How to get what you want in the Iterated Prisoner's Dilemma

Maud Nabben
5483875

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Written under supervision of Dr. M. Ruijgrok

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Department of Mathematics

Utrecht University

Contents

1 Introduction

Normally in game theory we assume that a player wants to maximize its own payoff. In 2012, Dyson and Press wrote an article [2] about the prisoner's dilemma, where they didn't look at the maximum payoff of the player, but at the payoff of the co-player. The article looks at the iterated prisoner's dilemma with memory-one strategies, and shows that there are strategies such that you can enforce a constraint for the payoff of the co-player, regardless of which strategy the co-player is using.

In a one-shot prisoner's dilemma there is a dominant strategy, but that strategy is not Pareto optimal. If we look at the iterated prisoner's dilemma, it is possible to get a Nash equilibrium. Assume that both players start with cooperating. If one of them will defect, then the other will also defect, to get a higher payoff. But mutual defection gives a smaller payoff for both players than mutual cooperation. In this game, it's quite remarkable that one player can enforce something.

We will first give some basic notations. After that we get a Lemma that is the foundation for this article. Then we differentiate between people who are nice to each other and people who are cautious, where we get a space of strategies. With those strategies we can enforce a certain payoff or a constraint for the payoff of the co-player. With zero-determinant strategies we can even be more specific; we can enforce a linear relationship between the two payoffs. This thesis is based on an article from Hilbe, Traulsen and Sigmund [1].

2 Memory-one strategy

We first have to introduce some notations and definitions, beginning with just looking at the one-shot prisoner's dilemma. When we look at the payoff matrix we see that the two players can choose between cooperation and defection. If they both choose to cooperate, they'll both get a payoff of R. If they both choose to defect, they'll both get a payoff of P. If the first player chooses to cooperate and the second player chooses to defect, then the first player will receive a payoff of S and the second player a payoff of T. Here we assume that $T > R > P > S$, such that defection is the dominant strategy.

	C	D
C	R,R	S,T
D	T,S	P,P

If the prisoner's dilemma is played more than once, and the players remember previous outcomes and change their strategy based on that, the game is called an iterated prisoner's dilemma. If $2R < T + S$, then it's best for the players to alternate between cooperation and defection in order to get the highest payoff $(CD, DC, CD, DC, \ldots)$. We're just looking at a memory-one strategy, which means that the players will just remember the previous outcome of the game. If the previous outcome is DD, then both players won't know who didn't play C, since they only remember DD. In this article, we will therefore assume that $2R > T + S$, such that we won't have this issue.

We want to define the players' expected payoffs in round t, with the help of some vectors. First we say that $v_a(t)$ is the probability that $a \in \{CC, CD, DC, DD\}$ is the outcome in round t, and we use the vector notation $\mathbf{v}(t) = (v_{CC}(t), v_{CD}(t), v_{DC}(t), v_{DD}(t))$. We also define the vectors $\mathbf{g}_{\mathrm{I}}$ and $\mathbf{g}_{\mathrm{II}}$ as the possible payoffs for player I and player II, so

$$\begin{aligned} \mathbf{g}_{\mathrm{I}} &= (R, S, T, P) \\ \mathbf{g}_{\mathrm{II}} &= (R, T, S, P) \end{aligned} \tag{1}$$

With these notations we can now define the players' expected payoffs in round t, since that is equal to the probability on a certain outcome times the payoff for that outcome, so $\pi_{\text{I}}(t) = \mathbf{g}_{\text{I}} \cdot \mathbf{v}(t)$ and $\pi_{\text{II}}(t) = \mathbf{g}_{\text{II}} \cdot \mathbf{v}(t)$. We use a discount factor δ, which is the probability that the game will continue. We differentiate between $\delta < 1$ and $\delta = 1$, since you might play a different strategy if you're sure that the game will continue, than when you're not sure that it will continue.

With this factor, the (normalized) expected payoffs can be defined as

$$\pi_{\text{I}} = (1-\delta)\sum_{t=0}^{\infty} \delta^t \pi_{\text{I}}(t) = \mathbf{g}_{\text{I}} \cdot \mathbf{v}, \tag{2}$$

and similarly $\pi_{\text{II}} = \mathbf{g}_{\text{II}} \cdot \mathbf{v}$, where $\mathbf{v} = (v_{CC}, v_{CD}, v_{DC}, v_{DD})$ refers to the mean distribution

$$\mathbf{v} = (1-\delta)\sum_{t=0}^{\infty} \delta^t \mathbf{v}(t). \tag{3}$$

In the case when $\delta = 1$, the payoff is given by

$$\pi_{\text{I}} = \lim_{\tau \to \infty} \frac{1}{\tau + 1} \sum_{t=0}^{\infty} \pi_{\text{I}}(t) \tag{4}$$

(if this limit exists), and a similarly for π_{II}.

We want to define a vector equal to the probability that the player will play C or D in the next round. Since we look at memory-one strategies, we only need to consider the outcome of the previous round in this probability vector. For example, we assume that player I applies a Tit-For-Tat strategy. Due to the definition of the TFT strategy, we know that player I will cooperate in the first round. After that, he will play what the co-player did in the previous round. So if the co-player played C in the previous round, the player will play C in this round, and vice versa. Now let p_0 be the probability that player I cooperates in the first round, which for TFT is equal to p_0 being 1. For the other rounds, we define p_a as the probability that player I will cooperate in the next round, where $a \in \{CC, CD, DC, DD\}$ is the outcome of the previous round. Here the first letter in the subscript refers to the player's own action in the previous round, and the second letter to the co-player's action in the previous round.

- If the outcome of the previous round is CC, then both players cooperated. Since player I plays what player II did in the previous round, player I will play C in the next round. So the probability that player I will cooperate in the next round is 1.
- If the outcome of the previous round is CD, then player I cooperated, but player II didn't. Since player I plays what player II did in the previous round, player I will play D in the next round. So the probability that player I will cooperate in the next round is 0.
- If the outcome of the previous round is DC, then player I didn't cooperate, but player II did. Since player I plays what player II did in the previous round, player I will play C in the next round. So the probability that player I will cooperate in the next round is 1.
- If the outcome of the previous round is DD, then both players defected. Since player I plays what player II did in the previous round, player I will play D in the next round. So the probability that player I will cooperate in the next round is 0.

We see that we can write the TFT strategy as a vector $\mathbf{p} = (p_{CC}, p_{CD}, p_{DC}, p_{DD}; p_0) = (1, 0, 1, 0; 1)$. In general we let p_a, with $a \in \{CC, CD, DC, DD\}$, be the probability that the player will cooperate in the next round, when a was the outcome of the previous round. Since at $t = 0$ we don't have a previous round, we define p_0 as the probability to cooperate in the first round. We can write this as $\mathbf{p} = (p_{CC}, p_{CD}, p_{DC}, p_{DD}; p_0)$. Again the first letter in the subscript refers to the player's own action in the previous round, and the second letter to the co-player's action in the previous round. We define this vector without the p_0 as $\tilde{\mathbf{p}} := (p_{CC}, p_{CD}, p_{DC}, p_{DD})$.

