Phenomenological Theory of Spin Hydrodynamic Generation

A.E. Baerends, BSc.

Master Thesis

Under the supervision of

prof. dr. R.A. Duine

Institute of Theoretical Physics, Utrecht University

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Abstract

In a recent experiment, it was found that mercury (Hg) flowing through a pipe lined with platinum (Pt) strips generated electrical voltage across the strips. This is thought to be due to spin hydrodynamic generation (SHG) from the coupling between the fluid vorticity and internal spins of the particles that make up the fluid, such as the electron's spin. A theory for this coupling is developed in a phenomenological manner from irreversible thermodynamics, and hydrodynamical modes and simple spin density solutions for flows between plates and inside pipes are obtained. These spin density solutions are then applied to the experimental setting motivating our research in order to investigate interface effects. We find and confirm that the spin current generated inside the viscous fluid induces spin transport across the Hg-Pt interface, and, by means of the inverse spin Hall effect, an electrical potential. We also discuss an alternative interpretation based on momentum, rather than spin, transfer.

Contents

Chapter 1

Introduction

One of the most prospective fields of research—both theoretical and practical—in condensed matter physics is that of spintronics: the use of the internal spin of the electron for data storage and transport[1,2]. It is this manipulation and use of electron spins which allows for novel, efficient computational devices, that has given the field of spintronics considerable attention over the past years.

Among the several ways to detect the spin currents inside such devices, the *inverse spin Hall effect* (ISHE) is one of them, and conceptually fairly simple. Whereas the spin Hall effect (SHE) lets an external electric field induce spin currents[3,4], the ISHE allows one to detect spin currents. In a system of two or more layers of different layers exhibiting the SHE, a spin current through one of the materials induces spin transport into the other layer, generating electrical current by means of the ISHE[5,6].

Recently, an experiment by Takahashi *et al.*[7] found that a flowing viscous fluid—in this case mercury (Hg)—inside a pipe lined with platinum (Pt)—a metal with sizeable SHE and ISHE—induces an electric charge current; see Fig. 1.1 for a schematic overview of the set-up. Takahashi *et al.* have proposed that a coupling between the fluid vorticity and electron spins occurs inside the mercury, which then generates a

Figure 1.1: Set-up as used by Takahashi *et al.* Image reference: *ibid.*

spin current. At the Hg-Pt interface, this spin current is injected into the platinum strips, where the ISHE subsequently occurs, providing the measured electric voltages. This observed effect has been named *spin hydrodynamic generation* (SHG) by Takahashi *et al.* due to its similarity to magnetohydrodynamic generation[8]; see Fig. 1.2 for an overview and comparison of
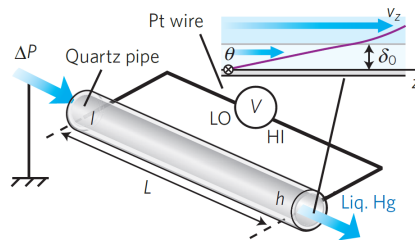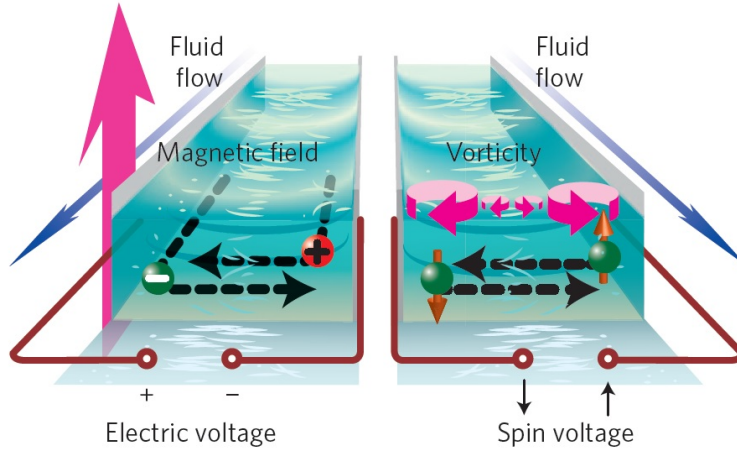
Figure 1.2: Magnetohydrodynamic generation (l.) and SHG (r.). Image reference: Takahashi *et al.* [7]

both phenomena. The main difference between magnetohydrodynamic generation and SHG, is that the latter does not require an external magnetic field.

While several of the collaborators of Takahashi *et al.* have presented a theory on this specific result based on quantum kinetic theory [9], it does not take in account phenomena which might be present in viscous fluids, such as advective currents or non-equilibrium thermodynamics. This motivates us to present a phenomenological theory of SHG, based on the non-equilibrium thermodynamics of flowing viscous fluids with internal spin. The fluids in particular do not necessarily need to consist of a mixture of nuclei and their electrons such as mercury, but generally encompass fluids where spin-vorticity coupling is present, such as viscous electron fluids [10].

The main idea of our phenomenological theory is that net average spin is generated in the fluid flow by means of rotational viscosity, which is the parameter coupling of the fluid vorticity to the internal fluid spin, in a microscopical setting by means of spin-orbit coupling. Next to the net spin, this mechanism also provides a spin current inside the fluid, directed perpendicular to the direction of the fluid's proper velocity, in the same manner as an electrical current through a conducting wire generates a magnetic field. While this spin current is normally bounded by the outermost limits of the fluid flow—*i.e.* no spins venture beyond the boundaries of the flow—we add strips of metals with sizeable SHE and ISHE, such that the generated spin current acts as spin transport in the metal, which by means of the *inverse spin Hall effect* (ISHE) generate electrical current inside the metal strips, and therefore electric potential.

Although this Thesis is mainly focused on a novel subject within condensed matter theory, the phenomenon of SHG—and by extension, that of

the coupling between fluid vorticity and spin-orbit coupling—is not necessarily confined to table-top experiments at room temperature. In a recent experiment[11], the STAR collaboration used the phenomenon of SHG in the hydrodynamic description of a type of fluid well-known in the field of high-energy physics: quark-gluon plasmas, products of heavy nuclei collisions. It was found that the quark-gluon plasma might possibly exhibit the largest vorticity yet observed, opening up new regimes in hydrodynamic descriptions of the quark-gluon plasma, whereas its vortical structure might aid in developing new insights on quantum chromodynamics[11,12]. It is this connection with high-energy physics, next to the obvious one with the condensed matter field of spintronics, that makes SHG interesting to research.

In Chapter 2, we shall set up a phenomenological theory of spin in a fluid flow based on a similar treatment for classical angular momentum given by De Groot & Mazur[13], while enabling for spin flux in the system as per Snider & Lewchuk[14]. This phenomenological theory is then modified by the presence of electromagnetic fields, in order to create a proper theory of spin hydrodynamics, where (internal) spin is generated by interaction with the fluid vorticity through rotational viscosity. As a first application of this theory, we extract hydrodynamic modes in Chapter 3, in order to determine the role of the fluid's proper velocity, vorticity, and viscosity—both of the bulk and rotational variants—in the aforementioned hydrodynamic modes. Next, we consider simple, practical fluid flows, such as the Poiseuille flow, in Chapter 4, where the spin per unit mass generated in these flows is analytically determined. The boundary conditions to be used in such systems—both for the spin itself as well as for the spin current—and systems with external electromagnetic forces are also discussed. Finally, Chapter 5 mainly concerns with the inverse spin Hall effect generated at the interface between the viscous spin fluid flow and metal strips along this flow, as a means of simulating flowing mercury sidelined with platinum strips. The generated voltage due to this effect is determined both analytically as well as numerically, and compared in Chapter 6 with the findings by Takahashi *et al.* and further discussed, with voltages due to electronic drag and materials, other than platinum, with sizeable SHE and ISHE.

Chapter 2

Theory

In this Chapter, we reproduce the results given by De Groot & Mazur[13], Chapters II, XII, and XIII, with some alterations on the spin current by Snider & Lewchuk[14] to account for the presence of spin diffusion in the spin hydrodynamic generation system. In Section 2.1, we obtain a set of phenomenological equations for hydrodynamic quantities under non-equilibrium thermodynamics, for a viscous fluid possessing internal spin. We then add electromagnetism to our viscous fluid in Section 2.2, and derive new phenomenological equations in Section 2.3 for multi-component fluids.

In contrast to the notation used by De Groot & Mazur, we shall use a tensorial index notation for vectors, tensors, and derivatives together with the Einstein summation convention. Let a be a quantity, then we denote its $\hat{x}$-component as a_x. For indices, we shall use Greek letters starting from α; the indices always note one of the spatial dimensions, such that a_α means either a_x, a_y, or a_z in a Cartesian system. Likewise, the inner product of two vectors a, b is denoted $a^\alpha b_\alpha$, and the vector product as $\epsilon_{\alpha\beta\gamma} a^\alpha b^\alpha$ where $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita tensor.

2.1 Viscous fluid with spin

Let a—whether a scalar, a vector, or a tensor—be a given quantity per unit mass; its total time derivative is defined as[13]:

$$\rho \frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\partial}{\partial t}(\rho a) + \frac{\partial}{\partial x_\alpha}(\rho a v^\alpha), \qquad (2.1)$$

where ρ is the fluid density, and v^α the fluid velocity. The final term of the right hand side of Eq. (2.1) denotes the divergence of the advective term for the quantity density a. The mass density ρ is conserved, by means of the continuity equation, as follows:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_\alpha}(\rho v^\alpha) = 0, \qquad (2.2)$$

which is used together with the chain rule in order to yield Eq. (2.1). Let us define the total system mass m as $m = \rho\nu$, where ν is the system's specific volume. Plugging this into Eq. (2.1) and again using the chain rule, we obtain a conservation law for mass:

$$\rho\frac{\mathrm{d}\nu}{\mathrm{d}t} = \frac{\partial}{\partial t}(\rho\nu) + \frac{\partial}{\partial x_\alpha}(\rho\nu v^\alpha) = \frac{\partial v^\alpha}{\partial x_\alpha}. \tag{2.3}$$

For a fluid with uniform mass density ρ in the absence of external forces, let the conservation laws of energy density ρe and momentum density ρv^α be:

$$\frac{\partial}{\partial t}(\rho e) = -\frac{\partial}{\partial x_\alpha}\left(j_\mathrm{e}^\alpha + \rho e v^\alpha\right), \tag{2.4}$$

$$\frac{\partial}{\partial t}(\rho v^\alpha) = -\frac{\partial}{\partial x_\beta}\left(P^{\alpha\beta} + \rho v^\alpha v^\beta\right), \tag{2.5}$$

where $P^{\alpha\beta}$ is the pressure tensor—here considered to contain an anti-symmetric part in addition to the symmetric part and the trace—and j_e^α the fluid energy flow which itself is proportional to the fluid heat flow j_q^α. The fluid also possesses an angular momentum per unit mass J^α, as an axial vector. This axial vector, along with other axial vectors present in this discourse, can be written in terms of anti-symmetric tensors as

$$J^\alpha = \epsilon^{\alpha\beta\gamma}J_{\beta\gamma}, \tag{2.6}$$

such that $J^3 = J_{12} = -J_{21}$. Using this tensorial notation, the total angular momentum can now be split into 'external' angular momentum per unit mass $L_{\alpha\beta}$, and 'internal' angular momentum per unit mass $S_{\alpha\beta}$ as $J_{\alpha\beta} = L_{\alpha\beta} + S_{\alpha\beta}$. The 'external' component uses the classical definition of angular momentum as the wedge product of distance and velocity:

$$L_{\alpha\beta} = r_\alpha v_\beta - r_\beta v_\alpha, \tag{2.7}$$

while the 'internal' component is, for our purposes, being equated with the quantum-mechanical spin per unit mass $S_{\alpha\beta}$

From a classical definition of the total angular momentum per unit mass and its conservation law, one might deduce the total time derivative of $S_{\alpha\beta}$ as being wholly dependent on the pressure tensor:

$$\rho\frac{\mathrm{d}S_{\alpha\beta}}{\mathrm{d}t} = -2P_{\alpha\beta}^\mathrm{a}, \ \ i.e. \ \ \rho\frac{\mathrm{d}S_\alpha}{\mathrm{d}t} = -2P_\alpha^\mathrm{a}, \tag{2.8}$$

which De Groot & Mazur have obtained. However, as noted by Snider & Lewchuk[14], this approach ignores the contribution of internal angular momentum flux to the internal angular momentum's conservation equation, and therefore Eq. (2.8) is not an adequate equation for systems where the

internal angular momentum is flowing. We therefore add an extra term, the divergence of the spin flux tensor $j_{s,\alpha\beta}$, to Eq. (2.8):

$$\rho\frac{\mathrm{d}S_\alpha}{\mathrm{d}t} = -2P_\alpha^{\mathrm{a}} - \frac{\partial}{\partial x^\beta}j_{s,\alpha\beta}, \qquad (2.9)$$

or written as the local time derivative of spin using Eq. (2.1):

$$\frac{\partial}{\partial t}\left(\rho S_\alpha\right) = -2P_\alpha^{\mathrm{a}} - \frac{\partial}{\partial x^\beta}\left(j_{s,\alpha\beta} + \rho S_\alpha v_\beta\right). \qquad (2.10)$$

The total energy per unit mass e can be divided into three separate parts as follows:

$$e = \frac{1}{2}v_\alpha v^\alpha + u_r + u, \qquad (2.11)$$

where the first term is (obviously) the kinetic term, while the second term, u_r denotes the 'rotational energy per unit mass', and the final term denotes the internal energy per unit mass. As expected, the internal energy per unit is quadratically proportional to the spin:

$$u_r = \frac{S^2}{2\chi_{\mathrm{s}}} = \frac{1}{2\chi_{\mathrm{s}}}S^\alpha S_\alpha, \qquad (2.12)$$

where χ_{s} is the spin susceptibility[1]. The (correct) balance equation for the rotational energy can be obtained using Eq. (2.10) and the chain rule:

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho u_r) &= \frac{\partial}{\partial t}\left(\frac{\rho}{2\chi_{\mathrm{s}}}S_\alpha S^\alpha\right) = \frac{S^\alpha}{\chi_{\mathrm{s}}}\frac{\partial}{\partial t}\left(\rho S_\alpha\right) \\
&= -\frac{2}{\chi_{\mathrm{s}}}S^\alpha P_\alpha^{\mathrm{a}} - \frac{S^\alpha}{\chi_{\mathrm{s}}}\frac{\partial}{\partial x^\beta}\left(j_{s,\alpha\beta} + \rho S_\alpha v_\beta\right) \\
&= -\frac{2}{\chi_{\mathrm{s}}}S^\alpha P_\alpha^{\mathrm{a}} - \frac{\partial}{\partial x^\beta}\left(\frac{S^\alpha}{\chi_{\mathrm{s}}}j_{s,\alpha\beta}\right) + \frac{j_{s,\alpha\beta}}{\chi_{\mathrm{s}}}\frac{\partial S^\alpha}{\partial x^\beta} - \frac{\partial}{\partial x^\beta}\left(\frac{\rho}{2\chi_{\mathrm{s}}}S^\alpha S_\alpha v_\beta\right) \\
&= -\frac{2}{\chi_{\mathrm{s}}}S^\alpha P_\alpha^{\mathrm{a}} - \frac{\partial}{\partial x^\beta}\left(\frac{S^\alpha}{\chi_{\mathrm{s}}}j_{s,\alpha\beta}\right) + \frac{j_{s,\alpha\beta}}{\chi_{\mathrm{s}}}\frac{\partial S^\alpha}{\partial x^\beta} - \frac{\partial}{\partial x_\alpha}\left(\rho u_r v^\alpha\right).
\end{aligned} \qquad (2.13)$$

Finally, the conservation law equation for the kinetic term $\frac{1}{2}v_\alpha v^\alpha$ follows from using the chain rule in Eq. (2.5):

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\rho}{2}v_\alpha v^\alpha\right) &= v_\alpha\frac{\partial}{\partial t}\left(\rho v^\alpha\right) \\
&= -v_\alpha\frac{\partial}{\partial x_\beta}\left(P^{\alpha\beta} + \rho v^\alpha v^\beta\right) \\
&= -\frac{\partial}{\partial x_\alpha}\left(P^{\alpha\beta}v_\beta\right) + P^{\alpha\beta}\frac{\partial v_\alpha}{\partial x_\beta} - \frac{\partial}{\partial x_\alpha}\left(\frac{\rho}{2}v^2 v^\alpha\right). \qquad (2.14)
\end{aligned}$$

---

[1]Note that for Eq. (2.12) to remain dimensionally correct as energy per unit mass on both sides, we require χ_{s} to possess the dimension of [length]^2. A practical calculation of χ_{s} can be seen in the Appendix, Section A.2, where the precise nature of χ_{s} is also further discussed.

