Poisson Manifolds of Compact Types

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Abstract

We explain a general strategy for constructing Poisson manifolds of strong compact type, and carry out in detail an application of this contruction using K3 surfaces, following the idea in [Mar13]. In the end, we obtain several PMSCT, with leaf spaces the circle and two different integral affine 2-tori.

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Contents

1 Introduction

Hamiltonian dynamics is one of the cornerstones of classical mechanics, built upon the idea of describing a physical system in terms of a set of differential equations on phase space depending only on the energy of the system: for a system in $\mathbb{R}^n$, the phase space is $\mathbb{R}^{2n}$ with configuration and momentum coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$, the energy is given by a smooth function $H : \mathbb{R}^{2n} \to \mathbb{R}$ and then the evolution of the system is determined by Hamilton's equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}.$$

In physics, an important problem is finding preserved quantities of such a system, i.e. smooth functions that are killed by the Hamiltonian vector field of H. The Poisson bracket made its first appearance as a method of constructing new preserved quantities: given two preserved quantities f and g, another can be obtained by setting

$$\{f, g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right).$$

Hamilton's equations can also be expressed in terms of this bracket:

$$\dot{q}^i = \{H, q^i\}, \qquad \dot{p}_i = \{H, p_i\}.$$

In fact, many of the fundamental concepts in Hamiltonian dynamics can be phrased in terms of the just defined bracket. The bracket also has some interesting properties on its own: it is bilinear, skew-symmetric, satisfies the Jacobi identity and is also a derivation in both arguments. With these properties in hand, one can thus axiomise the bracket, which leads to the notion of a *Poisson bracket*: on a smooth manifold M, such a bracket is a Lie bracket on the space of smooth functions $C^\infty(M)$, which is also a biderivation. A manifold equipped with a Poisson bracket is called a Poisson manifold, and Poisson geometry deals with the study of these manifolds.

In general, not a lot can be said about Poisson manifolds: for instance, every manifold admits a Poisson structure by setting the bracket to be identically zero. Therefore it is common to restrict one's attention to a specific class of Poisson manifolds. The class we will focus on in this thesis is that of Poisson manifolds "of compact type". The situation is analogous to that of Lie algebras, where we say they are compact if there is a compact Lie group integrating then. The question is then of course what the corresponding "integrating object" is for Poisson manifolds. One might try to integrate the Lie algebra structure on $C^\infty(M)$, but since this is an infinite-dimensional vector space, this is not the way to go. Instead, one considers a generalisation of Lie groups and Lie algebras, namely one looks at Lie groupoids and Lie algebroids, for which there is a similar notion of "integrating" a Lie algebroid to a Lie groupoid. One can associate to any Poisson manifold a Lie algebroid, and we call the Poisson manifold integrable if there exists a Lie groupoid integrating this Lie algebroid. If this is the case, the Lie groupoid carries extra structure making it into a symplectic groupoid: these are the "global objects" corresponding to Poisson manifolds, the "infinitestimal objects".

Contrary to the case of Lie groups and Lie algebras, there is more than one compactness type for Lie groupoids. There is also still the distinction between general integrations and "simply connected" integrations, and it turns out that in total there are six compactness types for Poisson manifolds. The strongest of these is the notion of *strong compactness*, and producing non-trivial[1] examples of Poisson manifolds of strong compact type has turned out to be rather difficult. The first such example was given in [Mar13], using the theory of K3 surfaces. In this thesis, we work out the construction in detail, providing several examples of Poisson manifolds of strong compact types.

1.1 Outline of this thesis

In Section 2 we work out the basics regarding Poisson manifolds. We discuss several different points of view for Poisson geometry, and give plenty of interesting examples.

In Section 3 we move on to Lie groupoids and Lie algebroids, giving their definitions and basic properties. We then describe several results regarding integration of Lie algebroids, looking specifically at the case of integrating Poisson manifolds.

We are then ready to give the definitions regarding compactness types in Poisson geometry in Section 4. We also prove a general result for constructing Poisson manifolds of strong compact type, which we will later apply to the construction using K3 surfaces.

In Section 5 we discuss K3 surfaces. We give their definition, discuss their basic properties and the necessary results allowing us to use them in our construction.

Finally, in Section 6 we give explicit examples of Poisson manifolds of strong compact type.

[1] Any 1-connected, compact symplectic manifold is trivially a Poisson manifold of strong compact type.

2 Poisson manifolds

In this section we introduce the concept of a Poisson manifold and exhibit several alternative ways of describing their structure. We provide several examples and discuss the symplectic foliation of a Poisson manifold. We mainly follow [FM15].

2.1 The classic definition

The most commonly used method of describing the Poisson structure on a manifold is endowing the space of smooth functions with a special kind of Lie bracket.

Definition 2.1. A *Poisson bracket* on a manifold M is a Lie bracket on $C^\infty(M)$ which also satisfies the Leibniz identity. More specifically, it is an $\mathbb{R}$-bilinear map $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying

- skew-symmetry: $\{f, g\} = -\{g, f\}$
- Jacobi identity: $\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0$
- Leibniz identity: $\{f, gh\} = \{f, g\}h + \{f, h\}g$

for all $f, g, h \in C^\infty(M)$. A *Poisson manifold* is a pair $(M, \{\cdot, \cdot\})$ consisting of a manifold M and a Poisson bracket $\{\cdot, \cdot\}$ on M. A *Poisson map* between two Poisson manifolds $(M, \{\cdot, \cdot\})$ and $(M', \{\cdot, \cdot\}')$ is a smooth map $\varphi : M \to M'$ that intertwines the brackets:

$$\{f \circ \varphi, g \circ \varphi\} = \{f, g\}' \circ \varphi \tag{2.1}$$

for all $f, g \in C^\infty(M')$. ♢

Let us briefly mention some examples before going deeper into the theory.

Example 2.2.

(i) On every manifold M we have the "zero Poisson structure", given by $\{f, g\} = 0$ for all $f, g \in C^\infty(M)$.

(ii) On $\mathbb{R}^{2n}$ we have the so-called *canonical Poisson structure* given by

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_{i+n}} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_{i+n}} \right). \tag{2.2}$$

More generally, on $\mathbb{R}^n$ we have Poisson structures

$$\{f, g\} = \sum_{i,j=1}^{n} c_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j},$$

where c_{ij} are constants such that $c_{ij} = -c_{ji}$.

(iii) For a more interesting example, recall that a *symplectic manifold* is a pair (S, ω) consisting of a manifold S and a closed, nondegenerate 2-form $\omega \in \Omega^2(S)$. Nondegeneracy gives us the notion of a Hamiltonian vector field associated to some $f \in C^\infty(S)$; it is the unique vector field $X_f \in \mathfrak{X}(S)$ satisfying
$$i_{X_f}\omega = df.$$
The bracket $\{\cdot,\cdot\}$ defined by
$$\{f, g\} := -\omega(X_f, X_g)$$
is a Poisson bracket on S. In other words, every symplectic manifold comes with an induced Poisson structure. For example, it is easy to see that the canonical Poisson structure (2.2) is induced by the canonical symplectic structure on $\mathbb{R}^{2n}$.

We will provide more details about these examples, and more examples in general, in Section 2.4. △

The Leibniz identity allows us to generalise the notion of Hamiltonian vector fields to any Poisson manifold.

Definition 2.3. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and let $f \in C^\infty(M)$. The *Hamiltonian vector field* associated to f is the vector field $X_f \in \mathfrak{X}(M)$ defined by
$$\mathcal{L}_{X_f}(g) = \{f, g\}, \quad g \in C^\infty(M). \qquad \diamondsuit$$

We see immediately that for a symplectic manifold, the original definition in Example 2.2 of a Hamiltonian vector field coincides with the definition for a Poisson manifold.

The following is an immediate consequence of the Jacobi identity.

Proposition 2.4. *Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold. Then for all $f, g \in C^\infty(M)$ we have*
$$X_{\{f,g\}} = [X_f, X_g]. \tag{2.3}$$
In other words, the map $C^\infty(M) \to \mathfrak{X}(M)$ given by $f \mapsto X_f$ is a Lie algebra homomorphism.

Remark 2.5. In fact, the notion of a Hamiltonian vector field still makes sense if one drops the Jacobi identity from the definition of a Poisson bracket. It is easy to see that Equation (2.3) is then equivalent to the Jacobi identity for $\{\cdot,\cdot\}$.

2.1.1 Locality of Poisson brackets

Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and let $U \subset M$ be open. Then there is an induced Poisson bracket $\{\cdot,\cdot\}_U$ on U satisfying
$$\{f, g\}|_U = \{f|_U, g|_U\}_U \tag{2.4}$$
for all $f, g \in C^\infty(M)$. This follows from the fact that
$$\mathrm{supp}(\{f, g\}) \subset \mathrm{supp}(f) \cap \mathrm{supp}(g)$$

for all $f, g \in C^\infty(M)$, which in turn is an easy consequence of the Leibniz identity. This is particularly interesting when we take $\big(U, (x_1, \ldots, x_n)\big)$ a coordinate domain; as we know, every derivation $C^\infty(U) \to C^\infty(U)$ must then be of the form

$$f \mapsto \sum_{i=1}^{n} c_i \frac{\partial f}{\partial x_i}$$

for some $c_i \in C^\infty(U)$. Since $\{\cdot, \cdot\}_U$ is a derivation in both arguments, it follows that it must be of the form

$$\{f, g\}_U = \sum_{i,j=1}^{n} \pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \tag{2.5}$$

for some $\pi_{ij} \in C^\infty(U)$.

Remark 2.6. Of course, (2.5) does not define a Poisson bracket for just any $\pi_{ij} \in C^\infty(U)$. In fact, since $\partial x_i / \partial x_j = \delta_{ij}$ we see that $\pi_{ij} = \{x_i, x_j\}$ and we can easily deduce the conditions for π_{ij} from the properties of Poisson brackets applied to the functions x_i. Indeed, skew-symmetry is equivalent to $\pi_{ij} = -\pi_{ji}$ $(i, j = 1, \ldots, n)$, while the Jacobi identity is equivalent to the system of equations

$$\sum_{l=1}^{n} \left(\pi_{lk} \frac{\partial \pi_{ij}}{\partial x_l} + \pi_{lj} \frac{\partial \pi_{ki}}{\partial x_l} + \pi_{li} \frac{\partial \pi_{jk}}{\partial x_l} \right) = 0 \quad (i, j, k = 1, \ldots, n). \tag{2.6}$$

2.2 Alternative approaches to Poisson structures

As mentioned before, there are many different ways of describing Poisson structures. We mention some of these now, all of which will be useful for us later.

2.2.1 Poisson bivectors

One way of describing a Poisson structure is using a particular type of bivector. This description is often the most convenient in practice.

Recall first the notion of a *multivector field* on a manifold M; essentially being the covariant analogue of a differential form, a k-vector field is simply a section of $\Lambda^k TM$. We denote by $\mathfrak{X}^k(M) := \Gamma(\Lambda^k TM)$ the space of k-vector fields on M, and we note that for $k = 0$ this is simply the space of smooth functions on M and that for $k = 1$ we recover the "standard" vector fields. As is well known, certainly in the case of the differential forms, we now have two ways of looking at a k-vector field $\xi \in \mathfrak{X}^k(M)$; not only as a (smooth) family of $\mathbb{R}$-multilinear, skew-symmetric maps

$$\xi_x : \overbrace{T_x^* M \times \cdots \times T_x^* M}^{k \text{ times}} \to \mathbb{R}, \quad (x \in M)$$

but also as a $C^\infty(M)$-multilinear, skew-symmetric map

$$\xi : \overbrace{\Omega^1(M) \times \cdots \times \Omega^1(M)}^{k \text{ times}} \to \mathbb{R}.$$

We will use the same notation for both.

Before we discuss how a bivector can encode a Poisson structure, let us first mention some operations involved with multivector fields, all of which will be familiar from either vector fields or differential forms.

- We have the wedge product $\wedge : \mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \to \mathfrak{X}^{k+l}(M)$ that satisfies all the familiar relations we know for differential forms.
- Where for differential forms we have interior multiplication by vector fields, for multivector fields we have interior multiplication by a 1-form; $i_\alpha : \mathfrak{X}^k(M) \to \mathfrak{X}^{k-1}(M)$ given by

$$i_\alpha(\xi)(\alpha_1, \ldots, \alpha_{k-1}) = \xi(\alpha, \alpha_1, \ldots, \alpha_{k-1})$$

  for $\alpha \in \Omega^1(M)$. Note that this is actually a pointwise operation.
- As for vector fields we have a pushforward operation, or more generally a notion of "relatedness" by smooth maps: for a smooth map $f : M \to M'$ we have an induced map $df(x)_* : \Lambda^k T_x M \to \Lambda^k T_{f(x)} M'$, and two multivector fields $\xi \in \mathfrak{X}^k(M)$ and $\xi' \in \mathfrak{X}^k(M')$ are called *f-related* if $\xi'_{f(x)} = df(x)_*(\xi_x)$ for all $x \in M$. When ξ determines ξ' uniquely (e.g. when f is a diffeomorphism) we call it the *pushforward* of ξ by f and write $\xi' = f_*(\xi)$.
- With the notion of pushforward comes the notion of Lie derivative along a vector field: for any $X \in \mathfrak{X}(M)$ this is an operation $\mathcal{L}_X : \mathfrak{X}^k(M) \to \mathfrak{X}^k(M)$, again satisfying the relations we know from differential forms.

To discover the relation between Poisson structures and bivectors, we need to discuss a third way of looking at multivector fields. It is well known that (1-)vector fields are in one-to-one correspondence with derivations $C^\infty(M) \to C^\infty(M)$ through the association $X \mapsto \mathcal{L}_X$. It turns out that a similar correspondence holds for higher order vector fields. Recall first the concept of a *multiderivation*; for $k \in \mathbb{N}$ a k-derivation is an $\mathbb{R}$-multilinear map

$$\overbrace{C^\infty(M) \times \cdots \times C^\infty(M)}^{k \text{ times}} \to C^\infty(M)$$

which is a derivation in each argument.

Proposition 2.7. *For $k \in \mathbb{N}$ there is a one-to-one correspondence between k-vector fields and skew-symmetric k-derivations on M, given by the map $\xi \mapsto \mathcal{L}_\xi$, where the latter is defined as*

$$\mathcal{L}_\xi(f_1, \ldots, f_k) := \xi(df_1, \ldots, df_k). \tag{2.7}$$

This proposition gives us what we want; indeed, a Poisson bracket is (in particular) a skew-symmetric biderivation and hence for any such bracket $\{\cdot,\cdot\}$ on M we have an associated bivector field $\pi \in \mathfrak{X}^2(M)$ such that $\mathcal{L}_\pi = \{\cdot,\cdot\}$. Of course, not any bivector field induces a Poisson bracket; we need to keep in mind the Jacobi identity. To describe the corresponding condition for bivector fields we need a new operation on multivector fields; this will be the *Schouten-Nijenhuis bracket*, which is an extension of the well-known Lie bracket of vector fields. It can be seen as the analogue of the exterior derivative of differential forms, and much like the exterior derivative the nicest way of describing it is by listing its main properties and noting that these properties determine it uniquely.

Theorem 2.8. *There exists and is unique a collection of bilinear maps* $[\cdot,\cdot] : \mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \to \mathfrak{X}^{k+l-1}(M)$ *with the following properties.*

(i) In low degrees we have the following:

- *The case* $k = l = 0$*: we have* $[f, g] = 0$ *for all* $f, g \in \mathfrak{X}^0(M) = C^\infty(M)$*;*
- *The case* $k = 1$*,* $l = 0$*: we have* $[X, f] = \mathcal{L}_X(f)$*, the standard Lie derivative, for* $X \in \mathfrak{X}^1(M) = \mathfrak{X}(M)$ *and* $f \in \mathfrak{X}^0(M) = C^\infty(M)$*;*
- *The case* $k = l = 1$*: we have that* $[X, Y]$ *is the usual Lie bracket for* $X, Y \in \mathfrak{X}^1(M) = \mathfrak{X}(M)$*.*

(ii) Skew-symmetry: for $\xi_1 \in \mathfrak{X}^k(M)$ *and* $\xi_2 \in \mathfrak{X}^l(M)$ *we have*

$$[\xi_1, \xi_2] = -(-1)^{(k-1)(l-1)}[\xi_2, \xi_1].$$

(iii) Leibniz identity: for $\xi_1 \in \mathfrak{X}^k(M), \xi_2 \in \mathfrak{X}^l(M)$ *and* $\xi_3 \in \mathfrak{X}^m(M)$ *we have*

$$[\xi_1, \xi_2 \wedge \xi_3] = [\xi_1, \xi_2] \wedge \xi_3 + (-1)^{(k-1)l}\xi_2 \wedge [\xi_1, \xi_3].$$

The proof is not very interesting and is omitted. In light of Proposition 2.7, there is an associated operation on skew-symmetric multiderivations. For that we need a notion of "composition" of multiderivations: for a k-derivation D and an l-derivation D' we define their composition $D \circ D'$, which will be of order $k + l - 1$, by

$$(D \circ D')(f_1, \ldots, f_{k+l-1}) := \sum_{\sigma \in S_{k,l-1}} (-1)^\sigma D\big(D'(f_{\sigma(1)}, \ldots, f_{\sigma(l)}), f_{\sigma(l+1)}, \ldots, f_{\sigma(k+l-1)}\big) \tag{2.8}$$

Here we denote by $S_{k,l-1} \subset S_{k+l-1}$ the group of $(k, l-1)$-shuffles. Note that this is in general not an actual multiderivation. However, with this composition we can define a commutator bracket by setting

$$[D, D'] := D \circ D' - (-1)^{(k-1)(l-1)} D' \circ D,$$

and this will in fact be a $(k + l - 1)$-derivation. We leave the proof as an exercise to the reader. The main point of all this is the following.

Proposition 2.9. *For all* $\xi_1 \in \mathfrak{X}^k(M)$ *and* $\xi_2 \in \mathfrak{X}^l(M)$ *we have that*

$$\mathcal{L}_{[\xi_1,\xi_2]} = [\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}].$$

In other words, under the correspondence from Proposition 2.7 the Schouten-Nijenhuis bracket and the commutator bracket on multiderivations are intertwined (indeed, sometimes the Schouten-Nijenhuis bracket is just defined by means of Proposition 2.9). Now we can finally formulate the Jacobi identity for bivectors.

Corollary 2.10. *For any* $\pi \in \mathfrak{X}^2(M)$*,* $\{\cdot,\cdot\} := \mathcal{L}_\pi$ *is a Poisson bracket if and only if* $[\pi, \pi] = 0$*.*

Proof. We show that

$$\mathcal{L}_{[\pi,\pi]}(f,g,h) = 2\big(\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\}\big).$$

To see this, note that by Proposition 2.9 we have $\mathcal{L}_{[\pi,\pi]} = 2(\mathcal{L}_\pi \circ \mathcal{L}_\pi)$ and that writing out the definition yields

$$(\mathcal{L}_\pi \circ \mathcal{L}_\pi)(f,g,h) = \{\{f,g\},h\} - \{\{f,h\},g\} + \{\{g,h\},f\}. \qquad \square$$

A bivector as in Corollary 2.10 is called a *Poisson bivector*. As a summary of the above, we know that on any manifold M there is a one-to-one correspondence between Poisson bivectors $\pi \in \mathfrak{X}^2(M)$ and Poisson brackets $\{\cdot,\cdot\}$, exhibited specifically by setting

$$\{f,g\} := \pi(df,dg), \quad f,g \in C^\infty(M),$$

given $\pi \in \mathfrak{X}^2(M)$. From now on, we will most often specify a Poisson structure by giving the corresponding bivector, denoting a Poisson manifold as the pair (M,π). Of course, we can formulate some concepts from Section 2.1 from this new point of view: the condition for a smooth map $\varphi : M_1 \to M_2$ between Poisson manifolds (M_1,π_1) and (M_2,π_2) being a Poisson map is that π_1 and π_2 are φ-related, and the Hamiltonian vector field associated to some $f \in C^\infty(M)$ can be written as

$$X_f = i_{df}(\pi). \tag{2.9}$$

In fact, we will soon see that the map $\alpha \mapsto i_\alpha(\pi)$ will play a large role, and we will denote it by $\pi^\# : \Omega^1(M) \to \mathfrak{X}(M)$. Again, it is useful to remember that this is induced by a vector bundle map $T^*M \to TM$, which we will also denote by $\pi^\#$. With this new notation, we get that $X_f = \pi^\#(df)$ for all $f \in C^\infty(M)$. We can also rewrite the Poisson map condition in terms of $\pi^\#$. Indeed, it is easy to verify that a smooth map $\varphi : M_1 \to M_2$ between Poisson manifolds (M_1,π_1) and (M_2,π_2) is a Poisson map iff the diagram

$$\begin{tikzcd} T_x^*M_1 \arrow[r, "\pi^\#_{1,x}"] & T_xM_1 \arrow[d, "d\varphi(x)"] \\ T^*_{\varphi(x)}M_2 \arrow[u, "d\varphi(x)^*"] \arrow[r, "\pi^\#_{2,\varphi(x)}"] & T_{\varphi(x)}M_2 \end{tikzcd} \tag{2.10}$$

commutes for all $x \in M_1$.

Remark 2.11. In any coordinate chart $\big(U,(x_1,\dots,x_n)\big)$ we can write π locally as

$$\pi|_U = \sum_{i<j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \tag{2.11}$$

We see immediately that the coefficients $\pi_{ij} \in C^\infty(U)$ must be the same as the ones we found earlier (see Eq. (2.5)).

2.2.2 The symplectic foliation

In this section we describe the symplectic foliation of a Poisson manifold; it will be a partition of the manifold into connected, initial submanifolds, each equipped with a symplectic structure compatible with the Poisson structure.

Remark 2.12. Recall that an *initial submanifold* of M is an immersed submanifold $N \subset M$ such that for every smooth manifold M' and every smooth map $f : M' \to M$ such that $f(M') \subset N$, the induced map $\widetilde{f} : M' \to N$ is smooth. It is also useful to recall that any subset $N \subset M$ has at most one smooth structure that makes it into an initial submanifold.

We need the following.

Lemma 2.13. *The relation on M defined by*

$$x \sim y \text{ iff there exist } f_1, \dots, f_k \in C^\infty(M) \text{ such that } y = \left(\varphi^1_{X_{f_1}} \circ \cdots \circ \varphi^1_{X_{f_k}}\right)(x) \tag{2.12}$$

is an equivalence relation. Here $\varphi^1_{X_{f_i}}$ denotes the time-1 flow of the Hamiltonian vector field X_{f_i}.

Proof. Reflexivity follows by considering $f = 0$, for which $X_f = 0$ and thus $\varphi^1_{X_f} = \mathrm{id}$. Transitivity is immediate. For symmetry it suffices to note that for all $f \in C^\infty(M)$ such that the flow $\varphi^t_{X_f}$ of X_f is defined at $t = 1$, the flow of X_{-f} is also defined at time 1, and in particular given by $\varphi^{-1}_{X_f} = \left(\varphi^1_{X_f}\right)^{-1}$. □

The previous lemma allows us to define the symplectic foliation.

Definition 2.14. The *symplectic leaves* of (M, π) are the equivalence classes of the relation (2.12). The *symplectic foliation* is simply the family of symplectic leaves $\mathcal{S} = \{S \mid S \text{ is a symplectic leaf}\}$. ♢

Of course, for now this is just a partition of M into subsets. The following theorem gives it the structure we mentioned earlier.

Theorem 2.15. *Every symplectic leaf S is a connected, initial submanifold of M, satisfying $T_xS = \mathrm{im}(\pi^\#_x)$ for all $x \in S$. Moreover, it admits a symplectic form ω_S given by*

$$\omega_{S,x}(\pi^\#_x(\alpha), \pi^\#_x(\beta)) = -\pi_x(\alpha, \beta) \tag{2.13}$$

for $x \in S$ and $\alpha, \beta \in T^*_xM$.

For a detailed proof of this result, see [FM15, Chapter 3].

The definition of the leaves in terms of the equivalence relation (2.12) is not very convenient in practice. Thankfully, there are several results that help determine the symplectic foliation. One of the most useful ones is given by the following.

Proposition 2.16. *Let $\mathcal{S}'$ be a partition of M into connected, immersed submanifolds such that $T_xS = \mathrm{im}(\pi^\#_x)$ for all $x \in S$ and $S \in \mathcal{S}'$. Then $\mathcal{S}'$ is the symplectic foliation.*

Another easily verified fact is the following: let $f \in C^\infty(M)$ such that $X_f = 0$. Then the symplectic leaves are contained in the regular level sets of f. More precisely, if $x \in M$ is in a regular level set of f, then the symplectic leaf through x is contained in that level set. A function $f \in C^\infty(M)$ such that $X_f = 0$ is called a *Casimir function.*

2.2.3 The Poisson Lie algebroid

In this section we show that for every Poisson manifold (M, π), its cotangent bundle T^*M has the structure of a *Lie algebroid*. We will define and describe these in detail later in Section 3.2, but for now a short description will have to suffice; a Lie algebroid A over a manifold M is a vector bundle $A \to M$ together with a Lie bracket $[\cdot,\cdot]_A$ on its space of smooth section $\Gamma(A)$ and a vector bundle map $\rho : A \to TM$, called the *anchor map*, satisfying the Leibniz identity

$$[\alpha, f\beta]_A = f[\alpha,\beta]_A + \mathcal{L}_{\rho(\alpha)}(f)\beta \tag{2.14}$$

for all $\alpha, \beta \in \Gamma(A)$ and $f \in C^\infty(M)$. It is a general fact about Lie algebroids (see Proposition 3.17) that $\rho : \Gamma(A) \to \mathfrak{X}(M)$ is now a Lie algebra map, i.e. that $\rho([\alpha,\beta]_A) = [\rho(\alpha), \rho(\beta)]$ for all $\alpha, \beta \in \Gamma(A)$, where on the right hand side we have the Lie bracket of vector fields.

Let us now describe the data from which we build the Poisson Lie algebroid. Given a Poisson manifold (M, π), we define the following.