Using the previous definitions, we can find a formula to determine the resulting mean distribution $\mathbf{v}$, which we will use later on. Assume that player I uses the memory-one strategy $\mathbf{p} = (p_{CC}, p_{CD}, p_{DC}, p_{DD}; p_0)$ and player II uses the memory-one strategy $\mathbf{q} = (q_{CC}, q_{CD}, q_{DC}, q_{DD}; q_0)$. We get

$$\mathbf{v} = (1-\delta)\mathbf{v}(0) \cdot (I_4 - \delta M)^{-1}. \tag{5}$$

Here is $\mathbf{v}(0)$ the probability that player I and player II will play C or D in the first round, which is equal to $\mathbf{v}(0) = (p_0 q_0, p_0(1-q_0), (1-p_0)q_0, (1-p_0)(1-q_0))$. I_4 is, as normal, the 4×4 identity matrix, and M is the transition matrix of the Markov chain given by

$$M = \begin{pmatrix} p_{CC}q_{CC} & p_{CC}(1-q_{CC}) & (1-p_{CC})q_{CC} & (1-p_{CC})(1-q_{CC}) \\ p_{CD}q_{DC} & p_{CD}(1-q_{DC}) & (1-p_{CD})q_{DC} & (1-p_{CD})(1-q_{DC}) \\ p_{DC}q_{CD} & p_{DC}(1-q_{CD}) & (1-p_{DC})q_{CD} & (1-p_{DC})(1-q_{CD}) \\ p_{DD}q_{DD} & p_{DD}(1-q_{DD}) & (1-p_{DD})q_{DD} & (1-p_{DD})(1-q_{DD}) \end{pmatrix} \tag{6}$$

Until now we've only introduced a few notations. We defined (5) such that it only yields when both players apply a memory-one strategy. In the next Lemma, we will show that there is still a strong connection between $\mathbf{p}$ and $\mathbf{v}$ when just one of the players applies a memory-one strategy. This Lemma is the foundation for this article.

Lemma 1 ([1], Lemma 1)**.** *Suppose player* I *applies a memory-one strategy* $\mathbf{p}$*, and let the strategy of player* II *be arbitrary, but fixed.*

(i) *In the case with discounting* ($\delta < 1$)*, let* $\mathbf{v}$ *denote the mean distribution of the repeated game. Then*

$$(\delta p_{CC} - 1)v_{CC} + (\delta p_{CD} - 1)v_{CD} + \delta p_{DC}v_{DC} + \delta p_{DD}v_{DD} = -(1-\delta)p_0, \tag{7}$$

or in vector notation, $(\delta\tilde{\mathbf{p}} - \mathbf{g_0}) \cdot \mathbf{v} = -(1-\delta)p_0$*, where* $g_0 = (1, 1, 0, 0)$.

(ii) *In the case without discounting, we have*

$$\lim_{\tau\to\infty} \frac{1}{\tau+1} \sum_{t=0}^{\tau} (\tilde{\mathbf{p}} - \mathbf{g_0}) \cdot \mathbf{v}(t) = 0. \tag{8}$$

The proof of this Lemma is shown in Appendix A. We've introduced a vector $\mathbf{g_0} = (1, 1, 0, 0)$, which we use for the proof. It is important to notice that this Lemma is really general. It doesn't matter what the strategy of player II is, and it doesn't depend on the constraints of the prisoner's dilemma.

In (5) we saw that $\mathbf{v}$ depends on $\mathbf{p}$ and $\mathbf{q}$, but Lemma 1 only depends on $\mathbf{p}$. It's quite exceptional that it doesn't matter what the strategy of player II is. It doesn't even have to be a memory-one strategy. Because of this, there are some consequences, which will be shown in Section 4.

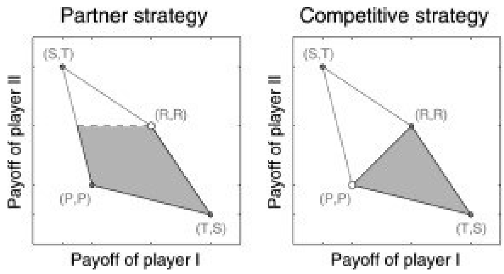

Fig. 1. Here we see a schematic representation of partner strategies and competitive strategies. The grey area are the potential payoffs for player I and player II, where player I plays the strategy that is mentioned. The white dot is the payoff that the players will get when they use the same strategy.

3 Nice and cautious strategies

As said in the introduction, we will differentiate between people who are nice and people who are cautious. First we define a partner strategy and a competitive strategy, which we'll use to get a space of strategies with certain constraints for the payoff of the co-player.

Definition 1 ([1], Def. 2)**.** A player's strategy is *nice*, if the player is never the first to defect, so for a memory-one strategy $p_0 = p_{CC} = 1$. A player's strategy is *cautious* if the player is never the first to cooperate, so for a memory-one strategy $p_0 = p_{DD} = 0$.

In Section 2 we have shown that vector notation for the strategy TFT is $(1, 0, 1, 0; 1)$, which is nice, according to the above definition. Another example is the strategy $AllD$, which means that the player will always play D, so the vector notation is $(0, 0, 0, 0; 0)$. This is a cautious strategy, according to the definition. We'll now introduce a Lemma which will help us to give some constraints for the payoff of the co-player.

Lemma 2 ([1], Lemma 2)**.** *If $2R > T + S$, then the payoffs satisfy $\pi_{\mathrm{I}} + \pi_{\mathrm{II}} \geq 2R$ if and only if $\pi_{\mathrm{I}} = \pi_{\mathrm{II}} = R$ (which for $\delta < 1$ is equivalent to both players being nice). Similarly, if $2P < T+S$, then $\pi_{\mathrm{I}} + \pi_{\mathrm{II}} \leq 2P$ if and only if $\pi_{\mathrm{I}} = \pi_{\mathrm{II}} = P$ (which for $\delta < 1$ is equivalent to both players being cautious).*

Proof. From (2) we see that $\pi_{\mathrm{I}} + \pi_{\mathrm{II}} = (\mathbf{g}_{\mathrm{I}} + \mathbf{g}_{\mathrm{II}}) \cdot \mathbf{v} = (2R, T + S, T + S, 2P) \cdot \mathbf{v}$. Now $\pi_{\mathrm{I}} + \pi_{\mathrm{II}} \geq 2R > T+S$ implies that π_{I} and π_{II} must both be R, so $v_{CC} = 1$. For $\delta < 1$, this means that both players should cooperate in every round. For $\delta = 1$, it only requires the players to cooperate in almost every round. Similarly, for a prisoner's dilemma with $\pi_{\mathrm{I}} + \pi_{\mathrm{II}} \leq 2P < T+S$, it implies that π_{I} and π_{II} must both be P, so $v_{DD} = 1$. □

Definition 2 ([1], Def. 3)**.**

(i) A *partner strategy* for player I is a nice strategy such that, irrespective of the co-player's strategy,

$$\pi_{\mathrm{I}} < R \Rightarrow \pi_{\mathrm{II}} < R. \tag{9}$$

(ii) A *competitive strategy* for player I is a strategy such that, irrespective of the co-player's strategy,

$$\pi_{\mathrm{I}} \geq \pi_{\mathrm{II}}. \tag{10}$$

If the two strategy classes exist, then we see a schematic representation of the potential payoffs for player I and player II in Fig. 1. We're not sure that every outcome is reached, if a partner strategy or competitive strategy is played.

