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Master Thesis

Graviton 1-loop corrections from massive non-minimally coupled scalar fields in de-Sitter background

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Abstract

Quantum one-loop corrections to various propagators on a de-Sitter background have been extensively studied by Woodard, Park, Prokopec and collaborators . In the current master thesis, we investigate the one-loop corrections to the graviton propagator due to the presence of massive non-minimally coupled scalar fields. Our background geometry is the open conformal coordinate patch of de-Sitter spacetime. We define the graviton to be a small fluctuation around the de-Sitter background while the massive non-minimally coupled scalar propagator is the so-called Chernikov-Tagirov propagator. Two Feynman diagrams contribute to the self-energy of the graviton. The quartic contribution, which corresponds to local contribution, has already been renormalized by R. Koelewijn ("Gravitons on de Sitter modified by quantum fluctuations of a nonminimally coupled massive scalar. MS thesis. 2017." [1]). We focus our attention to the cubic contribution (non-local) which is proportial to the TT correlator computed by R.Koelewijn. Our aim is to reorganize the expression for the TT correlator in such a way that it will allow us to renormalize it, using dimensional regularization scheme. In particular, we localize the divergences that appear onto local δ-function terms, and subtract them by using the local counterterms R^2 and C^2 introduced in the 80's by G. t'Hooft and M.Veltman. Since our final goal is to quantum correct the linearized Einstein's equations, the final form of the graviton self energy should contain only terms which are integrable in $D = 4$ dimensions and finite.

Contents

Introduction

During the last 3 decades, cosmology undergoes a scientific revolution. It has entered a new era, the so-called "Precision Cosmology" era [2][3] in which ground and space based experiments provide us with a huge amount of data. Under those circumstances, cosmological models are ruled out by observations, while others are favoured. This situation has led cosmology to progress rapidly, transforming the field from a "speculative" into a "hard" science.

Primordial inflation, a short period in the cosmic history during which the Universe expanded exponentially, is an example of a theory whose parameters are highly constrained by observations [4][5]. Inflation was introduced during the 90's by Guth, Starobinski and Linde [6][7][8] in order to solve some of the Big Bang puzzles. The importance of the specific model is related to the fact that, nowadays, we believe that the large structure we observe in the Universe (stars, galaxies, clusters of galaxies etc) have their origin in the physics of early Universe. In particular, the evolution of the quantum fluctuations produced during inflation, create density fluctuations as well as fluctuations in the metric in the period of decoupling, roughly 380.000 years after the Big Bang, which we observe today as temperature anisotropies in the Cosmic Microwave Background map (CMB).
This situation, i.e the direct contact of these quantum effects with the observations, motivates cosmologists to extensively study the quantum phenomena during inflation. Moreover, it is known that inflation produces a vast ensemble of particles. This idea can be traced back to the first half of the twentieth centure, where Erwin Schrodinger suggested that the expansion of the Universe can spontaneously create particles out of the vacuum [9]. Parker and Grishchuk were the first to show that for an accelerating expanding Universe, the effect of cosmological particle production is maximized from massless and non-conformally coupled fields (gravitons and scalars) [10][11][12][13]. We believe that these particles are the ones that comprise the primordial scalar and tensor fluctuations. The former has already been observed while for the latter, a discovery is anticipated within the next years.

Gravitons and massless minimally coupled scalars (MMC) produced during inflation, interact with themselves as well as with other particles that exist this period. Woodard, Park, Prokopec et al, have extensively studied 1-loop corrections to various propagators on de-Sitter background [14][15][16][17][18] [19][20][21][22][23][24][25]. Some of their work is

Figure 1: Taken from https://www.spacetelescope.org/. Large scale structure. In this picture, we can see roughly 10000 galaxies of different ages and distances from us (redshift). We believe that the initial conditions which seeded the formation of stars, galaxies and all the structure we observe today, have their origin in the very early Universe. In particular, inflation stretched the primordial quantum fluctuations which at later times induced density fluctuations. The latter, grew via gravitational instability into this beautiful structure depicted above.

- Interactions between gravitons and MMC scalar fields [14][15]
- MMC scalar quantum electrodynamics [26][27]
- MMC scalar fields with a quartic self interaction [28]

Woodard and Park [14][15] studied the effect of massless minimally coupled (MMC) scalar fields on the propagation of free gravitons during inflation. In particular, they computed the renormalized 1-loop contribution of MMC scalars to the graviton self-energy $\Sigma^{(Ren)}_{\mu\nu\rho\sigma}(x;x')$ and used the result in order to quantum correct the linearized Einstein's equation using the so-called Schwinger-Keldysh formalism [29]. Based on this computation, Woodard, Park, Leonard and Prokopec [21] showed that the presence of MMC scalar fields during the period of primordial inflation do not affect the propagation of gravitons at 1 loop order.

In [16], Woodard, Park and Prokopec , using the renormalized result for the graviton self energy, computed the quantum corrections to gravitational potentials caused by a static massive particle. They found the important result that the gravitational potentials exhibit secular effects which grow with time and can eventually become large at late times. The form of the potentials at late times is given in [16]

$$\phi_{dS}(x) = -\frac{GM}{\alpha r}\left[1+\frac{\hbar G}{20\pi c^3}\frac{1}{(\alpha r)^2}+\frac{\hbar G H^2}{\pi c^5}\left(-\frac{1}{30}\ln(\alpha)-\frac{3}{10}\ln\left(\frac{H\alpha r}{c}\right)\right)+O\left(G^2,\frac{1}{\alpha^3}\right)\right]$$

$$\psi_{dS}(x) = -\frac{GM}{\alpha r}\left[1-\frac{\hbar G}{60\pi c^3}\frac{1}{(\alpha r)^2}+\frac{\hbar G H^2}{\pi c^5}\left(-\frac{1}{30}\ln(\alpha)-\frac{3}{10}\ln\left(\frac{H\alpha r}{c}\right)+\frac{2H\alpha r}{3c}\right)+O\left(G^2,\frac{1}{\alpha^3}\right)\right]$$

Notice the $\propto \ln(\alpha)$ terms in the expressions above. To obtain some intuition above this result, we refer the flat space results [30][31][32]

$$\phi_{flat}(x) = -\frac{GM}{r}\left[1+\frac{\hbar G}{20\pi c^3}\frac{1}{r^2}+O(G^2)\right]$$

$$\psi_{flat}(x) = -\frac{GM}{r}\left[1-\frac{\hbar G}{60\pi c^3}\frac{1}{r^2}+O(G^2)\right]$$

Comparing the expressions for the potentials on de-Sitter with the corresponding flat space results, we conclude that the terms $\propto GH^2$ are de-Sitter corrections which, although are suppressed by a small number $\frac{\hbar G H^2}{\pi c^5} \to 10^{-10}$, they depend on $\ln(\alpha)$ which grow with time.

The previous work, done by the authors mentioned above, motivates the topic of this thesis. The idea is to generalize the results for the case of massive non minimally coupled scalar fields (MNMC). In other words, we are interested in loop corrections to the graviton propagator on a de-Sitter background, in the presence of MNMC scalars. The aim is to renormalize the graviton self-energy due to the presence of MNMC scalars, using dimensional regularization scheme [33] and by using Schwinger-Keldysh [29] formalism to solve the quantum corrected linearized Einstein's equation

$$\sqrt{-g}D^{\mu\nu\rho\sigma}h_{\rho\sigma}(x) - \int d^4x' \Sigma^{\mu\nu\rho\sigma}_{Ren}(x;x')h_{\rho\sigma}(x') = T^{\mu\nu}_{lin}(x)$$

where $D^{\mu\nu\rho\sigma}$ is the so-called Lichnerowicz operator on de-Sitter and $h_{\mu\nu}(x)$ is the graviton field. The graviton is defined as a small fluctuation around a de-Sitter background while the form of the Lichnerowicz operator can be obtained by expanding the Einstein-Hilbert action up to first order in the fluctuations $h_{\mu\nu}$. Solving the linearized Einstein equation, in terms of perturbation theory, will allow us to study whether MNMC scalar fields during the period of inflation induce a significant effect on gravitons at late times.

This thesis is organised in the following way. In chapter 1, we introduce some basic knowledge of quantum field theory in curved spacetime. In chapter 2, the classical theory of inflation will be briefly introduced. In particular, we will see how the Big Bang puzzles are solved by introducing a period of accelerating exponential expansion. In chapter 3, quantum fluctuations produced during inflation will be studied. More specifically, after briefly introducing some features of linear cosmological perturbation theory, the computation of primordial scalar and tensor power spectra will be sketched. In chapter 4, we study the effect of loop corrections to the tree level results. In particular, we quantify how large these corrections are compared to the tree level results and discuss the possibility of secular growth effects to occur (section 4.2). In chapter 5, we tackle the main issue of this thesis. Renormalization of the graviton self energy due to presence of massive non minimally coupled scalar fields during inflation. In chapter 6, we discuss our results and adress the future work which should be done. Finally, we include a rich bibliography which intends to help the reader map the field of study. Some appendices are also included in order to provide the reader with some relevant calculations performed.

Chapter 1

Quantum Field Theory In Curved Space Time

In this chapter we follow closely [34] [35] [36]

1.1 Quantum Field theory In Minkowski spacetime

We consider a classical, real scalar field in Minkowski backround. For future convenience, we will try to keep the analysis as general as possible and work in D spacetime dimensions. The action of the massive scalar field is given by

$$S_\phi = \int d^D x \left[-\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right] \tag{1.1}$$

where $\eta^{\mu\nu}$ is the usual Minkowski metric $diag[+1, -1, -1, -1]$ and m is the mass of the scalar field ϕ. The equation of motion for the dynamical field ϕ can be obtained by extremizing the action (1.1) , i.e

$$\delta S_\phi = 0 \Rightarrow \left(\Box - m^2 \right) \phi = 0 \tag{1.2}$$

where $\Box \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu$. This is the famous Klein-Gordon equation for a scalar field propagating in a flat Minkowski backround.

Equation (1.2) admits plane waves solutions, i.e

$$u_{\boldsymbol{k}}(t, \boldsymbol{x}) = A e^{i \boldsymbol{k} \boldsymbol{x} - \imath \omega t} \tag{1.3}$$

where A is a normalization constant which will be specified soon, $\boldsymbol{x}$ and $\boldsymbol{k}$ are $D-1$ dimensional vectors and the frequency ω satisfies the dispersion relation

$$\omega = \sqrt{k^2 + m^2} \tag{1.4}$$

where $k \equiv |\boldsymbol{k}|$.

Minkowski spacetime contains a huge amount of symmetry. The line element $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ is invariant under the action of Poincare group. The symmetries of the metric tensor are called Isometries. Since the coefficients of Minkowski metric $\eta_{\mu\nu}$ do not depend on time t, the vector $K = \partial_t$ generates temporal translations and satisfies the Killing equation. Therefore, $K = \partial_t$ is a Killing vector.

As a result, in Minkowski spacetime, it is possible to define positive and negative frequency modes with respect to t in a universal fashion, namely positive and negative frequency modes for all inertial observers. These modes $u_{\boldsymbol{k}}(t, \boldsymbol{x})$ and $u^*_{\boldsymbol{k}}(t, \boldsymbol{x})$ are eigenfunctions of Killing vector $K = \partial_t$, i.e

$$\partial_t u_{\boldsymbol{k}}(t, \boldsymbol{x}) = -\imath\omega u_{\boldsymbol{k}}(t, \boldsymbol{x}) \tag{1.5}$$

$$\partial_t u^*_{\boldsymbol{k}}(t, \boldsymbol{x}) = +\imath\omega u^*_{\boldsymbol{k}}(t, \boldsymbol{x}) \tag{1.6}$$

where in both equations $\omega > 0$.

Remark: The argument above does not hold for a general curved spacetime with metric $g_{\mu\nu}$. As we will see shortly, in curved spacetimes there is not such a Killing vector which generates time translations in a universal way. This subtelty will be analyzed in detail in a subsequent section.

The inner product of two different solutions ϕ_1,ϕ_2 of (1.2) is defined as :

$$(\phi_1, \phi_2) = -\imath \int_{\Sigma_t} d^{D-1}x \, (\phi_1 \partial_t \phi_2^* - \phi_2^* \partial_t \phi_1) \tag{1.7}$$

where Σ_t is a hypersurface of constant time t. The set of modes $u_{\boldsymbol{k}}(t, \boldsymbol{x})$ are orthormal and by choosing $A = [2\omega(2\pi)^{D-1}]^{-\frac{1}{2}}$, they satisfy

$$(u_{\boldsymbol{k}}, u_{\boldsymbol{k'}}) = \delta^{D-1}(\boldsymbol{k} - \boldsymbol{k'}) \tag{1.8}$$

We now proceed with the quantization of the field $\phi(t, \boldsymbol{x})$. We will use the canonical quantization approach. The field $\phi(t, \boldsymbol{x})$ and its conjugate momentum $\pi(t, \boldsymbol{x})$ are promoted to operators, $\hat{\phi}(t, \boldsymbol{x})$ and $\hat{\pi}(t, \boldsymbol{x})$ satisfying the equal time commutation relations

$$[\hat{\phi}(t, \boldsymbol{x}), \hat{\pi}(t, \boldsymbol{x'})] = \imath\delta^{D-1}(\boldsymbol{x} - \boldsymbol{x'}) \tag{1.9}$$

$$[\hat{\phi}(t, \boldsymbol{x}), \hat{\phi}(t, \boldsymbol{x'})] = 0 \tag{1.10}$$

$$[\hat{\pi}(t, \boldsymbol{x}), \hat{\pi}(t, \boldsymbol{x'})] = 0 \tag{1.11}$$

The field $\phi(t, \boldsymbol{x})$ can be expanded in terms of the complete set of mode functions $u_{\boldsymbol{k}}(t, \boldsymbol{x})$ and $u^*_{\boldsymbol{k}}(t, \boldsymbol{x})$.

$$\hat{\phi}(t, \boldsymbol{x}) = \sum_{\boldsymbol{k}} \left(\hat{a}_{\boldsymbol{k}} u_{\boldsymbol{k}}(t, \boldsymbol{x}) + \hat{a}^\dagger_{\boldsymbol{k}} u^*_{\boldsymbol{k}}(t, \boldsymbol{x}) \right) \tag{1.12}$$

where $\hat{a}_{\boldsymbol{k}}$ and $\hat{a}^{\dagger}_{\boldsymbol{k'}}$ are the creation and annihilation operators satisfying following commutation relations

$$[\hat{a}_{\boldsymbol{k}}, \hat{a}^{\dagger}_{\boldsymbol{k'}}] = \delta_{\boldsymbol{kk'}} \tag{1.13}$$

$$[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k'}}] = [\hat{a}^{\dagger}_{\boldsymbol{k}}, \hat{a}^{\dagger}_{\boldsymbol{k'}}] = 0 \tag{1.14}$$

The next natural step is to define the vacuum state $|0\rangle$. The vacuum is the state that is annihilated when we act with the operators $\hat{a}_{\boldsymbol{k}}$, i.e

$$\hat{a}_{\boldsymbol{k}}|0\rangle = 0 \tag{1.15}$$

for all the values of $\boldsymbol{k}$.

Acting once with the creation operator $\hat{a}^{\dagger}_{\boldsymbol{k}}$ on the vacuum, we get the state $|1_{\boldsymbol{k}}\rangle$. This is a one-particle state characterized by the quantum number $\boldsymbol{k}$. More generally, by acting on the vacuum with $\hat{a}^{\dagger}_{\boldsymbol{k_i}}$, where $\boldsymbol{k_i}$ for $\imath = 1, 2, ...n$ denotes different momenta, we obtain the so-called multiparticle states which are eigenstates of the number operator

$$\hat{N} = \sum_i n\hat{_{\boldsymbol{k_i}}} = \sum_i \hat{a}^{\dagger}_{\boldsymbol{k_i}} \hat{a}_{\boldsymbol{k_i}} \tag{1.16}$$

Those multiparticle states, denoted as $|n_i\rangle$, construct the Fock basis of the theory.

1.2 Quantum Field Theory In a General Curved backround

In the scope of this thesis, we will be interested in 1-loop quantum corrections to the graviton propagator during inflation. As already mentioned, the loop diagrams will contain massive scalar fields which couple non-minimally to gravity. The aim of this chapter is to briefly introduce the MNMC scalar field on a general curved backround. In a following section, we will specialize our results in the de-Sitter case. In anticipation of dimensional regularization, we will keep working in D-dimensions.

1.2.1 The Action

The action of this theory is given by

$$S_\phi = \int d^D x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{2} \xi R \phi^2 \right] \tag{1.17}$$

where m is the mass of the scalar field, R is the Ricci scalar, ∇_μ is the covariant derivative and ξ is a dimensionless parameter.

In the case of $\xi = 0$, the matter content of the theory is coupled to the gravity sector through

$\sqrt{-g}$. This is called minimally coupling.

For $\xi = \frac{D-2}{4(D-1)}$, we have the so called conformal coupling and if $m = 0$ the theory is invariant under local Weyl rescalings of the metric tensor.

It is obvious that the non-minimally coupling term in the action contributes to the mass of the scalar field. Therefore we define the effective mass

$$m_{eff}^2 = m^2 + \xi R \tag{1.18}$$

Varying the action (1.17) we obtain the equation of motion for the scalar field ϕ :

$$\left(\Box - m_{eff}^2\right)\phi = 0 \tag{1.19}$$

where $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant D'Alembertian. This is the so-called Klein-Gordon equation.

The scalar product of two solutions ϕ_1, ϕ_2 of the Klein-Gordon equation is defined as

$$(\phi_1, \phi_2) = -\imath \int_\Sigma \left(\phi_1 \nabla_\mu \phi_2^* - \phi_2^* \nabla_\mu \phi_1\right) \sqrt{-g_\Sigma} n^\mu d\Sigma \tag{1.20}$$

where Σ is a spacelike hypersurface and n^μ is a unit normal vector orthogonal to Σ.

1.2.2 Quantization

We procced now to canonically quantize the theory. We promote the field $\phi(\eta, x)$ and the conjugate momentum $\pi(\eta, x)$ to operators $\hat{\phi}(\eta, x)$ and $\hat{\pi}(\eta, x)$ which satisfy the equal-time commutation relations

$$[\hat{\phi}(\eta, x), \hat{\pi}(\eta, x')] = \imath\delta^{D-1}(x - x') \tag{1.21}$$

$$[\hat{\phi}(\eta, x), \hat{\phi}(\eta, x')] = 0 \tag{1.22}$$

$$[\hat{\pi}(\eta, x), \hat{\pi}(\eta, x')] = 0 \tag{1.23}$$

Following the same procedure as in the case of Minkowski background, we will expand the field $\hat{\phi}$ in terms of the modes $u_i(x)$. The reason why we can do that, is that these modes are orthonormal in the product of solutions (ϕ_1, ϕ_2) and form a complete set. Therefore, any solution $\hat{\phi}$ can be expanded in terms of modes functions , i.e

$$\hat{\phi}(x) = \sum_i \left(\hat{a}_i u_i(x) + \hat{a}_i^\dagger u_i^*(x)\right) \tag{1.24}$$

where $\hat{a}_i$ and $\hat{a}_i^\dagger$ are the creation and annihilation operators satisfying the usual commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \tag{1.25}$$

$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \tag{1.26}$$

We define the vacuum state $|0\rangle$ as the state which , when the annihilation operators act upon, give zero, i.e $\hat{a}_i|0\rangle = 0$, $\forall i$. Acting N-times on the vacuum with the creation operators $\hat{a}_i^\dagger$, for different values of $\imath$, we construct N-particle states which constitute the basis of the Fock space.

In the previous chapter, for Minkowski spacetime, we were able to define a natural set of modes and expand the field in terms of those modes. Such a decomposition was possible due to the symmetries of Minkowski spacetime. The action of Poincare group (translations,rotations and boosts) leaves the Minkowski line element invariant. In other words, the vector ∂_t is a Killing vector in Minkowski spacetime with eigenvalues $\pm\imath\omega$ for positive values of ω. Namely,

$$\partial_t u_i(x) = -\imath\omega u_i(x) \tag{1.27}$$

$$\partial_t u_i^*(x) = +\imath\omega u_i^*(x) \tag{1.28}$$

The $u_i(x)$ are called positive frequency modes whereas the $u_i^*(x)$ negative frequency modes. Therefore, we were able to define positive modes corresponding to the killing vector. All inertial observes will agree on the definition of those modes and consequently on the vacuum $|0\rangle$ as a zero particle state.

In a general curved spacetime, the above analysis is no longer valid. We know that in curved spacetime ∂_t is not a Killing vector since Poincare group in no longer a symmetry group. Therefore, the price we pay when we make the transition from flat to curved spacetimes, is that we cannot define positive and negative frequency modes in a unique fashion. This implies that there is not a unique mode decomposition of the field $\hat{\phi}$ and thus no unique definition of the vacuum state.

1.2.3 Bogoliubov Coefficients

In order to make the above arguments more clear, we decompose the field $\hat{\phi}$ in terms of the complete set of mode functions $\upsilon_j(x)$, i.e

$$\hat{\phi}(x) = \sum_j \left[\hat{b}_j \upsilon_j(x) + \hat{b}_j^\dagger \upsilon_j^*(x)\right] \tag{1.29}$$

In terms of this decomposition, the vacuum is defined as the state in which the action of $\hat{b}_j$ will give zero, i.e $\hat{b}_j|\bar{0}\rangle = 0$ for every value of j. Since both sets of mode functions are complete, we can relate each other, namely we can express the new mode functions $\upsilon_j(x)$ in terms of $u_i(x)$ and vice versa.

After a short computation, it can be shown that the creation and annihilation operators $\hat{a}_i$, $\hat{b}_j$, $\hat{a}_i^\dagger$, $\hat{b}_j^\dagger$ are related according to

$$\hat{a}_i = \sum_j \left[\alpha_{ji}\hat{b}_j + \beta_{ji}^*\hat{b}_j^\dagger\right] \tag{1.30}$$

$$\hat{b}_j = \sum_i \left[\alpha_{ji}^*\hat{a}_i - \beta_{ji}^*\hat{a}_i^\dagger\right] \tag{1.31}$$

This is the famous Bogoliubov transformation and the coefficients α_{ji} and β_{ji} are called Bogoliubov coefficients.

We can thus conclude that the annihilation operators $\hat{a}_i$ are different from $\hat{b}_j$ as long as $\beta_{ji} \neq 0$. The action of $\hat{a}_i$ on the vacuum $|\bar{0}\rangle$ gives

$$\hat{a}_i|\bar{0}\rangle = \sum_j \left[\alpha_{ji}\hat{b}_j + \beta_{ji}^*\hat{b}_j^\dagger\right]|\bar{0}\rangle = \sum_j \beta_{ji}^*\hat{b}_j^\dagger|\bar{0}\rangle = \sum_j \beta_{ji}^*|\bar{1}\rangle \; \neq 0 \tag{1.32}$$

On the other hand, by the way we defined $|0\rangle$ above, $\hat{a}_i|0\rangle = 0$.

Going a bit further and calculating the number of particles in $|0\rangle$, we find that

$$\langle 0|\hat{b}_j^\dagger\hat{b}_j|0\rangle = \sum_i |\beta_{ij}|^2 \neq 0 \tag{1.33}$$

whereas

$$\langle 0|\hat{a}_i^\dagger\hat{a}_i|0\rangle = 0 \tag{1.34}$$

Let's assume that a particular observer (let's call him Bob) decomposes the field $\hat{\phi}$ in terms of the mode functions $u_i(x)$. This means that Bob, defines the vacuum state $|0\rangle$ as the state that gives zero every time he "acts" on it with the annihilation operator $\hat{a}_i$. Another observer (let's call her Alice) decomposes the field with respect to the complete set of modes $\upsilon_j(x)$. Alice, labels the vacuum state, $|\bar{0}\rangle$, and the action of the operator $\hat{b}_j|\bar{0}\rangle$ gives zero. From Equations (1.33) and (1.34) we conclude that Bob measures no particles when he takes the expectation values of the number operator with respect to the vacuum state $|0\rangle$ while Alice measures $|\beta_{ij}|^2$ particles.

The upshot of the previous analysis is that, in quantum field theory in curved spacetime, the concept of particles has not a universal applicability. Different observers who make different field decompositions will define different vacua. The underlying reason why in Minkowski spacetime we were able to define the notion of particle is that the Minkowski vacuum is invariant under the action of Poincare group. The same holds for the observers in the Minkowski spacetime. They are related to each other with translations, rotations and boosts. In curved spacetime, the vacuum is not invariant under the Poincare group, ∂_t is not a Killing vector and there is not a universal definition of positive and negative modes.

In many cosmological models, we assume that we have an asymptotically flat (Minkowski or conformally Minkowski) spacetime. This means that at $t \to \pm\infty$ (in and out regions) we can define the Minkowski vacuum for all intertial observers. In that case, the notion of particle is well defined and all inertial observers will agree on the particle content of a specific measurement. Later on, when we discuss about cosmological particle production, we will see that in the asymptotically flat regions (in and out regions), inertial observers measure different particle content in the vacuum. The solution to this puzzle will lead to the idea that the expansion of the universe has spontaneously created particles.

1.3 Non-minimally coupled massive scalar field on a Friedmann-Lemaitre-Robertson-Walker background

Having the machinery of the previous section at our disposal, we are now ready to tackle the case of a massive scalar field in a curved FLRW background. After briefly introducing the FLRW spacetime, we will focus on a particular case which will be relevant for the thesis. The de-Sitter space and the corresponding propagator, i.e the Chernikov-Tagirov propagator.

1.3.1 Friedmann-Lemaitre-Robertson-Walker Metric

Current observations show that on large scales, roughly 100Mpc, the Universe is spatially homogeneous and isotropic. These observational information drive us to describe the universe on large scales as a manifold $M = \boldsymbol{R} \times \Sigma$ where Σ is a 3-dimensional maximally symmetric manifold equipped with a metric g_{ij} and $\boldsymbol{R}$ represents the temporal direction. Maximally symmetric spaces contain the maximum number of Killing vectors. In D dimensions, the number of Killing vectors is $\frac{1}{2}D(D+1)$.

Therefore, taking into account homogeneity and Isotropy of Σ, it can be shown that the most general invariant line element of this geometry can be put in the form

$$ds^2 = -dt^2 + \alpha^2(t)\delta_{ij}dx^i dx^j \tag{1.35}$$

where t is the physical time and $\alpha(t)$ is the scale factor. This is the so-called Friedmann-Lemaitre-Robertson-Walker (FRW) metric.

The evolution of the Universe described by (1.35) is fully determined as soon as a specific expression for the scale factor is obtained by solving the Einstein equations. The form of this expression is also related to the matter content of the universe, i.e the stress-energy tensor on the right hand side of Einstein's equation.