- The vector bundle will simply be $A = T^*M \to M$, so that $\Gamma(A) = \Omega^1(M)$.
- The anchor map will be $\rho = \pi^\# : T^*M \to TM$.
- We define a bracket $[\cdot,\cdot]_\pi$ on $\Omega^1(M)$ as follows:

$$[\alpha,\beta]_\pi := \mathcal{L}_{\pi^\#(\alpha)}(\beta) - \mathcal{L}_{\pi^\#(\beta)}(\alpha) - d\big(\pi(\alpha,\beta)\big) \tag{2.15}$$

  for $\alpha, \beta \in \Omega^1(M)$.

Remark 2.17. It is easily verified that for exact 1-forms we have

$$[df, dg]_\pi = d\{f, g\}, \tag{2.16}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket corresponding to π. In fact, (2.16) determines $[\cdot,\cdot]_\pi$ uniquely.

Proposition 2.18. *The triple $(T^*M, [\cdot,\cdot]_\pi, \pi^\#)$ is a Lie algebroid.*

Proof. It is clear that we only need to check the Jacobi identity for $[\cdot,\cdot]_\pi$ and the Leibniz identity (2.14). The latter is easily checked:

$$\begin{aligned}
[\alpha, f\beta]_\pi &= \mathcal{L}_{\pi^\#(\alpha)}(f\beta) - \mathcal{L}_{\pi^\#(f\beta)}(\alpha) - d\big(\pi(\alpha, f\beta)\big)\\
&= \mathcal{L}_{\pi^\#(\alpha)}(f)\beta + f\mathcal{L}_{\pi^\#(\alpha)}(\beta) - f\mathcal{L}_{\pi^\#(\beta)}(\alpha) - df \wedge i_{\pi^\#(\beta)}(\alpha) - d\big(f\pi(\alpha,\beta)\big)\\
&= \mathcal{L}_{\pi^\#(\alpha)}(f)\beta + f\mathcal{L}_{\pi^\#(\alpha)}(\beta) - f\mathcal{L}_{\pi^\#(\beta)}(\alpha) - df \wedge \pi(\beta,\alpha) - df \wedge \pi(\alpha,\beta) - fd\big(\pi(\alpha,\beta)\big)\\
&= f[\alpha,\beta]_\pi + \mathcal{L}_{\pi^\#(\alpha)}(f)\beta
\end{aligned}$$

for all $\alpha, \beta \in \Omega^1(M)$ and $f \in C^\infty(M)$. For the Jacobi identity we will not go into detail here, but the general idea is to first note that by (2.16), Jacobi holds for exact 1-forms, and to then use the Leibniz identity to deduce that it must hold for all 1-forms. □

It is a general fact about Lie algebroids that the following is well-defined (see Lemma 3.21).

Definition 2.19. The *isotropy Lie algebra* at $x \in M$ is $(\mathfrak{g}_x, [\cdot,\cdot]_{\mathfrak{g}_x})$, where $\mathfrak{g}_x := \ker(\pi_x^\#)$ and where $[\cdot,\cdot]_{\mathfrak{g}_x}$ is the Lie bracket uniquely determined by the fact that

$$[\alpha(x), \beta(x)]_{\mathfrak{g}_x} = [\alpha, \beta]_\pi(x)$$

for all $\alpha, \beta \in \Omega^1(M)$ such that $\alpha(x), \beta(x) \in \mathfrak{g}_x$. $\diamondsuit$

We actually have an alternative description of the isotropy Lie algebras in terms of the symplectic foliation. Indeed, let $x \in M$ and let S be the symplectic leaf through x. Then since $T_xS = \mathrm{im}(\pi_x^\#)$, it is immediate that $\mathfrak{g}_x = \nu_x^*(S) = (T_xS)^\circ$, the conormal space to S at x.

2.3 Regular Poisson manifolds

In this thesis we will mostly be working with *regular* Poisson manifolds. To explain what this means, we need the following definition.

Definition 2.20. For a Poisson manifold (M, π), the *rank* of π at $x \in M$ is just the rank of $\pi_x^\#$, i.e. the dimension of $\mathrm{im}(\pi_x^\#) \subset T_xM$. $\diamondsuit$

One can show that the rank at any point must be even, and that the rank of π must be constant along the symplectic leaves (in particular, it must be equal to the dimension of the leaf in question); indeed, in view of Theorem 2.15 these facts must necessarily hold.

Definition 2.21. A Poisson manifold (M, π) is called *regular* if the rank of π is constant along M. In this case, the rank of the Poisson manifold is defined to be the rank of π at any point. $\diamondsuit$

The results about the symplectic foliation can be proven in a much easier way for regular Poisson manifolds. Indeed, if the rank of $\pi_x^\#$ is constant, it is easy to see that $\pi^\#(T^*M) \subset TM$ is a smooth distribution. In fact, this distribution is involutive, which follows from Proposition 2.3 since $\pi^\#(T^*M)$ is spanned by Hamiltonian vector fields. But then Theorem 2.15 just follows from the Frobenius theorem.

A very important fact is the following. Say we are given a foliation on a manifold, with a leafwise symplectic form (i.e. a collection of symplectic forms on the leaves which "varies smoothly"). One might wonder whether this patches together to a Poisson structure, and it turns out it does.

Proposition 2.22 ([FM15, Proposition 3.11])**.** *Given a foliation $\mathcal{F}$ on a manifold M, together with a leafwise symplectic form, there exists a unique regular Poisson structure on M whose symplectic foliation is $\mathcal{F}$.*

In fact, a similar result holds for singular foliations, but we will not go into that here.

The importance of this proposition is that it gives us an alternative way of building Poisson manifolds, one that we will use in our construction of PMSCT.

Before we move on, let us mention one more important result regarding regular Poisson manifolds.

Lemma 2.23. *Let (M, π) be a regular Poisson manifold. Then for all $x \in M$, the isotropy Lie algebra $\mathfrak{g}_x$ is abelian.*

Proof. Let $\alpha, \beta \in \mathfrak{g}_x$ and choose $f, g \in C^\infty(M)$ such that $df(x) = \alpha$, $dg(x) = \beta$. Since (M, π) is regular, we can choose f and g in such a way that there exists a neighbourhood $U \subset M$ of x such that for all $y \in U$ we have

$$df(y), dg(y) \in \nu_y^*(S_y)$$

where S_y denotes the symplectic leaf through y. But then we have

$$\{f, g\} = dg(X_f) \equiv 0$$

on U since X_f is tangent to the symplectic leaves. Hence

$$[\alpha, \beta]_{\mathfrak{g}_x} = [df, dg]_\pi(x) = d\{f, g\}(x) = 0$$

and we see that $\mathfrak{g}_x$ is abelian. □

2.4 Examples

In this section we will recall some of the examples we mentioned earlier and relate them to the notions we introduced after. We also provide a new, interesting class of examples; the linear Poisson structures.

Example 2.24. On any manifold M we have the zero Poisson structure $\pi = 0$ (obviously corresponding to the abelian Lie algebra structure on $C^\infty(M)$). The Poisson Lie algebroid T^*M becomes quite trivial, since $[\cdot, \cdot]_\pi = 0$ and $\pi^\# = 0$. Hence the isotropy Lie algebras $\mathfrak{g}_x = T_x^*M$ are abelian for all $x \in M$. The symplectic foliation is just the collection of singletons $\big\{\{x\} \mid x \in M\big\}$. △

Example 2.25. We already know that on $\mathbb{R}^n$ any bivector π has the form

$$\pi = \sum_{i<j} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

and that this is a Poisson bivector iff it satisfies (2.6). We also mentioned already a special case of this, namely the case that the functions π_{ij} are constant. For a more interesting example, we can look at the case where they are instead linear, i.e.

$$\pi_{ij} = \sum_{k=1}^n c_{ij}^k x_k.$$

It is easy to see that (2.6) is now equivalent to the constants c_{ij}^k being structure constants of a Lie algebra structure on $\mathbb{R}^n$.[2] In other words, there is a one-to-one correspondence between these "linear" Poisson structures on $\mathbb{R}^n$ and Lie algebra structures on $\mathbb{R}^n$.

There is a more general, coordinate-free way of describing linear Poisson structures. Indeed, let $(\mathfrak{g}, [\cdot, \cdot])$ be any (finite dimensional) Lie algebra. We can define a Poisson bracket on $\mathfrak{g}^*$ as follows. Since we can canonically identify $T_\xi^*(\mathfrak{g}^*) \cong (\mathfrak{g}^*)^* \cong \mathfrak{g}$ for all $\xi \in \mathfrak{g}^*$, we can view $df(\xi)$ as an element of $\mathfrak{g}$ for all $f \in C^\infty(\mathfrak{g}^*)$. With this in mind, we define a bracket by

$$\{f, g\}_{\mathfrak{g}^*}(\xi) := \xi\big([df(\xi), dg(\xi)]\big), \quad \xi \in \mathfrak{g}^*, f, g \in C^\infty(\mathfrak{g}^*).$$

[2] Recall that a Lie bracket $[\cdot, \cdot]$ is determined by its structure constants $[e_i, e_j] = c_{ij}^k e_k$, where $\{e_i\}$ is a basis.

It follows immediately that this is in fact a Poisson bracket. It can easily be checked that, after choosing coordinates, this comes down to the situation on $\mathbb{R}^n$ we described above.

Now, the cotangent bundle of $\mathfrak{g}^*$ is identified with $\mathfrak{g}^* \times \mathfrak{g}$, and the space of sections then becomes $C^\infty(\mathfrak{g}^*, \mathfrak{g})$, the space of smooth functions from $\mathfrak{g}^*$ to $\mathfrak{g}$. Letting π be the bivector associated to $\{\cdot,\cdot\}_{\mathfrak{g}^*}$, it follows that the bracket $[\cdot,\cdot]_\pi$ on $C^\infty(\mathfrak{g}^*, \mathfrak{g})$ is given by

$$[f,g]_\pi(\xi) = [f(\xi), g(\xi)], \quad \xi \in \mathfrak{g}^*, f, g \in C^\infty(\mathfrak{g}^*, \mathfrak{g}),$$

while the anchor $\pi^\#_\xi : \mathfrak{g} \to \mathfrak{g}^*$ at any $\xi \in \mathfrak{g}^*$ is given by

$$v \mapsto -\mathrm{ad}^*_v(\xi).$$

Recall here that $\mathrm{ad}^*_v(\xi) = -\xi \circ \mathrm{ad}_v = -\xi \circ [v, \cdot]$. The isotropy Lie algebra at $\xi \in \mathfrak{g}^*$ is given by

$$\mathfrak{g}_\xi = \{v \in \mathfrak{g} \mid \mathrm{ad}^*_v(\xi) = 0\},$$

with Lie bracket inherited from $\mathfrak{g}$.

For the symplectic leaves, we have the following nice description. Choosing a connected Lie group G integrating $\mathfrak{g}$, the symplectic leaves of $\mathfrak{g}^*$ are exactly the coadjoint orbits of G. $\triangle$

Example 2.26. As we mentioned before, every symplectic manifold (S, ω) comes with an induced Poisson structure. In the context of Poisson bivectors, this Poisson structure can be expressed quite nicely. Indeed, we know that interior multiplication yields an isomorphism $\omega^\flat : TM \to T^*M$. Hence we can define a bivector $\pi \in \mathfrak{X}^2(S)$ by setting $\pi^\# := \left(\omega^\flat\right)^{-1}$; by the Koszul formula we see that the condition $[\pi, \pi] = 0$ is equivalent to $d\omega = 0$, which implies that π is actually a Poisson bivector. It is easy to see that this is the same Poisson structure as the one we described in Example 2.2(iii). From the construction above it is not surprising that π is sometimes called the "inverse of ω".

The class of Poisson manifolds that come from symplectic manifolds can also be described in a simple manner: we say that a Poisson manifold (M, π) is *nondegenerate* if $\pi^\# : T^*M \to TM$ is an isomorphism of vector bundles. Inverting the procedure above we then obtain a symplectic form $\omega \in \Omega^2(M)$, whose induced Poisson structure is clearly π. To summarise, on every manifold there is a one-to-one correspondence between symplectic structures and nondegenerate Poisson structures.

The other viewpoints become quite simple in the nondegenerate case; the anchor map of the Poisson Lie algebroid T^*M is an isomorphism, and hence the bracket on $\Omega^1(M)$ is the one corresponding, through this isomorphism, to the standard Lie bracket of vector fields. The isotropy Lie algebras are all trivial. Finally, the symplectic leaves are just the connected components of M. $\triangle$

3 Lie groupoids and Lie algebroids

In this section we introduce Lie groupoids and Lie algebroids, which generalise Lie groups and Lie algebras. We will discover that, as in the case of the latter, there is a "global versus infinitesimal" principle present; to every Lie groupoid we can associate a Lie algebroid, and thus the question arises of when a Lie algebroid integrates to a Lie groupoid. Applying these new concepts to Poisson manifolds, we see that the global counterpart of a Poisson manifold is a so-called *symplectic groupoid*. For the introduction to Lie groupoids and algebroids we mainly follow [MM03, Sections 5 & 6], and for the discussion about integrability we follow [CF03] and [CF04].

3.1 Lie groupoids

As mentioned above, a Lie groupoid is a generalisation of a Lie group, just as a Lie algebroid (which we briefly introduced in Section 2.2.3) generalises a Lie algebra. A short, informal description of a groupoid is the following: a group in which not every two elements are necessarily multiplicable. For those familiar with category theory, there is actually a very short *formal* definition of a groupoid.

Definition 3.1. A *groupoid* is a small category all whose arrows are invertible. $\diamondsuit$

For clarity, we will now explicitly describe all the data present: a groupoid, usually denoted $\mathcal{G} \rightrightarrows M$, consists mainly of a set M of *objects* and a set $\mathcal{G}$ of *arrows*. Moreover, there are several *structure maps*:

- Closely related are the *source map* $\mathbf{s} : \mathcal{G} \to M$ and the *target map* $\mathbf{t} : \mathcal{G} \to M$, which associate to an arrow $g \in \mathcal{G}$ its source and target. When $\mathbf{s}(g) = x$ and $\mathbf{t}(g) = y$, we say that g is an arrow from x to y, and we write $g : x \to y$ or $y \xleftarrow{g} x$.

- As mentioned above, not every two arrows can be multiplied (or *composed*); two arrows g and h are composable precisely when $\mathbf{s}(g) = \mathbf{t}(h)$. Hence, setting

$$\mathcal{G}_2 := \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid \mathbf{s}(g) = \mathbf{t}(h)\} \subset \mathcal{G} \times \mathcal{G},$$

  we have a *composition map* $m : \mathcal{G}_2 \to \mathcal{G}$. We often denote $m(g, h) = gh$. As should be clear intuitively, we require that $\mathbf{s}(gh) = \mathbf{s}(h)$ and that $\mathbf{t}(gh) = \mathbf{t}(g)$. In a picture, this becomes

$$z \xleftarrow{g} y \xleftarrow{h} x \implies z \xleftarrow{gh} x.$$

  As in the case of groups, this multiplication must be associative: for all $g, h, k \in \mathcal{G}$ such that $(g, h), (h, k) \in \mathcal{G}_2$ we must have $(gh)k = g(hk)$.

- Unlike in a group, in a groupoid there are actually multiple units: for every object $x \in M$ there is a unit 1_x, for which we have $\mathbf{s}(1_x) = \mathbf{t}(1_x) = x$, and of course $g1_x = g$ for all $g \in \mathbf{s}^{-1}(x)$ and $1_x g = g$ for all $g \in \mathbf{t}^{-1}(x)$. This defines the *unit map* $u : M \to \mathcal{G}, u(x) := 1_x$.

- Lastly, we have the *inverse map* $i : \mathcal{G} \to \mathcal{G}$, which we denote $i(g) = g^{-1}$. If g is an arrow from x to y, g^{-1} is an arrow from y to x, and we have $gg^{-1} = 1_y$ and $g^{-1}g = 1_x$.

Definition 3.2. Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ be groupoids. A *morphism of groupoids* between the two is a pair of maps $\Phi : \mathcal{G} \to \mathcal{H}$ and $\varphi : M \to N$ that is compatible with the structure maps. Explicitly:

- Compatibility with source and target: for $\mathcal{G} \ni g : x \to y$ we have $\Phi(g) : \varphi(x) \to \varphi(y)$.
- Compatibility with composition: for $(g, h) \in \mathcal{G}_2$ we have $\Phi(gh) = \Phi(g)\Phi(h)$.
- Compatibility with units: for $x \in M$ we have $\Phi(1_x) = 1_{\varphi(x)}$.
- Compatibility with inversion: for $g \in G$ we have $\Phi(g^{-1}) = \Phi(g)^{-1}$.

When $M = N$ and $\varphi = \mathrm{id}_M$ we call Φ a morphism of groupoids over M. $\diamondsuit$

Remark 3.3. Note that, like in the case of groups, the fourth condition actually follows from the first three.

Before we move on, we will mention some more useful notions associated to a groupoid $\mathcal{G} \rightrightarrows M$. First of all, just like in a group, we can talk about left or right multiplication by an arrow $g \in \mathcal{G}$. However, unlike in a group, this does not work on the entire set $\mathcal{G}$, but only on target and source fibres, respectively. To be more specific, let $\mathcal{G} \ni g : x \to y$. Then right multiplication by g gives us a bijection $R_g : \mathbf{s}^{-1}(y) \to \mathbf{s}^{-1}(x)$, and left multiplication gives us a bijection $L_g : \mathbf{t}^{-1}(x) \to \mathbf{t}^{-1}(y)$.

Next, for every $x \in M$ we have the *isotropy group at* x, defined as $\mathcal{G}_x := \mathbf{s}^{-1}(x) \cap \mathbf{t}^{-1}(x)$: it is the set of arrows that start and end at x. As might be expected from the name, this becomes a group when we restrict the multiplication from the groupoid.

Finally, the groupoid structure also induces a partition of the base M: there is an obvious equivalence relation on M, defined by declaring $x, y \in M$ equivalent iff there exists some $g \in \mathcal{G}$ such that $\mathbf{s}(g) = x$ and $\mathbf{t}(g) = y$. For $x \in M$, we denote the corresponding equivalence class by $\mathcal{O}_x$. Note that $\mathcal{O}_x = \mathbf{t}(\mathbf{s}^{-1}(x))$.

Let us now define Lie groupoids: they are essentially just "smooth" groupoids.

Definition 3.4. A *Lie groupoid* is a groupoid $\mathcal{G} \rightrightarrows M$ such that $\mathcal{G}$ and M are (smooth) manifolds, the structure maps $\mathbf{s}, \mathbf{t}, m, u, i$ are smooth, and $\mathbf{s}$ and $\mathbf{t}$ are, in addition, also submersions. A morphism of Lie groupoids is just a morphism of groupoids, such that both maps involved are smooth. $\diamondsuit$

Remark 3.5. Note that the condition that $\mathbf{s}$ and $\mathbf{t}$ are submersions ensures that the source and target fibres are all submanifolds of $\mathcal{G}$, and also that $\mathcal{G}_2$ is a submanifold of $\mathcal{G} \times \mathcal{G}$. The smoothness condition imposed on m is with respect to the submanifold structure on $\mathcal{G}_2$.

Remark 3.6. In general, one does not require $\mathcal{G}$ to be Hausdorff (or 2^{nd} countable), only the base M and the source and target fibres. However, as we shall see, the groupoids we will encounter in this thesis are all automatically Hausdorff, and hence we will not need to worry about this.

The following proposition establishes some basic facts about Lie groupoids.

Proposition 3.7. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Then the following hold.*

(i) The unit map $u : M \to \mathcal{G}$ is an embedding.

(ii) For every $x \in M$, the isotropy group $\mathcal{G}_x$ is a submanifold of $\mathcal{G}$, and hence becomes a Lie group.

(iii) For every $x \in M$, the orbit $\mathcal{O}_x$ admits a unique smooth structure such that it becomes an immersed submanifold of M, and such that $\mathbf{t} : \mathbf{s}^{-1}(x) \to \mathcal{O}_x$ becomes a principal $\mathcal{G}_x$-bundle.

Part (i) is not difficult. For a proof of parts (ii) and (iii), see [MM03, Theorem 5.4].

In practice, one often restricts to a class of Lie groupoids whose source fibres are more well-behaved.

Definition 3.8. A Lie groupoid is called *source connected* if all its source fibres are connected and it is called *source 1-connected* if they are also simply connected. $\diamondsuit$

We will now give some important examples.

Example 3.9. The simplest example of a Lie groupoid is obtained by letting the base M consist of a single point, $M = \{\star\}$. In this case, $\mathcal{G}$ is just a Lie group, without complications arising from different sources or targets. We see that Lie groups are exactly Lie groupoids over a point. Of course, source (1-)connectedness of the groupoid is equivalent to (1-)connectedness of the Lie group $\mathcal{G}$. $\triangle$

Example 3.10. A more interesting example is a so-called *bundle of Lie groups*; this is a Lie groupoid $\mathcal{G} \rightrightarrows M$ for which the source and target maps coincide, $\mathbf{s} = \mathbf{t}$. In this case, $\mathcal{G}$ is the disjoint union of the isotropy groups $\mathcal{G}_x$ ($x \in M$), and the orbit through any point is just the point itself. A good example of a bundle of Lie groups is a vector bundle $E \xrightarrow{\pi} M$, with $\mathbf{s} = \mathbf{t} = \pi$, the unit map equal to the zero section and the group structure just being the abelian group structure underlying the vector spaces E_x ($x \in M$). Clearly, every vector bundle is source 1-connected. $\triangle$

Example 3.11. The simplest example with nontrivial source and target maps is the *pair groupoid*. Letting M be any manifold, we obtain a Lie groupoid $\mathcal{G} \rightrightarrows M$ by setting $\mathcal{G} := M \times M$, $\mathbf{s} = \mathrm{pr}_2$, $\mathbf{t} = \mathrm{pr}_1$, $m : \big((z,y),(y,x)\big) \mapsto (z,x)$, $u : x \mapsto (x,x)$ and $i : (y,x) \mapsto (x,y)$. The intuitive definition is more clear: we simply think of a pair (y,x) as being an arrow from x to y. The previous definitions should be clear when keeping this intuitive picture in mind.

Now, we verify immediately that the isotropy groups are just the singletons $\mathcal{G}_x = \{(x,x)\}$ ($x \in M$), while there is a single orbit, M itself. The pair groupoid is source (1-)connected iff M itself is (1-)connected. $\triangle$

Example 3.12. We can make the previous example more interesting, and perhaps even more intuitive, by considering the *fundamental groupoid* of a manifold M, usually denoted $\Pi_1(M)$ or just $\Pi(M)$. The space $\mathcal{G}$ of arrows now consists almost literally of arrows: its elements are homotopy classes of paths in M (with fixed end points): for such a class $[\gamma] \in \mathcal{G}$, we set $\mathbf{s}([\gamma]) := \gamma(0)$ and $\mathbf{t}([\gamma]) := \gamma(1)$. Composition is simply concatenation of paths, and for $x \in M$ we define $u(x)$ to be the homotopy class of the constant path at x. Finally, inversion is the usual "inverse reparametrisation" of a path. Clearly, this data determines a groupoid structure; since M is a manifold, it can actually (naturally) be made into a Lie groupoid.

Many well-known concepts arise when we look into the groupoid structure; indeed, the isotropy group at some $x \in M$ is the fundamental group $\pi_1(M, x)$, and the orbit through x is the path component of M containing x. The fundamental groupoid is always source 1-connected: indeed, for $x \in M$, $\mathbf{s}^{-1}(x)$ is the universal cover of the path component of M containing x.

As a nice remark, note that this example is closely related to the previous one: when M is 1-connected, every two paths with the same end points are homotopic to each other: hence for any two points in M there is exactly one homotopy class of paths between them. Clearly, the fundamental groupoid coincides, in this case, with the pair groupoid. △

Example 3.13. Another interesting example is the *action groupoid*. Let M be a manifold and let G be a Lie group acting smoothly on M, say from the left. We obtain a Lie groupoid over M, usually denoted $G \ltimes M$, with space of arrows $G \times M$, and the following structure maps:

$$\begin{aligned}
&\mathbf{s} : (g, x) \mapsto x, \\
&\mathbf{t} : (g, x) \mapsto gx, \\
&m : \big((h, gx), (g, x)\big) \mapsto (hg, x), \\
&u : x \mapsto (e, x), \\
&i : (g, x) \mapsto (g^{-1}, gx).
\end{aligned}$$

Intuitively, one should think of a pair (g, x) as an arrow pointing from x to gx, i.e. as a visualisation of how g acts on x. The isotropy groups and orbits coincide with the ones we know from the theory of group actions: for $x \in M$, $\mathcal{G}_x$ consists of pairs (g, x) for which also $\mathbf{t}(g, x) = gx = x$, and thus it is identified with the standard isotropy group $\{g \in G \mid gx = x\}$, and the orbit through x is simply $\mathcal{O}_x = \mathbf{t}(\mathbf{s}^{-1}(x)) = \{gx \mid g \in G\}$, which is exactly the standard orbit we obtain from the group action. Finally, source (1-)connectedness of the action groupoid is clearly equivalent to the group G being (1-)connected. △

3.2 Lie algebroids

We already briefly discussed Lie algebroids in Section 2.2.3; let us recall the definition.

Definition 3.14. Let M be a manifold. A *Lie algebroid* over M is a triple $(A, [\cdot, \cdot]_A, \rho)$ consisting of a vector bundle $A \to M$, a Lie bracket $[\cdot, \cdot]_A$ on its space of sections $\Gamma(A)$ and a vector bundle map $\rho : A \to TM$, called the *anchor map*, such that the Leibniz identity

$$[\alpha, f\beta]_A = f[\alpha, \beta]_A + \mathcal{L}_{\rho(\alpha)}(f)\beta \tag{3.1}$$

holds for all $\alpha, \beta \in \Gamma(A)$ and $f \in C^\infty(M)$. ◇

Remark 3.15. One can think of a Lie algebroid as a "generalised tangent bundle": it is a vector bundle over M with a Lie bracket on its space of sections, just like for vector fields, and the anchor map ρ relates the vector bundle A to TM in a concrete way. The Leibniz identity (3.1) is the obvious analogue of the relation

$$[X, fY] = f[X, Y] + \mathcal{L}_X(f)Y$$

we know holds for vector fields $X, Y \in \mathfrak{X}(M)$ and functions $f \in C^\infty(M)$.