For the partner strategy we see that when the payoff of player I is smaller than R, the payoff of player II also has to be smaller than R. The grey area is therefore bounded by R as the payoff of player II. When both players apply a partner strategy, they will both never be the first to defect, since a partner strategy is a nice strategy. Therefore the payoff of both players will in that case be R, which is shown by the white dot. For player II the best response to player I, who applies a partner strategy, is to also cooperate to get the highest payoff.

For the competitive strategy we see that the payoff of player I will always be greater or equal to the payoff of player II. We can see that player I has to play a cautious strategy, otherwise, if player II plays D, it's possible that player I receives a payoff S and player II receives T, which means that the payoff of player I is not greater or equal to the payoff of player II.

By the definition of a partner strategy, the best reply to a player applying a partner strategy is to also apply a partner strategy. We see that condition (9) is equivalent to $(\pi_{\mathrm{I}} \geq R) \Rightarrow (\pi_{\mathrm{II}} \geq R)$, and due to Lemma 2 we get $\pi_{\mathrm{II}} \geq R$ implies $\pi_{\mathrm{I}} = \pi_{\mathrm{II}} = R$. This means that a player with a partner strategy will always get the payoff R, if the co-player applies a strategy that results in the best outcome for the co-player.

4 Space of strategies

We have defined a partner and a competitive strategy, but only in the general case. If the strategy classes exist, we want to find the strategies $\mathbf{p}$ for which the constraints from Def. 2 yield. Therefore we will derive two propositions, with the help of Lemma 1.

Proposition 1 ([1], Prop. 1). *For a player* I *with a nice memory-one strategy* $\mathbf{p}$, *the following are equivalent:*

(i) $\mathbf{p}$ *is a partner strategy;*

(ii) *If the co-player uses either AllD or the strategy* $(0,1,1,1;0)$, *then* $\pi_{\mathrm{II}} < R$;

(iii) *The two inequalities* $B_1 < 0$ *and* $B_2 < 0$ *hold, with*

$$\begin{aligned} B_1 &= \delta(T-R)p_{DD} - \delta(R-P)(1-p_{CD}) + (1-\delta)(T-R) \\ B_2 &= \delta(T-R)p_{DC} - \delta(R-S)(1-p_{CD}) + (1-\delta)(T-R). \end{aligned} \tag{11}$$

For example, we can compute the constraints of δ to let TFT $(1,0,1,0;1)$ and $WSLS$ $(1,0,0,1;1)$ be partner strategies. We begin with $B_1 < 0$ for a player with a TFT strategy.

$$\begin{aligned} B_1 &< 0 \\ \delta(T-R)p_{DD} - \delta(R-P)(1-p_{CD}) + (1-\delta)(T-R) &< 0 \\ \delta(T-R)\cdot 0 - \delta(R-P)(1-0) + (1-\delta)(T-R) &< 0 \\ \delta P - \delta R + T - R - \delta T + \delta R &< 0 \\ \delta(T-P) &> T-R \\ \delta &> \frac{T-R}{T-P} \end{aligned}$$

We compute the other constraints in a similar way. This implies that TFT is a partner strategy if and only if $\delta > \frac{T-R}{T-P}$ and $\delta > \frac{T-R}{R-S}$, whereas WSLS is a partner strategy if and only if $\delta > \frac{T-R}{R-P}$ and $\delta > \frac{T-R}{T-S}$.

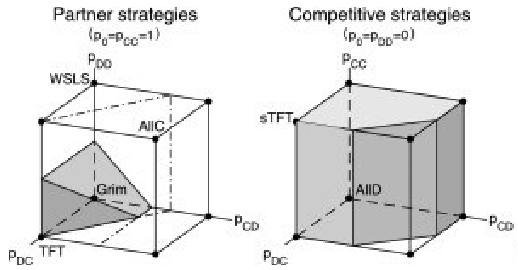

Fig. 2. Here we see the space of partner strategies and competitive strategies. The grey block is the set of strategies such that you are playing the strategy that is mentioned. We set $\delta = \frac{2}{3}$, and $T = 5, R = 3, P = 1, S = 0$.

Proposition 2 ([1], Prop. 2)**.** *Suppose player* I *applies the memory-one strategy* $\mathbf{p}$*. Then the following are equivalent:*

(i) $\mathbf{p}$ *is a competitive strategy;*

(ii) *If the co-player uses either AllD or the strategy* $(0, 0, 0, 1; 0)$*, then* $\pi_{\mathrm{I}} \geq \pi_{\mathrm{II}}$*;*

(iii) *The entries of* $\mathbf{p}$ *satisfy* $p_0 = p_{DD} = 0$ *and* $\delta(p_{CD} + p_{DC}) \leq 1$*.*

The proof of Proposition 1 is shown in Appendix B, and the proof of Proposition 2 is shown in Appendix C.

With the help of these propositions, we see that the strategy classes exist. The space of partner strategies and competitive strategies is shown in Fig. 2. We see that the grey block is bounded by the constraints given in Proposition 1 for partner strategies and the constraints given in Proposition 2 for competitive strategies. So this figure shows the strategies you can apply such that (9) yields for the partner strategies, and (10) yields for the competitive strategies. With these strategies you can enforce the payoff of your co-player.

5 ZD strategies

We have now found some strategies for which you can enforce the payoff of your co-player. In this paragraph we will be even more specific; we can enforce a linear relationship between the payoffs from player I and player II. First we define what a zero-determinant strategy is.

Definition 3 ([1], Def. 4)**.** A memory-one strategy $\mathbf{p}$ is said to be a *ZD strategy* if there exist constants α, β, γ such that

$$\delta\tilde{\mathbf{p}} = \alpha\mathbf{g}_{\mathrm{I}} + \beta\mathbf{g}_{\mathrm{II}} + (\gamma - (1-\delta)p_0)\,\mathbf{1} + \mathbf{g_0}. \tag{12}$$

With this definition we can get a linear relationship between the payoffs of both players.

