. Since $Y_1 \to X$ is a finite covering, and compositions of finite coverings are again finite coverings, we deduce that $Y_1 \times_{Y_3} Y_2 \to X$ is a finite covering map. We conclude that R_f/X has pullbacks.

(G2) If $Y \to X$ and $Z \to X$ are finite coverings, then the coproduct $Y \sqcup Z \to X$ is also a finite covering, hence it is also the coproduct in R_f/X. The initial object of R_f/X is $\emptyset \to X$.

For quotients by finite group actions, let a finite group G act on a finite covering $p\colon Y \to X$. The quotient Y/G in $\widehat{\mathbf{S}}$, which is obtained by taking the quotient Y_n/G levelwise, comes with a canonical map $Y/G \to X$ as $Y \to X$ factors through this quotient. We see that $Y/G \to X$ is the quotient in $\widehat{\mathbf{S}}/X$. Hence, if we can show that $Y/G \to X$ is an object in R_f/X, then we are done. The unique lifting property expressed in diagram (4.2) of Definition 4.69 can be easily verified. To see that $Y_0/G \to X_0$ is a covering space, let $x \in X_0$ be given. Note that the quotient Y_0/G is computed in **Top** by Lemma 2.58. Define, for every $g \in G$, the map L_g by $L_g(y) = gy$. Then $L_g\colon Y_0 \to Y_0$ is a map over X_0. By Lemma 4.72, there is an open neighborhood U_g of x such that L_g trivializes over U_g. Pick such a neighborhood U_g for every $g \in G$ and set $U = \bigcap_{g \in G} U_g$. As G is finite, U is an open neighborhood of x. The group action of G is trivial over U, meaning that $p^{-1}(U) \cong U \times S$, where S is discrete and where G only acts on S. We therefore see that $p^{-1}(U)/G \cong U \times (S/G)$, hence that $Y_0/G \to X_0$ trivializes over U. We conclude that $Y/G \to X$ is a finite covering.

(G3) Let $p\colon Y \to X$ and $q\colon Z \to X$ be finite covering spaces, and let $f\colon Y \to Z$ be a morphism in R_f/X. We will show that $\mathrm{im}(f) \subseteq Z$ is also a covering space of X when q is restricted to $\mathrm{im}(f)$. The epi-mono factorization is then given by $Y \to \mathrm{im}(f) \to Z$. To see that $\mathrm{im}(f)$ is indeed a finite covering space, note that $f_0\colon Y_0 \to Z_0$ is a finite covering of topological spaces, hence its image is open and closed. Since the restriction of a finite covering of topological spaces to a clopen subspace is again a finite covering, we see that $\mathrm{im}(f_0) \to X_0$ is a finite covering of topological spaces. One can also easily deduce the lifting property of Definition 4.69 for $\mathrm{im}(f) \subseteq Z$, hence $\mathrm{im}(f) \to X$ is a finite covering, and we conclude that any morphism in R_f/X can be factored as an epimorphism followed by a monomorphism.

To see that any monomorphism $f\colon Y \to Z$ is an isomorphism with a direct summand of Z, we need to show that monomorphisms in R_f/X are levelwise injections. To see that this is the case, assume we are given a monomorphism $f\colon Y \to Z$ with $f(y) = f(y')$ for some $y, y' \in Y_n$, for some n. Define the pullback $Y' = Y \times_Z Y$ by

$$\begin{array}{ccc} Y' & \xrightarrow{p_1} & Y \\ \Big\downarrow{\scriptstyle p_2} & & \Big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{f} & Z. \end{array}$$

We see that $fp_1 = fp_2$, so $p_1 = p_2$, so $y = p_1(y, y') = p_2(y, y') = y'$. We conclude that f is injective. To see that f is an isomorphism with a direct summand of Z, note that $\mathrm{im}(f) \subseteq Z$ is a clopen subset in each degree, since f is a covering of topological spaces in each degree. Define $W_n := Z_n \setminus \mathrm{im}(f_n)$. Using the lifting property of Definition 4.69, one easily shows that $d_i(W_n) \subseteq W_{n-1}$ and $s_i(W_n) \subseteq W_{n+1}$ for every i and n, so the profinite sets W_n form a profinite space W satisfying $\mathrm{im}(f) \sqcup W \cong Z$. One easily verifies that the restriction $q|_W$ of $q\colon Z \to X$ to W is a finite covering, so $\mathrm{im}(f)$ is a direct summand of Z. Since f is injective in each degree, we see that $f\colon Y \to \mathrm{im}(f)$ is an isomorphism.

(G4) F preserves the terminal object and pullbacks, hence all finite limits.

(G5) F preserves finite coproducts since these are computed underlying in $\widehat{\mathbf{S}}$. To see that epimorphisms are preserved, note that epimorphisms are maps that are surjective in each degree. This follows from the proof of (G3), namely that for $f\colon Y \to Z$, $\mathrm{im}(f)$ is a direct summand of Z. Indeed, if $Z = \mathrm{im}(f) \sqcup W$, then the maps $i_1, i_2\colon Z \to \mathrm{im}(f) \sqcup W \sqcup W$ which map W to either the first or second copy of W satisfy $i_1 f = i_2 f$. If f is epi, then this implies $i_1 = i_2$, hence $W = \emptyset$. Now if $f\colon Y \to Z$ is surjective in each degree, then it is also surjective in each fiber, hence $F_x(f)\colon F_x(Y) \to F_x(Z)$ is surjective.

To see that F_x commutes with quotients by finite group actions, let G act on a finite covering $p\colon Y \to X$. Then the quotient Y/G is computed by computing the quotients Y_n/G in **Top**. We therefore see that $F_x(Y/G)$ is the fiber of $Y_0/G \to X_0$ over x, which is $F_x(Y)/G$.

(G6) Let $p\colon Y \to X$ and $q\colon Z \to X$ be finite coverings, let $f\colon Y \to Z$ be a fiber-preserving morphism, and assume $F_x(f)$ is an isomorphism. We will show that $f_0\colon Y_0 \to Z_0$ is a homeomorphism. This will imply that f is an isomorphism, since one can deduce from the pullback squares of Lemma 4.70 that $f_n\colon Y_n \to Z_n$ is a homeomorphism for all $n \geq 0$ if f_0 is so. Note that f_0 is a homeomorphism if it is a bijection, and that it is a bijection if it is a bijection in each fiber, i.e. if $F_{x'}(f)\colon p^{-1}(x') \to q^{-1}(x')$ is an isomorphism for all $x' \in X_0$. To see that it is a bijection in each fiber, define $V \subseteq X_0$ by

$$V = \{x' \in X_0 \mid F_{x'}(f) \text{ is an isomorphism}\}.$$

Then V is an open subset of X_0. To see this, let $x' \in V$ be given. By Lemma 4.72, there is an open U around x' such that p_0, q_0 and f_0 are trivial above U. This implies that $F_{x''}(f)$ is a bijection for every $x'' \in U$, hence $U \subseteq V$. By a similar argument, $X_0 \setminus V$ is open as well, so V is a clopen subset of X_0. Define the map $s\colon X_0 \to \{0,1\}$ by $s(x') = 1$ for $x' \in V$ and $s(x') = 0$ for $x' \notin V$. This map is continuous since V is clopen. If we are given a 1-simplex $u \in X_1$, then lifting this 1-simplex at different starting points gives us natural bijections $p^{-1}(d_0u) \cong p^{-1}(d_1u)$ and $q^{-1}(d_0u) \cong q^{-1}(d_1u)$, so $F_{d_0u}(f)$ is a bijection if and only if $F_{d_1u}(f)$ is so. This implies that the map s coequalizes $X_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} X_0$, hence it factors through the coequalizer of these maps in **Stone**, which is $\pi_0(X)$. Since π_0X is a single point by assumption (X is connected), we see that s is constant. As $x \in V$, we know that $s(x) = 1$, so $V = X_0$ and therefore $F_{x'}(f)$ is an isomorphism for all $x' \in X_0$. We conclude that f is an isomorphism. ∎

The fact that R_f/X is a Galois category gives us many useful properties. For stating these properties, we need to define the notions of a connected object and a Galois object in a Galois category.

Definition 4.75. Let $\mathbf{C}$ be a Galois category. An object X of $\mathbf{C}$ is called *connected* if, whenever we can write $X = Y \sqcup Z$ for $Y, Z \in \mathbf{C}$, then precisely one of Y and Z is an initial object. An object X of $\mathbf{C}$ is called *Galois* if it is connected and the quotient $X/\operatorname{Aut}(X)$ is the terminal object of $\mathbf{C}$. ♢

We have the following characterization of connected and Galois objects in R_f/X.

Proposition 4.76. *Let X be a connected profinite space.*

(i) A finite covering $Y \to X$ is a connected object in R_f/X precisely if Y is connected as a profinite space.

(ii) A finite covering $Y \to X$ is a Galois object in R_f/X precisely if it is a connected principal G-bundle for some finite group G.

Proof. (i) Let $Y \to X$ be a finite covering. If Y is a connected profinite space, then $Y = Z \sqcup W$ implies that precisely one of Z and W is the initial object of $\widehat{\mathbf{S}}$. Since finite coproducts in R_f/X are computed underlying in $\widehat{\mathbf{S}}$, we conclude that $Y \to X$ is a connected object of R_f/X. Conversely, assume that $Y \to X$ is a connected object of R_f/X. If Y is not a connected profinite space, then either Y is empty, contradicting that $Y \to X$ is connected in R_f/X, or $Y = Z \sqcup W$ for two nonempty profinite spaces Z and W. The inclusions $Z, W \hookrightarrow Y$ induce morphisms $Z \to X$ and $W \to X$. These are finite coverings. To see this, note that the unique lifting property of diagram (4.2) is clearly satisfied. We also have that $Y_n = Z_n \sqcup W_n$ levelwise, so Z_0 and W_0 are clopen subsets of Y_0. This implies that $Z_0 \to X_0$ and $W_0 \to X_0$ are finite coverings, as the restriction of a finite covering map to a clopen subset is again a finite covering. We therefore see that $Y = Z \sqcup W$ in R_f/X, contradicting that Y is a connected object.

(ii) We already saw that principal G-bundles, with G finite, are finite coverings. If $E \to X$ is a principal G-bundle, then G is a subgroup of $\operatorname{Aut}(E)$, where $\operatorname{Aut}(E)$ are the automorphisms of E as space over X. Since $E/G \cong X$, we see that $E/G \to X$ is the terminal object of R_f/X, hence $E/\operatorname{Aut}(E) \to X$ is the terminal object of R_f/X as well. We conclude that a connected principal G-bundle is a Galois object. For the converse, assume that $Y \to X$ is a Galois object. By part (ii) of the next proposition (whose proof does not rely on this proposition), we see that the automorphism group $\operatorname{Aut}(Y)$ must act freely on Y. Namely, if $\phi\colon Y \to Y$ has a fixed pont, then, as it is fully determined by what it does to this point, it must be the identity. Now note that $Y/\operatorname{Aut}(Y) \to X$ is the terminal object of R_f/X precisely if $Y/\operatorname{Aut}(Y) \cong X$, which means that $\operatorname{Aut}(Y)$ acts transitively on the fibers of $Y \to X$. This in particular implies that $Y \to X$ is a principal $\operatorname{Aut}(Y)$-bundle. ■

The following proposition and theorem follow from the axioms of a Galois category, and are proved in [Len97] and [Gro+71]. They are translated to the context of finite coverings of profinite spaces.