The balance equation for the internal energy per unit mass can then be obtained by subtracting Eqs. (2.13) and (2.14) from Eq. (2.4):

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\rho u\right) &= \frac{\partial}{\partial t}\left(\rho e\right) - \frac{\partial}{\partial t}\left(\frac{\rho}{2}v_\alpha v^\alpha\right) - \frac{\partial}{\partial t}\left(\rho u_r\right) \\
&= -\frac{\partial}{\partial x_\alpha}\left(j_{\mathrm{e}}^\alpha + \rho e v^\alpha\right) + \frac{\partial}{\partial x_\alpha}\left(P^{\alpha\beta}v_\beta\right) - P^{\alpha\beta}\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial}{\partial x_\alpha}\left(\frac{\rho}{2}v^2 v^\alpha\right) \\
&\quad + \frac{2}{\chi_{\mathrm{s}}}S^\alpha P_\alpha^{\mathrm{a}} + \frac{\partial}{\partial x_\alpha}\left(\frac{S_\beta}{\chi_{\mathrm{s}}}j_{\mathrm{s}}^{\alpha\beta}\right) - \frac{j_{\mathrm{s},\alpha\beta}}{\chi_{\mathrm{s}}}\frac{\partial S^\alpha}{\partial x^\beta} + \frac{\partial}{\partial x_\alpha}\left(\rho u_r v^\alpha\right) \\
&= -\frac{\partial}{\partial x_\alpha}\left(j_{\mathrm{q}}^\alpha + \rho u v^\alpha\right) - P^{\alpha\beta}\frac{\partial v_\alpha}{\partial x_\beta} + \frac{2}{\chi_{\mathrm{s}}}S^\alpha P_\alpha^{\mathrm{a}} - \frac{j_{\mathrm{s},\alpha\beta}}{\chi_{\mathrm{s}}}\frac{\partial S^\alpha}{\partial x^\beta}, \quad (2.15)
\end{aligned}
$$

where we have defined the thermal energy flow j_{q}^α as:

$$
j_{\mathrm{q}}^\alpha = j_{\mathrm{e}}^\alpha - P^{\alpha\beta}v_\beta - \frac{S_\beta}{\chi_{\mathrm{s}}}j_{\mathrm{s}}^{\alpha\beta}, \tag{2.16}
$$

and used Eq. (2.11) to rewrite various divergent terms into the divergence of the advective term $\rho u v^\alpha$. In the total time derivative form of Eq. (2.1), Eq. (2.15) becomes:

$$
\begin{aligned}
\rho\frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{\partial}{\partial t}\left(\rho u\right) + \frac{\partial}{\partial x_\alpha}\left(\rho u v^\alpha\right) \\
&= -\frac{\partial}{\partial x_\alpha}j_{\mathrm{q}}^\alpha - P^{\alpha\beta}\frac{\partial v_\alpha}{\partial x_\beta} + \frac{2}{\chi_{\mathrm{s}}}S^\alpha P_\alpha^{\mathrm{a}} - \frac{j_{\mathrm{s},\alpha\beta}}{\chi_{\mathrm{s}}}\frac{\partial S^\alpha}{\partial x^\beta} \quad (2.17)
\end{aligned}
$$

It is now possible to exploit the degrees of freedom contained within the pressure tensor $P^{\alpha\beta}$: it can be split into a scalar equilibrium pressure p and a viscous pressure tensor $\Pi^{\alpha\beta}$ by writing $P^{\alpha\beta} = p\delta^{\alpha\beta} + \Pi^{\alpha\beta}$, where $\delta^{\alpha\beta}$ is a Kronecker delta. The viscous pressure tensor $\Pi^{\alpha\beta}$ can then be further split into a scalar part $\Pi = \frac{1}{3}\mathrm{Tr}(\Pi^{\alpha\beta})$, a symmetric part with zero trace $\left(\Pi^{\alpha\beta}\right)^{\mathrm{s},0}$, and an anti-symmetric part $\left(\Pi^{\alpha\beta}\right)^{\mathrm{a}}$. The pressure tensor can then be rewritten into scalar, symmetric, and anti-symmetric components:

$$
P^{\alpha\beta} = (p + \Pi)\delta^{\alpha\beta} + \left(\Pi^{\alpha\beta}\right)^{\mathrm{s},0} + \left(\Pi^{\alpha\beta}\right)^{\mathrm{a}}. \tag{2.18}
$$

Since the axial vector P_α^{a} is the pressure tensor's anti-symmetric component, we can use $P_\alpha^{\mathrm{a}} = \Pi_\alpha^{\mathrm{a}}$. A similar decomposition into scalar, symmetric, and anti-symmetric components may also be performed onto the velocity gradient term present in the right-hand side of Eq. (2.15):

$$
\frac{\partial v^\alpha}{\partial x^\beta} = \frac{1}{3}\frac{\partial v^\gamma}{\partial x_\gamma}\delta^{\alpha\beta} + \left(\frac{\partial v^\alpha}{\partial x^\beta}\right)^{\mathrm{s},0} + \left(\frac{\partial v^\alpha}{\partial x^\beta}\right)^{\mathrm{a}}. \tag{2.19}
$$

Using Eqs. (2.18) and (2.19), and noting that cross products between two different types of decomposition (*i.e.* symmetric with scalar) yield zero, the

internal energy balance equation (2.15) is rewritten as:

$$\frac{\partial}{\partial t}(\rho u) = -\frac{\partial}{\partial x_\alpha}\left(j_q^\alpha + \rho u v^\alpha\right) - \frac{j_{s,\alpha\beta}}{\chi_s}\frac{\partial S^\alpha}{\partial x^\beta} + \frac{1}{2\chi_s}\epsilon^{\alpha\beta\gamma}\epsilon_{\alpha\delta\epsilon}S_{\beta\gamma}\left(\Pi^{\delta\epsilon}\right)^{\mathrm{a}}$$

$$-\left(p+\Pi\right)\frac{\partial v^\alpha}{\partial x_\alpha} - \left(\Pi^{\alpha\beta}\right)^{\mathrm{s},0}\left(\frac{\partial v_\alpha}{\partial x_\beta}\right)^{\mathrm{s},0} - \left(\Pi^{\alpha\beta}\right)^{\mathrm{a}}\left(\frac{\partial v_\alpha}{\partial x_\beta}\right)^{\mathrm{a}}$$

$$= -\frac{\partial}{\partial x_\alpha}\left(j_q^\alpha + \rho u v^\alpha\right) - \frac{j_{s,\alpha\beta}}{\chi_s}\frac{\partial S^\alpha}{\partial x^\beta} - \left(p+\Pi\right)\frac{\partial v^\alpha}{\partial x_\alpha}$$

$$-\Pi_{\alpha\beta}^{\mathrm{s},0}\left(\frac{\partial v^\alpha}{\partial x^\beta}\right)^{\mathrm{s},0} - \Pi_\alpha^{\mathrm{a}}\left(\epsilon^{\alpha\beta\gamma}\frac{\partial v_\beta}{\partial x_\gamma} - \frac{S^\alpha}{\chi_s}\right). \tag{2.20}$$

Using Eq. (2.3), Eq. (2.20) is rewritten as

$$\frac{\partial}{\partial t}(\rho u) + p\frac{\partial}{\partial t}(\rho\nu) = -\frac{\partial}{\partial x_\alpha}\left(j_q^\alpha + \rho\left(u + p\nu\right)v^\alpha\right) - \frac{j_{s,\alpha\beta}}{\chi_s}\frac{\partial S^\alpha}{\partial x^\beta} - \Pi\frac{\partial v^\alpha}{\partial x_\alpha}$$

$$-\Pi_{\alpha\beta}^{\mathrm{s},0}\left(\frac{\partial v^\alpha}{\partial x^\beta}\right)^{\mathrm{s},0} - \Pi_\alpha^{\mathrm{a}}\left(\epsilon^{\alpha\beta\gamma}\frac{\partial v_\beta}{\partial x_\gamma} - \frac{S^\alpha}{\chi_s}\right). \tag{2.21}$$

We are now ready to set up equations for the entropy production inside the system. For monocomponent systems, the Gibbs relation states that[2]:

$$T\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t} + p\frac{\mathrm{d}\nu}{\mathrm{d}t}, \tag{2.22}$$

where T is the system's temperature and σ the system's entropy per unit mass. Using Eq. (2.1), Eq. (2.22) is readily rewritten as a local time derivative of the entropy density as:

$$T\frac{\partial}{\partial t}(\rho\sigma) = \frac{\partial}{\partial t}(\rho u) + p\frac{\partial}{\partial t}(\rho\nu) + \frac{\partial}{\partial x_\alpha}\left[\rho(u + p\nu - T\sigma)v^\alpha\right], \tag{2.23}$$

such that the entropy balance equation is obtained with use of Eqs. (2.21) and (2.23) as:

$$\frac{\partial}{\partial t}(\rho\sigma) = -\frac{\partial}{\partial x_\alpha}j_\sigma^\alpha + \zeta, \tag{2.24}$$

where

$$j_\sigma^\alpha = \frac{j_q^\alpha}{T} - \rho\sigma v^\alpha, \tag{2.25}$$

is the entropy flow and

$$\zeta = -\frac{j_q^\alpha - \rho T\sigma v^\alpha}{T^2}\frac{\partial T}{\partial x_\alpha} - \frac{\Pi}{T}\frac{\partial v^\alpha}{\partial x_\alpha} - \frac{\Pi_{\alpha\beta}^{\mathrm{s},0}}{T}\left(\frac{\partial v^\alpha}{\partial x^\beta}\right)^{\mathrm{s},0}$$

$$-\frac{\Pi_\alpha^{\mathrm{a}}}{T}\left(\epsilon^{\alpha\beta\gamma}\frac{\partial v_\beta}{\partial x_\gamma} - \frac{S^\alpha}{\chi_s}\right) - \frac{j_{s,\alpha\beta}}{T\chi_s}\frac{\partial S^\alpha}{\partial x^\beta}, \tag{2.26}$$

---

[2]In systems or fluids with only one mass component, the balance equation for the chemical potential energy equals zero, and thus does not need to be considered in this case. For multi-component systems, as we shall see in Section 2.3, the Gibbs relation requires another term to account for the (electro)chemical potential balance equation.

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the entropy production, for which $\zeta \geq 0$.

In Eq. (2.26), the tensorial decompositions done in the previous paragraph show their use: one clearly distinguishes scalar, polar vector, symmetric and anti-symmetric terms in the right-hand side. For each of these terms, we derive phenomenological equations for the quantities which contribute to entropy production:

$$j_{\mathrm{q}}^{\alpha} = -\lambda \frac{\partial T}{\partial x^{\alpha}} + \rho T \sigma v^{\alpha}; \tag{2.27}$$

$$\Pi = -\eta_v \frac{\partial v^{\alpha}}{\partial x_{\alpha}}; \tag{2.28}$$

$$\Pi_{\alpha\beta}^{\mathrm{s},0} = -2\eta \left(\frac{\partial v_{\alpha}}{\partial x_{\beta}} \right)^{\mathrm{s},0}; \tag{2.29}$$

$$\Pi_{\alpha}^{\mathrm{a}} = -\eta_r \left(\epsilon_{\alpha\beta\gamma} \frac{\partial v^{\beta}}{\partial x^{\gamma}} - \frac{S_{\alpha}}{\chi_{\mathrm{s}}} \right); \tag{2.30}$$

$$j_{\mathrm{s},\alpha\beta} = -\frac{D_{\mathrm{s}}}{\chi_{\mathrm{s}}} \frac{\partial S_{\alpha}}{\partial x_{\beta}}, \tag{2.31}$$

where λ is the heat conductivity, η_v the volume viscosity, η the shear viscosity, η_r the spin ('rotational') viscosity, and D_{s} a spin diffusion constant[3]. As one would intuitively expect, the thermal flow of Eq. (2.27), the fluid's pressure of Eqs. (2.28) to (2.30), and the spin diffusion of Eq. (2.31) contribute to the spin fluid's entropy production.

2.2 Viscous fluid with spin in an electromagnetic field

Up until now, we considered a viscous fluid with spin and its entropy production and how the spin diffusion contributes to the entropy production. The preceding treatment omitted the presence of electromagnetic fields—and by extension the electromagnetic forces acting on the spin fluid—and therefore yields an incomplete theoretical background for the experimental set-up as used by Takahashi *et al.*[7], and its immediate results. We shall assume the fluid to be polarized under the presence of such external electromagnetic fields, such that next to the electric field E_{α} and the magnetic field B_{α}, we also gain the electric and magnetic polarization fields, Z_{α} and M_{α} respectively. The addition of these fields in our motion and balance equations for the viscous fluid discussed in Section 2.1 alters these equations in various manners.

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[3]The quantity of 'spin diffusion constant' as measured and cited usually possesses the dimension of $[\mathrm{length}]^2[\mathrm{time}]^{-1}$; a quick dimensional analysis shows that D_{s} possesses the dimension of $[\mathrm{mass}][\mathrm{length}][\mathrm{time}]^{-1}$. This dimensional problem is overcome if we regard the ratio $D_{\mathrm{s}}/\chi_{\mathrm{s}}$ as the mass density ρ times the 'true' spin diffusion constant $\hat{D}_{\mathrm{s}}$, *viz.* the Appendix, Section A.1 for an elaboration on the 'true' spin diffusion constant.

The viscous fluid is primarily being polarized and magnetized by the electromagnetic fields, such that next to the electromagnetic fields themselves one has the polarization and magnetization fields, Z_α and M_α respectively. The electric and magnetic displacement fields, D_α and H_α respectively, are defined as:

$$D_\alpha = Z_\alpha + E_\alpha;$$
$$H_\alpha = B_\alpha - M_\alpha. \tag{2.32}$$

With the displacement fields of Eq. (2.32), the Maxwell equations relevant to our general inquiry are:

$$\frac{\partial D^\alpha}{\partial x_\alpha} = \rho z; \tag{2.33}$$

$$\frac{\partial B^\alpha}{\partial x_\alpha} = 0; \tag{2.34}$$

$$\frac{\partial D^\alpha}{\partial t} = c\epsilon^{\alpha\beta\gamma}\frac{\partial H_\beta}{\partial x_\gamma} - I^\alpha; \tag{2.35}$$

$$\frac{\partial B^\alpha}{\partial t} = -c\epsilon^{\alpha\beta\gamma}\frac{\partial E_\beta}{\partial x_\gamma}, \tag{2.36}$$

where z is the charge per unit mass, c the speed of light, and

$$I^\alpha = \rho z v^\alpha + j_{\rm c}^\alpha, \tag{2.37}$$

the electric current density, where $j_{\rm c}^\alpha$ is the charge flow.

We define the magnetization field M_α as

$$M_\alpha = -\gamma_{\rm m}\rho S_\alpha, \tag{2.38}$$

i.e. the gyromagnetic ratio $\gamma_{\rm m}$ times the spin density. Physically, the gyromagnetic ratio consists of two parts: the orbital gyromagnetic ratio $\gamma_{\rm o}$, which is considered as the 'classical' magnetization, and the spin gyromagnetic ratio $\gamma_{\rm s}$, due to the spin magnetic moment coupling to the magnetic field, such that we have $\gamma = \gamma_{\rm o} + \gamma_{\rm s}$. Microscopically, the spin gyromagnetic ratio is defined as

$$\gamma_{\rm s} = \frac{gz}{2m}, \tag{2.39}$$

where g is a particle-dependent number, z the electric charge of the particle, and m the mass of the charged particle. The spin magnetic moment effectively creates another energy term in the system's Hamiltonian[15]; more precisely, the spin magnetic interaction induces a magnetization to the Pauli equation[16]. For now, we shall consider the spin gyromagnetic ratio $\gamma_{\rm s}$ as a constant — such that the time derivative of the magnetization field is proportional to that of the spin density — postponing the use of Eq. (2.39) until necessary for our balance equations.

The electromagnetic fields and their polarization and displacement counterparts give rise to the Maxwell stress tensor $T^{\alpha\beta}$, which is defined as:

$$T^{\alpha\beta} = D^\alpha E^\beta + B^\alpha H^\beta + \frac{v^\alpha \epsilon^{\beta\gamma\delta}}{c} (Z_\gamma B_\delta - M_\gamma E_\delta)$$
$$- \left(\frac{1}{2} E_\gamma E^\gamma + \frac{1}{2} B_\gamma B^\gamma - M_\gamma B^\gamma \right) \delta^{\alpha\beta}, \qquad (2.40)$$

where we define the diagonal terms as the electromagnetic field energy density $\rho\psi$:

$$\rho\psi = \frac{1}{2} \left(E^\alpha E_\alpha + B^\alpha B_\alpha \right) - M^\alpha B_\alpha, \qquad (2.41)$$

which is added to the total energy budget ρe of Eq. (2.11), such that the total energy budget becomes:

$$e = \frac{1}{2} v_\alpha v^\alpha + u_r + u + \psi, \qquad (2.42)$$

The Maxwell stress tensor itself acts as a *radiative* pressure term, and therefore terms containing the *mechanical* pressure tensor term $P^{\alpha\beta}$ should be corrected to account for the presence of the radiative pressure. The first affected balance equation is that of the conservation of momentum of Eq. (2.5), which is therefore no longer valid, instead being replaced by the following balance equation:

$$\frac{\partial}{\partial t} (\rho v^\alpha) = -\frac{\partial}{\partial x_\beta} \left(P^{\alpha\beta} - T^{\alpha\beta} + \rho v^\alpha v^\beta \right)$$
$$= -\frac{\partial}{\partial x_\beta} \left(P^{\alpha\beta} + \rho v^\alpha v^\beta \right) + F^\alpha, \qquad (2.43)$$

where the force term F^α is defined as:

$$F^\alpha = \frac{\partial}{\partial x_\beta} T^{\alpha\beta} = F_{\mathrm{L}}^\alpha + F_{\mathrm{R}}^\alpha, \qquad (2.44)$$

where

$$F_{\mathrm{L}}^\alpha = \rho z \left(E^\alpha + \frac{1}{c} \epsilon^{\alpha\beta\gamma} v_\beta B_\gamma \right) \qquad (2.45)$$

is the Lorentz force, and

$$F_{\mathrm{R}}^\alpha = Z_\beta \frac{\partial E^\alpha}{\partial x^\beta} + M_\beta \frac{\partial B^\alpha}{\partial x^\beta}$$
$$+ \frac{\epsilon^{\alpha\beta\gamma}}{c} \left[\frac{\partial}{\partial t} (Z_\beta B_\gamma - M_\beta E_\gamma) + \frac{\partial}{\partial x^\delta} (Z_\beta B_\gamma v_\delta - M_\beta E_\gamma v_\delta) \right] \quad (2.46)$$

are the remaining force terms. In the non-polarized case, Eq. (2.44) yields the Lorentz force terms only. As one might imagine, the force term of Eq.