The next step is to introduce the conformal time η which is given by $d\eta = \frac{dt}{\alpha(t)}$. The invariant

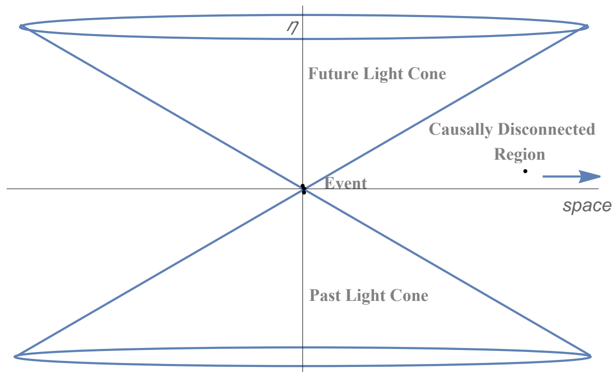


Figure 1.1: Causal structure of FLRW geometry in terms of conformal time η. Light travel at 45^o on the so called null geodesics. The set of all photon trajectories which pass through the event is called lightcone. Regions inside the light cone are causally connected to the event whereas points which lie outside the lightcone are causally disconnected.

line element can be rewritten, in terms of the conformal time, as

$$ds^2 = \alpha^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j) \tag{1.36}$$

The advantage of introducing the conformal time is that (1.36) is manifestly conformal to Minkowski spacetime. This means that the causal structure of FLRW spacetime is determined in terms of Minkowskian light cones by solving the equation $ds^2 = 0$. Therefore, the null geodesics that light follows, satisfy

$$\boldsymbol{x}(\eta) = \pm\eta + constant \tag{1.37}$$

The lightcone is given in figure 1.1. In terms of the physical time t, the lightcones in a curved backround wouldn't be straight lines. Two extremely important quantities are the Hubble parameter $H(t) \equiv \frac{\dot{\alpha}(t)}{\alpha}$ and the first slow roll parameter $\epsilon(t) \equiv -\frac{\dot{H}(t)}{H^2(t)}$. More details on the physical significance of those two parameters will be given in chapter 2, when we introduce the classical theory of inflation.

Our goal is to solve the equation of motion for the field $\hat{\phi}$. We decompose the field into Fourier modes, i.e

$$\hat{\phi}(\eta, x) = \frac{\alpha^{\frac{2-D}{2}}}{(2\pi)^{\frac{D-1}{2}}} \int d^{D-1}k \left[e^{ikx} U(k,\eta)\hat{b}(k) + e^{-ikx} U^*(k,\eta)\hat{b}^\dagger(k) \right] \tag{1.38}$$

Plugging equation (1.38) in (1.19), we find the equation of motion for the mode function $U(k, \eta)$.

$$\left[\frac{d^2}{d\eta^2} + k^2 + \left(m^2 + (\xi - \xi_c)R \right) \alpha^2 \right] U(k,\eta) = 0 \tag{1.39}$$

This is an equation of motion for an harmonic oscillator with a time-dependent mass. We notice that in case we have massless conformally coupled fields ,i.e $m = 0$ and $\xi = \xi_c = \frac{D-2}{4(D-1)}$, the problem simplifies tremenderously. It is just an harmonic oscillator equation with constant frequency! Nevertheless, the problem we have to face in this thesis (loop corrections of graviton propagator due to the presence of massive non-minimally coupled scalar fields) drive us to consider a non-zero value for ξ.

1.3.2 Case: de-Sitter Space

Next, we specialize our analysis to de-Sitter spacetime. de-Sitter spacetime is maximally symmetric. This implies that if we calculate the curvature at one point on the manifold, by making use of the isometries, it has the same value everywhere. The Ricci scalar in D-dimensions is given by

$$R = D(D-1)H^2 = const \tag{1.40}$$

Plugging this into (1.39), we find that

$$\left[\frac{d^2}{d\eta^2} + k^2 + M_\eta^2\right] U(k,\eta) = 0 \tag{1.41}$$

where

$$M_\eta^2 = \alpha^2 H^2 \left(\frac{1}{4} - \nu_D^2\right) \tag{1.42}$$

with ν_D given by

$$\nu_D = \sqrt{\frac{(D-1)^2}{4} - \left[\frac{m^2}{H^2} + D(D-1)\xi\right]} \tag{1.43}$$

The solution to (1.41) is given in terms of the Hankel functions of first and second kind $H_\nu^{(1)}$ and $H_\nu^{(2)}$ [1][37]

$$U(k,\eta) = \alpha_k \sqrt{\frac{-\pi\eta}{4}} H_\nu^{(1)}(-k\eta) + \beta_k \sqrt{\frac{-\pi\eta}{4}} H_\nu^{(2)}(-k\eta) \tag{1.44}$$

where α_k and β_k are the Bogoliubov coefficients. Since equation (1.41) is a second order ordinary differential equation, we need two boundary conditions in order to specify the solution. The first boundary condition is (1.20). This is satified if and only if

$$|\alpha_k|^2 - |\beta_k|^2 = 1 \tag{1.45}$$

The second boundary condition is the so-called vacuum choice. In the asymptotic region, we want the solution to reduce to the Bunch-Davies vacuum.
Imposing these boundary conditions, we have to choose

$$\begin{aligned} \alpha_k &= 1 \\ \beta_k &= 0 \end{aligned} \tag{1.46}$$

Therefore, the mode functions are given by

$$U(k,\eta) = \sqrt{\frac{-\pi\eta}{4}} H_\nu^{(1)}(-k\eta) \tag{1.47}$$

Since the mode functions $U(k,\eta)$ are specified, the solution $\hat{\phi}$ to equation (1.41) is obtained, i.e

$$\hat{\phi}(\eta,x) = \frac{\alpha^{\frac{2-D}{2}}}{(2\pi)^{\frac{D-1}{2}}} \int d^{D-1}k \sqrt{\frac{-\pi\eta}{4}} \left[e^{ikx} H_\nu^{(1)}(-k\eta)\hat{b}(k) + e^{-ikx} H_\nu^{*(1)}(-k\eta)\hat{b}^\dagger(k) \right] \tag{1.48}$$

Using the fact that $H_\nu^{*(1)}(-k\eta) = H_\nu^{(2)}(-k\eta)$, we finally get

$$\hat{\phi}(\eta,x) = \frac{\alpha^{\frac{2-D}{2}}}{(2\pi)^{\frac{D-1}{2}}} \int d^{D-1}k \sqrt{\frac{-\pi\eta}{4}} \left[e^{ikx} H_\nu^{(1)}(-k\eta)\hat{b}(k) + e^{-ikx} H_\nu^{(2)}(-k\eta)\hat{b}^\dagger(k) \right] \tag{1.49}$$

1.3.3 Chernikov-Tagirov Propagator

Having an expression for $\hat{\phi}$, we are now ready to calculate the propagator of the massive non minimally coupled scalar field. The following analysis turns out to be extremely relevant for our work since the loop corrections to the graviton propagator will contain MNMC scalar fields. Therefore, an explicit expression for the MNMC scalar field propagator is required.

The two-point function we want to compute is defined as

$$\imath\Delta(x,x') = \langle T\{\hat{\phi}(x)\hat{\phi}(x')\}\rangle \tag{1.50}$$

where

$$T\{\hat{\phi}(x)\hat{\phi}(x')\} = \theta(\eta-\eta')\hat{\phi}(x)\hat{\phi}(x') + \theta(\eta'-\eta)\hat{\phi}(x')\hat{\phi}(x) \tag{1.51}$$

Plugging (1.49) into (1.50), it can be shown [1][35] that the propagator has the form

$$\imath\Delta(x,x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(\frac{D-1}{2}-\nu_D)\Gamma(\frac{D-1}{2}+\nu_D)}{\Gamma(\frac{D}{2})} \cdot {}_2F_1\left(\frac{D-1}{2}+\nu_D, \frac{D-1}{2}-\nu_D; \frac{D}{2}; \frac{1}{2}\left(1-\frac{y}{4}\right)\right) \tag{1.52}$$

where ${}_2F_1\left(\frac{D-1}{2}+\nu_D, \frac{D-1}{2}-\nu_D; \frac{D}{2}; \frac{1}{2}(1-\frac{y}{4})\right)$ is the Gauss hypergeometric function.

At this point it should be stressed that we are assuming positive effective mass, i.e

$$m_{eff}^2 \equiv m^2 + \xi R > 0 \tag{1.53}$$

For this values, the propagator is de-Sitter invariant [38]. We will return back to some issues of the MNMC scalar field on chapter 5.

1.4 Cosmological Particle Production

In this section, the cosmological particle production in an expanding universe will be discussed. The aim is to illustrate the basic concepts and implications using a simple toy-model Universe. To prevent confusion with the bogoliubov coefficient α, notice that in this subsection, the scale factor is denoted as $\rightarrow a$ instead of α that is denoted in the other chapters.

We assume an homogeneous and isotropic two dimensional universe described by the metric g_{ij}. For simplicity, we consider one expanding spatial dimension x. The line element is given by

$$ds^2 = g_{ij}dx^i dx^j = dt^2 - a^2(t)dx^2 \tag{1.54}$$

Introducing the conformal time η, defined by $d\eta = \frac{dt}{a(t)}$ the corresponding spacetime will be conformal to Minkowski, i.e

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2) \tag{1.55}$$

We assume that the scale factor has the form of

$$a^2(\eta) = c_1 + c_2 \tanh(c_3\eta) \tag{1.56}$$

where c_1, c_2 and c_3 are constants. For $\eta \rightarrow +\infty$, $a^2(\eta) \rightarrow c_1 + c_2$ while for $\eta \rightarrow -\infty$, $a^2(\eta) \rightarrow c_1 - c_2$, i.e the scale factor takes constant values at future and past infinity [figure 1.2]. This implies that the in and out regions are Minkowski regions. Therefore, since we have an asymptotically flat spacetime, we expect that inertial observers will agree on the definition of the vacuum state as well as the particle content of a specific state.

We now assume that the matter fields living in the specific spacetime are minimally coupled, massive scalar fields. Following exactly the same procedure as before, namely expanding the field in terms of mode functions $u_k(\eta, x)$ we find that they satisfy the harmonic oscillator differential equation with time dependent frequency $\omega^2(\eta) = a^2(\eta)m^2$, i.e

$$\left[-\frac{d^2}{dx^2} + \frac{d^2}{d\eta^2} + \omega^2(\eta)\right] u_k(\eta, x) = 0 \tag{1.57}$$

Homogeneity of the spacetime implies that we can split the mode function $u_k(\eta, x)$ into a product of a spatial and a temporal function.

$$u_k(\eta, x) = e^{ikx}\upsilon_k(\eta) \tag{1.58}$$

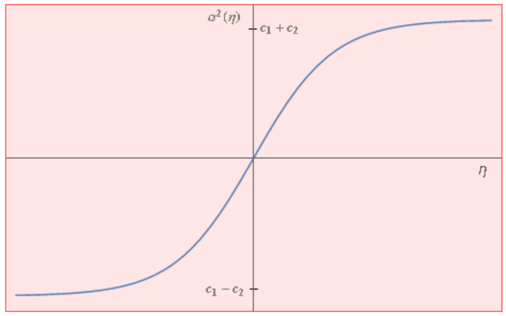


Figure 1.2: Scale factor as a function of the conformal time η. In the far past and future $\eta \to \pm\infty$ the scale factor has constant values. In this regions, spacetime is asymptotically Minkowski.

Thus, we have to solve

$$\left[\frac{d^2}{d\eta^2} + k^2 + \omega^2(\eta)\right] \upsilon_k(\eta) = 0 \tag{1.59}$$

The solution to this equation is known in bibliography [35]. It is given in terms of the Gauss hypergeometric functions ${}_2F_1(a, b; c; z)$.

$$\begin{aligned} u_k(\eta, x)_{in} =& (4\pi\omega_{in})^{-\frac{1}{2}} exp\left(ikx - \imath\omega_+\eta - \imath\frac{\omega_-}{c_3}\ln[2\cosh(\eta c_3)]\right)\times \\ & \times {}_2F_1\left(1 + \imath\frac{\omega_-}{c_3}, \imath\frac{\omega_-}{c_3}; 1 - \imath\frac{\omega_{in}}{c_3}; \frac{1}{2}(1 + \tanh c_3\eta)\right) \end{aligned} \tag{1.60}$$

$$\begin{aligned} u_k(\eta, x)_{out} =& (4\pi\omega_{out})^{-\frac{1}{2}} exp\left(ikx - \imath\omega_+\eta - \imath\frac{\omega_-}{c_3}[2\cosh(\eta c_3)]\right)\times \\ & \times {}_2F_1\left(1 + \imath\frac{\omega_-}{c_3}, \imath\frac{\omega_-}{c_3}; 1 + \imath\frac{\omega_{out}}{c_3}; \frac{1}{2}(1 - \tanh c_3\eta)\right) \end{aligned} \tag{1.61}$$

where

$$\omega_{in} = \left[k^2 + m^2(c_1 - c_2)\right]^{\frac{1}{2}} \tag{1.62}$$

$$\omega_{out} = \left[k^2 + m^2(c_1 + c_2)\right]^{\frac{1}{2}} \tag{1.63}$$

$$\omega_{\pm} = \frac{1}{2}\left(\omega_{out} \pm \omega_{in}\right) \tag{1.64}$$

We are interested in finding solutions in the asymptotic regions, namely at $\eta \to \pm\infty$.

For the in region ($\eta \to -\infty$)

$$u_k(\eta, x)_{in} \to (4\pi\omega_{in})^{-\frac{1}{2}} e^{\imath(kx - \omega_{in}\eta)} \tag{1.65}$$

whereas for the out region ($\eta \to +\infty$)

$$u_k(\eta, x)_{out} \to (4\pi\omega_{out})^{-\frac{1}{2}} e^{\imath(kx - \omega_{out}\eta)} \tag{1.66}$$

From (1.65) and (1.66) it is clear that the mode functions in the two asymptotic regions are not the same (since $\omega_{in} \neq \omega_{out}$). But what does this imply?

Any change in the mode functions $u_k(\eta, x)$, that leaves the solution for the field $\phi(\eta, x)$ the same, induces a change in the definition of creation and annihilation operators and hence a change in the definition of vacuum state. Therefore, the message of the above result is clear. The bogoliubov coefficients are different from zero. The explicit expressions for the latter are given by

$$|\alpha_k|^2 = \frac{\sinh^2(\frac{\pi\omega_+}{c_3})}{\sinh(\frac{\pi\omega_{in}}{c_3})\sinh(\frac{\pi\omega_{out}}{c_3})} \tag{1.67}$$

$$|\beta_k|^2 = \frac{\sinh^2(\frac{\pi\omega_-}{c_3})}{\sinh(\frac{\pi\omega_{in}}{c_3})\sinh(\frac{\pi\omega_{out}}{c_3})} \tag{1.68}$$

It has already been said that in the far past ($\eta \to -\infty$) , the spacetime is described by the Minkowski metric. Therefore, all the inertial observers will define the vacuum state $|0\rangle$ in terms of $u_k(\eta, x)_{in}$. In other words, if we prepare the system in the state $|in, 0\rangle$ all inertial observers in the spacetime will agree that the state contains no particle excitations.

In the far future ($\eta \to +\infty$), the inertial observers denote the vacuum state as $|out, 0\rangle$ in terms of the $u_k(\eta, x)_{out}$ modes. But the field in the out region will be still in the $|in, 0\rangle$ state (if we assume that the evolution of the quantum field is in the Heisenberg picture). This means that inertial observers at $\eta \to +\infty$ will register particles in the vaccum $|in, 0\rangle$. The upshot of this analysis is that inertial observers, although should agree on the definition of the physical vacuum, it turns out that they don't. The solution to this puzzle is that the expansion of the universe has spontaneously created k mode excitations (particles) given by (1.68).

In the limit that $\omega_- \to 0$ (and assuming $m \neq 0$) it follows that $\omega_{in} = \omega_{out} \Rightarrow c_2 = 0$. This means that the Bogoliubov coefficient $|\beta_k|^2$ vanishes when $c_2 = 0$. Therefore, in a non-expanding universe ($c_2 = 0$), there is a uniquely defined vacuum state for all inertial observers and no spontaneous particle production occurs. We conclude that the rate of expansion of the universe plays a significant role in cosmological particle creation. In the limit the rate goes to zero, there is an exponential decline of the particle production. This can be quantified with the parameter $\frac{c_3}{\omega_{in}}$, i.e

$$|\beta_k|^2 \to e^{\frac{-2\pi\omega_{in}}{c_3}} \to 0 \tag{1.69}$$

as $\frac{c_3}{\omega_{in}} \to 0$. The smallness of this parameter is related to the magnitude of the frequency ω_{in}. For large values of ω_{in} (or equilavently for large mass and/or momenta), the particle production is exponentially suppressed. Therefore, the expansion of the universe excite insufficiently high mass and momenta particles. Even though the reader can point out that these conclusions were withrawn using a simplified toy model (namely a 2-dimensional Universe with an ansatz scale factor $a(\eta)$ of our choice), the above features hold in any FLRW metric with a smooth scale factor. During inflation, a period of rapid expansion of the universe (see chapter 2), low energy and mass excitations are favoured, too.

The above feature will turn to be extremely relevant in our problem, where we will assume that the quantum loop corrections to the graviton propagator during inflation is due to the presence of an enseble of infrared scalars. This will allow us to perform gravitational computations in a perturbative way despite the fact the theory is not renormalizable and we do not have a fundamental theory of quantum gravity (effective field theory approach).

Let's discuss one last remark. In the case we are dealing with massless scalar fields, we see that $\omega_{-} = 0$. This means again that the Bogoliubov coefficient $|\beta_k|^2$ vanishes. This is something that we expect since the spacetime is conformal to Minkowski and the massless scalar field is also invariant under conformal transformations. The upshot of this is that spontaneous particle production occurs when we introduce mass to the scalar fields, i.e when conformal symmetry is broken. This becomes more clear if we think what does it technically mean "add mass to the scalar field". When we couple the massive scalar field to gravity, in semiclassical approximation (matter fields are quantized whereas gravity is treated classically), there exist vertices that couple gravity (in other words spacetime expansion) to scalar fields via the mass term. Thus, the time dependent expansion of the universe transfers energy to the scalar modes.

The lessons taught in this section are the following :

1) Cosmological particle production occurs in a smoothly expanding universe.
2) Low mass and momenta particle production is favoured. In constract, high mass and momentum excitations are exponentially supressed.

Chapter 2

Classical Theory Of Inflation

In this chapter we follow closely [34] [39]

2.1 Big Bang Theory

In 1927, Georges Lemaitre proposed that the universe is expanding. This idea, even though physicists nowadays are familiar with, was a rather revolutionary idea for that period. The main idea is that since the Universe is expanding, if we rewind the situation, in the far past it should have been emerged from a state with incredibly huge density and temperature. This state of the Universe is called Big Bang. From current observations of supernovas type IIA as well as the cosmic microwave backround, we know that this happened approximately 13.7 billion years ago. This idea gained some reputation in 1929 when the American astronaumer Edwin Hubble, by studing various galaxies and their relation between distances and redshift, made the observation that all the galaxies are recceding from us. This observation was consistent with the idea of Lemaitre and played a crucial role in convincing the physicists to adopt it as the main theory for the birth of the Universe.
The Bing Bang model relies on two important assumptions. First of all, it is assumed that the initial state was homogeneous and isotropic. This is called cosmological principle. It turns out to be a really crucial property since it allows us to build cosmological models and recreate the history of the Universe. Secondly, we assume that the laws of physics, and especially Einstein's theory of general relativity, are valid.

However, the Bing Bang model couldn't give explanation to some crucial problems of cosmology:

1) Flatness problem
2) Horizon problem

The theory which comes to the rescue, and gives theoretical explanations to those problems, is called cosmic inflation.

2.2 Cosmic Inflation to the rescue

2.2.1 The Basics

In section 1.3.1 , the FRW metric was briefly introduced. The form of the line element (1.35) is determined by the assumption of spatial isotropy and homogeneity. Moreover, the Hubble parameter H as well as the first slow roll parameter $\epsilon(t)$ were introduced. In this section, our aim is to find the equations that determine the evolution of the scale factor $\alpha(t)$. In the next step, we will figure out how inflation solves the horizon and flatness problems.

In order to determine the dynamics of the scale factor, we first introduce the Einstein's equations.

$$G_{\mu\nu} = T_{\mu\nu} \tag{2.1}$$

where $G_{\mu\nu}$ and $T_{\mu\nu}$ are the Einstein and stress-energy tensors respectively. Note that we set $8\pi G = 1$. Eistein Tensor can be written in terms of the Ricci tensor $R_{\mu\nu}$ and Ricci scalar R, i.e

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{2.2}$$

where

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} \tag{2.3}$$

and

$$R = g^{\mu\nu} R_{\mu\nu} \tag{2.4}$$

Moreover, $\Gamma^\alpha_{\beta\gamma}$ are the so-called Christoffel symbols.

As a source of energy and momentum, we assume a perfect fluid. The stress-energy tensor takes the form

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu} \tag{2.5}$$

where ρ and p are energy and momentum densities respectively (assuming that we work at the rest frame of the fluid) and U_μ is the fluid's 4-velocity.

Recall that the FRW metric has components $g_{\mu\nu}$ =diag$[1, -\alpha(t), -\alpha(t), -\alpha(t)]$.

Having all the ingredients at our disposal, we are now ready to solve the Einstein's equations and find the dynamical equations which determine the evolution of the scale parameter α.

The components of the Ricci tensor and the expression for the Ricci scalar for the FRW geometry are given by

$$R_{ij} = \left[2\dot{\alpha}^2 + \alpha\ddot{\alpha} + \frac{2k}{\alpha^2}\right]\delta_{ij} \tag{2.6}$$

$$R_{00} = -3\frac{\ddot{\alpha}}{\alpha} \tag{2.7}$$

$$R = 6\left(\frac{\ddot{\alpha}}{\alpha} + \left(\frac{\dot{\alpha}}{\alpha}\right)^2 + \frac{k}{\alpha^2}\right) \tag{2.8}$$

Solving the (00) component of (2.1), i.e $G_{00} = T_{00}$, we obtain

$$H^2 \equiv \left(\frac{\dot{\alpha}}{\alpha}\right)^2 = \frac{1}{3}\rho - \frac{k}{\alpha^2} \tag{2.9}$$

This is the so-called 1^{st} Friedmann equation.

Taking the trace of (2.1), the 2^{nd} Friedmann equation is derived, i.e

$$\dot{H} + H^2 \equiv \frac{\ddot{\alpha}}{\alpha} = -\frac{1}{6}(\rho + 3p) \tag{2.10}$$

Making use of the equation $\nabla_\mu T^{\mu\nu} = 0$, i.e the covariant conservation of the stress energy tensor, we get the continuity equation:

$$\dot{\rho} = -3H(1+w)\rho \tag{2.11}$$

where w is defined as

$$w = \frac{p}{\rho} \tag{2.12}$$

For constant values of the parameter w, we can solve the continuity equation and obtain

$$\rho \propto \alpha^{-3(1+w)} \tag{2.13}$$

Since we obtained an equation for the energy density ρ, we can substitute it in the 1^{st} Friedmann equation and find the solutions for the evolution of the scale factor. These solutions will depend on the value of the parameter w as well as the spatial curvature of the Universe k.

For a flat Universe, i.e $k = 0$, and $w \neq -1$

$$\alpha(t) \propto t^{\frac{2}{3(1+w)}} \tag{2.14}$$

while for a flat universe with $w = -1$

$$\alpha(t) \propto e^{Ht} \tag{2.15}$$

For a cosmological fluid consisting of non-relativistic and collisionless particles , $w = 0$. This implies that $\rho \propto \alpha^{-3}$ and the scale factor increases as $t^{2/3}$. For a universe dominated by radiation, $w = \frac{1}{3}$. The energy density falls off as $\rho \propto \alpha^{-4}$ and the scale factor increases as $t^{1/2}$. The fact that the energy density declines faster is due the redshift that photons suffer from. Finally, if we assume a Universe driven by the cosmological constant Λ, we have that $w = -1$ and the scale factor is given by $a(t) = e^{Ht}$.

2.2.2 Two Puzzles are "begging" for solution

The aim of this subsection is to briefly discuss two problems that arise from the Big Bang theory. The flatness problem and the horizon problem. The next step is to see how a period of cosmic inflation in the beginning of the universe actually solves these problems in an elegant and simple way.

Flatness problem

In order to shed some light to the flatness problem, we consider the first Friedmann equation, i.e

$$H^2 = \frac{1}{3}\rho(\alpha) - \frac{k}{\alpha^2} \tag{2.16}$$

Dividing the equation above with H^2 we get

$$1 - \Omega(a) = \frac{-k}{(\alpha H)^2} \tag{2.17}$$

where

$$\Omega(a) \equiv \frac{\rho(\alpha)}{3H^2}$$

In case we assume that the strong energy condition (SEC) is not violated, i.e $1 + 3w > 0$, the so-called comoving Hubble horizon increases monotonically with time. Namely,

$$\frac{d}{dt}(\alpha H)^{-1} > 0 \tag{2.18}$$

This implies that $1-\Omega(\alpha_0)$ should diverge today. Instead, what we currently observe is $\Omega(\alpha_0)$ of order one. In order to explain this observation, one should impose extremely fine tuned initial conditions for Ω in the begining of the Universe. The fact that we should impose by hand the initial conditions is not a paradox of the Big Bang theory. However, many physicists feel a bit unconfortable with that. A slight deviation of the initial conditions would lead to a much different Universe than the one we live in. Therefore, a mechanism that explains this extreme fine tuning would be highly appreciated.

Horizon Problem

In order to present this problem, we define an important quantity, the so called comoving particle horizon. This is related to the Hubble comoving horizon via the relation

$$\tau = \int_0^{\alpha} \frac{d\alpha'}{\alpha'} \frac{1}{\alpha' H} \tag{2.19}$$

where the primes above indicate the integration variables. Assuming a Universe described by a perfect fluid, the Hubble radius is given by

$$(\alpha H)^{-1} = H_0^{-1} \alpha^{\frac{1}{2}(1+3w)} \tag{2.20}$$

For $w > -\frac{1}{3}$, we clearly see that $(\alpha H)^{-1}$ increases with time. In other words, if the strong energy condition is valid, the comoving Hubble horizon is a monotonically increasing function with time. From (2.19), we can also conclude that the comoving particle horizon increases with time. Extrapolating back to the period of decoupling, the particle horizon was much smaller. This implies that during that period, there should exist comoving scales outside the horizon. The puzzle arises by taking into consideration the high degree of homogeneity of the cosmic microwave backround.

How is it possible to observe such an homogeneous Universe (up to 10^{-5} accuracy) taking into account that there were around 10^4 causally disconnected regions?

Introducing Inflation

One crucial thing that we need to stress at this point is the behaviour of the comoving Hubble horizon. The fact that is monotonically increasing leads to the horizon and flatness problems. Therefore, it seems natural to wonder what if there was period in the cosmic history in which the comoving Hubble horizon was decreasing ? Which would be the implications?

Starting with the flatness problem, the decreasing comoving Hubble radius leads to a flat universe. This can be concluded by observing the relation (2.17). Clearly, for decreasing values of $(\alpha H)^{-1}$, $\Omega(a) \to 1$. This is amazing news. If inflation last for an adequate amount of time, whatever curvature the universe had in the beginning, it will become negligible at late times. Therefore, the flatness we observe today need not an extreme fine tuning of the initial conditions.

Let's see which are the implications for the horizon problem.

By assuming a period of the Universe in which $(\alpha H)^{-1}$ was decreasing implies that scales which now enter the horizon, in the early universe, they were inside the horizon. This means that there were in causal contact in the far past, eliminating all the inhomogeneities that could in principle have been created.

Thus, we see that this assumption solves the two puzzles in a really simple and elegant way. The next step, is to create a model which will quantify the above considerations and lead to predictions which can be tested. Before that, one last thing which is important to refer at this point are the conditions which should be satisfied in order to have an inflationary period.

We already have at our disposal the initial condition, i.e $(\alpha H)^{-1}$ decreasing with for a sufficient period of time. By using the relation

$$\frac{d}{dt}(\alpha H)^{-1} = -\frac{\ddot{\alpha}}{(\alpha H)^2} \tag{2.21}$$

we conclude that during the period of Inflation, we have accelerating expansion, i.e

$$\ddot{\alpha} > 0 \tag{2.22}$$

Another equivalent condition for inflation comes from the second Friedmann equation. Accelerating expansion implies that

$$\rho + 3p < 0 \tag{2.23}$$

i.e the strong energy condition is violated.