Proposition 4.77. *Let X be a connected profinite space and let $x \in X_0$. The following hold.*

(i) Let Y be a finite connected covering of X, and let Z be any nonempty finite covering of X. Then any fiber-preserving map $Z \to Y$ is an epimorphism.

(ii) Let Y be a finite connected covering of X, and let Z be any finite covering of X. Then any fiber-preserving map $Y \to Z$ is fully determined by what it does on one vertex.

(iii) For any $Y \in R_f/X$, there exists a finite Galois cover Z such that $\operatorname{Hom}_{R_f/X}(Z, Y) \cong F_x(Y)$.

*(iv) The fiber functor $F_x\colon R_f/X \to$ **FinSet** is pro-representable by a pro-object of finite Galois coverings.*

Remark 4.78. The second property of the above theorem should actually say that for a Galois category $\mathbf{C}$, any map $f\colon Y \to Z$ with Z connected is fully determined by what $F(f)$ does with one element of the finite set $F(Y)$. However, noting that the pair $(R_f/X, F_x)$ is a Galois category for any $x \in X$, property (ii) follows. ◇

The pro-object which pro-represents F_x can be constructed in the following way. Let I be the category whose objects are pairs (Y, y), where Y is a finite Galois covering of X, and $y \in F_x(Y)$. Let the arrows $(Y, y) \to (Z, z)$ be the fiber-preserving morphisms $f : Y \to Z$ that satisfy $f(y) = z$. By the above proposition, only one such morphism can exist. One can show that I is a cofiltered category (and hence a codirected set). Therefore the diagram $I \to R_f/X$ which maps objects (Y, y) to the finite Galois cover Y is a pro-object in R_f/X. We will denote this pro-object by $\widetilde{X}$, and call it the *universal cover* of X. This object pro-represents F_x, meaning that for any $Y \in R_f/X$, there is a natural bijection $\mathrm{Hom}_{\mathrm{Pro}(R_f/X)}(\widetilde{X}, Y) \cong F_x(Y)$. If we are given a map $Y \to Z$ between finite Galois coverings, then one can show that this induces a unique map $\mathrm{Aut}(Y) \to \mathrm{Aut}(Z)$, so one can also define a diagram $I \to \mathbf{FinGrp}$, which maps (Y, y) to $\mathrm{Aut}(Y)$. Taking the limit of this diagram gives a profinite group, and one can show that this is precisely $\mathrm{Aut}(\widetilde{X})$. Since $\mathrm{Aut}(\widetilde{X})$ acts on $\widetilde{X}$, it also acts on $\mathrm{Hom}_{\mathrm{Pro}(R_f/X)}(\widetilde{X}, Y) \cong F_x(Y)$, giving $F_x(Y)$ the structure of a finite $\mathrm{Aut}(\widetilde{X})$-set. We can therefore view F_x as a functor $R_f/X \to \mathbf{FinSet}_G$ The following theorem holds.

Theorem 4.79. *Let X be a profinite space with universal cover $\widetilde{X}$, and view $G := \mathrm{Aut}(\widetilde{X})$ as a profinite set. Then $F_x : R_f/X \to \mathbf{FinSet}_G$ is an equivalence of categories.*

The above construction of the universal cover $\widetilde{X}$ depends on the basepoint x (although only up to a non-canonical isomorphism), hence the construction $\mathrm{Aut}(\widetilde{X})$ also depends on the basepoint x. Similar to the construction of the étale fundamental group in algebraic geometry, Quick in [Qui08] defines $\mathrm{Aut}(\widetilde{X})$ to be the fundamental group of X at x in [Qui08]. Therefore we will sometimes denote $\mathrm{Aut}(\widetilde{X})$ by $\pi_1^Q(X, x)$. After we have studied coverings of profinite spaces in further detail, we will see that $\pi_1^Q(X, x)$ agrees with the fundamental group defined in section 4.2. We will even see that one can retrieve the fundamental groupoid $\Pi_1 X$ from the universal cover $\widetilde{X}$, providing a different construction for the fundamental groupoid of a connected profinite space. This construction is closer to the one proposed by Quick in [Qui08]. However, to give this construction, we should first study the category $\mathrm{Pro}(R_f/X)$ more closely.

4.6.3 Profinite coverings

By the above theorem and Theorem 2.56, we see that $\mathrm{Pro}(R_f/X) \simeq \widehat{\mathbf{Set}}_G$, where the equivalence is obtained by extending the fiber functor F_x to a functor between pro-categories. We will show that $\mathrm{Pro}(R_f/X)$ is a full subcategory of $\widehat{\mathbf{S}}/X$, and that the extension of the fiber functor to $\mathrm{Pro}(R_f/X)$ corresponds to the restriction of the fiber functor $\widehat{\mathbf{S}}/X \to \widehat{\mathbf{Set}}$ to this full subcategory. We need the following lemma.

Lemma 4.80. *Let $X = \lim_i X_i$ be a projective limit of connected profinite space and let $Y \to X$ be a finite covering. Then for some i, there exists a finite cover $Y' \to X_i$ such that $Y \cong X \times_{X_i} Y'$.*

Proof. We will first show this for the case that Y is a finite Galois covering. Then Y is a principal G-bundle for some finite group G, hence there is a map $X \to BG$ such that $Y \cong X \times_{BG} EG$. Since BG is a coskeletal simplicial finite set, $X \to BG$ factors as $X \to X_i \to BG$ for some i, and we see that $Y \cong X \times_{X_i} Y'$ where $Y' = X_i \times_{BG} EG$.

Now assume that Y is connected. By property (iii) of Proposition 4.77, there exists a finite Galois covering $Z \to X$ with a map $Z \to Y$. This map is surjective by (i) of Proposition 4.77, so Y is a quotient of Z. Let $H \leq \mathrm{Aut}(Z)$ be the subgroup of those automorphisms ϕ that satisfy $f\phi = f$. Then $Z/H = Y$. We can write $Z = X \times_{X_i} Z'$ as above, for some i and some finite covering $Z' \to X_i$, since Z is Galois. Then $\mathrm{Aut}(Z) = \mathrm{Aut}(Z')$, as both Z and Z' are principal G-bundles with $G = \mathrm{Aut}(Z)$. In particular H acts on Z'. We then see that $Y = Z/H = X \times_{X_i} (Z'/H)$, so the lemma holds for connected coverings as well.

If Y is not connected, then either $Y = \emptyset$ or $Y = Y_1 \sqcup \ldots \sqcup Y_n$ with $Y_1, \ldots, Y_n$ connected. In the first case, the lemma clearly holds. For the second case, let covers $Z_1 \to X_{i_1}, \ldots, Z_n \to X_{i_n}$ with $Y_k \cong X \times_{X_{i_k}} Z_k$ for $1 \leq k \leq n$ be given. By picking j such that $j \leq i_k$ for all k, and pulling back Z_k to a cover of X_j, we can assume without loss of generality $i_1 = \ldots = i_n = j$. We then see that Y is the pullback of the covering $Z_1 \sqcup \ldots \sqcup Z_n$ of X_j, proving the case where Y is not connected. ■

We now show that the inclusion $R_f/X \to \widehat{\mathbf{S}}/X$ extends to a fully faithful functor $\mathrm{Pro}(R_f/X) \to \widehat{\mathbf{S}}/X$, using Proposition 2.50.

Theorem 4.81. *The inclusion* $R_f/X \hookrightarrow \widehat{\mathbf{S}}/X$ *extends to a cofiltered limit-preserving and fully faithful functor* $\mathrm{Pro}(R_f/X) \to \widehat{\mathbf{S}}/X$. *The extension of the fiber functor* F_x *to a functor* $\mathrm{Pro}(R_f/X) \to \widehat{\mathbf{Set}}$ *corresponds to the restriction of the fiber functor* $F_x \colon \widehat{\mathbf{S}}/X \to \widehat{\mathbf{Set}}$ *to the image of* $\mathrm{Pro}(R_f/X)$ *under this fully faithful functor* $\mathrm{Pro}(R_f/X) \to \widehat{\mathbf{S}}/X$.

Proof. Note that a functor is fully faithful if it gives an equivalence between its domain and its (essential) image. We therefore apply Proposition 2.50, but instead of considering the codomain $\widehat{\mathbf{S}}/X$, we take as codomain the (essential) image of the functor $\mathrm{Pro}(R_f/X) \to \widehat{\mathbf{S}}/X$. This in particular means that we do not have to prove the second assumption of Proposition 2.50.

Of course, we first have to show that $\widehat{\mathbf{S}}/X$ has all projective limits. Let a projective diagram $\{Y_i\}$ in $\widehat{\mathbf{S}}/X$ be given. Write p_i for the maps $Y_i \to X$, and write f^i_j for the maps $Y_i \to Y_j$ whenever $i \leq j$. We can compute the limit $\lim_i Y_i$ in $\widehat{\mathbf{S}}$. This limit comes with a canonical map $\lim_i Y_i \to X$, since the map $\lim_i Y_i \to Y_j \to X$ is independent of j. The limit of the diagram $\{Y_i\}$ in $\widehat{\mathbf{S}}/X$ is just $\lim_i Y_i$ together with this canonical map.

The map $R_f/X \to \widehat{\mathbf{S}}/X$ is fully faithful by definition. The only thing that we therefore still need to check is the cocompactness of objects of R_f/X. Let $\{Y_i\}$ be a projective diagram of profinite spaces over X, and let a finite cover $Z \to X$ be given. Note that there exist a coskeletal simplicial finite set X', a map $X \to X'$ and a finite cover $Z' \to X'$ such that $Z = X \times_{X'} Z'$. This follows from the previous lemma by noting that $X = \lim_i X_i$ for X_i coskeletal simplicial finite sets. By the universal property of the pullback, we see that

$$\mathrm{Hom}_{\widehat{\mathbf{S}}/X}(Y, Z) \cong \mathrm{Hom}_{\widehat{\mathbf{S}}/X'}(Y, Z').$$

Since X' is coskeletal and Z' is a finite cover, we see that Z' is coskeletal as well. We therefore have that $\mathrm{Hom}_{\widehat{\mathbf{S}}}(Y, Z') \cong \mathrm{colim}_i \mathrm{Hom}_{\widehat{\mathbf{S}}}(Y_i, Z')$. This implies $\mathrm{Hom}_{\widehat{\mathbf{S}}/X'}(Y, Z') \cong \mathrm{colim}_i \mathrm{Hom}_{\widehat{\mathbf{S}}/X'}(Y_i, Z')$ as well, which we leave to the reader to check. Now using the universal property of the pullback again, we see that

$$\mathrm{Hom}_{\widehat{\mathbf{S}}/X'}(Y, Z') \cong \operatorname*{colim}_i \mathrm{Hom}_{\widehat{\mathbf{S}}/X'}(Y_i, Z') \cong \operatorname*{colim}_i \mathrm{Hom}_{\widehat{\mathbf{S}}/X}(Y_i, Z),$$

so we conclude that Z is cocompact.