(2.44) profoundly alters the balance equation for the kinetic term in Eq. (2.14) which now becomes:

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2}v_\alpha v^\alpha\right) = -\frac{\partial}{\partial x_\alpha}\left(P^{\alpha\beta}v_\beta + \frac{\rho}{2}v^2 v^\alpha\right) + P^{\alpha\beta}\frac{\partial v_\alpha}{\partial x_\beta} + v_\alpha F^\alpha. \qquad (2.47)$$

The second balance equation affected by the addition of radiative pressure is Eq. (2.10), the spin balance equation. Analogous to the anti-symmetric pressure term, the additional term in the balance equation for the spin is proportional to the anti-symmetric component of the Maxwell stress tensor T_α^{a}, defined as

$$T_\alpha^{\text{a}} = \frac{\epsilon_{\alpha\beta\gamma}}{2}T^{\beta\gamma} = \frac{\epsilon_{\alpha\beta\gamma}}{2}\left(D^\beta E^\gamma + B^\beta H^\gamma\right). \qquad (2.48)$$

With the addition of the anti-symmetric part of the Maxwell stress tensor as an additional pressure-like term, Eq. (2.10) becomes:

$$\frac{\partial}{\partial t}\left(\rho S_\alpha\right) = -2\left(P_\alpha^{\text{a}} - T_\alpha^{\text{a}}\right) - \frac{\partial}{\partial x^\beta}\left(j_{\text{s},\alpha\beta} + \rho S_\alpha v_\beta\right). \qquad (2.49)$$

Using Eq. (2.49) instead of Eq. (2.10) necessarily alters the rotational energy balance equation:

$$\frac{\partial}{\partial t}\left(\rho u_r\right) = -\frac{2}{\chi_{\text{s}}}S^\alpha\left(P_\alpha^{\text{a}} - T_\alpha^{\text{a}}\right) - \frac{\partial}{\partial x_\alpha}\left(\frac{S_\beta}{\chi_{\text{s}}}j_{\text{s}}^{\alpha\beta} + \rho u_r v^\alpha\right) + \frac{j_{\text{s},\alpha\beta}}{\chi_{\text{s}}}\frac{\partial S^\alpha}{\partial x^\beta}. \qquad (2.50)$$

For the electromagnetic energy density balance equation, the partial time derivative (or Poynting equation) of Eq. (2.41) is readily calculated using Eqs. (2.35) and (2.36), and Eq. (2.32):

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho\psi) &= E_\alpha\frac{\partial E^\alpha}{\partial t} + (B_\alpha - M_\alpha)\frac{\partial B^\alpha}{\partial t} - B_\alpha\frac{\partial M^\alpha}{\partial t} \\
&= c\epsilon^{\alpha\beta\gamma}E_\alpha\frac{\partial H_\beta}{\partial x_\gamma} - c\epsilon^{\alpha\beta\gamma}(-H_\alpha)\frac{\partial E_\beta}{\partial x_\gamma} \\
&\quad - E_\alpha\frac{\partial Z^\alpha}{\partial t} - B_\alpha\frac{\partial M^\alpha}{\partial t} - E_\alpha I^\alpha \\
&= c\epsilon^{\alpha\beta\gamma}\frac{\partial}{\partial x_\alpha}(E_\beta H_\gamma) - E_\alpha\left(\frac{\partial Z^\alpha}{\partial t} + I^\alpha\right) - B_\alpha\frac{\partial M^\alpha}{\partial t}. \qquad (2.51)
\end{aligned}$$

As anticipated earlier, the use of Eq. (2.38) allows us rewrite the term $-B_\alpha\partial_t M^\alpha$ after using Eqs. (2.38) and (2.49) as:

$$\begin{aligned}
-B_\alpha\frac{\partial M^\alpha}{\partial t} &= \gamma_{\text{m}}B^\alpha\frac{\partial}{\partial t}\left(\rho S_\alpha\right) \\
&= 2\gamma_{\text{m}}B^\alpha\left(P_\alpha^{\text{a}} - T_\alpha^{\text{a}}\right) + \gamma_{\text{m}}\frac{\partial}{\partial x_\alpha}\left[B_\beta\left(j_{\text{s}}^{\alpha\beta} + \rho S^\beta v^\alpha\right)\right] \\
&\quad - \gamma_{\text{m}}\left(j_{\text{s},\alpha\beta} + \rho S_\alpha v_\beta\right)\frac{\partial B^\alpha}{\partial x^\beta}, \qquad (2.52)
\end{aligned}$$

which lets us rewrite Eq. (2.51) as:

$$\frac{\partial}{\partial t}\left(\rho\psi\right) = 2\gamma_{\mathrm{m}}B^{\alpha}\left(P_{\alpha}^{\mathrm{a}} - T_{\alpha}^{\mathrm{a}}\right) + \gamma_{\mathrm{m}}\frac{\partial}{\partial x_{\alpha}}\left[B_{\beta}\left(j_{\mathrm{s}}^{\alpha\beta} + \rho S^{\beta}v^{\alpha}\right)\right]$$

$$- \gamma_{\mathrm{m}}\left(j_{\mathrm{s},\alpha\beta} + \rho S_{\alpha}v_{\beta}\right)\frac{\partial B^{\alpha}}{\partial x^{\beta}} + c\epsilon^{\alpha\beta\gamma}\frac{\partial}{\partial x_{\alpha}}\left(E_{\beta}H_{\gamma}\right)$$

$$- E_{\alpha}\left(\frac{\partial Z^{\alpha}}{\partial t} + I^{\alpha}\right). \tag{2.53}$$

Analogous to the previous Section, the balance equation for the internal energy is obtained by means of using Eq. (2.42); subtracting Eqs. (2.47), (2.50), and (2.53) from Eq. (2.4) yields:

$$\frac{\partial}{\partial t}(\rho u) = \frac{\partial}{\partial t}(\rho e) - \frac{\partial}{\partial t}\left(\frac{\rho}{2}v^{2}\right) - \frac{\partial}{\partial t}(\rho u_{r}) - \frac{\partial}{\partial t}\left(-\rho\psi\right)$$

$$= -\frac{\partial}{\partial x_{\alpha}}(j_{\mathrm{e}}^{\alpha} + \rho e v^{\alpha}) + \frac{\partial}{\partial x_{\alpha}}\left(P^{\alpha\beta}v_{\beta} + \frac{\rho}{2}v^{2}v^{\alpha}\right) - P^{\alpha\beta}\frac{\partial v_{\alpha}}{\partial x_{\beta}}$$

$$- v_{\alpha}F^{\alpha} - \frac{j_{\mathrm{s},\alpha\beta}}{\chi_{\mathrm{s}}}\frac{\partial S^{\alpha}}{\partial x^{\beta}} + 2\left(\frac{S^{\alpha}}{\chi_{\mathrm{s}}} - \gamma_{\mathrm{m}}B^{\alpha}\right)\left(P_{\alpha}^{\mathrm{a}} - T_{\alpha}^{\mathrm{a}}\right)$$

$$+ \frac{\partial}{\partial x_{\alpha}}\left(\frac{S_{\beta}}{\chi_{\mathrm{s}}}j_{\mathrm{s}}^{\alpha\beta} + \rho u_{r}v^{\alpha}\right) - \gamma_{\mathrm{m}}\frac{\partial}{\partial x_{\alpha}}\left[B_{\beta}\left(j_{\mathrm{s}}^{\alpha\beta} + \rho S_{\mathrm{s}}^{\beta}v^{\alpha}\right)\right]$$

$$+ \gamma_{\mathrm{m}}\left(j_{\mathrm{s},\alpha\beta} + \rho S_{\alpha}v_{\beta}\right)\frac{\partial B^{\alpha}}{\partial x^{\beta}} - c\epsilon^{\alpha\beta\gamma}\frac{\partial}{\partial x_{\alpha}}\left(E_{\beta}H_{\gamma}\right)$$

$$+ E_{\alpha}\left(\frac{\partial Z^{\alpha}}{\partial t} + I^{\alpha}\right). \tag{2.54}$$

Defining the heat flow j_{q}^{α} in this case as:

$$j_{\mathrm{q}}^{\alpha} = j_{\mathrm{e}}^{\alpha} - P^{\alpha\beta}v_{\beta} - \left(\frac{S_{\beta}}{\chi_{\mathrm{s}}} - \gamma_{\mathrm{m}}B_{\beta}\right)j_{\mathrm{s}}^{\alpha\beta} + c\epsilon^{\alpha\beta\gamma}E_{\beta}H_{\gamma}. \tag{2.55}$$

permits us to rewrite Eq. (2.54) as:

$$\frac{\partial}{\partial t}\left(\rho u\right) = -\frac{\partial}{\partial x_{\alpha}}\left[j_{\mathrm{q}}^{\alpha} + \rho\left(u - \psi\right)v^{\alpha}\right] - P^{\alpha\beta}\frac{\partial v_{\alpha}}{\partial x_{\beta}} - v_{\alpha}F^{\alpha}$$

$$- \left[j_{\mathrm{s},\alpha\beta}\frac{\partial}{\partial x^{\beta}} - 2\left(P_{\alpha}^{\mathrm{a}} - T_{\alpha}^{\mathrm{a}}\right)\right]\left(\frac{S^{\alpha}}{\chi_{\mathrm{s}}} - \gamma_{\mathrm{m}}B^{\alpha}\right)$$

$$+ E_{\alpha}\left(\frac{\partial Z^{\alpha}}{\partial t} + I^{\alpha}\right) + \gamma_{\mathrm{m}}B_{\alpha}\frac{\partial}{\partial x_{\beta}}\left(\rho S^{\alpha}v^{\beta}\right). \tag{2.56}$$

Performing the same tensorial decompositions of Eqs. (2.18) and (2.19) done in Subsection 2.1, and using Eq. (2.3), the internal energy balance

Equation (2.56) is rewritten as:

$$\frac{\partial}{\partial t}(\rho u) + p\frac{\partial}{\partial t}(\rho \nu) = -\frac{\partial}{\partial x_\alpha}\left[j_q^\alpha + \rho\left(u + p\nu - \psi\right)v^\alpha\right] - v_\alpha F^\alpha$$
$$-\Pi\frac{\partial v^\alpha}{\partial x_\alpha} - \left(\Pi^{\alpha\beta}\right)^{s,0}\left(\frac{\partial v_\alpha}{\partial x_\beta}\right)^{s,0}$$
$$-\Pi_\alpha^a\left[\epsilon^{\alpha\beta\gamma}\frac{\partial v_\beta}{\partial x_\gamma} - \left(\frac{S^\alpha}{\chi_s} - \gamma_m B^\alpha\right)\right]$$
$$-\left(j_{s,\alpha\beta}\frac{\partial}{\partial x^\beta} + 2T_\alpha^a\right)\left(\frac{S^\alpha}{\chi_s} - \gamma_m B^\alpha\right)$$
$$+E_\alpha\left(\frac{\partial Z^\alpha}{\partial t} + I^\alpha\right) + \gamma_m B_\alpha\frac{\partial}{\partial x_\beta}\left(\rho S^\alpha v^\beta\right). \quad (2.57)$$

2.3 Multi-component fluid entropy production with electromagnetism

If we would consider our fluid to be monocomponent, the Gibbs energy balance Equation (2.57), used in the Gibbs relation Equation (2.22) yields us the entropy balance equation and subsequently the entropy production, which in this case would contain much more terms than our previous case in Section 2.1. This assumption, however, implies that our fluid consists purely of particles of the same mass, and of the same electric charge, thus making it unusable for our purposes.

Fortunately, the only required modifications of Eq. (2.57) are rewriting terms containing the charge z as terms with particle-dependent charges z_k for fluid component k, and then changing the other quantities in terms containing z_k accordingly[4]. The first, rather obvious, change is using the component-dependent form of the Lorentz force of Eq. (2.45):

$$F_L^\alpha = \sum_k \rho_k z_k\left(E^\alpha + \frac{1}{c}\epsilon^{\alpha\beta\gamma}v_{k,\beta}B_\gamma\right). \quad (2.58)$$

The second change, a multi-component form of the electric current density of Eq. (2.37), is equally trivial as well:

$$I^\alpha = v^\alpha\sum_k \rho_k z_k + j_c^\alpha. \quad (2.59)$$

The third and most profound change involves the spin gyromagnetic

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[4]That only terms containing the average charge z are rewritten to accommodate particle charges z_k is a consequence of a deliberate choice to regard the 'chargeless' fluid of Section 2.1 as being a component-averaged fluid, *i.e.* the quantities present in the fluid of Section 2.1, including spin per unit mass S^α, are averages among the components.

ratio $\gamma_{\rm s}$ as introduced in Eq. (2.39), which at first is rewritten as:

$$\gamma_{\rm s} = \sum_k \frac{g_k z_k}{2m_k}, \tag{2.60}$$

where g_k and m_k are the gyromagnetic factor and mass, respectively, for particles of fluid component k. While Eq. (2.60) is valid for any fluid with multiple charged particle components, we are able to further rewrite this in a more sophisticated and aesthetically pleasing manner: since we consider our fluid to consist of nuclei and (free) electrons, we only need to consider the electron contribution to Eq. (2.60). This is due to the heavy nuclei masses—nuclei masses are *at least* three order of magnitudes higher than the electron mass, even more so for *mercury nuclei*—which render the nuclei contributions to the spin magnetic moment energy to become relatively small compared to that of the electrons. In short, we simply replace Eq. (2.60) with

$$\gamma_{\rm s} = \gamma_{\rm e}, \tag{2.61}$$

where $\gamma_{\rm e} = g_{\rm e} z_{\rm e}/2m_{\rm e}$ is the electron gyromagnetic ratio.

With these changes in mind, and ignoring the orbital contribution $\gamma_{\rm o}$ to the gyromagnetic ratio $\gamma_{\rm m}$, the multi-component form of the internal energy balance equation Eq. (2.57) is written as:

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho u) + p\frac{\partial}{\partial t}(\rho\nu) = {} & -\frac{\partial}{\partial x_\alpha}\left[j_{\rm q}^\alpha + \rho\left(u + p\nu + \psi\right)v^\alpha\right] \\
& - v_\alpha F_{\rm R}^\alpha - \epsilon^{\alpha\beta\gamma}\frac{v_\alpha}{c}\sum_k \rho_k z_k v_{k,\beta} B_\gamma \\
& - \Pi\frac{\partial v^\alpha}{\partial x_\alpha} - \left(\Pi^{\alpha\beta}\right)^{\rm s,0}\left(\frac{\partial v_\alpha}{\partial x_\beta}\right)^{\rm s,0} - \Pi_\alpha^{\rm a}\left[\epsilon^{\alpha\beta\gamma}\frac{\partial v_\beta}{\partial x_\gamma} - \left(\frac{S^\alpha}{\chi_{\rm s}} - \gamma_{\rm e} B^\alpha\right)\right] \\
& - \left(j_{{\rm s},\alpha\beta}\frac{\partial}{\partial x^\beta} + 2T_\alpha^{\rm a}\right)\left(\frac{S^\alpha}{\chi_{\rm s}} - \gamma_{\rm e} B^\alpha\right) \\
& + E_\alpha\left(\frac{\partial Z^\alpha}{\partial t} + j_{\rm c}^\alpha\right) + \gamma_{\rm e} B_\alpha\frac{\partial}{\partial x_\beta}\left(\rho S^\alpha v^\beta\right).
\end{aligned} \tag{2.62}$$

With our internal energy balance equation rewritten for a multi-component fluid, we shall use the multi-component form of the Gibbs equation which includes the contribution of the (electro-)chemical potential energy:[5]

$$T\frac{{\rm d}\sigma}{{\rm d}t} = \frac{{\rm d}u}{{\rm d}t} + p\frac{{\rm d}\nu}{{\rm d}t} - \sum_k \mu_k\frac{{\rm d}c_k}{{\rm d}t}, \tag{2.63}$$

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[5]The derivation and motivation of using Eq. (2.63) is based on the absence of equilibrium values for E_α and B_α; for non-negligible equilibrium values, one has to substitute E_α with $E_\alpha - E_{{\rm eq},\alpha}$ and similar for B_α in the relevant Equations in this Section, *cf.* De Groot and Mazur.[13]

where μ_k is the chemical potential for component k, and c_k the component's mass fraction. Similar to our previous treatment in Section 2.1, we rewrite Eq. (2.63) using Eq. (2.1) as a local time derivative of the entropy density:

$$T\frac{\partial}{\partial t}(\rho\sigma) = \frac{\partial}{\partial t}(\rho u) + p\frac{\partial}{\partial t}(\rho\nu) - \sum_k \mu_k \frac{\partial}{\partial t}(\rho c_k)$$
$$+ \frac{\partial}{\partial x_\alpha}\left[\rho\left(u + p\nu - T\sigma - \sum_k \mu_k c_k\right)v^\alpha\right]. \qquad (2.64)$$

Unsurprisingly, using Eq. (2.64) into Eq. (2.62) yields us the entropy balance equation of Eq. (2.24), but now we have for the entropy flow j_σ^α:

$$j_\sigma^\alpha = \frac{1}{T}\left[j_q^\alpha - \rho\left(T\sigma + \sum_k \mu_k c_k - \psi\right)v^\alpha\right], \qquad (2.65)$$

while the entropy production ζ becomes:

$$\zeta = -\frac{1}{T^2}\left[j_q^\alpha - \rho\left(T\sigma + \sum_k \mu_k c_k - \psi\right)v^\alpha\right]\frac{\partial T}{\partial x_\alpha}$$
$$- \frac{\Pi}{T}\frac{\partial v^\alpha}{\partial x_\alpha} - \frac{\Pi_{\alpha\beta}^{s,0}}{T}\left(\frac{\partial v^\alpha}{\partial x^\beta}\right)^{s,0} - \frac{\Pi_\alpha^a}{T}\left[\epsilon^{\alpha\beta\gamma}\frac{\partial v_\beta}{\partial x_\gamma} - \left(\frac{S^\alpha}{\chi_s} - \gamma_e B^\alpha\right)\right]$$
$$- \frac{1}{T}\left(j_{s,\alpha\beta}\frac{\partial}{\partial x^\beta} + 2T_\alpha^a\right)\left(\frac{S^\alpha}{\chi_s} - \gamma_e B^\alpha\right)$$
$$+ \frac{E_\alpha}{T}\left(\frac{\partial Z^\alpha}{\partial t} + j_c^\alpha\right) + \gamma_e\frac{B_\alpha}{T}\frac{\partial}{\partial x_\beta}\left(\rho S^\alpha v^\beta\right). \qquad (2.66)$$

Since we assume the system to be in mechanical equilibrium, the force term $v_\alpha F_R^\alpha$ is omitted from the entropy production; similarly, the term $\epsilon^{\alpha\beta\gamma}v_\alpha(cT)^{-1}\sum_k \rho_k z_k v_{k,\beta} B_\gamma$ is also omitted for this reason, with the additional argument of the Prigogine theorem[17] being applied such that one is free to choose $v_{k,\beta} = 0$.[6]

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[6] *cf.* De Groot and Mazur, Sections V.2 and XIII.4[13]

The resulting phenomenological equations are then:

$$j_{\mathrm{q}}^\alpha = -\lambda \frac{\partial T}{\partial x^\alpha} + \rho \left(T\sigma + \sum_k \mu_k c_k - \psi \right) v^\alpha; \qquad (2.67)$$

$$\Pi = -\eta_v \frac{\partial v^\alpha}{\partial x_\alpha}; \qquad (2.68)$$

$$\Pi_{\alpha\beta}^{\mathrm{s},0} = -2\eta \left(\frac{\partial v_\alpha}{\partial x_\beta} \right)^{\mathrm{s},0}; \qquad (2.69)$$

$$\Pi_\alpha^{\mathrm{a}} = -\eta_r \left[\epsilon_{\alpha\beta\gamma} \frac{\partial v^\beta}{\partial x^\gamma} - \left(\frac{S_\alpha}{\chi_{\mathrm{s}}} - \gamma_{\mathrm{e}} B_\alpha \right) \right]; \qquad (2.70)$$

$$j_{\mathrm{s},\alpha\beta} = -\left(D_{\mathrm{s}} \frac{\partial}{\partial x_\beta} + 2T_\beta^{\mathrm{a}} \right) \left(\frac{S_\alpha}{\chi_{\mathrm{s}}} - \gamma_{\mathrm{e}} B_\alpha \right); \qquad (2.71)$$

$$j_{\mathrm{c}}^\alpha + \frac{\partial Z^\alpha}{\partial t} = \sigma_{\mathrm{e}} E^\alpha; \qquad (2.72)$$

$$\gamma_{\mathrm{e}} \frac{\partial}{\partial x^\beta} \left(\rho S_\alpha v_\beta \right) = \mu_{\mathrm{m}} B_\alpha. \qquad (2.73)$$

where—as in the conclusion of Section 2.1—λ is the heat conductivity, η_v the volume viscosity, η the shear viscosity, η_r the spin ('rotational') viscosity, and D_{s} a spin diffusion constant. An additional phenomenological equation, (2.72), results in Ohm's law of electrical resistance, where σ_{e} is the electrical conductivity, whereas another additional equation, (2.73), yields an analogous equation for the magnetic field, where μ_{m} is the magnetic permeability. Compared to their counterparts (2.27)–(2.31), Eqs. (2.67) to (2.69) remain the same, while Eqs. (2.70) and (2.71) are redefined to accommodate the presence of electromagnetic fields; setting $T_\alpha^{\mathrm{a}} = E_\alpha = B_\alpha = 0$ in (2.70) and (2.71) yields their non-electromagnetic counterparts, Eqs. (2.30) and (2.31).