2.2.3 Scalar Inflationary Models

We consider a scalar field ϕ minimally coupled to gravity. The action of this theory is given by

$$S_\phi = \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \tag{2.24}$$

Varying the action (2.24) with respect to the field ϕ we obtain the equation of motion for the scalar field ϕ :

$$\Box\phi + \frac{dV}{d\phi} = 0 \tag{2.25}$$

where the box operator in a general curved backround is given by $\Box = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu)$.

The variation of the action with respect to the metric tensor $g^{\mu\nu}$ gives the stress-energy tensor, i.e

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\rho\phi\partial_\rho\phi - V(\phi)\right) \tag{2.26}$$

In a FLRW backround, the scalar field ϕ as well as the scale factor α depend only on time t. Therefore, the equation of motion for the scalar field (2.25) becomes

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \tag{2.27}$$

Calculating the (00) and (ii) components of $T_{\mu\nu}$ we can find the expressions for the energy and momentum densities in terms of the field ϕ and the potential $V(\phi)$, i.e

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{2.28}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2.29}$$

From the first Friedmann equation (2.9), assuming that $k = 0$ (reasonable assumption based on observations) and equation (2.28), we find that the scale factor obeys

$$H^2 = \frac{1}{3}\rho = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{2.30}$$

2.2.4 Single Field Slow-Roll Inflation

Recall that relation (2.23) is a condition for an accelerating expansion. Combining this with equations (2.28) and (2.29) we get that accelerated expansion occurs if

$$\dot{\phi}^2 < V(\phi) \tag{2.31}$$

We introduce the first slow roll parameter ϵ,

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{1}{2}\frac{\dot{\phi}^2}{H^2} \tag{2.32}$$

Accelerated expansion occurs when the kinetic term $\dot{\phi}^2$ is smaller than the energy which drives inflation H^2. Therefore, the first slow roll parameter ϵ should satisfy

$$\epsilon < 1 \tag{2.33}$$

A condition for exponential expansion can be found by writting down the equation of state:

$$w = \frac{p}{\rho} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{2.34}$$

Exponential expansion occurs when $w = -1$, which implies that

$$\dot{\phi}^2 << V(\phi) \tag{2.35}$$

In both cases, the conclusion is that in order to create a model for inflation, we need a potential $V(\phi)$ which dominates upon the kinetic energy of the scalar field. This condition ensures that inflation will happen at some time t.

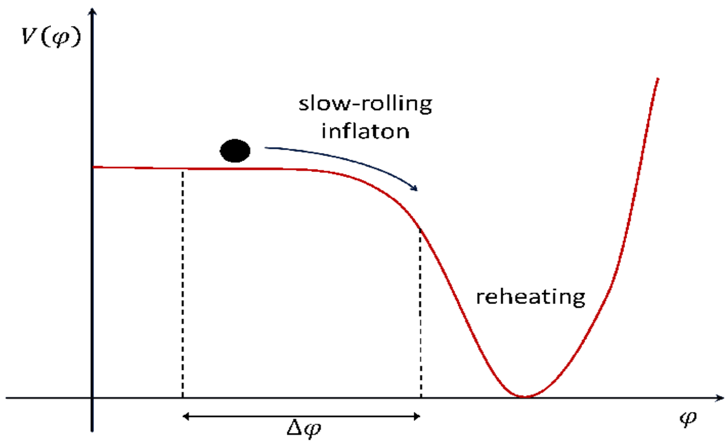


Figure 2.1: [39] The inflaton field ϕ slowly rolls down the potential. $\Delta\phi$ denotes the displacement of the field.

Furthermore, it turns out that one more condition is required if we want inflation to last for a sufficient amount of time. This condition can be deduced by observing the equation of motion (2.27) for the scalar field ϕ. We want the acceleration of the scalar field $\ddot{\phi}$ to be much smaller than $3H\dot{\phi}$ and $\frac{dV}{d\phi}$. In that case, no kinetic energy is induced and the condition (2.31) is satisfied. If $\ddot{\phi}$ becomes large enough, it will "feed" the kinetic energy and therefore (2.31) will be violated signaling the end of inflation. This second condition is quantified by introducing the second slow roll parameter η,

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} < 1 \tag{2.36}$$

Those two slow roll parameters can be rewritten in terms of the potential $V(\phi)$ as

$$\epsilon(V) \equiv \frac{M_{pl}^2}{2}\left(\frac{\frac{dV}{d\phi}}{V}\right)^2 \tag{2.37}$$

$$\eta(V) \equiv M_{pl}^2 \frac{\frac{d^2V}{d\phi^2}}{V} \tag{2.38}$$

Comment

In this chapter we introduced the classical theory of inflation. The way the topic was presented would leave the reader with the feeling that inflation is an extremely successful phenomenological theory solving the Big Bang puzzles in a simple and elegant way suffering from no ambiguities. However, one could in principle argue that inflation solves these puzzles "by construction". This is a reasonable argument. After realizing that in the heart of the Big Bang puzzles lies the monotonically increasing comoving Hubble radius, we postulated a period during cosmic history in which it was shrinking. This solved the flatness and horizon problems.

In the next chapter, we will combine quantum mechanics with inflation. This marriage turns out to be extremely fruitful. Primordial quantum fluctuations produced during inflation will leave an imprint in the patterns of temperature anisotropies and polarization on the CMB map. Quantum phenomena will costraint the space of possible solutions of the Big Bang puzzles. It turns out that most quantum mechanical observations (for example, the acoustic peaks appearing in the temperature-temperature correlators) related to the CMB, are "crying out" for inflation.

Chapter 3

Tree Level Inflationary Quantum Fluctuations

In this chapter we follow closely [39] [40]

The goal of this chapter is to obtain the primordial power spectra of scalar and tensor fluctuations created during the period of cosmic inflation. In the first section, a brief overview of linear cosmological perturbation theory is given. In particular, we introduce the metric and matter perturbations as well as discuss ambiguity related to the gauge choice. In section 2, we sketch the calculation of the primordial scalar spectrum . At this point, some basics concepts of quantum field theory is de-Sitter spacetime is required (Chapter 1). In the last section, the power spectrum of primordial tensor fluctuations is computed. We end the chapter with some comments which encapture some essentials features of the primordial scalar and tensor fluctuations.

3.1 Linear cosmological perturbation theory in a nutshell

One of the most important cosmological observations is considered to be the CMB anisotropies map. During the period of decoupling, just before the moment when electrons and protons coupled to form neutral hydrogen atoms, the Universe was nearly homogeneous. Small inhomogeneities exist at 10^{-5} level. These inhomogeneities can be studied within linear cosmological perturbation theory. The way this can be done is by expanding all the relevant physical quantities $X(t, \boldsymbol{x})$ around a FLRW backround plus small fluctuations, i.e

$$X(t, \boldsymbol{x}) \equiv \bar{X}(t) + \delta X(t, \boldsymbol{x}) \tag{3.1}$$

The overbar quantities refer to background values. Notice that the background quantity $\bar{X}(t)$ is independent of the spatial coordinates due to the isometries of the FLRW spacetime. In contrast, fluctuations are assumed in general to depend both in time and space. The smallness of the fluctuations $\delta X(t, \boldsymbol{x})$ implies that the solution can be found using the linearized version of Einstein equations, i.e

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \tag{3.2}$$

3.1.1 Scalar-Vector-Tensor decomposition

Linear cosmological perturbation theory is a really powerful method of computing small fluctuations around a background value. One of the reasons is the so-called Scalar-Vector-Tensor (SVT) decomposition.

A perturbation $\delta X(t, \boldsymbol{x})$ can be expanded in Fourier modes as

$$\delta X_{\boldsymbol{k}}(t) = \int d^3\boldsymbol{x} \delta X(t, \boldsymbol{x}) e^{i\boldsymbol{k}\boldsymbol{x}} \tag{3.3}$$

Under a rotation of the coordinate system around the direction of the mode $\boldsymbol{k}$ by angle θ , the Fourier mode $\delta X_{\boldsymbol{k}}$ changes as

$$\delta X_{\boldsymbol{k}} \to e^{im\theta} \delta X_{\boldsymbol{k}} \tag{3.4}$$

Depending on the value of m, the perturbation is characterized as scalar ($m = 0$) vector ($m = \pm 1$) or tensor ($m = \pm 2$). Thus, the modes are decomposed according to the transformation under θ. An important implication is that different type of perturbations (Scalar, vector and tensor type) evolve independently without mixing, at linear level. This is a result of rotational invariance of the linear equation of motion. Away from linear order, this is no longer valid. Similarly, it can be shown that due to translation invariance of linear Einstein's equations (3.2), different modes $\boldsymbol{k}$ do not mix [39]. To clarify this, let's assume for example two scalar modes with $\boldsymbol{k_1}$ and $\boldsymbol{k_2}$ respectively. These scalar modes, do not mix at linear level.

These two features of the linear perturbation theory, simplify the analysis a lot and will turn out to be an extremely useful tool in studying primordial fluctuations produced during the period of Inflation.

3.1.2 Fluctuations non uniquely defined-Choice of gauge

An important issue which should be stressed at this point is that the decomposition (3.1) is not unique, namely there is not a unique way of splitting a cosmological quantity $X(t, \boldsymbol{x})$ into background and fluctuations. It is highly dependent on the way we choose our coordinates of spacetime. This means that different slicing (i.e different way of choosing hypersurfaces constant time) and different threading (i.e different choice of worldlines of constant $\boldsymbol{x}$) define different fluctuations. It is possible to show that one can choose such a slicing of the spacetime that ***real*** fluctuations vanish (wrongly). Equivalently, we can also create fictitious perturbations by choosing a corresponding coordinate system. This ambiguity should be resolved. The way this can be done is by introducing gauge invariant quantities, i.e quantities that do not depend on coordinate transformations (slicing and threading of spacetime). Furthemore, the matter and metric fluctuations are related to each other via the linearized Einstein equations (3.2). This means that after a coordinate transformation, metric fluctuations can be transferred to matter sector or vice versa. Therefore, it is crucial to keep all the fluctuations in order to keep track of this interplay between matter and gravity sectors.

3.1.3 Metric and Matter Perturbations

At this point, we get a bit more specific by discussing metric and matter fluctuations around the FLRW background. As far as metric perturbations are concerned, the most general way to write them is

$$ds^2 = -(1+2\Phi)dt^2 + 2\alpha B_i dx^i dt + \alpha^2[(1-2\Psi)\delta_{ij} + E_{ij}]dx^i dx^j \tag{3.5}$$

where

$$B_i = \partial_i B - S_i \tag{3.6}$$

and

$$E_{ij} = 2\partial_i\partial_j E + \partial_i F_j + h_{ij} \tag{3.7}$$

By demanding that h_{ij} is transverse and traceless, it can be shown that it corresponds to a tensor perturbation, i.e it transforms under a rotation by an angle θ as $\delta h_{ij} \to e^{\pm 2i\theta} h_{ij}$. Moreover, S_i and F_i transform as vectors under rotation so they correspond to vector (divergence-free) perturbations.

Check

In principle, a metric has 10 dynamical degrees of freedom.

4 scalars $\to (B, \Psi, \Phi, E) = 4$ degrees of freedom.
2 divergence-free vectors $\to (S_i, F_i) = 2 \times 2 = 4$ degrees of freedom.
1 transverse traceless tensor $\to (h_{ij}) = 2$ degrees of freedom.

Indeed, the way we decomposed the metric has the correct amount of degrees of freedom. In case the stress energy tensor does not contain anisotropic stress, it can be shown that vector perturbations are subdominant during inflation and will not be relevant here. Taking this into account, the most general metric fluctuations can be written in form

$$ds^2 = -(1+2\Phi)dt^2 + 2\alpha\partial_i B dx^i dt + \alpha^2[(1-2\Psi)\delta_{ij} + \partial_i\partial_j E + h_{ij}]dx^i dx^j \tag{3.8}$$

Notice that (3.8) contains 6 dynamical degrees of freedom instead if 10 (since we got rid of the vector perturbations). Primordial scalar fluctuations are sourced by the scalar part of (3.8) (it is not exactly true, someone needs to take account also the matter perturbations which also source the scalar flutuations. We soon return back to that point), i.e

$$ds^2 = -(1+2\Phi)dt^2 + 2\alpha\partial_i B dx^i dt + \alpha^2[(1-2\Psi)\delta_{ij} + \partial_i\partial_j E]dx^i dx^j \tag{3.9}$$

whereas primordial tensor perturbations are entirely sourced by h_{ij}

$$ds^2 = -dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j \tag{3.10}$$

Focusing our attention to (3.9) we immediately see that there are 4 scalar dynamical degrees of freedom. At the same time, we have the freedom to choose our coordinates in such a way that 2 dynamical degrees of freedom are gauged (gauge choice). Finally, assuming that there in no anisotropic stress, we end up with 1 dynamical degree of freedom [39].

Under gauge/coordinate transformations

$$t \to t + a \tag{3.11}$$

$$x^i \to x^i + \delta^{ij}\partial_j \beta \tag{3.12}$$

scalar quantities that appear in the metric (3.9) transform as

$$\delta\Phi = -\dot{a} \tag{3.13}$$

$$\delta E = -\beta \tag{3.14}$$

$$\delta\Psi = Ha \tag{3.15}$$

$$\delta B = a\alpha^{-1} - \alpha\dot{\beta} \tag{3.16}$$

where α is the scale factor that appears in the FLRW metric.

In constrast, tensor fluctuations h_{ij} do not change under gauge transformations (3.11) and (3.12). It is a gauge invariant object.

Finally, the perturbations to the matter sector can be written as

$$T^0_0 = -(\bar{\rho} + \delta\rho) \tag{3.17}$$

$$T^0_i = (\bar{\rho} + \bar{p})\alpha\upsilon_i \tag{3.18}$$

$$T^i_0 = -(\bar{\rho} + \bar{p})\alpha^{-1}(\upsilon_i - B_i) \tag{3.19}$$

$$T^i_j = \delta^i_j(\bar{p} + \delta p) + \Sigma^i_j \tag{3.20}$$

where again we mention that the overbar quantities refer to background values, ρ is the energy density, p is pressure, υ_i the three velocity, α the scale factor and Σ^i_j is the so-called anisotropic stress. For future convenience, we also introduce the quantity $\partial_i \delta q = (\bar{\rho} + \bar{p})\upsilon_i$. Scalar quantities such as q, p, ρ transform under gauge transformations (3.11) and (3.12) as

$$\delta p \to \delta p - \dot{\bar{p}}a \tag{3.21}$$

$$\delta\rho \to \delta\rho - \dot{\bar{\rho}}a \tag{3.22}$$

$$\delta q \to \delta q + (\bar{\rho} + \bar{p})a \tag{3.23}$$

3.1.4 Comoving Curvature Perturbation

As we have already mentioned, it is crucial in our analysis to construct gauge invariant quantities, i.e quantities that do not transform under coordinate transformations (3.11)-(3.12). An important quantity which we will use in the calculation of the primordial power spectra is the so-called comoving curvature perturbation ζ. It is defined

$$\zeta = \Psi - \frac{H}{\bar{\rho} + \bar{p}}\delta q \tag{3.24}$$

It is easy to check that under gauge transformations, Ψ and δq transform in such a way that the comoving curvature perturbation remains invariant. Moreover, it is crucial to notice that this quantity is a combination of metric (Ψ) and matter (δq) perturbations. The importance of this quantity is that is remains constant on superhorizon scales ($k < \alpha H$). This implies that one can calculate this quantity at horizon crossing and be sure that when reenters the horizon at later times, the evolution of the modes at superhorizon scales was insignificant. This freezing of the comoving curvature perturbation will be relevant in the next section.

3.2 Primordial Scalar Power Spectrum

3.2.1 Importance

We now proceed to the one of the two main calculations of this chapter. The primordial power spectrum of the scalar fluctuations produced during the period of inflation. Before moving on, it is important to realize the importance of this calculation.

Primordial scalar fluctuations induce a local time delay δt at which the inflation ends. This implies that different patches of the universe inflate more or less than other patches. In other words, inflation does not stop globally at a specific time. In turn, this means that different cosmological patches experience slightly different evolutions. This can be translated in a difference in the energy density $\delta\rho$ of the Universe which will finally be encoded as temperature fluctuations on the CMB map that we have observed.

Of equal (or probably of bigger) importance is the calculation for the primordial tensor fluctuations. The latter induce primordial gravitational waves, i.e gravitational waves directly from the period of inflation. Their signature is expected to be encoded in the polarization of the CMB (more specifically to the so-called B-modes of the CMB polarization) [41] [42]. Their discovery is expected with great anticipation from the scientific community since their it will contain important information about the energy scale that cosmic inflation occured.

Therefore, due to the contact with observations, the following calculation is of a great significance.

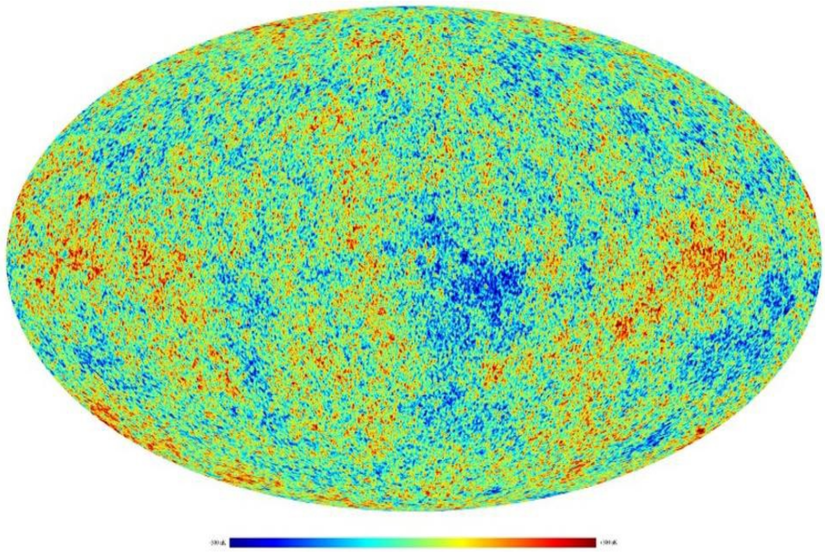


Figure 3.1: Taken from https://www.wikipedia.org/. This is the famous CMB map. Blue spots have lower temperature than the red and yellow spots. The difference in temperature is roughly of order 10^{-4} K.

3.2.2 The calculation

We assume an inflationary model of a single inflaton field which rolls slowly a potential $V(\phi)$. The primordial scalar fluctuations are sourced by a scalar field action minimally coupled to gravity (one could in principle assume non-minimal coupling. However, with field redefinition of the theory, it can always be transformed to minimal coupled gravity at least at the classical level. For cosmological quantum perturbations, we refer the reader to [43][44][45]).

$$S_\phi = \frac{1}{2}\int d^4x\sqrt{-g}\left[R - (\nabla\phi)^2 - 2V(\phi)\right] \tag{3.25}$$

We choose to work in the comoving gauge. In this gauge, the inflaton field in unperturbed and the dynamical degrees of freedom reside in the metric, i.e

$$\delta\phi = 0 \rightarrow \delta q = 0 \tag{3.26}$$

$$E = 0 \tag{3.27}$$

$$g_{ij} = \alpha^2(1-2\Psi)\delta_{ij} \tag{3.28}$$

From (3.24) we immediately see that for $\delta q = 0 \rightarrow \zeta = \Psi$. Therefore, (3.28) can be rewritten as

$$g_{ij} = \alpha^2(1-2\zeta)\delta_{ij} \tag{3.29}$$

Moreover B and Φ that appear in (3.9) are related to ζ via the linearized Einstein's equation. So the only dynamical degree of freedom is ζ.

Expanding the action to second order in ζ we obtain [46]

$$S_{(2)} = \frac{1}{2}\int d^4x\alpha^3\frac{\dot{\phi}^2}{H^2}\left[\dot{\zeta}^2 - \alpha^{-2}(\partial_i\zeta)^2\right] \tag{3.30}$$

Important Comment:

At this point we should stress that the action above is a free field action. The calculation we are performing is a tree level calculation, i.e we include no loop corrections to the primordial scalar fluctuations. In chapter 1, it is shown that an expanding universe spontaneously produces IR scalars and gravitons. In turn, these particles will enter in loops in the primordial fluctuations. Therefore, it is of primary importance to study whether these loop corrections have significant late time effects on the propagation of the dynamical primordial scalars and gravitons. This issue is lying in the heart of this thesis. We return at this in chapter 4 and 5.

Let's go back to the calculation. We observe that the kinetic term in the action $S_{(2)}$ is multiplied with a time dependent coefficient. In other words, it is not in canonical form. Therefore, in order to faciliate the quantization procedure, it is useful to introduce the so-called Mukhanov variable

$$\upsilon \equiv z\zeta \tag{3.31}$$

where

$$z^2 \equiv \alpha^2\frac{\dot{\phi}^2}{H^2} = 2\alpha^2\epsilon \tag{3.32}$$

Recall that the last equally is due to the fact that the slow roll parameter ϵ is equal to $\frac{1}{2}\frac{\dot{\phi}^2}{H^2}$.

Rewritting the action (3.30) in terms of the Mukhanov variable, we get

$$S_{(2)} = \frac{1}{2}\int d\tau d^3x\left[(\upsilon')^2 + (\partial_i\upsilon)^2 + \frac{z''}{z}\upsilon^2\right] \tag{3.33}$$

where primes correspond to derivatives with respect to the conformal time τ. Notice that by introducing the Mukhanov variable, the kinetic term has indeed been put in canonical. The price we pay for that, is that now there is a time dependent mass term.

The next step is to expand $\upsilon(t,\boldsymbol{x})$ in Fourier modes.

$$\upsilon(t,\boldsymbol{x}) = \int\frac{d^3k}{(2\pi)^3}\upsilon_{\boldsymbol{k}}(\tau)e^{i\boldsymbol{k}\boldsymbol{x}} \tag{3.34}$$

The mode functions depend only on time τ due to isotropy and homogeneity of spatial dimensions. Variation of the action, gives the equation of motion for the $\upsilon_{\boldsymbol{k}}(\tau)$

$$\upsilon_k''(\tau) + \left(k^2 - \frac{z''}{z}\right)\upsilon_k(\tau) = 0 \tag{3.35}$$

This is an equation for motion for an harmonic oscillator with time dependent frequency. It is crucial to notice that z is a function of the slow roll parameter ϵ. In single field slow roll inflation, ϵ is written in terms of the potential and the first derivative (check relation (2.37)). This means that the solutions for the modes υ_k depend on the dynamics of the background. One can in principle solve (3.35) numerically. Instead, we will try to find an approximation scheme which will lead to analytical solutions.

Since we are dealing with a second order differential equation, we need two boundary conditions to uniquely specify the mode solutions.

First Boundary Condition

$$< \upsilon_k, \upsilon_k > \equiv \frac{i}{\hbar}(\upsilon_k' \upsilon_k^* - \upsilon_k^{*\prime}\upsilon_k) = 1 \tag{3.36}$$

This boundary condition results from canonically quantizing the modes υ_k and demanding that the creation and anihilation operators satisfy the usual commutation relations (1.25).

Second Boundary Condition

The second boundary condition is the so-called vacuum selection. As we discussed in chapter 1, in quantum field theory in curved backround there is not a uniquely defined vacuum as opposed to the Minkowski background, in which all inertial observers agree on the definition of the vacuum, and subsequently on the particle content of a specific state.

A similar situation holds when we deal with the quantum mechanics of an harmonic oscillator with time dependent frequency. Vacuum in not uniquely defined. By assuming that the slow roll parameter is slowly varying (derivatives with respect to time vanish ,i.e $\dot{\epsilon} = 0$), we find that

$$\frac{z''}{z} = \frac{\alpha''}{\alpha} = \frac{2}{\tau^2} \tag{3.37}$$

We see that in the fast past $\tau \to -\infty$

$$\frac{z''}{z} \to 0$$

This means that deep inside the horizon, the modes do not feel the expansion of the Universe and it is as if they live in a Minkowski background. In Minkowski space, there is a preferred vacuum , the one which minimizes the energy of the system. The choice of this preferred vacuum uniquely fixes the mode functions. Therefore the second boundary condition corresponds to the so called Bunch-Davies vacuum, i.e

$$\lim_{\tau \to -\infty} \upsilon_k = \frac{e^{-ik\tau}}{\sqrt{2k}} \tag{3.38}$$

The unique solution of (3.35) satisfying the two boundary conditions is

$$\upsilon_k = \frac{e^{-ik\tau}}{\sqrt{2k}}(1 - \frac{i}{k\tau}) \tag{3.39}$$

Since we have obtained the mode functions, we can plug the solutions in (3.31) and find the solution for the comoving curvature perturbation ζ.

The primordial scalar power spectrum $P_\zeta(k)$ is related to the 2-point function of the fluctuations ζ. Namely,

$$< \zeta_{\boldsymbol{k}} \zeta^*_{\boldsymbol{k'}} >= (2\pi)^3 \delta(\boldsymbol{k} - \boldsymbol{k'}) P_\zeta(k) \tag{3.40}$$

The dimensionless power spectrum for the scalar fluctuations computed at horizon crossing, $k = \alpha H$, is given by

$$\Delta^2_\zeta \equiv \frac{k^3}{2\pi^2} P_\zeta(k) = \frac{\hbar G H^2}{\pi c^5} \frac{1}{\epsilon}\bigg|_{k=\alpha H} \tag{3.41}$$

This result is the famous power spectrum of the primordial scalar fluctuations produced during cosmic inflation.

We imediately realize the dependence of the power spectrum from the energy scale of inflation H as well as the model of inflation we use for the calculation (hidden into the definition of ϵ). Using a different potential for inflation, leads to a change in ϵ which in turn changes the power spectrum.

An important feature of the scalar power spectrum is that it appears to be almost scale invariant.

$$P_\zeta(k) = A_s \left(\frac{k}{k_*}\right)^{n_s - 1} \tag{3.42}$$

where A_s is the amplitude, k_* is a pivot scale and n_s the so-called spectral index which measures the deviation from scale invariance. The observed values for these parameters are the following [4][5]

$$A_s \to 10^{-9}$$
$$n_s \to 0.96$$

Note that $n_s = 1$ corresponds to a perfect scale invariant power spectrum.

3.3 Primordial Tensor Power Spectrum- A rigorous derivation

The calculation for the tensor fluctuations turns out to be much simpler. The reason is that tensor fluctuations are gauge invariant. Under a general gauge transformation (namely under a change of coordinates) they remain invariant. This was not the case for the scalar fluctuations. In this section we are interested in computing the power spectrum of tensor fluctuations slightly more rigorously than the way we did for the scalar fluctuations. In other words, we will compute the spectrum up to first order in the slow roll ϵ. For extra details, the reader is advised to review [39][40].