Proposition 3 ([1], Prop. 5)**.** *Let* $\delta < 1$*, and suppose player* I *applies a memory-one strategy* $\mathbf{p}$ *satisfying Eq. (12). Then, irrespective of the strategy of the co-player,*

$$\alpha\pi_{\mathrm{I}} + \beta\pi_{\mathrm{II}} + \gamma = 0. \tag{13}$$

The same relation holds for $\delta = 1$*, provided that the payoffs* π_{I} *and* π_{II} *exist.*

The proof of Proposition 3 is shown in Appendix D. It's not really clear yet that there is a linear relationship between both payoffs. That's why we will introduce new parameters: $\alpha = \phi\chi$, $\beta = -\phi$, and $\gamma = \phi\kappa(1-\chi)$. This can be put in (12), and we get

$$\delta\tilde{\mathbf{p}} = \phi\chi\mathbf{g}_{\mathrm{I}} - \phi\mathbf{g}_{\mathrm{II}} + (\phi\kappa(1-\chi) - (1-\delta)p_0)\,\mathbf{1} + \mathbf{g_0}$$
$$\delta\tilde{\mathbf{p}} = \phi\left[(1-\chi)(\kappa\mathbf{1} - \mathbf{g}_{\mathrm{I}}) + (\mathbf{g}_{\mathrm{I}} - \mathbf{g}_{\mathrm{II}})\right] - (1-\delta)p_0\mathbf{1} + \mathbf{g_0} \tag{14}$$

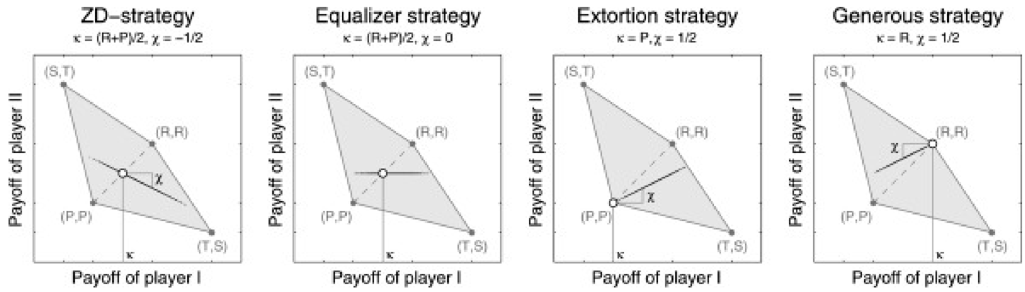

Fig. 3. Schematic payoffs according to (15), where the grey area is the set of possible payoffs. Here the strategy of player I is fixed, and the strategy of player II is the mean of 1000 random memory-one strategies. The white dots are the expected payoffs for player I and player II, if player I plays the strategy that is mentioned.
From (15) we see that the payoffs when a ZD-strategy is played, lie on a line where the slope is equal to χ and where they intersect at κ. For equalizer strategies we assume that $\chi = 0$, which means that the payoff of player II is equal to κ, independent of his strategy. For extortion strategies we assume that $\kappa = P$ and $0 < \chi < 1$, and for generous strategies we assume that $\kappa = R$ and $0 < \chi < 1$. We set $\delta = \frac{4}{5}$, and $T = 5, R = 3, P = 1, S = 0$.

We can now rewrite (13) as

$$\begin{aligned} &\phi\chi\pi_{\mathrm{I}} - \phi\pi_{\mathrm{II}} + \phi\kappa(1-\chi) = 0 \\ &\chi(\pi_{\mathrm{I}} - \kappa) - \pi_{\mathrm{II}} + \kappa = 0 \\ &\pi_{\mathrm{II}} - \kappa = \chi(\pi_{\mathrm{I}} - \kappa) \end{aligned} \tag{15}$$

The schematic representation of (15) is shown in Fig. 3. We also define a few other strategies beside a ZD-strategy, where we have restrictions for κ and χ.

Players cannot always enforce the co-players payoff with (15). The equation depends on $\tilde{\mathbf{p}}$, so κ, χ and ϕ need to have some restrictions. Therefore we have the following definition.

Definition 4 ([1], Def. 5)**.** For a given δ, we call a payoff relationship $(\kappa, \chi) \in \mathbb{R}^2$ *enforceable* if there are $\chi \in \mathbb{R}$ and $p_0 \in [0, 1]$ such that each entry of the continuation vector $\tilde{\mathbf{p}}$ according to (14) is in $[0, 1]$. We refer to the set of all enforceable payoff relationships as $\mathcal{E}_\delta$.

With this definition we get the following proposition.

Proposition 4 ([1], Prop. 6)**.**

(i) *The set of enforceable payoff relationships is monotonically increasing in the discount factor: if $\delta' \leq \delta''$, then $\mathcal{E}_{\delta'} \subseteq \mathcal{E}_{\delta''}$.*

(ii) *There is a $\delta < 1$ such that $(\kappa, \chi) \in \mathcal{E}_\delta$ if and only if $-1 < \chi < 1$ and*

$$\max\left\{P, \frac{S - T\chi}{1-\chi}\right\} \leq \kappa \leq \min\left\{R, \frac{T - S\chi}{1-\chi}\right\}, \tag{16}$$

with at least one inequality (16) in being strict.

The proof of Proposition 4 is shown in Appendix E. Due to Definition 4 and the first part of Proposition 4 we see that (15) is easier to enforce when δ is small, so when the players are patient. In the second part of Proposition 4, we found restrictions for κ. With those restrictions we can make a graph with all enforceable pairs (κ, χ). This is shown in Fig 4. We can also see for which restrictions for κ and χ the strategies from Fig. 3 yield.

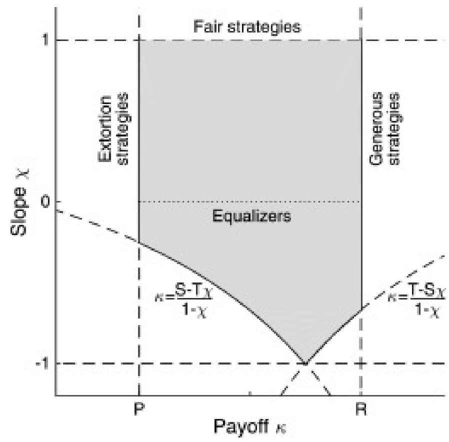

Fig. 4. Here the grey area represents all pairs (κ, χ) that are enforceable according to Proposition 4, when δ is close enough to one. In this figure we see the strategies introduced in Fig. 3, with the restrictions for κ and χ. A new strategy here is the fair strategy, where $\chi = 1$, such that $\pi_{II} - \kappa = \pi_I - \kappa$. This strategy only exist when $\delta = 1$. Again we set $T = 5, R = 3, P = 1, S = 0$.