Using the phenomenological equations (2.67)–(2.72), the conservation laws used for our system is rewritten in terms of known, measurable quantities. To begin with, let us rewrite the current density of Eq. (2.37) using Eq. (2.72):

$$I^\alpha = \rho z v^\alpha + \sigma_{\mathrm{e}} E^\alpha - \frac{\partial Z^\alpha}{\partial t}. \qquad (2.74)$$

With the phenomenological pressure equations (2.68)–(2.70), the momentum balance equation (2.43) is rewritten as a Navier-Stokes equation:

$$\begin{aligned}
\frac{\partial}{\partial t} \left(\rho v_\alpha \right) &= -\frac{\partial}{\partial x^\beta} \left[(p + \Pi) \delta_{\alpha\beta} + (\Pi_{\alpha\beta})^{\mathrm{s},0} + 2\epsilon_{\alpha\beta\gamma} \Pi^{\mathrm{a},\gamma} + \rho v_\alpha v_\beta \right] + F_\alpha \\
&= -\frac{\partial p}{\partial x_\alpha} + \eta \frac{\partial^2 v_\alpha}{\partial x^\beta \partial x_\beta} + \left(\frac{1}{3}\eta + \eta_v \right) \frac{\partial^2 v^\beta}{\partial x_\alpha \partial x_\beta} \\
&\quad - \frac{\partial}{\partial x^\beta} \left[2\eta_r \epsilon_{\alpha\beta\gamma} \left(\frac{S^\gamma}{\chi_{\mathrm{s}}} - \gamma_{\mathrm{e}} B^\gamma \right) + \rho v_\alpha v_\beta \right] + F_\alpha. \qquad (2.75)
\end{aligned}$$

Rewriting the spin balance equation (2.49) using Eqs. (2.70) and (2.71) then yields:

$$\frac{\partial}{\partial t}(\rho S_\alpha) = 2\left\{\eta_r\left[\epsilon_{\alpha\beta\gamma}\frac{\partial v^\beta}{\partial x^\gamma} - \left(\frac{S_\alpha}{\chi_s} - \gamma_e B_\alpha\right)\right] + T_\alpha^a\right\}$$
$$+ \frac{\partial}{\partial x^\beta}\left[\left(D_s\frac{\partial}{\partial x_\beta} + 2T_\beta^a\right)\left(\frac{S_\alpha}{\chi_s} - \gamma_e B_\alpha\right) - \rho S_\alpha v_\beta\right]. \quad (2.76)$$

Finally, the conservation law of energy, Eq. (2.4), is rewritten using Eqs. (2.55), (2.67)–(2.71) as:

$$\frac{\partial}{\partial t}(\rho e) = -\frac{\partial}{\partial x_\alpha}\left(j_q^\alpha + P^{\alpha\beta}v_\beta + \left(\frac{S_\beta}{\chi_s} - \gamma_e B_\beta\right)j_s^{\alpha\beta} - c\epsilon^{\alpha\beta\gamma}E_\beta H_\gamma + \rho e v^\alpha\right)$$
$$= \frac{\partial}{\partial x_\alpha}\left\{\lambda\frac{\partial T}{\partial x^\alpha} + \left[\eta_v\frac{\partial v^\alpha}{\partial x^\beta} + 2\eta\left(\frac{\partial v^\alpha}{\partial x^\beta}\right)^{s,0} - 2\eta_r\epsilon^{\alpha\beta\gamma}\left(\frac{S_\gamma}{\chi_s} - \gamma_e B_\gamma\right)\right]v_\beta\right.$$
$$+ \left(\frac{S_\beta}{\chi_s} - \gamma_e B_\beta\right)\left(D_s\frac{\partial}{\partial x^\beta} + 2T^{a,\beta}\right)\left(\frac{S_\alpha}{\chi_s} - \gamma_e B_\alpha\right)$$
$$\left.+ c\epsilon^{\alpha\beta\gamma}E_\beta H_\gamma - \rho\left(e + \frac{p}{\rho} + T\sigma + \sum_k \mu_k c_k - \psi\right)v^\alpha\right\}. \quad (2.77)$$

Chapter 3

Hydrodynamic modes

The balance equations and conservation laws obtained in Chapter 2—specifically Eqs. (2.2), (2.75), (2.76), and (2.77)—form a coherent, phenomenological description of the spin fluid, its properties, and its behaviour under both internal and external forces. Before we put these equations into practice, we shall regard the hydrodynamic modes of such spin fluids in this Chapter. In Section 3.1 we linearize the equations as average values with small perturbations, such that we are able to extract hydrodynamic modes from these linearizations in Section 3.2.

3.1   Linearizing the balance equations

In order to obtain the hydrodynamic modes, the aforementioned equations ought to be linearized first, *i.e.* the hydrodynamic variables are to be treated as the sum of an averaged value and of a small, linear fluctuation. In order to establish the notation used in this Chapter and onwards, let a be (again) a quantity to be linearized; then, we write $\bar{a}$ as the average value of a, and $\widetilde{a}(x_\alpha, t)$ as its fluctuation, such that

$$a(x_\alpha, t) = \bar{a} + \widetilde{a}(x_\alpha, t), \tag{3.1}$$

and subsequently

$$\frac{\partial \bar{a}}{\partial t} = \frac{\partial \bar{a}}{\partial x_\alpha} = 0; \qquad \mathcal{O}\left[(\widetilde{a}(x_\alpha, t))^2 \right] = \widetilde{a}(x_\alpha, t)\widetilde{b}(x_\alpha, t) = 0,$$

where $\widetilde{b}(x_\alpha, t)$ is the fluctuation of a different quantity, in this case b.

Now that the linearization formalism to be used is set, let us turn our attention to the present hydrodynamic variables: these are the mass density ρ, the velocity v_α, the spin S_α and the temperature T. Important to note is that the system's energy e is not a fundamental quantity; rather, it is dependent on the four variables mentioned. Furthermore, we assume the

average velocity and spin of the fluid to be zero, *i.e.*

$$\overline{v}_\alpha = \overline{S}_\alpha = 0_\alpha, \tag{3.2}$$

since these quantities are balanced. In general, Eq. (3.2) combined with the linear nature of the fluctuations yield

$$v^2 = S^2 = v_\alpha S_\beta = 0. \tag{3.3}$$

With these considerations in mind, the fundamental hydrodynamic variables of our system are linearized as:

$$\begin{aligned}
\rho(x_\alpha, t) &= \overline{\rho} + \widetilde{\rho}(x_\alpha, t); & v_\alpha(x_\beta, t) &= \widetilde{v}_\alpha(x_\beta, t); \\
S_\alpha(x_\beta, t) &= \widetilde{S}_\alpha(x_\beta, t); & T(x_\alpha, t) &= \overline{T} + \widetilde{T}(x_\alpha, t).
\end{aligned} \tag{3.4}$$

From here on, we shall drop the explicit spatiotemporal dependence of the fluctuations for the sake of readability.

The first equation to undergo the linearization procedure is the continuity equation, as cited in Eq. (2.2):

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x^\alpha}(\rho v_\alpha), \tag{3.5}$$

which after using the linearizations of Eq. (3.9) becomes:

$$\frac{\partial \widetilde{\rho}}{\partial t} = -\overline{\rho}\frac{\partial \widetilde{v}^\alpha}{\partial x_\alpha}. \tag{3.6}$$

The second equation to be linearized is the momentum balance equation (2.75), and for this Equation two points have to be addressed before any meaningful linearization might occur. Firstly, the advective term $-\partial/\partial x^\beta(\rho v_\alpha v_\beta)$ of the right hand side vanishes due to Eq. (3.3). Secondly, the pressure p is not a fundamental hydrodynamic variable in itself, since it effectively depends on the mass density ρ and the temperature T. The pressure gradient term of Eq. (2.75) therefore needs to be rewritten using the ρ and T dependencies and the chain rule:

$$\frac{\partial \widetilde{p}}{\partial x_\alpha} = K\left(\frac{\partial \widetilde{\rho}}{\partial x_\alpha} + \alpha\frac{\partial \widetilde{T}}{\partial x_\alpha}\right), \tag{3.7}$$

where $K = \partial\widetilde{p}/\partial\rho$ is the bulk modulus density per unit mass[18], and $\alpha = \partial\widetilde{\rho}/\partial T$ the thermal expansion coefficient of the spin fluid. Using Eq. (3.7), the momentum balance equation (2.75) is linearized as:

$$\begin{aligned}
\overline{\rho}\frac{\partial \widetilde{v}_\alpha}{\partial t} = {}&-K\left(\frac{\partial \widetilde{\rho}}{\partial x_\alpha} + \alpha\frac{\partial \widetilde{T}}{\partial x_\alpha}\right) + \eta\frac{\partial^2 \widetilde{v}_\alpha}{\partial x^\beta \partial x_\beta} + \left(\frac{1}{3}\eta + \eta_v\right)\frac{\partial^2 \widetilde{v}^\beta}{\partial x_\alpha \partial x_\beta} \\
&- 2\eta_r\epsilon_{\alpha\beta\gamma}\frac{\partial}{\partial x^\beta}\left(\frac{\widetilde{S}^\gamma}{\chi_s} - \gamma_e B^\gamma\right) + F_\alpha,
\end{aligned} \tag{3.8}$$

Compared to the momentum balance equation (2.75), the third equation to be linearized, Eq. (2.76), is relatively simple to linearize: Eq. (3.3) lets the advective term $-\partial/\partial x^\beta(\rho S_\alpha v_\beta)$ vanish, such that linearization yields:

$$\overline{\rho}\frac{\partial \widetilde{S}_\alpha}{\partial t} = 2\eta_r\left[\epsilon_{\alpha\beta\gamma}\frac{\partial \widetilde{v}^\beta}{\partial x^\gamma} - \left(\frac{\widetilde{S}_\alpha}{\chi_{\rm s}} - \gamma B_\alpha\right)\right] + 2T_\alpha^{\rm a}$$
$$+ \frac{\partial}{\partial x^\beta}\left[\left(D_{\rm s}\frac{\partial}{\partial x_\beta} + 2T_\beta^{\rm a}\right)\left(\frac{\widetilde{S}_\alpha}{\chi_{\rm s}} - \gamma_{\rm e}B_\alpha\right)\right]. \qquad (3.9)$$

As one would expect, the fourth and last equation in our linearization procedure, the energy balance equation (2.77), is the most laborious one to linearize. This is due to the aforementioned fact that energy per unit mass, e, is not a proper fundamental hydrodynamic variable; rather, it is dependent on quantities mentioned in Eq. (3.9), such that one would naively be tempted to write $e(v_\alpha, S_\alpha, T)$. However, kinetic energy is proportional to the square of velocity, and Eq. (2.12) tells us the same with rotational energy and the square of the fluid spin. Eq. (3.3) lets us ignore the dependence on v^2 and S^2 of a linearized energy e, such that the left hand term of Eq. (2.77) is easily rewritten using Eq. (3.5), the chain rule and temperature as the variable:

$$\frac{\partial}{\partial t}(\rho e) = \overline{e}\frac{\partial \widetilde{\rho}}{\partial t} + \overline{\rho}\frac{\partial \widetilde{e}}{\partial t} = \overline{\rho}\frac{\partial T}{\partial t}\frac{\partial \widetilde{e}}{\partial T} - \overline{\rho}\overline{e}\frac{\partial \widetilde{v}^\alpha}{\partial x_\alpha} = \overline{\rho}c_V\frac{\partial \widetilde{T}}{\partial t} - \overline{\rho}\overline{e}\frac{\partial \widetilde{v}^\alpha}{\partial x_\alpha}, \qquad (3.10)$$

where $c_V = \partial\widetilde{e}/\partial T$ is the fluid's specific heat density. Note that the final (advective) term on the right hand side of Eq. (3.10) cancels a similar advective term on the right hand side of Eq. (2.77), such that the energy balance equation is completely rewritten as a *temperature balance equation*. After performing the necessary linearizations, the temperature balance equation thus becomes:

$$\overline{\rho}\frac{\partial \widetilde{T}}{\partial t} = \frac{1}{c_V}\frac{\partial}{\partial x_\alpha}\left\{\lambda\frac{\partial \widetilde{T}}{\partial x^\alpha} + 2\eta_r\gamma_{\rm e}\epsilon^{\alpha\beta\gamma}\widetilde{v}_\beta B_\gamma + \gamma_{\rm e}^2 B_\beta\left(D_{\rm s}\frac{\partial}{\partial x^\beta} + 2T^{{\rm a},\beta}\right)B^\alpha\right.$$
$$- \frac{\gamma_{\rm e}}{\chi_{\rm s}}\widetilde{S}_\beta\left(D_{\rm s}\frac{\partial}{\partial x^\beta} + 2T^{{\rm a},\beta}\right)B^\alpha - \frac{\gamma_{\rm e}}{\chi_{\rm s}}B_\beta\left(D_{\rm s}\frac{\partial}{\partial x^\beta} + 2T^{{\rm a},\beta}\right)\widetilde{S}^\alpha$$
$$\left.+ c\epsilon^{\alpha\beta\gamma}E_\beta H_\gamma - \overline{p}\widetilde{v}^\alpha - \left(\overline{T}\overline{\sigma} + \sum_k \mu_k c_k - \psi\right)\overline{\rho}\widetilde{v}^\alpha\right\}. \qquad (3.11)$$

3.2   Obtaining the hydrodynamic modes

Compared to their counterparts in Chapter 2, the linearized equations obtained in Section 3.1 possess the distinct advantages of being differential equations of the thermodynamical variables at work in the fluid system,

having no superfluous advective terms present, and depending only on the external electromagnetic forces next to the thermodynamic variables themselves. This invites us to consider plane waves for these variables in order to obtain the hydrodynamic modes of the system as dispersion relations of the set of equations. Since hydrodynamic modes form an internal property of the fluid irrespective of outside influences, the external electromagnetic force and field terms may be safely ignored in obtaining the dispersion relations, further simplifying the equations substantially.

Let $\widetilde{a}(x_\alpha, t)$ be the fluctuation of a given quantity a as defined in Eq. (3.1), and suppose it is a plane wave which is written as:

$$\widetilde{a}(x_\alpha, t) = a_0 \exp\left\{i\left(k^\alpha x_\alpha - \omega t\right)\right\}, \tag{3.12}$$

where a_0 is the plane wave amplitude, k^α the wavenumber, and ω the wave frequency. Applying the plane wave form of Eq. (3.12) to the fluctuations $\widetilde{\rho}, \widetilde{v}_\alpha, \widetilde{S}_\alpha, \widetilde{T}$ and ignoring external electromagnetic forces then yields for the linearized equations Eqs. (3.5), (3.8), (3.9), (3.11):

$$\omega\overline{\rho}\widetilde{\rho} = \overline{\rho}^2 k^\alpha \widetilde{v}_\alpha; \tag{3.13}$$

$$\omega\overline{\rho}\widetilde{v}_\alpha = K k_\alpha \widetilde{\rho} - i\eta k^2 \widetilde{v}_\alpha - i\left(\frac{1}{3}\eta + \eta_v\right) k_\alpha k_\beta \widetilde{v}^\beta$$
$$+ 2\frac{\eta_r}{\chi_\mathrm{s}}\epsilon_{\alpha\beta\gamma}k^\beta \widetilde{S}^\gamma + \alpha K k_\alpha \widetilde{T}; \tag{3.14}$$

$$\omega\overline{\rho}\widetilde{S}_\alpha = 2\eta_r \epsilon_{\alpha\beta\gamma}k^\beta \widetilde{v}^\gamma - \frac{i}{\chi_\mathrm{s}}\left(2\eta_r + D_\mathrm{s}k^2\right)\widetilde{S}_\alpha; \tag{3.15}$$

$$\omega\overline{\rho}\widetilde{T} = \frac{1}{c_V}\left[\overline{p}k^\alpha + \overline{\rho}k^\alpha\left(\overline{T}\overline{\sigma} + \mu_k c_k\right)\right]\widetilde{v}_\alpha - \frac{i\lambda}{c_V}k^2\widetilde{T}. \tag{3.16}$$

Eqs. (3.13)–(3.16) describe a set of linear equations which is written as:

$$\omega\overline{\rho}\xi = A\xi, \tag{3.17}$$

where $\xi = (\widetilde{\rho}, \widetilde{v}_x, \widetilde{v}_y, \widetilde{v}_z, \widetilde{S}_x, \widetilde{S}_y, \widetilde{S}_z, \widetilde{T})^\mathrm{T}$ is an eight-vector of the eight degrees of freedom for the quantities[1], and A an 8×8 matrix, whose eigenvalues describe the hydrodynamic dispersion relations $\omega\overline{\rho}$ of the system.

