Automorphic Forms Related to Moduli Spaces of Curves

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Abstract

In the experimental calculation of the cohomology of moduli spaces of curves of genus g, some parts have been found that relate to Siegel modular forms of degree g, see [4] by Bergström, Faber and Van der Geer. However, there are also other parts of the cohomology that are not understood. Some of these parts seem to correspond to automorpic representations that Thomas Mégarbané studied in [26]. Our task is to give an explanation for this correspondence.

In this thesis we describe the setting of the problem, and give the argument for the correspondence with Siegel modular forms. We will also explain the way to view these modular forms as automorphic forms and representations. Finally, we try to give an interpretation for some of the unexplained parts of the cohomology by studying automorphic forms associated to even lattices.

Contents

Part I

Introduction

In order to make the structure of the thesis clearer the chapters are grouped together in parts. This part contains the chapters that introduce the reader to the contents and the form of this thesis. These chapters are specifically:

—Preface and Goals (p. 11).

—Prerequisites and Conventions (p. 15).

—Ch. 1, Overview of the Thesis (p. 17), where we give a concise explanation of the most important things in the thesis. It might be a bit difficult to fully comprehend everything at once, but it is meant as a quick way to get to know the theory.

—Ch. 2, The Case of Elliptic Curves (p. 23). This chapter discusses the example of modular forms occuring in the cohomology of the moduli space of elliptic curves. As these curves are familiar for many mathematicians, it is a good introduction to the subject. This example leads us to the first interesting results.

Preface and Goals

This is the master thesis of Koen Morel for finishing the studies Mathematical Sciences at Utrecht University.

The point of this chapter is first to give a brief history of the research on moduli spaces of curves and automorphic representations. After that, we will state the goals of this thesis, and give an outline of the way we want to reach these goals. A more thorough introduction to the mathematics in this thesis will be given in Ch. 1.

Moduli Spaces of Curves

As a set, a moduli space is the set of isomorphism classes of certain (geometric) objects. Our main focus of study is the moduli space of curves of a certain genus. We are especially interested in the genus 3 case. (The genus 1 and genus 2 moduli spaces are better understood for our purposes.) Of course the set of these curves is interesting already, but the moduli spaces must be seen as geometric objects themselves, to capture how curves vary in families. We therefore study them as algebraic varieties, schemes, or even stacks.

Moduli spaces of curves have been studied for a long time. Pierre Deligne and David Mumford came with a famous article in 1969 that established the irreducibility of these spaces [11]. As the moduli spaces of curves are notoriously complicated, for example in the sense that they are not smooth, this is already quite an accomplishment. Other research revolved around the different ways of constructing them, as well as trying to compactify them.

These moduli spaces are interesting objects in their own right, but they are particularly interesting to study as they give us more insight in the theory of algebraic curves. For example, a very basic property of a moduli space is its dimension. The dimension of the moduli spaces of curves of genus g is 1 for $g = 1$ and $3g - 3$ for $g > 1$. This exemplifies how many variables there are to specify a curve of a certain genus.

A more specific reason with respect to this thesis for studying moduli spaces is that it gives us a compelling example of a correspondence that is a central theme in the Langlands program. This conjectured correspondence is between Galois representations on the one hand and automorphic representations on the other. Galois representations are found in a natural way in the cohomology of the moduli spaces, and we relate them to automorphic representations that on a first glance seem to have nothing to do with moduli spaces of curves.

Another situation where curves and automorphic representations (in the form of classical modular forms) are related is in the Modularity Theorem which allowed Andrew Wiles to prove Fermat's Last Theorem. However, the modular forms that play a role in this theorem are of weight 2 and therefore of level > 1, and we will focus on modular forms (and automorphic representations) of level 1. However, we do find level > 1 if we consider curves with level structure.

Anyway, it is clear that there is an intruiging and perhaps unexpected connection between algebraic curves and automorphic representations, which are objects that are studied in completely different areas of mathematics. This should give researchers the motivation to continue studying this connection, and it gives us a reason for pursuing the subject of this thesis.

For more information on curves and their moduli spaces, we refer to Ch. 3 and Ch. 4.

History of Modular Forms and Automorphic Representations

As explained above, besides moduli spaces we also need to treat automorphic representations. The first example of automorphic representations is given by modular forms, which have been studied since the nineteenth century, well before the generalization to automorphic forms and representations was mentioned in the literature.

Classical modular forms are holomorphic functions on the upper half plane (complex numbers with positive imaginary part) which satisfy a certain transformation property. One of the interesting applications of the theory of modular forms is in number theory. It gives an easy proof of certain identities involving sums of powers of divisors. In this thesis they will occur as parts in the cohomology of the moduli space of curves of genus 1 (elliptic curves).

A generalization of these classical modular forms that is useful for the study of moduli spaces is Siegel modular forms, introduced by Carl Ludwig Siegel in 1939 [31]. Bluntly speaking these are modular forms where the upper half plane is replaced by a higher dimensional analogue. These modular forms occur quite naturally in the cohomology of moduli spaces of abelian varieties. Because there is a map with the right properties between the moduli space of curves of genus g and the moduli space of abelian varieties of dimension g, some of these Siegel modular forms will occur in the cohomology of moduli spaces of curves.

However, as stated in the abstract, not all parts of the cohomology of the moduli spaces of curves can be explained using Siegel modular forms. Therefore, we may need an even further generalization of modular forms.

These are called automorphic forms. Siegel modular forms are automorphic forms for symplectic groups, but we can also define automorphic forms for other linear algebraic groups. Most notably, we can consider certain (special) orthogonal groups, and we will study the automorphic forms for these groups using some theory of lattices.

We can realize automorphic forms as representations of the considered algebraic group on a suitable vector space. We then speak of automorphic representations. This point of view is particularly useful as it translates automorphic forms to the language of the Langlands program. Recent research has adapted the perspective of automorphic representations.

The exact definitions of modular forms, automorphic forms and automorphic representations are given in Ch. 6, 7 and 8.

Recent Research

Faber and others have studied moduli spaces and their cohomology by calculating the number of isomorphism classes of curves of a certain genus over finite fields [12, 4]. We will explain the exact method in Sec. 5.3.

Chenevier-Lannes [8] and Chenevier-Renard [7] have done important work in establishing what all automorphic representations are of small weight.

Mégarbané has found a way to calculate the trace of Hecke operators acting on automorphic representations for certain special orthogonal groups [26]. These Hecke traces are exactly the same as values that Faber finds in parts of his calculations on the cohomology of moduli spaces of curves of genus 3.