6 Conclusion

In Section 2 we began with some basic notations, working from the one-shot prisoner's dilemma to the iterated prisoner's dilemma. We derived a few vector notations, and with those we got to Lemma 1.

In Section 3 we differentiated between nice and cautious people, and defined a partner and a competitive strategy. A schematic representation of the potential payoffs for player I and player II was shown in Fig. 1. We used two propositions in Section 4 to prove that those strategy classes exist. With the propositions we've shown the space of partner strategies and competitive strategies. From Fig. 2 we see that, for a certain space of strategies, we can enforce that either $\pi_I < R \Rightarrow \pi_{II} < R$ or $\pi_I \geq \pi_{II}$ yields.

In Section 5 we first defined zero-determinant strategies. After that, we introduced a few parameters, in order to get to (15). From that equation we see that there is a linear relationship between π_I and π_{II}, shown in Fig. 3. With the restrictions in Proposition 4, we made a graph with all enforceable pairs (κ, χ), shown in Fig. 4. It is important to note that we didn't make any assumptions on the payoff values, so these results can also be used for other strategic games.

In conclusion we discovered how much control player I has over the outcome of the game and thus the payoffs of player I and player II, regardless of the strategy of player II.

A Proof of Lemma 1

Proof. We begin with the first part of the Lemma. Suppose $\delta < 1$, and let $q_{\mathrm{I}}(t)$ be the probability that player I cooperates in round t. Then $q_{\mathrm{I}}(t) = (v_{CC}(t), v_{CD}(t), 0, 0) = \mathbf{g_0} \cdot \mathbf{v}(t)$ and $q_{\mathrm{I}}(t+1) = (p_{CC}v_{CC}(t), p_{CD}v_{CD}(t), p_{DC}v_{DC}(t), p_{DD}v_{DD}(t)) = \tilde{\mathbf{p}} \cdot \mathbf{v}(t)$. It follows that $w(t) := \delta q_{\mathrm{I}}(t+1) - q_{\mathrm{I}}(t)$ is given by

$$\begin{aligned} w(t) &= \delta q_{\mathrm{I}}(t+1) - q_{\mathrm{I}}(t) \\ &= \delta\left(\tilde{\mathbf{p}} \cdot \mathbf{v}(t)\right) - \mathbf{g_0} \cdot \mathbf{v}(t) \\ &= (\delta\tilde{\mathbf{p}} - \mathbf{g_0}) \cdot \mathbf{v}(t). \end{aligned} \tag{17}$$

Multiplying each $w(t)$ by $(1-\delta)\delta^t$ and summing up over $t = 0, \ldots, \tau$ yields

$$\begin{aligned} (1-\delta)\sum_{t=0}^{\tau} \delta^t w(t) &= (1-\delta)\left(\delta q_{\mathrm{I}}(1) - q_{\mathrm{I}}(0) + \delta^2 q_{\mathrm{I}}(2) - \delta q_{\mathrm{I}}(1) \cdots + \delta^{\tau+1} q_{\mathrm{I}}(\tau+1) - \delta^{\tau} q_{\mathrm{I}}(\tau)\right) \\ &= (1-\delta)\delta^{\tau+1} q_{\mathrm{I}}(\tau+1) - (1-\delta) q_{\mathrm{I}}(0). \end{aligned} \tag{18}$$

When $\tau \to \infty$, we see that the first part of the equation goes to 0, since $\delta^{\tau+1} \to 0$. The second part of the equation goes to p_0, since $q_{\mathrm{I}}(0)$ is the probability that player I cooperates in round 0, which is p_0. So

$$(1-\delta)\sum_{t=0}^{\tau} \delta^t w(t) \to -(1-\delta)p_0. \tag{19}$$

On the other hand, due to equation (3) and (9),

$$(1-\delta)\sum_{t=0}^{\tau} \delta^t w(t) = (1-\delta)\sum_{t=0}^{\tau} \delta^t (\delta\tilde{\mathbf{p}} - \mathbf{g_0}) \cdot \mathbf{v}(t) \to (\delta\tilde{\mathbf{p}} - \mathbf{g_0}) \cdot \mathbf{v} \tag{20}$$

As both limits need to coincide, we have confirmed equation (7).

For the case without discounting, we have an analogous calculation as in equation (10).

$$\frac{1}{\tau+1}\sum_{t=0}^{\tau} w(t) = \frac{1}{\tau+1}\sum_{t=0}^{\tau} (q_{\mathrm{I}}(t+1) - q_{\mathrm{I}}(t)) \to 0. \tag{21}$$

This holds since $\frac{1}{\tau+1} \to 0$ when $\tau \to \infty$. Equation (12) becomes

$$\frac{1}{\tau+1}\sum_{t=0}^{\tau} w(t) = \frac{1}{\tau+1}\sum_{t=0}^{\tau} (\tilde{\mathbf{p}} - \mathbf{g_0}) \cdot \mathbf{v}(t). \tag{22}$$

It follows that the limit of $\frac{1}{\tau+1}\sum_{t=0}^{\tau}(\tilde{\mathbf{p}} - \mathbf{g_0}) \cdot \mathbf{v}(t)$ for $\tau \to \infty$ exists and equals zero. □

B Proof of Proposition 1

Proof.

(i) ⇒ (ii) Assume to the contrary that $\pi_{\mathrm{II}} \geq R$. Then the definition of partner strategies implies that $\pi_{\mathrm{I}} = \pi_{\mathrm{II}} = R$. Since all players use memory-one strategies, this would require that everyone cooperates after mutual cooperation, which is neither true for *AllD* $= (0,0,0,0;0)$ nor for the strategy $(0,1,1,1;0)$.

(ii) ⇒ (iii) Against a player using a nice memory-one strategy $\mathbf{p}$ (with $p_0 = p_{CC} = 1$), the payoff of an *AllD* co-player is given by

$$\begin{aligned}\pi_{\mathrm{II}} &= \mathbf{g}_{\mathrm{II}} \cdot \mathbf{v} \\ &= Rv_{CC} + Tv_{CD} + Sv_{DC} + Pv_{DD} \end{aligned} \tag{23}$$

We know that the co-player is playing an *AllD*-strategy $(0,0,0,0;0)$, so the co-player will never cooperate, thus $v_{CC} = v_{DC} = 0$. In order to establish the payoff of the co-player, we have to determine v_{CD} and v_{DD}. We will do this by using (5).

$$\begin{aligned}\mathbf{v} &= (1-\delta)\mathbf{v}(0) \cdot (I_4 - \delta M)^{-1} \\ &= (0, 1-\delta, 0, 0) \cdot \begin{pmatrix} 1 & \delta p_{CC} & 0 & \delta(1-p_{CC}) \\ 0 & 1-\delta p_{CD} & 0 & \delta(1-p_{CD}) \\ 0 & \delta p_{DC} & 1 & \delta(1-p_{DC}) \\ 0 & \delta p_{DD} & 0 & 1-\delta(1-p_{DD}) \end{pmatrix}^{-1}\end{aligned}$$