To see that the extension $\widetilde{F}_x\colon \mathrm{Pro}(R_f/X) \to \widehat{\mathbf{Set}}$ of the fiber functor of finite coverings is the restriction of the usual fiber functor $F_x\colon \widehat{\mathbf{S}}/X \to \widehat{\mathbf{Set}}$, one needs to show that this second functor preserves projective limits. This can be shown by considering that projective limits are computed underlying in $\widehat{\mathbf{S}}$, as we saw above. This is left to the reader. ■

The above theorem motivates the following definition of a profinite covering.

Definition 4.82. Let X be a connected profinite space. We call $p\colon Y \to X$ a *profinite covering* if it is in the essential image of $\mathrm{Pro}(R_f/X) \to \widehat{\mathbf{S}}/X$, i.e. if it is a projective limit of finite coverings of X. The category of profinite coverings of X is denoted $\widehat{R}/X$. ♢

Remark 4.83. The above definition might appear somewhat ad-hoc, as we define profinite covering spaces through two properties that we want them to have; namely that any profinite covering is a projective limit of finite ones, and that $\widehat{R}/X \simeq \widehat{\mathbf{Set}}_G$, where G is the automorphism group of the universal cover. The author has not found a more "intrinsic" (and equivalent) definition of a profinite covering, unfortunately. ♢

Recall that the universal cover, in $\mathrm{Pro}(R_f/X)$, was the pro-object that pro-represents the fiber functor $F_x\colon R_f/X \to \mathbf{FinSet}$. This, together with the equivalence $\mathrm{Pro}(R_f/X) \simeq \widehat{R}/X$, motivates the following definition.

Definition 4.84. Let X be a connected profinite space and let $p\colon Y \to X$ be a profinite covering. Then Y is called a *universal cover* of X if, for any $x \in X_0$, any $y \in p^{-1}(y)$ and any finite cover $Z \to X$, the map

$$\mathrm{Hom}_{\widehat{R}/X}(Y, Z) \to F_x(Z); \quad f \mapsto f(y)$$

is a bijection. ♢

One can show that if the above definition holds for one $x \in X_0$ and one $y \in p^{-1}(y)$, then it holds for every $x \in X_0$ and every $y \in p^{-1}(y)$. Furthermore, one can show that there exists an isomorphism between any two universal covers, as they represent isomorphic functors. However, this isomorphism is not canonical, as it depends on a choice of basepoint of both the universal covers. We therefore usually speak about *a* universal cover, instead of *the* universal cover.

Since a map of connected profinite spaces $f\colon X \to Y$ induces a map $f^*\colon R_f/Y \to R_f/X$ by pullback, it also induces a map $\widetilde{f}^*\colon \mathrm{Pro}(R_f/Y) \to \mathrm{Pro}(R_f/X)$. Since projective limits and pullbacks commute, we see that this corresponds to the restriction of the pullback functor $f^*\colon \widehat{\mathbf{S}}/Y \to \widehat{\mathbf{S}}/X$ to $\widehat{R}/Y$. In particular we see that a pullback of a profinite covering is again a profinite covering.

The next proposition assures that profinite coverings have the same lifting properties as finite coverings.

Proposition 4.85. *Let X be a connected profinite space, and let $p\colon Y \to X$ be a profinite covering. Then the map p also has the unique lifting property expressed through diagram (4.2).*

Proof. Let $Y = \lim_i Y_i$ with $Y_i \to X$ finite coverings. Since maps between finite coverings are again finite coverings, we see that all maps involved in this limit have the unique lifting property of diagram (4.2). Combining this with the universal property of the limit shows that $p\colon Y \to X$ has the same unique lifting property. ■

As in the case of finite coverings, the image of a morphism between profinite coverings is again a profinite covering.

Proposition 4.86. *Let $Y, Z \to X$ be profinite coverings of a connected profinite space X, and let a morphism $f\colon Y \to Z$ of profinite coverings be given. Then $f(Y)$, the levelwise image of $Y \to Z$, is again a profinite covering. In particular, axiom (G3) of a Galois category is satisfied.*

Proof. Write $Y = \lim_i Y_i$ and $Z = \lim_j Z_j$ with Y_i and Z_j finite coverings of X. Then $f_j\colon Y \to Z_j$ factors through some Y_i for every j. In particular the image of $Y \to Z_j$ is the image of $f'_j\colon Y_i \to Z_j$. Since Y_i and Z_j are finite coverings, we know that the image $f_j(Y) = f'_j(Y_i) \subseteq Z_j$ is a finite covering of X. Then $f(Y) \subseteq Z$ is equal to $\lim_j f_j(Y)$, which we leave to the reader to verify. Since $f_j(Y) \to X$ is a finite covering for every j, we see that $f(Y)$ is a projective limit of finite coverings, hence a profinite covering. ■

One can of course define connected objects and Galois objects in $\widehat{R}/X$ in a similar way as in R_f/X.

Definition 4.87. Let X be a connected profinite space. An object $Y \to X$ of $\widehat{R}/X$ is called *connected* if, whenever we can write $Y = Z \sqcup W$ for $Y, Z \in \widehat{R}/X$, then precisely one of Y and Z is an initial object. Y is called *Galois* if it is connected and $Y/\operatorname{Aut}(Y) = X$, i.e. if $\operatorname{Aut}(Y)$ acts transitively on every fiber. ♢

One can show that properties (i) and (ii) of Proposition 4.77 also hold for profinite coverings. Connected objects and Galois objects can also be characterized in $\widehat{R}/X$ in the same way as in R_f/X.

Proposition 4.88. *Let Y and Z be profinite coverings of X.*

(i) Assume $Y \to X$ is a connected profinite covering and Z is nonempty. Then any fiber-preserving map $Z \to Y$ is an epimorphism.

(ii) Assume $Y \to X$ is a connected profinite covering. Then any fiber-preserving map $Y \to Z$ is fully determined by what it does on one vertex.

(iii) $Y \to X$ *is a connected object in* R_f/X *precisely if* Y *is connected as a profinite space.*

(iv) $Y \to X$ *is a Galois object in* R_f/X *precisely if it is a connected principal* G*-bundle for some profinite group* G*.*

Proof. Before proving these properties, we show that if $Y = \lim_i Y_i$ with Y_i finite coverings of X and $p_i\colon Y \to Y_i$ the projections, then we may assume that $\mathrm{im}(p_i) = Y_i$ for every i. To see this, we define for every i the finite covering $Z_i \subseteq Y_i$ by $Z_i := \bigcap_{j\le i} p_i^j(Y_i)$ (the intersection taken levelwise). It might not be clear that this is a finite covering (or even a profinite space). However, to see that this is indeed the case, write $Y_i = W_1 \sqcup \ldots \sqcup W_n$ as a disjoint sum of connected finite coverings. Since a morphism to a connected finite covering is always an epimorphism, we see that for $1 \le k \le n$, either $W_k \subseteq p_i^j(Y_i)$ or W_k is disjoint from $p_i^j(Y_i)$. We therefore see that $Z_i = \bigcap_{j\le i} p_i^j(Y_i)$ must be of the form $W_{k_1} \sqcup \ldots \sqcup W_{k_l}$ for some $1 \le k_1 < \ldots < k_l \le n$, which is a finite covering. By arguments similar to the proof of Proposition 2.47 and Corollary 2.49, we see that $\mathrm{im}(p_i) = Z_i$ for all i, that $p_i^j(Z_j) = Z_i$ for all $j \le i$ and that $Y = \lim_i Z_i$.

Now assume that in the above case, Y is a connected object of $\widehat{R}/X$. Then Z_i is connected for every i as well. To see this, assume Z_i is not connected for some i, and write $Z_i = V_i \sqcup W_i$. Define, for every $j \le i$, $V_j := (p_i^j)^{-1}(V_i)$ and $W_j := (p_i^j)^{-1}(W_i)$. Then $Z_j = V_j \sqcup W_j$ for every $j \le i$, hence

$$Y = \lim_{j\le i} Z_j = (\lim_{j\le i} V_j) \sqcup (\lim_{j\le i} W_j).$$

Since $p_j\colon Y \to Z_j$ is levelwise surjective, we see that V_j and W_j are always nonempty, hence the two projective limits on the right-hand side are nonempty as well. We therefore see that Y is not connected, which is a contradiction. We conclude that for any profinite covering Y, we can write $Y = \lim_i Y_i$ with Y_i a finite covering of X for each i, such that the projections $p_i\colon Y \to Y_i$ are levelwise surjections, and such that the finite coverings Y_i are connected if Y is so. We now turn to the proofs of the above four properties.

(i) Let $f\colon Z \to Y$ be a map of profinite coverings with Y connected. As the image of f is a profinite covering by Proposition 4.86 and Z is nonempty, we see that f must be levelwise surjective, hence an epimorphism.

(ii) Let Y and Z be connected profinite coverings with Y connected. Since $Z = \lim_i Z_i$ for finite coverings Z_i, we see that any map $f\colon Y \to Z$ is uniquely determined by maps $f_i\colon Y \to Z_i$. Therefore, in showing that any map $f\colon Y \to Z$ is fully determined by what it does on one vertex, we can assume that Z is a finite covering. Now write $Y = \lim_i Y_i$ with the projections $p_i\colon Y \to Y_i$ levelwise surjective, and in particular with Y_i a connected finite covering for every i. Let $y \in Y$ be given and let $f, g\colon Y \to Z$ satisfy $f(y) = g(y)$. Both f and g factor through Y_i for some i, i.e. there exist maps $f_i, g_i\colon Y_i \to Z$ such that $g = g_i p_i$ and $f = f_i p_i$ for some i. Since Y_i is a finite connected covering, and $f_i(p_i(y)) = g_i(p_i(y))$, we see that $f_i = g_i$, hence $f = g$.

(iii) First assume Y is a connected object in $\widehat{R}/X$. Then we can write $Y = \lim_i Y_i$ with Y_i connected finite coverings. We see that $\pi_0(Y) = \lim_i \pi_0(Y_i) = \{*\}$, so Y is connected as a profinite space. Conversely, assume that $\pi_0(Y) = \{*\}$. Then Y is not empty, so if Y is not a connected profinite covering, then $Y = Z \sqcup W$ with Z and W both nonempty. This contradicts $\pi_0(Y) = \{*\}$, so Y is a connected profinite covering.