Goals of this Thesis

The goal of this thesis is to examine the role of automorphic forms and automorphic representations in relation to the cohomology of moduli spaces of curves and of abelian varieties. We tried to address this goal by pursuing the following sub-goals:

—review the role of Siegel modular forms in the cohomology of these moduli spaces,

—explain the automorphic representation point of view as used in the articles of Chenevier & Lannes [8] and Chenevier & Renard [7], and

—try to explain the connection between the cohomology of the moduli space of curves of genus 3 and the automorphic representations found by Mégarbané [26].

In Ch. 1 we will quickly sketch in what way we want to achieve these goals and we will state the main results of the thesis. We will also explain what the reader can expect in each of the chapters of the thesis.

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But especially I want to thank my supervisor, Carel Faber, for handing me such a fascinating subject, for his patience, and for sharing some of his interesting ideas.

Prerequisites and Conventions

Prerequisites

We will assume undergraduate mathematics such as linear algebra, group theory, topology, ring theory, etcetera. If the reader wants to review parts of these subjects, there are more than enough books to find, and moreover all that we really need can be found on the internet as well.

We will also assume knowledge of algebraic geometry including schemes and sheaf cohomology. As this is the only part of the prerequisites that is quite advanced, we might explain some basic things in Pt. II of the thesis. Otherwise we will refer to Algebraic Geometry by Hartshorne [20], as this contains all the things that we want to use.

The reader does not need to be familiar with classical modular forms, as the basics of the theory are explained in this thesis, mostly in Ch. 2. However, the readers who do know a little bit about the matter have an advantage, as they do not need to get used to the theory and might better appreciate the advances that are made in the thesis. We encourage the reader to indulge in modular forms, as it has some surprising and nice applications and it has become a major part of the modern mathematical research.

Some chapters in this thesis also have as prerequisite that earlier chapters have to be read! For example the chapters 3–4–5 have to be read in order, and the same holds for 6–7–8 and 9–10–11.

Style of the Thesis

As explained before, the chapters are grouped together in parts. Moreover, many paragraphs of the text have a title, in order to make sure that it is clear what the contents are of that piece of text. The precise hierarchy is part–chapter–section–paragraph. Parts are enumerated by Roman numerals, chapters by numbers, and sections by the number of the chapter that it is in followed by a dot and the number of the section. Paragraphs are not enumerated.

The chapters are written with priority for motivation of the theory and intuition, so some details might be missing at some points. In that case there are references that can be consulted if the reader is not happy with the lack of details.

At the end of the thesis you can find the references (p. 93) and the index, containing keywords of the thesis (p. 97). The references are listed in alphabetical order. In the index there are also symbols included, and they are found alphabetically by how they are pronounced.

Notation

We use the symbol $\subset$ to indicate that we have a strict inclusion. The symbol $\subseteq$ indicates an inclusion that might be an equality.

The complement of a set with respect to another set is denoted by $\smallsetminus$.

We may use a / as a division sign in order to manage vertical spacing. Moreover, with a/bc we mean $a/(bc)$ and not $(a/b)c$.

The imaginary part of a complex number z is denoted by $\operatorname{Im} z$ and the real part by $\operatorname{Re} z$.

In formulas abbreviations are denoted upright and variables in italic, at least most of the time.

Equalities that give a definition are notated by $:=$, isomorphisms by $\cong$, and modulo congruences by $\equiv$.

If we need to write a 2×2-matrices within the text, we may write the matrix as $(a, b; c, d)$. The ; indicates the start of a new matrix row.

We write $\mathbb{A}$ to denote the ring of adeles and $\widehat{\mathbb{Z}}$ the finite adelic integers.

Other Conventions

If we talk about a ring, we assume it is commutative, it has a unit, and that a ring morphism sends the unit to the unit of the target.

With the natural numbers we mean the positive integers, and we denote this set by $\mathbb{N}$.

1 Overview of the Thesis

As stated before, the goal of the thesis is to examine the role of automorphic forms and automorphic representations in relation to the cohomology of moduli spaces of curves and of abelian varieties. This is rephrased in sub-goals:

—review the role of Siegel modular forms in the cohomology of these moduli spaces,

—explain the automorphic representation point of view as used in the articles of Chenevier & Lannes [8] and Chenevier & Renard [7], and

—try to explain the connection between the cohomology of the moduli space of curves of genus 3 and the automorphic representations found by Mégarbané [26].

In this chapter we will give an overview of the most important things that will be treated in the thesis in order to understand the problem and get closer to these goals. We start with the statement of the definition of moduli spaces of curves in Sec. 1.1. Then in Sec. 1.2 we introduce Siegel modular forms, which occur "naturally" in the cohomology of moduli spaces of curves. In Sec. 1.3 we speak of other automorphic forms that we can consider and might play a role as well. We end the chapter with Teichmüller modular forms in Sec. 1.4, a variant of Siegel modular forms that is defined directly on the moduli spaces of curves.

1.1 Moduli Spaces of Curves

Here we will give a quick definition of the coarse moduli space of curves of genus g (as a scheme). Moduli spaces of curves are defined using families of curves.

Definition 1.1. A family of curves of genus g is a morphism of schemes such that each fiber is a curve of genus g.

With a curve we mean a complete and connected variety of dimension 1. Let k be an algebraically closed field. The genus of a smooth curve C over k is defined as $\dim H^0(C, \Omega_{C/k})$, where $\Omega_{C/k}$ is the sheaf of relative differentials of C over k. For the details we point forward to Ch. 3.

Definition 1.2. The (coarse) moduli space of curves of genus g, denoted by $\mathcal{M}_g$, is a variety satisfying:

(a) There is a bijection between $\mathcal{M}_g$ and the set of isomorphism classes of smooth curves of genus g.

(b) Let $X \to T$ be a flat family of curves of genus g. Then we have a morphism $f : T \to \mathcal{M}_g$ in such a way that for all $t \in T$ the curve X_t is in the isomorphism class that corresponds with the point $f(t) \in \mathcal{M}_g$.

Universal Family

We speak of a coarse moduli space, as these moduli spaces miss a certain property that we would like it to have: we want to have a universal family $\mathcal{C} \to \mathcal{M}_g$, but unfortunately for all g this is not possible. However, a universal family does exist if we define $\mathcal{M}_g$ as a stack.

The Jacobian

An interesting fact of a curve of genus g is that we can construct from it a principally polarized abelian variety of dimension g. The moduli space of these abelian varieties is denoted by $\mathcal{A}_g$. The resulting map $t : \mathcal{M}_g \to \mathcal{A}_g$ is called the Torelli map, and it will play an important role in understanding the automorphic forms occuring in the cohomology of $\mathcal{M}_g$.

In Ch. 4 we will also talk about various related moduli spaces of curves as well and also about the moduli space of principally polarized abelian varieties.