Note that the term proportional to $k_\alpha k^\beta \widetilde{v}_\beta$ would allow for 'cross-wavenumber' terms where $\alpha \neq \beta$ to exist, yet we practically shall ignore these contributions to the matrix A. This is primarily because we consider our fluid, mercury, to be incompressible, where $\partial_t \widetilde{\rho} = 0$ such that $\partial_\alpha \widetilde{v}^\alpha = 0$ according to Eq. (3.6), and therefore the velocity divergence term in Eq. (3.8) . Furthermore, we expect the motion in fluid not directed along the main flow to be the result of rotational viscosity coupling with the fluid spins, as evidenced by terms proportional to η_r. Finally, the main goal of this Section is to obtain dispersion relations for the fluid in order to establish the

---

[1]Each of the three Cartesian components of $\widetilde{v}_\alpha$ and $\widetilde{S}_\alpha$ form a degree of freedom.

properties sound waves and similar phenomena in the fluid might have—a task vastly made more difficult if we were to include the 'cross-wavenumber' terms, since it would inhibit calculating the necessary eigenvalues.[2]

The simplest case considered is when only the diagonal elements of A contribute to the dispersion relations. Since the mass density equation does not possess such a diagonal element, we obtain three dispersion relations for three distinct modes originating from different physical properties[3]:

$$\omega_{v_\alpha} = \frac{-i}{\overline{\rho}} \left(\frac{4}{3}\eta + \eta_v \right) k^2; \tag{3.18}$$

$$\omega_{S_\alpha} = \frac{-i}{\overline{\rho}\chi_{\mathrm{s}}} (2\eta_r + D_{\mathrm{s}}k^2); \tag{3.19}$$

$$\omega_T = \frac{-i\lambda}{\overline{\rho}c_V} k^2, \tag{3.20}$$

which are the dispersion relations for the diffusive momentum, diffusive spin, and diffusive heat modes respectively[19]. We identify the ratio $(\frac{4}{3}\eta + \eta_v)/\overline{\rho}$ of Eq. (3.18) as the kinematic version of the second coefficient of viscosity[19,20]; in an incompressible fluid, Eq (3.18) simplifies to

$$\omega_{v_\alpha} = -i\eta_k k^2, \tag{3.21}$$

where $\eta_k = \eta/\overline{\rho}$ is the kinematic viscosity[19,20]. The ratio $\lambda/\overline{\rho}c_V$ can be regarded as the thermal diffusion constant D_T. Furthermore, the diffusive spin relation (3.19) has a term independent of the wavenumber which originates from the spins' proper orbital angular momentum, indicating the fluid's rotational viscosity η_r provides a wavenumber-independent damping term to the spin's proper dispersion relations.

More interesting results obviously occur when one allows non-diagonal matrix elements to be considered. As a first, simple example, let the fluid's dispersion depend on mass density and velocity only; the set of plane wave equations (3.13)–(3.16) then simplifies to:

$$\omega\overline{\rho}\widetilde{\rho} = \overline{\rho}^2 k^\alpha \widetilde{v}_\alpha;$$
$$\omega\overline{\rho}\widetilde{v}_\alpha = Kk_\alpha\widetilde{\rho} - i\overline{\rho}\eta_k k^2 \widetilde{v}_\alpha,$$

which yields the following dispersion relations for diffusive momentum modes:

$$\omega_1 = -i\eta_k k^2; \tag{3.22}$$

$$\omega_{2,3} = \frac{-i}{2}\eta_k k^2 \pm \sqrt{Kk^2 - \frac{1}{4}\left(\eta_k k^2\right)^2}. \tag{3.23}$$

---

[2]We were not able to find any sensible eigenvalues for the simplest non-trivial example, *i.e.* the dispersion relations Eqs. (3.22)–(3.23).

[3]Here, we include compressible fluids for the sake of completeness.

Since mercury as an incompressible fluid possesses a large bulk modulus[4] and low kinematic viscosity in part due to its heavy density, expanding the square root in the right hand side of Eq. (3.23) for small η_k yields us:

$$\omega_{2,3} \simeq \frac{-i}{2}\eta_k k^2 \pm \sqrt{K}|k| \left(1 - \frac{1}{8K}\eta_k^2 k^2\right). \qquad (3.24)$$

Note that Eq. (3.22), equal to the diagonal-only result Eq. (3.18), is a special non-zero case of Eq. (3.23) (and Eq. (3.24)) if $K \to 0$, *i.e.* if the fluid is infinitely compressible. Alternatively, using $\eta_k \to 0$ in Eq. (3.23) for a non-viscous fluid results in a real solution $\omega = \pm\sqrt{K}|k|$, *i.e.* a sound wave. Dividing this result with the wavenumber vector length yields us the familiar result of the speed of sound waves in a fluid as the square root of the bulk modulus[18]:

$$v_s = \left|\frac{\omega}{k}\right| = \sqrt{K} \qquad (3.25)$$

From this we observe (and confirm) that both compressibility and viscosity influence the fluid's dispersion relations — and therefore the fluid's sound velocity — in the manner observed in classical fluid dynamics: compressibility (or its inverse, the bulk modulus) determines the speed of sound waves, while the fluid's viscosity — an imaginary term to the dispersion relations — regulates the damping of such sound waves.

Next we add the rotational viscosity term η_r to the system, effectively (partially) adding the fluid spin $\widetilde{S}_\alpha$ to the set of equations, which become:

$$\omega\overline{\rho}\widetilde{\rho} = \overline{\rho}^2 k^\alpha \widetilde{v}_\alpha;$$

$$\omega\overline{\rho}\widetilde{v}_\alpha = Kk_\alpha\widetilde{\rho} - i\overline{\rho}\eta_k k^2 \widetilde{v}_\alpha + \frac{2\eta_r}{\chi_s}\epsilon_{\alpha\beta\gamma}k^\beta \widetilde{S}^\gamma;$$

$$\omega\overline{\rho}\widetilde{S}_\alpha = 2\eta_r\epsilon_{\alpha\beta\gamma}k^\beta\widetilde{v}^\gamma - \frac{i}{\chi_s}\left(2\eta_r + D_s k^2\right)\widetilde{S}_\alpha.$$

Since the equations for both $\widetilde{v}_\alpha$ and $\widetilde{S}_\alpha$ are related to each other by an anti-symmetric term proportional to η_r, the Cartesian components of $\widetilde{v}_\alpha$ and $\widetilde{S}_\alpha$ need to be considered separately[5]. This yields us a 7×7 matrix of which

---

[4]The bulk modulus is inversely proportional to the compressibility, *cf.* Marder[27]

[5]In the previous case, the Cartesian components of $\widetilde{v}_\alpha$ also had been treated separately. However, the absence of anti-symmetric, cross-Cartesian terms radically simplifies calculations.

the eigenvalues are:

$$\omega_1 = -\frac{i}{\overline{\rho}\chi_{\mathrm{s}}}\left(2\eta_r + D_{\mathrm{s}}k^2\right) \tag{3.26}$$

$$\omega_{2,3} = \frac{-i}{2}\eta_k k^2 \pm \sqrt{Kk^2 - \frac{1}{4}\left(\eta_k k^2\right)^2}; \tag{3.27}$$

$$\omega_{4,5} = \frac{-i}{2}\left[\frac{1}{\overline{\rho}\chi_{\mathrm{s}}}\left(2\eta_r + D_{\mathrm{s}}k^2\right) + \eta_k k^2\right]$$

$$\pm\sqrt{\left[\left(\frac{2\eta_r}{\overline{\rho}}\right)^2 - \frac{2\eta_r}{\overline{\rho}}\eta_k - D_{\mathrm{s}}\eta_k k^2\right]\frac{k^2}{\chi_{\mathrm{s}}} - \left[\frac{1}{\overline{\rho}\chi_{\mathrm{s}}}\left(2\eta_r + D_{\mathrm{s}}k^2\right) + \eta_k k^2\right]^2}, \tag{3.28}$$

where $\omega_{4,5}$ are each double degenerate values. While Eq. (3.26) is the diagonal-only result for spin and Eq. (3.27) describes the general dispersion relation (3.23) of the previous case, Eq. (3.28) describes a new mode with both types of fluid viscosities present. Furthermore, one might interpret the spin diffusion constant D_{s} occurring in Eqs. (3.26) & (3.28) as a new pseudo-viscosity term for the spin dispersion modes.

Chapter 4

Non-turbulent fluid flows

With the theory of spin hydrodynamics set up in Chapter 2 and its hydrodynamic modes investigated in Chapter 3, we are now ready to consider simple fluid flows, before dealing with the actual spin hydrodynamic generation. This is firstly done to obtain an overview of the fluid's spin behaviour in a fluid flow, and secondly as another means of testing the results of Ch. 2. In this Chapter, we investigate simple fluid flows in the absence of any electromagnetic fields in Section 4.1, while in Section 4.2 we consider the presence of an effective magnetic field generated by the fluid's proper flow.

4.1 Fluid flows and spin, without electromagnetic fields

As a first, simple application, we consider a laminar fluid flowing unidirectionally and steadily between two parallel plates, known as the *Poiseuille flow*[9,20]. Let $\hat{x}$ be the Cartesian direction of the fluid current, and the plates be positioned at $\pm y_0$ on the $xz-$plane, such that the distance between them equals y_0. Choosing $v_x(y = \pm y_0) = 0, v_x(y = 0) = v_0$ as our boundary conditions, we obtain for the fluid's velocity:

$$v_x(y) = v_0 \left[1 - \left(\frac{y}{y_0} \right)^2 \right]; \qquad v_y = v_z = 0, \qquad (4.1)$$

i.e. the fluid velocity adheres to a parabolic profile. Since we consider our fluid to be incompressible, the fluid velocity to be constant—*i.e.* $\partial_t(\rho v_\alpha) = 0$—and ignore external (electromagnetic) forces in this case, conservation

laws and balance equations (2.75)–(2.77) become:

$$0 = -\frac{\partial p}{\partial x} + \eta \frac{\partial^2 v_x}{\partial y^2} - \frac{2\eta_r}{\chi_s} \frac{\partial S_z}{\partial y}; \tag{4.2}$$

$$0 = \left(\frac{D_s}{\chi_s} \frac{\partial^2}{\partial x^\alpha \partial x_\alpha} - \frac{2\eta_r}{\chi_s} \right) S_z + 2\eta_r \frac{\partial v_x}{\partial y}; \tag{4.3}$$

$$0 = \left(\frac{D_s}{\chi_s} \frac{\partial^2}{\partial x^\alpha \partial x_\alpha} - \frac{2\eta_r}{\chi_s} \right) S_{x,y}. \tag{4.4}$$

By using symmetry arguments, we expect the spin density S_α to depend solely on the y coordinate, since the system is physically bounded by the $xz-$planes formed by the two plates. By these same arguments, we expect $S_{x,y}$ to be non-existent. Altogether, using the symmetry arguments and Eq. (4.1) allows us to simplify Eq. (4.3) as:

$$0 = \left(\frac{\partial^2}{\partial y^2} - \frac{1}{\ell_s^2} \right) S_z + \xi_y y, \tag{4.5}$$

where

$$\frac{1}{\ell_s^2} = \frac{2\eta_r}{D_s} \qquad \xi_y = -\frac{4\eta_r \chi_s v_0}{D_s y_0^2}. \tag{4.6}$$

We identify the quantity ℓ_s as the *spin diffusion length* for fluids.

The spin density equation (4.5) is obviously to be recognized as an inhomogeneous second-order differential equation. To solve it, let us consider the homogeneous case $\xi_y = 0$; the general solution is then:

$$S_z^h(y) = A e^{y/\ell_s} + B e^{-y/\ell_s},$$

where A, B are constants. Since we require

$$S_z(y = -y_0) = -S_z(y = y_0); \qquad S_z(y = 0) = 0 \tag{4.7}$$

as the boundary conditions for our spin density, we obtain $B = -A$ such that

$$S_z^h(y) = 2A \sinh \left(\frac{y}{\ell_s} \right). \tag{4.8}$$

For the general, inhomogeneous case, we simply add an inhomogeneous term Cy—where C is a constant—to our homogeneous solution (4.8); plugging this into Eq. (4.5) yields $C = \xi_y \ell_s^2$, *i.e.* we obtain as general solution:

$$S_z(y) = 2A \sinh \left(\frac{y}{\ell_s} \right) - \frac{2\chi_s v_0}{y_0^2} y, \tag{4.9}$$

such that the boundary conditions (4.7) are satisfied. Note that for either $1/\ell_s \to 0$ or $y_0 \gg \ell_s$, S_z depends solely on the linear term as:

$$S_z(y)|_{1/\ell_s \to 0} = -\frac{2\chi_s v_0}{y_0^2} y. \tag{4.10}$$

30

The constant A present in Eq. (4.9) is also determined using suitable boundary conditions, in this case those pertaining to the spin current. Using Eq. (2.71), the relevant z-polarized spin current $j_{\mathrm{s},zy}$ is:

$$
\begin{aligned}
j_{\mathrm{s},zy} &= -\frac{D_{\mathrm{s}}}{\chi_{\mathrm{s}}}\frac{\partial S_z}{\partial y} \\
&= -\frac{2D_{\mathrm{s}}A}{\chi_{\mathrm{s}}\ell_{\mathrm{s}}}\cosh\left(\frac{y}{\ell_{\mathrm{s}}}\right) + \frac{2D_{\mathrm{s}}v_0}{y_0^2}.
\end{aligned} \tag{4.11}
$$

With the boundary condition that $j_{\mathrm{s},zy}(y = \pm y_0) = 0$, *i.e.* the spins cannot venture beyond the plates, we obtain for A:

$$
A = \frac{\ell_{\mathrm{s}}\chi_{\mathrm{s}}v_0}{y_0^2\cosh\left(y_0/\ell_{\mathrm{s}}\right)}
$$

such that the spin density (4.9) is written as:

$$
S_z(y) = -\frac{2\chi_{\mathrm{s}}v_0}{y_0^2}\left[y - \ell_{\mathrm{s}}\frac{\sinh\left(y/\ell_{\mathrm{s}}\right)}{\cosh\left(y_0/\ell_{\mathrm{s}}\right)}\right]. \tag{4.12}
$$

A plot of this spin density is given in Figure 4.1, for $y_0/\ell_{\mathrm{s}} = 1$ and $y_0/\ell_{\mathrm{s}} = 100$; note that the latter case yields the linear approximation of Eq. (4.10).
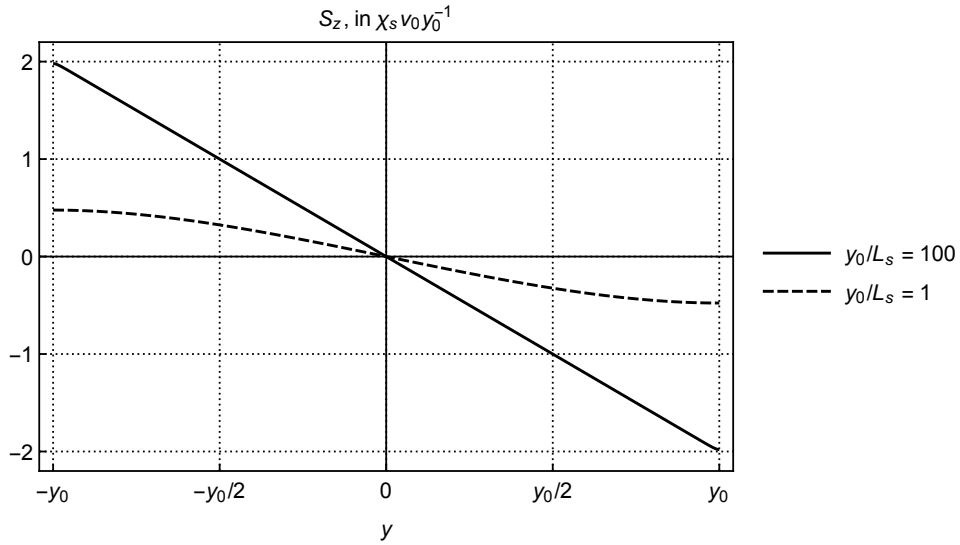

Figure 4.1: Plot for the spin density solution of Eq. (4.12) inside Poiseuille flows, for two different values of y_0/ℓ_{s}.

Using the general spin density solution (4.9) into Eq. (4.2) yields for the pressure gradient along the direction of the flow:

$$
\frac{\partial p}{\partial x} = -\frac{2v_0}{y_0^2}\left\{\eta - 2\eta_r\left[1 - \frac{\cosh\left(y/\ell_{\mathrm{s}}\right)}{\cosh\left(y_0/\ell_{\mathrm{s}}\right)}\right]\right\}. \tag{4.13}
$$

Since $\cosh(y/\ell_\mathrm{s}) \geq 1$ for $y \neq 0$, Eq. (4.13) implies the pressure gradient is smaller along the plates than in the middle of the flow; for $1/\ell_\mathrm{s} \to 0$ however, the pressure gradient becomes constant between the plates.