Remark 3.16. Sometimes we will denote a Lie algebroid just by $A \to M$; often the base manifold is more important to explicitely mention than the bracket and the anchor.

The following might have been expected in the definition, but is actually a consequence of the other properties.

Proposition 3.17. *Let $(A, [\cdot,\cdot]_A, \rho)$ be a Lie algebroid. Then the anchor map $\rho : \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra map, i.e.*

$$\rho([\alpha, \beta]_A) = [\rho(\alpha), \rho(\beta)]$$

for all $\alpha, \beta \in \Gamma(A)$, where on the right hand side we have the standard Lie bracket of vector fields.

Remark 3.18. In fact, if we let $(A, [\cdot,\cdot]_A, \rho)$ be a triple consisting of a vector bundle $A \to M$, a bilinear, skew-symmetric bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$ and a vector bundle map $\rho : A \to TM$, satisfying the relation (3.1), then it is easy to show that the Jacobi identity for $[\cdot,\cdot]_A$ is equivalent to $\rho : \Gamma(A) \to \mathfrak{X}(M)$ preserving brackets.

The notion of a morphism of Lie algebroids *over the same base* is quite clear.

Definition 3.19. Let $(A_1, [\cdot,\cdot]_{A_1}, \rho_1)$ and $(A_2, [\cdot,\cdot]_{A_2}, \rho_2)$ be Lie algebroids over the same base M. A *morphism of Lie algebroids* is a vector bundle map $F : A_1 \to A_2$ (covering the identity) that interwines the anchors;

$$\begin{tikzcd} \Gamma(A_1) \arrow[rr, "F"] \arrow[dr, "\rho_1"'] & & \Gamma(A_2) \arrow[dl, "\rho_2"] \\ & TM & \end{tikzcd} \tag{3.2}$$

and which is a Lie algebra map $F : \Gamma(A_1) \to \Gamma(A_2)$, i.e. which preserves the brackets $[\cdot,\cdot]_{A_1}$ and $[\cdot,\cdot]_{A_2}$. ♢

Remark 3.20. For Lie algebroids over different bases M_1 and M_2 there is still the notion of a morphism, but there is a complication. Obviously, we now need to consider a pair (F, f), where $F : A_1 \to A_2$ is a vector bundle map covering $f : M_1 \to M_2$. Compatibility with the anchors is quite clear; the diagram (3.2) now becomes

$$\begin{tikzcd} \Gamma(A_1) \arrow[r, "F"] \arrow[d, "\rho_1"'] & \Gamma(A_2) \arrow[d, "\rho_2"] \\ TM_1 \arrow[r, "df"] & TM_2 \end{tikzcd}$$

but the problem lies with the compatibility of the brackets. Indeed, a section of A_1 can not always be pushed forward to a section of A_2, and so the condition becomes quite ugly. Since we do not need it in this thesis, we will not formulate it here.

Before we move on to examples, we mention some important notions associated to a general Lie algebroid. It turns out that every fibre A_x has a subspace which inherits a Lie algebra structure from the bracket $[\cdot,\cdot]_A$. We denote $\mathfrak{g}_x := \ker(\rho_x) \subset A_x$ for $x \in M$.

Lemma 3.21. *For every $x \in M$ there is a Lie bracket $[\cdot,\cdot]_{\mathfrak{g}_x}$ on $\mathfrak{g}_x$, uniquely determined by the fact that*

$$[\alpha(x), \beta(x)]_{\mathfrak{g}_x} = [\alpha, \beta]_A(x) \tag{3.3}$$

for all $\alpha, \beta \in \Gamma(A)$ such that $\alpha(x), \beta(x) \in \mathfrak{g}_x$.

Proof. Clearly, we can take (3.3) as a definition, and all we need to is show that this is independent of the choice of α and β: the Lie algebra properties then follow directly from the corresponding ones of $[\cdot,\cdot]_A$. The key remark is that by the Leibniz identity (3.1) we have

$$[\alpha, f\beta]_A(x) = f(x)[\alpha, \beta]_A(x)$$

for all $f \in C^\infty(M)$ and all $\alpha, \beta \in \Gamma(A)$ such that $\alpha(x), \beta(x) \in \mathfrak{g}_x = \ker(\rho_x)$. Well-definedness of (3.3) then follows from a standard argument in differential geometry. $\square$

Now we can state the following definition.

Definition 3.22. For $x \in M$, $(\mathfrak{g}_x, [\cdot,\cdot]_{\mathfrak{g}_x})$ is called the *isotropy Lie algebra of* A at x. $\diamondsuit$

Just like with Lie groupoids, a Lie algebroid $(A, [\cdot,\cdot]_A, \rho)$ induces a partition of the base manifold M. More precisely, we have a partition of M into connected, immersed submanifolds, which satisfy

$$T_x\mathcal{O}_x = \operatorname{im}(\rho_x) \tag{3.4}$$

for all $x \in M$, where $\mathcal{O}_x$ denotes the member of the partition containing x (often called the orbit through x). We will not give the proof of this statement, but if the Lie algebroid is regular, meaning that the rank of ρ_x is constant, it easily follows from the Frobenius theorem, as in Section 2.3.

Let us now give some examples.

Example 3.23. As before, for the simples example we look at the case $M = \{\star\}$. A vector bundle A over a point is of course just a vector space, and its space of sections $\Gamma(A)$ is identified with A itself. Hence A inherits a Lie algebra structure. The Leibniz identity is always satisfied, so that we can conclude that a Lie algebroid over a point is just a Lie algebra. $\triangle$

Example 3.24. Next, we consider a so-called *bundle of Lie algebras*; this is a Lie algebroid $A \to M$ for which the anchor map ρ is just the zero map. In this case, we have $\mathfrak{g}_x = A_x$, and thus the fibres A_x are all Lie algebras themselves, hence the name. For every $x \in M$ we clearly have $\mathcal{O}_x = \{x\}$. $\triangle$

Example 3.25. Another quite trivial example is the tangent bundle itself: for a manifold M, we can form a Lie algebroid by setting $A = TM$, letting $[\cdot,\cdot]_A$ be the standard Lie bracket of vector fields, and setting $\rho = \mathrm{id} : TM \to TM$. As we already mentioned, the Leibniz identity is automatically satisfied, and thus the above data defines a Lie algebroid, called the *tangent Lie algebroid*. The orbits are now precisely the connected components of M. $\triangle$

Example 3.26. For a more interesting example, consider a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ and assume that it acts on a manifold M, i.e. that we have a Lie algebra map $\rho : \mathfrak{g} \to \mathfrak{X}(M)$. We form a Lie algebroid as follows: the bundle $A = M \times \mathfrak{g}$ is simply the trivial vector bundle over M with fibre $\mathfrak{g}$. The anchor $M \times \mathfrak{g} \to TM$ is induced by the action ρ in the obvious way. The bracket is defined as

$$[\alpha, \beta]_A(x) := [\alpha(x), \beta(x)]_{\mathfrak{g}} + \nabla_{\rho(\alpha)}(\beta) - \nabla_{\rho(\beta)}(\alpha), \qquad x \in M, \alpha, \beta \in \Gamma(A).$$

Here ∇ denotes the canonical flat connection on A. This Lie algebroid is called the *action Lie algebroid* and is usually denoted $\mathfrak{g} \ltimes M$. $\triangle$

Remark 3.27. It is easily checked that the above bracket is the unique Lie bracket which satisfies the Leibniz identity and for which we have

$$[c_u, c_v]_A = c_{[u,v]_{\mathfrak{g}}}, \qquad u, v \in \mathfrak{g}.$$

Here $c_u \in \Gamma(A)$ denotes the constant section with value $u \in \mathfrak{g}$.

Let us not forget the most important example, at least in the context of this thesis.

Example 3.28. We recall from Section 2.2.3 the construction of the *Poisson Lie algebroid* of a Poisson manifold (M, π): the vector bundle is the cotangent bundle T^*M, the anchor is $\pi^\# : T^*M \to TM$, and the bracket $[\cdot, \cdot]_\pi$ on $\Omega^1(M)$ is given by

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\#(\alpha)}(\beta) - \mathcal{L}_{\pi^\#(\beta)}(\alpha) - d\big(\pi(\alpha, \beta)\big), \qquad \alpha, \beta \in \Omega^1(M).$$

The foliation induced by this Lie algebroid is of course just the symplectic foliation of the Poisson manifold (M, π), and the isotropy Lie algebra at a point $x \in M$ is $\nu_x^*(S)$, the conormal space to the symplectic leaf S through x. $\triangle$

3.2.1 The Lie algebroid of a Lie groupoid

The definitions and examples given above indicate that Lie groupoids and Lie algebroids indeed generalise the notions of Lie groups and Lie algebras. It should therefore not be a surprise that the well-known "Lie functor", which associates to a Lie group its Lie algebra, also exists for Lie groupoids and Lie algebroids. In this section, we will describe this functor in detail.

First, it is useful to recall the construction of the Lie algebra of a Lie group, since the Lie algebroid of a Lie groupoid is constructed similarly. So let G be a Lie group. Then its associated Lie algebra, denoted $\mathrm{Lie}(G)$, has as underlying vector space the tangent space at the identity T_1G. The bracket is obtained by noting that T_1G is isomorphic to the space of right-invariant vector fields on G; the bracket on the latter space is then transported to T_1G, so that it becomes a Lie algebra.

Now, let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. We construct a Lie algebroid $(A, [\cdot, \cdot]_A, \rho)$ over M as follows. First, the vector bundle $A \to M$ is defined as

$$A := u^*\big(\ker(d\mathbf{s})\big).$$

Specifically, this means that the fibre at $x \in M$ is given by $A_x = T_{1_x}\big(\mathbf{s}^{-1}(x)\big)$. Next, the anchor map at $x \in M$ is given by

$$\rho_x := d\mathbf{t}(1_x) : T_{1_x}\big(\mathbf{s}^{-1}(x)\big) \to T_xM,$$

i.e. the map obtained by first restricting $d\mathbf{t} : T\mathcal{G} \to TM$ to $\ker(d\mathbf{s})$ and then pulling the restricted map back via u. For the bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$, we use a similar trick as with Lie groups and Lie algebras. However, the situation here is a bit trickier; as already mentioned, we do not have globally defined right multiplication. For $\mathcal{G} \ni g : x \to y$, the right multiplication by g is a diffeomorphism

$$R_g : \mathbf{s}^{-1}(y) \to \mathbf{s}^{-1}(x).$$

This inspires the following definition.

Definition 3.29. A vector field $X \in \mathfrak{X}(\mathcal{G})$ is called *right-invariant* if

(i) it is tangent to the $\mathbf{s}$-fibres, and

(ii) it satisfies

$$dR_g(h)\big(X_h\big) = X_{hg}$$

for all $(h, g) \in \mathcal{G}_2$.

The space of right-invariant vector fields on $\mathcal{G}$ is denoted $\mathfrak{X}^R(\mathcal{G})$. $\diamondsuit$

The first important remark is that $\mathfrak{X}^R(\mathcal{G})$ is closed under the Lie bracket $[\cdot,\cdot]$ of vector fields on $\mathcal{G}$. The second is that $\mathfrak{X}^R(\mathcal{G})$ is isomorphic to $\Gamma(A)$, through the following isomorphism: for $\alpha \in \Gamma(A)$, we define a vector field $\alpha^R \in \mathfrak{X}^R(\mathcal{G})$ by

$$\alpha^R(g) := dR_g(1_{\mathbf{t}(g)})\big(\alpha(\mathbf{t}(g))\big), \qquad g \in \mathcal{G}.$$

Putting the two remarks together we obtain a Lie bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$, determined by

$$\big([\alpha,\beta]_A\big)^R = [\alpha^R, \beta^R], \qquad \alpha, \beta \in \Gamma(A).$$

Proposition 3.30. *The above data $(A, [\cdot,\cdot]_A, \rho)$ defines a Lie algebroid over M, denoted* $\mathrm{Lie}(\mathcal{G})$.

Proof. Clearly, we only need to check the Leibniz identity. So let $\alpha, \beta \in \Gamma(A)$ and $f \in C^\infty(M)$. We need to show that

$$\big([\alpha, f\beta]_A\big)^R = \big(f[\alpha,\beta]_A\big)^R + \big(\mathcal{L}_{\rho(\alpha)}(f)\beta\big)^R.$$

There are three statements we need here.

(i) First of all, it is easily verified that

$$(g\gamma)^R = (g \circ \mathbf{t})\gamma^R$$

for all $g \in C^\infty(M)$ and $\gamma \in \Gamma(A)$. Applied to our case, this means that

$$(f\beta)^R = (f \circ \mathbf{t})\beta^R, \quad \big(f[\alpha,\beta]_A\big)^R = (f \circ \mathbf{t})\big([\alpha,\beta]_A\big)^R, \quad \big(\mathcal{L}_{\rho(\alpha)}(f)\beta\big)^R = \big(\big(\mathcal{L}_{\rho(\alpha)}(f)\big) \circ \mathbf{t}\big)\beta^R.$$

(ii) Secondly, we have that

$$\big(\mathcal{L}_{\rho(\alpha)}(f)\big) \circ \mathbf{t} = \mathcal{L}_{\alpha^R}(f \circ \mathbf{t}).$$

Indeed, for any $g \in \mathcal{G}$ we see that

$$\begin{aligned}
\mathcal{L}_{\alpha^R}(f \circ \mathbf{t})(g) &= d(f \circ \mathbf{t})(g)\big(\alpha^R(g)\big) \\
&= \big(df(\mathbf{t}(g)) \circ d\mathbf{t}(g) \circ dR_g(1_{\mathbf{t}(g)})\big)\big(\alpha(\mathbf{t}(g)\big) \\
&= \big(df(\mathbf{t}(g)) \circ d\mathbf{t}(1_{\mathbf{t}(g)})\big)\big(\alpha(\mathbf{t}(g)\big) \\
&= \mathcal{L}_{\rho(\alpha)}(f)(\mathbf{t}(g)).
\end{aligned}$$

(iii) Finally, we have the standard Leibniz rule for vector fields. Applied to our case, it says that

$$[\alpha^R, (f \circ \mathbf{t})\beta^R] = (f \circ \mathbf{t})[\alpha^R, \beta^R] + \mathcal{L}_{\alpha^R}(f \circ \mathbf{t})\beta^R.$$

Putting all this together, with also the definition of the bracket $[\cdot,\cdot]_A$, we see that the Leibniz identity indeed holds. □

Of course, to complete the functor, we need to associate to every morphism of Lie groupoids a morphism of Lie algebroids. We will only give the definition here. So let $(\Phi, \varphi) : (\mathcal{G}_1 \rightrightarrows M_1) \to (\mathcal{G}_2 \rightrightarrows M_2)$ be a morphism of Lie groupoids. Because Φ and φ are compatible with the source maps, the differential $d\Phi(1_x) : T_{1_x}\mathcal{G}_1 \to T_{1_{\varphi(x)}}\mathcal{G}_2$ restricts to a map between $\mathrm{Lie}(\mathcal{G}_1)_x \to \mathrm{Lie}(\mathcal{G}_2)_{\varphi(x)}$; this defines the morphism of Lie algebras, denoted $\mathrm{Lie}(\Phi)$.

Before we move on to some examples, let us mention two results which relate some of the concepts associated with Lie groupoids and algebroids. The proofs are not difficult.

Proposition 3.31. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, and let $A = \mathrm{Lie}(\mathcal{G})$ be its Lie algebroid. Then the following hold.*

(i) For all $x \in M$, the isotropy Lie algebra $\mathfrak{g}_x \subset A_x$ is the Lie algebra of the isotropy group $\mathcal{G}_x$.

(ii) If $\mathcal{G}$ is source connected, the orbits in M determined by the Lie groupoid coincide with those determined by the Lie algebroid.

As might be expected, some of the examples we gave are also related by the Lie functor.

Example 3.32.

(i) Of course, applied to a Lie group, the Lie functor just yields the Lie algebra of the Lie group in question.

(ii) If $\mathcal{G} \rightrightarrows M$ is a bundle of Lie groups, then $\mathrm{Lie}(\mathcal{G})$ is a bundle of Lie algebras (since $\mathbf{s} = \mathbf{t}$, the anchor $d\mathbf{t}$ vanishes on $\ker(d\mathbf{s})$). Looking back at Proposition 3.31(i), we can view the Lie functor, in this case, as fibrewise applying the classic Lie functor between Lie groups and Lie algebras.

(iii) For M any manifold, the Lie algebroids of both the pair groupoid and the fundamental groupoid are the tangent Lie algebroid of M. This is obvious for the pair groupoid, and for the fundamental groupoid we note that, for every $x \in M$, $\mathbf{s}^{-1}(x)$ is the universal cover of the connected component of M containing x, with the target map $\mathbf{t}$ as the canonical projection. Hence $d\mathbf{t}$ induces an isomorphism between $\mathrm{Lie}(\Pi_1(M))$ and TM.

(iv) If $\mathcal{G} = G \ltimes M$ is an action groupoid, then $\mathrm{Lie}(\mathcal{G})$ is the action Lie algebroid $\mathfrak{g} \ltimes M$, where $\rho : \mathfrak{g} \to \mathfrak{X}(M)$ is the infinitesimal action corresponding to the action of G on M, meaning that $\mathfrak{g}$ is the Lie algebra of G and that ρ is given by

$$\rho(v)_x = \left.\frac{d}{dt}\right|_{t=0} \exp(tv)x, \qquad v \in \mathfrak{g}, x \in M. \qquad \triangle$$

3.3 Integration of Lie algebroids

Now that we have a way of associating to a Lie groupoid a Lie algebroid, the natural question arises of whether, given a Lie algebroid, there exists a Lie groupoid integrating it.

Definition 3.33. A Lie algebroid $A \to M$ is called *integrable* if there exists a Lie groupoid $\mathcal{G} \rightrightarrows M$ (not necessarily Hausdorff, see Remark 3.6) such that $\mathrm{Lie}(\mathcal{G})$ is isomorphic to A. In this case, we also say that $\mathcal{G} \rightrightarrows M$ integrates A. Similar terminology holds for morphisms of Lie groupoids and Lie algebroids. $\diamondsuit$

Of course, this notion of "inverting the Lie functor" is well-known from the case of Lie groups and Lie algebras. In fact, a lot of the results carry over to Lie groupoids and Lie algebroids. The following are the clear analogues of Lie's first and second theorems for Lie algebras.

Theorem 3.34 (Lie I)**.** *If a Lie algebroid A is integrable, then there exists a unique (up to isomorphism) source 1-connected Lie groupoid integrating A.*

The idea of the construction here is fairly simple: given any Lie groupoid $\mathcal{G}$ integrating A, one builds a new groupoid whose source fibres are the universal covers of the source fibres of $\mathcal{G}$.

Theorem 3.35 (Lie II)**.** *Let $F : A_1 \to A_2$ be a morphism of Lie algebroids, and assume that A_1 and A_2 are integrated by Lie groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. If $\mathcal{G}_1$ is source 1-connected, then there exists a necessarily unique Lie groupoid morphism $\Phi : \mathcal{G}_1 \to \mathcal{G}_2$ integrating F.*

For proofs of these theorems, see [MM02, Propositions 3.3 & 3.5].

Of course, the best-known is Lie's third theorem for Lie algebras, stating that *every* (finite dimensional) Lie algebra is integrable by a Lie group. Unfortunately, this result does not hold for Lie algebroids; there exist Lie algebroids that are not integrable, not even by non-Hausdorff Lie groupoids. The upside is that the conditions for integrability are quite well-known. Indeed, as we will see in the next section, we can construct for every Lie algebroid $A \to M$ a "candidate integrating groupoid", called the *Weinstein groupoid*: this will be a groupoid which, if A is integrable, can be given a smooth structure such that it becomes the unique source 1-connected Lie groupoid integrating A. In other words, integrability of A is equivalent to this candidate groupoid being smooth and integrating it. The integrability criteria can then be made even more explicit using the so-called *monodromy groups* of A.

3.3.1 The Weinstein groupoid and monodromy groups

The basic idea is to define a special class of paths $[0,1] =: I \to A$, called A-paths, and a suitable notion of homotopy between them, called A-homotopy.

Definition 3.36. Let $A \xrightarrow{\pi} M$ be a Lie algebroid. An *A-path* is a path $a : I \to A$ such that

$$\rho(a(t)) = \frac{d\gamma}{dt}(t) \tag{3.5}$$

for all $t \in I$, where $\gamma := \pi \circ a : I \to M$. We denote the set of all A-paths by $\mathcal{P}(A)$. $\diamondsuit$

Remark 3.37. There is a more intuitive ways to think about A-paths: for any manifold M, given a path $\gamma : I \to M$ one has an associated path $I \to TM$ which is just "the derivative" of γ:

$$t \mapsto \frac{d\gamma}{dt}(t).$$

As we mentioned before, we can think of Lie algebroids as a kind of generalised tangent bundle, related to the standard tangent bundle by the anchor map; we can then think of an A-path as a pair (γ, a), where $\gamma : I \to M$ is a path in M and a is the "A-derivative" of γ, which is related to the standard derivative of γ through the anchor map.

Now, we will not go into detail about the A-homotopy, since the definition is relatively involved (see [CF03, Section 1.3]). At its core however, it is just a collection of paths $a = a(\epsilon, t) : I \times I \to A$ such that the base paths γ_ϵ have fixed end points, satisfying some extra condition involving the Lie algebroid structure. We will write $a \sim_A a'$ when $a, a' \in \mathcal{P}(A)$ are A-homotopic.

Let us now describe the Weinstein groupoid. First of all, the space of arrows is simply the set of A-paths modulo A-homotopy, denoted

$$\mathcal{G}(A) := \mathcal{P}(A)/\sim_A .$$

Since A-homotopy is just standard path-homotopy on the base paths, we can define source and target $\mathbf{s}, \mathbf{t} : \mathcal{G}(A) \to M$ by

$$\mathbf{s}([a]) := (\pi \circ a)(0), \qquad \mathbf{t}([a]) := (\pi \circ a)(1).$$

Multiplication is more involved; for $g = [a]$ and $h = [b]$ with $\mathbf{s}(g) = \mathbf{t}(h)$, we want to define gh as a kind of concatenation. Specifically, for A-paths a and b such that $(\pi \circ a)(0) = (\pi \circ b)(1)$, we define a new path $a \star b$ by

$$(a \star b)(t) = \begin{cases} 2b(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ 2a(2t-1) & \text{if } \frac{1}{2} < t \le 1. \end{cases}$$

Of course, the problem is that this path is in general not smooth at $t = \frac{1}{2}$. This problem is solved by using a reparametrisation: for $\tau : I \to I$ a reparametrisation, one can show that any A-path $a : I \to A$ is A-homotopic to its reparametrisation a^τ defined by $a^\tau(t) = \tau'(t)a(\tau(t))$. Hence for $g, h \in \mathcal{G}(A)$ with $\mathbf{s}(g) = \mathbf{t}(h)$ we simply use suitable representatives for which the concatenation defined above yields a smooth path, and let $gh \in \mathcal{G}(A)$ be the class of the resulting A-path. One can also show that this class is independent of the representatives chosen.

Finally, the unit map $u : M \to \mathcal{G}(A)$ is defined by letting $u(x)$ be the constant path mapping to $0_x \in A_x$, and inversion $i : \mathcal{G}(A) \to \mathcal{G}(A)$ maps a path $a : I \to A$ to the path $t \mapsto -a(1-t)$.

Proposition 3.38 ([CF03, Theorem 2.1])**.** *The above data defines a groupoid $\mathcal{G}(A) \rightrightarrows M$. When A is integrable, it admits a unique smooth structure making it the source 1-connected Lie groupoid integrating A.*

Even though we have not described the homotopy explicitely, we will see that this description of an integrating groupoid in terms of paths will be quite useful later.

Remark 3.39. The space of all A-paths $\mathcal{P}(A)$ naturally has the structure of a Banach manifold. The desired smooth structure on $\mathcal{G}(A) = \mathcal{P}(A)/\sim_A$ from the above proposition is then the one for which the quotient $\mathcal{P}(A) \to \mathcal{G}(A)$ is a submersion. As we know, at most one of such smooth structures exists.

From now on, we will simply say that "$\mathcal{G}(A)$ is smooth" when the conclusion of the above proposition holds.

As we mentioned before, the condition that $\mathcal{G}(A)$ is smooth can be further rephrased in terms of the so-called monodromy groups. Let us give the definition of these now.

Definition 3.40. Let $x \in M$. Then the *monodromy group* at x is defined as

$$N_x(A) := \{v \in Z(\mathfrak{g}_x) \mid v \text{ and } 0_x \text{ are } A\text{-homotopic}\}.$$

In words, it is subgroup of $\mathfrak{g}_x$, the isotropy Lie algebra of A at x, consisting of vectors $v \in Z(\mathfrak{g}_x)$ for which the constant A-path mapping to v is A-homotopic to the constant A-path mapping to 0_x. $\diamondsuit$

We will see that when A is the Poisson Lie algebroid associated to a regular Poisson manifold, there is a nicer alternative description of the monodromy groups. For now, let us mention the precise criterion concerning smoothness of the Weinstein groupoid. In order to phrase it, we introduce the following notation. Given a Lie algebroid $A \to M$, we fix some norm on the vector bundle A and define a function $r : M \to [0, \infty]$ by setting $r(x) := d(0_x, N_x(A) \setminus \{0_x\})$, with the convention that $d(0_x, \emptyset) = \infty$.

Theorem 3.41 ([CF03, Theorem 4.1])**.** *Let $A \to M$ be a Lie algebroid. Then the Weinstein groupoid $\mathcal{G}(A)$ is smooth if and only if the monodromy groups $N_x(A)$ are* locally uniformly discrete*, meaning that*

(i) the monodromy group $N_x(A)$ is discrete, i.e. $r(x) > 0$ for all $x \in M$;

(ii) for all $x \in M$ we have $\liminf_{y \to x} r(y) > 0$.