1.2 Siegel Modular Forms

Let $g \in \mathbb{N}$ and define the Siegel upper half space $\mathbb{H}_g$ as the set of symmetric complex $g \times g$-matrices with positive definite imaginary part. The symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on $\mathbb{H}_g$ by $\gamma Z = (CZ + D)(AZ + B)^{-1}$ for $\gamma = (A, B; C, D) \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and $Z \in \mathbb{H}_g$. We use this action to define Siegel modular forms.

Definition 1.3. Let $\rho : \mathrm{GL}_g(\mathbb{C}) \to \mathrm{GL}(V)$ be a representation with V a finite dimensional $\mathbb{C}$-vector space. We mean with a Siegel modular form of weight ρ and degree n a function $f : \mathbb{H}_g \to \mathbb{C}$ with the following properties:

(a) the function satisfies the transformation condition

$$f(\gamma Z) = \rho(CZ + D)f(Z)$$

for all $\gamma = (A, B; C, D) \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and all $Z \in \mathbb{H}_n$.

(b) the function f is holomorphic and

(c) it is holomorphic at infinity.

We write $M_\rho(\mathrm{Sp}_{2g}(\mathbb{Z}))$ or $M_\rho(\mathrm{Sp}_{2g})$ for the vector space of Siegel modular forms of weight ρ and degree g. If we take $g = 1$, we see that this definition just gives classical modular forms for $\mathrm{SL}_2(\mathbb{Z})$.

We can view the moduli space of principally polarized abelian varieties of dimension g as $\mathcal{A}_g \cong \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$. For this reason, we can also define Siegel modular forms as global sections of certain sheaves on $\mathcal{A}_g$.

1.3 Other Automorphic Forms

It turns out that Siegel modular forms are part of a more general notion of automorphic forms. The definition of an automorphic form is as follows.

Definition 1.4. Let G be any of the following groups:

$$\mathrm{SL}_n, \quad \mathrm{GL}_n, \quad \mathrm{Sp}_{2g}, \quad \mathrm{SO}_n, \quad \mathrm{SO}_{p,q},$$

for some $g, n, p, q \in \mathbb{N}$. Then we define an automorphic form φ for G as a function

$$\varphi : G(\mathbb{Q})\backslash G(\mathbb{A})/G(\widehat{\mathbb{Z}}) \to \mathbb{C}$$

such that φ is square integrable with respect to the measure coming from the unique $G(\mathbb{A})$-invariant Radon measure on $G(\mathbb{Q})\backslash G(\mathbb{A})$.

Note that automorphic forms can also be defined as a function $\varphi : G(\mathbb{Z})\backslash G(\mathbb{R}) \to \mathbb{C}$, and that these definitions are equivalent. The adelic definition is used in most modern literature as it is useful in the context of automorphic representations.

Siegel modular forms of degree g will fit in this definition if we take $G = \mathrm{Sp}_{2g}$, and they are a notorious example of automorphic forms. However, we will also consider other automorphic forms, coming from the group $G = \mathrm{SO}_n$. For this we need to study lattices.

Lattices

Let $n \in \mathbb{N}$ and let $(x_i) \cdot (y_i) = \sum x_i y_i$ be the inner product on $\mathbb{R}^n$. We denote by $q : \mathbb{R}^n \to \mathbb{R}$ the quadratic form $x \mapsto x \cdot x/2$.

Definition 1.5. A lattice $L \subset \mathbb{R}^n$ is a free abelian group of rank n (that is a subgroup of $\mathbb{R}^n$ with respect to addition) which also spans $\mathbb{R}^n$ as a vector space, together with a quadratic form coming from the quadratic form q on $\mathbb{R}^n$.

Definition 1.6. A lattice $L \subset \mathbb{R}^n$ is called even if $x \cdot x \in 2\mathbb{Z}$ for all $x \in L$.

We denote by $\det L$ the determinant of L, which is defined as the determinant of the Gram matrix of an arbitrary $\mathbb{Z}$-base of L.

For $n \equiv 0 \pmod 8$ the collection of even lattices $L \subset \mathbb{R}^n$ with $\det L = 1$ is denoted by $\mathcal{L}_n$. For $n \equiv \pm 1 \pmod 8$ it is the notation for the collection of even lattices $L \subset \mathbb{R}^n$ with $\det L = 2$.

Lattices can be transformed by automorphisms on $\mathbb{R}^n$, which are given by elements of $\mathrm{GL}_n(\mathbb{R})$. However, in general these do not preserve the quadratic form defined on the lattice. The group of linear transformations that do preserve the quadratic form is the orthogonal group $\mathrm{O}_n(\mathbb{R})$. In other words

$$\mathrm{O}_n(\mathbb{R}) = \{\gamma \in \mathrm{GL}_n(\mathbb{R}) : q \circ \gamma = q\}.$$

The subgroup of elements of $\mathrm{O}_n(\mathbb{R})$ that have determinant 1 is denoted by $\mathrm{SO}_n(\mathbb{R})$ and is called the special orthogonal group.

Definition 1.7. Let (ρ, W) be a finite dimensional representation of $\mathrm{SO}_n(\mathbb{R})$ over $\mathbb{C}$. Then we can define automorphic forms of weight W for SO_n as functions $f : \mathcal{L}_n \to W$ for which $f(\gamma \cdot L) = \rho(\gamma) \cdot f(L)$ for all $\gamma \in \mathrm{SO}_n(\mathbb{R})$ and $L \in \mathcal{L}_n$. The vector space of such automorphic forms is denoted by $M_\rho(\mathrm{SO}_n)$.

Of course we need to compare this definition with the general definition of automorphic forms given in Def. 1.4.

Proposition 1.8. *Let $n \in \mathbb{N}$ such that $n \equiv -1, 0, 1 \pmod 8$. Then we have the following description of the collection of even lattices $\mathcal{L}_n$ as defined in Def. 1.6:*

$$\mathcal{L}_n \cong \mathrm{SO}_n(\mathbb{Q}) \backslash \mathrm{SO}_n(\mathbb{A}) / \mathrm{SO}_n(\widehat{\mathbb{Z}}).$$

This shows us that the functions in Def. 1.7 can be viewed as functions on $G(\mathbb{Q}) \backslash G(\mathbb{A}) / G(\widehat{\mathbb{Z}})$ for $G = \mathrm{SO}_n$, which is required in Def. 1.4 (a). Of course we have to establish that these functions are square integrable as well (Def. 1.4 (b)).

1.4 Teichmüller Modular Forms

Teichmüller modular forms are defined similar to the definition of Siegel modular forms as certain global sections on the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g. We only need to replace $\mathcal{A}_g$ with $\mathcal{M}_g$.