Next, we consider a more realistic approach to the Poiseuille flow described earlier; instead of two infinitely large plates along the $xz-$plane, we now consider a flow through an infinitely long, circular pipe of radius r_0, known as the *Poiseuille-Hagen flow* [9,20]. Instead of Cartesian coordinates, it is more natural to use cylindrical coordinates to describe the system, in this case radius r, angle θ, and length x. The fluid's velocity profile is then analogous to Eq. (4.1):

$$v_x(r) = v_0 \left[1 - \left(\frac{r}{r_0} \right)^2 \right]; \qquad v_r = v_\theta = 0. \tag{4.14}$$

In cylindrical coordinates, and using symmetric arguments similar to the Cartesian case—$S_r = S_x = 0$ and S_θ only depends on r, since the system is physically bounded by the $\hat{x}$-aligned pipe—Eqs. (4.2) and (4.5) effectively become:

$$0 = -\frac{\partial p}{\partial x} + \eta \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) v_x - \frac{2\eta_r}{\chi_\mathrm{s} r} \left[\frac{\partial}{\partial r} (r S_\theta) \right]; \tag{4.15}$$

$$0 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} - \frac{1}{\ell_\mathrm{s}^2} \right) S_\theta + \xi_r r, \tag{4.16}$$

where ℓ_s^{-2} is as defined in Eq. (4.6), and ξ_r is similar to ξ_y as described in Eq. (4.6), but with the pipe radius r_0 instead of the center-to-plate distance y_0. The solution to the homogeneous version of Eq. (4.16)—*i.e.* when $\xi_r = 0$—is given by a linear combination of modified Bessel functions:

$$S_\theta^\mathrm{h}(r) = A I_1 \left(\frac{r}{\ell_\mathrm{s}} \right) + B K_1 \left(\frac{r}{\ell_\mathrm{s}} \right), \tag{4.17}$$

where A, B are constants to be determined by means of boundary conditions, and I_1, K_1 the modified Bessel functions of the first and second kind, respectively. Since we require $S_\theta(r = 0) = 0$ as an essential boundary condition and $K_1(z)$ exhibits asymptotic behaviour for $z \to 0$, we obtain $B = 0$. Similar to the laminar flow case, we now add an inhomogeneous term Cr to the homogeneous solution (4.17); plugging this back into the inhomogeneous equation (4.16) then yields $C = \xi_r \ell_\mathrm{s}^2 = -2\chi_\mathrm{s} v_0 r_0^{-2}$. The general solution for S_θ then becomes:

$$S_\theta(r) = A I_1 \left(\frac{r}{\ell_\mathrm{s}} \right) - \frac{2\chi_\mathrm{s} v_0}{r_0^2} r. \tag{4.18}$$

Rather unsurprisingly, we use the θ-polarized spin current $j_{\mathrm{s},\theta r}$ to determine the constant A in Eq. (4.18) in the manner similar to the laminar flow case:

$$j_{\mathrm{s},\theta r}(r) = -\frac{A D_\mathrm{s}}{2\ell_\mathrm{s} \chi_\mathrm{s}} \left[I_0 \left(\frac{r}{\ell_\mathrm{s}} \right) + I_2 \left(\frac{r}{\ell_\mathrm{s}} \right) \right] + \frac{2 D_\mathrm{s} v_0}{r_0^2} \tag{4.19}$$

yields us for the boundary condition $j_{s,\theta r}(r = r_0) = 0$:

$$A = \frac{4\ell_s \chi_s v_0}{r_0^2} \left[I_0 \left(\frac{r_0}{\ell_s} \right) + I_2 \left(\frac{r_0}{\ell_s} \right) \right]^{-1},$$

such that the spin density inside a viscous fluid flowing through a pipe of radius r_0 is as follows:

$$S_\theta(r) = -\frac{2\chi_s v_0}{r_0^2} \left[r - 2\ell_s \frac{I_1 (r/\ell_s)}{I_0 (r_0/\ell_s) + I_2 (r_0/\ell_s)} \right]. \tag{4.20}$$

Note that we obtain a result similar to the laminar flow case for $1/\nu \to 0$:

$$S_\theta(r)|_{1/\ell_s \to 0} = -\frac{2\chi_s v_0}{r_0^2} r.$$

The similarities with the laminar flow case become even more apparent when the solution of Eq. (4.20) is plotted in Figure 4.2; again we use two different values of r_0/ℓ_s, and the resulting plot is near-indistinguishable from that of Figure 4.1.
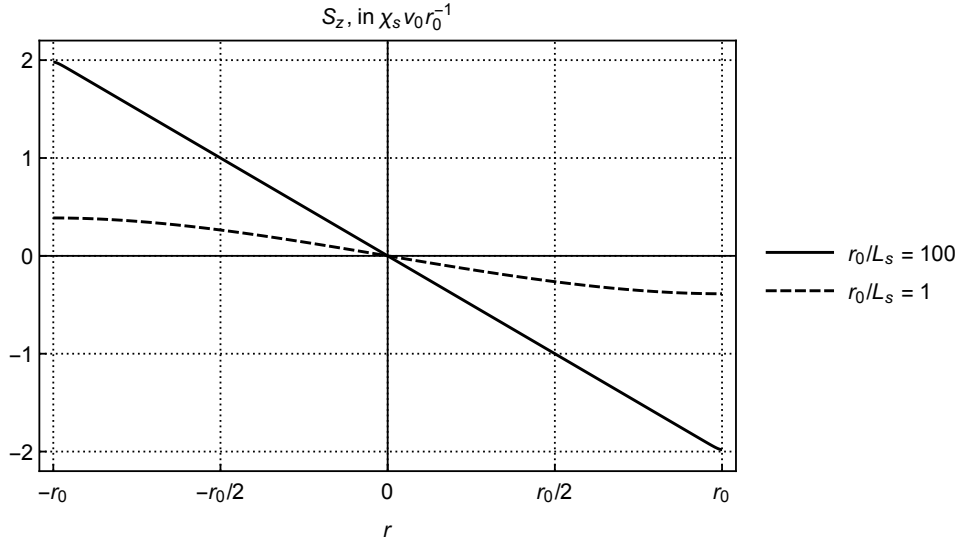

Figure 4.2: Plot for the spin density solution of Eq. (4.20) inside Poiseuille-Hagen flows, for two different values of r_0/ℓ_s.

To close this Section on non-turbulent flows, let us put Eqs. (4.14) and (4.20) into Eq. (4.15) to yield the pressure gradient inside the pipe:

$$\frac{\partial p}{\partial x} = -\frac{4v_0}{r_0^2} \left\{ \eta - \eta_r \left[2 - \frac{I_0(r/\ell_s) + I_2(r/\ell_s) + 2\ell_s r^{-1} I_1(r/\ell_s)}{I_0(r_0/\ell_s) + I_2(r_0/\ell_s)} \right] \right\}, \tag{4.21}$$

which as in the laminar flow case becomes constant over the whole pipe for $1/\ell_s \to 0$.

4.2 Fluid flows, spin, and effective magnetism

We now consider a monocomponent fluid of spins, which generate an effective magnetic field B_α internally, flowing through a circular pipe with circumstances similar to the latter part of Section 4.1. Since the spins themselves generate the magnetic field, we may utilize for the magnetic field the same symmetry arguments used for the spins, such that we have $B_\theta(r)$ as the effective magnetic field, *i.e.* the magnetic field is polarized in the θ direction and is dependent only on the radius r. In this situation, Eqs. (4.15) and (4.16) become:

$$0 = -\frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)v_x - \frac{2\eta_r}{\chi_s r}\left[\frac{\partial}{\partial r}\left(rS_\theta - r\gamma_e\chi_s B_\theta\right)\right]; \qquad (4.22)$$

$$0 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} - \frac{1}{\ell_s^2}\right)[S_\theta - \gamma_m\chi_s B_\theta] + \xi_r r, \qquad (4.23)$$

where ℓ_s and ξ_r are as defined in Eq. (4.6), with in the latter case y_0 substituted for r_0. We note that we may choose to regard the term $[S_\theta - \gamma_e\chi_s B_\theta]$ in the right hand side of Eq. (4.23) as a 'pseudo-quantity' adhering to a linear differential equation with an inhomogeneous term, similar to how S_θ was treated in Section 4.1. Using the same arguments which lead to Eq. (4.18), the general solution for this pseudo-quantity is:

$$S_\theta(r) - \gamma_m\chi_s B_\theta(r) = A I_1\left(\frac{r}{\ell_s}\right) - \frac{2\chi_s v_0}{r_0^2}r, \qquad (4.24)$$

where A is again a constant yet to be determined.

The most notable unknown quantity in Eq. (4.24) is obviously the θ-polarized magnetic field $B_\theta(r)$. The Maxwell equations relevant in determining B_θ are Eqs. (2.35) and (2.36), which after using Eq. (2.74) become:

$$\frac{\partial B_\alpha}{\partial t} = -c\epsilon_{\alpha\beta\gamma}\frac{\partial E^\beta}{\partial x^\gamma} = 0; \qquad (4.25)$$

$$\sigma_e E_\alpha = c\epsilon_{\alpha\beta\gamma}\frac{\partial B^\beta}{\partial x^\gamma} - \rho z v_\alpha, \qquad (4.26)$$

where we assumed $\partial_t E_\alpha = \partial_t B_\alpha = 0$ since our fluid is flowing with constant speed. In cylindrical coordinates, Eq. (4.26) becomes

$$\sigma_e E_x = -\frac{c}{r}\frac{\partial(rB_\theta)}{\partial r} - \rho z v_x, \qquad (4.27)$$

which differentiated with respect to r and after using Eq. (4.25) becomes

$$\underbrace{\frac{1}{r}\frac{\partial(rE_x)}{\partial r}}_{\propto\partial_t B_\theta = 0} = -\frac{\rho z}{\sigma_e r}\frac{\partial(rv_x)}{\partial r} - \frac{c}{\sigma_e r}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)\right]$$

$$\Rightarrow 0 = \frac{3\rho z v_0}{r_0^2}r - c\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)B_\theta. \qquad (4.28)$$

Integrating Eq. (4.28) twice over the radius r, and setting $B_\theta(r = 0) = 0$ as our boundary condition, we obtain as a first result for the effective magnetic field:

$$B_\theta(r) = \frac{3\rho z v_0}{8 c r_0^2} r^3 + C r, \tag{4.29}$$

where C is an integration constant yet to be determined. Plugging in Eq. (4.29) into Eq. (4.24) then yields for the spin density:

$$S_\theta(r) = A I_1\left(\frac{r}{\ell_s}\right) - \frac{2\chi_s v_0}{r_0^2} r + \frac{3\gamma_m \chi_s \rho z v_0}{8 c r_0^2} r^3 + \gamma_e \chi_s C r, \tag{4.30}$$

such that the (relevant) spin current $j_{s,\theta r}$ is as follows:

$$j_{s,\theta r}(r) = -\frac{D_s A}{2\chi_s \ell_s}\left[I_0\left(\frac{r}{\ell_s}\right) + I_2\left(\frac{r}{\ell_s}\right)\right] + \frac{2 D_s v_0}{r_0^2}$$
$$- \frac{9\gamma_m D_s \rho z v_0}{8 c}\left(\frac{r}{r_0}\right)^2 - \gamma_m D_s C. \tag{4.31}$$

Similar to the cases considered in Section 4.1, setting $j_{s,\theta r}(r = r_0) = 0$ yields us the exact values of the integration constants present; in this case, we obtain the values for A and C simultaneously:

$$A = 4\ell_s \frac{\chi_s v_0}{r_0^2}\left[I_0\left(\frac{r_0}{\ell_s}\right) + I_2\left(\frac{r_0}{\ell_s}\right)\right]^{-1}; \qquad C = -\frac{9\rho z v_0}{8 c}, \tag{4.32}$$

which then yields for our effective magnetic field and spin density:

$$B_\theta(r) = -\frac{3\rho z v_0}{8 c}\left[3 - \left(\frac{r}{r_0}\right)^2\right] r; \tag{4.33}$$

$$S_\theta(r) = -\frac{\chi_s v_0}{r_0^2}\left[r - 2\ell_s \frac{I_1(r/\ell_s)}{I_0(r_0/\ell_s) + I_2(r_0/\ell_s)}\right]$$
$$- \frac{3\gamma_m \chi_s \rho z v_0}{8 c}\left[3 - \left(\frac{r}{r_0}\right)^2\right] r. \tag{4.34}$$

For the Poiseuille case in the first half of Section 4.1, it is sufficient to redo the above calculations in a Cartesian system. This yields for the effective magnetic field B_z and spin per unit mass S_z:

$$B_z(y) = -\frac{\rho z v_0}{3 c}\left[3 - \left(\frac{y}{y_0}\right)^2\right] y; \tag{4.35}$$

$$S_z(y) = -\frac{2\chi_s v_0}{y_0^2}\left[y - \ell_s \frac{\sinh(y/\ell_s)}{\cosh(y_0/\ell_s)}\right] - \frac{\rho z \gamma_m \chi_s v_0}{3 c}\left[3 - \left(\frac{y}{y_0}\right)^2\right] y. \tag{4.36}$$

Chapter 5

The Inverse Spin Hall Effect

In Chapter 4, we have seen the spins' behaviour in a fluid flow, and how the spin current is affected by it. In this Chapter, we shall now consider a setup similar to the one used by Takahashi *et al.*[7], *i.e.* a Poiseuille flow-like system as considered in the first half of Section 4.1, where both plates are externally lined with a platinum (Pt) strip of thickness y_{p}. These strips are then situated parallel along the fluid flow, for a system length L. Section 5.1 deals with deriving the altered spins per unit masses and spin currents for a system with an Hg-Pt interface, while in Section 5.2 an expression for the resulting voltage is analytically determined. We then close this Chapter in Section 5.3 with a numerical calculation of this voltage.

5.1   The ISHE and the Hg-Pt Interface

An electric field transversally applied to a collection of spins, yields a spin current perpendicular to the electric field[3], exhibiting what is now commonly known as the *spin Hall effect*, often abbreviated as SHE. In the absence of (effective) magnetic fields, the spin current of Eq. (2.71) is rewritten to accommodate the SHE as[3,5]:

$$j_{\mathrm{s},\alpha\beta} = -\frac{D_{\mathrm{s}}}{\chi_{\mathrm{s}}}\frac{\partial S_{\alpha}}{\partial x_{\beta}} - \frac{\hbar}{2e}\theta_{\mathrm{SH}}\sigma_{\mathrm{e}}\epsilon_{\alpha\beta\gamma}E^{\gamma}, \qquad (5.1)$$

where $\hbar$ is the reduced Planck's constant, e the electron charge, and θ_{SH} the so-called *spin Hall angle*. The value of the spin Hall angle depends on the material used; for platinum, we have $\theta_{\mathrm{SH,Pt}} \simeq 0.05$ as an experimentally deduced minimal value for bulk materials[21], while for interfaces under ISHE conditions a value of $\theta_{\mathrm{SH,Pt}} \simeq 0.026$ was obtained[22]. Next to the regular SHE, a spin current is also able to induce a charge current under specific circumstances; this effect is then known as the *inverse spin Hall effect*[6], abbreviated ISHE. Rewriting the charge current of Eq. (2.72) to include the

ISHE then yields[5,6]:

$$j_c^\alpha = \sigma_e E^\alpha - \frac{2e}{\hbar}\frac{D_s}{\chi_s}\theta_{\mathrm{SH}}\epsilon^{\alpha\beta\gamma}\frac{\partial S_\beta}{\partial x_\gamma}.$$ (5.2)

In our setup, the electrical circuit is closed such that the total charge current due to the ISHE is zero-valued. Eq. (5.2) then yields us an equation for the effective electric field:

$$E_\alpha^{\mathrm{eff}} = \frac{2e}{\hbar\sigma_e}\frac{D_s}{\chi_s}\theta_{\mathrm{SH}}\epsilon_{\alpha\beta\gamma}\frac{\partial S^\beta}{\partial x^\gamma}.$$ (5.3)

Plugging in Eq. (5.3) into Eq. (5.1) yields a spin current equation for closed electrical circuits as ISHE systems:

$$j_{s,\alpha\beta} = -\frac{D_s}{\chi_s}(1+\theta_{\mathrm{SH}}^2)\frac{\partial S_\alpha}{\partial x_\beta}.$$ (5.4)

Since the mercury fluid spin per unit mass $S_z^{\mathrm{Hg}}(y)$ is polarized in the $\hat{z}$-direction, and dependent on y transversal to the $\hat{x}$-aligned fluid velocity, we expect the platinum strip spin density $S_z^{\mathrm{Pt}}(y)$ to adhere to the same polarization, with a dependence on y transversal to the direction of the expected $\hat{x}$-aligned ISHE electric current. We expect the platinum strip density to adhere to the Valet-Fert equation[23], *i.e.*

$$\left(\frac{\partial^2}{\partial y^2} - \frac{1}{(\ell_s^{\mathrm{Pt}})^2}\right)S_z^{\mathrm{Hg}}(y) = 0,$$ (5.5)

where ℓ_s^{Pt} is the experimentally deduced spin diffusion length of platinum. Using Eqs. (4.9) and (5.5), the *Ansätze* for the spins per unit masses are

$$S_z^{\mathrm{Hg}}(y) = 2A_1 \sinh\left(\frac{y_0}{\ell_s^{\mathrm{Hg}}}\right) - \frac{2\chi_s^{\mathrm{Hg}}v_0}{y_0^2}y;$$ (5.6)

$$S_z^{\mathrm{Pt}}(y) = A_2 e^{y/\ell_s^{\mathrm{Pt}}} + A_3 e^{-y/\ell_s^{\mathrm{Pt}}},$$ (5.7)

where A_1, A_2, A_3 are constants to be determined from boundary conditions. For the respective spin currents, Eqs. (4.11) and (5.4) yield us:

$$j_{s,zy}^{\mathrm{Hg}} = -\frac{D_s^{\mathrm{Hg}}}{\chi_s^{\mathrm{Hg}}}\frac{\partial S_z^{\mathrm{Hg}}}{\partial y};$$ (5.8)

$$j_{s,zy}^{\mathrm{Pt}} = -\frac{D_s^{\mathrm{Pt}}}{\chi_s^{\mathrm{Pt}}}(1+\theta_{\mathrm{SH}}^2)\frac{\partial S_z^{\mathrm{Pt}}}{\partial y},$$ (5.9)

where θ_{SH} from now on refers to the platinum spin Hall angle.