3.4 Integration of Poisson manifolds: symplectic groupoids

After the excursion to Lie groupoids and algebroids, it is now time to return to Poisson geometry. Putting the results from the previous sections together, there is already a clear guess of what it means for a Poisson manifold (M, π) to be integrable; we simply look for a Lie groupoid $\mathcal{G} \rightrightarrows M$ that integrates the Poisson Lie algebroid T^*M. It turns out however that for the groupoid $\mathcal{G}$ to fully encode the Poisson structure on M, we need some extra data on it.

Definition 3.42. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A differential form $\omega \in \Omega^k(\mathcal{G})$ is called *multiplicative* if

$$m^*\omega = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega.$$

Here $m, \mathrm{pr}_1, \mathrm{pr}_2 : \mathcal{G}_2 \to \mathcal{G}$ denote the multiplication map and the restrictions of the projections $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$. $\diamondsuit$

Definition 3.43. A *symplectic groupoid* is a pair $(\mathcal{G}, \omega)$ consisting of a Lie groupoid $\mathcal{G} \rightrightarrows M$ and a multiplicative symplectic form $\omega \in \Omega^2(\mathcal{G})$. If we want to emphasise the base manifold, we write $(\mathcal{G}, \omega) \rightrightarrows M$. $\diamondsuit$

The following properties of symplectic groupoids are easily verified.

Proposition 3.44. *Let $(\mathcal{G}, \omega) \rightrightarrows M$ be a symplectic groupoid. Then the following hold.*

(i) The unit map $u : M \to \mathcal{G}$ embeds M as a Lagrangian submanifold in $\mathcal{G}$. Consequently, we have $\dim(\mathcal{G}) = 2\dim(M)$.

(ii) The source and target fibres are symplectically orthogonal, i.e.

$$\ker(d\mathbf{s}) = \ker(d\mathbf{t})^{\omega}.$$

As mentioned above, we are looking for Lie groupoids which encode the Poisson structure on the base. The following theorem shows that symplectic groupoids satisfy this condition.

Theorem 3.45. *Let $(\mathcal{G}, \omega) \rightrightarrows M$ be a symplectic groupoid. Then there is a unique Poisson structure π on M, called the* induced Poisson structure*, such that*

$$\mathbf{t} : (\mathcal{G}, \omega) \to (M, \pi)$$

is a Poisson map. Moreover, the Lie algebroid A of $\mathcal{G} \rightrightarrows M$ is isomorphic to the Poisson Lie algebroid T^*M associated to π, through the map

$$\sigma_\omega : A \to T^*M, \qquad (x, \alpha) \mapsto du(x)^* \left(\omega^\flat_{1_x}(\alpha) \right).$$

Of course, we still need the converse: given a Poisson manifold (M, π) whose Poisson Lie algebroid is integrable, we want to find a *symplectic* groupoid integrating it. This converse is given by the following theorem.

Theorem 3.46. *Let (M, π) be a Poisson manifold whose Poisson Lie algebroid is integrable, let $\mathcal{G} \rightrightarrows M$ be a source 1-connected integration of the latter and fix an isomorphism $\sigma : A \to T^*M$ between the Lie algebroid A of $\mathcal{G} \rightrightarrows M$ and the Poisson Lie algebroid of (M, π). Then there exists a unique closed and multiplicative form $\omega \in \Omega^2(\mathcal{G})$ such that $\sigma_\omega = \sigma$. Moreover, $(\mathcal{G}, \omega)$ is a symplectic groupoid whose induced Poisson structure on M is π.*

The above theorems 3.45 and 3.46 were stated in this form in [Cra17].

We see in particular that the Weinstein groupoid becomes a symplectic groupoid in this way. It is common to denote the Weinstein groupoid corresponding to a Poisson manifold (M, π) by $\Sigma(M, \pi) := \mathcal{G}(T^*M)$. Furthermore, T^*M-paths are often called *Poisson paths*, or *cotangent paths*.

Theorems 3.45 and 3.46 have now confirmed that the symplectic groupoids are the "right" integrations of Poisson manifolds. Indeed, they show that given a Poisson manifold (M, π), the Poisson Lie algebroid T^*M is integrable if and only if there exists a symplectic groupoid $(\mathcal{G}, \omega) \rightrightarrows M$ whose induced Poisson structure on M is π. If the latter holds, we also say that $(\mathcal{G}, \omega)$ integrates (M, π).

3.4.1 The regular case

For now, let (M,π) be a regular Poisson manifold. As mentioned before, there is now a nicer alternative description of the monodromy groups, which we now denote by $N_x(M,\pi)$. For a proof of the results mentioned in this section, see [CF04, Proposition 5]. We fix $x \in M$, and denote by $S = S_x$ the symplectic leaf through x. We recall that the isotropy Lie algebra is now $\mathfrak{g}_x = \nu_x^*(S)$, and that it is abelian, i.e. equal to its centre, by Lemma 2.23. This already simplifies the monodromy group $N_x(M,\pi)$, where we only consider elements of the centre. It turns out that there is actually an exact sequence of groups

$$\pi_2(S,x) \xrightarrow{\partial} \nu_x^*(S) \xrightarrow{\exp} \Sigma_x(M,\pi) \xrightarrow{p} \pi_1(S,x) \longrightarrow 0, \tag{3.6}$$

and that in this exact sequence we have $N_x(M,\pi) = \operatorname{im}(\partial) = \ker(\exp)$. We will not prove exactness of the sequence, but we will explain the maps involved.

Firstly, the map $p : \Sigma_x(M,\pi) \to \pi_1(S,x)$ simply sends a class $[a] \in \Sigma_x(M,\pi)$ to the class of its base path $\gamma = \pi \circ a$. Of course, this is well-defined since elements of $\Sigma_x(M,\pi)$ have loops as their base path, since the base paths of Poisson paths clearly stay inside the symplectic leaves and since Poisson homotopy comes down to standard path homotopy on the base paths.

Next, $\exp : \nu_x^*(S) \to \Sigma_x(M,\pi)$ sends an element $\alpha \in \nu_x^*(S) \subset T_x^*M$ to the constant Poisson path mapping to α. From this, it is clear that $N_x(M,\pi) = \ker(\exp)$.[3]

Finally, and most interestingly, we turn to $\partial : \pi_2(S,x) \to \nu_x^*(S)$, often called the *monodromy map*. So let $[\sigma] \in \pi_2(S,x)$, where $\sigma : (S^2, p_N) \to (S,x)$ is a smooth representative, i.e. a smooth map sending the north pole p_N to x. We need to define how $\partial([\sigma])$ acts on some $v \in \nu_x(S) = T_xM/T_xS$. One can show that there exists a smooth path $t \mapsto x_t$, defined on $(-\epsilon,\epsilon)$ for some $\epsilon > 0$, such that $x_0 = x$ and such that $v = [\dot{x}_0]$, and that there exists a smooth family of maps $\sigma_t : (S^2,p_N) \to (S_t,x_t)$ such that $\sigma_0 = \sigma$ (here we denote by (S_t,ω_t) the symplectic leaf through x_t). With this data in place, we set

$$\partial([\sigma])v := \left.\frac{d}{dt}\right|_{t=0} \int_{S^2} \sigma_t^*\omega_t. \tag{3.7}$$

One can show that this is independent of the choices, and hence this defines the monodromy map ∂.

As a result, we know have a description of the monodromy group $N_x(M,\pi) = \operatorname{im}(\partial)$ which is much more geometric; we essentially only use the symplectic foliation of (M,π). We will see that this description is quite useful for determining the monodromy groups.

[3] The notation exp is explained by the following. As we know, if the Weinstein groupoid $\Sigma(M,\pi)$ is smooth, then $\Sigma_x(M,\pi)$ is a Lie group, whose Lie algebra is $\nu_x^*(S)$. It can be shown that the map $\nu_x^*(S) \to \Sigma_x(M,\pi)$ defined here is in that case actually the exponential map.

4 Poisson manifolds of compact types

After introducing the necessary basic concepts, we can now finally turn to the definition of the main objects of study in this thesis: Poisson manifolds of compact types. Just like in the case of Lie algebras, which we call compact if it is integrated by a compact Lie group, compactness of a Poisson manifold is defined in terms of Lie groupoids. We mainly follow [CFMT15] and [CFMT16].

4.1 Compactness types of Lie groupoids

Unlike in the case of Lie groups, there are actually multiple notions of compactness for Lie groupoids.

Definition 4.1. A Hausdorff, source connected Lie groupoid $\mathcal{G} \rightrightarrows M$ is called

- *proper* if the map $(\mathbf{s}, \mathbf{t}) : \mathcal{G} \to M \times M$ is proper,
- $\mathbf{s}$-*proper*, or *source-proper*, if the source map $\mathbf{s} : \mathcal{G} \to M$ is proper, and
- *compact* if the space of arrows $\mathcal{G}$ is compact. $\diamondsuit$

Remark 4.2. It follows immediately that any compact Lie groupoid is also $\mathbf{s}$-proper, and that any $\mathbf{s}$-proper one is also proper.

The following lemma will prove useful for determining whether a Lie groupoid is $\mathbf{s}$-proper.

Lemma 4.3. *Let $\mathcal{G} \rightrightarrows M$ be a source connected Lie groupoid. If the source fibres are in addition all compact, then the source map $\mathbf{s} : \mathcal{G} \to M$ is proper.*

The proof is purely topological, and is inspired by some exercises in [WD79, Section B.II].

Proof. It suffices to show that $\mathbf{s} : \mathcal{G} \to M$ is closed. So let $K \subset \mathcal{G}$ be a closed set. To show that $\mathbf{s}(K)$ is closed, it suffices to show that $\mathbf{s}^{-1}(\mathbf{s}(K))$ is closed. The latter set is just the union of all fibres of $\mathbf{s}$ meeting K, and its complement is the union of all fibres not meeting K, i.e. precisely those fibres contained in the open set $U := \mathcal{G} \setminus K$. We will now show that the union of all fibres contained in U is open, which will complete the proof.

We will show that if $\mathbf{s}^{-1}(x)$ is any fibre, we can find an open set containing it which only meets fibres contained in U. This will imply that the union of all fibres contained in U is open, since we can cover it by such opens.

So let $\mathbf{s}^{-1}(x)$ be any fibre, which by assumption is compact and connected. Since $\mathcal{G}$ is a manifold, there exists a precompact open neighbourhood V of $\mathbf{s}^{-1}(x)$; indeed, every point in $\mathbf{s}^{-1}(x)$ has a precompact neighbourhood, and by compactness $\mathbf{s}^{-1}(x)$ is covered by finitely many of them. Then the union of these is the desired V. Setting now $C := \partial(U \cap V)$, we have that $\mathbf{s}(C) \subset M$ is closed, since C is compact and M is Hausdorff. Now $W := U \cap V \cap \mathbf{s}^{-1}(M \setminus \mathbf{s}(C))$ is the desired open set: if $\mathbf{s}^{-1}(y)$ is any fibre not contained in U, it is in particular not contained in $U \cap V$. So if W were to meet $\mathbf{s}^{-1}(y)$, then in particular $U \cap V$ and $\mathbf{s}^{-1}(y)$ meet, meaning that $\mathbf{s}^{-1}(y)$ must also meet C, since it is connected. But that implies that $y \in \mathbf{s}(C)$, meaning that $\mathbf{s}^{-1}(y) \subset \mathbf{s}^{-1}(\mathbf{s}(C))$, contradicting the fact that W meets $\mathbf{s}^{-1}(y)$. $\square$

Of course, when the base M is a point, all three notions coincide. It turns out that this holds in greater generality.

Proposition 4.4. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid.*

(i) If the orbits of $\mathcal{G} \rightrightarrows M$ are all compact, then $\mathcal{G} \rightrightarrows M$ is proper iff it is $\mathbf{s}$*-proper.*

(ii) If the base M is compact, then all three notions of compactness coincide.

Proof. To show (i), we only need to show the implication " $\Longrightarrow$ " (recall Remark 4.2). To do this, we only need to show that the source fibres are compact, by Lemma 4.3. Recall that for any $x \in M$, $\mathbf{t} : \mathbf{s}^{-1}(x) \to \mathcal{O}_x$ is a principal $\mathcal{G}_x$-bundle, where $\mathcal{O}_x$ denotes the orbit through x. By assumption $\mathcal{O}_x$ is compact, and since $\mathcal{G}_x = (\mathbf{s}, \mathbf{t})^{-1}(x, x)$, this is also compact by properness of $(\mathbf{s}, \mathbf{t})$. This means that $\mathbf{s}^{-1}(x)$ is a fibre bundle over a compact base and with compact fibre; hence $\mathbf{s}^{-1}(x)$ is compact itself.

For (ii), we note that since $\mathcal{G} = \mathbf{s}^{-1}(M)$, clearly compactness and $\mathbf{s}$-properness are equivalent. To complete the proof, it suffices again to show that properness implies $\mathbf{s}$-properness. So assume that $\mathcal{G} \rightrightarrows M$ is proper. If we can show that the orbits are compact, then we can mimic the argument from (i) and conclude that $\mathcal{G} \rightrightarrows M$ is $\mathbf{s}$-proper. To prove this, it suffices the show that the orbits are closed, since M is compact. To see this, let $x \in M$. Clearly $\mathbf{s}^{-1}(x)$ is closed, and since $(\mathbf{s}, \mathbf{t})$ is proper, it is also a closed map (because $M \times M$ is locally compact and Hausdorff). Hence $(\mathbf{s}, \mathbf{t})(\mathbf{s}^{-1}(x)) = \{x\} \times \mathcal{O}_x \subset M \times M$ is closed, which of course implies that $\mathcal{O}_x \subset M$ is. $\square$

Let us now turn to some examples.

Example 4.5. As already mentioned, when $M = \{\star\}$, all three notions of compactness are equivalent. Interpreting a Lie groupoid over a point as a Lie group, they are all equivalent to compactness of the Lie group. $\triangle$

Example 4.6. If $\mathcal{G} \rightrightarrows M$ is a bundle of Lie groups, i.e. when $\mathbf{s} = \mathbf{t}$, all orbits consist of just a single point, meaning that properness and $\mathbf{s}$-properness are equivalent, and clearly these two are equivalent to compactness (and connectedness) of the isotropy groups $\mathcal{G}_x$. Finally, $\mathcal{G}$ is compact if in addition M is compact. $\triangle$

Example 4.7. For the pair groupoid $M \times M \rightrightarrows M$, the map $(\mathbf{s}, \mathbf{t}) : M \times M \to M \times M$ is a homeomorphism, meaning that it is always proper. Hence, if M is connected, the pair groupoid is always proper. Clearly, it is compact iff it is $\mathbf{s}$-proper iff M is compact (and connected). $\triangle$

Example 4.8. Consider now the fundamental groupoid $\Pi_1(M) \rightrightarrows M$. Viewing it as the gauge groupoid of the universal cover $\widetilde{M} \to M$, it follows that it is Hausdorff. It is proper iff the fundamental group $\pi_1(M)$ is finite, $\mathbf{s}$-proper iff $\pi_1(M)$ is finite and M has compact path components and compact iff $\pi_1(M)$ is finite and M is compact. $\triangle$

Example 4.9. Perhaps the most insightful example is that of the action groupoid $G \ltimes M$. As is immediate from the definitions, this groupoid is proper iff the action of G on M is proper, $\mathbf{s}$-proper iff G is compact and compact iff both G and M are compact. $\triangle$

4.2 Compactness types of Poisson manifolds

The generalisation of the compactness types from the previous section to Poisson manifolds is now clear. However, we can distinguish even more compactness types for Poisson manifolds. Indeed, recall from the case of Lie algebras that even when a Lie algebra is integrated by a compact Lie group, its unique 1-connected integration is not necessarily compact. The same holds for Poisson manifolds.

Definition 4.10. Let $\mathcal{C} \in \{\text{proper}, \mathbf{s}\text{-proper}, \text{compact}\}$. An integrable Poisson manifold (M, π) is said to be

- *of $\mathcal{C}$ type* if it is integrated by a Lie groupoid with property $\mathcal{C}$, and
- *of strong $\mathcal{C}$ type* if its 1-connected integration $\Sigma(M, \pi)$ has property $\mathcal{C}$. $\diamondsuit$

It follows from the discussion in the previous section that for any Poisson manifold (M, π) we have a diagram of implications

$$\begin{tikzcd}
\text{strong compact} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{strong } \mathbf{s}\text{-proper} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{strong proper} \arrow[d, Rightarrow] \\
\text{compact} \arrow[r, Rightarrow] & \mathbf{s}\text{-proper} \arrow[r, Rightarrow] & \text{proper.}
\end{tikzcd}$$

From Proposition 4.4 it follows that when the symplectic leaves of (M, π) are compact, this diagram becomes

$$\begin{tikzcd}
\text{strong compact} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{strong } \mathbf{s}\text{-proper} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \text{strong proper} \arrow[d, Rightarrow] \\
\text{compact} \arrow[r, Rightarrow] & \mathbf{s}\text{-proper} \arrow[r, Leftrightarrow] & \text{proper,}
\end{tikzcd}$$

and that when M is compact we get

$$\begin{tikzcd}
\text{strong compact} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \text{strong } \mathbf{s}\text{-proper} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \text{strong proper} \arrow[d, Rightarrow] \\
\text{compact} \arrow[r, Leftrightarrow] & \mathbf{s}\text{-proper} \arrow[r, Leftrightarrow] & \text{proper.}
\end{tikzcd}$$

As we can see, the strongest conditions belong to Poisson manifolds of strong compact type, or PMSCT for short. In this thesis we focus on these PMSCT. For interesting examples of Poisson manifolds of the other compactness types, see e.g. [CFMT15, Section 4]. For now, there is one easy example we can give of a PMSCT.

Example 4.11. Let (S, ω) be a symplectic manifold. We have seen earlier in Example 2.26 that its Poisson Lie algebroid is isomorphic to the tangent Lie algebroid through the map $\omega^\flat$. We have already seen in Example 3.32 (iii) that this is the Lie algebroid associated the the (source 1-connected!) fundamental groupoid $\Pi_1(S) \rightrightarrows S$, and it is not difficult to see that endowed with

$$\Omega := \mathbf{t}^*\omega - \mathbf{s}^*\omega,$$

$(\Pi_1(S), \Omega)$ is the (symplectic) Weinstein groupoid of (S, ω). Hence from Example 4.8 we see that when S is compact and has finite fundamental group, (S, ω) is a PMSCT. $\triangle$

It turns out that besides nondegenerate Poisson manifolds, examples of PMSCT are difficult to produce. In the next section, we will state and prove a proposition giving a general construction of PMSCT.

4.3 Construction of PMSCT

The construction we will outline uses the concept of an *integral affine structure* on a manifold, so we will first explore such structures in some detail.

First, let $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^q)$ denote the group of integral affine maps $\mathbb{R}^q \to \mathbb{R}^q$, i.e. of maps of the form

$$x \mapsto Ax + v,$$

with $A \in \mathrm{GL}_{\mathbb{Z}}(\mathbb{R}^q)$ and $v \in \mathbb{R}^q$. Here $\mathrm{GL}_{\mathbb{Z}}(\mathbb{R}^q)$ denotes the group of invertible linear maps $A : \mathbb{R}^q \to \mathbb{R}^q$ for which, as matrices, both A and A^{-1} have integer entries. An integral affine structure is now just an integrable $\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^q)$-structure.

Definition 4.12. An *integral affine structure* on a manifold B is a maximal atlas $\{(U_i, \varphi_i)\}_{i \in I}$ on B with the property that the transition functions

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

are restrictions of integral affine maps, for all $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$. Any element of the atlas is called an *integral affine chart.* $\diamondsuit$

Note that if we can cover B by charts that are compatible in the above sense, we can extend these to an integral affine structure on B in the same way as for standard smooth atlases.

An equivalent, and for us more useful definition can be given in terms of lattices. In a (real, finite dimensional) vector space V, a *lattice* Λ is simply a discrete additive subgroup of maximal rank. It is easy to show that any lattice is isomorphic to $\mathbb{Z}^q \subset \mathbb{R}^q$, where $q = \dim(V)$, i.e. that we can find a basis $\{v_1, \ldots, v_q\}$ of V such that

$$\Lambda = \mathbb{Z}\, v_1 + \cdots + \mathbb{Z}\, v_q.$$

We have a similar notion for vector bundles: a lattice in a vector bundle $E \to B$ is simply a subbundle

$$\Lambda = \bigcup_{b \in B} \Lambda_b \subset E,$$

such that $\Lambda_b \subset E_b$ is a lattice for all $b \in B$. Such a lattice is called *smooth* if around all $b_0 \in B$ we have an open $U \subset B$ and smooth local sections $s_1, \ldots, s_q \in \Gamma(E|_U)$ such that

$$\Lambda_b = \mathbb{Z}\, s_1(b) + \cdots + \mathbb{Z}\, s_q(b)$$

for all $b \in U$. Here $q = \dim(B)$.

The following now gives an alternative description of integral affine structures.

Proposition 4.13. *Let B be a q-dimensional manifold. Then there is a one-to-one correspondence between integral affine structures and smooth lattices $\Lambda \subset T^*B$ all whose local sections are closed 1-forms. This correspondence associates to an integral affine structure the lattice defined by*

$$\Lambda_b := \mathbb{Z}\, dx_1(b) + \cdots + \mathbb{Z}\, dx_q(b),$$

where $(x_1, \ldots, x_q)$ are the coordinates of some integral affine chart around $b \in B$.

Proof. First, given an integral affine structure, note that the definition of Λ is clearly independent of the chart used by the very definition of integral affine structures. Hence Λ is well-defined, and by its definition it is smooth and all its local sections are closed 1-forms.

Now, given a smooth lattice $\Lambda \subset T^*B$ all whose local sections are closed 1-forms, we construct an integral affine structure as follows. By assumption, around every $b \in B$ we can find an open $U \subset B$ and local sections $s_i \in \Omega^1(U)$ such that

$$\Lambda_b = \mathbb{Z}\, s_1(b) + \cdots + \mathbb{Z}\, s_q(b)$$

and such that every s_i is closed. Shrinking U if necessary, we can assume that the s_i are exact, i.e. $s_i = dx_i$ for $x_i \in C^\infty(U)$. Since $\{dx_1(b), \ldots, dx_q(b)\}$ is now linearly independent, it follows from the inverse function theorem that $(x_1, \ldots, x_q)$ form a chart on some neighbourhood of b. Again, it is easy to see that for any other charts obtained in this way, the transition functions must be integral affine maps. Hence we obtain an integral affine structure on M. This construction is clearly inverse to the above construction, which proves the proposition. $\square$

Example 4.14. An important example is that of so-called *complete* integral affine manifolds. Let $\Gamma \subset \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^q)$ be a discrete subgroup, and assume that its obvious action on $\mathbb{R}^q$ is free and proper (such a subgroup is sometimes called *smooth*). Then $B := \mathbb{R}^q/\Gamma$ is a smooth manifold, which actually comes with an induced integral affine structure. This can be seen in several ways, using the fact that the projection $\mathbb{R}^q \to B$ is a local diffeomorphism. For instance, we know that the (continuous) local sections of $\mathbb{R}^q \to B$ can be used as charts for the smooth structure on B, and we see that the transition maps for these charts belong to Γ, meaning that they determine an integral affine structure on B. Some simple examples of complete integral affine manifolds are the following.

(i) The subgroup $\Gamma \subset \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R})$ generated by $x \mapsto x + 1$ endows $S^1 = \mathbb{R}/\Gamma$ with an integral affine structure.

(ii) The subgroup $\Gamma \subset \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^2)$ generated by $(x, y) \mapsto (x+1, y)$ and $(x, y) \mapsto (x, y+1)$ endows $\mathbb{T}^2 = \mathbb{R}^2/\Gamma$ with an integral affine structure.

(iii) For the subgroup $\Gamma \subset \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^2)$ generated by $(x, y) \mapsto (x+1, y)$ and $(x, y) \mapsto (x+y, y+1)$, the quotient $\mathbb{R}^2/\Gamma$ is still (diffeomorphic to) the torus $\mathbb{T}^2$, but with a different induced integral affine structure (see [Mis96, Theorem A]).

The integral affine structures in parts (i) and (ii) are often called the standard integral affine structures for S^1 and $\mathbb{T}^2$, respectively. $\triangle$

Remark 4.15. There is a more general notion of integral affine structures for foliated manifolds. Indeed, a *transverse integral affine structure* on a foliated manifold $(M, \mathcal{F})$ is just a (maximal) foliation atlas all whose transition maps are integral affine maps. In terms of lattices, a transverse integral affine structure is a lattice in the conormal bundle $\nu^*(\mathcal{F})$ which is locally spanned by closed, basic 1-forms. It is easy to see that if the foliation is simple, i.e. induced by a submersion $p : M \to B$, the above is equivalent to an integral affine structure on B (pullback by p gives an isomorphism between T^*B and $\nu^*(\mathcal{F})$).

The construction we will outline is inspired by two properties all PMSCT share, and which we will outline now.

The first is the following. Let $(\mathcal{G}, \omega)$ be a proper integration of a regular Poisson manifold (M, π). This integration induces a transverse integral affine structure on the symplectic foliation $\mathcal{F}_\pi$ as follows. It is easily verified that for all $x \in M$ the isomorphism σ_ω from Theorem 3.45 identifies the isotropy Lie algebra $\mathfrak{g}_x$ with $\nu_x^*(\mathcal{F}_\pi)$. Since (M, π) is regular, $\mathfrak{g}_x$ is abelian and thus the kernel of the exponential map $\mathfrak{g}_x \to \mathcal{G}_x$ defines a lattice in $\mathfrak{g}_x$, and we can transport this lattice through the above isomorphism to obtain a lattice in $\nu_x^*(\mathcal{F}_\pi)$. It is shown in [CFMT16, Theorem 3.3.1] that these lattices together form a transverse integral affine structure on $\mathcal{F}_\pi$, denoted $\Lambda_\mathcal{G}$. If (M, π) is *strong* proper, we actually have $\Lambda_{\mathcal{G},x} = N_x(M, \pi)$ for all $x \in M$; this follows directly from the desciption of $N_x(M, \pi)$ given in Section 3.4.1.