Ichikawa introduced Teichmüller modular forms in [21]. However, he only defined scalar-valued ones. Because vector-valued Siegel modular forms are found in the cohomology of $\mathcal{A}_g$, it is also more natural to define vector-valued Teichmüller modular forms. For the definition of these vector-valued Teichmüller modular forms, we refer to Ch. 12. To give an idea, we already introduce scalar-valued Teichmüller forms here.

Before we do that, we first need to introduce

$$\mathbb{E} = \pi_*(\Omega_{\mathcal{C}/\mathcal{M}_g}), \quad \lambda = \wedge^g \mathbb{E}.$$

Here $\pi : \mathcal{C} \to \mathcal{M}_g$ is the universal curve and $\Omega_{\mathcal{C}/\mathcal{M}_g}$ the relative sheaf of differentials. This means that we should view $\mathcal{M}_g$ in this context as a stack. The vector bundle $\mathbb{E}$ is called the Hodge bundle.

Definition 1.9. Let $g > 1$, and let $\mathcal{M}_g$ denote the moduli stack of smooth curves of genus g. Then we define the space of Teichmüller modular forms of weight h and genus g as

$$T_{g,h} = H^0(\mathcal{M}_g \otimes \mathbb{C}, \lambda^{\otimes h}).$$

It turns out that all Teichmüller modular forms for $g = 2$ can be constructed using pullbacks of Siegel modular forms of degree 2, using the Torelli map $t : \mathcal{M}_2 \to \mathcal{A}_2$. In a similar way we find Teichmüller modular forms for $g = 3$, but not all Teichmüller modular forms are explained by Siegel modular forms anymore, since $t : \mathcal{M}_3 \to \mathcal{A}_3$ is 2-to-1 as a morphism of stacks. One such Teichmüller modular form not coming from a Siegel modular form is χ_9 of weight 9 for genus 3. Its square is equal to χ_{18}, (a pullback of) a certain Siegel modular form of weight 18.

To explain these other Teichmüller modular forms, it may be useful to find maps similar to the Torelli map t, in order to lift other automorphic forms. We are already quite certain that automorphic forms for SO_7 or $\mathrm{SO}_{4,3}$ occur in the higher cohomology groups of $\mathcal{M}_3$, so they may be found as Teichmüller modular forms (which are elements of H^0-groups) as well.

2 The Case of Elliptic Curves

Elliptic curves are one of the most studied objects in algebraic geometry and number theory. They are very interesting in their own right, but they also have many applications such as in number theory for the proof of Fermat's Last Theorem or in cryptography. Moreover, they occur in the subject of the thesis as the first example that motivates the theory. Because it is such an important example, we will give the basics of elliptic curves. For a more complete introduction to elliptic curves we refer to [28].

Definitions of Elliptic Curves

There are plenty of ways to define elliptic curves. Here we give the constructions that are used for our goal. Note that for now we take elliptic curves to be defined over $\mathbb{C}$, the complex numbers. The theory in subsequent chapters does not require this restriction, but for the objective of this chapter, it is more convenient.

Definition 2.1. An elliptic curve over $\mathbb{C}$ can be defined in the following ways.

(a) A smooth curve over $\mathbb{C}$ of genus 1 together with a marked point. For our practical purposes, a curve is smooth if it does not have singular points, which are points on the curve where all partial derivatives of the defining polynomial vanish. A curve of genus 1 is a smooth curve that can be defined in $\mathbb{P}^2$ as the zero set of a degree 3 polynomial.

(b) A curve in the complex projective plane $\mathbb{P}^2$ isomorphic to the zero set of the polynomial $Y^2Z = X^3 + aXZ^2 + bZ^3$, for certain $a, b \in \mathbb{C}$ such that the discriminant $\Delta := 4a^3 + 27b^2 \neq 0$. The marked point is the point $(0:1:0)$.

(c) The quotient $\mathbb{C}/L$, where L is a lattice in $\mathbb{C}$. A lattice in $\mathbb{C}$ can be defined as a sub-$\mathbb{Z}$-module of rank 2 such that $L \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{C}$. In practice this means that it is of the form $L = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ with $\omega_1, \omega_2 \in \mathbb{C}$ such that $\omega_1/\omega_2 \notin \mathbb{R}$. The marked point is the image of the origin $0 \in \mathbb{C}$.

Proposition 2.2. *Definitions (a), (b) and (c) of an elliptic curve as given above in Def. 2.1 are equivalent.*

The above statement is not trivial at all. Due to Def. 2.1 (c) we know that the points of an elliptic curve form a group! In this case it comes from the group structure on $\mathbb{C}$, but we can also define the group structure on an elliptic curve in the context of Def. 2.1 (a) and (b).

Content of the Chapter

Now that we know what elliptic curves are, we will construct the moduli space of elliptic curves in Sec. 2.1. In Sec. 2.2 we introduce classical modular forms, and we will already see that there are some parallels with the way we define elliptic curves. In Sec. 2.3 we reveal the connection between the moduli space of elliptic curves and these modular forms by calculation of the cohomology of certain local systems on the moduli space. It shows that we succeed in this case to give the connection, but we will explain in Sec. 2.4 what the complications are if we consider curves of higher genus.

2.1 Moduli Space of Elliptic Curves

To construct the moduli space of elliptic curves, we have to keep in mind that the points of the moduli space should correspond to isomorphism classes of elliptic curves. We have to identify when two elliptic curves are isomorphic. To do this, we use description (c) of Def. 2.1. Keep in mind though that this construction does not meet the official definition of a moduli space that we will give in Ch. 4.

Isomorphic Elliptic Curves

With the following results we want to establish a way of finding all isomorphism classes of elliptic curves, because this then allows us to construct the moduli space.

Proposition 2.3. *Two elliptic curves $\mathbb{C}/L$ and $\mathbb{C}/L'$ are isomorphic if and only if $L' = \lambda L$, where $\lambda \in \mathbb{C}^*$ is a scalar.*

Proof. "$\Longleftarrow$" This direction is easy, as we can construct an explicit isomorphism, namely that of multiplying with the scalar λ.