Equations (5.6) and (5.7) provide us with three unknown constants; fortunately, we may use three reasonable boundary conditions in order to deduce these constants. Since the system extends from $-y_0 - y_{\mathrm{p}}$ to $y_0 + y_{\mathrm{p}}$,

we shall first give the boundary conditions for the 'upper half' $y \geq 0$ and the resulting values for the constant, and then translate these for the 'lower half' $y \leq 0$. Firstly, we expect that no spins venture beyond the system, in this case beyond the exterior of the platinum strip at $y = y_0 + y_\mathrm{p}$, *i.e.* for the spin current we have:

$$j_{\mathrm{s},zy}^{\mathrm{Pt}}(y = y_0 + y_\mathrm{p}) = 0. \tag{5.10}$$

Secondly, we expect the spin current on the mercury-platinum interface at $y = y_0$ to be continuous, such that

$$j_{\mathrm{s},zy}^{\mathrm{Pt}}(y = y_0) = j_{\mathrm{s},zy}^{\mathrm{Hg}}(y = y_0). \tag{5.11}$$

From this a third boundary condition equating the mercury and platinum spins per unit masses at the interface naturally arises:

$$S_z^{\mathrm{Hg}}(y = y_0) = S_z^{\mathrm{Pt}}(y = y_0). \tag{5.12}$$

The boundary conditions (5.10)–(5.12) then yield us for A_1, A_2, A_3 in the region $y \geq 0$:

$$A_1 = -\frac{\chi_\mathrm{s}^{\mathrm{Hg}}\ell_\mathrm{s}^{\mathrm{Hg}}v_0}{y_0^2 \sinh\left(y_0/\ell_\mathrm{s}^{\mathrm{Hg}}\right)}$$

$$\times \left[\frac{(1 + \theta_{\mathrm{SH}}^2)(e^{2y_\mathrm{p}/\ell_\mathrm{s}^{\mathrm{Pt}}} - 1)y_0 D_\mathrm{s}^{\mathrm{Pt}}\chi_\mathrm{s}^{\mathrm{Hg}} - (e^{2y_\mathrm{p}/\ell_\mathrm{s}^{\mathrm{Pt}}} + 1)D_\mathrm{s}^{\mathrm{Hg}}\ell_\mathrm{s}^{\mathrm{Pt}}\chi_\mathrm{s}^{\mathrm{Pt}}}{(1 + \theta_{\mathrm{SH}}^2)(e^{2y_\mathrm{p}/\ell_\mathrm{s}^{\mathrm{Pt}}} - 1)D_\mathrm{s}^{\mathrm{Pt}}\ell_\mathrm{s}^{\mathrm{Hg}}\chi_\mathrm{s}^{\mathrm{Hg}} + (e^{2y_\mathrm{p}/\ell_\mathrm{s}^{\mathrm{Pt}}} + 1)\frac{D_\mathrm{s}^{\mathrm{Hg}}\ell_\mathrm{s}^{\mathrm{Pt}}\chi_\mathrm{s}^{\mathrm{Pt}}}{\tanh(y_0/\ell_\mathrm{s}^{\mathrm{Hg}})}}\right]; \tag{5.13}$$

$$A_2 = -\frac{2v_0\chi_\mathrm{s}^{\mathrm{Hg}}}{y_0^2}e^{-y_0/\ell_\mathrm{s}^{\mathrm{Pt}}}\left[y_0 - \ell_\mathrm{s}^{\mathrm{Hg}}\tanh(y_0/\ell_\mathrm{s}^{\mathrm{Hg}})\right]$$

$$\times \left[(1 + \theta_{\mathrm{SH}}^2)(e^{2y_\mathrm{p}/\ell_\mathrm{s}^{\mathrm{Pt}}} - 1)\tanh\left(\frac{y_0}{\ell_\mathrm{s}^{\mathrm{Hg}}}\right)\frac{D_\mathrm{s}^{\mathrm{Pt}}\ell_\mathrm{s}^{\mathrm{Hg}}\chi_\mathrm{s}^{\mathrm{Hg}}}{D_\mathrm{s}^{\mathrm{Hg}}\ell_\mathrm{s}^{\mathrm{Pt}}\chi_\mathrm{s}^{\mathrm{Pt}}} + e^{2y_\mathrm{p}/\ell_\mathrm{s}^{\mathrm{Pt}}} + 1\right]^{-1}; \tag{5.14}$$

$$A_3 = A_2 \exp\left\{\frac{2(y_0 + y_\mathrm{p})}{\ell_\mathrm{s}^{\mathrm{Pt}}}\right\}. \tag{5.15}$$

For the region $y \leq 0$, the boundary conditions (5.10)–(5.12) become

$$j_{\mathrm{s},zy}^{\mathrm{Pt}}(y = -y_0 - y_\mathrm{p}) = 0;$$
$$j_{\mathrm{s},zy}^{\mathrm{Pt}}(y = -y_0) = j_{\mathrm{s},zy}^{\mathrm{Hg}}(y = -y_0);$$
$$S_z^{\mathrm{Hg}}(y = -y_0) = S_z^{\mathrm{Pt}}(y = -y_0), \tag{5.16}$$

which yield the same value for A_1 as in Eq. (5.13), yet A_2 and A_3 switch places, and reverse signs:

$$A_3\big|_{y\leq 0} = +\frac{2v_0\chi_s^{\text{Hg}}}{y_0^2}e^{-y_0/\ell_s^{\text{Pt}}}\left[y_0 - \ell_s^{\text{Hg}}\tanh(y_0/\ell_s^{\text{Hg}})\right]$$
$$\times\left[(1+\theta_{\text{SH}}^2)(e^{2y_{\text{p}}/\ell_s^{\text{Pt}}}-1)\tanh\left(\frac{y_0}{\ell_s^{\text{Hg}}}\right)\frac{D_s^{\text{Pt}}\ell_s^{\text{Hg}}\chi_s^{\text{Hg}}}{D_s^{\text{Hg}}\ell_s^{\text{Pt}}\chi_s^{\text{Pt}}} + e^{2y_{\text{p}}/\ell_s^{\text{Pt}}}+1\right]^{-1}$$
$$(5.17)$$

$$A_2\big|_{y\leq 0} = A_3\big|_{y\leq 0}\exp\left\{\frac{2(y_0+y_{\text{p}})}{\ell_s^{\text{Pt}}}\right\}. \tag{5.18}$$

In general, the solution for the platinum spin per unit mass is described using Eqs. (5.7) and (5.14) as:

$$S_z^{\text{Pt}}(y) = \text{sgn}(y)A_2\left[\exp\left\{\frac{|y|}{\ell_s^{\text{Pt}}}\right\} + \exp\left\{2\frac{y_0+y_{\text{p}}}{\ell_s^{\text{Pt}}}\right\}\exp\left\{-\frac{|y|}{\ell_s^{\text{Pt}}}\right\}\right]. \tag{5.19}$$

Figure 5.1 shows the spins per unit masses S_z for two different ratios of y_0/ℓ_s^{Hg} and $y_{\text{p}}/\ell_s^{\text{Pt}}$ as a function of y; one might observe that a higher ratio between plate thickness or flow diameter and spin diffusion lengths results in a more sawtooth-like distribution of S_z.
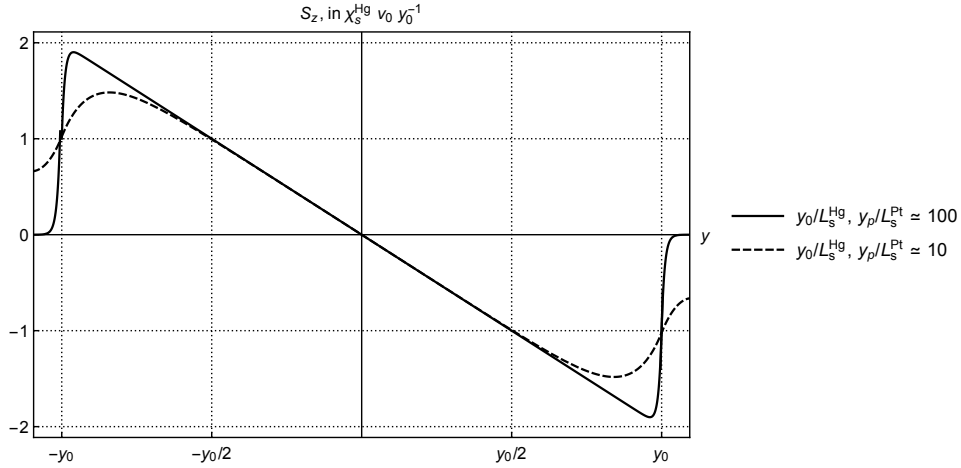

Figure 5.1: Comparison of spins per unit masses S_z for two different ratios between plate thickness or flow diameter, and spin diffusion lengths.

5.2 The ISHE Voltage

In Section 5.1, we used $j_c^\alpha = 0$ for a closed electrical circuit in Eq. (5.2), in order to substitute the term proportional to the electric field in Eq. (5.1)

and subsequently derive Eq. (5.4). Now that the spins per unit masses for both the mercury flow and the platinum strip have been derived, we proceed to the derivation of the ISHE voltage V_{ISHE}. For $j_{\mathrm{c}}^{\alpha} = 0$, Eq. (5.3) yields us the effective electric field $E_{\alpha}^{\mathrm{eff}}$, which in the case of our platinum strip is $\hat{x}$-polarized and must be evaluated in the 'upper half' strip for $y_0 + y_{\mathrm{p}} \geq y \geq y_0$:

$$E_x^{\mathrm{eff}} = \frac{2eD_{\mathrm{s}}^{\mathrm{Pt}}\theta_{\mathrm{SH}}}{\hbar\chi_{\mathrm{s}}^{\mathrm{Pt}}\sigma_{\mathrm{e}}^{\mathrm{Pt}}} \left[\left.\frac{\partial S_z^{\mathrm{Pt}}}{\partial y}\right|_{y=y_0+y_{\mathrm{p}}} - \left.\frac{\partial S_z^{\mathrm{Pt}}}{\partial y}\right|_{y=y_0} \right]. \tag{5.20}$$

The resulting ISHE voltage V_{ISHE} is then calculated in the usual manner, *i.e.* as a line integral of the electric field:

$$
\begin{aligned}
V_{\mathrm{ISHE}} &= \int \mathrm{d}l_{\alpha}\, E_{\alpha}^{\mathrm{eff}} \\
&= \frac{2eD_{\mathrm{s}}^{\mathrm{Pt}}\theta_{\mathrm{SH}}}{\hbar\sigma_{\mathrm{e}}^{\mathrm{Pt}}\chi_{\mathrm{s}}^{\mathrm{Pt}}} \int_0^L \mathrm{d}x \left[\left.\frac{\partial S_{\mathrm{s}}^{\mathrm{Pt}}}{\partial y}\right|_{y=y_0+y_{\mathrm{p}}} - \left.\frac{\partial S_{\mathrm{s}}^{\mathrm{Pt}}}{\partial y}\right|_{y=y_0} \right] \\
&= \frac{2eD_{\mathrm{s}}^{\mathrm{Pt}}\theta_{\mathrm{SH}}L}{\hbar\sigma_{\mathrm{e}}^{\mathrm{Pt}}\chi_{\mathrm{s}}^{\mathrm{Pt}}} \frac{A_2}{\ell_{\mathrm{s}}^{\mathrm{Pt}}} \left[e^{(y_0+y_{\mathrm{p}})/\ell_{\mathrm{s}}^{\mathrm{Pt}}} - e^{2(y_0+y_{\mathrm{p}})/\ell_{\mathrm{s}}^{\mathrm{Pt}}} e^{-(y_0+y_{\mathrm{p}})/\ell_{\mathrm{s}}^{\mathrm{Pt}}} \right. \\
&\qquad\qquad \left. - e^{y_0/\ell_{\mathrm{s}}^{\mathrm{Pt}}} + e^{2(y_0+y_{\mathrm{p}})/\ell_{\mathrm{s}}^{\mathrm{Pt}}} e^{-y_0/\ell_{\mathrm{s}}^{\mathrm{Pt}}} \right] \\
&= \frac{2eD_{\mathrm{s}}^{\mathrm{Pt}}\theta_{\mathrm{SH}}L}{\hbar\sigma_{\mathrm{e}}^{\mathrm{Pt}}\chi_{\mathrm{s}}^{\mathrm{Pt}}} A_2 e^{y_0/\ell_{\mathrm{s}}^{\mathrm{Pt}}} \left[e^{2y_{\mathrm{p}}/\ell_{\mathrm{s}}^{\mathrm{Pt}}} - 1 \right]. \tag{5.21}
\end{aligned}
$$

Writing out the constant A_2 using Eq. (5.14) allows us to rewrite Eq. (5.21) as:

$$
\begin{aligned}
V_{\mathrm{ISHE}} = &-\frac{4ev_0 D_{\mathrm{s}}^{\mathrm{Pt}} D_{\mathrm{s}}^{\mathrm{Hg}} \chi_{\mathrm{s}}^{\mathrm{Hg}} L\theta_{\mathrm{SH}}}{\hbar\sigma_{\mathrm{e}}^{\mathrm{Pt}} y_0^2} \\
&\cdot \left[\frac{y_0 - \ell_{\mathrm{s}}^{\mathrm{Hg}} \tanh\left(y_0/\ell_{\mathrm{s}}^{\mathrm{Hg}}\right)}{\left(\frac{e^{2y_{\mathrm{p}}/\ell_{\mathrm{s}}^{\mathrm{Pt}}}+1}{e^{2y_{\mathrm{p}}/\ell_{\mathrm{s}}^{\mathrm{Pt}}}-1}\right) D_{\mathrm{s}}^{\mathrm{Hg}} \ell_{\mathrm{s}}^{\mathrm{Pt}} \chi_{\mathrm{s}}^{\mathrm{Pt}} + (1+\theta_{\mathrm{SH}}^2) D_{\mathrm{s}}^{\mathrm{Pt}} \ell_{\mathrm{s}}^{\mathrm{Hg}} \chi_{\mathrm{s}}^{\mathrm{Hg}}} \right]. \tag{5.22}
\end{aligned}
$$

Under the reasonable limit of $y_0 \gg \ell_{\mathrm{s}}^{\mathrm{Hg}}$, Eq. (5.22) yields:

$$\lim_{y_0/\ell_{\mathrm{s}}^{\mathrm{Hg}}\to 0} V_{\mathrm{ISHE}} = -\frac{4ev_0 D_{\mathrm{s}}^{\mathrm{Pt}} \chi_{\mathrm{s}}^{\mathrm{Hg}} L\theta_{\mathrm{SH}}}{\hbar\sigma_{\mathrm{e}}^{\mathrm{Pt}} y_0 \ell_{\mathrm{s}}^{\mathrm{Pt}} \chi_{\mathrm{s}}^{\mathrm{Pt}}} \left(\frac{e^{2y_{\mathrm{p}}/\ell_{\mathrm{s}}^{\mathrm{Pt}}} - 1}{e^{2y_{\mathrm{p}}/\ell_{\mathrm{s}}^{\mathrm{Pt}}} + 1} \right). \tag{5.23}$$

For large x, the function $(e^{2x}-1)/(e^{2x}+1)$ quickly reaches its asymptotic value of 1; for $y_{\mathrm{p}} > \ell_{\mathrm{s}}^{\mathrm{Pt}}$ we then obtain for the limit ISHE voltage $V_{\mathrm{ISHE}}^{\mathrm{lim}}$:

$$V_{\mathrm{ISHE}}^{\mathrm{lim}} = \lim_{\substack{y_0/\ell_{\mathrm{s}}^{\mathrm{Hg}}\to 0 \\ y_{\mathrm{p}}/\ell_{\mathrm{s}}^{\mathrm{Pt}}\to 0}} V_{\mathrm{ISHE}} = -\frac{4ev_0 D_{\mathrm{s}}^{\mathrm{Pt}} \chi_{\mathrm{s}}^{\mathrm{Hg}} L\theta_{\mathrm{SH}}}{\hbar\sigma_{\mathrm{e}}^{\mathrm{Pt}} y_0 \ell_{\mathrm{s}}^{\mathrm{Pt}} \chi_{\mathrm{s}}^{\mathrm{Pt}}}. \tag{5.24}$$

Figure 5.2 shows V_{ISHE} for various values of the ratios y_0/ℓ_s^{Hg} and y_p/ℓ_s^{Pt}; one might observe that the ratio y_0/ℓ_s^{Hg} is of much greater importance on the resulting ISHE voltage than y_p/ℓ_s^{Pt}. Furthermore, Figure 5.2 shows the accuracy of $V_{\text{ISHE}}^{\text{lim}}$ of Eq. (5.24) compared to V_{ISHE} of Eq. (5.22) for $y_p/\ell_s^{\text{Pt}} \geq 5$ and $y_0/\ell_s^{\text{Hg}} \geq 100$. Since $y_P \simeq 10^{-4}$ m in usual experimental settings[7], and $\ell_s^{\text{Pt}} \simeq 3.7 \cdot 10^{-9}$ m under ISHE conditions[22], the condition $y_P/\ell_s^{\text{Pt}} \geq 5$ is easily satisfied. Similarly, we have $y_0 \simeq 10^{-4}$ m for our pipe half-diameter[7], and we may also estimate the mercury spin diffusion length ℓ_s^{Hg} to be of the order of nanometers, such that $y_0/\ell_s^{\text{Hg}} \geq 1000$ also applies to our situation. Therefore, we may use $V_{\text{ISHE}}^{\text{lim}}$ as an appropriate approximation for V_{ISHE}.
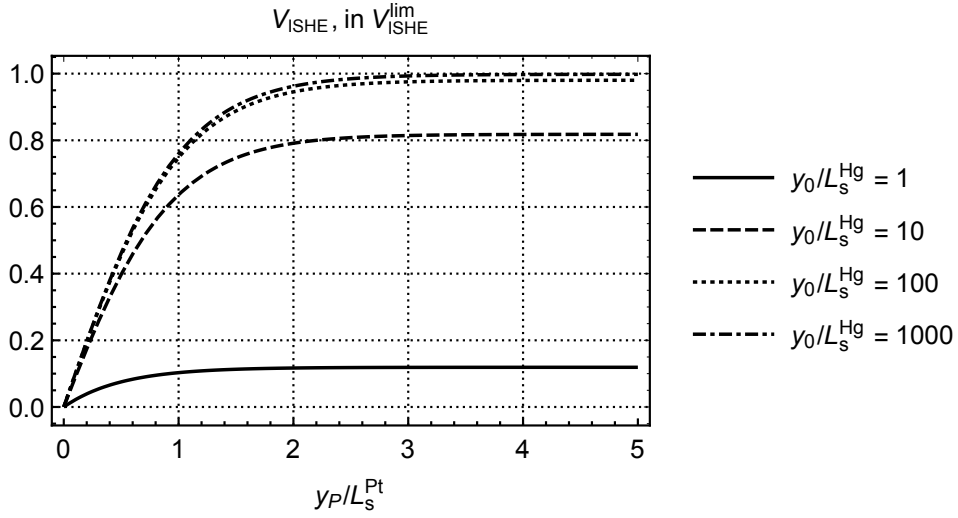

Figure 5.2: V_{ISHE} compared to $V_{\text{ISHE}}^{\text{lim}}$ for various values of $y_0/\ell_s^{\text{Hg}}, y_p/\ell_s^{\text{Pt}}$.