(iv) First assume $Y \to X$ is a Galois covering. Then $\mathrm{Aut}(Y)$ acts transitively on each fiber. We also see that $\mathrm{Aut}(Y)$ acts freely on Y by part (ii), so $Y \to X$ is by definition a principal $\mathrm{Aut}(Y)$-bundle. Conversely, let $E \to X$ be a principal G-bundle, with G a profinite group and E connected. Then $G \leq \mathrm{Aut}(E)$, where $\mathrm{Aut}(E)$ is the group of fiber-preserving automorphisms of E, as space over X. Since $E/G \cong X$, we see that $E/\mathrm{Aut}(E) \cong X$ as well. To see that $E \to X$ is a profinite covering, we need to show that it is a projective limit of finite coverings. Since quotients of profinite spaces are computed levelwise, we see by Lemma 2.59 that $E = \lim_N E/N$, where N ranges over all open normal subgroups of G. If N is a normal subgroup of G, we obtain an induced G/N-action on E/N, which makes $E/N \to X$ into a principal G/N-bundle. If N is open, then G/N is finite, hence $E/N \to X$ is a finite Galois covering. In particular, we have written E as a projective limit of finite coverings. Now note that E is connected by assumption, so by part (iii), $E \to X$ is a connected object in $\widehat{R}/X$. We conclude that $E \to X$ is a Galois covering. ∎

4.6.4 Connected coverings and closed subgroups of π_1

The next proposition follows by studying connected objects and Galois objects in $\widehat{\mathbf{Set}}_G$, for G a profinite group. It rests on the observation that for a profinite covering $Y \to X$, the category of profinite coverings $\widehat{R}/Y$ is simply $(\widehat{R}/X)/Y$.

Proposition 4.89. *Let X be a connected profinite space, and let $x \in X_0$ be a basepoint. Then there is a 1-1 correspondence between conjugacy classes of closed subgroups of $\pi_1^Q(X,x)$ and profinite connected coverings $Y \to X$, such that*

(i) the closed subgroup corresponding to $Y \to X$ is isomorphic to $\pi_1^Q(Y,y)$, where y is any point in Y_0 above x;

(ii) the universal cover of X corresponds to the trivial subgroup of $\pi_1^Q(X,x)$; in particular, $\widetilde{X} \to X$ is a universal cover if and only if $\pi_1^Q(\widetilde{X},\widetilde{x}) = \{*\}$ for some $\widetilde{x} \in \widetilde{X}_0$;

(iii) Galois coverings $Y \to X$ correspond to closed normal subgroups of $\pi_1^Q(X,x)$; and

(iv) for the closed normal subgroup N corresponding to the Galois covering $Y \to X$, we have $\mathrm{Aut}(Y) \cong \pi_1^Q(X,x)/N$.

Proof. We begin by describing connected objects in $\widehat{\mathbf{Set}}_G$, where G is a profinite group. These are the objects to which the profinite connected coverings $Y \to X$ correspond under the equivalence $\widehat{R}/X \simeq \widehat{\mathbf{Set}}_{\pi_1^Q(X,x)}$. One easily sees that a nonempty $S \in \widehat{\mathbf{Set}}_G$ is a connected object (i.e. it cannot be written as $T_1 \sqcup T_2$ with T_1, T_2 both nonempty) if and only if G acts transitively on S. If we pick $s \in S$, then $H := \mathrm{Stab}(s)$ is a closed subgroup of G, and the canonical map $G/H \to S$ defined by $gH \mapsto g \cdot s$ is a well-defined, continuous, G-equivariant bijection, hence an isomorphism of profinite G-sets. We see that H depends on our choice of $s \in S$, but only up to conjugacy. We also see that $G/H \cong G/H'$ as profinite G-sets if H and H' are conjugate closed subgroups. So we conclude that there is a 1-1 correspondence between conjugacy classes of closed subgroups of $\pi_1^Q(X,x)$ and profinite connected coverings $Y \to X$. We will now prove the four above-mentioned properties of this 1-1 correspondence.

(i) Let $Y \to X$ be a connected cover. Pick a universal cover $\widetilde{X} \to X$ with basepoint $\widetilde{x}$ above x. Then $F_x(Y) \cong \mathrm{Hom}(\widetilde{X}, Y)$, and the action of $\pi_1^Q(X,x) = \mathrm{Aut}(\widetilde{X})$ on $F_x(Y) \cong \mathrm{Hom}(\widetilde{X}, Y)$ is given by $\phi \cdot f = f \circ \phi^{-1}$. If we pick a point $y \in F_x(Y)$, then the stabilizer of this group action is computed by picking the corresponding map $f \colon \widetilde{X} \to Y$, and then considering the automorphisms ϕ satisfying $f \circ \phi = f$. These are precisely the automorphisms of $f \colon \widetilde{X} \to Y$ in $\widehat{R}/Y$. By Lemma 4.90, $\widetilde{X} \to Y$ is a universal covering, so the group corresponding to Y is isomorphic to $\pi_1^Q(Y,y)$.

(ii) Note that a connected object S in $\widehat{\mathbf{Set}}_G$ corresponds to a universal cover if and only if G acts freely and transitively on S. This means that the stabilizer of any $s \in S$ is trivial, hence a universal cover corresponds to the trivial subgroup of $\pi_1^Q(X,x)$ under the above 1-1 correspondence. The result now follows from (i).

(iii) A connected covering $Y \to X$ is a profinite Galois covering if $\mathrm{Aut}(Y)$ acts transitively on each fiber. This is equivalent to saying that $\mathrm{Aut}(F_x(Y))$ acts transitively on $F_x(Y)$ in $\widehat{\mathbf{Set}}_G$, where $G = \pi_1^Q(X,x)$. We know that $F_x(Y)$ is isomorphic to G/H for some closed subgroup H of G. Now let $\phi \colon G/H \to G/H$ be a G-equivariant homeomorphism. Then $\phi(H) = gH$ for some $g \in G$. We see that $\phi(g'H) = g'\phi(H) = g'gH$ must hold for every $g' \in G$. We know that for every $g \in G$, there is some ϕ such that $\phi(H) = gH$, since $\mathrm{Aut}(G/H)$ acts transitively on G/H. Now let $g \in G$ and $h \in H$ be given, and let ϕ_g be the automorphism satisfying $\phi_g(H) = gH$. We then see that $H = \phi_g(g^{-1}H) = \phi_g(g^{-1}hH) = g^{-1}hgH$, hence $g^{-1}hg \in H$. We conclude that H is a closed normal subgroup of G.

Conversely, assume the profinite connected covering $Y \to X$ corresponds to a closed normal subgroup H of $G = \pi_1^Q(X,x)$. One easily sees that $\mathrm{Aut}(G/H)$ acts transitively on G/H in $\widehat{\mathbf{Set}}_G$. In particular, the quotient $(G/H)/\mathrm{Aut}(G/H)$ is the terminal object of $\widehat{\mathbf{Set}}_G$, so by the equivalence $\widehat{R}/X \simeq \widehat{\mathbf{Set}}_G$ we see that $Y/\mathrm{Aut}(Y) \cong X$, i.e. Y is a Galois covering.

(iv) This follows from the fact that for $H \trianglelefteq G$ a closed normal subgroup, we have $\mathrm{Aut}(G/H) \cong G/H$ in $\widehat{\mathbf{Set}}_G$. ∎

Lemma 4.90. *Let X be a connected profinite space and let $Y \to X$ be a connected covering. Then $\widehat{R}/Y \simeq (\widehat{R}/X)/Y$. In particular, if $\widetilde{X} \to X$ is a universal covering over X, then any fiber-preserving morphism $\widetilde{X} \to Y$ is a universal covering of Y.*

Proof. There is a canonical functor $\widehat{R}/Y \to (\widehat{R}/X)/Y$, where $(\widehat{R}/X)/Y$ is the category of profinite coverings of X with a morphism to Y. This canonical functor is given by sending a covering $Z \to Y$ to $Z \to Y \to X$. There is also a functor in the opposite direction, which sends an object $Z \to Y \to X$ of $(\widehat{R}/X)/Y$ to the covering $Z \to Y$. These functors are clearly inverse to each other. However, we do need to show that they are well defined. Namely, we need to show that a composition of profinite coverings is again a profinite covering, and that a morphism between profinite coverings is a profinite covering as well.

For the first case, let $Z \to Y$ be a profinite covering. We need to show that $Z \to Y \to X$ is a projective limit of finite coverings of X. Note that Z is a projective limit of finite coverings of Y, so it suffices to prove this statement for finite coverings of Y. Write $Y = \lim_i Y_i$ with Y_i finite coverings of X. We see by Lemma 4.80 that $Z = Y \times_{Y_j} Z'$ for some j and for some finite covering $Z' \to Y_j$. We now see that

$$Z = (\lim_{i \leq j} Y_i) \times_{Y_j} Z' = \lim_{i \leq j}(Y_i \times_{Y_j} Z').$$

Since the right-hand side is a projective limit of finite coverings of X, we are done.

For the second case, write $p\colon Y \to X$ and $q\colon Z \to Y$. Assume p and pq are both profinite coverings. We need to show that $q\colon Z \to Y$ is a projective limit of finite coverings of Y. Noting that $\widehat{R}/X \simeq \mathrm{Pro}(R_f/X)$, we can write $Z = \lim_i Z_i$ and $Y = \lim_i Y_i$ both as projective limits of finite coverings of X, such that q is represented by natural morphisms $q_i\colon Z_i \to Y_i$ (i.e., we pick a level representation of q, see Definition 2.25). Since Y_i and Z_i are finite coverings, we see that q_i is a finite covering for every i as well by Corollary 4.73, hence the pullback $Y \times_{Y_i} Z_i$ is a finite covering of Y. We now see that

$$Z \cong Z \times_Y Y \cong \lim_i Z_i \times_{\lim_i Y_i} Y = \lim_i (Z_i \times_{Y_i} Y).$$

As the right-hand side is a projective limit of finite coverings of Y, we see that $Z \to Y$ is a profinite covering.

For the last claim of this lemma, let $\widetilde{X} \to X$ be a universal cover, and pick any morphism $\widetilde{X} \to Y$. To see that $\widetilde{X} \to Y$ is a universal cover, we will show that it represents the fiber functor. Let $\widetilde{x} \in \widetilde{X}_0$ be a basepoint, and let $y \in Y_0$ and $x \in X_0$ the corresponding basepoints of Y and X. Let $Z \to Y$ be a profinite covering. We see that $\mathrm{Hom}_{\widehat{R}/X}(\widetilde{X}, Z) \cong F_x(Z)$, by mapping a morphism $f\colon \widetilde{X} \to Z$ to $f(\widetilde{x})$. We also see that $\mathrm{Hom}_{\widehat{R}/Y}(\widetilde{X}, Z)$ is the subset of $\mathrm{Hom}_{\widehat{R}/X}(\widetilde{X}, Z)$ consisting of those morphisms $f\colon \widetilde{X} \to Z$ such that

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "f"] \arrow[dr] & & Z \arrow[dl] \\ & Y & \end{tikzcd}$$

commutes. Since $\widetilde{X}$ is connected, any morphism $\widetilde{X} \to Y$ is fully determined by where it maps $\widetilde{x}$. We therefore see that this diagram commutes precisely if $f(\widetilde{x})$ lies in the fiber above y, i.e. if $f(\widetilde{x}) \in F_y(Z)$. Hence the bijection $\mathrm{Hom}_{\widehat{R}/X}(\widetilde{X}, Z) \cong F_x(Z)$ restricts to a bijection $\mathrm{Hom}_{\widehat{R}/Y}(\widetilde{X}, Z) \cong F_y(Z)$. ■