"$\Longrightarrow$" Suppose we have an isomorphism between two elliptic curves $\mathbb{C}/L \to \mathbb{C}/L'$. Then this isomorphism must lift to a group isomorphism $\mathbb{C} \to \mathbb{C}$ with respect to the additive group structure on $\mathbb{C}$. This can only be $z \mapsto \lambda z$ for some $\lambda \in \mathbb{C}^*$. But then we must also have $L' = \lambda L$. Note that for this argument we use the fact that an elliptic curve has a group structure. □

The Moduli Space

We define the following subspace of the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, called the fundamental domain:

$$D = \left\{ z \in \mathbb{H} : |z| \geq 1 \text{ and } -\frac{1}{2} \leq \operatorname{Re} z \leq \frac{1}{2} \right\}.$$

Theorem 2.4. *The moduli space of elliptic curves over $\mathbb{C}$ is given by $D/\sim$, where the equivalence relation $\sim$ is given by $z \sim -\overline{z}$ for all z on the boundary of D.*

Proof. Because of Prop. 2.3, an isomorphism class of elliptic curves can be represented by $\mathbb{C}/L$, with $L = \mathbb{Z} \oplus \tau\mathbb{Z}$ and $\tau \in \mathbb{H}$. Moreover, we can choose τ in such a way that it lies in the fundamental domain D. If τ is on the boundary, then either $\operatorname{Re} \tau = \pm 1/2$ or $|\tau| = 1$. In both cases we see that $\mathbb{Z} \oplus \tau\mathbb{Z} = \mathbb{Z} \oplus (-\overline{\tau})\mathbb{Z}$. This proves our theorem. □

We denote this moduli space by $\mathcal{M}_{1,1}$, where the first "1" emphasizes that it is the moduli space of curves of genus 1, and the second "1" emphasizes that 1 point is marked. By Def. 2.1 (a) these curves are all elliptic curves. The theorem tells us that $\mathcal{M}_{1,1} \cong D/\sim$, with the equivalence relation $\sim$ explained above.

We have constructed our desired moduli space of elliptic curves. It is the fundamental domain with the edges glued together in a certain way. Topologically speaking this constructed space is homeomorphic to an open disk. We will mention in Sec. 4.2 that $\mathcal{M}_{1,1}$ is isomorphic to the affine line, and this has the topology of an open disk as well.

An Alternative Construction of $\mathcal{M}_{1,1}$

Alternatively, we can construct the moduli space as $\mathcal{M}_{1,1} \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, where $\mathbb{H}$ is the upper half plane and $\mathrm{SL}_2(\mathbb{Z})$ is the group of invertible 2×2-matrices with entries in $\mathbb{Z}$ and determinant 1. These matrices act on an element $z \in \mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

Of course we have to check some things to make sure this is a well-defined action, but this will be done later in Sec. 2.2, because we will coincidentally also need this action for the definition of modular forms.

The lattices that define an elliptic curve only scale under the action of $\mathrm{SL}_2(\mathbb{Z})$. So they define isomorphic elliptic curves. To see this, consider

$L = \mathbb{Z} \oplus \tau\mathbb{Z}$, with $\tau \in \mathbb{H}$. The action of $(a, b; c, d) \in \mathrm{SL}_2(\mathbb{Z})$ brings it to $(a\tau+b)/(c\tau+d)$, and this defines another lattice $L' = \mathbb{Z} \oplus (a\tau+b)/(c\tau+d)\mathbb{Z}$. If we scale this by $\lambda = c\tau+d$, we get $\lambda L' = (c\tau+d)\mathbb{Z} \oplus (a\tau+b)\mathbb{Z}$, and this basis gives the same lattice L, since the determinant of $(a, b; c, d)$ is $ad - bc = 1$.

To see that this basis gives the same lattice, we calculate

$$d(a\tau + b) - b(c\tau + d) = ad\tau + bd - bc\tau - bd = (ad - bc)\tau = \tau$$

and

$$-c(a\tau + b) + a(c\tau + d) = -ac\tau - bc + ac\tau + ad = -bc + ad = 1.$$

Now, using Prop. 2.3, we know that two lattices $L = \mathbb{Z} \oplus \tau\mathbb{Z}$ and $L' = \mathbb{Z} \oplus \tau'\mathbb{Z}$ with $\tau, \tau' \in \mathbb{H}$ define isomorphic elliptic curves if and only if $L = \lambda L'$, and by the discussion above, this happens if and only if $\tau' = \gamma\tau$ with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. So we conclude that we can also construct the moduli space of elliptic curves as $\mathcal{M}_{1,1} \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

Compactification of the Moduli Space

We end this section with an informal discussion of the compactification of $\mathcal{M}_{1,1}$. This becomes useful later on in Sec. 2.3.

As $\mathcal{M}_{1,1}$ is homeomorphic to an open disc, we know we can compactify it by adding one point. This point corresponds to the lattice $L = \mathbb{Z} \oplus \tau\mathbb{Z}$ with $\tau \to \infty$, so say $L = \mathbb{Z} \subset \mathbb{C}$. So our "curve" will come from $\mathbb{C}/L$, which now is a strip $\{z \in \mathbb{C} : 0 \leq z \leq 1\}$ with the two edges $\{z = 0\}$ and $\{z = 1\}$ glued together. This does not meet the requirement of a curve, because it is not compact yet. However, by adding the point at infinity, we get something that is compact. Unfortunately, it has a singular point (namely the one at infinity), and this makes it a stable curve. We will define stable curves in Def. 3.7.

The resulting space is denoted by $\overline{\mathcal{M}}_{1,1} \cong \overline{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}$, the stable compactification of $\mathcal{M}_{1,1}$.

2.2 Classical Modular Forms

The starting point of the theory of automorphic forms is the notion of classical modular forms (also called elliptic modular forms). Let us briefly recall the definition.

The Action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$

First some well-known notation: let R be a commutative ring, at first this will be the integers $\mathbb{Z}$ or the real numbers $\mathbb{R}$, but later we will also take R to be the field of p-adic numbers, the ring of adeles, or the complex numbers. Then we denote by $\mathrm{GL}_n(R)$ the group of invertible $n \times n$-matrices with entries in R, alternatively the matrices that have determinant that is a unit in R. We call this the general linear group. The subgroup $\mathrm{SL}_n(R)$ of matrices with determinant 1 is called the special linear group.

Let $\gamma = (a, b; c, d) \in \mathrm{SL}_2(\mathbb{Z})$. The matrix γ acts on the upper half plane $\mathbb{H}$, the set of complex numbers with positive imaginary part, by sending a point $z \in \mathbb{H}$ to $(az + b)/(cz + d)$. We must not forget to check that the denominator is non-zero and that the image of this map lies in $\mathbb{H}$, because otherwise this action is not well-defined. The first follows from the fact that $-d/c \in \mathbb{R}$ does not lie in $\mathbb{H}$ if $c \neq 0$. This implies $z \neq -d/c$ and therefore $cz + d \neq 0$. In the case $c = 0$ then $\det \gamma = 1$ implies that $d \neq 0$ so again $cz + d \neq 0$. The second follows from the fact that $\operatorname{Im} \gamma z = \operatorname{Im} z/|cz + d|^2$ and $\operatorname{Im} z > 0$.

Definition of Modular Forms

We can use this action to define a classical modular form.