Tensor fluctuations are derived from the Einstein-Hilbert action by expanding up to second order in the fluctuations.

$$S_{(2)} = \frac{M_{pl}^2}{8} \int d\tau d^3x \alpha^2 \left[(h'_{ij})^2 - (\partial_l h_{ij})^2 \right] \tag{3.43}$$

where prime denotes derivative ith respect to conformal time. Going to Fourier space, the graviton field can be decomposed in terms of creation and anihilation operators, i.e

$$h_{ij}(x) = \frac{2}{M_{pl}} \sum_{s=x,+} \int \frac{d^3k}{(2\pi)^3} \left[\epsilon_{ij}^s(k) h^s(k,\tau) \hat{a}_{\boldsymbol{k},s} e^{i\boldsymbol{kx}} + \epsilon_{ij}^{*s}(k) h^{s*}(k,\tau) \hat{a}_{\boldsymbol{k},s}^\dagger e^{-i\boldsymbol{kx}} \right] \tag{3.44}$$

where s denotes the 2 different values of the polarization for the tensor modes. $\hat{a}_{\boldsymbol{k},s}^\dagger$ and $\hat{a}_{\boldsymbol{k},s}$ are the creation and anihilation operators respectively which satisfy the usual commutation relations (check chapter 1).

Defining

$$\upsilon^s(k,\tau) \equiv \frac{\alpha}{2} M_{pl} h^s(k,\tau) \tag{3.45}$$

we can rewrite the action in the following way

$$S_{(2)} = \sum_s \frac{1}{2} \int d\tau d^3k \left[\Big((\upsilon^s(k,\tau))' \Big)^2 - \left(k^2 - \frac{\alpha''}{\alpha} \right) \Big(\upsilon^s(k,\tau) \Big)^2 \right] \tag{3.46}$$

Given this action, the equation of motion can be easily found

$$\left[\partial_\eta^2 + k^2 - \frac{\alpha''}{\alpha}\right]\upsilon(k,\tau) = 0 \tag{3.47}$$

where $\frac{\alpha''}{\alpha}$ is equal to [40]

$$\frac{\alpha''}{\alpha} = \frac{2-\epsilon}{(1-\epsilon)^2}\frac{1}{\tau^2} + O(\epsilon') \tag{3.48}$$

Note that we are interested in solving (3.47) in power law inflation and for small values of ϵ, i.e $\epsilon << 1$ and the scale factor to be of the form,

$$\alpha(\tau) = \Big((\epsilon-1)H_0\tau\Big)^{\frac{1}{\epsilon-1}} \tag{3.49}$$

where

$$H_0 = H\alpha^\epsilon \tag{3.50}$$

Moreover, we assume that the change of ϵ with time is negligible. In other words, $\epsilon' << 1$.

Plugging (3.48) into (3.47) and introducing a new variable $\tilde{\tau} = -k\tau$ we find

$$\left[\frac{d^2}{d\tilde{\tau}^2} + 1 - \frac{2-\epsilon}{(1-\epsilon)^2}\frac{1}{\tilde{\tau}^2}\right]\upsilon(k,\tau) = 0 \tag{3.51}$$

Notice that for $\epsilon = 0$, i.e for perfect de Sitter, the scale factor (3.49) and the equation of motion (3.51) reduce to the well know results [47]. The differential equation (3.50) is the so-called Bessel differential equation. The solution of this equation is given in terms of the Hankel functions

$$\upsilon(k,\tau) = \sqrt{\frac{-\pi\tau}{4}}H_\nu^{(1)}(k\tau) \tag{3.52}$$

and

$$\upsilon^*(k,\tau) = \sqrt{\frac{-\pi\tau}{4}}H_\nu^{(2)}(k\tau) \tag{3.53}$$

where

$$\nu = \frac{3-\epsilon}{2(1-\epsilon)} \tag{3.54}$$

The power spectrum for tensor fluctuations is given as usual by computing the 2-point function [40], i.e

$$< h_{ij}(\boldsymbol{x},\tau)h_{kl}(\boldsymbol{x'},\tau) >\equiv \int \frac{dk}{k}\Delta_h(k,\tau)\frac{\sin(kr)}{kr}\frac{1}{4}[tensor]_{ijkl} \tag{3.55}$$

where $r = |\boldsymbol{x} - \boldsymbol{x'}|$ and the tensor structure is the following

$$Tensor_{ijkl} = \Big[P_{ik}P_{jl} + P_{il}P_{jk} - P_{ij}P_{kl}\Big] \tag{3.56}$$

with P_{ik} to be the so called transverse operator in k-space [40]

$$P_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2} \tag{3.57}$$

The dimensionless power spectrum for the tensors is given by

$$\Delta_h(k,\tau) = \frac{4k^3}{\pi^2}\frac{|h(k,\tau)|^2}{M_{pl}^2} \tag{3.58}$$

As we have already discussed in the beginning of this chapter, when fluctuation modes enter on superhorizon scales, they freeze. We are interested in computing the power spectrum on superhorizon scales. One can expand expand the Hankel functions in terms of Bessel functions and after some computation [40], by using some identities for the Bessel functions, we end up with an expression for the primordial tensor power spectrum

$$\Delta_h(k,\tau) = \frac{2H_0^2}{\pi^2 M_{pl}^2}2^{\frac{3-\epsilon}{1-\epsilon}}\Gamma^2\left(\frac{3-\epsilon}{2(1-\epsilon)}\right)(1-\epsilon)^{\frac{2}{1-\epsilon}}\left(\frac{k}{H_0}\right)^{-\frac{2\epsilon}{1-\epsilon}} \tag{3.59}$$

Keeping terms up to the leading order in ϵ we finally find that

$$\Delta_h(k,\tau) = \Delta_{h*}\left(\frac{k}{k_*}\right)^n \tag{3.60}$$

where

$$\Delta_{h*} = \frac{2H_0^2}{\pi^2 M_{pl}^2}\Big[1 + 2\epsilon(1-\gamma_E - \ln 2)\Big] \tag{3.61}$$

and

$$n = -2\epsilon \tag{3.62}$$

Comments

• The first thing we should stress once more is that the result (3.61) is given for a power law inflation, i.e for a scale factor satisfying (3.49), for small non vanishing slow roll ϵ. One can of course recover the de-Sitter space result, i.e $\epsilon = 0$. The computation for the de-Sitter case can be found in [39][40].

• Another important feature of the tensor power spectrum which we should notice is the dependence of (3.61) upon the slow roll ϵ. The 0^{th} order expansion is ϵ is

$$P_* = \frac{2H_0^2}{\pi^2 M_{pl}^2} \tag{3.63}$$

Recall that in the case of scalar fluctuations, the power spectrum is enchanced by a factor $\frac{1}{\epsilon}$. Considering a value for $\epsilon < 0.01$ [4][5], we conclude that the signal of primordial scalar fluctuations is at least 100 times bigger than the tensorial one. This is the reason why we have measured the scalars and not the tensor modes.

• This result shows that the power spectrum is directly related to H^2. This means that detection of the primordial tensor fluctuations (gravitational waves from the early Universe) gives a direct piece of information about the energy scale that inflation occured. Combining with the spectrum of primordial scalar fluctuations, can also provide information about the inflationary model consistent with the observational values.

• The equation of motion for the tensor fluctuations is given by

$$u^{''} + 3Hu^{'} + \frac{k^2}{\alpha^2}u = 0 \tag{3.64}$$

where the derivatives are with respect to the conformal time τ. Recall that the equation of motion for the scalar fluctuations υ is the following

$$\upsilon^{''} + \left[3H + \frac{\dot{\epsilon}}{\epsilon}\right]\upsilon^{'} + \frac{k^2}{\alpha^2}\upsilon = 0 \tag{3.65}$$

In perfect de-Sitter space, i.e $\epsilon = 0$, the equations of motion for the scalar and tensor fluctuations coincide [40]. In other words, if we compute the power spectrum of scalar modes, it is a trivial exercise two generalize it for the tensors (tensors are equivalent to two copies of MMC scalars. [39])

• One comment about scalar fluctuations. By following exactly the same procedure as for the tensors, we find that the primordial power spectrum for scalars is

$$P_\zeta(k,\tau) = P_\zeta^*\left(\frac{k}{(1-\epsilon)\alpha H}\right)^{n_s-1} \tag{3.66}$$

where

$$P_\zeta^* = \frac{\Big[1 + 2\epsilon(5 - 3\ln 2 - 3\gamma_E) - 2\eta(2 - \ln 2 - \gamma_E)\Big] H^2}{8M_{pl}^2\pi^2\epsilon} \tag{3.67}$$

and

$$n_s - 1 = (-6\epsilon + 2\eta) + \frac{2}{3}\Big(-13\epsilon^2 + 4\epsilon\eta + \eta^2\Big) \tag{3.68}$$

The result of this computation is more general that the one we found in section 3.2.2. At order $O(\frac{1}{\epsilon})$, there is an agreement between the two results.

• One last comment. At leading order in the slow roll, the spectral index for the scalar fluctuations is

$$n_s - 1 = -6\epsilon + 2\eta \tag{3.69}$$

while for the tensor fluctuations is

$$n_t = -2\epsilon \tag{3.70}$$

The scalar to tensor ration is equal to

$$r = 16\epsilon \tag{3.71}$$

Combining (3.70) and (3.71), we end up with the consistency relation

$$r = -8n_t \tag{3.72}$$

Chapter 4

Loop corrections to primordial power spectra

In this chapter we start the discussion of loop corrections to primordial power spectra. In particular, we quantify the magnitude of the loop corrections and discuss the motivation for studing those effects. For further information, the reader is advised to review [48] [49].

4.1 ϵ-suppression patterns

In the previous chapter, the primordial scalar and tensor power spectra were computed. The computations were based on free actions (3.30) and (3.43). What we are now interested in, is to identify the dependence of the action as well as the power spectra upon the slow roll parameter ϵ.

Recall that the free field action (3.30) is suppressed by a factor $\frac{\dot{\phi}^2}{H^2}$ which is related to ϵ via

$$\epsilon = \frac{1}{2}\frac{\dot{\phi}^2}{H^2} \tag{4.1}$$

Therefore,

$$S_{(2)} = \int d^4x\alpha^3\boldsymbol{\epsilon}\left[\dot{\zeta}^2 - \alpha^{-2}(\partial_i\zeta)^2\right] \tag{4.2}$$

The scalar power spectrum computed using the free action (4.2), after reintroducing units, is given by

$$\Delta_\zeta^2 = \frac{\hbar G H^2}{\pi c^5}\frac{1}{\boldsymbol{\epsilon}} \tag{4.3}$$

For the tensor power spectrum, recall that the free action which was the calculation based on, was the following

$$S_{(2)} = \frac{M_{pl}^2}{8}\int d\tau d^3x\alpha^2\left[(h_{ij}')^2 - (\partial_l h_{ij})^2\right] \tag{4.4}$$

The resulting power spectrum was found to be

$$\Delta_t^2 = \frac{16\hbar G H^2}{\pi c^5} \tag{4.5}$$

We also introduced the so-called tensor to scalar ration r which is given by

$$r = \frac{\Delta_t^2}{\Delta_\zeta^2} = 16\epsilon$$

Remark

A crucial difference between the primordial scalar and tensor power spectra is that the former is suppressed by $\frac{1}{\epsilon}$ whereas the latter is independent of ϵ. Assuming a value of $\epsilon \sim 0.07$ [4][5] during primordial inflation, we conclude that the scalar power spectrum is enhanced by a factor of 100 compared to tensor power spectrum. This is the reason why we have measured the signal of scalar modes coming from the early universe and still haven't detected tensor modes (B mode polarization). Moreover, the appearance of the slow roll parameter in the actions $S_{(2)}$ also deserves a comment. In perfect de Sitter space ($\epsilon = 0$) the action vanishes. This corresponds to a pure gauge mode [46]. In order to make a non trivial calculation, we need the slow roll parameter ϵ to have a small non-zero value. In other words, we perform the calculation in a quasi-de Sitter space. In physical terms, for $\epsilon = 0$ (i.e perfect de-Sitter space), inflation cannot occur since the potential is flat (no-slow roll).

Finally, as already mentioned, the primordial power spectra computed in chapter 3, correspond to tree level results. They are just free propagators which we denote them as

Graviton

scalar

ζ self interactions

In 2002, J.Maldacena was the first one to calculate the 3-point function $< \zeta_{\boldsymbol{k_1}} \zeta_{\boldsymbol{k_2}} \zeta_{\boldsymbol{k_3}} >$, the so-called primordial non-Gaussianity [46]. In 2006, the 4-point function was computed $< \zeta_{\boldsymbol{k_1}} \zeta_{\boldsymbol{k_2}} \zeta_{\boldsymbol{k_3}} \zeta_{\boldsymbol{k_4}} >$ [50]. Finally, 5 and 6-point interactions were calculated [51].

In order to find the contributions of ζ self couplings to the primordial scalar power spectrum, we need to have the explicit expressions for the Lagrangians from which we will deduce the Feynman rules. The corresponding expressions are [46][50][51]

$$\frac{1}{\hbar}\mathcal{L}_{\zeta^3} \propto \frac{c^4\boldsymbol{\epsilon}^2\alpha^{D-1}}{16\hbar\pi G}\zeta(\partial\zeta)^2 \tag{4.6}$$

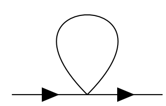
$$\frac{1}{\hbar}\mathcal{L}_{\zeta^4} \propto \frac{c^4\boldsymbol{\epsilon}^2\alpha^{D-1}}{16\hbar\pi G}\zeta^2(\partial\zeta)^2 \tag{4.7}$$

$$\frac{1}{\hbar}\mathcal{L}_{\zeta^5} \propto \frac{c^4\boldsymbol{\epsilon}^3\alpha^{D-1}}{16\hbar\pi G}\zeta^3(\partial\zeta)^2 \tag{4.8}$$

$$\frac{1}{\hbar}\mathcal{L}_{\zeta^6} \propto \frac{c^4\boldsymbol{\epsilon}^3\alpha^{D-1}}{16\hbar\pi G}\zeta^4(\partial\zeta)^2 \tag{4.9}$$

Note that the ζ^3 and ζ^4 interactions are suppressed by a factor ϵ^2 whereas the 5 and 6 point interactions are suppressed by ϵ^3.

The Feynman diagrams which contribute to 1-loop quantum corrections to the primordial scalar power spectrum are the following :



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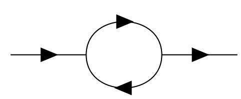
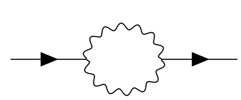


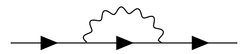
$\boldsymbol{\zeta}$-h interactions

In case we are interested in interactions between the scalar ζ and the graviton h, the corresponding Lagrangians are the following [52]

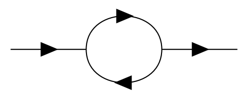
$$\frac{1}{\hbar}\left[\mathcal{L}_{\zeta h^2} + \mathcal{L}_{\zeta^2 h} + \mathcal{L}_{\zeta^2 h^2}\right] \propto \frac{c^4\boldsymbol{\epsilon}\alpha^{D-1}}{16\hbar\pi G}\left[\zeta(\partial h)^2 + h(\partial\zeta)^2 + h^2(\partial\zeta)^2\right] \tag{4.11}$$

These interaction vertices give rise to the following Feynman diagrams which contribute to the scalar power spectrum





Since we have explicit expressions for the interactions that appear in the Lagrangians above, we can quantify the maginitude of the loop corrections. Let's take as a first example the Feynman diagram



(4.12)

The mathematical expression for this Feynman diagram has the form

$$\int d^D y \int d^D y' \imath\Delta_\zeta(x;y)\imath\Delta_\zeta(x';y')\left[\imath\Delta_\zeta(y;y')\right]^2 V_{\zeta^3}(y')V_{\zeta^3}(y) \tag{4.13}$$

where points x and x' are fixed and points y and y' are integrated over since they correspond to vertices. Moreover, $V_{\zeta^3}(y)$ and $V_{\zeta^3}(y')$ correspond to vertex functions which can be deduced from the corresponding Lagrangian with 3-point interactions.

Quantifying the corrections

We can now evaluate the strength of each diagram taking into account the following rules

• For every scalar propagator Δ_ζ that appear in mathematical expressions similar to (4.13), we include a factor $\frac{\hbar G H^2}{c^5}\frac{1}{\boldsymbol{\epsilon}}$.
• For every graviton propagator Δ_h, we include a factor of the form $\frac{\hbar G H^2}{c^5}$.
• For each vertex with any number of graviton lines and $2N$ or $2N-1$ scalar lines ζ, we include a factor $\frac{c^5\boldsymbol{\epsilon}^N}{\hbar G H^2}$.

Case: ζ self-coupling corrections to scalar power spectrum

Given these rules, for the Feynman diagram, which contains vertices with 4 scalar lines (4.12), we find that

$$\left[\frac{\hbar G H^2}{c^5}\frac{1}{\boldsymbol{\epsilon}}\right]^4 \times \left[\frac{c^5\boldsymbol{\epsilon}^2}{\hbar G H^2}\right]^2 = \left[\frac{\hbar G H^2}{c^5}\frac{1}{\boldsymbol{\epsilon}}\right] \times \left[\frac{\hbar G H^2}{c^5}\boldsymbol{\epsilon}\right] \equiv \Delta_\zeta^2 \times \left[\frac{\hbar G H^2}{c^5}\boldsymbol{\epsilon}\right] \tag{4.14}$$

where Δ_ζ^2 is the tree level 2 point function we found above (4.3).

Similar computation for the other Feynman diagram (4.10), which contains one vertex with 4 scalar lines, gives a contribution to the scalar power spectrum of the form

$$\left[\frac{\hbar G H^2}{c^5}\frac{1}{\boldsymbol{\epsilon}}\right]^3 \times \left[\frac{c^5\boldsymbol{\epsilon}^2}{\hbar G H^2}\right]^1 = \left[\frac{\hbar G H^2}{c^5}\frac{1}{\boldsymbol{\epsilon}}\right] \times \left[\frac{\hbar G H^2}{c^5}\right] \equiv \Delta_\zeta^2 \times \left[\frac{\hbar G H^2}{c^5}\right] \tag{4.15}$$

Case: ζ-h interaction corrections to scalar power spectrum

Taking into account vertices which arise from the Lagrangians (4.11), i.e interactions of the form

- ζh^2
- $\zeta^2 h$
- $\zeta^2 h^2$

we get the following contributions to the scalar power spectrum.

For the $\zeta^2 h$ vertex:

$$\left[\frac{\hbar G H^2}{c^5}\frac{1}{\boldsymbol{\epsilon}}\right]^3 \times \left[\frac{\hbar G H^2}{c^5}\right]^1 \times \left[\frac{c^5 \boldsymbol{\epsilon}}{\hbar G H^2}\right]^2 = \left[\frac{\hbar G H^2}{c^5}\frac{1}{\boldsymbol{\epsilon}}\right] \times \left[\frac{\hbar G H^2}{c^5}\right] \equiv \Delta_\zeta^2 \times \left[\frac{\hbar G H^2}{c^5}\right] \tag{4.16}$$

For the $\zeta^2 h^2$ vertex:

$$\left[\frac{\hbar G H^2}{c^5}\frac{1}{\boldsymbol{\epsilon}}\right]^2 \times \left[\frac{\hbar G H^2}{c^5}\right]^1 \times \left[\frac{c^5 \boldsymbol{\epsilon}}{\hbar G H^2}\right]^1 = \left[\frac{\hbar G H^2}{c^5}\frac{1}{\boldsymbol{\epsilon}}\right] \times \left[\frac{\hbar G H^2}{c^5}\right] \equiv \Delta_\zeta^2 \times \left[\frac{\hbar G H^2}{c^5}\right] \tag{4.17}$$

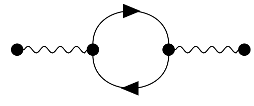
For the ζh^2 vertex:

$$\left[\frac{\hbar G H^2}{c^5}\frac{1}{\boldsymbol{\epsilon}}\right]^2 \times \left[\frac{\hbar G H^2}{c^5}\right]^2 \times \left[\frac{c^5 \boldsymbol{\epsilon}}{\hbar G H^2}\right]^2 = \left[\frac{\hbar G H^2}{c^5}\frac{1}{\boldsymbol{\epsilon}}\right] \times \left[\frac{\hbar G H^2}{c^5}\boldsymbol{\epsilon}\right] \equiv \Delta_\zeta^2 \times \left[\frac{\hbar G H^2}{c^5}\boldsymbol{\epsilon}\right] \tag{4.18}$$

We thus conclude that the corrections to the scalar power spectrum due to interactions between ζ and h are suppressed by the factor $\frac{\hbar G H^2}{c^5}$. In case of ζh^2 vertex, there is an extra ϵ suppression.

ζ-h interaction corrections to tensor power spectrum

Following the same rules as before, we can find the strength of loop corrections to tensor power spectrum due to interactions between ζ and gravitons. The corresponding Feynman diagrams are the following:



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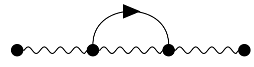
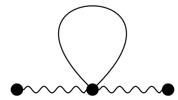


(4.20)



(4.21)

As an example to illustrate, we compute the contribution of the Feynman diagrams (4.19) and (4.21) to the tensor power spectrum and we find that

For the $h\zeta^2$ vertex:

$$\left[\frac{\hbar G H^2}{c^5}\frac{1}{\boldsymbol{\epsilon}}\right]^2 \times \left[\frac{\hbar G H^2}{c^5}\right]^2 \times \left[\frac{c^5\boldsymbol{\epsilon}}{\hbar G H^2}\right]^2 = \left[\frac{\hbar G H^2}{c^5}\right] \times \left[\frac{\hbar G H^2}{c^5}\right] \equiv \Delta_h^2 \times \left[\frac{\hbar G H^2}{c^5}\right] \tag{4.22}$$

For the $h^2\zeta^2$ vertex:

$$\left[\frac{\hbar G H^2}{c^5}\frac{1}{\boldsymbol{\epsilon}}\right]^1 \times \left[\frac{\hbar G H^2}{c^5}\right]^2 \times \left[\frac{c^5\boldsymbol{\epsilon}}{\hbar G H^2}\right]^1 = \left[\frac{\hbar G H^2}{c^5}\right] \times \left[\frac{\hbar G H^2}{c^5}\right] \equiv \Delta_h^2 \times \left[\frac{\hbar G H^2}{c^5}\right] \tag{4.23}$$

For the $h^2\zeta$ vertex:

$$\left[\frac{\hbar G H^2}{c^5}\frac{1}{\boldsymbol{\epsilon}}\right]^1 \times \left[\frac{\hbar G H^2}{c^5}\right]^3 \times \left[\frac{c^5\boldsymbol{\epsilon}}{\hbar G H^2}\right]^2 = \left[\frac{\hbar G H^2}{c^5}\right] \times \left[\frac{\hbar G H^2}{c^5}\boldsymbol{\epsilon}\right] \equiv \Delta_h^2 \times \left[\frac{\hbar G H^2}{c^5}\boldsymbol{\epsilon}\right] \tag{4.24}$$

The conclusion of the analysis above is clear. Loop corrections to primordial power spectra experience a suppression compared to the tree level scalar and tensor propagators by a factor $\boldsymbol{\frac{\hbar G H^2}{c^5} \sim 10^{-10}}$. Some diagrams, on the top of this suppression, they get suppressed by the slow roll parameter ϵ. Therefore, 1-loop corrections to the tree level results are extremely small.

4.2 Secular growth effects

Taking into account the analysis of the previous section, it is reasonable to ask whether there is an importance in studying loop corrections to the tree level primordial power spectra. The 1-loop corrections experience a huge suppression compared to the tree level results which make it impossible to be experimentally proved soon. However, in some cases loop corrections to various propagators can create the so-called 'secular growth effects'. In order to understand where these effects come from, recall that the equations of motion for the scalar and tensor modes υ and u are given by

- Tensor modes

$$u^{''} + 3Hu^{'} + \frac{k^2}{\alpha^2}u = 0 \tag{4.25}$$

- Scalar modes

$$\upsilon^{''} + \left[3H + \frac{\dot{\epsilon}}{\epsilon}\right]\upsilon^{'} + \frac{k^2}{\alpha^2}\upsilon = 0 \tag{4.26}$$

At first horizon crossing, at time $t = t_*$, and for simplicity assuming that $\dot{\epsilon} = 0$, the mode functions are given by [53]

$$|u(t,k)|^2 \rightarrow \frac{H^2(t_*)}{2k^3}\frac{1}{\pi}\Gamma^2\left(\frac{1}{2}+\frac{1}{1-\epsilon}\right)[2(1-\epsilon)]^{\frac{2}{1-\epsilon}} \tag{4.27}$$

$$|\upsilon(t,k)|^2 \rightarrow \frac{H^2(t_*)}{2\epsilon(t_*)k^3}\frac{1}{\pi}\Gamma^2\left(\frac{1}{2}+\frac{1}{1-\epsilon}\right)[2(1-\epsilon)]^{\frac{2}{1-\epsilon}} \tag{4.28}$$

Initially, during inflation and before they experience the first horizon crossing, the scalar and tensor modes are oscillating [48]

$$u(k,t) \propto \frac{1}{\alpha(t)\sqrt{2k}}e^{-ik\int\frac{dt'}{\alpha(t')}} \tag{4.29}$$

$$\upsilon(k,t) \propto \frac{1}{\alpha(t)\sqrt{2k\epsilon(t)}}e^{-ik\int\frac{dt'}{\alpha(t')}} \tag{4.30}$$

This transition from the oscillating behaviour at early times, to a constant field configuration after first horizon crossing is the so-called freezing that primordial scalar and tensor modes experience. This is the reason why these modes can survive at later times [53].

As we already mentioned in chapters 2 and 3, during Inflation, the comoving Hubble horizon is shrinking. This means that as long as inflation occurs, more and more oscillating modes experience first horizon crossing and freeze. This results in secular growth to the scalar and tensor 2-point functions, i.e

$$i\Delta_h(x;x')|_{secular} = \frac{1}{4\pi^2}\int^{H\alpha}\frac{dk}{k}H^2(t_*)\Gamma^2\left(\frac{1}{2}+\frac{1}{1-\epsilon}\right)[2(1-\epsilon)]^{\frac{2}{1-\epsilon}} \tag{4.31}$$

$$i\Delta_\zeta(x;x')|_{secular} = \frac{1}{4\pi^2}\int^{H\alpha}\frac{dk}{k}\frac{H^2(t_*)}{\epsilon(t_*)}\Gamma^2\left(\frac{1}{2}+\frac{1}{1-\epsilon}\right)[2(1-\epsilon)]^{\frac{2}{1-\epsilon}} \tag{4.32}$$

For perfect de-Sitter space, it was first found [53]

$$i\Delta_h(x;x')|_{secular} = \frac{H^2}{4\pi^2}\ln(\sqrt{\alpha(t)\alpha(t')}) \tag{4.33}$$

Woodard, Park, Prokopec and collaborators, have calculated quantum loop corrections to various propagators. The secular effects can occur in case the particles under study can keep on interacting for a sufficient period of time.

- For a graviton loop correction to the MMC scalar propagator, no secular growth occurs [54][55].

This is an expected result since graviton couples to the MMC field via its kinetic energy. Therefore, the vertex is suppressed in the IR limit (kinetic energy redshifts as the Universe expands) and we expect no significant late time effects.

• For a fermion propagator with a graviton loop, there is significant interaction due to the spin that both particles possess [17][56]

• Interactions between photons and gravitons is persisting at late times due to the spin that both particles carry. This results in secular growth effects. [18][22][57]

Finally, as already mentioned in the introduction, the potentials in de-Sitter acquire corrections proportional to $\ln \alpha$ which grow with time.

Therefore, we are interested in loop corrections to the tree level results since they may come with logarithms of the scale factor. This means that these effects may be stronger than we initially thought.