4.6.5 An alternative construction of the fundamental groupoid

We will now give an alternative construction of the fundamental groupoid, following Quick's ideas in [Qui08]. We will denote, for a profinite space X, this groupoid by $\Pi_1^Q X$. The idea in [Qui08] is to construct, given vertices $x, y \in X_0$, the profinite set $(\Pi_1^Q X)(x, y)$ as the group of fiber-preserving morphisms $\widetilde{X}_x \to \widetilde{X}_y$, where $\widetilde{X}_x$ and $\widetilde{X}_y$ are the universal covers constructed using the fiber functors F_x and F_y, respectively. All of these morphisms are isomorphisms, and it is obvious how this defines a groupoid. There is however no immediate topology on $\mathrm{Ar}(\Pi_1^Q X)$, so this does not immediately give a Stone groupoid, let alone a profinite groupoid. As any connected Stone groupoid is of the form $G[S]$ with G a profinite group and S a profinite set, we could simply give $\Pi_1^Q X$ the topology of $\pi_1^Q(X, x)[X_0]$. However, this feels somewhat unnatural, as the proof that any connected Stone groupoid is of this form does not give a canonical isomorphism, and it depends on the axiom of choice. We therefore take a somewhat different approach to obtain a topology on $\mathrm{Ar}(\Pi_1^Q X)$. Recall from the usual theory of covering spaces (either of topological spaces or simplicial sets) that if given a universal cover $\widetilde{X}$ of X, two points x to y in X, and a point $\widetilde{x}$ in the fiber above x, then there is a 1-1 correspondence between homotopy classes of paths from x to y, and points in the fiber above y. This 1-1 correspondence comes from lifting the path to $\widetilde{X}$, starting at $\widetilde{x}$. One can therefore represent a path in X (up to homotopy) by a pair of points $(\tilde{x}, \tilde{y})$ in $\widetilde{X}$. Two pairs $(\tilde{x}, \tilde{y})$ and $(\tilde{x}', \tilde{y}')$ represent the same path precisely if there is an automorphism ϕ of $\widetilde{X}$ such that $\phi(\tilde{x}) = \tilde{x}'$ and $\phi(\tilde{y}) = \tilde{y}'$. This idea can also be used to construct the fundamental groupoid from the universal cover.

Definition 4.91. Let X be a connected profinite space, and let $p\colon \widetilde{X} \to X$ be a universal cover of X. Then define a Stone groupoid $\Pi_1^Q X$ by

(i) $\mathrm{Ob}(\Pi_1^Q X) = X_0$;

(ii) $\mathrm{Ar}(\Pi_1^Q X) = (\widetilde{X}_0 \times \widetilde{X}_0)/\operatorname{Aut}(\widetilde{X})$, where $\operatorname{Aut}(\widetilde{X})$ acts on $\widetilde{X}_0 \times \widetilde{X}_0$ by $\phi(x, y) = (\phi(x), \phi(y))$;

(iii) the source and target maps are given by $s([x, y]) = p(x)$ and $t([x, y]) = p(y)$;

(iv) the multiplication map m is given by $m([y', z'], [x, y]) = [\phi(x), z']$, where $[x, y]$ denotes the class of the pair (x, y) and where $\phi(y) = y'$ (note that a unique such ϕ exists if $p(y') = p(y)$); and

(v) the inverse map is given by $i([x, y]) = [y, x]$. ♢

It might not be immediately clear that the above definition is independent of the choice of universal cover, let alone that it defines a functor. However, one can show that, given another universal covering $\widetilde{X}'$ of X, any isomorphism $\widetilde{X}' \to \widetilde{X}$ induces the same isomorphism between the fundamental groupoids. One can also show that, given a map of profinite spaces $X \to Y$, there always exists a map between universal coverings $\widetilde{X} \to \widetilde{Y}$

such that the square

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

commutes, and furthermore that any such map induces the same morphism of Stone groupoids $\Pi_1^Q X \to \Pi_1^Q Y$. We state the existence of such a map $\widetilde{X} \to \widetilde{Y}$ as a lemma.

Lemma 4.92. *Assume we are given a map $f\colon X \to Y$ between connected profinite spaces, let $x \in X$, let $\widetilde{X} \to X$ be a universal cover and let $Z \to Y$ be any profinite covering. For any $\widetilde{x} \in X$ above x and any $z \in Z$ above $f(x)$, there exists a unique map $\widetilde{f}\colon \widetilde{X} \to Z$ such that $\widetilde{f}(\widetilde{x}) = z$ and*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & Z \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

Proof. Note that maps $\widetilde{f}\colon \widetilde{X} \to Z$ such that the above diagram commutes are in 1-1 correspondence with maps $\widetilde{X} \to X \times_Y Z$ over X, by the universal property of the pullback. The result now follows since $\widetilde{X}$ represents the fiber functor $F_x\colon \widetilde{R}/X \to \widehat{\mathbf{Set}}$. ■

Proposition 4.93. *The above construction defines a functor $\Pi_1^Q\colon \widehat{\mathbf{S}}_0 \to \mathbf{StoneG}_0 \simeq \widehat{\mathbf{G}}_0$, where $\widehat{\mathbf{S}}_0$ denotes the category of connected profinite spaces, $\mathbf{StoneG}_0$ the category of connected Stone groupoids and $\widehat{\mathbf{G}}_0$ the category of connected profinite groupoids.*

Proof. We leave the verification that $\Pi^Q X$ is a well-defined Stone groupoid to the reader, i.e. the verification that all the maps involved in the definition of $\Pi_1^Q X$ are indeed well-defined and continuous.

To see that Π_1^Q defines a functor, we first need to show that $\Pi_1^Q X$ does not depend on the choice of universal cover. For a universal cover $\widetilde{X} \to X$, we for now denote the corresponding Stone groupoid by $(\Pi_1^Q X)_{\widetilde{X}}$. We need to show that for any two universal covers $\widetilde{X} \to X$ and $\widetilde{X}' \to X$, there exists a canonical isomorphism $(\Pi_1^Q X)_{\widetilde{X}} \cong (\Pi_1^Q X)_{\widetilde{X}'}$. Note that there exists a (non-canonical) isomorphism $\widetilde{X} \to \widetilde{X}'$. This isomorphism induces a well-defined map $(\widetilde{X}_0 \times \widetilde{X}_0)/\operatorname{Aut}(\widetilde{X}) \to (\widetilde{X}'_0 \times \widetilde{X}'_0)/\operatorname{Aut}(\widetilde{X}')$, i.e. a map $\operatorname{Ar}((\Pi_1^Q X)_{\widetilde{X}}) \to \operatorname{Ar}((\Pi_1^Q X)_{\widetilde{X}'})$. This map is an isomorphism of Stone groupoids. To see that this map is canonical, we need to show that this isomorphism does not depend on our choice of map $\widetilde{X} \to \widetilde{X}'$. This follows from what we prove next, taking $f = \mathrm{id}_X$.

Let a map $f\colon X \to Y$ between connected profinite spaces be given. We show that f induces a canonical map $\Pi_1^Q X \to \Pi_1^Q Y$. Let $\widetilde{X} \to X$ and $\widetilde{Y} \to Y$ be universal covers. Then by the above lemma, there exists a map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ above f. Pick such a map $\widetilde{f}$. This map induces a continuous map $(\widetilde{X}_0 \times \widetilde{X}_0)/\operatorname{Aut}(\widetilde{X}) \to (\widetilde{Y}_0 \times \widetilde{Y}_0)/\operatorname{Aut}(\widetilde{Y})$, i.e. a map $\operatorname{Ar}(\Pi_1^Q X) \to \operatorname{Ar}(\Pi_1^Q Y)$, which is a map of Stone groupoids. To see that the induced map of Stone groupoids does not depend on the choice of $\widetilde{f}$, assume $\widetilde{f}'\colon \widetilde{X} \to \widetilde{Y}$ is also a map above f. Pick a point $\widetilde{x} \in \widetilde{X}_0$. Then there is a unique automorphism $\phi \in \mathrm{Aut}(\widetilde{Y})$ such that $\widetilde{f}(\widetilde{x}) = \phi \widetilde{f}'(\widetilde{x})$. Since $\widetilde{Y}$ is connected, this implies that $\widetilde{f} = \phi \widetilde{f}'$. Now $[x, y] = [\phi(x), \phi(y)]$ for any $[x, y] \in \mathrm{Ar}(\Pi_1^Q Y)$, so we see that the induced map $\Pi_1^Q X \to \Pi_1^Q Y$ does not depend on the choice of $\widetilde{f}$. In particular Π_1^Q is well-defined.

To see that Π_1^Q is indeed a functor, note that if we are given $f : X \to Y$ and $g : Y \to Z$, and if $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ and $\widetilde{g} : \widetilde{Y} \to \widetilde{Z}$ lie above f and g, respectively, then $\widetilde{g}\widetilde{f}$ lies above fg. ■

4.6.6 Comparing the constructions of the fundamental groupoid

We will now prove that this functor Π_1^Q is naturally isomorphic to Π_1 from section 4.2. The proof consists of two steps, namely proving that Π_1^Q and Π_1 agree for connected simplicial finite sets, and then showing that Π_1^Q preserves projective limits of (connected) simplicial finite sets.

Lemma 4.94. *Let X be a connected simplicial finite set. Then $\Pi_1^Q X \cong \Pi_1 X$.*

Proof. Denote the "classical" fundamental groupoid functor $\mathbf{S} \to \mathbf{G}$ by Π_1^c. Then $\Pi_1 X = \widehat{\Pi_1^c X}$ by definition, i.e. $\Pi_1 X$ is the profinite completion of the classical fundamental groupoid of X. As $\Pi_1^Q X$ is connected, it is a profinite groupoid by Proposition 3.21. In particular, giving a morphism $\Pi_1 X \to \Pi_1^Q X$ is the same as giving a morphism $\Pi_1^c X \to \Pi_1^Q X$, by the universal property of the profinite completion. Recall that in the classical theory of coverings of simplicial sets, a map $Y \to X$ is called a covering if for any solid arrow diagram of the form

$$\begin{tikzcd} \Delta^1 \arrow[r] \arrow[d] & Y \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted, "\exists!"] & X, \end{tikzcd}$$

there exists a unique lift $\Delta^n \to Y$ (with no assumptions on the map $Y_0 \to X_0$). Such a covering is called finite if the fibers of the map $Y \to X$ are all finite. One obtains a category R/X of "classical" coverings of X, with the full subcategory R_f/X of finite coverings. For a simplicial finite set, there is no difference between the classical and the profinite notion of a finite covering. In particular, the categories R/X and $\widehat{R}/X$ both have R_f/X as a full subcategory. By Theorem 2.62 there exists a profinite completion functor $R/X \to \widehat{R}/X$, left adjoint to the functor $\widehat{R}/X \to R/X$ which forgets the profinite structure on a profinite covering $Y \to X$ (i.e. the Stone topology on Y_n). Also recall from the classical covering theory that $R/X \simeq \mathbf{Set}_{\pi_1^c(X,x)}$, where $\pi_1^c(X, x)$ is the "classical" fundamental group of X at a basepoint x. We also see that $R_f/X \simeq \mathbf{FinSet}_{\pi_1^c(X,x)}$. Write $G = \pi_1^c(X, x)$ and write $\widehat{G}$ for the profinite completion of G. As a finite G-set is the same as a finite $\widehat{G}$-set, we see that

$$\mathrm{Pro}(\mathbf{FinSet}_G) \simeq \mathrm{Pro}(\mathbf{FinSet}_{\widehat{G}}) \simeq \widehat{\mathbf{Set}}_{\widehat{G}}.$$

There is, by a similar argument as above, also a profinite completion functor $\mathbf{Set}_G \to$

$\widehat{\mathbf{Set}}_{\widehat{G}}$, so we obtain a commutative diagram

$$\begin{tikzcd} R/X \arrow[r, "F_x", "\simeq"'] \arrow[d, "\widehat{(\cdot)}"] & \mathbf{Set}_G \arrow[d, "\widehat{(\cdot)}"] \\ \widehat{R}/X \arrow[r, "F_x", "\simeq"'] & \widehat{\mathbf{Set}}_{\widehat{G}} \end{tikzcd}$$

where symbols F_x represent the fiber functors. Note that the classical universal cover $\widetilde{X}^c$ corresponds to G, acting on itself from the left, in $\mathbf{Set}_G$, and similarly that the profinite universal cover $\widetilde{X}$ corresponds to $\widehat{G}$.