If we calculate the inverse-matrix, we get

$$\begin{aligned}v_{CD} &= (1-\delta) \cdot -\frac{-\delta(-p_{DD}+1)+1}{(1-\delta)(\delta p_{CD} - \delta p_{DD} - 1)} \\ v_{DD} &= (1-\delta) \cdot \frac{\delta(-p_{CD}+1)}{(1-\delta)(\delta p_{CD} - \delta p_{DD} - 1)}\end{aligned}$$

We put this in (23) and we get

$$\begin{aligned}\pi_{\mathrm{II}} &= Tv_{CD} + Pv_{DD} \\ &= T \cdot -\frac{(1-\delta)(-\delta(-p_{DD}+1)+1)}{(1-\delta)(\delta p_{CD} - \delta p_{DD} - 1)} + P \cdot \frac{(1-\delta)(\delta(-p_{CD}+1))}{(1-\delta)(\delta p_{CD} - \delta p_{DD} - 1)} \\ &= \frac{-T + \delta T - \delta T p_{DD} + \delta P - \delta P p_{CD}}{\delta p_{CD} - \delta p_{DD} - 1} \\ &= \frac{-(1-\delta)T - \delta T p_{DD} + \delta P - \delta P p_{CD}}{\delta p_{CD} - \delta p_{DD} - 1} \\ &= \frac{(1-\delta)T + \delta T p_{DD} + \delta P - \delta P p_{CD}}{1 + \delta(p_{DD} - p_{CD})} \end{aligned} \tag{24}$$

Now $p_{CD} < 1$, otherwise player I would always cooperate. In that case, the co-player would get payoff $T > R$, which is not possible by (ii). This means that π_{II} is also defined when $\delta = 1$.
We see that

$$\begin{aligned}B_1 &= \delta(T-R)p_{DD} - \delta(R-P)(1-p_{CD}) + (1-\delta)(T-R) \\ &= (1-\delta)T + \delta T p_{DD} + \delta P - \delta P p_{CD} - R\left(1 + \delta(p_{DD} - p_{CD})\right) \\ &= (1 + \delta(p_{DD} - p_{CD}))(\pi_{\mathrm{II}} - R) \end{aligned} \tag{25}$$

In particular, B_1 has the same sign as $\pi_{\text{II}} - R$. Due to (ii), both will be smaller than zero.
On the other hand, if the co-player uses the strategy $(0, 1, 1, 1; 0)$, we can determine the payoff in the same way as above. We get

$$\pi_{\text{II}} = \frac{(1-\delta)T + \delta S + \delta\left((1-\delta)R - S\right)p_{CD} + \delta(T+\delta R)p_{DC}}{1 + \delta^2(p_{DC} - p_{CD}) + \delta p_{DC}} \tag{26}$$

and

$$B_2 = \left(1 + \delta^2(p_{DC} - p_{CD}) + \delta p_{DC}\right)(\pi_{\text{II}} - R). \tag{27}$$

Here B_2 has also the same sign as $\pi_{\text{II}} - R$.

(iii) $\Rightarrow$ (i) Assume that $B_1 < 0$ and $B_2 < 0$, and that $\pi_{\text{II}} \geq R$. In order to proof that $\mathbf{p}$ is a partner strategy, we need to show that $\pi_{\text{I}} = \pi_{\text{II}} = R$. Due to (23), we note that $\pi_{\text{II}} \geq R$ is equivalent to

$$(T-R)v_{CD} - (R-S)v_{DC} - (R-P)v_{DD} \geq 0. \tag{28}$$

Using the linear equation $\mathbf{1} \cdot \mathbf{v} = v_{CC} + v_{CD} + v_{DC} + v_{DD} = 1$ and (i) of Lemma 1 with $p_0 = p_{CC} = 1$, we get

$$\begin{aligned}
&(\delta-1)v_{CC} + (\delta p_{CD} - 1)v_{CD} + \delta p_{DC}v_{DC} + \delta p_{DD}v_{DD} = -(1-\delta)\\
&(\delta-1)v_{CC} + (\delta p_{CD} - 1)v_{CD} + \delta p_{DC}v_{DC} + \delta p_{DD}v_{DD} = (\delta-1)\cdot 1\\
&(\delta-1)v_{CC} + (\delta p_{CD} - 1)v_{CD} + \delta p_{DC}v_{DC} + \delta p_{DD}v_{DD}\\
&\quad = (\delta-1)(v_{CC} + v_{CD} + v_{DC} + v_{DD})\\
&(\delta-1)v_{CD} - (1-\delta p_{CD})v_{CD} = \delta p_{DC}v_{DC} + \delta p_{DD}v_{DD} - (\delta-1)(v_{DC} + v_{DD})\\
&(1-p_{CD})\delta v_{CD} = \left(1-(1-p_{DC})\delta\right)v_{DC} + \left(1-(1-p_{DD})\delta\right)v_{DD}\\
&v_{CD} = \frac{\left(1-(1-p_{DC})\delta\right)v_{DC} + \left(1-(1-p_{DD})\delta\right)v_{DD}}{(1-p_{CD})\delta}
\end{aligned} \tag{29}$$

We know that $B_1 < 0$, so $p_{CD} < 1$ which implies that the denominator of v_{CD} is positive. Plugging (29) into (28) gives us

$$\begin{aligned}
0 &\leq (T-R)\cdot\frac{\left(1-(1-p_{DC})\delta\right)v_{DC} + \left(1-(1-p_{DD})\delta\right)v_{DD}}{(1-p_{CD})\delta}\\
&\quad - (R-S)v_{DC} - (R-P)v_{DD}\\
&\leq (T-R)\left(\left(1-(1-p_{DC})\delta\right)v_{DC} + \left(1-(1-p_{DD})\delta\right)v_{DD}\right)\\
&\quad - (R-S)(1-p_{CD})\delta v_{DC} - (R-P)(1-p_{CD})\delta v_{DD}\\
&\leq \left((T-R)(1-(1-p_{DC})\delta) - (R-S)(1-p_{CD})\delta\right)v_{DC}\\
&\quad + \left((T-R)(1-(1-p_{DD})\delta) - (R-P)(1-p_{CD})\delta\right)v_{DD}\\
&\leq \left(T - R - \delta T + \delta R + \delta T p_{DC} - \delta R p_{DC} - \delta R + \delta S + \delta R p_{CD} - \delta S p_{CD}\right)v_{DC}\\
&\quad + \left(T - R - \delta T + \delta R + \delta T p_{DD} - \delta R p_{DD} - \delta R + \delta P + \delta R p_{CD} - \delta P p_{CD}\right)v_{DD}\\
&\leq \left(\delta(T-R)p_{DC} - \delta(R-S)(1-p_{CD}) + (1-\delta)(T-R)\right)v_{DC}\\
&\quad + \left(\delta(T-R)p_{DD} - \delta(R-P)(1-p_{CD}) + (1-\delta)(T-R)\right)v_{DD}
\end{aligned}$$

This shows, due to (11) that $\pi_{\text{II}} \geq R$ if and only if

$$B_2 v_{DC} + B_1 v_{DD} \geq 0 \tag{30}$$

Thus the assumptions $B_1 < 0$ and $B_2 < 0$ show that $v_{DC} = v_{DD} = 0$, and by that $v_{CD} = 0$. We can conclude that $v_{CC} = 1$, so $\pi_{\text{I}} = \pi_{\text{II}} = R$ and $\mathbf{p}$ is a partner strategy. □

C Proof of Proposition 2

Proof.