Chapter 5

Graviton self-energy 1 loop corrections due to Massive non minimally coupled scalar fields

5.1 Graviton self energy

The aim of this section is to find the form of the 1-loop graviton self energy due to the presence of non minimally massive scalar fields. Recall from chapter 1 the following scalar action

$$S_\phi = \int d^D x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{2} \xi R \phi^2 \right] \tag{5.1}$$

We are interested in performing the calculation is de-Sitter space. Therefore the graviton $h_{\mu\nu}$ is defined as a fluctuation around the classical de-Sitter background $\bar{g}_{\mu\nu}$, i.e

$$\kappa h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu} \tag{5.2}$$

where $\kappa = \sqrt{16\pi G}$, G is the Newton constant and $g_{\mu\nu}$ is the full metric.

In order to find the interaction vertices between scalars and gravitons, we have to expand the action (5.1) up to second order in the graviton $h_{\mu\nu}$. This can be done by expanding the metric $g_{\mu\nu}$, the Ricci scalar R and the determinant of the metric $\sqrt{-g}$ with respect to (5.2). After a tedious computation one obtains [1][58]

$$S = \bar{S} + S_{(3)} + S_{(4)} \tag{5.3}$$

where $\bar{S}$ is the background action, namely the action (5.1) with the substitution $g \to \bar{g}$, $R \to \bar{R}$, $g_{\mu\nu} \to \bar{g}_{\mu\nu}$.

The expansion up to first order in the graviton field $h_{\mu\nu}$, i.e expansion to order $O(\kappa)$, leads to the following form for the action

$$S_{(3)} = \frac{\kappa}{2} \int d^D x \sqrt{-\bar{g}} h^{\mu\nu}(x) \bar{T}_{\mu\nu}(x) \tag{5.4}$$

where $\bar{T}_{\mu\nu}(x)$ is the background stress energy tensor. Varying the action with respect to the graviton field and setting it to zero at the end of the calculation ($h_{\mu\nu} \to 0$), we find following expression

$$\bar{T}_{\mu\nu}(x) = \partial_\mu \phi(x) \partial_\nu \phi(x) - \frac{1}{2} \bar{g}_{\mu\nu} \partial_\alpha \phi(x) \partial^\alpha \phi(x) - \frac{1}{2} \left[\bar{g}_{\mu\nu} m^2 + \xi \Big(\bar{G}_{\mu\nu}(x) + \bar{g}_{\mu\nu} \Box - \bar{\nabla}_\mu \bar{\nabla}_\nu \Big) \right] \phi^2(x) \tag{5.5}$$

Note that in the massless ($m = 0$) and minimally coupled case ($\xi = 0$) we recover the well know result for the stress energy tensor in a general background $g_{\mu\nu}$ [34].

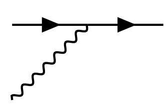
Finally, the $S_{(4)}$ action comes from the expansion up to second order in the fluctuations, i.e up to $O(\kappa^2)$

$$S_{(4)} = \frac{\kappa^2}{2} \int d^D x \int d^D x' \sqrt{-\bar{g}} h^{\mu\nu}(x) V_{\mu\nu\rho\sigma}(x - x') h^{\rho\sigma}(x') \tag{5.6}$$

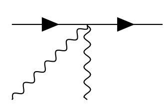
where $V_{\mu\nu\rho\sigma}(x - x')$ is a rather long expression which can be found at [1]. This expression will not be needed in the present work.

The corresponding vertices are the following:

- 3-point interaction



- 4-point interaction



where the wiggly lines correspond to gravitons and dash lines to scalar particles. By making use of the effective action, we find that [1]

$$-\imath \left[\Sigma_{\mu\nu\rho\sigma} \right](x; x') = -\imath \left[\Sigma_{\mu\nu\rho\sigma} \right]_{3pt}(x; x') + -\imath \left[\Sigma_{\mu\nu\rho\sigma} \right]_{4pt}(x; x') \tag{5.7}$$

where the three and four point contributions are given by [1]

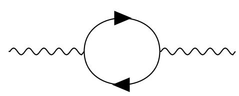
$$-\imath \left[\Sigma_{\mu\nu\rho\sigma}\right]_{3pt}(x;x') = -\frac{\kappa^2}{4}\sqrt{-\bar{g}(x)}\sqrt{-\bar{g}(x')} < \bar{T}_{\mu\nu}(x)\bar{T}_{\rho\sigma}(x') > \tag{5.8}$$

$$-\imath \left[\Sigma_{\mu\nu\rho\sigma}\right]_{4pt}(x;x') = \imath\kappa^2\sqrt{-\bar{g}(x)} < V_{\mu\nu\rho\sigma}(x-x') > \tag{5.9}$$

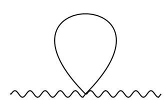
The TT correlator $< \bar{T}_{\mu\nu}(x)\bar{T}_{\rho\sigma}(x') >$ that appears in (5.8) is extremely relevant for our work. The aim of this thesis to rewrite this TT correlator in a form which could be renormalized. From (5.8) we conclude that renormalization of TT correlator is equivalent to renormalization of the cubic contribution to the graviton self energy.

Given the Feynman vertices above, the diagrams that contribute to the graviton self energy at 1-loop loop order are

• 3-point non local contribution



• 4-point local contribution



5.1.1 Graviton self energy ansatz

In the previous section, we obtained an expression for the self energy in terms of the TT correlator and the $V_{\mu\nu\rho\sigma}(x-x')$. We are now interested in obtaining another represention of the graviton self energy. Under symmetry considerations, it was shown in [14] [21], that assuming

• Traversality
• Symmetry under $\mu \leftrightarrow \nu$ and $\rho \leftrightarrow \sigma$
• Symmetry under $x \leftrightarrow x'$

the graviton self energy can be expressed as

$$\begin{aligned}-\imath \left[\Sigma^{\mu\nu\rho\sigma}\right](x;x') = &\sqrt{-\bar{g}(x)}\sqrt{-\bar{g}(x')}P^{\mu\nu}(x)P^{\rho\sigma}(x')F_0(y)+ \\ &+\sqrt{-\bar{g}(x)}\sqrt{-\bar{g}(x')}P^{\mu\nu}_{\alpha\beta\gamma\delta}(x)P^{\rho\sigma}_{\kappa\lambda\theta\phi}(x')T^{\alpha\kappa}T^{\beta\lambda}T^{\gamma\theta}T^{\delta\phi}\left(\frac{D-2}{D-3}\right)F_2(y)\end{aligned} \tag{5.10}$$

where $F_0(y)$ and $F_2(y)$ are the so called spin-0 and spin-2 structure functions which depend only on the de Sitter length function $y(x;x')$ which has the form

$$y(x;x') = \alpha\alpha' H^2 \left[||x-x'||^2 - (|\eta-\eta'| - i\epsilon)^2\right] \tag{5.11}$$

The latter is related to the geodesic distance $l(x;x')$, which connects two points x^μ and x'^μ on de Sitter, via the relation

$$y(x;x') = 4sin^2\left(\frac{H}{2}l(x;x')\right) \tag{5.12}$$

Moreover, the $T^{\alpha\kappa}(x;x')$ that appears in the self-energy ansatz is given by

$$T^{\alpha\kappa}(x;x') = -\frac{1}{2H^2}\frac{\partial^2 y(x;x')}{\partial x_\alpha \partial x'_\kappa} \tag{5.13}$$

$P^{\mu\nu}(x)$ and $P^{\mu\nu}_{\alpha\beta\gamma\delta}(x)$ are second order differential operators which can be found by expanding the scalar Ricci curvature and Weyl curvature around de Sitter background (5.2), i.e

$$R - D(D-1)H^2 = P^{\mu\nu}\kappa h_{\mu\nu} + O(\kappa^2 h^2) \tag{5.14}$$

$$C_{\alpha\beta\gamma\delta} = P^{\mu\nu}_{\alpha\beta\gamma\delta}\kappa h_{\mu\nu} + O(\kappa^2 h^2) \tag{5.15}$$

The form that $P^{\mu\nu}$ operator has is [14]

$$P^{\mu\nu} = \nabla^\mu\nabla^\nu - \bar{g}^{\mu\nu}\left[\Box + (D-1)H^2\right] \tag{5.16}$$

with ∇^μ correspond to the covariant derivative in de Sitter space and $\Box = \nabla^\mu\nabla_\mu$.

For the $P^{\mu\nu}_{\alpha\beta\gamma\delta}(x)$, we get [14]

$$\begin{aligned} P^{\mu\nu}_{\alpha\beta\gamma\delta} = \mathcal{D}^{\mu\nu}_{\alpha\beta\gamma\delta} + \frac{1}{D-2}\left[\bar{g}_{\alpha\delta}\mathcal{D}^{\mu\nu}_{\beta\gamma} - \bar{g}_{\beta\delta}\mathcal{D}^{\mu\nu}_{\alpha\gamma} - \bar{g}_{\alpha\gamma}\mathcal{D}^{\mu\nu}_{\beta\delta} + \bar{g}_{\beta\gamma}\mathcal{D}^{\mu\nu}_{\alpha\delta}\right] + \\ + \frac{1}{(D-1)(D-2)}\left[\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}\right]\mathcal{D}^{\mu\nu} \end{aligned} \tag{5.17}$$

where

$$\mathcal{D}^{\mu\nu}_{\alpha\beta\gamma\delta} \equiv \frac{1}{2}\left[\delta^{(\mu}_{\alpha}\delta^{\nu)}_{\delta}\nabla_\gamma\nabla_\beta - \delta^{(\mu}_{\beta}\delta^{\nu)}_{\delta}\nabla_\gamma\nabla_\alpha - \delta^{(\mu}_{\alpha}\delta^{\nu)}_{\gamma}\nabla_\delta\nabla_\beta + \delta^{(\mu}_{\beta}\delta^{\nu)}_{\gamma}\nabla_\delta\nabla_\alpha\right] \tag{5.18}$$

Contracting (5.18) with $\bar{g}^{\alpha\gamma}$ and $\bar{g}^{\alpha\gamma}\bar{g}^{\beta\delta}$ we get the following expressions for the other two operators $\mathcal{D}^{\mu\nu}_{\beta\delta}$ and $\mathcal{D}^{\mu\nu}$ that appear in (5.17)

$$\mathcal{D}^{\mu\nu}_{\beta\delta} = \frac{1}{2}\left[\delta^{(\mu}_{\delta}\nabla^{\nu)}\nabla_\beta + \delta^{(\mu}_{\beta}\nabla_\delta\nabla^{\nu)} - \delta^{(\mu}_{\beta}\delta^{\nu)}_{\delta}\Box - \bar{g}^{\mu\nu}\nabla_\delta\nabla_\beta\right] \tag{5.19}$$

$$\mathcal{D}^{\mu\nu} = \nabla^{(\mu}\nabla^{\nu)} - \bar{g}^{\mu\nu}\Box \tag{5.20}$$

5.2 Renormalization of the quartic contribution

As far as the four point contribution to the self energy is concerned, this has already been renormalized in [1] using dimensional regularization [33] to extract $\frac{1}{D-4}$ terms from the $< V_{\mu\nu\rho\sigma}(x-x') >$ which at the end are subtracted with the addition of appropriate counterterms. Here, we just give the final expression for the renormalized four point function and a brief discussion. For detailed analysis of the computation we refer the reader to [1]

The renormalized quartic contribution to the graviton self-energy is

$$-\imath \left[\Sigma^{ren}_{\mu\nu\rho\sigma}\right]_{4pt}(x;x') = [Tensor_1] \times \left[\frac{H^4}{64\pi^2}\frac{m^2}{H^2}\left(\frac{m^2}{H^2}+12\xi-2\right)+2\imath\Delta\left(\frac{\Lambda}{16\pi G}\right)_f\right] \times \delta^D(x-x') - \\ - [Tensor_2] \times \imath\Delta\left(\frac{1}{16\pi G}\right)_f \times \delta^4(x-x') \tag{5.21}$$

where

$$Tensor_1 = \imath\kappa^2\alpha^4\left[\frac{1}{4}\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma}-\frac{1}{2}\bar{g}_{\mu(\rho}\bar{g}_{\sigma)\nu}\right] \tag{5.22}$$

$$Tensor_2 = \imath\kappa^2\alpha^4\left[\frac{1}{2}\bar{g}_{\mu(\rho}\bar{g}_{\sigma)\nu}\Box-\frac{1}{2}\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma}\Box+\frac{1}{2}\left(\bar{g}_{\mu\nu}\bar{\nabla}_{(\rho}\bar{\nabla}_{\sigma)}+\bar{g}_{\rho\sigma}\bar{\nabla}_{(\mu}\bar{\nabla}_{\nu)}\right)-\bar{g}_{(\mu|(\rho}\bar{\nabla}_{\sigma)}\bar{\nabla}_{|\nu)}\right] \tag{5.23}$$

and the parameters $\imath\Delta\left(\frac{\Lambda}{16\pi G}\right)_f$, $\imath\Delta\left(\frac{1}{16\pi G}\right)_f$ are finite coefficients that can be found in [1].

Note that the counterterms which are added are the cosmological constant counterterm and the inverse Newton contanst G. In other words, the counterterm action is

$$S_{ct} = \int d^Dx\sqrt{-g(x)}\left[\Delta\left(\frac{1}{16\pi G}\right)R-2\Delta\left(\frac{\Lambda}{16\pi G}\right)\right]$$

Explicit expressions for the coefficientsin S_{ct} can be found in [1].
In the case of $m=0$ and $\xi=0$, considering that the finite parameters vanish in that limit (see page 33, [1]), the quartic vertex does not contribute to the self energy , i.e

$$-\imath\left[\Sigma^{ren}_{\mu\nu\rho\sigma}\right]_{4pt}(x;x') = 0 \tag{5.24}$$

This result is in agreement with the result of Woodard and Park [14] for the case of minimally coupled massless scalar fields.

5.3 Towards the renormalization of cubic vertex

5.3.1 Chernikov-Tagirov propagator: A closer look

In chapter 1, we found an expression for the so called Chernikov-Tagirov propagator, which describes the propagation of a massive non minimally coupled scalar field in a de Sitter background.

$$\imath\Delta(x,x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(\frac{D-1}{2}-\nu_D)\Gamma(\frac{D-1}{2}+\nu_D)}{\Gamma(\frac{D}{2})}\,{}_2F_1\left[\frac{D-1}{2}+\nu_D, \frac{D-1}{2}-\nu_D; \frac{D}{2}; 1-\frac{y}{4}\right] \tag{5.25}$$

By making use of the formula

$$\begin{aligned}{}_2F_1\left[\frac{D-1}{2}+\nu_D, \frac{D-1}{2}-\nu_D; \frac{D}{2}; 1-\frac{y}{4}\right] &= \frac{\Gamma(\frac{D}{2})\Gamma(1-\frac{D}{2})}{\Gamma(\frac{1}{2}-\nu_D)\Gamma(\frac{1}{2}+\nu_D)}\times \\ \times\, {}_2F_1\left[\frac{D-1}{2}+\nu_D, \frac{D-1}{2}-\nu_D; \frac{D}{2}; \frac{y}{4}\right] &+ \left(\frac{y}{4}\right)^{1-\frac{D}{2}}\frac{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-1)}{\Gamma(\frac{D-1}{2}-\nu_D)\Gamma(\frac{D-1}{2}+\nu_D)}\times \\ {}_2F_1\left[\frac{1}{2}+\nu_D, \frac{1}{2}-\nu_D; 2-\frac{D}{2}; \frac{y}{4}\right]\end{aligned} \tag{5.26}$$

and expanding the Gauss hypergeometric functions in power series,

$${}_2F_1\left[a,b;c;z\right] = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n \tag{5.27}$$

where $(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}$ denotes the so-called Pochhammer symbol, we obtain an expression for the scalar propagator written in terms of two infinite series. One containing D-dependent powers of y and the other D-independent. Namely,

$$G(y) = A\sum_{n=0}^{\infty}A_n\left(\frac{y}{4}\right)^n + B\sum_{n=0}^{\infty}B_n\left(\frac{y}{4}\right)^{(n+1-\frac{D}{2})} \tag{5.28}$$

where

$$A = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(\frac{D-3}{2}-\nu_D)\Gamma(\frac{D-3}{2}+\nu_D)}{\Gamma(\frac{1}{2}-\nu_D)\Gamma(\frac{1}{2}+\nu_D)}\left[\frac{(D-3)^2}{4}-\nu_D^2\right]\Gamma\left(1-\frac{D}{2}\right) \tag{5.29}$$

$$B = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(\frac{D}{2}-1\right) \tag{5.30}$$

$$A_n = \frac{\left(\frac{D-1}{2}+\nu_D\right)_n\left(\frac{D-1}{2}-\nu_D\right)_n}{\left(\frac{D}{2}\right)_n}\frac{1}{n!} \tag{5.31}$$

$$B_n = \frac{\left(\frac{1}{2}+\nu_D\right)_n \left(\frac{1}{2}-\nu_D\right)_n}{\left(2-\frac{D}{2}\right)_n} \frac{1}{n!} \tag{5.32}$$

$$\nu_D = \sqrt{\frac{(D-1)^2}{4} - \left(\frac{m^2}{H^2} + D(D-1)\xi\right)} \tag{5.33}$$

We should notice at this point that the propagator depends only on the de Sitter length function $y(x;x')$, which is related to the geodesic distance between two points x and x' in de Sitter. Moreover, in order to write the propagator as a function of $y(x;x')$, we have to assume that the vaccum state is invariant under the de Sitter group $O(1,D)$. For further details on vaccum states for massless and massive scalar fields on de-Sitter, we refer the reader to review the really nice paper by Bruce Allen [38].

***Important remark* :**
A careful reader may pointed out the offensive term $\Gamma\left(1-\frac{D}{2}\right)$ which resides inside the coefficient A, which in turn is part of the scalar propagator $G(y)$. Expanding around $D=4$ dimensions, we get

$$\Gamma\left(1-\frac{D}{2}\right) = \frac{2}{D-4}\left[1+\frac{D-4}{2}\left(-1+\gamma_E\right)\right] \tag{5.34}$$

where γ_E is the so-called Euler-Mascheroni constant. The propagator $G(y)$ seems to diverge in the $D=4$ limit. But this cannot be true! A divergent propagator is meaningless. A more careful inspection of the situation shows that the coefficients B_n also contain $\frac{1}{D-4}$ divergences for every $n>0$. At every order in powers of y, the coefficients in front of the divergent terms coming from the two sums exactly cancel. This property is preserved when we take derivatives of $G(y)$ with respect to y. In appendix A, a proof of the above statement can be found.

In our computation, it turns out that we will need the exact values of A_n for $n=0,1,2,3$ and B_n for $n=0,1,2,3,4$. These are given on the table 1 below

Table 1:Values of A_n and B_n for relevant n	
A_0	1
B_0	1
A_1	$\frac{\left(\frac{D-1}{2}\right)^2-\nu_D^2}{\frac{D}{2}}$
A_2	$\frac{\left[\left(\frac{D-1}{2}\right)^2-\nu_D^2\right]\left[\left(\frac{D+1}{2}\right)^2-\nu_D^2\right]}{D(\frac{D}{2}+1)}$
A_3	$\frac{\left[\left(\frac{D-1}{2}\right)^2-\nu_D^2\right]\left[\left(\frac{D+1}{2}\right)^2-\nu_D^2\right]\left[\left(\frac{D+3}{2}\right)^2-\nu_D^2\right]}{3D(\frac{D}{2}+1)(\frac{D}{2}+2)}$
B_1	$\frac{\frac{1}{4}-\nu_D^2}{2-\frac{D}{2}}$
B_2	$\frac{\left[\frac{1}{4}-\nu_D^2\right]\left[\frac{9}{4}-\nu_D^2\right]}{6-D}\ \frac{1}{2-\frac{D}{2}}$
B_3	$\frac{\left[\frac{1}{4}-\nu_D^2\right]\left[\frac{9}{4}-\nu_D^2\right]\left[\frac{25}{4}-\nu_D^2\right]}{6(3-\frac{D}{2})(4-\frac{D}{2})}\ \frac{1}{2-\frac{D}{2}}$
B_4	$\frac{\left[\frac{1}{4}-\nu_D^2\right]\left[\frac{9}{4}-\nu_D^2\right]\left[\frac{25}{4}-\nu_D^2\right]\left[\frac{49}{4}-\nu_D^2\right]}{24(3-\frac{D}{2})(4-\frac{D}{2})(5-\frac{D}{2})}\ \frac{1}{2-\frac{D}{2}}$

(5.35)

Notice that A_n do not contain $\frac{1}{D-4}$ whereas B_n for $n > 0$ contain.

5.3.2 TT-correlator

As it has already been mentioned above, the object we are interested in renomalizing is the TT correlator $< \bar{T}_{\mu\nu}(x)\bar{T}_{\rho\sigma}(x') >$. After a rather lengthy computation performed in [1], the final result is

$$\begin{aligned}
\frac{(4\pi)^D}{H^{2D-4}} < \bar{T}_{\mu\nu}(x)\bar{T}_{\rho\sigma}(x') >= & \left[\alpha_1 G\frac{d^4G}{dy^4} + \alpha_2\frac{dG}{dy}\frac{d^3G}{dy^3} + \alpha_3\left(\frac{d^2G}{dy^2}\right)^2\right] \times \left[\partial_\mu y\partial_\nu y\partial'_\rho y\partial'_\sigma y\right] + \\
& + \left[\beta_1 G\frac{d^3G}{dy^3} + \beta_2\frac{dG}{dy}\frac{d^2G}{dy^2}\right] \times \left[\partial_{(\mu} y\partial_{\nu)}\partial'_{(\rho} y\partial'_{\sigma)} y\right] + \\
& + \left[\gamma_1 G\frac{d^2G}{dy^2} + \gamma_2\left(\frac{dG}{dy}\right)^2\right] \times \left[\partial_{\mu)}\partial'_{(\rho} y\partial'_{\sigma)}\partial_{(\nu} y\right] + \\
& + H^2\left[(2-y)\delta_{11} G\frac{d^3G}{dy^3} + (2-y)\delta_{12}\frac{dG}{dy}\frac{d^2G}{dy^2} + \delta_{21}\left(\frac{dG}{dy}\right)^2 + \delta_{22} G\frac{d^2G}{dy^2}\right] \times \\
& \times \left[\bar{g}'_{\rho\sigma}\partial_\mu y\partial_\nu y + \bar{g}_{\mu\nu}\partial'_\rho y\partial'_\sigma y\right] + H^4\left[\bar{g}_{\mu\nu}\bar{g}'_{\rho\sigma}\right] \times \\
& \left[\epsilon_{31} G\frac{d^2G}{dy^2} + \epsilon_{33}\left(\frac{dG}{dy}\right)^2 + (4y-y^2)\epsilon_{11}\left(\frac{dG}{dy}\right)^2 + (2-y)\epsilon_{21} G\frac{dG}{dy} + \epsilon_{34} G^2\right]
\end{aligned} \tag{5.36}$$

where the coefficients which appear in the expectation value of the TT correlator are given in table 2.

Table 2: Explicit values of the coefficients that appear in TT correlator (5.36)	
α_1	$4\xi^2$
α_2	$-8\xi(1-2\xi)$
α_3	$2(1-4\xi+6\xi^2)$
β_1	$16\xi^2$
β_2	$4(1-8\xi+12\xi^2)$
γ_1	$8\xi^2$
γ_2	$2(1-2\xi)^2$
δ_{11}	$4\xi^2$
δ_{12}	$D-2-8\xi(D-1)+4\xi^2(4D-1)$
δ_{21}	$1-M^2+4\xi(D-1+2M^2)-4\xi^2(3D-1+4M^2)$
δ_{22}	$-M^2+8M^2\xi-4\xi^2(D+1+4M^2)$
ϵ_{11}	$\frac{1}{2}\Big[-(D-1)^2+2M^2+4\xi\Big((2D-1)(D-1)-4M^2\Big)-8\xi^2(2D^2-2D+1-4M^2)\Big]$
ϵ_{31}	$16\xi^2$
ϵ_{34}	$M^4-2M^2\xi(D-1+4M^2)+2\xi^2\Big((D-1)^2+2M^2(2D-3)+8M^4\Big)$
ϵ_{33}	$2(D^2-D-4)-16\xi(D^2-3)+16\xi^2(2D^2+2D-3)$
ϵ_{21}	$-(D-1)M^2+8DM^2\xi-4\xi^2(D-1+4(D+1)M^2)$

where

$$M^2 = \frac{m^2}{H^2} + D(D-1)\xi \tag{5.37}$$

***3 comments* :**

First, we notice that the coefficients in the table depend on the physical parameters of our problem (ξ, D, m and H) and they are independent of $y(x;x')$. This will be relevant in a bit and we will return back to this point.

Second, notice that the TT correlator has been written in terms of 5 tensor structures each one multiplied a coefficient which contains products of the scalar propagator and derivatives of it. As we will soon show, these coefficients can be further simplified and collected in a clever way in order to make the computation more tractable.

Third, the highest power derivative of the scalar propagator $G(y)$ that appears in the TT correlator is four. The existence of higher order derivatives than two, implies that the cubic vertex contribution to graviton self energy will contain many more divergent terms compared to the massless minimally coupled case. In the latter, third and fourth order derivatives were absent. This can be immediately seen from the TT correlator above. Taking the limit $\xi = 0$ and $m = 0$, all the coefficients which multiply $\frac{d^3G}{dy^3}$ and $\frac{d^4G}{dy^4}$, vanish.

5.3.3 Spin zero structure

Taking the trace of (5.10) , the tensor structure that multiplies the spin-2 structure function $F_2(y)$ vanishes , so we obtain an expression for the spin-0 part of the graviton self energy, i.e

$$\frac{\bar{g}_{\mu\nu}\bar{g}^{'}_{\rho\sigma}}{\sqrt{-g}\sqrt{-g^{'}}}(-\imath)\left[\Sigma^{\mu\nu\rho\sigma}\right](x;x') \equiv \bar{g}_{\mu\nu}\bar{g}^{'}_{\rho\sigma}P^{\mu\nu}(x)P^{\rho\sigma}(x')F_0(y) = (D-1)^2\left[\Box + DH^2\right]^2 F_0(y) \quad (5.38)$$

Note that in the last equality above of (5.38), we do not care about the primed and unprimed box operators since they act in the same way on the function $y(x;x')$.

Taking the trace of (5.7) or in other words, tracing the TT correlator, is a long computation which can be found in Appendix B. Note that (5.9) is a local contribution to the self energy, so at this point we do not need to take the trace of it. It will not contribute to the spin-zero part of the self energy. The result of this computation is

$$\begin{aligned}
&\frac{\bar{g}^{\mu\nu}}{\sqrt{-g}}\frac{\bar{g}^{'\rho\sigma}}{\sqrt{-g^{'}}}(-\imath)\left[\Sigma^{\mu\nu\rho\sigma}\right](x;x') \equiv -\frac{\kappa^2}{4}\bar{g}^{\mu\nu}\bar{g}^{'\rho\sigma} < \bar{T}_{\mu\nu}(x)\bar{T}_{\rho\sigma}(x') >= \\
&= -\frac{\kappa^2}{4}\frac{H^{2D}}{(4\pi)^D}\Bigg[(4y-y^2)^2\alpha_R(y) + (4y-y^2)(2-y)\beta_R(y) + \Big(4D-(4y-y^2)\Big)\gamma_R(y) + \qquad (5.39)\\
&+ 2D(4y-y^2)\delta_R(y) + D^2\epsilon_R(y)\Bigg]
\end{aligned}$$

where the coefficients $\alpha_R(y)$, $\beta_R(y)$, $\gamma_R(y)$, $\delta_R(y)$, $\epsilon_R(y)$ are given in the table 3 below.