By the universal property of the profinite completion functor, it follows that the profinite completion of $\widetilde{X}^c$ pro-represents the fiber functor $F_x \colon R_f/X \to \mathbf{FinSet}$, hence it is a profinite universal cover of X. This also implies that $\widehat{G}$ is the profinite completion of G, under the profinite completion functor $\widehat{(\cdot)} \colon \mathbf{Set}_G \to \widehat{\mathbf{Set}}_{\widehat{G}}$. Now note that we can construct $\Pi_1^c X$ from $\widetilde{X}^c$ exactly like in Definition 4.91. In particular the morphism $\widetilde{X}^c \to \widetilde{X}$ coming from the fact that $\widetilde{X}$ is the profinite completion of $\widetilde{X}^c$ induces a morphism $\Pi_1^c X \to \Pi_1^Q X$, which induces a morphism $\Pi_1 X = \widehat{\Pi_1^c X} \to \Pi_1^Q X$. This morphism is the identity on objects, so to see that it is an isomorphism, we need to show that it is fully faithful. This follows if the induced map $(\widehat{\Pi_1^c X})(x) \to (\Pi_1^Q X)(x)$ is an isomorphism. However, as $(\Pi_1^c X)(x) \cong G$ and $(\Pi_1^Q X)(x) \cong \widehat{G}$, this map is just the identity $\mathrm{id}_{\widehat{G}}$. We conclude that $\Pi_1 X$ is canonically isomorphic to $\Pi_1^Q X$ for connected simplicial finite sets X. ■

To see that the functor Π_1^Q preserves projective limits, we need to show that the universal cover of $X = \lim_i X_i$ can be obtained from those of X_i.

Lemma 4.95. *Let $X = \lim_i X_i$ be a projective limit of profinite spaces, and let $\widetilde{X}_i$ be universal covers of X_i. Then $\lim_i \widetilde{X}_i$ is a universal cover of X. Furthermore,* $\mathrm{Aut}(\widetilde{X}) = \lim_i \mathrm{Aut}(\widetilde{X}_i)$.

Proof. We first need to specify which maps $\widetilde{X}_i \to \widetilde{X}_j$ we pick for $i \le j$. For this, pick a basepoint $x \in X$. Then $x = (x_i)_i$ with x_i a basepoint of X_i, for every i. Pick a point $\widetilde{x}_i$ in $\widetilde{X}_i$ above x_i for every i. By Lemma 4.92, there are unique maps $\widetilde{f}^i_j \colon \widetilde{X}_i \to \widetilde{X}_j$ above f^i_j satisfying $\widetilde{f}^i_j(\widetilde{x}_i) = \widetilde{x}_j$, for $i \le j$. This uniqueness implies $\widetilde{f}^j_k \widetilde{f}^i_j = \widetilde{f}^i_k$ for $i \le j \le k$, so we obtain a projective diagram $\{\widetilde{X}_i\}$. Note that the maps $\widetilde{X}_i \to X_i$ induce a map $\lim_i \widetilde{X}_i \to X$. We now see that

$$\lim_i \widetilde{X}_i = X \times_X \lim_i \widetilde{X}_i \cong X \times_{\lim_i X_i} \lim_i \widetilde{X}_i \cong \lim_i (X \times_{X_i} \widetilde{X}_i).$$

Since the right-hand side is a projective limit of profinite coverings of X, we see that $\lim_i \widetilde{X}_i$ is a profinite covering of X. Denote $\lim_i \widetilde{X}_i$ by $\widetilde{X}$, and denote the point $(\widetilde{x}_i)_i$ by $\widetilde{x}$.

To see that $\widetilde{X}$ is a universal cover, we need to show that it represents the fiber functor F_x. Define, for $Z \to X$ a finite covering, $\phi \colon \mathrm{Hom}_{\widehat{R}/X}(\widetilde{X}, Z) \to F_x(Z)$ by $f \mapsto f(\widetilde{x})$. To see that this is an isomorphism, note that $Z = X \times_{X_i} Z'$ for some i and some finite covering $Z' \to X_i$. For every $z \in F_{x_i}(Z') = F_x(Z)$ there exists a unique map $f\colon \widetilde{X}_i \to Z'$ satisfying $f(\widetilde{x}_i) = z$. By precomposing this map with the projection $\widetilde{X} \to \widetilde{X}_i$, and using the universal property of the pullback to obtain a map $\widetilde{X} \to Z$, we see that ϕ is surjective.

To see that ϕ is injective, note that $\pi_0(\widetilde{X}) = \lim_i \pi_0(\widetilde{X}_i) = \{*\}$, so $\widetilde{X}$ is a connected profinite covering by property (iii) of Proposition 4.88. In particular, if for two maps $f, g\colon \widetilde{X} \to Z$ we have $f(\widetilde{x}) = g(\widetilde{x})$, then $f = g$ by property (ii) of Proposition 4.88.

To see that $\operatorname{Aut}(\widetilde{X}) = \lim_i \operatorname{Aut}(\widetilde{X}_i)$, note that there are natural homeomorphisms $\operatorname{Aut}(\widetilde{X}) \cong F_x(\widetilde{X})$ and $\operatorname{Aut}(\widetilde{X}_i) \cong F_{x_i}(\widetilde{X}_i)$. Since $\widetilde{X} = \lim_i \widetilde{X}_i$, we see that $F_x(\widetilde{X}) = \lim_i F_{x_i}(\widetilde{X})$ by restricting to the fibers. In particular $\operatorname{Aut}(\widetilde{X}) = \lim_i \operatorname{Aut}(\widetilde{X}_i)$. ■

Theorem 4.96. *The functors Π_1 and Π_1^Q agree for all connected profinite spaces.*

Proof. We already know that Π_1 and Π_1^Q agree for connected simplicial finite sets, so we are left with proving that Π_1^Q preserves projective limits, as any connected profinite space is a projective limit of connected simplicial finite sets, and we already know that Π_1 preserves projective limits. So let $X = \lim_i X_i$ be given. Assume we are given universal covers $\widetilde{X}_i$ of X_i. By the above lemma, $\widetilde{X} := \lim_i \widetilde{X}_i$ is a universal cover of X. The canonical map $\Pi_1^Q X \to \lim_i \Pi_1^Q X_i$ coming from $X = \lim_i X_i$ is, on arrows, given by the canonical map

$$(\widetilde{X}_0 \times \widetilde{X}_0)/\operatorname{Aut}(\widetilde{X}) \to \lim_i \Big(((\widetilde{X}_i)_0 \times (\widetilde{X}_i)_0)/\operatorname{Aut}(\widetilde{X}_i) \Big) .$$

Using that $\operatorname{Aut}(\widetilde{X}) = \lim_i \operatorname{Aut}(\widetilde{X}_i)$, the next lemma implies that the above map is a homeomorphism, hence the map $\Pi_1^Q X \to \lim_i \Pi_1^Q X_i$ is an isomorphism of profinite groupoids. ■

Lemma 4.97. *Let $G = \lim_i G_i$ be a projective limit of profinite groups, and let $X = \lim_i X_i$ be a projective limit of profinite G_i-sets, where the maps $X_i \to X_j$ are compatible with the G_i-actions, meaning that*

$$\begin{array}{ccc} G_i \times X_i & \longrightarrow & G_j \times X_j \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X_j \end{array}$$

commutes for all $i \leq j$. Then $X/G \cong \lim_i (X_i/G_i)$.

Proof. The canonical map $\phi\colon X/G \to \lim_i X_i/G_i$ maps the class $[(x_i)_i]$ to $([x_i])_i$. We need to show that this map is a bijection. For surjectivity, let $([x_i])_i$ in $\lim_i X_i/G_i$ be given. Then $[x_i]$ is a closed subset of X_i for every i, since it is an orbit of a continuous group action by a profinite group. We therefore see that $[x_i]$ is a profinite set for every i. Since the G_i-actions on X_i are compatible, we see that the profinite sets $[x_i]$ form a projective diagram. Since projective limits of nonempty profinite sets are nonempty, we see that $\lim_i [x_i] \subseteq X$ is nonempty. Since any $x \in \lim_i [x_i]$ satisfies $\phi([x]) = ([x_i])_i$, we conclude that ϕ is surjective.

For injectivity, assume $\phi([(x_i)_i]) = \phi([(y_i)_i])$. Then $[x_i] = [y_i]$ for every i, so denote by $H_i \subseteq G_i$ the set of $g \in G_i$ satisfying $g \cdot x_i = y_i$. Then H_i is a closed subset of G_i, hence a profinite set. H_i is also nonempty for every i, as $[x_i] = [y_i]$. Therefore $\lim_i H_i$ is also nonempty. Since $\lim_i H_i \subseteq \lim_i G_i = G$, we see that there exists a $g = (g_i) \in G$ such that $g_i \cdot x_i = y_i$ for every i. We conclude that $[(x_i)_i] = [(y_i)_i]$, hence that ϕ is injective. ∎

In light of the above theorem, we will from now on denote Π_1^Q by Π_1 and no longer distinguish these two functors. By noting that $\pi_1^Q(X, x) = (\Pi_1^Q X)(x)$ and $\pi_1(X, x) = (\Pi_1 X)(x)$, we obtain a natural isomorphism $\pi_1^Q(X, x) \cong \pi_1(X, x)$ as well.

Corollary 4.98. *Let $\widetilde{X} \to X$ be a profinite covering, with X a connected profinite space. Then $\widetilde{X}$ is a universal covering precisely if $\Pi_1 \widetilde{X} = \mathrm{Codisc}(\widetilde{X}_0)$, or equivalently, if $\widetilde{X}$ is connected and $\pi_1(\widetilde{X}, x)$ is trivial for any choice of $x \in \widetilde{X}$.* ∎

This suggests a strategy for computing $\Pi_1 X$ for a connected profinite space X. Namely, if we can find a profinite covering $Y \to X$ such that $\Pi_1 Y$ is codiscrete, then $\Pi_1 X$ can be constructed as a quotient of $Y_0 \times Y_0$. Conversely, we can also compute a universal covering of X if we know $\Pi_1 X$.