The second property is slightly more complicated. Assume in addition that (M, π) is of $\mathbf{s}$-proper type and that the symplectic leaves are 1-connected; in this case the leaf space $B = M/\mathcal{F}_\pi$ is smooth and by the above property and Remark 4.15 it is actually an integral affine manifold. The assertion is then that, in some sense, the cohomology classes of the symplectic forms on the leaves of $\mathcal{F}_\pi$ "vary linearly" with respect to this integral affine structure as we move through the leaf space B. We will not go into the details for the general situation here (see [CFMT16, Sections 4.1 – 4.3]), but in the trivial case where $M = U \times S$, with $U \subset \mathbb{R}^q$ open and $\mathcal{F}_\pi = \{\{x\} \times S \mid x \in U\}$, it comes down to

$$[\omega_v] = [\omega_0] + v^1 c_1 + \cdots + v^q c_q \in H^2(S, \mathbb{R}), \tag{4.1}$$

where $c_i \in H^2(S, \mathbb{Z})$ are certain linearly independent integral classes.

The strategy is now to "invert" the above discussion. Specifically, we will show that a Poisson manifold which fibres "nicely" over a complete integral affine manifold and whose symplectic forms satisfy a (4.1)-like equation is always $\mathbf{s}$-proper.

So let $S \to E \to \mathbb{R}^q$ be a fibre bundle, with the typical fibre S compact and 1-connected, and assume that the total space E is a Poisson manifold whose symplectic leaves are precisely the fibres of $E \to \mathbb{R}^q$. Let us denote the symplectic leaf corresponding to $v \in \mathbb{R}^q$ by (S_v, ω_v). We denote by

$$\mathcal{H}^2 := \bigsqcup_{v \in \mathbb{R}^q} H^2(S_v, \mathbb{R})$$

the vector bundle of degree two-cohomologies of the fibre bundle (trivialisations of the fibre bundle $E \to \mathbb{R}^q$ induce trivialisations of $\mathcal{H}^2$). In this vector bundle we have the smooth lattice

$$\mathcal{H}^2_\mathbb{Z} := \bigsqcup_{v \in \mathbb{R}^q} H^2(S_v, \mathbb{Z})$$

and the associated flat connection ∇ called the *Gauss-Manin connection*, uniquely determined by its vanishing on sections of $\mathcal{H}^2_{\mathbb{Z}}$. Of course, the Poisson structure gives us a section ϖ of $\mathcal{H}^2$, given by $v \mapsto [\omega_v]$.

Next, let $\Gamma \subset \mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^q)$ be smooth (meaning that it is discrete and acts freely and properly on $\mathbb{R}^q$, see Example 4.14) and assume that it acts on E through Poisson maps and equivariantly with respect to the projection $E \to \mathbb{R}^q$. Then $B := \mathbb{R}^q/\Gamma$ is smooth and inherits an integral affine structure, as explained above, and because equivariance implies that also the action on E is free and proper, it follows that $M := E/\Gamma$ is smooth as well. It is also easily verified that the obvious projection $p : M \to B$ defines a fibre bundle with typical fibre S, and because Γ acts on E by Poisson maps, we get an induced Poisson structure π on M as well. The symplectic leaves are again just the fibres of $p : M \to B$. In conclusion, (M, π) is a regular Poisson manifold with leaf space B.

Proposition 4.16. *Let (M, π) be constructed as above. Assume that there exists a section $s \in \Gamma(\mathcal{H}^2)$ such that $\nabla s = 0$ and that there exist linearly independent sections $c_1, \ldots, c_q \in \Gamma(\mathcal{H}^2_{\mathbb{Z}})$ such that*

$$\varpi = s + \sum_{i=1}^{q} \mathrm{pr}_i c_i, \tag{4.2}$$

where $\mathrm{pr}_i : \mathbb{R}^q \to \mathbb{R}$ denotes the projection onto the i-th coordinate. Then (M, π) is of strong **s***-proper type. If in addition B is compact, (M, π) is a PMSCT.*

Proof. Note that pulling back the induced integral affine structure on B by p yields a transverse integral affine structure on M relative to the symplectic foliation $\mathcal{F}_\pi$, which we denote by $\Lambda \subset \nu^*(\mathcal{F}_\pi)$.

We first show that for any $x \in M$ the monodromy group $\mathcal{N}_x(M, \pi)$ is equal to the lattice $\Lambda_x \subset \nu^*_x(\mathcal{F}_\pi)$. To do this, we will reduce to a much simpler setting. First, we use a local section of $\mathbb{R}^q \to B$ around $p(x)$ to get back to the (restriction of the) original bundle $E \to \mathbb{R}^q$, and assuming without loss of generality that the domain of our local section is contractible, we trivialise this bundle. The setting is then as follows. Our Poisson manifold is then simply $U \times S$, with $U \subset \mathbb{R}^n$ open, with symplectic leaves the obvious fibres $\{y\} \times S$, $y \in U$. Moreover, (4.2) implies that the symplectic forms ω_y on the fibre $\{y\} \times S$ satisfy

$$[\omega_y] = \widetilde{s} + \sum_{i=1}^{q} y_i \widetilde{c_i}$$

in cohomology, where $\widetilde{s} \in H^2(S, \mathbb{R})$, and the $\widetilde{c_i} \in H^2(S, \mathbb{Z})$ are linearly independent. Also, identifying the conormal space to any of the leaves, in any point, with $(\mathbb{R}^q)^*$, the transverse integral affine structure on M now translates to the standard one given by $\mathbb{Z}\, e^1 + \cdots + \mathbb{Z}\, e^q$ where $\{e^1, \ldots, e^q\}$ is the standard dual basis of $\mathbb{R}^q$.

Let now $(y, z) \in U \times S$ correspond to the original $x \in M$, and identify the conormal space at (y, z) with $(\mathbb{R}^q)^*$. We will use Section 3.4.1 to determine the monodromy group. So let $v \in \mathbb{R}^q$ be an element of the normal bundle. Then we have an obvious choice of path $t \mapsto (y + tv, z)$ through (y, z) with derivative v, and given $[\sigma] \in \pi_2(\{y\} \times S, (y, z))$, i.e. a pointed map $\sigma : (S^2, p_N) \to (\{y\} \times S, (y, z))$, we obtain an obvious smooth family $\sigma_t : (S^2, p_N) \to (\{y + tv\} \times S, (y + tv, z))$

by essentially sliding σ along the path. Then we obtain

$$\partial([\sigma])v = \frac{d}{dt}\bigg|_{t=0} \int_{S^2} \sigma_t^* \omega_{y+tv} = \sum_{i=1}^{q} v_i \int_{S^2} \sigma^* \widetilde{c_i},$$

and we conclude that

$$\partial([\sigma]) = \sum_{i=1}^{q} \left[\int_{S^2} \sigma^* \widetilde{c_i}\right] e^i$$

To analyse the terms $\int_{S^2} \sigma^* \widetilde{c_i}$, we use the Hurewicz theorem, which states in particular that for a 1-connected topological space the map $\pi_2(X) \to H_2(X, \mathbb{Z})$ defined by $[\sigma] \mapsto \sigma_*[S^2]$ is an isomorphism. Combining this, in our case, with Poincaré duality, we get a perfect pairing $H^2(S, \mathbb{Z}) \times \pi_2(S) \to \mathbb{Z}$ given by

$$(\omega, [\sigma]) = \int_{S^2} \sigma^* \omega.$$

But then we see that for every $[\sigma] \in \pi_2(\{y\} \times S, (y, z))$ the numbers $\int_{S^2} \sigma^* \widetilde{c_i}$ are integers, and using that the $\widetilde{c_i}$ are linearly independent we see that, as $[\sigma]$ ranges through $\pi_2(\{y\} \times S, (y, z))$, we find all possible combinations of integers, i.e.

$$\mathrm{im}(\partial) = \mathbb{Z}\, e^1 + \cdots + \mathbb{Z}\, e^q.$$

This proves that $N_x(M, \pi) = \Lambda_x$ for all $x \in M$. This tells us two crucial things: first of all, since Λ is a smooth lattice, Theorem 3.41 tells us that (M, π) is integrable, i.e. that the Weinstein groupoid $\Sigma(M, \pi)$ is smooth. Secondly, the exact sequence (3.6) tells us that since $\pi_1(S) = 0$, the isotropy groups $\Sigma_x(M, \pi)$ are isomorphic to the q-dimensional torus, meaning that they are compact. But then, as we saw before, by Proposition 3.7 (iii) we can conclude that the source fibres of $\Sigma(M, \pi)$ are all compact, since S is also compact. Then by Lemma 4.3 we conclude that $\Sigma(M, \pi)$ is $\mathbf{s}$-proper, meaning that (M, π) is of $\mathbf{s}$-proper type.

Finally, if B is compact, then so is M, meaning that $\mathbf{s}$-properness is equivalent to compactness. □

Remark 4.17. From the proof we see that the induced integral affine structure on the leaf space B coincides with the integral affine structure coming from the quotient $\mathbb{R}^q/\Gamma$.

Our problem of producing concretes examples of PMSCT has now been reduced to finding the appropriate data to be able to apply Proposition 4.16. The main input is of course the fibre S, which should be a compact, 1-connected manifold with "sufficiently interesting" cohomology $H^2(S)$, in order to satisfy Equation 4.2. The second major part is finding the action on E. As it happens, we have a perfect way to obtain all this; the solution is given by *K3 surfaces*, which are certain complex surfaces. Not only do they have a big degree two cohomology group, there also exist strong theorems, known as the Torelli theorems, which provide a way of constructing diffeomorphisms out of maps on the cohomology groups. In the next section, we will see that the theory of K3 surfaces will provide us with all the data we need to construct a PMSCT.

5 K3 surfaces

In this section, we will provide the definition and basic properties of K3 surfaces, after which we will discuss the moduli space and universal family of K3 surfaces. Finally, we will show how we can turn this family into a Poisson manifold with symplectic leaves the K3 surfaces that are the fibres of the family. The main references for this section are [BPV84] and [Huy16].

5.1 Preliminaries

Before we get into the theory of K3 surfaces, let us first recall some facts and establish some terminology regarding lattices and complex geometry.

5.1.1 Lattices

Definition 5.1. A *lattice* will, in this section, be a free, finitely generated $\mathbb{Z}$-module Λ, endowed with a symmetric bilinear form

$$(\cdot,\cdot) : \Lambda \times \Lambda \to \mathbb{Z}.$$

A lattice is called

- *unimodular* if the map $\Lambda \to \Lambda^* = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ given by $a \mapsto (a, \cdot)$ is an isomorphism, or equivalently if the determinant of the matrix representing $(\cdot,\cdot)$ is ± 1;
- *even* if $(a, a) \in 2\mathbb{Z}$ for all $a \in \Lambda$, and *odd* otherwise;
- of *signature* (p, q) if it has an orthogonal basis $(e_1, \ldots, e_p, f_1, \ldots, f_q)$ such that $(e_i, e_i) > 0$ and $(f_i, f_i) < 0$.

We define the *index* of Λ to be $\tau(\Lambda) := p - q$. We say that Λ is *positive definite* (resp. *negative definite*) if $p = \mathrm{rank}(\Lambda)$ (resp. $q = \mathrm{rank}(\Lambda)$), and *indefinite* otherwise. $\diamondsuit$

There is an obvious notion of (iso)morphisms between lattices defined above, namely $\mathbb{Z}$-module morphisms which preserve the forms. Note also that we can produce new lattices out of given ones by taking direct sums and orthogonally extending the forms. Clearly, this construction preserves whether the lattices are unimodular, even etc. and the signature "adds up" in the sense that if lattices Λ_i have signature (p_i, q_i), then $\oplus_i \Lambda_i$ has signature $(\sum_i p_i, \sum_i q_i)$.

For us, the most important example is the following.

Example 5.2. Let X be a $4k$-dimensional, compact oriented manifold. Let us assume for simplicity that $H_{2k}(X, \mathbb{Z})$ and $H^{2k}(X, \mathbb{Z})$ are torsion-free. Then the cup product $H^{2k}(X, \mathbb{Z}) \times H^{2k}(X, \mathbb{Z}) \to H^{4k}(X, \mathbb{Z}) \cong \mathbb{Z}$ is a lattice since the cup product is symmetric in even degree. In fact, it is also unimodular, since the map $a \mapsto a \smile -$ is the composition of the isomorphism $H^{2k}(X, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_{2k}(X, \mathbb{Z}), \mathbb{Z})$, obtained from the Universal Coefficient Theorem, and the isomorphism $\mathrm{Hom}_{\mathbb{Z}}(H_{2k}(X, \mathbb{Z}), \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H^{2k}(X, \mathbb{Z}), \mathbb{Z})$ induced by Poincaré Duality.

In this example, we often denote the index by $\tau(X) := \tau(H^{2k}(X, \mathbb{Z}))$. $\triangle$

Let us also provide some more concrete examples, which will appear once we study K3 surfaces.

Example 5.3.

(i) The *hyperbolic plane* is the $\mathbb{Z}$-module $U := \mathbb{Z}^{\oplus 2}$ with the form given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Concretely, denoting by $\{u, v\}$ the standard basis of U, we have $(u, u) = (v, v) = 0$ and $(u, v) = 1$. This lattice is unimodular, even and has signature $(1, 1)$.

(ii) The E_8*-lattice* is the $\mathbb{Z}$-module $E_8 = \mathbb{Z}^{\oplus 8}$ with the form given by the matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

This lattice is unimodular, even and positive definite.

(iii) Combining the previous two examples yields the *K3 lattice* $L = U^{\oplus 3} \oplus (-E_8)^{\oplus 2}$. Here the minus in $-E_8$ means that we consider the lattice with -1 times the matrix given above. We see that L is unimodular, even and has signature $(3, 19)$. As might be expected from its name, L will serve as a concrete model for the cohomology of a K3 surface. △

The following uniqueness result will allow us to determine the cohomology of a K3 surface.

Theorem 5.4. *A unimodular, even, indefinite lattice is determined uniquely (up to isomorphism) by its rank and its signature.*

For a proof, see [MH73, Theorem 5.3].

5.1.2 Differential forms on complex manifolds

Next, let us establish some notation regarding complex geometry (see e.g. [Huy05]). When X is a complex manifold, the complexified tangent bundle $T_{\mathbb{C}}X$ splits as $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$. Often $T^{1,0}X$ is called the *holomorphic tangent bundle* and $T^{0,1}X$ the *anti-holomorphic tangent bundle*. Of course, the exterior powers split as well:

$$\bigwedge^k T^*_{\mathbb{C}}X = \bigoplus_{p+q=k} \bigwedge^{p,q} X,$$

where

$$\bigwedge^{p,q} X = \bigwedge^p (T^{1,0}X)^* \otimes \bigwedge^q (T^{0,1}X)^*.$$

Consequently the spaces of differential forms split similarly as

$$\Omega^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(X),$$

where

$$\Omega^{p,q}(X) = \Gamma\Big(\bigwedge^{p,q} X\Big).$$

Note that here we are just taking smooth sections. The vector bundles $\bigwedge^{p,0} X$ are actually holomorphic, and we will denote their space of holomorphic sections by $\Omega^p(X)$. If $n = \dim_{\mathbb{C}}(X)$, $\bigwedge^{n,0} X$ is called the *canonical bundle* of X and is often denoted by ω_X.

The exterior derivative $d : \Omega^\bullet(X, \mathbb{C}) \to \Omega^{\bullet+1}(x, \mathbb{C})$ splits as $d = \partial + \bar{\partial}$, where $\partial : \Omega^{\bullet,\bullet}(X) \to \Omega^{\bullet+1,\bullet}(X)$ and $\bar{\partial} : \Omega^{\bullet,\bullet}(X) \to \Omega^{\bullet,\bullet+1}(X)$. Since $\bar{\partial}^2 = 0$, we can take the *Dolbeault cohomology groups*

$$H^{p,q}(X) := \frac{\ker\big(\bar{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)\big)}{\operatorname{im}\big(\bar{\partial} : \Omega^{p,q-1}(X) \to \Omega^{p,q}(X)\big)}.$$

Note that for $q = 0$ we simply get $H^{p,0}(X) = \Omega^p(X)$. Dolbeault's theorem says that this holds more generally:

$$H^{p,q}(X) \cong H^q(X, \Omega^p)$$

for all p, q. Here the right hand side is the sheaf cohomology of the sheaf of holomorphic p-forms.

Now assume that X is a compact Kähler manifold. Then we have the famous *Hodge decomposition*

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X),$$

and under this isomorphism we have $\overline{H^{p,q}(X)} = H^{q,p}(X)$. Setting $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$, this means that $h^{p,q}(X) = h^{q,p}(X)$ and that we can express the Betti numbers of X as

$$b_k(X) = \sum_{p+q=k} h^{p,q}(X).$$

5.1.3 Divisors and line bundles

Let us now recall some facts about the group of divisors and the Picard group. We follow [Huy05, Sections 2.2 & 2.3].

Let X be a complex manifold. As we know, a *complex submanifold of codimension* k is a subset $Y \subset X$ that is locally given by the zero locus of a collection $\{f_1, \ldots, f_k\}$ of holomorphic functions, such that the Jacobian of the map $(f_1, \ldots, f_k)$ has maximal rank k. More generally, we have the notion of an *analytic subvariety*, which is defined in the same way except that we do not require the Jacobians involved to have maximal rank. A point on an analytic subvariety Y where the rank is not maximal is called *singular*, and we denote the collection of such points by Y_{sing}. Clearly, $Y_{\text{reg}} := Y \setminus Y_{\text{sing}}$ is just a complex submanifold. An analytic subvariety is called *irreducible* if it cannot be expressed as the proper union of two other subvarieties, and it is called a *hypersurface* if it has codimension one.

Definition 5.5. A *divisor* E on X is a finite formal combination of irreducible hypersurfaces, i.e.

$$D = \sum_{i=1}^{k} a_i Y_i$$

with $a_i \in \mathbb{Z}$ and Y_i an irreducible hypersurface. We denote by $\operatorname{Div}(X)$ the set of divisors, endowed with the obvious group structure. ♢

Remark 5.6. Traditionally, the definition is just a locally finite sum, but since we will only be dealing with compact complex manifolds, this definition works just as well.

Note that to any divisor we can associate a fundamental cohomology class, or Poincaré dual: indeed, if Y is an irreducible hypersurface, we obtain a class $[Y] \in H^2(X, \mathbb{R})$ by just taking the Poincaré dual of Y_{reg}, i.e. the class corresponding to integration over Y_{reg}.[4] Then we just extend this linearly, obtaining a cohomology class $[D]$ for every $D \in \operatorname{Div}(X)$. We want to determine a convenient criterion for when a cohomology class is the Poincaré dual of a divisor. For this, we need the Picard group.

Recall that a *holomorphic vector bundle* on X is just a (smooth) complex vector bundle with the extra property that the local trivialisations are required to be biholomorphic maps. As a consequence, holomorphic vector bundles are determined by their holomorphic cocycles, similar to the smooth case. In the case of line bundles, cocycles are of course just nonvanishing holomorphic functions. It is easy to see that the cocycles of the tensor product of two line bundles are just the products of the cocycles of the original bundles, and that the cocycles of the dual bundle of a line bundle are the inverses of the cocycles of the original line bundle. Together, this means that the following is well-defined.

Definition 5.7. We denote by $\operatorname{Pic}(X)$ the group of isomorphism classes of holomorphic line bundles, where the multiplication is induced by the tensor product, and the inversion is induced by taking the dual bundle. $\diamondsuit$

Of course, since a line bundle is determined by its cocycles, it follows that $\operatorname{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$. The first Chern class yields a group homomorphism $c_1 : \operatorname{Pic}(X) \to H^2(X, \mathbb{Z})$ and the identification $\operatorname{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$ allows us to write this in a different way: the short exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

gives us a long exact sequence of groups, part of which is

$$H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}).$$

It can be shown (see [Huy05, Corollary 2.3.10 & Proposition 4.4.12]) that the map $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$ corresponds to $c_1 : \operatorname{Pic}(X) \to H^2(X, \mathbb{Z})$.

The following now gives us what we want.

Proposition 5.8. *There exists a group homomorphism* $\operatorname{Div}(X) \to \operatorname{Pic}(X)$*, which we denote by* $D \mapsto \mathcal{O}_X(D)$*. For all* $D \in \operatorname{Div}(X)$ *we have*

$$c_1(\mathcal{O}_X(D)) = [D],$$

the Poincaré dual of D*.*

For a proof, see [Huy05, Corollary 2.3.10 & Proposition 4.4.13]. Now that we have a different description of the Poincaré dual, let us study the image of the map $\operatorname{Div}(X) \to \operatorname{Pic}(X) \to H^2(X, \mathbb{Z})$.

[4] In fact, one can define integration over Y by just integrating over Y_{reg} in the usual way, and one can show that integration descends to a map on the (de Rham) cohomology of X. Then $[Y]$ is the cohomology class corresponding to this integration.

From now on, we assume that X is a compact Kähler manifold. With the Hodge decomposition in hand, we define the *Néron-Severi lattice* to be

$$\mathrm{NS}(X) := H^{1,1}(X) \cap H^2(X, \mathbb{Z}).$$

Then we have the following.

Proposition 5.9 ([Huy05, Proposition 3.3.2]). *If X is a compact Kähler manifold, $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ has image $\mathrm{NS}(X)$.*

Since c_1 becomes the map $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$ from the exact sequence above under the isomorphism $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$, we get the following corollary.

Corollary 5.10. *If X is a compact Kähler manifold with $H^1(X, \mathcal{O}_X) = 0$, $c_1 : \mathrm{Pic}(X) \to \mathrm{NS}(X)$ is an isomorphism.*

We will see in the next section that this result holds for K3 surfaces. Hence we have established that, for K3 surfaces, the second part of $\mathrm{Div}(X) \to \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ is an isomorphism onto $\mathrm{NS}(X)$. Luckily, the image of the first map is well understood.

Proposition 5.11 ([Huy05, Proposition 2.3.18]). *For a line bundle $L \in \mathrm{Pic}(X)$, $H^0(X, L) \neq 0$ iff there exists an effective divisor $D \in \mathrm{Div}(X)$ such that $\mathcal{O}_X(D) = L$.*

5.1.4 Deformations of compact complex manifolds

In this section we will mention several results regarding families of complex manifolds, which we will use when constructing the moduli spaces of K3 surfaces. We mainly follow [BPV84, Section I.10] but also use some properties mentioned in [LP80, Section 5].

Definition 5.12. A *smooth family of compact complex manifolds* is a triple $(\mathfrak{X}, p, S)$ consisting of connected complex manifolds $\mathfrak{X}$ and S and a proper holomorphic map $p : \mathfrak{X} \to S$ which is everywhere of maximal rank, so that the fibres $\mathfrak{X}_s := p^{-1}(s)$ are compact complex submanifolds of $\mathfrak{X}$ for all $s \in S$. A *morphism* between families $(\mathfrak{X}_1, p_1, S_1)$ and $(\mathfrak{X}_2, p_2, S_2)$ is a pair of holomorphic maps $f : S_1 \to S_2$ and $F : \mathfrak{X}_1 \to \mathfrak{X}_2$ such that the diagram

$$\begin{tikzcd}
\mathfrak{X}_1 \arrow[r, "F"] \arrow[d, "p_1"'] & \mathfrak{X}_2 \arrow[d, "p_2"] \\
S_1 \arrow[r, "f"'] & S_2
\end{tikzcd}$$

commutes. ♢

Remark 5.13. We will only consider smooth families as defined above, but the general notion does not require this. Instead, X and S need only be *complex spaces*, i.e. ringed spaces which locally look like analytic subvarieties of domains in $\mathbb{C}^n$. However, it turns out that for K3 surfaces all relevant families will be smooth.

Remark 5.14. Note that for a smooth family as above, all fibres are diffeomorphic, by Ehresmann's theorem (in fact, the family is a fibre bundle). However, in general the fibres will not be isomorphic as complex manifolds.

As might be expected (and desired), there is a notion of pullback of these families along maps between bases. If $(\mathfrak{X}, p, S)$ is a smooth family and $f : S' \to S$ a holomorphic map, we set

$$\mathfrak{X}' := \mathfrak{X} \times_S S' = \{(x, s) \in \mathfrak{X} \times S' \mid p(x) = f(s)\}$$

and define $p' : \mathfrak{X}' \to S'$ by $p'(x, s) := p(x)$. Then $(\mathfrak{X}', p', S')$ is again a smooth family of compact complex manifolds. Note that we always have a morphism $(\mathfrak{X}', p', S') \to (\mathfrak{X}, p, S)$ covering f.

The main idea is to view these families as deformations of a specified fibre.

Definition 5.15. Let X be a compact complex manifold. A *smooth deformation* of X is a smooth family $(\mathfrak{X}, p, S)$ together with a basepoint $0 \in S$ and an isomorphism $X \cong \mathfrak{X}_0$. A smooth deformation $(\mathfrak{X}, p, S \ni 0)$ is called *complete* if for any other smooth deformation $(\mathfrak{X}', p', S' \ni 0')$ of X we have a holomorphic map $f : (S', 0') \to (S, 0)$ such that $(\mathfrak{X}', p', S')$ is isomorphic to the pullback of $(\mathfrak{X}, p, S)$ by f, and that this isomorphism is compatible with the isomorphisms $X \cong \mathfrak{X}_0$ and $X \cong \mathfrak{X}'_{0'}$. If in addition the map f is uniquely determined by $(\mathfrak{X}', p', S' \ni 0')$, then $(\mathfrak{X}, p, S \ni 0)$ is called *universal*. If just the differential $df(0') : T^{1,0}_{0'} S' \to T^{1,0}_0 S$ is uniquely determined, $(\mathfrak{X}, p, S \ni 0)$ is called *versal*. ♢

Remark 5.16. When we use the theory of deformations later on, we will have to shrink the bases of certain universal families. Thus the families we obtain in that way will only be locally universal, meaning that any other deformation will only locally be isomorphic to the pullback of the families in question. However, we will still call these families universal.

Associated to any deformation $(\mathfrak{X}, p, S \ni 0)$ of X we have the *Kodaira-Spencer map*, defined as follows. The normal bundle sequence of $X \cong \mathfrak{X}_0$ becomes

$$0 \to T^{1,0}X \to T^{1,0}\mathfrak{X}\big|_X \to X \times T^{1,0}_0 S \to 0$$

since the normal bundle is trivial. Taking the long exact sequence of the sheaf cohomology of the holomorphic sections of these bundles we obtain the Kodaira-Spencer map

$$\delta : T^{1,0}_0 S \to H^1(X, T^{1,0}X). \tag{5.1}$$

Let us now turn to the existence of these deformations. We summarise the for us most useful results in the following theorem.