From Eq. (A.15), we have

$$\frac{D_s^{\text{Pt}} \chi_s^{\text{Hg}}}{\chi_s^{\text{Pt}}} = \frac{\hbar^2 \sigma_e^{\text{Pt}}}{4e^2} \left(\frac{n_e^{\text{Pt}}}{n_e^{\text{Hg}}} \right)^{2/3}, \tag{5.25}$$

such that we obtain as the *final expression* for the ISHE voltage:

$$V_{\text{ISHE}} = -\frac{\hbar v_0 L \theta_{\text{SH}}}{e y_0 \ell_s^{\text{Pt}}} \left(\frac{n_e^{\text{Pt}}}{n_e^{\text{Hg}}} \right)^{2/3} \tag{5.26}$$

5.3 Numerical value of the ISHE Voltage

The conduction electron number density of an element X, n_e^X, is obtained using the element's basic physical properties as:

$$n_e^X = \frac{z^X \rho^X N_A}{m_u^X \cdot 10^{-3}}, \qquad (5.27)$$

where z^X is the element's primary valency, ρ^X the element's mass density, N_A the Avogadro constant, and m_u^X the element's atomic weight; the factor 10^{-3} is used to convert the atomic weight from its usual unit of grams per mole to the more suitable kilograms per mole. The valencies, mass densities, and atomic weights of platinum and mercury are[24]:

$$z_e^{Pt} = z_e^{Hg} = 2;$$
$$\rho_e^{Pt} = 21.5 \text{ kg m}^{-3}; \qquad m_u^{Pt} = 195.084 \text{ g mol}^{-1};$$
$$\rho_e^{Hg} = 13.534 \text{ kg m}^{-3}; \qquad m_u^{Hg} = 200.592 \text{ g mol}^{-1},$$

which putting into Eq. (5.27) yields:

$$n_e^{Pt} = 13.27 \cdot 10^{28} \text{ m}^{-3}; \qquad n_e^{Hg} = 8.126 \cdot 10^{28} \text{ m}^{-3}. \qquad (5.28)$$

For the properties of the fluid flow as used by Takahashi *et al.*, we use:

$$v_0 = 2.7 \text{ m s}^{-1}; \qquad L = 8 \cdot 10^{-2} \text{ m}; \qquad y_0 = 4 \cdot 10^{-4} \text{ m}. \qquad (5.29)$$

while for the spin diffusion length and spin Hall angle of platinum, we use[7]:

$$\ell_s^{Pt} \simeq 3.7 \cdot 10^{-9} \text{ m}; \qquad \theta_{SH,Pt} \simeq 0.026, \qquad (5.30)$$

respectively, as estimated under spin-pumping ISHE conditions[22].

Finally, we shall respectively use as numerical values for the natural constants $\hbar, e, m_e$[25]:

$$\hbar = 1.054571800 \cdot 10^{-34} \text{ Js}; \qquad e = -1.6021766208 \cdot 10^{-19} \text{ C};$$
$$m_e = 9.10938356 \cdot 10^{-31} \text{ kg}. \qquad (5.31)$$

Plugging in the numerical values of Eqs. (5.28)–(5.31) into Eq. (5.26) then yields for the numerical value of the ISHE voltage:

$$\boxed{V_{ISHE} = 3.464 \cdot 10^{-6} \text{ V}} \qquad (5.32)$$

Chapter 6

Conclusion, Discussion, and Outlook

In this Thesis, we have set up a phenomenological theory of SHG from the irreversible thermodynamics of the coupling between fluid vorticity and internal spin. After verifying the theory using the hydrodynamic modes inside the fluid, solutions for the spin density and spin current generated inside the flowing viscous fluid were obtained. This was followed by obtaining solutions for interfaces between solid metal and viscous fluid, and from this an expression for the electrical potential—*i.e.* voltage—due to the ISHE. The calculated ISHE voltage of Eq. (5.32) is approximately 3.5 microvolts, which is two orders of magnitude higher than the approximately 50 nanovolts measured by Takahashi *et al.*[7] under similar circumstances for fluid velocity, diameter, and length. The V_{ISHE} value calculated in Section 5.3 could primarily explain the voltages measured by Takahashi *et al.*, especially when we consider two possibilities to account for this discrepancy.

The first possible cause of the discrepancy is due to *electron mechanical drag*, in which the electrons are 'dragged along' with the moving fluid, and subsequent momentum transfer on the solid metal. Put into practice, we consider the mercury electrons to move at the same velocity as the mercury itself—*i.e.* with velocity v_0—up to the outer bounds y_0 of the fluid. There, at the Hg-Pt interface, the mercury electrons then transfer their momentum to the platinum electrons, thus causing an electrical current without any involvement of the ISHE. Let the charge current due to this mechanical drag and momentum transfer, $j_{\mathrm{c,md}}$ be defined as

$$j_{\mathrm{c,md}} = e n_{\mathrm{e}} v_0, \tag{6.1}$$

where e is the electron charge and n_{e} the electronic density. From Eq. (2.72) we then have, for zero polarization, $E_{\mathrm{md},z} = j_{\mathrm{c,md}}/\sigma_{\mathrm{e}}$. For a platinum strip

of length L, we then obtain for the resulting mechanical drag voltage:

$$V_{\text{md}} = \frac{e n_e^{\text{Pt}} v_0 L}{\sigma_e^{\text{Pt}}},\tag{6.2}$$

which with Eqs. (5.28), (5.29), (5.31), and the electrical conductance of platinum given as [24]:

$$\sigma_e^{\text{Pt}} = 9.259 \cdot 10^6 \text{ S m}^{-1},\tag{6.3}$$

yields the numerical result of:

$$V_{\text{md}} = -496 \text{ V}.\tag{6.4}$$

While this momentum transfer voltage is 8 orders higher than the calculated ISHE voltage of Eq. (5.32), it should be noted that the assumptions leading to Eq. (6.4)—-unbounded electrons—are vastly exaggerated, in part due to the fluid viscosity, the fluid velocity being zero at the interface, and the electrons being electrically attracted to their respective nuclei. However, the momentum transfer should not be completely dismissed, on grounds that the result of Eq. (6.4) is of opposite sign compared to Eq. (5.32), thus giving a possible cause of the discrepancy with Takahashi *et al.*

A second possible cause of the discrepancy—which as the first one is based on the coupling between fluid velocity and spins—involves viscous boundary layers near the Hg-Pt interface and the presence of turbulent flows. Both Takahashi *et al.* and Matsuo *et al.*[9] have considered such turbulent boundary layers with friction velocities, such that the fluid possesses two velocity profiles across its diameter. It is their expectation that this turbulent boundary layer plays a key role in spin current generation, since the intersection of the two velocity profiles would yield the largest vorticity gradient[9]. We did not include such turbulent boundary layers for the velocity profiles in Chapter 4, because the relevant Navier-Stokes equations (4.2), (4.15), and (4.22)—and their direct predecessor, Eq. (2.75)—contain a spin gradient term proportional to the fluid's rotational viscosity η_r. While globally addressing the cause of the viscous sublayer (viscosity), this η_r-proportional gradient term cannot simulate a dual velocity profile. The effect of turbulent boundary layers on the spin current profiles obtained in Ch. 4 is left for future investigation.

While unrelated to the discrepancy, another point to be addressed here is the spin Hall angle and spin diffusion length of the solid metal—in our case platinum—used in SHG experiments. An alternative to platinum is tantalum (Ta), since its spin Hall angle is roughly of the same order of magnitude of $\theta_{\text{SH,Pt}}$, but of opposite sign. The valency, mass density, and atomic weight of tantalum are, respectively [24]:

$$z_e^{\text{Ta}} = 5; \qquad \rho_e^{\text{Ta}} = 16.4 \text{ kg m}^{-3}; \qquad m_u^{\text{Ta}} = 180.95 \text{ g mol}^{-1},$$

which plugged into Eq. (5.27) yields for the electron number density of tantalum:

$$n_{\mathrm{e}}^{\mathrm{Ta}} = 27.290 \cdot 10^{28} \ \mathrm{m}^{-3}. \tag{6.5}$$

From measurements performed under ISHE conditions[26], we have for the spin Hall angle and spin diffusion length of β-tantalum:

$$\theta_{\mathrm{SH,Ta}} \simeq -0.0062; \qquad \ell_{\mathrm{s}}^{\mathrm{Ta}} \simeq 5.1 \cdot 10^{-9} \ \mathrm{m}. \tag{6.6}$$

Using the tantalum-specific quantities of Eqs. (6.5) & (6.6) in Eq.(5.26) instead of the platinum-specific quantities then yields for the ISHE voltage:

$$V_{\mathrm{ISHE}}^{\mathrm{Hg\text{-}Ta}} = -9.69 \cdot 10^{-7} \ \mathrm{V}, \tag{6.7}$$

i.e. a factor -0.28 of that found for the Hg-Pt interface. Were the experiment of Takahashi *et al.* to be repeated with β-tantalum substituting the platinum in the original set-up, one would therefore expect a measured voltage of reverse sign, and a quarter of the voltage magnitude, if this measured voltage was indeed primarily due to the ISHE.

Based on the considerations in the preceding paragraphs, future research on SHG should accommodate for the presence of momentum transfer as a 'correction' to V_{ISHE}, turbulent layers inside the fluid, and various substitutes for platinum with different spin Hall angles and spin diffusion lengths under ISHE circumstances.

Appendix A

Determination of spin diffusion and susceptibility

In this Appendix, we shall attempt to give an alternate expression to the ratio $D_\mathrm{s}^\mathrm{Pt}\chi_\mathrm{s}^\mathrm{Hg}/\chi_\mathrm{s}^\mathrm{Pt}$ as found in the ISHE voltage expression of Eq. (5.24); in particular, we shall see that this specific ratio, the product of the electron mass density in platinum times the true spin diffusion constant in platinum on one hand, and of the spin susceptibility in mercury on the other hand, can be written in terms of natural constants, a ratio of electron number densities, and electrical conductivity of (solid) platinum.

A.1   The spin diffusion constant

As noted in Section 2.1 for Eq. (2.31), the ratio $D_\mathrm{s}/\chi_\mathrm{s}$ can be (dimensionally) thought of as the mass density ρ times the 'true' spin diffusion constant $\hat{D}_\mathrm{s}$. For free electrons inside a Drude-model metal, the true spin diffusion constant is defined as[27]

$$\hat{D}_\mathrm{s} = \frac{\lambda_\mathrm{mfp}^2}{d\tau_\mathrm{s}}, \tag{A.1}$$

where d is the dimension of the system and $\lambda_\mathrm{mfp} = v_F\tau_\mathrm{s}$ is the electron mean free path, where v_F is the Fermi velocity for electrons and τ_s the spin diffusion time. For a three-dimensional metal with $d = 3$ we then obtain from Eq. (A.1):

$$\hat{D}_\mathrm{s} = \frac{v_F^2\tau_\mathrm{s}}{3}. \tag{A.2}$$

The Fermi velocity is easily obtained using $E_F = m_\mathrm{e}v_F^2/2$ such that

$$v_F = \frac{\hbar k_F}{m_\mathrm{e}} = \frac{\hbar}{m_\mathrm{e}}\left(3\pi^2 n_\mathrm{e}\right)^{1/3}, \tag{A.3}$$

where $k_F = (3\pi^2 n_\mathrm{e})^{1/3}$ is the Fermi wavenumber written in terms of the electron number density n_e[27]. The spin diffusion time τ_s is equated to the

Drude model electron collision time, such that it is written in terms of the electron charge e, the electron mass m_e, the electron number density n_e, and electrical conductivity σ_e as:

$$\tau_\mathrm{s} = \frac{m_\mathrm{e}\sigma_\mathrm{e}}{e^2 n_\mathrm{e}}. \tag{A.4}$$

Combining Eqs. (A.2)–(A.4) then yields for the ratio $D_\mathrm{s}/\chi_\mathrm{s}$:

$$\begin{aligned}
\frac{D_\mathrm{s}}{\chi_\mathrm{s}} &= \rho_\mathrm{e} v_F^2 \tau_\mathrm{s} \\
&= m_\mathrm{e} n_\mathrm{e} \cdot \frac{\hbar^2}{m_\mathrm{e}^2}\left(3\pi^2 n_\mathrm{e}\right)^{2/3} \cdot \frac{m_\mathrm{e}\sigma_\mathrm{e}}{e^2 n_\mathrm{e}} \\
&= \frac{\hbar^2 \sigma_\mathrm{e}}{3e^2}\left(3\pi^2 n_\mathrm{e}\right)^{2/3}.
\end{aligned} \tag{A.5}$$

A.2 The spin susceptibility

The rotational energy per unit mass u_r, as defined in Eq. (2.12), is inversely proportional to the square of a quantity χ_s which we have named the *spin susceptibility*. Similar to its magnetic counterpart, the spin susceptibility is the ratio of change in the (total) average spin, $\langle S_\alpha \rangle$ to the change in an external quantity, in this case the external magnetic field B_α. Since our main focus, in Chapter 5, is on electrons in metals approximated as Fermi gases, we shall mainly concern ourselves with the spin susceptibility of electrons.

The rotational energy of Eq. (2.12) for an electron changes in the presence of an external magnetic field as:

$$u_r = \frac{S^2}{2\chi_\mathrm{s}} - \gamma_\mathrm{e} B_\alpha S^\alpha, \tag{A.6}$$

where γ_e is the electron gyromagnetic ratio, as defined in Eq. (2.39). For a system with zero *internal* average spin, such as ours in Chapter 2, we then have:

$$\langle S_\alpha \rangle = \rho_\mathrm{e} \chi_\mathrm{s} \gamma_\mathrm{e} B_\alpha. \tag{A.7}$$

Note that the electron (mass) density $\rho_\mathrm{e} = m_\mathrm{e} n_\mathrm{e}$ is added to the right-hand side to ease the use of Eq. (A.7) in hydrodynamic units. In the presence of a $\hat{z}$-polarized external magnetic field B_z, the Fermi-Dirac distribution for an electron with energy $E(k) = \hbar^2 k^2/(2m)$ and spin $\sigma = \pm\hbar/2$ is written as:

$$n_F(E(k), \sigma) = \left[\exp\left\{\beta\left(\frac{\hbar^2(k^2 - k_F^2)}{2m} + \sigma\gamma_\mathrm{e} B_z\right)\right\} + 1\right]^{-1}, \tag{A.8}$$

where $\beta^{-1} = k_B T$, k_B being the Boltzmann constant and T the temperature. Eq. (A.8) lets us calculate the average spin by means of an integral in

reciprocal space as:

$$\langle S_z \rangle = \frac{\hbar}{2} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \left[n_F(E(k), +) - n_F(E(k), -) \right], \qquad (\text{A.9})$$

i.e. an integral of the difference of the Fermi-Dirac distributions for the two possible spin levels, times a factor $\hbar/2$ to account for the electron's native spin.

While the integral of Eq. (A.9) might be evaluated numerically, an analytical approach is both preferable and feasible. Since we are only interested in the change of the average spin under an infinitesimal change of the magnetic field, a series expansion of the difference of the Fermi-Dirac distributions $n_F(E(k), +) - n_F(E(k), -)$ around $\sigma\gamma_e B_z = 0$ would yield a first approximation; up to first order in $\sigma\gamma_e B_z$, we then obtain:s

$$[n_F(E(k), +) - n_F(E(k), -)] \simeq -2\frac{\hbar}{2}\gamma_e B_z \frac{\beta \exp\left\{\beta(E(k) - E_F)\right\}}{\left[\exp\left\{\beta(E(k) - E_F)\right\} + 1\right]^2}$$

$$= \hbar\gamma_e B_z \left(\frac{\partial n_F(E(k))}{\partial E(k)} \right). \qquad (\text{A.10})$$

The second approximation is based on the Fermi temperature T_F: since metals often possess Fermi temperatures of the order of 10^4 to 10^5 kelvin, we may use $T \to 0$ for room temperature conditions. The Fermi-Dirac distribution function $n_F(E(k))$ can be approximated by the Heaviside step function $\Theta(E(k) - E_F)$ for the limit $T \to 0$; since the derivative of the step function $\Theta(x)$ yields the Dirac delta function $\delta(x)$, we obtain

$$\lim_{T \to 0} \left(\frac{\partial n_F(E(k))}{\partial E(k)} \right) = \delta(E(k) - E_F). \qquad (\text{A.11})$$

Applying Eq. (A.10) and (A.11) to Eq. (A.9) yields us:

$$\lim_{T \to 0} \langle S_z \rangle = \frac{\hbar^2 \gamma_e B_z}{2} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^3} \delta(E(k) - E_F). \qquad (\text{A.12})$$

The remaining integral is then properly dealt with using spherical coordinates, and subsequently $k^2 \, \mathrm{d}k = m_e \sqrt{2m_e} \hbar^{-3} \sqrt{E} \, \mathrm{d}E$ as a substitution of integration variables:

$$\langle S_z \rangle = \frac{\hbar^2 \gamma_e B_z}{4\pi^2} \int \mathrm{d}k \, k^2 \delta(E(k) - E_F)$$

$$= \frac{\hbar^2 \gamma_e B_z}{4\pi^2} \cdot \frac{m_e \sqrt{2m_e}}{\hbar^3} \int \mathrm{d}E \, \sqrt{E} \delta(E(k) - E_F)$$

$$= \frac{m_e \sqrt{2m_e}}{4\pi^2 \hbar} \cdot \frac{\hbar k_F}{\sqrt{2m_e}} \gamma_e B_z$$

$$= \frac{m_e (3\pi^2 n_e)^{1/3}}{4\pi^2} \gamma_e B_z, \qquad (\text{A.13})$$

where we have used $k_F = (3\pi^2 n_{\mathrm{e}})^{1/3}$ to write the Fermi wavenumber in terms of the electron number density n_{e}. Comparing Eqs. (A.7) and (A.13) then yields us for the spin susceptibility of a metal for (room) temperatures well below the Fermi temperature:

$$\chi_{\mathrm{s}} = \frac{3^{1/3}}{4} \left(\frac{1}{\pi^2 n_{\mathrm{e}}} \right)^{2/3} . \qquad (\mathrm{A.14})$$

A.3 Diffusion and susceptibility combined

In the case of the ISHE voltage of Eq. (5.24), we find that the ratio $D_{\mathrm{s}}^X / \chi_{\mathrm{s}}^X$ of a metal X is multiplied with the spin susceptibility χ_{s}^Y of a different metal Y. For the product $D_{\mathrm{s}}^X \chi_{\mathrm{s}}^Y / \chi_{\mathrm{s}}^X$, we find using Eqs. (A.5) and (A.14):

$$\begin{aligned}
\frac{D_{\mathrm{s}}^X \chi_{\mathrm{s}}^Y}{\chi_{\mathrm{s}}^X} &= \frac{\hbar^2 \sigma_{\mathrm{e}}^X}{3e^2} \left(3\pi^2 n_{\mathrm{e}}^X \right)^{2/3} \cdot \frac{3^{1/3}}{4} \left(\frac{1}{\pi^2 n_{\mathrm{e}}^Y} \right)^{2/3} \\
&= \frac{\hbar^2 \sigma_{\mathrm{e}}^X}{4e^2} \left(\frac{n_{\mathrm{e}}^X}{n_{\mathrm{e}}^Y} \right)^{2/3} , \qquad (\mathrm{A.15})
\end{aligned}$$

i.e. we obtain an expression which largely depends on the ratio between the electron number densities of both metals.

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