Proposition 4.99. *Let X be a connected profinite set, and let $x \in X$ be a basepoint. Then*

$$X \times_{B\Pi_1 X} B(\Pi_1 X \downarrow x) \to X$$

is a universal cover of X.

Proof. Note that $B(\Pi_1 X \downarrow x) \to B\Pi_1 X$ is a universal cover. The easiest way to see this is to note that $B(\Pi_1 X \downarrow x)$ is a principal $\pi_1(X, x)$-bundle, hence a profinite (Galois) covering, and that is $\Pi_1(B(\Pi_1 X \downarrow x)) = \Pi_1 X \downarrow x$ is codiscrete.

Now note that the adjunction $\Pi_1 \dashv B$ induces a map $X \to B\Pi_1 X$, which in turn induces an isomorphism on the fundamental groupoid $\Pi_1 X \to \Pi_1 B\Pi_1 X \cong \Pi_1 X$, since the counit $\Pi_1 BA \to A$ is an isomorphism for any profinite groupoid A. If we construct $\Pi_1 X = \Pi_1^Q X$ from a universal cover $\widetilde{X} \to X$, then the map $\Pi_1 X \to \Pi_1 B\Pi_1 X$ is obtained by choosing a map f such that

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{f} & B(\Pi_1 X \downarrow x) \\ \downarrow & & \downarrow \\ X & \longrightarrow & B\Pi_1 X \end{array}$$

commutes (see the proof of Proposition 4.93 to recall how Π_1^Q was defined on arrows). Since we already know that the induced map $\Pi_1 X \to \Pi_1 B\Pi_1 X$ is an isomorphism, we see that f must give an isomorphism on the fiber above x by construction of Π_1^Q. By the universal property of the pullback, we obtain an induced map $f' \colon \widetilde{X} \to X \times_{B\Pi_1 X} B(\Pi_1 X \downarrow x)$, which must also be an isomorphism in the fiber above x. Since the fiber functor $F_x \colon \widehat{R}/X \to \widehat{\mathbf{Set}}_{\pi_1(X,x)}$ is an equivalence, we conclude that f' itself is an isomorphism. ∎

Appendix A

Fibrantly generated model categories

The aim of this section is to give a summary of some notions related to fibrantly generated model categories, which will be used when constructing model structures on pro-categories. Some familiarity with model categories is assumed, e.g. lifting properties and retracts. The reader not familiar with such notions should consult an introductory text, such as chapter 7 of [Hir03].

Fibrantly generated model categories are precisely the dual of cofibrantly generated model categories, which are more common in the literature. In a cofibrantly generated model category, the so-called *small object argument* is used to construct a functorial factorization of maps into a trivial cofibration followed by a fibration, or a cofibration followed by a trivial fibration. This argument requires that the domains of the generating (trivial) cofibrations are 'small' objects in a certain sense. However, in the context of pro-categories, 'cosmall' objects occur more naturally than small objects, so it is more natural to work with fibrantly generated model categories. We follow chapters 10 and 11 of [Hir03], dualizing all required definitions and statements.

Definition A.1. Let $\mathbf{C}$ be a category.

(i) Let $\mathbf{C}$ have all filtered colimits. A λ-sequence in $\mathbf{C}$, with λ an ordinal, is a diagram $X\colon \lambda \to \mathbf{C}$, i.e. a sequence

$$X_0 \to X_1 \to \cdots \to X_\beta \to X_{\beta+1} \to \cdots,$$

such that for all $0 < \gamma < \lambda$ with γ a limit ordinal, $\operatorname{colim}_{\beta<\gamma} X_\beta = X_\gamma$. We call the map $X_0 \to \operatorname{colim}_{\beta<\lambda} X_\beta$ the *(transfinite) composition* of the λ-sequence. Note that an ω-sequence is precisely an inductive diagram indexed by $\mathbb{N}$.

(ii) Let $\mathbf{C}$ have all cofiltered limits. A λ-tower in $\mathbf{C}$, with λ an ordinal, is a diagram $X\colon \lambda^{op} \to \mathbf{C}$, i.e. a sequence

$$X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_\beta \leftarrow X_{\beta+1} \leftarrow \cdots,$$

such that for all $0 < \gamma < \lambda$ with γ a limit ordinal, $\lim_{\beta<\gamma} X_\beta = X_\gamma$. We call the map $X_0 \to \lim_{\beta<\lambda} X_\beta$ the *(transfinite) precomposition* of the λ-tower. Note that an ω-tower is precisely a projective diagram indexed by $\mathbb{N}^{op}$. ♢

Definition A.2. Let $\mathbf{C}$ be a category, $\mathbf{D}$ a subcategory and S an object in $\mathbf{C}$.

(i) Let $\mathbf{C}$ have all filtered colimits. We say that S is *small relative to* $\mathbf{D}$ if, for every ordinal λ and every λ-sequence $\{X_\beta\}_{\beta<\lambda}$ such that $X_\beta \to X_{\beta+1}$ is in $\mathbf{D}$ for all $\beta+1<\lambda$, the canonical map

$$\operatorname*{colim}_{\beta<\lambda} \mathrm{Hom}_{\mathbf{C}}(S, X_\beta) \to \mathrm{Hom}_{\mathbf{C}}(S, \operatorname*{colim}_{\beta<\lambda} X_\beta)$$

is an isomorphism. This means in particular that any map $S \to \operatorname{colim}_{\beta<\lambda} X_\beta$ factors through X_β for some $\beta<\lambda$. We say that S is *small* if S is small relative to $\mathbf{C}$.

(ii) Let $\mathbf{C}$ have all cofiltered limits. We say that S is *cosmall relative to* $\mathbf{D}$ if, for every ordinal λ and every λ-tower $\{X_\beta\}_{\beta<\lambda}$ such that $X_{\beta+1} \to X_\beta$ is in $\mathbf{D}$ for all $\beta+1<\lambda$, the map

$$\operatorname*{colim}_{\beta<\lambda} \mathrm{Hom}_{\mathbf{C}}(X_\beta, S) \to \mathrm{Hom}_{\mathbf{C}}(\lim_{\beta<\lambda} X_\beta, S)$$

is an isomorphism. This means in particular that any map $\lim_{\beta<\lambda} X_\beta \to S$ factors through X_β for some $\beta<\lambda$. We say that S is *cosmall* if S is cosmall relative to $\mathbf{C}$. $\diamondsuit$

Remark A.3. What we define here as small and cosmall is sometimes called $\aleph_0$-small and $\aleph_0$-cosmall. An object is then called (co)small if it is κ-(co)small for some cardinal κ, which is weaker than the notion of (co)smallness defined above. However, the (co)small objects that we will encounter are always (co)small in the above, stronger, sense. For purposes of simplicity, we will use the above definition. $\diamondsuit$

Example A.4. This example is the main motivation for us to work with fibrantly generated model structures. Let $\mathbf{C}$ be any category. By Theorem 2.30, $\mathrm{Pro}(\mathbf{C})$ has all cofiltered limits, so in particular we can speak about cosmall objects. In $\mathrm{Pro}(\mathbf{C})$ all objects in the image of the inclusion $\iota\colon \mathbf{C} \hookrightarrow \mathrm{Pro}(\mathbf{C})$, i.e. the representables, are cosmall. To see this, note that

$$\mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}(\{D_i\}, \iota C) = \operatorname*{colim}_i \mathrm{Hom}_{\mathbf{C}}(D_i, C) \cong \operatorname*{colim}_i \mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}(\iota D_i, \iota C)$$

by definition. This means that for all C in $\mathbf{C}$, $\mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}(-, \iota C)$, as functor $\mathrm{Pro}(\mathbf{C}) \to \mathbf{Set}^{op}$, preserves cofiltered limits of representables. By arguments similar to those used in the proof of Theorem 2.30 and Proposition 2.31, this implies that $\mathrm{Hom}_{\mathrm{Pro}(\mathbf{C})}(-, \iota C)$ preserves all cofiltered limits. Since λ-towers are cofiltered limits, we see that ιC is cosmall in $\mathrm{Pro}(\mathbf{C})$ for all C in $\mathbf{C}$. $\diamondsuit$

Example A.5. The dual of the above statement of course holds for $\mathrm{Ind}(\mathbf{C})$, meaning that any object in the image of the inclusion $\mathbf{C} \hookrightarrow \mathrm{Ind}(\mathbf{C})$ is small. $\diamondsuit$

From now on, we will focus only on fibrantly generated model categories. All of the definitions, propositions and theorems stated below come from chapters 10 and 11 of [Hir03], where they are all stated in their dual form.

Definition A.6. Let $\mathbf{C}$ be a category and P a set of maps in $\mathbf{C}$.

(i) The maps having the left-lifting property with respect to every map in P by $\mathrm{llp}(P)$. These maps are sometimes called *P-projective maps*.

(ii) Similarly, the maps having the right-lifting property with respect to every map in P by $\mathrm{rlp}(P)$. These maps are sometimes called *P-injective maps*.

(iii) The class of *P-fibrations* is the class of maps defined by $\mathrm{fib}(P) = \mathrm{rlp}(\mathrm{llp}(P))$.

(iv) An object C is called *P-fibrant* if the map $C \to *$, where $*$ is the terminal object of $\mathbf{C}$, is a P-fibration. $\diamondsuit$

Definition A.7. Let $\mathbf{C}$ be a complete category, and let P be a set of maps in $\mathbf{C}$.

(i) The class of *relative P-cocell complexes* is the class of maps that can be constructed as a transfinite precomposition of pullbacks of products of maps in P. Such a map is also called *P-cocellular*. The P-cocellular maps form a subcategory of $\mathbf{C}$ with the same class of objects.

(ii) An object is a *P-cocell complex* if the map from the object to the terminal object is a relative P-cocell complex.