Theorem 5.17. *Let X be a compact complex manifold. Then we have the following.*

(i) If $H^2(X, T^{1,0}X) = 0$ then X has a smooth, versal deformation whose associated Kodaira-Spencer map is an isomorphism.

(ii) If $H^0(X, T^{1,0}X) = 0$ then any versal deformation is universal. In fact, it has the even stronger property that not only the map between the bases but also the map between the total spaces is unique.[5]

[5] Let us make this more precise: denote the versal deformation by $(\mathfrak{X}, p, S \ni 0)$ and let $(\mathfrak{X}', p', S' \ni 0')$ be another deformation. Then $(\mathfrak{X}', p', S' \ni 0')$ is (isomorphic to) the pullback of $(\mathfrak{X}, p, S \ni 0)$ by a map f, and the whole associated morphism $(\mathfrak{X}', p', S' \ni 0') \to (\mathfrak{X}, p, S \ni 0)$ is uniquely determined by $(\mathfrak{X}', p', S' \ni 0')$, not just the base map $f : S' \to S$.

(iii) If the map $s \mapsto \dim_{\mathbb{C}} H^1(\mathfrak{X}_s, T^{1,0}\mathfrak{X}_s)$ is constant on the base S of a versal deformation, then this deformation is actually versal for all of its fibres.

5.2 Definition and basic properties

With the prerequisites out of the way, we can now give the definition of a K3 surface and determine its cohomology lattice, as in Example 5.2.

Definition 5.18. A *K3 surface* is a 1-connected, compact complex surface X with trivial canonical bundle. $\diamondsuit$

Here "complex surface" means that $\dim_{\mathbb{C}}(X) = 2$, i.e. $\dim_{\mathbb{R}}(X) = 4$.

The triviality of the canonical bundle tells us that $\Omega^{2,0}(X) \cong C^\infty(X, \mathbb{C})$. Since every holomorphic function on X is constant by compactness of X, we see that $H^{2,0}(X) = \Omega^2(X) \cong \mathbb{C}$, i.e. $h^{2,0}(X) = 1$.

Although the definition might seem rather simple, it turns out that K3 surfaces have many strong properties, many of which we will take advantage of in constructing the PMSCT. A remarkable fact is the following: *for every two K3 surfaces, the underlying smooth manifolds are diffeomorphic* (see [BPV84, Corollary VIII.8.6]). This will allow us to speak of *the* underlying smooth manifold of any K3 surface. The following is the "nicest" concrete model of a K3 surface.

Example 5.19. The *Fermat quartic* $X \subset \mathbb{C}P^3$ given by

$$X := \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{C}P^2 \mid x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}$$

is an algebraic K3 surface (see [Huy16, Example 1.3 (i)]). $\triangle$

Let us now determine the cohomology of a K3 surface X. Firstly, we have $H^0(X, \mathbb{Z}) \cong \mathbb{Z}$ and $H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ as always. Next, since X is simply connected, we have $H_1(X, \mathbb{Z}) = H^1(X, \mathbb{Z}) = 0$. By Poincaré Duality, we obtain $H^3(X, \mathbb{Z}) = 0$, and $H_1(X, \mathbb{Z}) = 0$ also implies that $H^2(X, \mathbb{Z})$ is torsion-free. To obtain more information about $H^2(X, \mathbb{Z})$ we need to use the complex structure of X. A very useful fact is that *every K3 surface X is Kähler*. This follows directly from the following result about complex surfaces.

Theorem 5.20 ([BPV84, Theorem IV.3.1])**.** *A compact complex surface X is Kähler iff $b_1(X)$ is even.*

Now, we can use the Hirzebruch-Riemann-Roch theorem ([BPV84, Theorem I.5.4]) to determine the Euler characteristic of X, and consequently also $b_2(X)$. Indeed, applied to the trivial line bundle $X \times \mathbb{C}$, whose sheaf of sections is just Ω^0, the theorem reads

$$\dim_{\mathbb{C}} H^0(X, \Omega^0) - \dim_{\mathbb{C}} H^1(X, \Omega^0) + \dim_{\mathbb{C}} H^2(X, \Omega^0) = \frac{1}{12} \int_X \big(c_1(X)^2 + c_2(X)\big).$$

Of course, by Dolbeault's theorem and the Hodge decomposition the left hand side is just

$$h^{0,0}(X) - h^{0,1}(X) + h^{0,2}(X) = h^{0,0}(X) + h^{2,0}(X) = 2.$$

Moreover, we have $c_1(X) = c_1(T^{1,0}X) = -c_1((T^{1,0}X)^*) = -c_1(\omega_X) = 0$ since ω_X is trivial and since the first Chern class does not change when taking top degree exterior power. Since $c_2(X) = e(X)$, the formula above gives us

$$\chi(X) = \int_X e(X) = 12 \cdot 2 = 24.$$

From this we deduce that $b_2(X) = 24 - 2 = 22$, meaning that $H^2(X,\mathbb{Z})$ has rank 22. Let us now turn to the lattice structure on $H^2(X,\mathbb{Z})$. To determine the index, we use the Thom-Hirzebruch index theorem ([BPV84, Theorem I.3.1]), which in (real) dimension four takes the form

$$\tau(X) = \frac{1}{3}\int_X p_1(X).$$

Since X is a complex manifold, we have the formula $p_1(X) = c_1(X)^2 - 2c_2(X) = -2c_2(X)$, and we obtain that $\tau(X) = -16$, meaning that $H^2(X,\mathbb{Z})$ has signature $(3,19)$. Next, we wish to show that the cup product is even. To see this, consider first the cup product of cohomology with $\mathbb{Z}/2$ coefficients. By definition of the Wu classes, we then have $a \smile a = v_2 \smile a$ for all $a \in H^2(X,\mathbb{Z}/2)$, where $v_2 = v_2(X) \in H^2(X,\mathbb{Z}/2)$ is the second Wu class. Now, by Wu's formula ([MS74, Theorem 11.14]) we have $v_2(X) = w_1(X)^2 + w_2(X)$. But $w_1(X) = 0$ since $H^1(X,\mathbb{Z}/2) = 0$ and $w_2(X) = 0$ since it is the image of $c_1(X) = 0$ under the coefficient homomorphism $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z}/2)$. Hence we see that the cup product $H^2(X,\mathbb{Z}/2) \times H^2(X,\mathbb{Z}/2) \to \mathbb{Z}/2$ is identically zero. But since $H_1(X,\mathbb{Z}) = 0$ this means precisely that the cup product on $H^2(X,\mathbb{Z})$ is even. Putting everything together, using Theorem 5.4, we obtain the following description of the cohomology of X.

Proposition 5.21. *Let X be a K3 surface. Then*

$$H^k(X,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \textit{if } k = 0,4, \\ \mathbb{Z}^{\oplus 22} & \textit{if } k = 2 \\ 0 & \textit{else} \end{cases}$$

as $\mathbb{Z}$-modules. Moreover, as a lattice, $H^2(X,\mathbb{Z})$ is isomorphic to the K3 lattice L.

Remark 5.22. A nice way of writing down the Hodge decomposition of a compact Kähler manifold is the *Hodge diamond*. By our computations above, the Hodge diamond of a K3 surface is given by the following.

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} \quad = \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Before we move on, let us mention that since $H_1(X,\mathbb{Z}) = 0$ we have $H^2(X,\mathbb{R}) = H^2(X,\mathbb{Z}) \otimes \mathbb{R}$ and $H^2(X,\mathbb{C}) = H^2(X,\mathbb{Z}) \otimes \mathbb{C}$. Consequently, setting $L_\mathbb{R} := L \otimes \mathbb{R}$ and $L_\mathbb{C} := L \otimes \mathbb{C}$, we have that $H^2(X,\mathbb{R}) \cong L_\mathbb{R}$ and $H^2(X,\mathbb{C}) \cong L_\mathbb{C}$. Of course, we can extend the $\mathbb{Z}$-linear form on L to an $\mathbb{R}$-valued (resp. $\mathbb{C}$-valued) form on $L_\mathbb{R}$ (resp. $L_\mathbb{C}$), and the above isomorphisms still hold when we consider the cup product form on the left and the extended form on the right. Clearly, all properties regarding nondegeneracy, signature (in the real case) etc. still hold for these forms.

5.3 The Kähler cone

In this section, we roughly follow [BPV84, Section VIII.3] and [Huy16, Chapter 8].

Let X be a K3 surface. As mentioned above, X is Kähler; this means that it makes sense about the *Kähler cone* of X, defined as follows. Leaving the complex structure fixed, every Kähler metric g induces a Kähler form ω, which defines a cohomology class since it is closed. As we know, such a class lives in $H^{1,1}(X,\mathbb{R}) := H^{1,1}(X) \cap H^2(X,\mathbb{R})$. In addition, since for every Kähler form ω we have $([\omega],[\omega]) = \int_X \omega \wedge \omega > 0$, we see that every Kähler class is actually an element of the set

$$\{a \in H^{1,1}(X,\mathbb{R}) \mid (a,a) > 0\}.$$

Using the following lemma, we can say more about this set.

Lemma 5.23. *The signature of the cup product restricted to $H^{1,1}(X,\mathbb{R})$ is $(1,19)$.*

Proof. As mentioned in the previous section, the cup product on $H^2(X,\mathbb{R})$ again has signature $(3,19)$. Now, the Hodge decomposition $H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$ implies that we have a similar decomposition

$$H^2(X,\mathbb{R}) = H^{1,1}(X,\mathbb{R}) \oplus \big((H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X,\mathbb{R})\big).^6$$

The second summand consists of elements of the form $\sigma + \bar{\sigma}$ or $i(\sigma - \bar{\sigma})$, with $\sigma \in H^{2,0}(X)$. For such σ we have $(\sigma,\sigma) = (\bar{\sigma},\bar{\sigma})$ (for type reasons) and $(\sigma,\bar{\sigma}) > 0$ if $\sigma \neq 0$ (this is easy to see in local coordinates). It follows that

$$(\sigma + \bar{\sigma}, \sigma + \bar{\sigma}) = (i(\sigma - \bar{\sigma}), i(\sigma - \bar{\sigma})) = 2(\sigma,\bar{\sigma}) > 0,$$

i.e. that the cup product is positive definite on $(H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X,\mathbb{R})$. Since the decomposition above is clearly orthogonal, again for type reasons, we conclude. $\square$

From this lemma it follows that $\{a \in H^{1,1}(X,\mathbb{R}) \mid (a,a) > 0\}$ consists of two connected components, which are mapped to each other by $-\mathrm{id}$, and each connected component is a *convex cone*; a set which is closed under positive linear combinations. Recall that the set of Kähler classes is also a convex cone.

Definition 5.24. We define the *Kähler cone* K_X to be the set of all Kähler classes of the K3 surface X. The connected component of $\{a \in H^{1,1}(X,\mathbb{R}) \mid (a,a) > 0\}$ in which K_X is contained is called the *positive cone* and is denoted C_X. $\diamondsuit$

Of course, this description of the Kähler cone is quite tautological, and we will spend the rest of this section finding a more useful characterisation of Kähler classes. Our starting point is the following. It is a well-known fact in complex geometry that the restriction of a Kähler form to a complex submanifold is again Kähler. For a complex surface, this means that integrating the Kähler form over a complex curve, i.e. a codimension one subvariety, has to yield a positive number. As it turns out, this property characterises the Kähler cone.

[6] This follows directly from the following: if V is a real vector space and W a subspace of the complexification of V such that $W = \overline{W}$, then W is the complexification of $W \cap V$.

Theorem 5.25 ([DP04, Theorem 0.1]). *Let X be a compact Kähler manifold. Then K_X is a connected component of the set*

$$\left\{ a \in H^{1,1}(X,\mathbb{R}) \,\middle|\, \int_Y a^k > 0 \text{ for all irreducible subvarieties } Y, \text{ where } k = \dim Y \right\}.$$

So for a K3 surface X we can now conclude that

$$K_X = \left\{ a \in C_X \,\middle|\, \int_D a > 0 \text{ for all irreducible curves } D \right\}.$$

Using the discussion from Section 5.1.3 we can simplify this even more. Indeed, we have seen there that for a K3 surface X the first Chern class $c_1 : \mathrm{Pic}(X) \to \mathrm{NS}(X)$ is an isomorphism and that we have a map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ denoted $D \mapsto \mathcal{O}_X(D)$ such that $c_1(\mathcal{O}_X(D))$ is just the Poincaré dual of D. Let us introduce some terminology: we call a class $d \in \mathrm{NS}(X)$ *effective* if there exists an effective divisor $D \in \mathrm{Div}(X)$ such that $d = c_1(\mathcal{O}_X(D))$, and similarly for *irreducible*. Then we can describe K_X as the set of classes $a \in C_X$ for which $(a,d) > 0$ for all irreducible, and thus also effective classes $d \in \mathrm{NS}(X)$.

Let us set $\Delta_X := \{d \in \mathrm{NS}(X) \mid (d,d) = -2\}$, the set of *roots*. Then we have the following lemma.

Lemma 5.26. *Let $d \in \mathrm{NS}(X)$ be such that $(d,d) \geq -2$. Then d or $-d$ is effective.*

Proof. On a K3 surface, the Hirzebruch-Riemann-Roch theorem for line bundles reads

$$h^0(X,L) - h^1(X,L) + h^2(X,L) = 2 + \frac{1}{2}c_1(L)^2.$$

By Serre duality (and since the canonical bundle is trivial), $h^2(X,L) = h^0(X,L^*)$, where L^* the dual bundle of L. So we get the inequality

$$h^0(X,L) + h^0(X,L^*) \geq 2 + \frac{1}{2}c_1(L)^2.$$

Now let $L \in \mathrm{Pic}(X)$ be the line bundle corresponding to d. Then $c_1(L)^2 \geq -2$ and the above formula tells us that L or L^* has a global non-trivial section. Then using Proposition 5.11 we conclude, since L^* corresponds to $-d$. □

Using this lemma, setting $\Delta_X^+ := \{d \in \mathrm{NS}(X) \mid (d,d) = -2 \text{ and } d \text{ is effective}\}$, we obtain $\Delta_X = \Delta_X^+ \sqcup (-\Delta_X^+)$. The following is now the desired description of the Kähler cone.

Proposition 5.27. *For a K3 surface X, we have*

$$K_X = \{a \in C_X \mid (a,d) > 0 \text{ for all } d \in \Delta_X^+\}.$$

In particular, and this will be important later, we see that $K_X \subset \{a \in C_X \mid (a,d) \neq 0 \text{ for all } d \in \Delta_X\}$.

In the proof of Proposition 5.27, we will use the following lemmas.

Lemma 5.28. *Let $a, b \in \overline{C}_X$, the closure of the positive cone. Then $(a,b) \geq 0$, and if at least one of a and b is actually contained in C_X, we have $(a,b) > 0$.*

For a proof, see [BPV84, Corollary IV.7.2].

Lemma 5.29. *Let $d \in \mathrm{NS}(X)$ be irreducible. Then $(d,d) \geq -2$.*

Proof of Proposition 5.27. We have to show that for a class $a \in C_X$, $(a,d) > 0$ for all irreducible $d \in \mathrm{NS}(X)$ if and only if $(a,d) > 0$ for all $d \in \Delta_X^+$. The implication " $\Longrightarrow$ " is obvious since classes in Δ_X^+ are effective. For the converse, let $d \in \mathrm{NS}(X)$ be irreducible. By Lemma 5.29 we have $(d,d) \geq -2$ and since the cup product on $H^2(X,\mathbb{Z})$ is even, we have $(d,d) \geq 0$ or $(d,d) = -2$. In the former case, $d \in \overline{C}_X$ or $d \in -\overline{C}_X$. The second possibility is excluded by Lemma 5.28 since $(a,d) > 0$ for Kähler classes a, which are contained in C_X. Hence $d \in \overline{C}_X$ and again by Lemma 5.28 we automatically have that $(a,d) > 0$ for all $a \in C_X$. So for $d \in \mathrm{NS}(X)$ irreducible satisfying $(d,d) \geq 0$ the condition is void. Thus the case $(d,d) = -2$ remains, which proves the converse. $\square$

Let us end this section by saying a bit more about roots and the positive cone. The upshot is that the roots will induce a chamber decomposition of C_X. Let us be more specific.

For $d \in \Delta_X$ we define the isometry $s_d : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ by

$$s_d(x) := x + (x,d)d.$$

We also denote its $\mathbb{R}$-linear extension $H^2(X,\mathbb{R}) \to H^2(X,\mathbb{R})$ by s_d; the maps s_d are called *Picard-Lefschetz reflections.* Note that s_d is indeed just the reflection in the hyperplane $d^\perp$, since it acts as the identity on that hyperplane and $s_d(d) = -d$. In particular, $s_d^2 = \mathrm{id}$. Also, since $d \in \mathrm{NS}(X) \subset H^{1,1}(X)$, s_d restricts to an isometry on $H^{1,1}(X,\mathbb{R})$. In fact, it also preserves the positive cone C_X: it obviously preserves $\{a \in H^{1,1}(X,\mathbb{R}) \mid (a,a) > 0\}$, and for $a \in C_X$ we have

$$(a, s_d(a)) = (a,a) + (a,d)^2 > 0$$

so that by Lemma 5.28 we conclude that $s_d(a) \in C_X$. We define the *Weyl group* W_X to be the subgroup of automorphisms of $H^{1,1}(X,\mathbb{R})$ generated by $\{s_d \mid d \in \Delta_X\}$.

Let us look closer into what the Weyl group does on the positive cone. For $d \in \Delta_X$ the set of fixed points in the positive cone $d^\perp \cap C_X$ is called a *wall*; the connected components of

$$C_X \setminus \bigcup_{d \in \Delta_X} (d^\perp \cap C_X)$$

are called *chambers*. We claim that the action of W_X on C_X leaves the union of walls invariant. This follows since for all $d, d' \in \Delta_X$ we have that $s_d(d') \in \Delta_X$ and that $s_d(x) \in s_d(d')^\perp$ if $x \in (d')^\perp$. Both these statements are easily checked. It follows now that W_X also acts on the set of chambers. A well-known fact is the following.

Proposition 5.30 ([Huy16, Proposition 8.5.5])**.** *The action of W_X on the set of chambers is free and transitive.*

Before we move on, let us mention one more fact we will use. A choice of *positive roots* is a set $\Delta'_X \subset \Delta_X$ such that $\Delta_X = \Delta'_X \sqcup (-\Delta'_X)$. The one can show that such Δ'_X are in one-to-one correspondence with chambers of C_X through the map which assigns to a chamber $C'_X \subset C_X$ the set

$$\{d \in \Delta_X \mid (a,d) > 0 \text{ for all } a \in C'_X\}.$$

This shows that *the Kähler cone is a chamber*, corresponding to the effective roots Δ_X^+ defined earlier.

5.4 The Torelli theorems

In this section we will state the *Torelli theorem*. This is essentially a way of integrating suitable isomorphisms between the second cohomology groups of K3 surfaces to biholomorphisms of the K3 surfaces themselves. Ultimately, this theorem will allow us obtain the “action part” of the previously laid out construction of a PMSCT.

Definition 5.31. Let X, X' be K3 surfaces. A $\mathbb{Z}$-module isomorphism $H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is called a *Hodge isometry* if

(i) it preserves the cup product form $(\cdot, \cdot)$, i.e. if it is an isometry;

(ii) its $\mathbb{C}$-linear extension $H^2(X', \mathbb{C}) \to H^2(X, \mathbb{C})$ preserves the Hodge decomposition.

A Hodge isometry is called *effective* if it maps a Kähler class of X' to a Kähler class of X. $\diamondsuit$

Remark 5.32. Note that since $h^{2,0}(X) = 1$ and $H^{0,2}(X) = \overline{H^{2,0}(X)}$, an isometry $H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ will be a Hodge isometry iff its $\mathbb{C}$-linear extension preserves $H^{2,0}(X)$, i.e. maps a nowhere vanishing holomorphic 2-form of X' to one of X.

Remark 5.33. There are several different (equivalent) definitions of effectiveness, the one above being the weakest. It is not so difficult to see that for a Hodge isometry $H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ the following are equivalent (see [BPV84, Proposition VIII.3.11]).

(i) It preserves the positive cone and induced a bijection between the sets of effective classes of X' and X.

(ii) It maps the Kähler cone of X' to the one of X.

(iii) It maps a Kähler class of X' to a Kähler class of X.

For obvious reasons we will use the weakest of these in the sequel.

We can now state the Torelli theorem, which is often called the *strong Torelli theorem*, for reasons which will become clear shortly.

Theorem 5.34. *Let $\varphi : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ be an effective Hodge isometry between K3 surfaces X, X'. Then there is a unique biholomorphism $f : X \to X'$ such that $f^* = \varphi$.*

For a proof, see [BPV84, Theorem VIII.11.1] and the preceding sections.

From the strong Torelli theorem we get the following corollary, often called the *weak Torelli theorem*.

Corollary 5.35. *Two K3 surfaces X, X' are biholomorphic iff there is a Hodge isometry $H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$.*

To deduce this from the strong Torelli theorem, we will use the Picard-Lefschetz from the previous section.

Proof of Corollary 5.35. The implication " $\Longrightarrow$ " is immediate, since all diffeomorphisms preserve the cup product and all biholomorphisms preserve the Hodge decomposition. For the converse, let $\varphi : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ be a Hodge isometry and let $a \in C_{X'}$ be a Kähler class. By definition, since φ is an isometry and preserves the Hodge decomposition, we know that $\varphi(a) \in C_X \sqcup (-C_X)$. So by composing with $-\mathrm{id}$ if necessary, we can assume that $\varphi(a) \in C_X$ (note that $-\mathrm{id}$ is actually a Hodge isometry). Again, since φ is an isometry, we see that $\varphi(a)$ actually lies in a chamber of C_X, because a lies in a chamber of $C_{X'}$. Using Proposition 5.30 it follows that we can find $\psi \in W_X$ such that $\psi \circ \varphi$ maps a to a Kähler class of X. It is easily verified that the Picard-Lefschetz reflections, and thus all elements of the Weyl group W_X are Hodge isometries. Using then that $\psi \circ \varphi : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is now an *effective* Hodge isometry, the strong Torelli theorem 5.34 implies that X and X' are biholomorphic. $\square$

5.5 Moduli spaces, universal families and period maps

In this section we will investigate the moduli spaces and universal families associated to K3 surfaces, and we will define the associated period maps. As it turns out, it is convenient to consider *marked K3 surfaces* for this; these are K3 surfaces X together with a choice of isometry $H^2(X, \mathbb{Z}) \to L$. For these marked K3 surfaces there exists a smooth moduli space and universal family. Quite naturally arising is then the *period map*, which assigns to a marked K3 surface the image of the nowhere-vanishing holomorphic 2-form under the marking. After this we will consider marked K3 surfaces together with a specified Kähler class. It turns out that we get a moduli space and universal family for these objects too, as well as the *refined period map*. An important fact will be that this refined period map is a diffeomorphism. We will mainly follow [BPV84, Sections VIII.12 & VIII.14] and [Huy16, Sections 6.2 & 6.3].

5.5.1 Marked K3 surfaces and the period map

Before we define the moduli space and period map, let us give some motivation for the definitions.

We have mentioned already in Section 5.2 that there is only one smooth manifold underlying K3 surfaces. Furthermore, it can be shown that any complex structure on this manifold defines a K3 surface (see [FM94, Theorem S.9]). Thus a K3 surface can be seen as a pair (X_0, I) consisting of a fixed smooth manifold X_0, e.g. the Fermat quartic from Example 5.19, and a complex structure I on X_0. Then we have the following correspondence.

Proposition 5.36. *There is a bijection*

$$\{\text{complex structures on } X_0\} \longleftrightarrow \{\sigma \in \Omega^2(X_0, \mathbb{C}) \mid d\sigma = 0, \sigma \wedge \sigma = 0, \sigma \wedge \bar{\sigma} > 0\}/\mathbb{C}^*.$$

Proof. We have already seen that for a K3 surface we have, up to scalar multiplication, a unique nowhere vanishing holomorphic 2-form; this gives us a map to the right.

For the other map, we note first that an almost complex structure on X_0 is the same as a decomposition

$$T_{\mathbb{C}} X_0 = T^{1,0} X_0 \oplus T^{0,1} X_0$$

such that $\overline{T^{1,0} X_0} = T^{0,1} X_0$. So when $\sigma \in \Omega^2(X_0, \mathbb{C})$ as above is given, we set $T^{0,1} X_0 := \ker(\sigma)$, where we view σ as a linear map $T_{\mathbb{C}} X_0 \to \Omega^1(X_0, \mathbb{C})$. The dimension of $T^{0,1} X_0$ is even since σ is skew-symmetric, the condition $\sigma \wedge \sigma = 0$ implies that σ is degenerate, i.e. the dimension is nonzero, and the condition $\sigma \wedge \bar{\sigma} > 0$ implies that $\sigma \neq 0$. In conclusion, $T^{0,1}X_0$ has dimension two. In fact, the condition $\sigma \wedge \bar{\sigma} > 0$ also implies that $T^{0,1}X_0 \cap \overline{T^{0,1}X_0} = 0$, meaning that we obtain an almost complex structure on X_0. Since $d\sigma = 0$, the Koszul formula implies that $T^{0,1}X$ is integrable, meaning that the almost complex structure integrates to a *unique* complex structure on X_0. Finally, note that multiplying σ by a nonzero scalar does not change the complex structure, giving us our map to the left.

To see that these maps are inverse to each other, note first that in the construction of the map to the left, σ actually becomes a holomorphic 2-form for the obtained complex structure, since by definition it vanishes on the anti-holomorphic tangent bundle $T^{0,1}X_0$. So starting with a σ on the right, the induced complex structure defines a K3 surface, meaning that σ is the (modulo $\mathbb{C}^*$) unique nowhere vanishing holomorphic 2-form, so it gets mapped back to itself modulo $\mathbb{C}^*$. Conversely, starting with a complex structure on X_0, with anti-holomorphic tangent bundle $T^{0,1}X_0$, the complex structure induced by the nowhere vanishing holomorphic 2-form will have the same anti-holomorphic tangent bundle $T^{0,1}X_0$, and by uniqueness of integrating almost complex structures the two complex structure must be the same. This concludes the proof. □

The point of this proposition is to show that it will be a good strategy to try to classify K3 surfaces by their holomorphic 2-forms. As it turns out, to get a smooth structure on the moduli space, it is more convenient to consider *marked K3 surfaces*, defined below.