Definition 2.5. A function f from the upper half plane to the complex numbers is said to be a modular form of weight k if

(a) the function satisfies the transformation property:

$$f(\gamma z) = (cz + d)^k f(z),$$

where γ is again an element of $\mathrm{SL}_2(\mathbb{Z})$ with coefficients a, b, c, d;

(b) the function is holomorphic on $\mathbb{H}$ and

(c) the function is holomorphic at the cusp, in other words if $\operatorname{Im} z$ goes to infinity. Alternatively requirement (c) can be formulated as a condition on the growth of f as $\operatorname{Im} z$ grows.

The vector space of classical modular forms of weight k is denoted by $M_k(\mathrm{SL}_2(\mathbb{Z}))$ or M_k.

Example 2.6. The Eisenstein series of weight k is defined by $G_k(z) = \sum (nz + m)^{-k}$, where the sum is over al $(n, m) \in \mathbb{Z}^2 \smallsetminus \{(0, 0)\}$. It converges for all $k > 2$, and for odd k it is zero, as all the terms of the sum cancel. The Eisenstein series G_4 and G_6 are of particular importance, as they are the first examples of modular forms for $\mathrm{SL}_2(\mathbb{Z})$, and they are of weight 4 and 6 respectively.

We will also normalize these functions to $E_k = G_k/2\zeta(k)$, where ζ is the Riemann ζ-function. (We will later see why we want to normalize.)

The Vector Space of Modular Forms

Using the modular forms E_4 and E_6, we can generate all modular forms for $\mathrm{SL}_2(\mathbb{Z})$. Note that the product of modular forms f_1 and f_2 of weight k_1 and k_2 is again a modular form, but now of weight $k_1 + k_2$. This follows from the easy calculation

$$\begin{aligned}(f_1 f_2)(gz) &= f_1(gz) f_2(gz) = (cz+d)^{k_1} f_1(z)(cz+d)^{k_2} f_2(z) \\ &= (cz+d)^{k_1+k_2}(f_1 f_2)(z).\end{aligned}$$

Proposition 2.7. *The ring $\oplus_k M_k$ of modular forms is generated freely by E_4 and E_6. This gives a dimension formula for the space of classical modular forms of weight k for* $\mathrm{SL}_2(\mathbb{Z})$.

Sketch of proof. It should be clear that by multiplying combinations of E_4 and E_6 we can find generators of the space of classical modular forms of other weight. The proof that these generate all classical modular forms relies on studying the fundamental domain of $\mathbb{H}$ for the action of $\mathrm{SL}_2(\mathbb{Z})$, a subset of $\mathbb{H}$ such that each orbit of the action has a point that lies in this domain (p. 25). For more information on the fundamental domain, and a proof that E_4 and E_6 generate all classical modular forms, consult [34, p. 6 & 10]. Moreover we must know that E_4 and E_6 generate all modular forms freely, in the sense that there are no relations. This is shown in Lem. 2.8 below.

From this we calculate the dimension of the space of classical modular forms of weight k by finding all the generators. For example the space of classical modular forms of weight 12 is of dimension 2, as it is generated by E_4^3 and E_6^2, but that of weight 14 is of dimension 1, as it is only generated by $E_6 E_4^2$.

Lemma 2.8. *The modular forms E_4 and E_6 are algebraically independent.*

Proof. To show that E_4 and E_6 freely generate all modular forms, we have to show that these are algebraically independent. We will do this in 2 steps.

Step 1: we show that E_4^3 and E_6^2 are not scalar multiples. Suppose on the other hand that $E_4^3 = \lambda E_6^2$, for some $\lambda \in \mathbb{C}^*$. Then the function $f : E_6/E_4$ would satisfy (a) of Def. 2.5 for $k = 2$. We can write $f^2 = \lambda E_4$ due to our assumption. So f should be holomorphic and holomorphic at the cusp, which is not true due to the fact that there are no modular forms of weight 2. (This can be shown again using the fundamental domain [34, p. 10].)

Step 2: any two modular forms f_1, f_2 of the same weight that are no scalar multiples must be algebraically independent. Suppose on the contrary that there is a polynomial $p(X,Y) \in \mathbb{C}[X,Y]$ such that $p(f_1(z), f_2(z)) = 0$ for all $z \in \mathbb{H}$. The above polynomial will be the sum of modular forms of different weights. Modular forms of different weights cannot cancel, so $p_d(f_1(z), f_2(z)) = 0$ for all $z \in \mathbb{H}$ for each homogeneous part p_d of p. This follows from the fact that modular forms of different weights transform differently under the action of $\mathrm{SL}_2(\mathbb{Z})$, according to Def. 2.5 (a).

Then $p_d(f_1(z), f_2(z))/f_2(z)^d$ is a polynomial of one variable $f_1(z)/f_2(z)$. As a non-zero polynomial has only finitely many zero's, $f_1(z)/f_2(z)$ must be constant, so f_1 and f_2 are scalar multiples. Contradiction.

Applying step 2, E_4^3 and E_6^2 are algebraically independent and hence E_4 and E_6 are algebraically independent. (Because again modular forms can only cancel if they are of the same weight.) □

Prop. 2.7 says in particular that for each weight these spaces are finite dimensional. We have seen that the calculation of the dimension of vector spaces of classical modular forms can be done with elementary mathematics and is quite easy. However this turns out to be a lot harder in the generalization to Siegel modular forms, which we will explain in Sec. 6.2. In many situations the dimensions are not yet known, and in the case that there is a result it is due to considerable effort in modern research.

Cusp Forms

If we check the transformation of a classical modular form f for the matrix $\gamma = (1,1;0,1)$, we find $f(z) = f(z+1)$, so f is periodic with respect to translations of 1. This allows us to write f as a Fourier expansion $f(q) = \sum_{n=0}^{\infty} a_n q^n$, with $q = \exp 2\pi i z$. This is called the q-expansion of a classical modular form. The expansion starts at $n = 0$ because we required modular forms to be holomorphic at infinity.

Definition 2.9. A classical modular form is called a classical cusp form if the constant term of the Fourier expansion is zero. Alternatively, if $f(z) \to 0$ as $\operatorname{Im} z$ goes to infinity.

The point at infinity is called the cusp of the fundamental domain. So a classical modular form is called a cusp form if and only if it vanishes at the cusp.

The space of classical cusp forms is denoted by $S_k(\mathrm{SL}_2(\mathbb{Z}))$ or S_k.

Example 2.10. As E_4 is of weight 4 and E_6 is of weight 6 the function Δ defined by the equation $1728\Delta = E_4^3 - E_6^2$ is a classical modular form of weight 12 called the delta or discriminant cusp form. Because the constant term in the q-expansion vanishes (E_4 and E_6 have constant term 1, hence the normalization in Ex. 2.6) it is actually a cusp form as well by Def. 2.9, and the first to appear in terms of weight, since by Thm. 2.7 there is no way to combine E_4 and E_6 to produce one of lower weight. This in turn gives a way to find the dimensions of the space of cusp forms for $\mathrm{SL}_2(\mathbb{Z})$ for a certain weight. An alternative definition for this cusp form is by the q-expansion $\Delta(q) = q\prod_n(1-q^n)^{24}$. If we expand this product into a (Fourier) series, then the coefficients of this series are denoted by $\tau(n)$. The τ-function was studied in detail by Ramanujan.