Table 3: Coefficients of (5.39). Their origin can be traced back to TT correlator (5.36)
• $\alpha_R(y) = \alpha_1 GG^{''''} + \alpha_2 G^{'}G^{'''} + \alpha_3(G^{''})^2$
• $\beta_R(y) = \beta_1 GG^{'''} + \beta_2 G^{'}G^{''}$
• $\gamma_R(y) = \gamma_1 GG^{''} + \gamma_2(G^{'})^2$
• $\delta_R(y) = (2-y)\delta_{11}GG^{'''} + (2-y)\delta_{12}G^{'}G^{''} + \delta_{21}\left(G^{'}\right)^2 + \delta_{22}GG^{''}$
• $\epsilon_R(y) = \epsilon_{31}GG^{''} + \epsilon_{33}\left(G^{'}\right)^2 + (4y-y^2)\epsilon_{11}\left(G^{'}\right)^2 + (2-y)\epsilon_{21}GG^{'} + \epsilon_{34}G^2$

(5.40)

Note that they depend on $y(x;x')$ since they are written in terms of products of the scalar propagator and its corresponding derivatives which explicitly depend on $y(x;x')$.

Equating the relations (5.38) and (5.39) we find an equation for the spin zero structure

$$
\begin{aligned}
F_0(y) = &\frac{-\kappa^2 H^{2D}}{4(D-1)^2(4\pi)^D}\left[\frac{\Box}{H^2}+D\right]^{-2}\Bigg[G^2 Q_9(y) + G'^2\Big(Q_5(y)-Q_4(y)\Big)+ \\
&+(G'')^2\Big(Q_3(y)+Q_1(y)-Q_2(y)\Big)+(G^2)'\frac{Q_8(y)}{2}+(G^2)''\frac{Q_4(y)}{2}+(G^2)'''\frac{Q_6(y)}{2}+ \\
&+(G^2)''''\frac{Q_1(y)}{2}+\Big[(G')^2\Big]'\left(\frac{1}{2}Q_7(y)-\frac{3}{2}Q_6(y)\right)+\Big[(G')^2\Big]''\left(-2Q_1(y)+\frac{Q_2(y)}{2}\right)\Bigg]
\end{aligned} \tag{5.41}
$$

where the primes above denote derivates of the propagator $G(y)$ with respect to $y(x;x')$. In order to reach (5.41), we converted the expressions for the scalar propagator appearing in the coefficients $\alpha_R(y)$, $\beta_R(y)$, $\gamma_R(y)$, $\delta_R(y)$, $\epsilon_R(y)$ according to the following table

Table 4: Scalar propagator $\rightarrow$ derivative manipulations
• $GG' = \frac{1}{2}(G^2)'$
• $GG'' = \frac{1}{2}(G^2)'' - (G')^2$
• $GG''' = \frac{1}{2}(G^2)''' - \frac{3}{2}\left[(G')^2\right]'$
• $GG'''' = \frac{1}{2}(G^2)'''' - 2\left[(G')^2\right]'' + (G'')^2$
• $G'G'' = \frac{1}{2}\left[(G')^2\right]'$
• $G'G''' = \frac{1}{2}\left[(G')^2\right]'' - (G'')^2$

(5.42)

In other words, we expressed them in terms of G^2, $(G')^2$, $(G'')^2$ and their derivatives. The $Q_i(y)$'s that appear in (5.41), for $i = 1, 2, ..., 9$ are given in table 5. Note their explicit y dependence. Recall that the various α_i, β_i, γ_i, δ_{ij} and ϵ_{ij} do not depend on y (table 2).

Table 5: Values for the $Q_i's$ that appear in equation (5.41)
• $Q_1(y) = (4y-y^2)^2\alpha_1$
• $Q_2(y) = (4y-y^2)^2\alpha_2$
• $Q_3(y) = (4y-y^2)^2\alpha_3$
• $Q_4(y) = (4y-y^2)(-\gamma_1+2D\delta_{22}) + 4D\gamma_1 + D^2\epsilon_{31}$
• $Q_5(y) = (4y-y^2)(D^2\epsilon_{11}+2D\delta_{21}-\gamma_2) + 4D\gamma_2 + D^2\epsilon_{33}$
• $Q_6(y) = (4y-y^2)(2-y)(\beta_1+2D\delta_{11})$
• $Q_7(y) = (4y-y^2)(2-y)(\beta_2+2D\delta_{12})$
• $Q_8(y) = (2-y)\epsilon_{21}$
• $Q_9(y) = \epsilon_{34}$

(5.43)

Let's pause for a moment and reflect a bit on what we have already obtained. The spin-0 function $F_0(y)$, given by (5.41), is expressed in terms of G^2, $(G')^2$ and up to 4 and 2 derivatives acting on those respectively. It is also written in terms of $(G'')^2$. These expressions contain terms which are non integrable in $D = 4$ spacetime dimensions. Since our aim is to quantum correct the linearized Einstein equation

$$\sqrt{-g}D^{\mu\nu\rho\sigma}h_{\rho\sigma}(x) - \int d^4x'\Sigma^{\mu\nu\rho\sigma}_{Ren}(x;x')h_{\rho\sigma}(x') = T^{\mu\nu}_{lin}(x)$$

the graviton self energy should contain only terms which are integrable in $D = 4$ dimensions. Consequently, our next goal is to isolate those non-integrable terms that appear in the equation (5.41). The process we followed was to find the explicit expression for G^2 (by taking the square of (5.28)), identify the non-integrable terms and recursively find the non-integrable terms for the more complicated expressions of the propagator (for example for $(G^2)''''$).
The expression for G^2 is

$$G^2 = \sum_{l=0}^{\infty} G_l \left(\frac{y}{4}\right)^l + \sum_{l=0}^{\infty} F_l \left(\frac{y}{4}\right)^{l+1-\frac{D}{2}} + \sum_{l=0}^{\infty} C_l \left(\frac{y}{4}\right)^{l+2-D} \tag{5.44}$$

where the coefficients G_l, F_l and C_l are given in terms of A, B, A_n, B_n which are defined at (5.29)-(5.32).

$$G_l = A^2 \sum_{n=0}^{l} A_n A_{l-n} \tag{5.45}$$

$$C_l = B^2 \sum_{n=0}^{l} B_n B_{l-n} \tag{5.46}$$

$$D_l = 2AB \sum_{n=0}^{l} A_n B_{l-n} \tag{5.47}$$

Note that the coefficient G_l that appears in equation (5.45) has nothing to do with the propagator $G(y)$. For every value of l, it corresponds to just a y-independent coefficient. We should also stress once more that $\frac{1}{D-4}$ terms reside inside B_n and A. We will come back to this at a later point. The relevant values for our computation turn out to be the one given at Table 6 below.

Table 6: Values for the $D_l's$ and $C_l's$ for $l = 0, 1, 2, 3, 4$
• $D_0 = 2AB$
• $D_1 = 2AB(A_1 + B_1)$
• $D_2 = 2AB(A_2 + B_2 + A_1B_1)$
• $D_3 = 2AB(A_3 + B_3 + A_2B_1 + A_1B_2)$
• $C_0 = B^2$
• $C_1 = 2B^2B_1$
• $C_2 = B^2(2B_2 + B_1^2)$
• $C_3 = B^2(2B_3 + 2B_1B_2)$
• $C_4 = B^2(2B_4 + 2B_1B_3 + B_2^2)$

(5.48)

After a tedious calculation, we can rewrite equation (5.41) explicitly in terms of powers of y that are not integrable in $D=4$ dimensions, i.e terms of the form $(\frac{4}{y})^{D+i}$ for $i=-2,-1,0,...$ and $(\frac{4}{y})^{\frac{D}{2}+i}$ for $i=0,1,2...$ .These terms are the one's we are primarily interested in, since the process of making them integrable (by extracting d'Alembertian operators, check appendix E), will give rise to divergences. The result of the computation is

$$
\begin{aligned}
F_0(y) = \frac{-\kappa^2 H^{2D}}{4(D-1)^2(4\pi)^D}\left[\frac{\square}{H^2}+D\right]^{-2}\Bigg[& Z_0(y)\left(\frac{4}{y}\right)^{\frac{D}{2}} + Z_1(y)\left(\frac{4}{y}\right)^{\frac{D}{2}+1} + Z_2(y)\left(\frac{4}{y}\right)^{\frac{D}{2}+2} + \\
& + Z_3(y)\left(\frac{4}{y}\right)^{\frac{D}{2}+3} + Z_4(y)\left(\frac{4}{y}\right)^{D+1} + Z_5(y)\left(\frac{4}{y}\right)^{D+2} + Z_6(y)\left(\frac{4}{y}\right)^{D} + Z_7(y)\left(\frac{4}{y}\right)^{D-2} + \\
& + Z_8(y)\left(\frac{4}{y}\right)^{D-1}\Bigg]
\end{aligned} \tag{5.49}
$$

The coefficients $Z_i(y)$ for $i=0,1,2,...,8$ are complicated expressions of A, B, A_n's, B_n's, Q_i's, F_i's, and C_i's as well as on the number of spacetime dimensions D. The explicit expressions can be found in appendix C.

Important Remark **:**
Let's have a closer look on (5.49). One can notice two different "families" of powers, i.e

- Powers of the form $(\frac{4}{y})^{\frac{D}{2}+i}$ for $i=0,1,2,3$ and
- Powers of the form $(\frac{4}{y})^{D+i}$ for $i=-2,-1,0,1,2$.

The former powers were absent in the massless minimally coupled case. Their presence in our work is undesirable since they correspond to non-local contributions to the spin-0 structure function F_0(y). Our expectation is that these terms should disappear in our final formula for the spin-0 structure function $F_0(y)$ (with coefficients which do not depend on y). Our work in the next subsection is towards this direction.

As far as the $(\frac{4}{y})^{D+i}$ terms are concerned, the method we will use to make the coefficients $Z_i(y)$ independent of y, will reduce the powers of D. After this procedure, D'Alembertian operators can be extracted in order to end up with integrable powers of $(\frac{4}{y})^{D+i}$, i.e for $i<-2$. By doing this, divergences will appear which should be substracted off by local counterterms, as already performed for the massless minimally coupled case in [14] and partially performed (for the quartic contributions to the graviton self energy) in [1]. More details on the method of extracting d'Alembertian operators can be found in appendix E.

5.3.4 y-independent coefficients for the $\frac{D}{2}$ powers

As already mentioned, the spin zero structure $F_0(y)$ is written in terms of $\frac{D}{2}$-dependent powers. Moreover, the coefficients $Z_i(y)$ for $i = 1, 2, ..., 9$ contain the $Q_i(y)'s$ which depend on y (see table 5). Our aim is to obtain an expression for the spin zero structure $F_0(y)$ in powers of y multiplied with coefficients which are independent of y. To demostrate this, we take the part of (5.49) which we are interested in manipulating at this subsection, i.e

$$Z_0(y)\left(\frac{4}{y}\right)^{\frac{D}{2}} + Z_1(y)\left(\frac{4}{y}\right)^{\frac{D}{2}+1} + Z_2(y)\left(\frac{4}{y}\right)^{\frac{D}{2}+2} + Z_3(y)\left(\frac{4}{y}\right)^{\frac{D}{2}+3} =$$

$$= 4^{\frac{D}{2}}\left[Z_0(y)\left(\frac{1}{y}\right)^{\frac{D}{2}} + 4Z_1(y)\left(\frac{1}{y}\right)^{\frac{D}{2}+1} + 16Z_2(y)\left(\frac{1}{y}\right)^{\frac{D}{2}+2} + 64Z_3(y)\left(\frac{1}{y}\right)^{\frac{D}{2}+3}\right] \tag{5.50}$$

For each $Z_i(y)$, with $i = 0, 1, 2, 3$, we introduce a new notation which aims to show the explicit y-dependence of each coefficient.

$$Z_0(y) = Z_0^{(0)}y^0 + Z_0^{(1)}y^1 + Z_0^{(2)}y^2 + Z_0^{(3)}y^3 + Z_0^{(4)}y^4 \tag{5.51}$$

By doing this, the y dependence of the initial $Z_0(y)$ has been extracted, leaving coefficients $Z_0^{(i)}$ for $i = 0, 1, .., 4$, independent of y. The reason why in (5.51) up to 4^{th} order powers of y appear, is a result of a complicated computation done using Mathematica.

This decomposition allows us to reduce the $\frac{D}{2} + i$ powers, for $i = 0, 1, 2, 3$ that appear in the expression for the spin zero structure function $F_0(y)$ (5.49).

To make this explicit, the result of the decomposition (5.51) when multiplied with $\left(\frac{1}{y}\right)^{\frac{D}{2}}$ is :

$$Z_0(y)\left(\frac{1}{y}\right)^{\frac{D}{2}} = Z_0^{(0)}\left(\frac{1}{y}\right)^{\frac{D}{2}} + Z_0^{(1)}\left(\frac{1}{y}\right)^{\frac{D}{2}-1} + Z_0^{(2)}\left(\frac{1}{y}\right)^{\frac{D}{2}-2} + Z_0^{(3)}\left(\frac{1}{y}\right)^{\frac{D}{2}-3} + Z_0^{(4)}\left(\frac{1}{y}\right)^{\frac{D}{2}-4} \tag{5.52}$$

Following the same procedure for the remaining coefficients we get :

$$Z_1(y)\left(\frac{1}{y}\right)^{\frac{D}{2}+1} = Z_1^{(0)}\left(\frac{1}{y}\right)^{\frac{D}{2}+1} + Z_1^{(1)}\left(\frac{1}{y}\right)^{\frac{D}{2}} + Z_1^{(2)}\left(\frac{1}{y}\right)^{\frac{D}{2}-1} + Z_1^{(3)}\left(\frac{1}{y}\right)^{\frac{D}{2}-2} + Z_1^{(4)}\left(\frac{1}{y}\right)^{\frac{D}{2}-3} \tag{5.53}$$

$$Z_2(y)\left(\frac{1}{y}\right)^{\frac{D}{2}+2} = Z_2^{(1)}\left(\frac{1}{y}\right)^{\frac{D}{2}+1} + Z_2^{(2)}\left(\frac{1}{y}\right)^{\frac{D}{2}} + Z_2^{(3)}\left(\frac{1}{y}\right)^{\frac{D}{2}-1} + Z_2^{(4)}\left(\frac{1}{y}\right)^{\frac{D}{2}-2} \tag{5.54}$$

$$Z_3(y)\left(\frac{1}{y}\right)^{\frac{D}{2}+3} = Z_3^{(2)}\left(\frac{1}{y}\right)^{\frac{D}{2}+1} + Z_3^{(3)}\left(\frac{1}{y}\right)^{\frac{D}{2}} + Z_3^{(4)}\left(\frac{1}{y}\right)^{\frac{D}{2}-1} \tag{5.55}$$

The y-independent coefficients appearing in (5.52)-(5.55) has been obtained using Mathematica and are given in appendix D. Note that the coefficients $Z_2^{(0)}$, $Z_3^{(0)}$ and $Z_3^{(1)}$ do not appear in (5.54) and (5.55). This is a result coming from simplifying the expressions for $Z_2(y)$ and $Z_3(y)$ in Mathematica. Introducing this notation, there are reductions in the powers. In other words, $(\frac{4}{y})^{\frac{D}{2}+2}$ and $(\frac{4}{y})^{\frac{D}{2}+2}$ have disappeared.

Plugging (5.52) $-$ (5.55) into (5.50) and organizing in terms of powers $(\frac{1}{y})^{\frac{D}{2}+i}$ we get

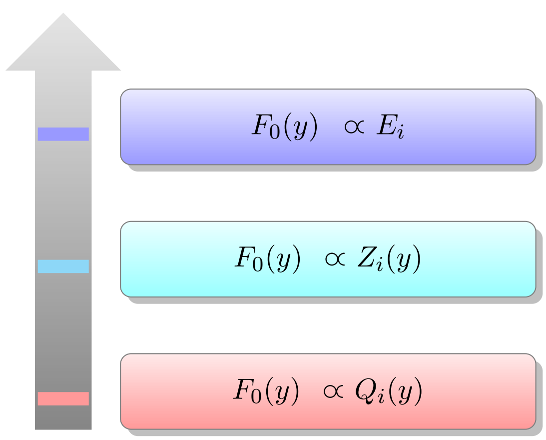
$$4^{\frac{D}{2}}\left[Z_0(y)\left(\frac{1}{y}\right)^{\frac{D}{2}}+4Z_1(y)\left(\frac{1}{y}\right)^{\frac{D}{2}+1}+16Z_2(y)\left(\frac{1}{y}\right)^{\frac{D}{2}+2}+64Z_3(y)\left(\frac{1}{y}\right)^{\frac{D}{2}+3}\right]=$$
$$=4^{\frac{D}{2}}\left[E_1\left(\frac{1}{y}\right)^{\frac{D}{2}+1}+E_2\left(\frac{1}{y}\right)^{\frac{D}{2}}+E_3\left(\frac{1}{y}\right)^{\frac{D}{2}-1}+E_4\left(\frac{1}{y}\right)^{\frac{D}{2}-2}+E_5\left(\frac{1}{y}\right)^{\frac{D}{2}-3}+E_6\left(\frac{1}{y}\right)^{\frac{D}{2}-4}\right] \tag{5.56}$$

where the $E_i's$ are given in the table below.

Table 6: y-independent coefficients for l=0,1,2,3,4
• $E_1 = 4Z_1^{(0)} + 16Z_2^{(1)} + 64Z_3^{(2)}$
• $E_2 = Z_0^{(0)} + 4Z_1^{(1)} + 16Z_2^{(2)} + 64Z_3^{(3)}$
• $E_3 = Z_0^{(1)} + 4Z_1^{(2)} + 16Z_2^{(3)} + 64Z_3^{(4)}$
• $E_4 = Z_0^{(2)} + 4Z_1^{(3)} + 16Z_2^{(4)}$
• $E_5 = Z_0^{(3)} + 4Z_1^{(4)}$
• $E_6 = Z_0^{(4)}$

(5.57)

• ***Recap***



The flow chart above shows the route we followed in order to obtain coefficients independent of y for the spin zero function $F_0(y)$. We first wrote $F_0(y)$ in terms of the $Q_i(y)'s$ (relation 5.41). Next, we expressed the $F_0(y)$ in terms of $\frac{D}{2}$ and D powers of y (relation 5.49) with coefficients $Z_i(y)$. Finally, decomposing each $Z_i(y)$ according to (5.51) we ended up to (5.56) which contains no y-dependent coefficients E_i.

Check

At this point we would like to check whether our result (5.56) recovers the result found by Woodard and Park [14]. Their result for the spin zero $F_0(y)$ contains no powers of the form $(\frac{1}{y})^{\frac{D}{2}+i}$ for $i = 0, 1$. By setting $m = 0$ and $\xi = 0$, the coefficients E_1 and E_2 indeed vanish. This can be easily seen by observing the coefficients in appendix D. Every term is multiplied either with ξ or with m. Therefore, setting them to zero, the result of [14] is recovered. The other terms, i.e $(\frac{1}{y})^{\frac{D}{2}+i}$ for $i = -4, -3, -2, -1$ are integrable in $D = 4$. Therefore, after checking that there are no $\frac{1}{D-4}$ multiplying them, we can set $D = 4$ and obtain a finite contribution to the spin zero structure function $F_0(y)$.

In case they contain $\frac{1}{D-4}$ divergences, then one can combine them with terms of the form $(\frac{1}{y})^{D+i}$. This is an important thing which needs more clarification.

Let's assume that a term of the following form appears

$$\frac{1}{D-4}\left(\frac{1}{y}\right)^{\frac{D}{2}-1} \tag{5.58}$$

Although the term is integrable in $D = 4$, obviously we cannot set $D = 4$ due to $\frac{1}{D-4}$ that appears. By working the terms $(\frac{1}{y})^{D+i}$, a term of the form

$$\frac{1}{D-4}\left(\frac{1}{y}\right)^{D-3} \tag{5.59}$$

may appear. Note that again, in this term, we cannot set $D = 4$ even though it is integrable in $D = 4$. Combining (5.58) and (5.59) we obtain

$$\frac{1}{D-4}\left[\left(\frac{1}{y}\right)^{D-3} - \left(\frac{1}{y}\right)^{\frac{D}{2}-1}\right] = \frac{1}{D-4}\left(\frac{1}{y}\right)^{\frac{D}{2}-1}\left[\left(\frac{1}{y}\right)^{\frac{D}{2}-2} - 1\right]$$

$$= \frac{1}{D-4}\left(\frac{4}{y}\right)^{\frac{D}{2}-1}\left[-\frac{(D-4)}{2}\ln\left(\frac{y}{4}\right)\right] = -\frac{1}{2}\left(\frac{4}{y}\ln\left(\frac{y}{4}\right)\right) \tag{5.60}$$

Therefore, we see that combining these two types of powers, (5.58) and (5.59), the divergences disappear. In particular, in appendix E, we show that combining these powers, we localize the divergences onto δ function terms. In turn, those terms can be subtracted by adding local counterterms. For further details, we refer the reader to appendix E.

Result

One important result we obtained is the fact that E_1 coefficient vanishes. This means that there are no non-local $(\frac{1}{y})^{\frac{D}{2}+1}$ contributions to $F_0(y)$. We expect that also E_2 should vanish. However, it turns out that, although the extremely complicated initial expression simplifies a lot, the result is still non zero. We suspect during the whole computation there exist a numerical mistake which leads to the non-zero coefficient E_2. The computation which we performed is extremely sensitive in the sense that just a numerical factor in front of an expression, can change significantly the result. Neverthless, the fact that E_1 indeed vanish is a good sign that our calculation is on the right way. Our starting point for future work is to identify the mistake and show that E_2 vanishes, as expected.

Chapter 6

Conclusions and Future work

In this thesis, we focused our attention on the cubic contribution to the graviton self energy due to massive non minimally coupled scalar fields in de-Sitter background. In particular, we made some important first steps towards the renormalization of the TT correlator. It turns out that the non minimally coupling as well as adding mass to the scalar field, make regularization procedure a cumbersome computation. Since our goal is to quantum correct the linearized Einstein's equations

$$\sqrt{-\bar{g}} D^{\mu\nu\rho\sigma} h_{\rho\sigma}(x) - \int d^4x' \Sigma^{\mu\nu\rho\sigma}_{Ren}(x;x') h_{\rho\sigma}(x') = T^{\mu\nu}_{lin}(x)$$

the renormalized self energy should be an integrable function in $D = 4$ dimensions [14]. To reach that point, one should extract d'Alembertian operators. It turns out that in the MNMC case, higher order derivatives contribute to the self energy (check first two lines in (5.36)) and in order to make those contributions integrable, we should extract many more d'Alembertians than the massless minimally coupled case. We managed to bring the spin zero structure function $F_0(y)$ which is almost ready for renormalization. There are two natural steps that should be done before. Firstly, we should show that the coefficient E_2 indeed vanishes (5.56). This term , $(\frac{1}{y})^{\frac{D}{2}}$, is a non-local contribution to the graviton self energy which we cannot renormalize in the BPHZ sense [59] [60], in other words, adding local counterterms to the initial action. The $(\frac{1}{y})^{\frac{D}{2}+1}$ term which appeared in our final reduction, indeed vanishes, as expected. This signals that our computation is on the right way. Next natural step, is to reduce all the D dependent powers appearing in (5.49). This can be done and we expect that no difficulties will arise.

Our final goal is to use the renormalized graviton self energy and see whether massive non minimally coupled scalar fields have a significant effect on gravitons at 1-loop order. This can be done using the so-called Schwinger-Keldysh formalism [29]. Then, since we only computed the 1-loop correction to the graviton self-energy, we can solve the linearized Einstein's equation perturbatively up to order $O(\kappa^2)$. In other words, we expand the graviton field $h_{\mu\nu}$ as well as the renormalized graviton self energy in powers of κ^2

$$h_{\mu\nu} = h^{(0)}_{\mu\nu} + \kappa^2 h^{(1)}_{\mu\nu} + O(\kappa^4)$$

$$\left[\Sigma^{\mu\nu\rho\sigma}_{ren}(x;x')\right]_{Retarded} = \kappa^2 \left[\Sigma^{\mu\nu\rho\sigma}_{(1)}(x;x')\right] + O(\kappa^4)$$

where

$$h^{(0)}_{\mu\nu} = \epsilon_{\mu\nu}(\boldsymbol{k}) u_0(\eta, \boldsymbol{k}) e^{i\boldsymbol{k}\boldsymbol{x}}$$

solves the classical Einstein equation (expansion up to zeroth order in κ) and $\epsilon_{\mu\nu}$ the polarization tensor of the graviton. The mode functions of the classical solution $u_0(\eta, \boldsymbol{k})$ are given by

$$u_0(\eta, \boldsymbol{k}) = \frac{H}{\sqrt{2k^3}} \left[1 - \frac{ik}{H\alpha}\right] e^{\frac{ik}{H\alpha}}$$

Or equivalently,

$$u_0(\eta, \boldsymbol{k}) = \frac{H}{\sqrt{2k^3}} \Big(1 + ik\eta\Big) e^{-ik\eta}$$

where in order to obtain the latter, we have set

$$\alpha = -\frac{1}{H\eta}$$

which is the form of the scale factor $\alpha(\eta)$ in a de-Sitter background.

Expanding up to first order in κ^2, we obtain an equation for the 1-loop correction (in absence of a stress-energy tensor on the right hand side)

$$\sqrt{-\bar{g}} D^{\mu\nu\rho\sigma} h^{(1)}_{\rho\sigma}(x) - \int d^4x' \Sigma^{\mu\nu\rho\sigma}_{(1)}(x;x') h^{(0)}_{\rho\sigma}(x') = 0$$

Woodard, Park, Prokopec and Leonard [15][61] showed that for the MMC case, there are no significant late time effects on gravitons. This is an expected result since MMC scalars and gravitons couple via a derivative term (the kinetic energy of the MMC scalar). But since inflation produce a vast ensemble of infrared scalars [10][11][12][13], this vertex experience a suppresion which naturally leads to the null effect found by the authors mentioned above.

For our case, i.e for a massive non minimally coupled scalars, we expect that there will be significant effect on the propagation of the dynamical gravitons. This is because there is a coupling between the gravitons and MNMC via the effective mass

$$m^2_{eff} = m^2 + \xi R$$

which does not redshift.

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I would like to thank Tomislav for giving me the opportunity to work on this project under his supervision. He was always willing to spend time with me, answering my questions and encouraging me to work hard. Apart from the technical knowledge that he provided me, it was extremely beneficial to me to observe the way he was trying to tackle a problem: Performing computations in a small piece of paper, trying to gain physical insight , making connections/analogies with things he already knows and understands. This was my first contact with research, and I am grateful that I had the chance to work with him. I would also like to thank Drazen and Stefano for their useful comments and the discussions we had. Whenever I needed, they were always willing to help me.

Away from university and physics, there is also life. I feel extremely lucky for all the amazing people that I met in Utrecht and the inummerable nice moments we have spent together. I thank all of them and wish them the best.

However, I cannot forget the place where everything have started. I would like to thank my friends back to Greece with whom, although separated since 2016, they are still an important part of me.

Last but not least, without the support of my family, I would have achieved much less.