(iii) A map is a *projection of P-cocell complexes* if it is a relative P-cocell complex whose domain is a P-cocell complex. $\diamondsuit$

Remark A.8. The above definition may need some clarification. By a pullback of a product of maps in P, we mean the following. If we are given a pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[d, "g"] \\ C \arrow[r] & D, \end{tikzcd}$$

then we say that f is a pullback of g. Hence a pullback of a product of maps in P is such a square where g is a product of maps in P. A transfinite precomposition of pullbacks of products of maps in P is then a map $f\colon \lim_{\beta<\lambda} X_\beta \to X_0$, where $\{X_\beta\}_{\beta<\lambda}$ is a λ-tower, and where any map $X_{\beta+1} \to X_\beta$ fits in a pullback diagram

$$\begin{tikzcd} X_{\beta+1} \arrow[r] \arrow[d] & \prod_j A_j \arrow[d, "\prod_j g_j"] \\ X_\beta \arrow[r] & \prod_j B_j \end{tikzcd}$$

for some maps $g_j \in P$. In particular, the class of relative P-cocell complexes is obtained by first adding all pullbacks, and then adding all transfinite compositions. Lastly, note that all identity morphisms are P-cocellular maps. To see this, let $\lambda = 1$ and define $X_0 = X$ for some $X \in \mathbf{C}$. Then $\lim_{\beta<\lambda} X_\beta \to X$ is the identity morphism. $\diamondsuit$

Remark A.9. In Proposition 10.2.7 of [Hir03], it is proved that coproducts of maps in P are transfinite compositions of pushouts of maps in P. Dualizing this, we see that products of maps in P are transfinite precompositions of pullbacks of maps in P. This means that any map that is a transfinite precomposition of pullbacks of products of maps in P, is also a transfinite precomposition of pullbacks of maps in P. $\diamondsuit$

Example A.10. To somewhat clarify (the dual of) the above definition, we will consider relative I-cell complexes in **Top**, where I is the set of inclusions $S^{n-1} \hookrightarrow D^n$. A relative I-cell complex is a map obtained by transfinite composition of pushouts of maps in I. The map $X_\beta \to X_{\beta+1}$ then attaches an n-cell to X_β for some n. A relative I-cell complex $X \to Y$ is an inclusion $X \hookrightarrow Y$, where Y is obtained by attaching arbitrarily many cells to X. Say that we are given a λ-sequence $\{X_\beta\}_{\beta<\lambda}$ with the extra property that for any β, if $X_\beta \to X_{\beta+1}$ attaches an n-cell, then no m-cell with $m > n$ has yet been attached. Then the pair $(\operatorname{colim}_{\beta<\gamma} X_\beta, X_0)$ is a relative CW-complex, and any relative CW-complex is of this form. $\diamondsuit$

Proposition A.11. *Every retract of a relative P-cocell complex is a P-fibration.*

Proof. Note that a map in P by definition has the right-lifting property with respect to any map in $\operatorname{llp}(P)$. By a standard argument, pullbacks and retracts of maps in $\operatorname{rlp}(J)$ are again in $\operatorname{rlp}(J)$, for any class of maps J. By transfinite induction, one can prove that $\operatorname{rlp}(J)$ is closed under transfinite precomposition. In particular any relative P-cocell complex is a P-fibration. Again by a standard argument, if a map is in $\operatorname{rlp}(J)$, then any retract of this map is also in $\operatorname{rlp}(J)$. $\blacksquare$

Definition A.12. Let $\mathbf{C}$ be a category and P a set of maps in $\mathbf{C}$. We say that an object is *cosmall relative to* P if it is cosmall relative to the subcategory of relative P-cocell complexes, in the sense of Definition $A.2$. $\diamondsuit$

The following definition and proposition are the essential ingredients in constructing a fibrantly generated model structure.

Definition A.13. Let $\mathbf{C}$ be a category and P a set of maps in $\mathbf{C}$. We say that P *permits the cosmall object argument* if the codomains of the elements of P are cosmall relative to P. $\diamondsuit$

Proposition A.14 (The cosmall object argument). *Let $\mathbf{C}$ be a complete category and let P be a set of maps in $\mathbf{C}$ that permits the cosmall object argument. Then there is a functorial factorization of every map in $\mathbf{C}$ into a P-projective map followed by a P-cocellular map.*

Proof. This proof is based on the proof of Proposition 10.5.16 in [Hir03]. We will construct the factorization of a map $f\colon X \to Y$ in $\mathbf{C}$ as $X \xrightarrow{j} Z_P \xrightarrow{p} Y$ by an ω-tower

$$\begin{tikzcd}
Y = Z_0 & Z_1 \arrow[l] & Z_2 \arrow[l] & \cdots \arrow[l] & Z_n \arrow[l] & \cdots \arrow[l] & (n \in \mathbb{N}), \\
 & & X \arrow[ull, "j_0"] \arrow[ul, "j_1"'] \arrow[u, "j_2"'] \arrow[urr, "j_n"'] & & & &
\end{tikzcd}$$

where each $Z_{n+1} \to Z_n$ is a pullback of products of maps in P, and we let $Z_P = \lim_n Z_n$. We set $j = \lim_n j_n$, and we let $p\colon Z_P \to Y$ be the transfinite precomposition. Then $Z_P \to Y$ is a P-cocellular map by definition. The idea is to "add" all the P-cocells to Y which are needed to make the map $j\colon X \to Z_P$ P-projective.

If $j\colon X \to Z_P$ is P-projective, then for any commutative diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "j"] & A \arrow[d, "k"] \\ Z_P \arrow[r] & B, \end{tikzcd}$$

with $k \in P$, there must exist a lift. Since B is assumed to be cosmall relative to P, we obtain a commutative solid arrow diagram of the form

$$\begin{tikzcd} & & X \arrow[r] \arrow[dll] \arrow[dl] \arrow[d] & A \arrow[d, "k"] \\ Z_P \arrow[r] & Z_{n+1} \arrow[r] \arrow[urr, dotted] & Z_n \arrow[r] & B \end{tikzcd}$$

for some $n \in \mathbb{N}$. The idea is now to construct Z_{n+1} in such a way from Z_n that a lift $Z_{n+1} \to A$ exists. Precomposition with the map $Z_P \to Z_{n+1}$ will then give the desired lift.

To achieve this, we will define the Z_n inductively. Let $Z_0 = Y$ and let $j_0 = f$. Now assume Z_n has been constructed. Let

$$P_n = \{(k_i, g, h) \in P \times \mathrm{Hom}_{\mathbf{C}}(X, A_i) \times \mathrm{Hom}_{\mathbf{C}}(Z_n, B_i) \mid k_i\colon A_i \to B_i \text{ and } kg = hj_n\}.$$

Note that we can also view this as

$$P_n = \coprod_{(k_i\colon A_i \to B_i) \in P} \mathrm{Hom}_{\mathbf{C}}(X, A_i) \times_{\mathrm{Hom}_{\mathbf{C}}(X,B)} \mathrm{Hom}_{\mathbf{C}}(Z_n, B_i).$$

We now define Z_{n+1} to be the pullback $(\prod A_i) \times_{(\prod B_i)} Z_n$ as in the diagram

$$\begin{tikzcd} Z_{n+1} \arrow[r] \arrow[d] & \displaystyle\prod_{(k_i,g,h) \in P_n} A_i \arrow[d] \\ Z_n \arrow[r] & \displaystyle\prod_{(k_i,g,h) \in P_n} B_i. \end{tikzcd}$$

Here the vertical map on the right is given by k_i on each component, and the bottom map is the product of the maps $h\colon Z_n \to B_i$. We construct j_{n+1} from j_n through the universal property of the pullback, where the map $X \to \prod A_i$ is given by $g\colon X \to A_i$ on each component.

Now let $Z_P = \lim_n Z_n$. One directly sees from the discussion above that $X \to Z_P$ is P-projective, and as stated above, $Z_P \to Y$ is P-cocellular by definition. Functoriality is left to the reader. ■

Corollary A.15. *Let* $\mathbf{C}$ *be a category and let* P *be a set of maps permitting the cosmall object argument. Then the class of* P*-fibrations equals the class of retracts of relative* P*-cocell complexes.*

Proof. We already proved that a retract of a relative P-cocell complex is a P-fibration. For the converse, let $g\colon X \to Y$ be a P-fibration. Factor g as $X \xrightarrow{j} Z_P \xrightarrow{p} Y$ with j a P-projective map and p a relative P-cocell complex. Then $j \in \mathrm{llp}(P)$ and $g \in \mathrm{rlp}(\mathrm{llp}(P))$, so there exists a map $f\colon Z_P \to X$ satisfying $fj = \mathrm{id}_X$ and $gf = p$. In particular the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[rr, bend left, "\mathrm{id}_X"] \arrow[d, "g"] & Z_P \arrow[r, "f"] \arrow[d, "p"] & X \arrow[d, "g"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd}$$

commutes, so we conclude that g is a retract of p. ∎

Definition A.16. A *fibrantly generated model category* is a model category $\mathbf{C}$ for which there exist two sets of maps P and Q such that:

(i) P and Q permit the cosmall object argument,

(ii) the class of trivial cofibrations is equal to $\mathrm{llp}(P)$, and

(iii) the class of cofibrations is equal to $\mathrm{llp}(Q)$. ♢

Remark A.17. The above definition implies that the class of fibrations in $\mathbf{C}$ is given by $\mathrm{fib}(P)$, and that the trivial fibrations are $\mathrm{fib}(Q)$. For that reason, P is the set of *generating fibrations* and Q the set of *generating trivial fibrations.* ♢

The following theorem, which (in its dual form) is attributed to D.M. Kan, is a useful tool in constructing model structures on pro-categories. It can be found in [Hir03] as Theorem 11.3.1. Recall that a class of maps W satisfies the "two out of three" axiom if, for two composible maps f, g, whenever two of the maps f, g and gf are in W, then so is the third.

Theorem A.18. *Let* $\mathbf{C}$ *be a complete and cocomplete category and let* W *be a class of maps in* $\mathbf{C}$ *that is closed under retracts and satisfies the "two out of three" axiom. If* P *and* Q *are sets of maps in* $\mathbf{C}$ *such that:*

1. *both* P *and* Q *permit the cosmall object argument,*

2. *every* Q*-fibration is both a* P*-fibration and an element of* W*,*

3. *every* P*-projective map is both* Q*-projective and an element of* W*, and*

4. *one of the following two conditions hold:*

   a. *a map that is both a* P*-fibration and an element of* W *is a* Q*-fibration, or* *b. a map that is both a Q-projective map and an element of W is P-projective,*

then there is a fibrantly generated model structure on $\mathbf{C}$ *in which W is the class of weak equivalences, P is a set of generating fibrations and Q is a set of generating trivial fibrations.*

Proof. Define the weak equivalences to be maps in W, the fibrations to be the P-fibrations and the cofibrations to be the Q-projectives. It is straightforward to check the axioms of a model category, using the cosmall object argument to prove the factorization axiom. The only part that might give rise to problems is the lifting axiom, so we will prove this assuming 4a. The proof assuming 4b is similar.

Assume we are given a commutative solid arrow diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y, \end{tikzcd}$$

then we want to construct a lift in two cases. The first case is when i is a cofibration (i.e. a Q-projective map) and p a trivial fibration. In this case i is a Q-fibration by assumption 4a, hence a lift exists by definition of a Q-fibration. The second case is when i is a trivial cofibration and p a fibration (i.e. a P-fibration). Factor i as $A \xrightarrow{j} C \xrightarrow{q} B$ with j a P-projective map and q a P-fibration. Then by assumption 3, j is a weak equivalence, hence q is a weak equivalence by the two out of three property. By assumption 4a, q is a Q-fibration. This means that q has the right-lifting property with respect to i, hence there exists a map $h\colon B \to C$ such that $hi = j$ and $qh = \mathrm{id}_B$. By a proof similar to that of Corollary $A.15$, we see that i is a retract of j, hence i has the left-lifting property with respect to maps in P, hence with respect to P-fibrations. Since p is a P-fibration, we conclude that a lift in the above diagram exists. ■

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