Definition 5.37. A *marked K3 surface* is a pair (X, φ) consisting of a K3 surface X and a *marking* φ, i.e. an isometry $\varphi : H^2(X, \mathbb{Z}) \to L$. Two marked K3 surfaces (X, φ) and (X', φ') are *equivalent* if there exists a biholomorphism $f : X \to X'$ such that $\varphi \circ f^* = \varphi'$. We define the *moduli space of marked K3 surfaces* to be the set of equivalence classes

$$M_1 := \{(X, \varphi)\}/\sim . \qquad \diamondsuit$$

Remark 5.38. Note that when (X, φ) is a marked K3 surface, we can extend φ to linear maps $H^2(X, \mathbb{R}) \to L_{\mathbb{R}}$ and $H^2(X, \mathbb{C}) \to L_{\mathbb{C}}$. We will still denote these maps by φ.

Let us denote by $\mathbb{P}(L_{\mathbb{C}}) = L_{\mathbb{C}}/\mathbb{C}^*$ the projectivation of $L_{\mathbb{C}}$. Motivated by Proposition 5.36, we define the *period domain* to be

$$\Omega := \{[\sigma] \in \mathbb{P}(L_{\mathbb{C}}) \mid (\sigma, \sigma) = 0, (\sigma, \bar{\sigma}) > 0\}.$$

Note that the conditions make sense since they are invariant under (nonzero) scalar multiplication. Now we can define the period map.

Definition 5.39. We define the *period map* $\tau_1 : M_1 \to \Omega$ by

$$[(X, \varphi)] \mapsto [\varphi(\sigma_X)],$$

where σ_X is a nowhere vanishing holomorphic 2-form on X. ♢

Remark 5.40. Note that the period map is well-defined: if $[(X, \varphi)] = [(X', \varphi')]$, there is a biholomorphism $f : X \to X'$ such that $\varphi \circ f^* = \varphi'$. Hence $\sigma_X := f^*(\sigma_{X'})$ is a nowhere vanishing holomorphic 2-form on X if $\sigma_{X'}$ is one on X', meaning that

$$\tau_1([X', \varphi']) = [\varphi'(\sigma_{X'})] = [(\varphi \circ f^*)(\sigma_{X'})] = [\varphi(\sigma_X)] = \tau_1([X, \varphi]).$$

We have the following.

Theorem 5.41 ([Huy16, Theorem 6.3.1]). *The period map $\tau_1 : M_1 \to \Omega$ is surjective.*

One might wonder whether the period map is injective as well. As it turns out, it isn't. We have the following result, which is essentially a restatement of the weak Torelli theorem 5.35.

Corollary 5.42. *Let X, X' be K3 surfaces. Then X and X' are biholomorphic iff we can find markings $\varphi : H^2(X, \mathbb{Z}) \to L$ and $\varphi' : H^2(X', \mathbb{Z}) \to L$ such that $\tau_1([X, \varphi]) = \tau_1([X', \varphi'])$.*

Proof. We know that X and X' are biholomorphic iff there exists a Hodge isometry $\Phi : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$. So assume that we have such a Hodge isometry. Then for any marking $\varphi : H^2(X, \mathbb{Z}) \to L$ for X, we can set $\varphi' := \varphi \circ \Phi$ and then we clearly have $\tau_1([X, \varphi]) = \tau_1([X', \varphi'])$. Conversely, given markings φ, φ' such that $\tau_1([X, \varphi]) = \tau_1([X', \varphi'])$, we see that $\varphi^{-1} \circ \varphi'$ is a Hodge isometry, by Remark 5.32. □

Note that all the discussion so far has been purely set-theoretical: M_1 is just a set and τ_1 a map between sets. Let us now go into the smooth structures involved. As it turns out, the following discussion will also allow us to construct the universal family of marked K3 surfaces.

Smoothness of M_1 and τ_1 and the universal family

The basic idea is to put a complex structure on M_1 by gluing together local deformations of K3 surfaces. The following is an important fact.

Proposition 5.43 ([BPV84, Proposition IV.4.4]). *Let X be a K3 surface, and let $(\mathfrak{X}, p, S \ni 0)$ be a deformation of X. Then, after possibly shrinking the base S, every fibre of $p : \mathfrak{X} \to S$ is also a K3 surface.*

The following lemma will give us existence of deformations of K3 surfaces.

Lemma 5.44. *Let X be a K3 surface. Then*

(i) $H^0(X, T^{1,0}X) = 0$,

(ii) $\dim_{\mathbb{C}} H^1(X, T^{1,0}X) = 20$,

(iii) $H^2(X, T^{1,0}X) = 0$.

Proof. This follows directly from Serre duality and our computations in Section 5.2. □

Putting these results and Theorem 5.17 together, we get the following.

Corollary 5.45. *Let X be a K3 surface. Then there exists a smooth universal deformation of X, all whose fibres are K3 surfaces, which is universal for all of its fibres and whose associated Kodaira-Spencer map is an isomorphism (for all fibres). Consequently, the base has dimension 20.*

Of course, in order to work towards M_1 we need to somehow introduce markings to these families. So let (X, φ) be a marked K3 surface and let $(\mathfrak{X}, p, S \ni 0)$ be a deformation as in Proposition 5.43, and assume that S is contractible (or just simply connected). Then the marking for $X \cong X_0$ induces markings $\varphi_s : H^2(X_s, \mathbb{Z}) \to L$ for all fibres of $p : \mathfrak{X} \to S$, and these markings vary smoothly in the sense that their $\mathbb{R}$-linear extensions induce a trivialisation of vector bundles

$$\bigsqcup_{s \in S} H^2(X_s, \mathbb{R}) \cong S \times L_{\mathbb{R}}.$$

We can then define a local period map for this family, just like we did earlier, by setting

$$\tau : S \to \Omega, \quad s \mapsto [\varphi_s(\sigma_s)], \tag{5.2}$$

where σ_s is a nowhere vanishing holomorphic 2-form on X_s. The following is an important property of the local period map.

Proposition 5.46 ([Huy16, Propositions 2.3 & 2.4])**.** *The local period map (5.2) is holomorphic. Moreover, if the Kodaira-Spencer map associated to $(\mathfrak{X}, p, S \ni 0)$ is an isomorphism, then the differential $d\tau(0) : T_0^{1,0} S \to T_{\tau(0)}^{1,0} \Omega$ is an isomorphism.*

Putting this together with Corollary 5.45 we obtain the following.

Corollary 5.47. *Let (X, φ) be a K3 surface. After possibly shrinking the base, the family we get for X from Corollary 5.45 inherits markings for all its fibres, such that the associated local period map is a local biholomorphism. Thus, after possibly shrinking the base again, we can assume that the local period map is an embedding.*

We already saw that when two marked K3 surfaces are equivalent, they get mapped to the same point by the period map. Hence we know that in a family as in Corollary 5.47, no two fibres are equivalent as marked K3 surfaces. With this important remark in hand, we can start putting a complex structure on M_1.

Indeed, note first that we might just as well build M_1 as follows: we take the disjoint union, indexed over all marked K3 surfaces, of the bases of the families of Corollary 5.47, and we identify two points if the fibres over them are equivalent as marked K3 surfaces. Now the remark just made implies that the bases of the families from Corollary 5.47 all inject into M_1, so it makes sense to try to endow M_1 with a complex structure using the ones on the bases of the families. Indeed, since the families we are dealing with are universal, we know that for every class in M_1, and for any two families having a fibre belonging to that class, their bases must be locally biholomorphic around the corresponding points. This means that if we use the injections of the bases of the families from Corollary 5.47 to obtain charts for M_1, using two different families will still give holomorphically compatible charts; without going too far into the details, this essentially proves the following.

Theorem 5.48. *The moduli space M_1 has the structure of a* 20*-dimensional complex manifold.*

In fact, the discussion above shows more: the extra property from Theorem 5.17 (ii) allows us to also glue together the families in the same way we just did with the bases, and in fact the markings glue together to a global marking of the family we obtain. So we also get a smooth universal family over M_1 with smoothly varying markings for all fibres, and it is easy to see that the local period maps glue together to the global period map τ_1 defined earlier.

Corollary 5.49. *There exists a smooth universal family $\mathcal{U} \to M_1$ of marked K3 surfaces, such that for all $t \in M_1$ the fibre X_t is endowed with a marking φ_t for which $[(X_t, \varphi_t)] = t$. The period map $\tau_1 : M_1 \to \Omega$ is a local biholomorphism with respect to the complex structure on M_1.*

Note that once again these markings vary smoothly in the same sense as before.

With this, we have achieved quite a lot: we have a complex manifold parametrising the marked K3 surfaces, and the period map relating it to the (relatively) simple space Ω. However, we are not quite done. Firstly, M_1 can be shown *not* to be Hausdorff, and while the period map τ_1 is a surjective local biholomorphism, it is *not* injective.[7]

We will solve these problems by constructing more a more refined moduli space and period map, using Kähler classes. We saw already with the Torelli theorems that involving Kähler classes can get us stronger results, and the same will happen here.

5.5.2 Marked pairs and the refined period map

The idea here is quite simple: we take the moduli space M_1 and over every point we take all the Kähler classes of the fibre of $\mathcal{U} \to M_1$ over that point.

First, since by Ehresmann's theorem $\mathcal{U} \to M_1$ is a fibre bundle, we have the associated vector bundle

$$\mathcal{H}^2 = \bigsqcup_{t \in M_1} H^2(X_t, \mathbb{C}).$$

Then we have the following lemma.

Lemma 5.50 ([BPV84, Lemma VIII.9.3])**.** *The set*

$$\bigsqcup_{t \in M_1} H^{1,1}(X_t, \mathbb{R})$$

forms a real-analytic subbundle of $\mathcal{H}^2$. Moreover, the set M_2 of all Kähler classes is open in it.

From this lemma it follows that M_2 is a real-analytic manifold of dimension 60. A point of M_2 should be thought of as a class of marked K3 surfaces with a specified Kähler class.

Let us now turn to the refined period map. First, we need a new period domain. We already saw that for a Kähler form ω on a K3 surface we have $\omega \wedge \omega > 0$, and obviously for type reasons we have $\omega \wedge \sigma = 0$ for σ the holomorphic 2-form. Hence it makes sense to define

$$K\Omega := \{(k, [\sigma]) \in L_{\mathbb{R}} \times \Omega \mid (k, \sigma) = 0, (k, k) > 0\}.$$

However, we actually know more about the Kähler cone, by Proposition 5.27. Noting that, for any K3 surface X, Δ_X just consists of elements in $H^2(X, \mathbb{Z})$ which have self-intersection -2 and are perpendicular to $H^{2,0}(X)$, we define

$$K\Omega^0 := \{(k, [\sigma]) \in K\Omega \mid (k, d) \neq 0 \text{ for all } d \in L \text{ such that } (d, d) = -2 \text{ and } (d, \sigma) = 0\}. \quad (5.3)$$

One can show (see [BPV84, Lemma VIII.9.2]) that this is an open subset of $K\Omega$.

Now we can define the period map.

[7] In fact, these two inconveniences are related; if two points cannot be separated, they get mapped to the same point by the period map (see [Huy16, Proposition 7.2.1]).

Definition 5.51. The *refined period map* $\tau_2 : M_2 \to K\Omega^0$ is defined by

$$(t,k) \mapsto (\varphi_t(k), \tau_1(t)). \qquad \diamondsuit$$

It is quite easy to see that this is a real-analytic map, since the markings vary smoothly with t.

Denoting now by $\pi : M_2 \to M_1$ the restriction of the obvious projection $\mathcal{H}^2 \to M_1$, and by $\mathrm{pr} : K\Omega^0 \to \Omega$ the restriction of the projection $L_{\mathbb{R}} \times \Omega \to \Omega$, we have the obvious commutative diagram

$$\begin{tikzcd} M_2 \arrow[r, "\tau_2"] \arrow[d, "\pi"'] & K\Omega^0 \arrow[d, "\mathrm{pr}"] \\ M_1 \arrow[r, "\tau_1"'] & \Omega. \end{tikzcd}$$

As might be expected from the proof of Corollary 5.42, the strong Torelli theorem gives us injectivity of the refined period map.

Corollary 5.52. *The refined period map $\tau_2 : M_2 \to K\Omega^0$ is injective.*

Proof. Assume that $\tau_2(t,k) = \tau_2(t',k')$. Then $\varphi_t^{-1} \circ \varphi_{t'} : H^2(X_{t'}, \mathbb{Z}) \to H^2(X_t, \mathbb{Z})$ is an effective Hodge isometry, which by the strong Torelli theorem 5.34 integrates to a biholomorphism $f : X_t \to X_{t'}$, i.e. $f^* = \varphi_t^{-1} \circ \varphi_{t'}$. This means that $t = t'$, hence also $\varphi_t = \varphi_{t'}$, which implies that $k = k'$. $\square$

Thankfully, surjectivity continues to hold as well.

Theorem 5.53 ([BPV84, Theorem VIII.14.1])**.** *The refined period map $\tau_2 : M_2 \to K\Omega^0$ is a diffeomorphism.*

In particular, we see that M_2 is actually Hausdorff.

We obtain a family over $M_2 \cong K\Omega^0$ by pulling back the one over M_1.

Definition 5.54. We set

$$K\mathcal{U} := (\pi \circ \tau_2^{-1})^* \mathcal{U}. \qquad \diamondsuit$$

By itself, this is just a real-analytic family[8] over $K\Omega^0$, but we should remember that there is a lot of extra data floating around. Namely, every fibre, say over $(k, [\sigma])$, is a K3 surface for which we have a specified marking and Kähler class, so that the marking sends the Kähler class to k. Moreover, the markings still vary smoothly.

The next step is to turn $K\mathcal{U}$ into a Poisson manifold by choosing suitable representatives of the specified Kähler classes. This we will do in the next section, using the famous Calabi-Yau theorem.

[8]We will not really go into the precise definition, but the basic idea is the same as before. The difference is that the spaces and maps involved don't belong to the complex category: instead, the base space is real-analytic, as is the projection, and the total space is a so-called *Levi flat CR-manifold*. Roughly, this means that it is foliated by complex submanifolds, and in the case of a real-analytic family we want these submanifolds to be the fibres of the family.

5.6 The Calabi-Yau theorem and the Poisson structure on $K\mathcal{U}$

In simple terms, the Calabi-Yau theorem allows you to, under suitable assumptions, select a canonical respresentative for a given Kähler class. To state the theorem, we need some concepts from Riemannian geometry.

Let (X, g) be a Riemannian manifold, let ∇ be the Levi-Civita connection and let R be the associated curvature $(1, 3)$-tensor.

Definition 5.55. The *Ricci tensor* Ric is the covariant 2-tensor defined by

$$\operatorname{Ric}(X_1, X_2) := \operatorname{Tr}\big(X_3 \mapsto R(X_3, X_1)X_2\big).$$

We say that g is an *Einstein metric* if there exists $\lambda \in \mathbb{R}$ such that $\operatorname{Ric} = \lambda g$. If $\lambda = 0$, i.e. $\operatorname{Ric} = 0$, g is called *Ricci flat*. ♢

The situation becomes more interesting when we consider Kähler manifolds.

Definition 5.56. Let X be a Kähler manifold with Kähler metric g and almost complex structure J. The *Ricci form* ρ is the form associated to the Ricci tensor:

$$\rho(X_1, X_2) := \operatorname{Ric}(JX_1, X_2).$$

The metric g is said to be *Kähler-Einstein* if it is also Einstein, or equivalently if there exists $\lambda \in \mathbb{R}$ such that

$$\rho = \lambda\omega,$$

where ω is the Kähler form associated to g. If in addition $\lambda = 0$, g is called *Calabi-Yau*. ♢

The following are not too difficult general properties of the Ricci form.

Proposition 5.57 ([Bes87, Proposition 2.45, 2.47 & 2.75])**.** *The Ricci form of a Kähler manifold X is closed, of type $(1, 1)$ and represents the first Chern class of X.*

From this we see that on a compact Kähler manifold with vanishing first Chern class, a metric is Kähler-Einstein iff it is Calabi-Yau, i.e. Ricci flat.

The Calabi-Yau theorem essentially gives an inverse to Proposition 5.57.

Theorem 5.58 ([Yau78, Theorem 2])**.** *Let X be a compact Kähler manifold with Kähler metric g and Kähler form ω. Let ρ be a closed $(1, 1)$-form representing the first Chern class of X. Then there exists a unique Kähler metric g' on X whose Ricci form is ρ and whose associated Kähler form ω' is cohomologous to ω, i.e. $[\omega] = [\omega'] \in H^2(X, \mathbb{R})$.*

For manifolds with vanishing first Chern class we can take $\rho = 0$ and we obtain the following corollary, which we phrase in terms of Kähler classes.

Corollary 5.59. *Let X be a compact Kähler manifold with vanishing first Chern class. Then for any Kähler class $k \in K_X$ there exists a unique Ricci flat Kähler metric whose Kähler form belongs to k.*

Of course this corollary holds for any K3 surface, and thus we can use it to endow the fibres of $K\mathcal{U} \to K\Omega^0$ with symplectic forms. In fact, it is not too difficult to see that this establishes a Poisson structure on $K\mathcal{U}$: by construction, the Kähler classes on the fibres of $K\mathcal{U}$ "vary smoothly" when moving through $K\Omega^0$. Now, the metric we obtain from Corollary 5.59 is actually the solution of a differential equation (see [Yau78]), so that the symplectic forms on the fibres continue to vary smoothly in the sense that they define a symplectic foliation, and thus a Poisson structure by Proposition 2.22. So we have the following.

Proposition 5.60. *The family $K\mathcal{U}$ admits a regular Poisson structure $\pi_{K\mathcal{U}}$ such that the symplectic leaves are the fibres of $K\mathcal{U} \to K\Omega^0$ and such that the symplectic form on a fibre X over $(k,[\sigma])$, with marking φ, is the Kähler form associated to the unique Ricci flat Kähler metric on X with Kähler class $\varphi^{-1}(k)$.*

5.7 The action on $K\mathcal{U}$

Looking back at Proposition 4.16, we see that we have already obtained a large part of the required data: we have a "Poisson fibre bundle" whose base explicitely determines the class of the forms on the symplectic leaves. The last major ingredient is the action, which we will focus on in this section.

Let us denote by $O(L)$ the group of isometries of L, i.e. isomorphisms that preserve $(\cdot,\cdot)$. We have the obvious actions of $O(L)$ on $L_{\mathbb{R}}$ and Ω, and it is easily verified that $K\Omega^0$ is an invariant subset of $L_{\mathbb{R}} \times \Omega$ endowed with the diagonal action.

Obtaining an action of $O(L)$ on $K\mathcal{U}$ is more work. The following gives us all we want, setting us up to start the construction in the next section.

Proposition 5.61. *There is a smooth, equivariant Poisson action of $O(L)$ on $K\mathcal{U}$, i.e. $O(L)$ acts equivariantly with respect to the projection $K\mathcal{U} \to K\Omega^0$ and through Poisson maps.*

Proof. Let $\gamma \in O(L)$. First, we will construct for any $(k,[\sigma]) \in K\Omega^0$ a map between the fibres over $(k,[\sigma])$ and $\gamma\cdot(k,[\sigma]) = (\gamma(k),[\gamma(\sigma)])$. So let X be the K3 surface over $(k,[\sigma])$, with marking φ, Kähler class k_X and nowhere vanishing holomorphic 2-form ω_X, and let $X', \varphi', k_{X'}, \sigma_{X'}$ be the same objects over $(\gamma(k),[\gamma(\sigma)])$. Then since

$$\varphi'(k_{X'}) = \gamma(k) = (\gamma \circ \varphi)(k_X),$$
$$\varphi'(\omega_{X'}) = \gamma(\sigma) = (\gamma \circ \varphi)(\omega_X),$$

we see that $\varphi^{-1} \circ \gamma^{-1} \circ \varphi' : H^2(X',\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is an effective Hodge isometry. Thus from the strong Torelli theorem 5.34 we get a unique biholomorphism $f_\gamma : X \to X'$ integrating it. We already saw that on any K3 surface there are no global holomorphic vector fields (Lemma 5.44 (i)), and then it follows from [Mee09] that the fiberwise biholomorphisms f_γ together form an automorphism $F_\gamma : K\mathcal{U} \to K\mathcal{U}$. The uniqueness part of the strong Torelli theorem implies that $F_{\gamma\circ\gamma'} = F_\gamma \circ F_{\gamma'}$ for $\gamma, \gamma' \in O(L)$ so that we actually get an action on $K\mathcal{U}$. By construction it is obvious that this action is equivariant, and to check that F_γ is a Poisson map is equivalent to showing that it preserves the symplectic forms on the fibres.

So let $f_\gamma : X \to X'$ (and all other notation) be as above and let $\omega_X, \omega_{X'}$ be the symplectic forms on X and X' respectively. Since f_γ is a biholomorphism, to show that $f_\gamma^*\omega_{X'} = \omega_X$, it suffices to show that $f_\gamma^* g' = g$, where g, g' are the Kähler metrics associated to $\omega_X, \omega_{X'}$

respectively. From the uniqueness part of Corollary 5.59 we see that it suffices to show that $f_\gamma^* g'$ is a Ricci-flat Kähler metric on X with class k_X. But g' is a Ricci-flat Kähler metric on X', so since f_γ is a biholomorphism, $f_\gamma^* g'$ is one X. Finally, the Kähler class of $f_\gamma^* g'$ is $f^* k_{X'} = k_X$, and we are done. $\square$

6 Examples of PMSCT

We are now in position to construct explicit examples of PMSCT. The underlying idea is very simple using the results about K3 surfaces.

Proposition 6.1. *Assume that we have an embedding $f : \mathbb{R}^q \to K\Omega^0$ and a subgroup $\Gamma \subset O(L)$ such that*

(i) the first component of f has the form

$$(x_1, \dots, x_q) \mapsto a + \sum_{i=1}^{q} x_i a_i$$

for fixed $a, a_1, \dots, a_q \in L$, with $\{a_1, \dots, a_q\}$ linearly independent;

(ii) the action of Γ on $K\Omega^0$ preserves the image of f;

(iii) the induced action on $\mathbb{R}^q$ is free and proper and by integral affine maps.

Then $M := f^*K\mathcal{U}/\Gamma$ is a Poisson manifold of strong **s***-proper type with leaf space $B := \mathbb{R}^q/\Gamma$. If B is compact, M is a PMSCT.*

Proof. This is basically a direct consequence of Proposition 4.16, and the discussion before it. Indeed, $f^*K\mathcal{U}$ is by definition a Poisson manifold whose symplectic leaves are the fibres of $f^*K\mathcal{U} \to \mathbb{R}^q$. In fact, the fibres are K3 surfaces, and we still have smoothly varying markings for them. Moreover denoting the fibre over $x \in \mathbb{R}^q$ by X_x, with marking φ_x, we know that the cohomology class of the symplectic form on X_x is given by $\varphi_x^{-1}(f_1(x))$, where f_1 denotes the first component of f.

The construction of the action of $O(L)$ on $K\mathcal{U}$ shows that the action of Γ restricts to $f^*K\mathcal{U}$, and that this action is still equivariant with respect to $f^*K\mathcal{U} \to \mathbb{R}^q$, and that the action is still by Poisson maps. So we are in the situation of Proposition 4.16, and the assumption about the first component of f allows us to apply said proposition. Indeed, letting $\mathcal{H}^2$, $\mathcal{H}^2_{\mathbb{Z}}$ and ϖ be as in Section 4.3, and defining $s \in \Gamma(\mathcal{H}^2)$ by $x \mapsto \varphi_x^{-1}(a)$ and defining $c_i \in \Gamma(\mathcal{H}^2_{\mathbb{Z}})$ by $x \mapsto \varphi_x^{-1}(a_i)$ we obtain

$$\varpi = s + \sum_{i=1}^{q} \mathrm{pr}_i c_i$$

by what we said before. This concludes the proof. $\square$

We will now give three explicit examples. Let us establish some notation before we start. Recall that L has three copies of U and two copies of $-E_8$. We will denote the standard bases of the three copies of U by $\{u, v\}$, $\{x, y\}$ and $\{z, t\}$. This means that $(u, v) = (x, y) = (z, t) = 1$ and the rest of the combinations between them yield zero. Recall also that $-E_8$ is negative definite. Finally, we need the following number theoretic fact, which can be found in [Bes40].

Theorem 6.2. *Let $k \in \mathbb{Z}_{\geq 1}$, let $p_1, \dots, p_k$ be distinct primes and let $n_1, \dots, n_k \in \mathbb{Z}_{\geq 2}$. For all $1 \leq i \leq k$, let x_i be a positive real root of $x^{n_i} - p_i = 0$. If P is a polynomial in k variables with rational coefficients, whose degree for the i-th variable is stricly less than n_i, then $P(x_1, \dots, x_k) = 0$ iff all coefficients of P are zero.*

We need this theorem to give us existence of the following set. We let $\{e_1, \dots, e_8\}$ be a set of real numbers such that the set

$$\{1, e_1, \dots, e_8, e_1^2, e_1e_2, e_1e_3, \dots, e_7^2, e_7e_8, e_8^2\},$$

consisting of $1, e_1, \dots, e_8$ and their pairwise products, is linearly independent over the integers, or equivalently over the rationals. We then set $e = (e_1, \dots, e_8) \in (-E_8)_{\mathbb{R}}$, where we scale if necessary so that $|(e,e)| \leq \frac{1}{2}$, and we set $a = (0, e) \in (-E_8)_{\mathbb{R}}^{\oplus 2}$, $b = (e, 0) \in (-E_8)_{\mathbb{R}}^{\oplus 2}$.