Appendix A

Massive scalar propagator in dS contains no divergences at D=4 limit

The scalar propagator is given by

$$G(y) = A\sum_{n=0}^{\infty} A_n \left(\frac{y}{4}\right)^n + B\sum_{n=0}^{\infty} B_n \left(\frac{y}{4}\right)^{(n+1-\frac{D}{2})} \tag{A.1}$$

where the expressions for A, B, A_n and B_n are given by (5.29)-(5.32). Recall that the terms which contain $\frac{1}{D-4}$ divergence are B_n for every $n > 0$ and A. Extracting the 0^{th} order from the second sum and renaming the index we can rewrite the propagator as

$$G(y) = BB_0 \left(\frac{y}{4}\right)^{1-\frac{D}{2}} + \sum_{n=0}^{\infty} \left[AA_n + BB_{n+1}\left(\frac{y}{4}\right)^{2-\frac{D}{2}}\right]\left(\frac{y}{4}\right)^n \tag{A.2}$$

But:

$$\left(\frac{y}{4}\right)^{2-\frac{D}{2}} = 1 - \frac{D-4}{2}\ln\left(\frac{y}{4}\right) + O\left((D-4)^2\right) \tag{A.3}$$

First of all, the first term in $(A.2)$ contains no divergences when we expand around $D = 4$ since $B_0 = 1$ and $B = \Gamma(\frac{D}{2} - 1)$. The expression in the square bracket of $(A.2)$ contains at most $\frac{1}{D-4}$ divergence due to the presence of A and B_{n+1}. Therefore, combining with $(A.3)$, and neglecting terms $O\left((D-4)^0\right)$ we get:

$$G(y) = \sum_{n=0}^{\infty} \left[AA_n + BB_{n+1}\right]\left(\frac{y}{4}\right)^n \tag{A.4}$$

The next step is to expand each quantity that appears on $AA_n + BB_{n+1}$ around $D = 4$. To do so, we need to make use of the following

$$\frac{\Gamma(\frac{D-3}{2} - \nu_D)\Gamma(\frac{D-3}{2} + \nu_D)}{\Gamma(\frac{1}{2} - \nu_D)\Gamma(\frac{1}{2} + \nu_D)} = 1 + \frac{D-4}{2}\left[\psi\left(\frac{1}{2} + \nu\right) + \psi\left(\frac{1}{2} - \nu\right)\right] \tag{A.5}$$

$$\Gamma\left(1-\frac{D}{2}\right)=\frac{2}{D-4}\left[1+\frac{D-4}{2}\left(-1+\gamma_E\right)\right] \tag{A.6}$$

$$\Gamma\left(\frac{D}{2}-1\right)=1-\frac{D-4}{2}\gamma_E \tag{A.7}$$

$$\left(\frac{D-3}{2}\right)^2-\nu_D^2=\left(\frac{1}{4}-\nu^2\right)\left[1+\frac{D-4}{2}\left(\frac{1-\delta\nu^2}{\frac{1}{4}-\nu^2}\right)\right] \tag{A.8}$$

where $\psi(z)$ is the so-called digamma function which does not have divergence at $D=4$. Plugging $(A.5)$-$(A.8)$ back to $(A.4)$, we obtain

$$AA_n+BB_{n+1}=\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{2}{D-4}\left[A_n-\frac{1}{(n+1)!}\frac{\left(\frac{3}{2}+\nu\right)_n\left(\frac{3}{2}-\nu\right)_n}{(3-\frac{D}{2})_n}\right]=$$

$$=\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{2}{D-4}\frac{1}{n!}\left[\frac{\left(\frac{D-1}{2}+\nu_D\right)_n\left(\frac{D-1}{2}-\nu_D\right)_n}{\left(\frac{D}{2}\right)_n}\left(\frac{(D-3)^2}{4}-\nu_D^2\right)-\frac{\frac{1}{4}-\nu_D^2}{n+1}\frac{\left(\frac{3}{2}+\nu_D\right)_n\left(\frac{3}{2}-\nu_D\right)_n}{(3-\frac{D}{2})_n}\right] \tag{A.9}$$

where in the last line we used the identity for the Pochammer symbols $a_{n+1}=a(a+1)_n$ in order to show that

$$B_{n+1}=\frac{1}{(n+1)!}\frac{\left(\frac{1}{2}+\nu_D\right)_{n+1}\left(\frac{1}{2}-\nu_D\right)_{n+1}}{(2-\frac{D}{2})_{n+1}}=\frac{(\frac{1}{4}-\nu_D^2)(-2)}{n!(D-4)}\frac{1}{n+1}\frac{\left(\frac{3}{2}+\nu_D\right)_n\left(\frac{3}{2}-\nu_D\right)_n}{(3-\frac{D}{2})_n}$$

It is easy to check that taking the limit $D\to 4$ of the square bracket in $(A.9)$, gives zero for every $n=0,1,2,\ldots$. Therefore, the propagator $G(y)$ is finite (since the numerator tends to zero in the $D=4$ limit and the same for the denominator in (A.9)). We thus conclude that the graviton propagator contains no $\frac{1}{D-4}$ divergences when we take the limit $D\to 4$.

Appendix B

Tracing the TT-correlator

Recall the de Sitter length function $y(x;x')$ given in (5.11). In order to compute the trace of the TT correlator we need to make use of the following identities [1][62]

$$\partial_\mu y \partial^\mu y = \partial^{'}_\mu y \partial^{'\mu} y = H^2(4y - y^2) \tag{B.1}$$

$$(\partial_\mu y)(\partial^\mu \partial'^\nu y) = H^2(2-y)\partial^{'\nu} y \tag{B.2}$$

$$(\partial'_\sigma y)(\partial^\mu \partial'^\sigma y) = H^2(2-y)\partial^\mu y \tag{B.3}$$

$$(\partial^\mu \partial'_\sigma y)(\partial^{'\sigma}\partial^\rho y) = 4H^4 g^{\mu\rho} - H^2(\partial^\mu y)(\partial^\rho y) \tag{B.4}$$

$$(\partial^{'\mu}\partial_\sigma y)(\partial^\sigma \partial^{'\rho} y) = 4H^4 g^{'\mu\rho} - H^2(\partial^{'\mu} y)(\partial^{'\rho} y) \tag{B.5}$$

$$\nabla_\rho \partial^\mu y = H^2(2-y)\delta^\mu_\rho \tag{B.6}$$

$$\nabla^{'}_\rho \partial^{'\mu} y = H^2(2-y)\delta^\mu_\rho \tag{B.7}$$

$$\nabla_\mu \partial^\mu y \equiv \Box y = DH^2(2-y) \tag{B.8}$$

$$\nabla^{'}_\mu \partial^{'\mu} y \equiv \Box^{'} y = DH^2(2-y) \tag{B.9}$$

The TT correlator (5.36) consists of 5 tensor structures. Our goal is trace each tensor structure and end up with relation (5.39). In the derivation of the following we make use of the identities

$(B.1)$-$(B.9)$.

$$g^{\mu\nu}g^{'\rho\sigma}\left[\partial_\mu y\partial_\nu y\partial^{'}_\rho y\partial^{'}_\sigma y\right] = \partial_\mu y\partial^\mu y\partial^{'}_\rho y\partial^{'\rho} y = H^4(4y-y^2)^2 \tag{B.10}$$

$$g^{\mu\nu}g^{'\rho\sigma}\left[\partial_{(\mu} y\partial_{\nu)}\partial^{'}_{(\rho} y\partial^{'}_{\sigma)} y\right] = \partial^{(\nu} y\partial_{\nu)}\partial^{'}_{(\rho} y\partial^{'\rho)} y = H^2(2-y)\partial^{(\nu} y\partial_{\nu)} y = H^4(2-y)(4y-y^2) \tag{B.11}$$

$$g^{\mu\nu}g^{'\rho\sigma}\left[\partial_{\mu)}\partial^{'}_{(\rho} y\partial^{'}_{\sigma)}\partial_{(\nu} y\right] = 4DH^4 - H^2\partial^{\nu)} y\partial_{(\nu} y = H^4[4D-(4y-y^2)] \tag{B.12}$$

$$g^{\mu\nu}g^{'\rho\sigma}\left[H^4 g_{\mu\nu}g^{'}_{\rho\sigma}\right] = H^4D^2 \tag{B.13}$$

$$g^{\mu\nu}g^{'\rho\sigma}H^2\left[\partial_\mu y\partial_\nu y g^{'}_{\rho\sigma} + g_{\mu\nu}\partial^{'}_\rho y\partial^{'}_\sigma y\right] = 2DH^4(4y-y^2) \tag{B.14}$$

Therefore, the trace of the TT correlator is given by

$$-\frac{4}{\kappa^2}\frac{\bar{g}^{\mu\nu}}{\sqrt{-g}}\frac{\bar{g}^{'\rho\sigma}}{\sqrt{-g^{'}}}(-\imath)\left[\Sigma^{\mu\nu\rho\sigma}\right](x;x') \equiv \bar{g}^{\mu\nu}\bar{g}^{'\rho\sigma} < \bar{T}_{\mu\nu}(x)\bar{T}_{\rho\sigma}(x') >=$$
$$= \frac{H^{2D}}{(4\pi)^D}\left[(4y-y^2)^2\alpha_R + (4y-y^2)(2-y)\beta_R + \big(4D-(4y-y^2)\big)\gamma_R + 2D(4y-y^2)\delta_R + D^2\epsilon_R\right] \tag{B.15}$$

with the coefficients given at table 3.

Appendix C

Z(y) coefficients in (5.49)

In this appendix, the values of the coefficients that appear in (5.49) for the spin zero structure function $F_0(y)$ are given. As the reader can notice, these coefficients $Z_i(y)$ have pretty complicated expressions. They are written in terms of the coefficients $Q_i(y)$ that are given in table 5. In turn, these $Q_i(y)$ are combination of the coefficients in table 2. We are interested in isolating potentially divergent terms $\frac{1}{D-4}$. To do so, we need to identify which coefficients, when expanded around $D = 4$, can cause these divergences.

First of all, by observing table 2 and table 5, we immediately conclude that the $Q_i(y)$'s cannot lead to divergences. Moreover, A_n's are also free of divergences (check table 1). As far as the B_n coefficients are concerned, from table 1, we see that terms $\frac{1}{D-4}$ appear for every $n > 0$. The way we treated them in our computation was to redefine each coefficient B_n in the following way:

$$B_n = \bar{B}_n \frac{1}{2 - \frac{D}{2}} \tag{C.1}$$

for $i = 1, 2, 3, 4$. The explicit expressions for the $\bar{B}_i$'s are the following

$$\bar{B}_1 = \frac{1}{4} - \nu_D^2 \tag{C.2}$$

$$\bar{B}_2 = \frac{\left(\frac{1}{4} - \nu_D^2\right)\left(\frac{9}{4} - \nu_D^2\right)}{6 - D} \tag{C.3}$$

$$\bar{B}_3 = \frac{\left(\frac{1}{4} - \nu_D^2\right)\left(\frac{9}{4} - \nu_D^2\right)\left(\frac{25}{4} - \nu_D^2\right)}{6(3 - \frac{D}{2})(4 - \frac{D}{2})} \tag{C.4}$$

$$\bar{B}_4 = \frac{\left(\frac{1}{4} - \nu_D^2\right)\left(\frac{9}{4} - \nu_D^2\right)\left(\frac{25}{4} - \nu_D^2\right)\left(\frac{49}{4} - \nu_D^2\right)}{24(3 - \frac{D}{2})(4 - \frac{D}{2})(5 - \frac{D}{2})} \tag{C.5}$$

where the coefficients $\bar{B}_1$,$\bar{B}_2$,$\bar{B}_3$,$\bar{B}_4$ are explicitly free of divergences at $D = 4$ limit. Notice that these coefficients appear in almost all the $Z_i(y)$'s, i.e for almost every i.

As it has already been pointed out before (check remark in subsection (5.3.1)), the coefficient A, that appears in the $Z_i(y)$ for $i = 0, 1, 2, 3$, contains a term $\Gamma(1 - \frac{D}{2})$. This term, when expanded around $D = 4$, leads to a term $\propto \frac{1}{D-4}$. Notice that A appears in the $Z_i(y)$'s for $i = 0, 1, 2, 3$. Finally, the B coefficient which appears in the $Z_i(y)$ for $i = 4, 5, 6, 7, 8$, when expanded around $D = 4$, brings no $\frac{1}{D-4}$ terms (check appendix A, relation A.7).

A last thing which should be stressed is that the way we wrote down the coefficients is organized in terms of powers of $D - 4$. For instance, let's have a look on the coefficient $Z_0(y)$ (C.6). We can immediately see that when we expand A (which multiplies the whole expression in the square bracket) around $D = 4$, $Z_0(y)$ is written schematically in the form

$$Z_0(y) = \frac{1}{D-4} Z_0^{'}(y) + (D-4)^0 Z_0^{''}(y) + O(D-4)$$

where $Z_0^{'}(y)$ and $Z_0^{''}(y)$ some divergence-free coefficients which can be read off from $(C.6)$. The terms which are of order $O(D-4)$ have been neglected since when we set $D = 4$, they vanish. On the other hand, terms $O((D-4)^0)$ have been kept, since at the $D = 4$ limit, they correspond to a constant contribution to the spin zero structure function $F_0(y)$.

Above, the explicit coefficients are given:

$$\begin{aligned}
Z_0(y) = 2\boldsymbol{A}B\frac{(1-\frac{D}{2})}{4}\Bigg[&\Bigg(Q_5(y) - Q_4(y)\Bigg)\frac{A_1}{4} + \frac{1}{8}\Bigg(Q_1(y) + Q_3(y) - Q_2(y)\Bigg)\Bigg(3A_3\frac{(-\frac{D}{2})}{4} + A_2\frac{\bar{B}_1}{4}\Bigg) + \frac{Q_8(y)}{2} + \\
&+ \frac{Q_4(y)}{2}\frac{\bar{B}_1}{4} + \frac{Q_6(y)}{2}(A_1\bar{B}_1 + \bar{B}_2)\frac{(3-\frac{D}{2})}{16} + \frac{1}{2}\Bigg(Q_7(y) - 3Q_6(y)\Bigg)\Bigg(\frac{1}{16}A_1\bar{B}_1 + \frac{1}{8}A_2\frac{(1-\frac{D}{2})}{4}\Bigg) + \\
&+ \Bigg(-2Q_1(y) + \frac{Q_2(y)}{2}\Bigg)\frac{1}{16}A_1\bar{B}_2\frac{(3-\frac{D}{2})}{4} + \frac{Q_1(y)}{8}\frac{(4-\frac{D}{2})}{4}\frac{(3-\frac{D}{2})}{4}\Bigg(A_1\bar{B}_2 + A_2\bar{B}_1 + \bar{B}_3\Bigg) + \\
&+ \frac{\boldsymbol{(2-\frac{D}{2})}}{4}\Bigg(\frac{Q_4(y)}{2}A_1 + \Bigg(-2Q_1(y) + \frac{Q_2(y)}{2}\Bigg)\Bigg(\frac{1}{8}A_2\bar{B}_1 + \frac{6A_3}{8}\frac{(1-\frac{D}{2})}{4}\Bigg) + A_2\frac{Q_6(y)}{2}\frac{(3-\frac{D}{2})}{4} + \\
&+ A_3\frac{Q_1(y)}{2}\frac{(4-\frac{D}{2})}{4}\frac{(3-\frac{D}{2})}{4}\Bigg)\Bigg]
\end{aligned} \tag{C.6}$$

Notice that when we expand A around $D = 4$, $Z_0(y)$ will contain a term proportional to $\frac{1}{D-4}$ and a term proportial to $(D-4)^0$.

$$Z_1(y) = 2\boldsymbol{A}B\frac{(1-\frac{D}{2})}{4}\frac{(-\frac{D}{2})}{4}\Bigg[\frac{Q_4(y)}{2} + \frac{Q_1(y)}{2}\frac{(3-\frac{D}{2})}{16}\left(A_1\bar{B}_1 + \bar{B}_2\right) + \frac{Q_6(y)}{8}\bar{B}_1 + \\ + \left(Q_7(y) - 3Q_6(y)\right)\frac{A_1}{8} + \left(-2Q_1(y) + \frac{Q_2(y)}{2}\right)\left(\frac{1}{16}A_1\bar{B}_1 + \frac{A_2}{2}\frac{(1-\frac{D}{2})}{4}\right) + \\ + \left(Q_1(y) + Q_3(y) - Q_2(y)\right)\frac{A_2}{8} + \frac{\boldsymbol{(2-\frac{D}{2})}}{4}\left(\frac{Q_6(y)}{2}A_1 + \frac{Q_1(y)}{2}A_2\frac{(3-\frac{D}{2})}{4}\right)\Bigg] \tag{C.7}$$

$$Z_2(y) = 2\boldsymbol{A}B\frac{(1-\frac{D}{2})}{4}\frac{(-\frac{D}{2})}{4}\frac{(-1-\frac{D}{2})}{4}\Bigg[\frac{Q_6(y)}{2} + \frac{A_1}{4}\left(-2Q_1(y) + \frac{Q_2(y)}{2}\right) + \frac{Q_1(y)}{8}\bar{B}_1 + \\ + \frac{Q_1(y)A_1}{8}\frac{\boldsymbol{(2-\frac{D}{2})}}{4}\Bigg] \tag{C.8}$$

$$Z_3(y) = 2\boldsymbol{A}B\frac{(1-\frac{D}{2})}{4}\frac{(-\frac{D}{2})}{4}\frac{(-1-\frac{D}{2})}{4}\frac{(-2-\frac{D}{2})}{4}\frac{Q_1(y)}{2} \tag{C.9}$$

$$Z_4(y) = B^2\Bigg[\frac{Q_6(y)}{2}\frac{(2-D)}{4}\frac{(1-D)}{4}\frac{(-D)}{4} + \frac{1}{2}\left(Q_7(y) - 3Q_6(y)\right)\frac{(1-\frac{D}{2})^2}{16}\frac{(-D)}{4} + \\ + \left(-2Q_1(y) + \frac{Q_2(y)}{2}\right)\frac{(1-D)}{4}\frac{(-D)}{4}\frac{\bar{B}_1}{2}\frac{(1-\frac{D}{2})}{4} + \left(Q_1(y) + Q_3(y) - Q_2(y)\right)\frac{\bar{B}_1}{2}\frac{(-\frac{D}{2})}{4}\frac{(1-\frac{D}{2})^2}{16} + \\ + Q_1(y)\bar{B}_1\frac{1}{\boldsymbol{(2-\frac{D}{2})}}\frac{(3-D)}{4}\frac{(2-D)}{4}\frac{(1-D)}{4}\frac{(-D)}{4}\Bigg] \tag{C.10}$$

$$Z_5(y) = B^2\Bigg[\frac{Q_1(y)}{2}\frac{(2-D)}{4}\frac{(1-D)}{4}\frac{(-D)}{4}\frac{(-1-D)}{4} + \left(-2Q_1(y) + \frac{Q_2(y)}{2}\right)\frac{(-D)}{4}\frac{(-1-D)}{4}\frac{(1-\frac{D}{2})^2}{16} + \\ + \left(Q_1(y) + Q_3(y) - Q_2(y)\right)\frac{(-\frac{D}{2})^2}{16}\frac{(1-\frac{D}{2})^2}{16}\Bigg] \tag{C.11}$$

$$
\begin{aligned}
Z_6(y) = B^2 \Bigg[&\left(\frac{Q_4(y)}{2} + Q_1(y)\frac{\bar{B}_2}{2}\frac{(3-D)}{4} \right)\frac{(2-D)}{4}\frac{(1-D)}{4} + \Big(Q_5(y) - Q_4(y) \Big)\frac{(1-\frac{D}{2})^2}{16} + \\
&+ \Big(Q_7(y) - 3Q_6(y) \Big)\frac{\bar{B}_1}{4}\frac{(1-\frac{D}{2})}{4}\frac{(1-D)}{4} + \left(-2Q_1(y) + \frac{Q_2(y)}{2} \right)\frac{(2-D)}{4}\frac{(1-D)}{4}\frac{(\bar{B}_1)^2}{16} + \\
&+ \Big(Q_1(y) + Q_3(y) - Q_2(y) \Big)\left(\frac{(\bar{B}_1)^2}{16}\frac{(1-\frac{D}{2})^2}{16} + \frac{\bar{B}_2}{2}\frac{(1-\frac{D}{2})}{4}\frac{(-\frac{D}{2})}{4}\frac{(3-\frac{D}{2})}{4} \right) + \\
&+ \frac{(2-D)(1-D)}{16\boldsymbol{(2-\frac{D}{2})}}\Bigg(Q_1(y)\frac{(\bar{B}_1)^2}{4}\frac{(3-D)}{4} + Q_6(y)\bar{B}_1\frac{(3-D)}{4} + (-2Q_1(y) + \frac{Q_2(y)}{2})2\bar{B}_2 \\
&\frac{(1-\frac{D}{2})}{4}\frac{(3-\frac{D}{2})}{4} \Bigg) \Bigg]
\end{aligned} \tag{C.12}
$$

$$
\begin{aligned}
Z_7(y) = B^2 \Bigg[&Q_9(y) + \Big(Q_4(y) - Q_5(y) \Big)\frac{(\bar{B}_1)^2}{16} + \Big(Q_1(y) + Q_3(y) - Q_2(y) \Big)\frac{(\bar{B}_2)^2}{16}\frac{(3-\frac{D}{2})^2}{16} + \\
&+ \frac{Q_1(y)}{2}\bar{B}_4\frac{(3-D)}{4}\frac{(5-D)}{4}\frac{(6-D)}{4} + \frac{Q_4}{2}\frac{(3-D)}{4}\bar{B}_2 + \frac{Q_6(y)}{2}\frac{(5-D)}{4}\frac{(3-D)}{4}\bar{B}_3 + \\
&+ \frac{(3-D)}{4}\left(-2Q_1(y) + \frac{Q_2(y)}{2} \right)\left(\bar{B}_4\frac{(1-\frac{D}{2})}{4}\frac{(5-\frac{D}{2})}{4} + \frac{\bar{B}_1\bar{B}_3}{4}\frac{(4-\frac{D}{2})}{4} \right) + \\
&+ \frac{1}{\boldsymbol{(2-\frac{D}{2})}}\Bigg((Q_5(y) - Q_4(y))2\bar{B}_2\frac{(1-\frac{D}{2})}{4}\frac{(3-\frac{D}{2})}{4} + \Big(\frac{Q_6(y)}{2}\frac{(5-D)}{4}\bar{B}_1\bar{B}_2 + \frac{Q_4(y)(\bar{B}_1)^2}{4} + \\
&+ Q_8(y)\bar{B}_1 \Big)\frac{(3-D)}{4} + (Q_1(y) + Q_3(y) - Q_2(y))\frac{(1-\frac{D}{2})}{4}\frac{(4-\frac{D}{2})}{4}\left(\frac{(-\frac{D}{2})}{4}\frac{(5-\frac{D}{2})}{4}2\bar{B}_4 + \frac{(3-\frac{D}{2})}{4}\bar{B}_1\frac{\bar{B}_3}{2} \right) + \\
&+ (Q_7(y) - 3Q_6(y))\frac{(3-D)}{4}\left(\frac{(1-\frac{D}{2})}{4}\frac{(4-\frac{D}{2})}{4}\bar{B}_3 + \frac{(3-\frac{D}{2})}{4}\bar{B}_1\frac{\bar{B}_2}{4} \right) + (-2Q_1(y) + \frac{Q_2(y)}{2})\frac{(3-\frac{D}{2})}{8} \\
&(\bar{B}_2)^2\frac{(3-\frac{D}{2})^2}{16} + \frac{Q_1(y)}{2}\frac{(3-D)}{4}\frac{(5-D)}{4}\frac{(6-D)}{4}(\bar{B}_1\bar{B}_3 + \frac{(\bar{B}_2)^2}{2}) \Bigg) \Bigg]
\end{aligned} \tag{C.13}
$$

$$
\begin{aligned}
Z_8(y) = B^2 \Bigg[&(Q_4 - Q_5)\frac{(1-\frac{D}{2})}{4}\frac{\bar{B}_1}{2} + (Q_1 + Q_3 - Q_2)\frac{(1-\frac{D}{2})}{4}\frac{(3-\frac{D}{2})}{4}\frac{\bar{B}_1\bar{B}_2}{8} + \\
&+ \frac{Q_8}{2}\frac{(2-D)}{4} + \frac{Q_6}{2}\frac{(2-D)}{4}\frac{(3-D)}{4}\bar{B}_2 + \frac{Q_1}{2}\frac{(2-D)}{4}\frac{(3-D)}{4}\frac{(5-D)}{4}\bar{B}_3 + \\
&+ (Q_7 - 3Q_6)\frac{(\bar{B}_1)^2}{32}\frac{(2-D)}{4} + \frac{1}{\boldsymbol{(2-\frac{D}{2})}}\Bigg((Q_1 + Q_3 - Q_2) 2\bar{B}_3 \frac{(-\frac{D}{2})}{4}\frac{(1-\frac{D}{2})}{4}\frac{(3-\frac{D}{2})}{4}\frac{(4-\frac{D}{2})}{4} + \\
&+ Q_4\bar{B}_1\frac{(3-D)}{4}\frac{(2-D)}{4} + Q_6\frac{(\bar{B}_1)^2}{4}\frac{(3-D)}{4}\frac{(2-D)}{4} + \frac{Q_1}{2}\bar{B}_1\bar{B}_2\frac{(5-D)}{4}\frac{(3-D)}{4}\frac{(2-D)}{4} + \\
&+ (Q_7 - 3Q_6)\bar{B}_2\frac{(3-\frac{D}{2})}{4}\frac{(1-\frac{D}{2})}{4}\frac{(2-D)}{4} + (\frac{Q_2}{2} - 2Q_1)\frac{(3-D)}{4}\frac{(2-D)}{4}\frac{(3-\frac{D}{2})}{4}\Bigg(\frac{\bar{B}_1\bar{B}_2}{2} + \\
&+ \bar{B}_3\frac{(1-\frac{D}{2})}{2}\Bigg)\Bigg)\Bigg]
\end{aligned}
\tag{C.14}
$$

Appendix D

Z(y) decomposition

Recall that we decomposed each coefficient $Z_i(y)$ for $(i = 0, 1, 2, 3)$

$$Z_i(y) = Z_i^{(0)} y^0 + Z_i^{(1)} y^1 + Z_i^{(2)} y^2 + Z_i^{(3)} y^3 + Z_i^{(4)} y^4 \tag{D.1}$$