Proposition 2.11. *The first non-zero vector spaces of cusp forms are* S_{12}*,* S_{16}*,* S_{18}*,* S_{20} *and* S_{22}*, and these all have dimension* 1*. Using these dimensions together with the fact that* $\dim S_{k+12} = 1 + \dim S_k$ *gives us the dimensions for all weights* k*.*

There is a deep connection between classical modular forms and elliptic curves, which has for example become apparent in the modularity theorem. A nice introduction on this is given by [28, Ch. V].

More closely related to the subject of this thesis is the occurrence of classical modular forms in certain cohomology of the moduli space of elliptic curves, which for example has been shown in [10]. We will look at this connection in Sec. 2.3 below.

2.3 Modular Forms in the Cohomology

We have seen that elliptic curves and classical modular forms already have a lot in common: we encountered the action of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half plane $\mathbb{H}$ both times, and the fundamental domain plays a role as well. We now show the connection by calculating the cohomology of the moduli space $\mathcal{M}_{1,1}$. Not all details are given yet. They can be expected in subsequent chapters.

We first use that we can identify cusp forms $S_k(\mathrm{SL}_2(\mathbb{Z})) \cong H^0(\overline{\mathcal{M}}_{1,1}, \omega^{\otimes k})$ as global sections on $\overline{\mathcal{M}}_{1,1}$, where ω is the canonical line bundle. These global sections can be lifted to higher cohomology groups. We use the Shimura isomorphism

$$H^0(\overline{\mathcal{M}}_{1,1}, \omega^{\otimes k}) \oplus \overline{H^0(\overline{\mathcal{M}}_{1,1}, \omega^{\otimes k})} \to H^1_!(\mathcal{M}_{1,1}, \mathbb{V}_{k-2}),$$

to obtain that $H^1_!(\mathcal{M}_{1,1}, \mathbb{V}_{k-2}) \cong S_k \oplus \overline{S_k}$. With $\overline{S}$ we mean the space of complex conjugates of cusp forms of weight k. The cohomology group $H_!$ is called the inner cohomology and defined on p. 47, and $\mathbb{V}_k$ are local systems defined on p. 48.

2.4 Complications for Higher Genus

There are a few aspects that make the above calculations more complicated if we replace $\mathcal{M}_{1,1}$ by a moduli space of curves of higher genus.

Problem 1: for higher genus there is no isomorphism anymore between curves and abelian varieties, see Sec. 4.4. Abelian varieties are the higher dimensional generalization of Def. 2.1 (c). So while there is still a notion of modular forms on the moduli space of certain abelian varieties, there is no immediate way to define modular forms on moduli spaces of curves of higher genus.

Problem 2: we can partially still use abelian varieties, but the Siegel modular forms that belong to those are not so easy to find as classical modular forms. See Sec. 6.2.

Problem 3: in Sec. 2.3 we saw that modular forms can be seen as elements of H^0-groups. However, in the research the focus lies on finding modular forms in higher cohomology groups of (local systems of) moduli spaces. So how can we understand that modular forms appear in these higher cohomology groups as well?

Part II

Geometry

In this part are all the chapters that have something to do with algebraic geometry. They mainly revolve around the moduli space of curves of genus g. Below is a list of individual chapters.

—Ch. 3, Curves and their Jacobians (p. 35), where we introduce the objects of the moduli spaces that we will be studying.

—Ch. 4, Moduli Spaces of Curves and Abelian Varieties (p. 39). This chapter contains the definition of these moduli spaces, and gives a connection between them.

—Ch. 5, Cohomology of Local Systems (p. 45). We eventually want to calculate the cohomology of local systems of the moduli space of curves of a certain genus, so in this chapter we will describe how we can try to do that.

3 Curves and their Jacobians

A good way to start the discussion of moduli spaces of curves and of abelian varieties is to look at the objects of these moduli spaces themselves. In all the definitions below we work over an algebraically closed field k of characteristic 0, so we may as well take $k = \mathbb{C}$.

3.1 Curves

Definition 3.1. A curve is a complete and connected variety of dimension 1. Here we use the notion of a variety for an integral separated scheme of finite type over an algebraically closed field. The dimension of a variety is simply defined as the dimension of the variety as a topological space (as in the definition for any scheme).

Since varieties are defined as schemes over an algebraically closed field k, the same holds true for curves. Therefore we should always view a curve as a scheme together with a morphism from that scheme to $\operatorname{Spec} k$.

Example 3.2. —The most basic example of a curve is the projective line $\mathbb{P}^1$: it is complete and irreducible and of dimension 1.

—Another example is a plane curve. It lies in $\mathbb{P}^2$ and it is the variety given by the zero set of an irreducible homogeneous polynomial in $k[X, Y, Z]$ of degree ≥ 1.

Definition 3.3. A variety X over k is smooth of relative dimension n if and only if X is regular of dimension n [20, p. 268]. A scheme is regular if all of its local rings are regular local rings. So we will call a curve smooth if it is regular. A point is called a singular point if its local ring is not a regular local ring.

Definition 3.4. The genus of a smooth curve C is defined as the dimension of $H^0(C, \Omega_{C/k})$, where $\Omega_{C/k}$ is the sheaf of relative differentials of C over k [20, p. 175]. The plural of genus is genera.

For the definition and calculation of the sheaf cohomology of curves, we refer to [20, Ch. III and IV] and [15, Ch. 2].

Example 3.5. A smooth curve of genus 1 together with a point is called an elliptic curve. See Def. 2.1 and more generally Ch. 2. It can be embedded in $\mathbb{P}^2$ as a curve that is defined by a polynomial of degree 3 [20, p. 309]. This polynomial can be given in the form of the equation $Y^2Z = X^3 + aXZ^2 + bZ^3$, for some $a, b \in \mathbb{C}$ such that the discriminant $\Delta = 4a^3 + 27b^2 \neq 0$ [28, p. 50].

Definition 3.6. An n-pointed curve C is a tuple $(C, p_1, \ldots, p_n)$ consisting of the curve itself together with distinct points $p_1, \ldots, p_n \in C$. The fact that it is defined as a tuple stresses the fact that the points $p_1, \ldots, p_n \in C$ have to be ordered.

Definition 3.7. A stable curve is a curve that has only nodes as singularities and has only finitely many automorphisms. A node is a singular point with multiplicity 2 that has distinct tangent directions.