(i) ⇒ (ii) Since $\mathbf{p}$ is competitive, we know that $\pi_{\mathrm{I}} \geq \pi_{\mathrm{II}}$ against any co-player.

(ii) ⇒ (iii) If player π_{II} applies *AllD*, then the payoffs are given by

$$\begin{aligned} \pi_{\mathrm{I}} - \pi_{\mathrm{II}} &= (\mathbf{g}_{\mathrm{I}} - \mathbf{g}_{\mathrm{II}}) \cdot \mathbf{v} \\ &= (S-T)v_{CD} + (T-S)v_{DC} \end{aligned} \tag{31}$$

We know that the co-player is playing an *AllD*-strategy $(0,0,0,0;0)$, so the co-player will never cooperate, thus $v_{CC} = v_{DC} = 0$. In order to determine $\pi_{\mathrm{I}} - \pi_{\mathrm{II}}$, we have to calculate v_{CD}. We will do this by using (5).

$$\begin{aligned} \mathbf{v} &= (1-\delta)\mathbf{v}(0) \cdot (I_4 - \delta M)^{-1} \\ &= (0, (1-\delta)p_0, 0, (1-\delta)(1-p_0)) \cdot \begin{pmatrix} 1 & \delta p_{CC} & 0 & \delta(1-p_{CC}) \\ 0 & 1-\delta p_{CD} & 0 & \delta(1-p_{CD}) \\ 0 & \delta p_{DC} & 1 & \delta(1-p_{DC}) \\ 0 & \delta p_{DD} & 0 & 1-\delta(1-p_{DD}) \end{pmatrix}^{-1} \end{aligned}$$

If we calculate the inverse-matrix, we get

$$\begin{aligned} v_{CD} &= (1-\delta)p_0 \cdot -\frac{-\delta(-p_{DD}+1)+1}{(1-\delta)(\delta p_{CD} - \delta p_{DD} - 1)} \\ &\quad + (1-\delta)(1-p_0) \cdot \frac{\delta p_{DD}}{(1-\delta)(\delta p_{CD} - \delta p_{DD} - 1)} \\ &= \frac{p_0(\delta p_{DD} - \delta + 1)}{1+\delta(p_{DD} - p_{CD})} - \frac{\delta p_{DD}(1-p_0)}{1+\delta(p_{DD} - p_{CD})} \\ &= \frac{-\delta p_0 + \delta p_{DD} + p_0}{1+\delta(p_{DD} - p_{CD})} \end{aligned}$$

We put this in (31) and get

$$\begin{aligned} \pi_{\mathrm{I}} - \pi_{\mathrm{II}} &= (T-S)v_{CD} \\ &= (S-T) \cdot \frac{-\delta p_0 + \delta p_{DD} + p_0}{1+\delta(p_{DD} - p_{CD})} \\ &= -\frac{(T-S)\left((1-\delta)p_0 + \delta p_{DD}\right)}{1+\delta(p_{DD} - p_{CD})} \end{aligned} \tag{32}$$

We see that $\pi_{\mathrm{I}} \geq \pi_{\mathrm{II}}$ implies $p_0 = p_{DD} = 0$.

On the other hand, if the co-player uses the strategy $(0,0,0,1;0)$, we can determine the payoff in the same way as above. We get

$$\pi_{\mathrm{I}} - \pi_{\mathrm{II}} = \frac{\delta\,(T-S)\,(1-\delta(p_{CD} + p_{DC}))}{1+\delta\,(1-(1+\delta)p_{CD} + \delta p_{DC})} \tag{33}$$

From this we see that $\pi_{\mathrm{I}} - \pi_{\mathrm{II}} \geq 0$ if and only if $\delta\,(T-S)\,(1-\delta(p_{CD}+p_{DC})) \geq 0$, which is equivalent to $\delta(p_{CD}+p_{DC}) \leq 1$.

(iii) $\Rightarrow$ (i) Assume that $p_{DD} = p_0 = 0$ and $\delta(p_{CD} + p_{DC}) \leq 1$. Using (i) of Lemma 1, we get

$$
\begin{aligned}
&(\delta p_{CC} - 1)v_{CC} + (\delta p_{CD} - 1)v_{CD} + \delta p_{DC} v_{DC} = 0 \\
&\delta p_{DC} v_{DC} = (1 - \delta p_{CC})v_{CC} + (1 - \delta p_{CD})v_{CD} \\
&(1 - \delta p_{CD})v_{DC} \geq (1 - \delta p_{CC})v_{CC} + (1 - \delta p_{CD})v_{CD} \\
&(1 - \delta p_{CD})(v_{DC} - v_{CD}) \geq (1 - \delta p_{CC})v_{CC} \\
&v_{DC} - v_{CD} \geq \frac{(1 - \delta p_{CC})v_{CC}}{1 - \delta p_{CD}} \\
&v_{DC} \geq v_{CD} + \frac{(1 - \delta p_{CC})v_{CC}}{1 - \delta p_{CD}} \geq v_{CD} \qquad (34)
\end{aligned}
$$

Since $v_{DC} \geq v_{CD}$, $\pi_{\mathrm{I}} - \pi_{\mathrm{II}} = (T - S)(v_{DC} - v_{CD}) \geq 0$. So $\mathbf{p}$ is a competitive strategy. □