- For $i = 2$:

$$Z_2(y) = Z_2^{(0)} y^0 + Z_2^{(1)} y^1 + Z_2^{(2)} y^2 + Z_2^{(3)} y^3 + Z_2^{(4)} y^4 \tag{D.2}$$

where

$$Z_2^{(4)} = \frac{(D^2-4)\,\xi}{4096}\Big[(D-2)D^2\xi - 4M^2(D\xi-4)\Big] \tag{D.3}$$

$$Z_2^{(3)} = \frac{(D^2-4)\,\xi}{4096}\Big[-4(D-2)D^2\xi - 4(D+6)D^2\xi + 32M^2(D\xi-4) - 64D\xi\Big] \tag{D.4}$$

$$Z_2^{(2)} = \frac{(D^2-4)\,\xi}{4096}\Big[16(D+6)D^2\xi - 64M^2(D\xi-4) + 64(D+2)D\xi + 256D\xi\Big] \tag{D.5}$$

$$Z_2^{(1)} = -\frac{1}{16}\left(D^2-4\right)D(D+2)\xi^2 \tag{D.6}$$

$$Z_2^{(0)} = 0 \tag{D.7}$$

- For $i = 3$:

$$Z_3(y) = Z_3^{(0)} y^0 + Z_3^{(1)} y^1 + Z_3^{(2)} y^2 + Z_3^{(3)} y^3 + Z_3^{(4)} y^4 \tag{D.8}$$

where

$$Z_3^{(0)} = Z_3^{(1)} = 0 \tag{D.9}$$

$$Z_3^{(2)} = \frac{1}{128}(D^2-4)D(D+4)\xi^2 \tag{D.10}$$

$$Z_3^{(3)} = -\frac{1}{256}(D^2-4)D(D+4)\xi^2 \tag{D.11}$$

$$Z_3^{(4)} = \frac{(D^2-4)D(D+4)\xi^2}{2048} \tag{D.12}$$

• For $i = 1$:

$$Z_1(y) = Z_1^{(0)}y^0 + Z_1^{(1)}y^1 + Z_1^{(2)}y^2 + Z_1^{(3)}y^3 + Z_1^{(4)}y^4 \tag{D.13}$$

where

$$Z_1^{(0)} = \frac{1}{8}(D^2-4)D^2\xi^2 \tag{D.14}$$

$$Z_1^{(1)} = \frac{D-2}{32}\Big[-D\xi^2\left\{8+4D+D^2(8+D)\right\} + 4M^2\left\{1-8\xi-D+D\xi(4+\xi(D-6))\right\}\Big] \tag{D.15}$$

$$\begin{aligned} Z_1^{(2)} =& \frac{D-2}{1024(D+2)H^4}\Big[-32(D+2)(D^2-3D+3)H^4M^2 + 128m^2(DH^2+m^2) + \\ & (D+2)H^4\xi^2\Big(-16(D-12)M^4 + 16D(44-41D)M^2 + D^2(D+4)(D(D+16)-4) + 64D\Big) + \\ & 32\xi\Big(-4(D+2)H^4M^4 + (D+2)((9D^2-14D)+24)H^4M^2 + 4(D-2)m^2(DH^2+m^2)\Big)\Big] \end{aligned} \tag{D.16}$$

$$\begin{aligned} Z_1^{(3)} =& \frac{(D-2)}{2048(D+2)H^4}\Big[16(D^3-2D+4)H^4M^2 - 128m^2(DH^2+m^2) - \\ & (D+2)H^4\xi^2\Big(D^2(D-2(D+2)(D+4) - 16(D-12)M^4 + 16D(20-21D)M^2\Big) - \\ & -32\xi\Big(-4(D+2)H^4M^4 + (D+2)(5D^2-6D+8)H^4M^2 + 4m^2(D-2)(DH^2+m^2)\Big)\Big] \end{aligned} \tag{D.17}$$

$$\begin{aligned} Z_1^{(4)} =& \frac{(D-2)}{16384(D+2)H^4}\Big[128m^2(DH^2+m^2) - \\ & (D+2)H^4\xi^2\Big((D^2-2D-4M^2)(4M^2(D-12)+D(D+2)(D-4))\Big) - \\ & +32\xi\Big(-4(D+2)H^4M^4 + D(D+2)(D-2)H^4M^2 + 4m^2(D-2)(DH^2+m^2)\Big)\Big] \end{aligned} \tag{D.18}$$

• For $i = 0$:

$$Z_0(y) = Z_0^{(0)} y^0 + Z_0^{(1)} y^1 + Z_0^{(2)} y^2 + Z_0^{(3)} y^3 + Z_0^{(4)} y^4 \tag{D.19}$$

where

$$\begin{aligned} Z_0^{(0)} =& \frac{D-2}{16}\Bigg[\xi^2(D^4 - 4D^2 + 8D - 8) - 16DM^2\xi + \\ & + 2M^2\bigg(-D^3 + D^2 + 5D - 5 + 8\xi(D+2)(D-1)^2 + 2\xi^2(8 + 22D - 9D^2 - 8D^3)\bigg)\Bigg] \end{aligned} \tag{D.20}$$

$$\begin{aligned} Z_0^{(1)} =& -\frac{D-2}{128D(D+2)H^4}\Bigg[+\xi^2\bigg((D^5 + 16D^4 + 16D^2 - 64D^2 + 16 + 32D - 64)DH^4 \\ & - 256D^2(D-2)(D-1)H^2m^2 - 256D(D-2)(D-1)m^4 + 16(D+2)(31D^2 - 6D + 24)H^4M^4 - \\ & 16D(D+2)(16D^3 - 13D^2 + 42D - 52)H^4M^2\bigg) - 8(D-1)(D+2)(2D^3 - 3D^2 + D - 8)H^4M^2 + \\ & 32\xi\bigg(-4(2D^3 + 3D^2 + 4)H^4M^4 + (4D^5 + D^4 + 16D^2 - 16D + 16)H^4M^2 + \\ & 4(D-2)(D^2 - D + 2)m^2(DH^2 + m^2)\bigg) + 32(D+2)(D^2 - D + 1)H^4M^4 - \\ & 16(D-2)(D^2 - 2D + 2)m^2(DH^2 + m^2) - 64D^2(D+2)H^4M^2\xi\Bigg] \end{aligned} \tag{D.21}$$

$$\begin{aligned} Z_0^{(2)} =& \frac{(1 - \frac{D}{2})}{3072}\Bigg[-\frac{1536m^6}{(D^2 + 6D + 8)H^6} + \frac{1}{D}\bigg(-(D-2)D^2(D(D(D(D+32) + 164) - 176) - 192)\xi^2 - \\ & - 128M^6\xi((D-12)\xi + 6) + 16M^4\bigg((D(3(D-262)D + 320) - 672)\xi^2 + 24(D(17D - 14) + 20)\xi - \\ & - 24(D(2D-3) + 3)\bigg) + 16M^2\bigg(2D(D(7D(21D - 29) + 370) - 384)\xi^2 - \\ & - 3(D(D(D(49D - 108) + 172) - 144) + 64)\xi + 6(D-2)(D(3(D-1)D + 5) - 4)\bigg)\bigg) + \\ & + \frac{48m^2}{D(D+2)(D+4)H^4}\bigg(m^2 + DH^2\bigg) \times Coeff\Bigg] \end{aligned} \tag{D.22}$$

where

$$
\begin{aligned}
Coeff =&\Bigg(4(3D^4-2D^3-58D^2+8(D+4)M^2+60D-48)-\\
&-(D+4)\xi^2\Big((D-2)(D^3-206D^2+240D)-4(D-8)(D-6)M^2\Big)-\\
&96(D-2)(D+4)(D^2-D+2)\xi\Bigg)
\end{aligned}
\tag{D.23}
$$

$$
\begin{aligned}
Z_0^{(3)} =&\frac{1-\frac{D}{2}}{6144D}\Bigg[\frac{384m^2(m^2+DH^2)}{(D+2)(D+4)H^6}\Big(D^2(D+10)H^2+4Dm^2-4(D+4)H^2M^2\Big)+\\
&+48M^2\xi\Big((D^2-2D-4M^2)(D^2-2D-4M^2-8)\Big)+\frac{\xi^2(D^2-2D-4M^2)}{(D+2)H^4}\Bigg((D-6)\\
&\Big(D(D^2-16)(D+2)^2H^4+48(D-8)m^2(m^2+DH^2)\Big)+4(D+2)(192+56D+D^2(D-26))H^4M^2-\\
&-32(D-12)(D+2)H^4M^4\Bigg)+12(D+2)\xi^2(D^2-2D-4M^2)\Big(D(D+2)(D-4)\\
&+4(D-12)M^2\Big)+\frac{48}{(D+2)H^4}\Big(4(D-2)m^2(m^2+DH^2)+D(D^2-4)H^4M^2-4(D+2)H^4M^4\Big)\\
&\Big(2-16\xi+(-2D+D^2(1-4\xi)^2)+8\xi(1-2\xi)\Big)\Bigg]
\end{aligned}
\tag{D.24}
$$

$$
\begin{aligned}
Z_0^{(4)} =&\frac{1-\frac{D}{2}}{49152D}\Bigg[-\frac{384m^2(m^2+DH^2)}{(D+2)(D+4)H^6}\Big(D^2(D+10)H^2+4Dm^2-4(D+4)H^2M^2\Big)-\\
&-48M^2\xi\Big((D^2-2D-4M^2)(D^2-2D-4M^2-8)\Big)+\frac{\xi^2(-D^2+2D+4M^2)}{(D+2)H^4}\\
&\Bigg((D-6)\Big(D(D^2-16)(D+2)^2H^4+48(D-8)m^2(m^2+DH^2)\Big)\\
&+4(D+2)(192+56D+D^2(D-26))H^4M^2-32(D-12)(D+2)H^4M^4\Bigg)\Bigg]
\end{aligned}
\tag{D.25}
$$

Appendix E

Extracting D'Alembertians

The aim of this appendix is to outline the way we extract d'Alembertian operators. This method is extremely relevant for the purposes of this thesis, since the strategy is to extract d'Alembertian operators until the right hand side of (5.49) is integrable at $D = 4$. At this point, one can set $D = 4$ (Of course if there are no $\frac{1}{D-4}$ factor).

The master formula we use is

$$\left[\frac{\square}{H^2} + D\right] f(y) = (4y - y^2) f^{''}(y) + D(2 - y) f^{'} + Df \tag{E.1}$$

For $f(y) \propto (\frac{4}{y})^{\frac{D}{2}-1}$, this identity produces a delta function [1][63]

$$\left[\frac{\square}{H^2} - \frac{D(D-2)}{4}\right]\left(\frac{4}{y}\right)^{\frac{D}{2}-1} - \frac{(4\pi)^{\frac{D}{2}}}{\Gamma(\frac{D}{2}-1)} \frac{\imath\delta^D(x-x')}{H^D\sqrt{-\bar{g}}} = 0 \tag{E.2}$$

This identity is extremely useful since it allows us to localize the $\frac{1}{D-4}$ divergences onto local δ function terms. To illustrate this, during our computation there will be terms of the form

$$\frac{1}{D-4}\left[\frac{\square}{H^2} - \frac{D(D-2)}{4}\right]\left(\frac{4}{y}\right)^{D-3} \tag{E.3}$$

Adding zero in $(E.3)$ and using $(E.2)$, we end up with

$$\frac{1}{D-4}\left[\frac{\square}{H^2} - \frac{D(D-2)}{4}\right]\left(\left(\frac{4}{y}\right)^{D-3} - \left(\frac{4}{y}\right)^{\frac{D}{2}-1}\right) + \frac{(4\pi)^{\frac{D}{2}}}{\Gamma(\frac{D}{2}-1)} \frac{\imath\delta^D(x-x')}{H^D\sqrt{-\bar{g}}} \frac{1}{D-4} =$$

$$= -\frac{1}{2}\left[\frac{\square}{H^2} - 2\right]\left(\frac{4}{y}\ln\left(\frac{y}{4}\right)\right) + O(D-4) + \frac{(4\pi)^{\frac{D}{2}}}{\Gamma(\frac{D}{2}-1)} \frac{\imath\delta^D(x-x')}{H^D\sqrt{-\bar{g}}} \frac{1}{D-4} \tag{E.4}$$

In order to show $(E.4)$ we used that

$$\frac{1}{D-4}\left[\left(\frac{4}{y}\right)^{D-3}-\left(\frac{4}{y}\right)^{\frac{D}{2}-1}\right]=\frac{1}{D-4}\left(\frac{4}{y}\right)^{\frac{D}{2}-1}\left[\left(\frac{4}{y}\right)^{\frac{D}{2}-2}-1\right]$$

$$=\frac{1}{D-4}\left(\frac{4}{y}\right)^{\frac{D}{2}-1}\left[-\frac{(D-4)}{2}\ln\left(\frac{y}{4}\right)\right]=-\frac{1}{2}\left(\frac{4}{y}\ln\left(\frac{y}{4}\right)\right) \tag{E.5}$$

where in order to go from the second equality to the third, we expand the function $(\frac{4}{y})^{\frac{D}{2}-2}$ around $D=4-\epsilon$, for $\epsilon\to 0$.

$$\left(\frac{4}{y}\right)^{\frac{D}{2}-2}-1=e^{\ln(\frac{4}{y})(\frac{D}{2}-2)}-1=-\frac{D-4}{2}\ln\frac{4}{y}$$

Notice that the $\frac{1}{D-4}$ factor in $(E.3)$ has been localized onto a delta function term. By adding the appropriate local counterterms [14][1][64][65][33], we subtract these terms.

For $f(y)\propto(\frac{4}{y})^{p-2}$, by making use of $(E.1)$ and performing some algebra, we find the important formula

$$\left[\frac{\Box}{H^2}+D\right]^2\left(\frac{4}{y}\right)^{p-2}=(p-1)\,(p-2)\left(p-\frac{D}{2}\right)\left(p-1-\frac{D}{2}\right)\left(\frac{4}{y}\right)^{p}+$$

$$+(p-2)\left(p-1-\frac{D}{2}\right)\left[D(2p-1)-2(p-1)^2\right]\left(\frac{4}{y}\right)^{p-1}+(p-1)^2(D-p+2)^2\left(\frac{4}{y}\right)^{p-2} \tag{E.6}$$

Or equivalently, we can write $(E.5)$ in the following way

$$\left[\frac{\Box}{H^2}+D\right]^{-2}\left(\frac{4}{y}\right)^{p}=\frac{1}{(p-1)(p-2)(p-\frac{D}{2})(p-1-\frac{D}{2})}\left(\frac{4}{y}\right)^{p-2}$$

$$-\left[\frac{\Box}{H^2}+D\right]^{-2}\left[\frac{[D(2p-1)-2(p-1)^2]}{(p-1)(p-\frac{D}{2})}\left(\frac{4}{y}\right)^{p-1}+\frac{(p-1)(D-p+2)^2}{(p-2)(p-\frac{D}{2})(p-1-\frac{D}{2})}\left(\frac{4}{y}\right)^{p-2}\right] \tag{E.7}$$

For different values of $p = (D-2, D-1, D, D+1, D+2, \frac{D}{2}+2, \frac{D}{2}+3)$, $(E.7)$ reduces the powers of $\frac{1}{y}$. One can recursively use $(E.7)$ in order to make the expressions integrable in $D=4$. It is apparent from $(E.7)$ that for $p = \frac{D}{2}$ and $p = \frac{D}{2}+1$ the above formula is not useful since the denominator blows up.

Reductions

At this point, we will use the formula $(E.7)$ in order to reduce the powers of various $(\frac{1}{y})^i$ for $i = (D, D-1, D-2)$.

- For $i = D$,

$$\left[\frac{\square}{H^2} + D\right]^{-2} \left(\frac{4}{y}\right)^{D} = k_1 \left(\frac{4}{y}\right)^{D-2} - \left[\frac{\square}{H^2} + D\right]^{-2} \left(k_2 \left(\frac{4}{y}\right)^{D-1} + k_3 \left(\frac{4}{y}\right)^{D-2}\right) \tag{E.8}$$

- For $i = D-1$,

$$\left[\frac{\square}{H^2} + D\right]^{-2} \left(\frac{4}{y}\right)^{D-1} = k_4 \left(\frac{4}{y}\right)^{D-3} - \left[\frac{\square}{H^2} + D\right]^{-2} \left(k_5 \left(\frac{4}{y}\right)^{D-2} + k_6 \left(\frac{4}{y}\right)^{D-3}\right) \tag{E.9}$$

- For $i = D-2$,

$$\left[\frac{\square}{H^2} + D\right]^{-2} \left(\frac{4}{y}\right)^{D-2} = k_7 \left(\frac{4}{y}\right)^{D-4} - \left[\frac{\square}{H^2} + D\right]^{-2} \left(k_8 \left(\frac{4}{y}\right)^{D-3} + k_9 \left(\frac{4}{y}\right)^{D-4}\right) \tag{E.10}$$

where,

Table 7:Values of k_i for $i = 1, 2, .., 9$

k_1	$\frac{4}{D(D-1)(D-2)^2}$
k_2	$\frac{2(3D-2)}{D(D-1)}$
k_3	$\frac{16(D-1)}{D(D-2)^2}$
k_4	$\frac{4}{(D-4)(D-3)(D-2)^2}$
k_5	$\frac{(5D-8)}{(D-2)^2}$
k_6	$\frac{36}{(D-3)(D-4)}$
k_7	$\frac{4}{(D-3)(D-6)(D-4)^2}$
k_8	$\frac{2(7D-18)}{(D-3)(D-4)}$
k_9	$\frac{64(D-3)}{(D-6)(D-4)^2}$

(E.11)

Finally, important reductions for our analysis are

$$\left(\frac{4}{y}\right)^{D} = \frac{4}{D(D-1)(D-2)^2}\left[\left(\frac{\Box}{H^2}\right)^2 - (D-2)\frac{\Box}{H^2}\right]\left(\frac{4}{y}\right)^{D-2} \tag{E.12}$$

$$\left(\frac{4}{y}\right)^{D-1} = \frac{2D}{D(D-2)^2}\left[\frac{\Box}{H^2} - (D-2)\right]\left(\frac{4}{y}\right)^{D-2} \tag{E.13}$$

In (E.13) we extract one d'Alembertian and the $D-1$ power is reduced to $D-2$, whereas in (E.12) we extracted two d'Alembertians and D reduced by two $\rightarrow D-2$.

Bibliography

[1] RL Koelewijn. Gravitons on de sitter modified by quantum fluctuations of a nonminimally coupled massive scalar. Master's thesis, 2017.

[2] Joel R Primack. Precision cosmology. *New Astronomy Reviews*, 49(2):25–34, 2005.

[3] Joel R Primack. Precision cosmology: successes and challenges. *Nuclear Physics B-Proceedings Supplements*, 173:1–5, 2007.

[4] N Aghanim, Y Akrami, M Ashdown, J Aumont, C Baccigalupi, M Ballardini, AJ Banday, RB Barreiro, N Bartolo, S Basak, et al. Planck 2018 results. vi. cosmological parameters. *arXiv preprint arXiv:1807.06209*, 2018.

[5] Y Akrami, F Arroja, M Ashdown, J Aumont, C Baccigalupi, M Ballardini, AJ Banday, RB Barreiro, N Bartolo, S Basak, et al. Planck 2018 results. x. constraints on inflation. *arXiv preprint arXiv:1807.06211*, 2018.

[6] Andrei D Linde. Chaotic inflation. *Physics Letters B*, 129(3-4):177–181, 1983.

[7] Alexei A Starobinsky. Dynamics of phase transition in the new inflationary universe scenario and generation of perturbations. *Physics Letters B*, 117(3-4):175–178, 1982.

[8] Alan H Guth. Inflationary universe: A possible solution to the horizon and flatness problems. *Physical Review D*, 23(2):347, 1981.

[9] Erwin Schro¨ dinger. The proper vibrations of the expanding universe. *Physica*, 6(7-12):899–912, 1939.

[10] Leonard Parker. Particle creation in expanding universes. *Physical Review Letters*, 21(8):562, 1968.

[11] Leonard Parker. Quantized fields and particle creation in expanding universes. i. *Physical Review*, 183(5):1057, 1969.

[12] Leonard Parker. Quantized fields and particle creation in expanding universes. ii. *Physical Review D*, 3(2):346, 1971.

[13] LP Grishchuk. Amplification of gravitational waves in an isotropic universe. *Soviet Journal of Experimental and Theoretical Physics*, 40:409, 1975.

[14] Sohyun Park and RP Woodard. Scalar contribution to the graviton self-energy during inflation. *Physical Review D*, 83(8):084049, 2011.

[15] Sohyun Park and R. P. Woodard. Inflationary scalars do not affect gravitons at one loop. *Phys. Rev. D*, 84:124058, Dec 2011.

[16] Sohyun Park, Tomislav Prokopec, and RP Woodard. Quantum scalar corrections to the gravitational potentials on de sitter background. *Journal of High Energy Physics*, 2016(1):74, 2016.

[17] Shun-Pei Miao and RP Woodard. The fermion self-energy during inflation. *Classical and Quantum Gravity*, 23(5):1721, 2006.

[18] D Glavan, T Prokopec, SP Miao, and RP Woodard. Electrodynamic effects of inflationary gravitons. *Classical and Quantum Gravity*, 31(17):175002, 2014.

[19] Katie E Leonard and RP Woodard. Graviton corrections to vacuum polarization during inflation. *Classical and Quantum Gravity*, 31(1):015010, 2013.

[20] SP Miao, T Prokopec, and RP Woodard. Scalar enhancement of the photon electric field by the tail of the graviton propagator. *Physical Review D*, 98(2):025022, 2018.

[21] Katie E Leonard, Sohyun Park, Tomislav Prokopec, and RP Woodard. Representing the graviton self-energy on de sitter background. *Physical Review D*, 90(2):024032, 2014.

[22] Katie E Leonard, Tomislav Prokopec, and Richard P Woodard. Representing the vacuum polarization on de sitter. *Journal of Mathematical Physics*, 54(3):032301, 2013.

[23] TM Janssen, SP Miao, T Prokopec, and RP Woodard. Infrared propagator corrections for constant deceleration. *Classical and Quantum Gravity*, 25(24):245013, 2008.

[24] D Glavan, SP Miao, Tomislav Prokopec, and RP Woodard. One loop graviton corrections to dynamical photons in de sitter. *Classical and Quantum Gravity*, 34(8):085002, 2017.

[25] SP Miao, T Prokopec, and RP Woodard. The graviton tail almost completely wags the dog. *arXiv preprint arXiv:1806.00742*, 2018.

[26] T Prokopec, NC Tsamis, and RP Woodard. Stochastic inflationary scalar electrodynamics. *Annals of Physics*, 323(6):1324–1360, 2008.

[27] Tomislav Prokopec, Nicholas C Tsamis, and Richard P Woodard. Two loop stress-energy tensor for inflationary scalar electrodynamics. *Physical Review D*, 78(4):043523, 2008.

[28] T Brunier, VK Onemli, and RP Woodard. Two-loop scalar self-mass during inflation. *Classical and Quantum Gravity*, 22(1):59, 2004.

[29] Leonid Veniaminovich Keldysh. Diagram technique for nonequilibrium processes. *Zh. Eksp. Teor. Fiz.*, 47:1018, 1964.

[30] Sohyun Park and RP Woodard. Solving the effective field equations for the newtonian potential. *Classical and Quantum Gravity*, 27(24):245008, 2010.

[31] Anja Marunovic and Tomislav Prokopec. Time transients in the quantum corrected newtonian potential induced by a massless nonminimally coupled scalar field. *Physical Review D*, 83(10):104039, 2011.

[32] Anja Marunović and Tomislav Prokopec. Antiscreening in perturbative quantum gravity and resolving the newtonian singularity. *Physical Review D*, 87(10):104027, 2013.

[33] Martinus Veltman et al. Regularization and renormalization of gauge fields. *Nuclear Physics B*, 44(1):189–213, 1972.

[34] Sean M Carroll. *Spacetime and geometry. An introduction to general relativity*. 2004.

[35] Nicholas David Birrell, Nicholas David Birrell, and PCW Davies. *Quantum fields in curved space*. Number 7. Cambridge university press, 1984.

[36] Michael E Peskin. *An introduction to quantum field theory*. CRC Press, 2018.

[37] Dionysios Anninos. de sitter musings. *International Journal of Modern Physics A*, 27(13):1230013, 2012.

[38] Bruce Allen. Vacuum states in de sitter space. *Physical Review D*, 32(12):3136, 1985.

[39] Daniel Baumann. Tasi lectures on inflation. *arXiv preprint arXiv:0907.5424*, 2009.

[40] Tomislav Prokopec. Lecture notes on cosmology. *Series of lectures given at the University of Utrecht, spring*, 75, 2005.

[41] Daniel Baumann, Mark G Jackson, Peter Adshead, Alexandre Amblard, Amjad Ashoorioon, Nicola Bartolo, Rachel Bean, Maria Beltrán, Francesco De Bernardis, Simeon Bird, et al. Probing inflation with cmb polarization. In *AIP Conference Proceedings*, volume 1141, pages 10–120. AIP, 2009.

[42] Wayne Hu and Martin White. A cmb polarization primer. *arXiv preprint astro-ph/9706147*, 1997.

[43] Tomislav Prokopec and Jan Weenink. Frame independent cosmological perturbations. *Journal of Cosmology and Astroparticle Physics*, 2013(09):027, 2013.

[44] Tomislav Prokopec and Jan Weenink. Uniqueness of the gauge invariant action for cosmological perturbations. *Journal of Cosmology and Astroparticle Physics*, 2012(12):031, 2012.

[45] Jan Weenink and Tomislav Prokopec. Gauge invariant cosmological perturbations for the non-minimally coupled inflaton field. *Physical Review D*, 82(12):123510, 2010.

[46] Juan Maldacena. Non-gaussian features of primordial fluctuations in single field inflationary models. *Journal of High Energy Physics*, 2003(05):013, 2003.

[47] NA Chernikov and EA Tagirov. Quantum theory of scalar field in de sitter space-time. In *Annales de l'IHP Physique théorique*, volume 9, pages 109–141. Gauthier-Villars, 1968.

[48] RP Woodard. Perturbative quantum gravity comes of age. *International Journal of Modern Physics D*, 23(09):1430020, 2014.

[49] SP Miao and RP Woodard. Issues concerning loop corrections to the primordial power spectra. *Journal of Cosmology and Astroparticle Physics*, 2012(07):008, 2012.

[50] David Seery, James E Lidsey, and Martin S Sloth. The inflationary trispectrum. *Journal of Cosmology and Astroparticle Physics*, 2007(01):027, 2007.

[51] Philip R Jarnhus and Martin S Sloth. The de sitter limit of inflation and non-linear perturbation theory. *Journal of Cosmology and Astroparticle Physics*, 2008(02):013, 2008.

[52] Wei Xue, Xian Gao, and Robert Brandenberger. Ir divergences in inflation and entropy perturbations. *Journal of Cosmology and Astroparticle Physics*, 2012(06):035, 2012.

[53] S Basu and RP Woodard. Testing an ansatz for the leading secular loop corrections from quantum gravity during inflation. *Classical and Quantum Gravity*, 33(20):205007, 2016.

[54] EO Kahya and RP Woodard. Quantum gravity corrections to the one loop scalar self-mass during inflation. *Physical Review D*, 76(12):124005, 2007.

[55] EO Kahya and RP Woodard. Scalar field equations from quantum gravity during inflation. *Physical Review D*, 77(8):084012, 2008.

[56] Shun-Pei Miao and RP Woodard. A simple operator check of the effective fermion mode function during inflation. *Classical and Quantum Gravity*, 25(14):145009, 2008.

[57] Katie E Leonard, T Prokopec, and RP Woodard. Covariant vacuum polarizations on de sitter background. *Physical Review D*, 87(4):044030, 2013.

[58] Charles W Misner, Kip S Thorne, John Archibald Wheeler, and David I Kaiser. *Gravitation*. Princeton University Press, 2017.

[59] George Leibbrandt. Introduction to the technique of dimensional regularization. *Reviews of Modern Physics*, 47(4):849, 1975.

[60] Klaus Sibold. Bogoliubov-parasiuk-hepp-zimmermann renormalization scheme. *Scholarpedia*, 5(5):7306, 2010.

[61] Katie E Leonard, Sohyun Park, Tomislav Prokopec, and RP Woodard. Representing the graviton self-energy on de sitter background. *Physical Review D*, 90(2):024032, 2014.

[62] Katie E Leonard, Tomislav Prokopec, and Richard P Woodard. Representing the vacuum polarization on de sitter. *Journal of Mathematical Physics*, 54(3):032301, 2013.

[63] Vakif K Onemli and Richard P Woodard. Super-acceleration from massless, minimally coupled ϕ4. *Classical and Quantum Gravity*, 19(17):4607, 2002.

[64] M Veltman. G.'t hooft. *Ann. Henri Poincaré*, 20(1), 1974.

[65] Gerardus t Hooft and Martinus JG Veltman. One-loop divergencies in the theory of gravitation. *Ann. Henri Poincaré*, 20(CERN-TH-1723):69–94, 1974.