6.1 A PMSCT with leaf space the circle

This is the example originally given in [Mar13]. We will construct a PMSCT whose leaf space is S^1 with its standard integral affine structure, i.e. we want the standard action of $\mathbb{Z}$ on $\mathbb{R}$ (recall Example 4.14 (i)). Consider the map $f : \mathbb{R} \to L_{\mathbb{R}} \times \mathbb{P}(L_{\mathbb{C}})$ defined by

$$s \mapsto (2u + v + sy, [x - su + 2y + a + i(z + 2t + b)])$$

and the map $\varphi : L \to L$ defined by $u \mapsto u, v \mapsto v + y$, $x \mapsto x - u, y \mapsto y$ on the first two copies of U and as the identity on the other summands of L.

Claim 1. The map f is an embedding $\mathbb{R} \to K\Omega^0$.

Claim 2. We have $\varphi \in O(L)$.

Assuming this for the moment, let $\Gamma := \langle \varphi \rangle \subset O(L)$. Let us verify the conditions of Proposition 6.1. Condition (i) is of course verified. Note then that

$$\begin{aligned}
\varphi \cdot f(s) &= (\varphi(2u + v + sy), [\varphi(x - su + 2y + a + i(z + 2t + b))]) \\
&= (2u + v + y + sy, [x - u - su + 2y + a + i(z + 2t + b)]) \\
&= (2u + v + (s+1)y, [x - (s+1)u + 2y + a + i(z + 2t + b)]) \\
&= f(s+1).
\end{aligned}$$

This implies that the image of f is invariant under the action of Γ, and also that the induced action on $\mathbb{R}$ is just the standard action of $\mathbb{Z}$ on $\mathbb{R}$. So conditions (ii) and (iii) are verified, and since $S^1 = \mathbb{R}/\mathbb{Z}$ is compact Proposition 6.1 implies that we obtain a PMSCT with leaf space S^1.

Proof of Claim 1. It suffices to show that f maps into $K\Omega^0$. Indeed, the first component of f is clearly an embedding into $L_{\mathbb{R}}$, so that f is an embedding into $L_{\mathbb{R}} \times \mathbb{P}(L_{\mathbb{C}})$ (it is clearly smooth). Since $K\Omega^0$ is an embedded submanifold of the latter, if f maps into $K\Omega^0$ it is also an embedding into $K\Omega^0$.

So let $s \in \mathbb{R}$. Setting $f_1(s) = 2u + v + sy$, $f_2(s) = x - su + 2y + a$ and $f_3(s) = z + 2t + b$, we see that

$$
\begin{aligned}
(f_2(s), f_2(s)) &= (x - su + 2y + a, x - su + 2y + a) \\
&= 4(x, y) + (a, a) \\
&= 4 + (e, e) \geq 3\frac{1}{2} > 0, \\
(f_3(s), f_3(s)) &= (z + 2t + b, z + 2t + b) \\
&= 4(z, t) + (b, b) \\
&= 4 + (e, e) \geq 3\frac{1}{2} > 0, \\
(f_2(s), f_3(s)) &= (x - su + 2y + a, z + 2t + b) \\
&= 0.
\end{aligned}
$$

These computations imply that $[f_2(s) + if_3(s)] \in \Omega$. Since

$$
\begin{aligned}
(f_1(s), f_2(s)) &= (2u + v + sy, x - su + 2y + a) = -s(v, u) + s(y, x) = -s + s = 0, \\
(f_1(s), f_3(s)) &= (2u + v + sy, z + 2t + b) = 0,
\end{aligned}
$$

we see that $f(s) \in K\Omega$. It remains to check the extra condition on $K\Omega^0$. So assume that we have $d \in L$ such that $(d, d) = -2$ and $(d, f_1(s)) = (d, f_2(s)) = (d, f_3(s)) = 0$. We need to find a contradiction. Let us write

$$
d = Au + Bv + Cx + Dy + Ez + Ft + d_1 + d_2,
$$

with $A, \ldots, F \in \mathbb{Z}$ and d_i in the i-th copy of $-E_8$. Since $-E_8$ is even and positive definite, we can write $(d_i, d_i) = -2n_i$, for $n_i \in \mathbb{Z}_{\geq 0}$. The above conditions then translate into three equations;

$$
AB + CD + EF = n_1 + n_2 - 1, \tag{6.1}
$$

$$
2B + A + Cs = 0, \tag{6.2}
$$

$$
D - Bs + 2C + (d_2, e) = 0, \tag{6.3}
$$

$$
F + 2E + (d_1, e) = 0. \tag{6.4}
$$

This is where the seemingly strange choice of e comes in. Indeed, (d_1, e) is just an integral linear combination of the coordinates $\{e_1, \ldots, e_8\}$ of e, and since these coordinates were chosen such that, in particular, $\{1, e_1, \ldots, e_8\}$ is linearly independent over the integers, it follows from Equation 6.4 that we must have $F + 2E = 0$ and that the coefficients in front of the e_i must be zero. The latter implies that $d_1 = 0$, and thus also $n_1 = 0$.[9] Next, we need to consider two cases, namely $C = 0$ and $C \neq 0$. In the first case, Equation 6.2 yields $2B + A = 0$, and Equation 6.1 becomes

$$
2B^2 + 2E^2 = 1 - n_2.
$$

Of course, this implies that $B = E = 0$ and $n_2 = 1$. But then $d_2 \neq 0$ and Equation 6.3 becomes

$$
D + (d_2, e) = 0,
$$

[9]This can be seen easily by writing out the explicit form, as given in Example 5.3 (ii).

which together with $d_2 \neq 0$ contradicts the "linear independence" assumption on e.

So let us consider the case $C \neq 0$. Then from Equation 6.2 we get

$$s = -\frac{2B + A}{C},$$

and substituting this into Equation 6.3 yields

$$AB + CD = -2C^2 - 2B^2 - (d_2, e).$$

Combining this with Equation 6.1 gives

$$2B^2 + 2C^2 + 2E^2 + C(d_2, e) = 1 - n_2.$$

From the properties of e we get $Cd_2 = 0$, implying that $d_2 = 0$ and thus also that $n_2 = 0$, so that we are left with

$$2B^2 + 2C^2 + 2E^2 = 1,$$

which is obviously impossible. $\square$

Proof of Claim 2. To see that φ is an isomorphism, note that we get an inverse by setting $u \mapsto u, v \mapsto v - y$ and $x \mapsto x + u, y \mapsto y$ on the first two copies of U and extending by the identity. Let us now check that φ preserves $(\cdot, \cdot)$. Since only the first two copies of U are involved, it suffices to check

$$\begin{aligned}
(\varphi(u), \varphi(u)) &= (u, u), & (\varphi(x), \varphi(x)) &= (x - u, x - u) = (x, x),\\
(\varphi(u), \varphi(v)) &= (u, v + y) = (u, v), & (\varphi(x), \varphi(y)) &= (x - u, y) = (x, y),\\
(\varphi(v), \varphi(v)) &= (v + y, v + y) = (v, v), & (\varphi(y), \varphi(y)) &= (y, y),\\
(\varphi(u), \varphi(x)) &= (u, x - u) = (u, x), & (\varphi(v), \varphi(x)) &= (v + y, x - u)\\
& & &= (v, x) + (y, x) - (v, u) = (v, x),\\
(\varphi(u), \varphi(y)) &= (u, y), & (\varphi(v), \varphi(y)) &= (v + y, y) = (v, y).
\end{aligned}$$

So indeed $\varphi \in O(L)$. $\square$

6.2 A PMSCT with leaf space the standard torus

It is not that difficult to adapt the above strategy to the 2-dimensional case, which will yield a PMSCT with leaf space the torus $\mathbb{T}^2$ with its standard integral affine structure from Example 4.14 (ii). Consider the map $f : \mathbb{R}^2 \to L_{\mathbb{R}} \times \mathbb{P}(L_{\mathbb{C}})$ defined by

$$(s, r) \mapsto (2u + v + sy + rt, [x - su + 2y + a + i(z - ru + 2t + b)]),$$

the map $\varphi : L \to L$ as in the previous example and the map $\psi : L \to L$ defined by $u \mapsto u, v \mapsto v + t$, $z \mapsto z - u, t \mapsto t$ on two copies of U and as the identity on the other summands of L.

Claim 1. The map f is an embedding $\mathbb{R}^2 \to K\Omega^0$.

Claim 2. We have $\varphi, \psi \in O(L)$.

Again, taking this for granted for now, set $\Gamma := \langle \varphi, \psi \rangle$. The other conditions are not difficult to check. Condition (i) is once again trivial, while (ii) and (iii) follow from the computations

$$\begin{aligned}
\varphi \cdot f(s,r) &= (\varphi(2u+v+sy+rt), [\varphi(x-su+2y+a+i(z-ru+2t+b))]) \\
&= (2u+v+y+sy+rt, [x-u-su+2y+a+i(z-ru+2t+b)]) \\
&= (2u+v+(s+1)y+rt, [x-(s+1)u+2y+a+i(z-ru+2t+b)]) \\
&= f(s+1,r), \\
\psi \cdot f(s,r) &= (\psi(2u+v+sy+rt), [\psi(x-su+2y+a+i(z-ru+2t+b))]) \\
&= (2u+v+t+sy+rt, [x-su+2y+a+i(z-u-ru+2t+b)]) \\
&= (2u+v+sy+(r+1)t, [x-su+2y+a+i(z-(r+1)u+2t+b)]) \\
&= f(s,r+1).
\end{aligned}$$

The above also shows that the induced action of Γ on $\mathbb{R}^2$ is the standard one of $\mathbb{Z}^2$, so that Proposition 6.1 yields a PMSCT with leaf space the standard torus $\mathbb{T}^2$.

Proof of Claim 1. Like in the previous case, it suffices to show that f maps into $K\Omega^0$. So let f_1, f_2, f_3 be the three "components" of f, as before, and let $(s,r) \in \mathbb{R}^2$. We compute

$$\begin{aligned}
(f_2(s,r), f_2(s,r)) &= (x-su+2y+a, x-su+2y+a) \\
&= 4(x,y)+(a,a) \\
&= 4+(e,e) \geq 3\frac{1}{2} > 0, \\
(f_3(s,r), f_3(s,r)) &= (z-ru+2t+b, z-ru+2t+b) \\
&= 4(z,t)+(b,b) \\
&= 4+(e,e) \geq 3\frac{1}{2} > 0, \\
(f_2(s,r), f_3(s,r)) &= (x-su+2y+a, z-ru+2t+b) \\
&= 0
\end{aligned}$$

and conclude that $[f_2(s,r) + i f_3(s,r)] \in \Omega$. Also,

$$\begin{aligned}
(f_1(s,r), f_2(s,r)) &= (2u+v+sy+rt, x-su+2y+a) = -s(u,v)+s(x,y) = -s+s = 0, \\
(f_1(s,r), f_3(s,r)) &= (2u+v+sy+rt, z-ru+2t+b) = -r(u,v)+r(z,t) = -r+r = 0
\end{aligned}$$

implies that $f(s,r) \in K\Omega$. To check the final condition, let $d \in L$ such that $(d,d) = -2$ and $(d, f_1(s)) = (d, f_2(s)) = (d, f_3(s)) = 0$ and write again

$$d = Au + Bv + Cx + Dy + Ez + Ft + d_1 + d_2,$$

and let $n_i \in \mathbb{Z}_{\geq 0}$ as before. Like before, we need to find a contradiction. The relevant equations now become

$$AB + CD + EF = n_1 + n_2 - 1, \tag{6.5}$$

$$2B + A + Cs + Er = 0, \tag{6.6}$$

$$D - Bs + 2C + (d_2, e) = 0, \tag{6.7}$$

$$F - Br + 2E + (d_1, e) = 0. \tag{6.8}$$

Let us first consider the case $B = 0$. Then the assumptions on e, together with Equations 6.7 and 6.8, imply that $d_1 = d_2 = 0$, so $n_1 = n_2 = 0$, and $D + 2C = F + 2E = 0$. But then Equation 6.5 becomes

$$2C^2 + 2E^2 = 1,$$

which is clearly not possible.

So consider the case $B \neq 0$. Then from Equations 6.7 and 6.8 we get

$$s = \frac{D + 2C + (d_2, e)}{B}, \qquad r = \frac{F + 2E + (d_1, e)}{B}.$$

Substituting this into Equation 6.6 gives

$$AB + CD + EF = -2B^2 - 2C^2 - 2E^2 - C(d_2, e) - E(d_1, e),$$

and combining this with Equation 6.5 we obtain

$$2B^2 + 2C^2 + 2E^2 + C(d_2, e) + E(d_1, e) = 1 - n_1 - n_2.$$

The assumptions on e imply that $Cd_2 + Ed_1 = 0$, so that this becomes

$$2B^2 + 2C^2 + 2E^2 = 1 - n_1 - n_2.$$

This implies that $B = C = E = 0$ and that $n_1 = 1, n_2 = 0$ or $n_1 = 0, n_2 = 1$. But $B = C = E = 0$ together with Equations 6.7 and 6.8 and the assumptions on e implies that $d_1 = d_2 = 0$, so $n_1 = n_2 = 0$, which gives a contradiction.

□

Proof of Claim 2. We already saw that $\varphi \in O(L)$. Showing that $\psi \in O(L)$ is exactly the same argument, just exchanging $\{x, y\} \leftrightarrow \{z, t\}$. □

6.3 A PMSCT with leaf space a non-standard torus

In this example we will construct a PMSCT whose leaf space is still a torus, but with a different induced integral affine structure, namely the one from Example 4.14 (iii). Consider the map $f : \mathbb{R}^2 \to L_{\mathbb{R}} \times \mathbb{P}(L_{\mathbb{C}})$ defined by

$$(s, r) \mapsto (2u + v + sy + rt, [x + (r^2 - s)u - rz + 2y + a + i(z - ru + 2t + 2ry + b)])$$

the map $\varphi : L \to L$ as in the previous examples and the map $\psi : L \to L$ defined by $u \mapsto u, v \mapsto v + t$, $x \mapsto x - z + u, y \mapsto y$, $z \mapsto z - u, t \mapsto t + y$ on the copies of U and as the identity on the other summands of L.

Claim 1. The map f is an embedding $\mathbb{R}^2 \to K\Omega^0$.

Claim 2. We have $\varphi, \psi \in O(L)$.

Again, we assume this for now, setting $\Gamma := \langle \varphi, \psi \rangle$. The verification of the conditions in Proposition 6.1 is the same as before, now building on the computation

$$
\begin{aligned}
\varphi \cdot f(s,r) &= \big(\varphi(2u+v+sy+rt), \\
&\qquad [\varphi(x+(r^2-s)u-rz+2y+a+i(z-ru+2t+2ry+b))]\big) \\
&= \big(2u+v+y+sy+rt, \\
&\qquad [x-u+(r^2-s)u-rz+2y+a+i(z-ru+2t+2ry+b)]\big) \\
&= \big(2u+v+(s+1)y+rt, \\
&\qquad [x-u+(r^2-(s+1))u-rz+2y+a+i(z-ru+2t+2ry+b)]\big) \\
&= f(s+1,r), \\
\psi \cdot f(s,r) &= \big(\psi(2u+v+sy+rt), \\
&\qquad [\psi(x+(r^2-s)u-rz+2y+a+i(z-ru+2t+2ry+b))]\big) \\
&= \big(2u+v+t+sy+rt+ry, \\
&\qquad [x-z+u+(r^2-s)u-rz+ru+2y+a+i(z-u-ru+2t+2y+2ry+b)]\big) \\
&= \big(2u+v+(s+r)y+(r+1)t, \\
&\qquad [x+((r+1)^2-(s+r))u-(r+1)z+2y+a+i(z-(r+1)u+2t+2(r+1)y+b)]\big) \\
&= f(s+r,r+1).
\end{aligned}
$$

With the same arguments as before, this gives us a PMSCT with leaf space the "non-standard" torus from Example 4.14 (iii).

Proof of Claim 1. Again, we need only show that f maps into $K\Omega^0$. Denote once more by f_1, f_2, f_3 the "components" of f, and let $(s,r) \in \mathbb{R}^2$. Since

$$
\begin{aligned}
(f_2(s,r), f_2(s,r)) &= (x+(r^2-s)u-rz+2y+a, x+(r^2-s)u-rz+2y+a) \\
&= 4(x,y)+(a,a) \\
&= 4+(e,e) \geq 3\frac{1}{2} > 0, \\
(f_3(s,r), f_3(s,r)) &= (z-ru+2t+2ry+b, z-ru+2t+2ry+b) \\
&= 4(z,t)+(b,b) \\
&= 4+(e,e) \geq 3\frac{1}{2} > 0, \\
(f_2(s,r), f_3(s,r)) &= (x+(r^2-s)u-rz+2y+a, z-ru+2t+2ry+b) \\
&= 2r(x,y)-2r(z,t) = 2r-2r = 0.
\end{aligned}
$$

we get that $[f_2(s,r)+if_3(s,r)] \in \Omega$. The computations

$$
\begin{aligned}
(f_1(s,r), f_2(s,r)) &= (2u+v+sy+rt, x+(r^2-s)u-rz+2y+a) \\
&= (r^2-s)(u,v)+s(x,y)-r^2(z,t) = r^2-s+s-r^2 = 0, \\
(f_1(s,r), f_3(s,r)) &= (2u+v+sy+rt, z-ru+2t+2ry+b) = -r(u,v)+r(z,t) = -r+r = 0
\end{aligned}
$$

show that $f(s, r) \in K\Omega$. To see that $f(s, r) \in K\Omega^0$, let again $d \in L$ such that $(d, d) = -2$ and $(d, f_1(s)) = (d, f_2(s)) = (d, f_3(s)) = 0$. Like before we write

$$d = Au + Bv + Cx + Dy + Ez + Ft + d_1 + d_2,$$

and we set $(d_i, d_i) = -2n_i$ with $n_i \in \mathbb{Z}_{\geq 0}$. The goal is again to find a contradiction. The main equations are now

$$AB + CD + EF = n_1 + n_2 - 1, \tag{6.9}$$
$$2B + A + Cs + Er = 0, \tag{6.10}$$
$$D + B(r^2 - s) - Fr + 2C + (d_2, e) = 0, \tag{6.11}$$
$$F - Br + 2E + 2Cr + (d_1, e) = 0. \tag{6.12}$$

First consider the case $B - 2C = 0$. Then Equation 6.12 tells us that $d_1 = 0$ and $F + 2E = 0$. We also know that $C \neq 0$, since $C = 0$ implies that $B = 0$, so that Equation 6.9 becomes

$$2E^2 = 1 - n_2.$$

This is only possible if $E = 0$ and $n_2 = 1$, but then also $F = 0$ and Equation 6.11 becomes

$$D + (d_2, e) = 0,$$

which would imply that $d_2 = 0$, contradicting $n_2 = 1$.

So we indeed know that $C \neq 0$. Then Equation 6.10 tells us that

$$s = -\frac{2B + A + Er}{C},$$

and with Equation 6.11 we obtain

$$Br^2 - 2Fr + 2A + 5B + D + (d_2, e) = 0.$$

Since $B \neq 0$ and $r \in \mathbb{R}$, we must have that

$$F^2 \geq 2AB + 5B^2 + BD + B(d_2, e).$$

But $F = -2E$ and $B = 2C$, so combining this with Equation 6.9 yields

$$1 - n_2 \geq 10C^2 + C(d_2, e).$$

Since $C \neq 0$, this is certainly impossible when C and (d_2, e) have the same parity. So let us assume that they have opposite parity, so that the equation becomes

$$1 - n_2 \geq 10C^2 - |C| \cdot |(d_2, e)|. \tag{6.13}$$

Now both d_2 and e lie in the same copy of $-E_8$, and since $(\cdot, \cdot)$ is negative definite on $-E_8$ we can use the Cauchy-Schwarz inequality to obtain

$$|(d_2, e)| \leq \sqrt{|(d_2, d_2)| \cdot |(e, e)|} = \sqrt{2 \cdot |(e, e)| n_2} \leq \sqrt{n_2},$$

using that we chose e such that $|(e,e)| \leq \frac{1}{2}$. Now, in order for Equation 6.13 to hold we certainly must have

$$n_2 \leq |C| \cdot |(d_2, e)|,$$

since $10C^2 > 1$. So we get

$$n_2 \leq |C| \cdot \sqrt{n_2},$$

i.e.

$$|C| \geq \sqrt{n_2}.$$

But then

$$10C^2 - |C| \cdot |(d_2, e)| \geq 10C^2 - n_2 \geq 10n_2 - n_2 = 9n_2 > 1 - n_2,$$

which contradicts Equation 6.13.

It remains to look at the case $B - 2C \neq 0$. In that case, we must also have $B \neq 0$. To see this, assume to the contrary that $B = 0$. Then we must have $F \neq 0$. Indeed, if we had $F = 0$, Equation 6.11 would become

$$D + 2C + (d_2, e) = 0,$$

meaning that $d_2 = 0$, so $n_2 = 0$, and $D + 2C = 0$. But then Equation 6.9 becomes

$$2C^2 = 1 - n_1,$$

which can only hold if $C = 0$ and $n_1 = 1$. But then Equation 6.12 becomes

$$2E + (d_1, e) = 0,$$

which implies $d_1 = 0$, contradicting $n_1 = 1$. So we see indeed that $F \neq 0$. But then Equations 6.11 and 6.12 yield

$$r = -\frac{F + 2E + (d_1, e)}{2C} = \frac{D + 2C + (d_2, e)}{F}.$$

This becomes

$$2CD + 4C^2 + 2C(d_2, e) + F^2 + 2EF + F(d_1, e) = 0,$$

and the assumptions on e imply that $2Cd_2 + Fd_1 = 0$ and

$$2CD + 4C^2 + F^2 + 2EF = 0.$$

Since $B = 0$, combining this with Equation 6.9 we obtain

$$4C^2 + F^2 = 2(1 - n_1 - n_2).$$

Both C and F are nonzero, meaning that this is impossible. So indeed $B \neq 0$. This means that we can write

$$r = \frac{F + 2E + (d_1, e)}{B - 2C}, \qquad s = \frac{D + Br^2 - Fr + 2C + (d_2, e)}{B}.$$

This yields

$$s = \frac{(B - 2C)^2(2C + D + (d_2, e)) + B(F + 2E + (d_1, e))^2 - F(B - 2C)(F + 2E + (d_1, e))}{B(B - 2C)^2}$$

and substituting this into Equation 6.10 gives

$$\begin{aligned}0 &= 2B^2(B-2C)^2 + AB(B-2C)^2 \\ &\quad + C\Big((B-2C)^2(2C+D+(d_2,e)) + B(F+2E+(d_1,e))^2 - F(B-2C)(F+2E+(d_1,e))\Big) \\ &\quad + BE(B-2C)(F+2E+(d_1,e)).\end{aligned}$$

Using the assumptions on e (actually, finally using them to their full potential), this reduces to

$$\begin{aligned}0 &= 2B^2(B-2C)^2 + AB(B-2C)^2 \\ &\quad + C\Big((B-2C)^2(2C+D) + B(F+2E)^2 - F(B-2C)(F+2E)\Big) \\ &\quad + BE(B-2C)(F+2E).\end{aligned}$$

Some rewriting turns this into

$$\begin{aligned}0 &= (B-2C)^2\big(2B^2+2C^2+AB+CD\big) \\ &\quad + BC(2E+F)^2 - CF(B-2C)(2E+F) + BE(B-2C)(F+2E),\end{aligned}$$

and some easy computations show that the second line is equal to

$$EF(B-2C)^2 + 2(BE+CF)^2,$$

so that altogether we obtain

$$(B-2C)^2\big(2B^2+2C^2+AB+CD+EF\big) + 2(BE+CF)^2 = 0.$$

Combining this with Equation 6.9 we get

$$2\Big((B-2C)^2(B^2+C^2) + (BE+CF)^2\Big) = (B-2C)^2(1-n_1-n_2).$$

But this is impossible, since $B-2C \neq 0$, $B \neq 0$ and $n_1, n_2 \geq 0$, giving us the desired contradiction. □

Proof of Claim 2. We know already that $\varphi \in O(L)$. For ψ, note that we can define an inverse by setting $u \mapsto u, v \mapsto v - t + y, x \mapsto x + z, y \mapsto y, z \mapsto z + u, t \mapsto t - y$ and extending by the identity. Then it suffices to check

$$\begin{aligned}(\varphi(u),\varphi(u)) &= (u,u), & (\varphi(x),\varphi(x)) &= (x-z+u, x-z+u) = (x,x),\\ (\varphi(u),\varphi(v)) &= (u,v+t) = (u,v), & (\varphi(x),\varphi(y)) &= (x-z+u, y) = (x,y),\\ (\varphi(v),\varphi(v)) &= (v+t,v+t) = (v,v), & (\varphi(y),\varphi(y)) &= (y,y),\\ (\varphi(z),\varphi(z)) &= (z-u,z-u) = (z,z), & &\\ (\varphi(z),\varphi(t)) &= (z-u,t+y) = (z,t), & &\\ (\varphi(t),\varphi(t)) &= (t+y,t+y) = (t,t), & &\end{aligned}$$

$$
\begin{aligned}
(\varphi(u),\varphi(x)) &= (u, x-z+u) = (u,x), & (\varphi(v),\varphi(x)) &= (v+t, x-z+u)\\
& & &= (v,x)+(v,u)-(t,z) = (v,x),\\
(\varphi(u),\varphi(y)) &= (u,y), & (\varphi(v),\varphi(y)) &= (v+t,y) = (v,y),\\
(\varphi(u),\varphi(z)) &= (u,z-u) = (u,z), & (\varphi(v),\varphi(z)) &= (v+t,z-u)\\
& & &= (v,z)-(v,u)+(t,z) = (v,z),\\
(\varphi(u),\varphi(t)) &= (u,t+y) = (u,t), & (\varphi(v),\varphi(t)) &= (v+t,t+y) = (v,t),\\
(\varphi(x),\varphi(t)) &= (x-z+u,t+y) & (\varphi(y),\varphi(z)) &= (y,z-u) = (y,z),\\
&= (x,t)+(x,y)-(z,t) = (x,t), & &\\
(\varphi(x),\varphi(z)) &= (x-z+u,z-u) = (x,z), & (\varphi(y),\varphi(t)) &= (y,t+y) = (y,t).
\end{aligned}
$$

So indeed $\psi \in O(L)$. $\square$

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