D Proof of Proposition 3

Proof. Using Def. 3 and Lemma 1, with the identities $\pi_{\mathrm{I}} = \mathbf{g}_{\mathrm{I}} \cdot \mathbf{v}$, $\pi_{\mathrm{II}} = \mathbf{g}_{\mathrm{II}} \cdot \mathbf{v}$, and $1 = \mathbf{1} \cdot \mathbf{v}$, we get

$$
\begin{aligned}
\delta\tilde{\mathbf{p}} &= \alpha\mathbf{g}_{\mathrm{I}} + \beta\mathbf{g}_{\mathrm{II}} + (\gamma - (1-\delta)p_0)\,\mathbf{1} + \mathbf{g_0} \\
\delta\tilde{\mathbf{p}} - \mathbf{g_0} &= \alpha\mathbf{g}_{\mathrm{I}} + \beta\mathbf{g}_{\mathrm{II}} + (\gamma - (1-\delta)p_0)\,\mathbf{1} \\
(\delta\tilde{\mathbf{p}} - \mathbf{g_0}) \cdot \mathbf{v} &= \alpha(\mathbf{g}_{\mathrm{I}} \cdot \mathbf{v}) + \beta(\mathbf{g}_{\mathrm{II}} \cdot \mathbf{v}) + (\gamma - (1-\delta)p_0)\,\mathbf{1} \cdot \mathbf{v} \\
-(1-\delta)p_0 &= \alpha\pi_{\mathrm{I}} + \beta\pi_{\mathrm{II}} + \gamma - (1-\delta)p_0 \\
\alpha\pi_{\mathrm{I}} + \beta\pi_{\mathrm{II}} + \gamma &= 0.
\end{aligned}
$$

□

E Proof of Proposition 4

Proof.

(i) According to Def. 4, $(\kappa, \chi) \in \mathcal{E}_\delta$ if and only if there are $\chi \in \mathbb{R}$ and $p_0 \in [0,1]$ such that $\mathbf{0} \leq \delta\tilde{\mathbf{p}} \leq \delta\mathbf{1}$, which is equal to $\mathbf{0} \leq \phi\,[(1-\chi)(\kappa\mathbf{1} - \mathbf{g}_{\mathrm{I}}) + (\mathbf{g}_{\mathrm{I}} - \mathbf{g}_{\mathrm{II}})] - (1-\delta)p_0\mathbf{1} + \mathbf{g_0} \leq \delta\mathbf{1}$. We will differentiate between the possible outcomes of the game.

$$
\begin{aligned}
0 \leq \phi\,[(1-\chi)(\kappa - R) + 0] - (1-\delta)p_0 + 1 &\leq \delta \\
-(1 - (1-\delta)p_0) \leq \phi(1-\chi)(\kappa - R) &\leq -((1-\delta)(1-p_0)) \\
(1-\delta)(1-p_0) \leq \phi(1-\chi)(R - \kappa) &\leq 1 - (1-\delta)p_0 \qquad (35a)
\end{aligned}
$$

$$
\begin{aligned}
0 \leq \phi\,[(1-\chi)(\kappa - S) + S - T] - (1-\delta)p_0 + 1 &\leq \delta \\
-(1 - (1-\delta)p_0) \leq \phi\,[(1-\chi)(\kappa - S) + S - T] &\leq -((1-\delta)(1-p_0)) \\
(1-\delta)(1-p_0) \leq \phi\,[(1-\chi)(S - \kappa) + T - S] &\leq 1 - (1-\delta)p_0 \qquad (35b)
\end{aligned}
$$

$$
\begin{aligned}
0 \leq \phi\,[(1-\chi)(\kappa - T) + T - S] - (1-\delta)p_0 &\leq \delta \\
(1-\delta)p_0 \leq \phi\,[(1-\chi)(\kappa - T) + T - S] &\leq \delta + (1-\delta)p_0 \qquad (35c)
\end{aligned}
$$

$$
\begin{aligned}
0 \leq \phi\,[(1-\chi)(\kappa - P) + 0] - (1-\delta)p_0 &\leq \delta \\
(1-\delta)p_0 \leq \phi(1-\chi)(\kappa - P) &\leq \delta + (1-\delta)p_0 \qquad (35d)
\end{aligned}
$$

From these equations we see that the left hand side is monotonically decreasing in δ, whereas the right hand side is monotonically increasing in δ. This means that if the conditions (35a) – (35d) are satisfied for some $\delta' \leq 1$, then they are also satisfied for any $\delta'' \geq \delta'$.

(ii) (⇒) Suppose $(\kappa, \chi) \in \mathcal{E}_\delta$, then the conditions (35a) – (35d) hold for $\phi \in \mathbb{R}$ and $p_0 \in [0,1]$. We sum up the first inequality of (35a) with the first inequality of (35d), which gives

$$\begin{aligned}(1-\delta)(1-p_0) + (1-\delta)p_0 &\leq \phi(1-\chi)(R-\kappa) + \phi(1-\chi)(\kappa - P) \\ 1-\delta &\leq \phi(1-\chi)(R-P)\end{aligned} \tag{36}$$

If we do the same for the first inequality of (35b) with the first inequality of (35c), we get

$$\begin{aligned}(1-\delta)(1-p_0) + (1-\delta)p_0 &\leq \phi\left[(1-\chi)(S-\kappa) + T - S\right] + \phi\left[(1-\chi)(\kappa - T) + T - S\right] \\ (1-\delta) &\leq \phi\left[S - \chi S + \chi T - T + 2T - 2S\right] \\ (1-\delta) &\leq \phi\left[T - S + \chi T - \chi S\right] \\ (1-\delta) &\leq \phi(1+\chi)(T-S)\end{aligned} \tag{37}$$

In particular, $0 < \phi(1+\chi)$ and $0 < \phi(1-\chi)$, and therefore $\phi > 0$ and $-1 < \chi < 1$. Applying this to the conditions (35a) – (35d), we get

$$\begin{aligned}0 &\leq \phi(1-\chi)(\kappa - R) \\ 0 &\leq \phi\left[(1-\chi)(S-\kappa) + T - S\right] \\ 0 &\leq \phi\left[(1-\chi)(\kappa - T) + T - S\right] \\ 0 &\leq \phi(1-\chi)(\kappa - P)\end{aligned} \tag{38}$$

with $\phi > 0$ and $\chi < 1$. This gives us respectively $\kappa \leq R$, $\kappa \leq \frac{T - S\chi}{1-\chi}$, $\kappa \geq \frac{S - T\chi}{1-\chi}$ and $\kappa \geq P$, which are exactly the restrictions in (16). If none of the restrictions was strict, then (35a) and (35b) would require $p_0 = 1$, whereas (35c) and (35d) would require $p_0 = 0$.

(⇐) Assume that $-1 < \chi < 1$ and $\max\left\{P, \frac{S-T\chi}{1-\chi}\right\} \leq \kappa < \min\left\{R, \frac{T-S\chi}{1-\chi}\right\}$. Then we know that the inequalities in (38) hold for $\phi > 0$, with the first two inequalities being strict. We need to show that the inequalities in (35a) – (35d) are satisfied. Therefore we choose ϕ sufficiently small such that each term on the right hand side of (38) is bounded by $\frac{1}{2}$. If we then set $p_0 = 0$ and choose δ close to one, we see that the inequalities in (35a) – (35d) are satisfied. Similarly, if we set $p_0 = 1$, we get the same for $\kappa = \min\left\{R, \frac{T-S\chi}{1-\chi}\right\}$. □

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