Example 3.8. —Smooth curves of genus $g \geq 2$ are stable curves. The case $g = 0$ does not satisfy the condition finitely many automorphisms: every such curve is isomorphic to $\mathbb{P}^1$ and the projective line has automorphism group $\mathrm{PGL}_2(k)$. Smooth curves of genus $g = 1$ are not stable either. Namely, let C be such a curve. Then we know that it admits a group structure. Using the group operation, we can define a map $P \mapsto P + Q$ that is an automorphism of C for every $Q \in C$. Curves of $g \geq 2$ fortunately do have finitely many automorphisms, and this case is treated in [20, Ex. IV.2.5].

—The curve in $\mathbb{P}^2$ defined by $XYZ = X^3 + Y^3$ is a stable curve that is not smooth. It has a node at $(0 : 0 : 1) \in \mathbb{P}^2$ [16, p. 50].

Definition 3.9. Let C be a stable curve. Then we define the genus $g = g(C)$ of C to be the dimension of $H^1(C, \mathcal{O}_C)$.

Actually, the genus g of a smooth curve is traditionally defined as the dimension of $H^0(C, \Omega_{C/k})$ as in Def. 3.4. However, by Serre duality [20, Ch. III.7], these coincide, so the definition of the genus of a stable curve makes perfect sense.

Let $n \geq 0$ and $g \neq 0$. It turns out that n-pointed stable curves of genus g only exist if $2g - 2 + n > 0$. This inequality follows from Ex. 3.8 together with the fact that curves of genus 0 need 3 marked points and curves of genus 1 need 1 marked point to make sure that there are only finitely many automorphisms. The genus as defined in Def. 3.4 is often called the geometric genus, and the genus as in Def. 3.9 the arithmetic genus.

3.2 Abelian Varieties

Now we jump to the next objects of study: abelian varieties. For this we have to recall that a group variety G over an algebraically closed field is a variety G, equipped with morphisms $m : G \times G \to G$ and $i : G \to G$ such that the set of closed points of G becomes a group under the multiplication that is defined by the morphism m and with inverse map induced by the morphism i.

Definition 3.10. An abelian variety is a smooth connected complete group variety over an algebraically closed field.

Example 3.11. —The first examples of abelian varieties are elliptic curves, which were already mentioned in Def. 2.1 and Ex. 3.5. They are curves, hence complete varieties, and we can define a group operation on them [20, p. 321], so they turn out to be group varieties as well.

—Consider, for some $g \geq 1$, the quotient $\mathbb{C}^g/L$, where L is a lattice of $\mathbb{C}^g$. A lattice is a discrete subgroup of $\mathbb{C}^g$ such that the quotient is compact. If we take $g = 1$, we retrieve the notion of a complex elliptic curve. It is an abelian variety if and only if there exists a positive definite Hermitian form H on $\mathbb{C}^g$ and an integral skew-symmetric form E on L such that $E = \operatorname{Im} H$, see [30, p. 267]. Every abelian variety of dimension g can be written as $\mathbb{C}^g/L$.

Principal Polarization

We need to look at certain abelian varieties: for now only those that have a principal polarization. A polarization of an abelian variety A is a morphism $\lambda : A \to A^\vee$ with certain properties. Here $A^\vee$ is the dual abelian variety of A given by $\operatorname{Pic}^0 A$. A polarization is called principal if it is an isomorphism. In the example above the Hermitian form gives the polarization of $\mathbb{C}^g/L$. For the details we can turn to [30, p. 271], [29, p. 120] or [14, p. 4].

An alternative way of defining the polarization on an abelian variety A, is by writing $A = \mathbb{C}^g/L$. We then define the polarization to be the Hermitian form H of Ex. 3.11. Moreover, we use the skew-symmetric form $E = \operatorname{Im} H$ to define the degree $d \in \mathbb{N}$ of the polarization. Namely, the number such that $d^2 = (-1)^g \det E$. If we restrict this form to $L \times L$ then we get a map with coefficients in $\mathbb{Z}$. Together with the fact that E is skew-symmetric, we know that such a d will exist. In the case that $d = 1$ we speak of a principal polarization.

3.3 The Jacobian of a Curve

Abelian varieties are not only interesting on their own right: there is a connection with curves. For each smooth complete curve C over an algebraically closed field k we can construct an abelian variety over k called the Jacobian variety of C.

The divisor class group $\operatorname{Cl} C$ of a smooth curve C is the quotient of the group of divisors on C by the group of principal divisors on C. The kernel of the degree map $\operatorname{Cl} C \to \mathbb{Z}$ is denoted by $\operatorname{Cl}^0 C$.

Definition 3.12. Let C be a smooth curve over an algebraically closed field k. Then the abelian variety whose group of closed points is isomorphic to $\mathrm{Cl}^0 C$ is called the Jacobian of C. We denote this abelian variety with $J(C)$.

We could define it using the Picard group as well. A problem of Def. 3.12 is that it does not guarantee that there exists a Jacobian for each curve. We will see below that for a curve of genus $g = 1$ the Jacobian does exist, but for higher genus this is much more difficult. See for example [29, Ch. 6] for a proof.

Proposition 3.13. *An elliptic curve is isomorphic to its Jacobian.*

This is essentially what we have used in Ch. 2. Of course this does not hold for curves of higher genus. One fact that shows this is: curves of genus > 1 do not admit a group structure, whereas the Jacobian always has a group structure by definition.

Proposition 3.14. *The Jacobian of a curve has the following two properties.*
(a) *The Jacobian of a smooth curve is principally polarized.*
(b) *The Jacobian of a smooth curve of genus g has dimension g.*

Proof. (a) The Jacobian of a curve C can be written as $J(C) = \mathbb{C}^g/L$, where $L \cong H_1(C, \mathbb{Z})$. Then the skew-symmetric form E can defined using the intersection pairing $H_1(C, \mathbb{Z}) \times H_1(C, \mathbb{Z}) \to \mathbb{Z}$. This intersection pairing is a bilinear form that is unimodular, and together with the fact that it is skew-symmetric, we have $\det E = (-1)^g$, so in the end we have $d = 1$. We conclude that $J(C)$ is principally polarized.

(b) Let C be a curve of genus g and let $J(C)$ be its Jacobian. Using the definition of the Jacobian, we can write:

$$\dim\, J(C) = \dim \mathrm{Cl}^0 C = \dim H^1(C, \mathcal{O}_C) = g,$$

where the last equality follows from Def. 3.9. So the genus of a curve is equal to the dimension of its Jacobian. $\square$

4 Moduli Spaces of Curves and Abelian Varieties

Moduli spaces are algebraic varieties such that the points of the moduli space correspond to certain objects in algebraic geometry. In this section we will give the definitions of the moduli spaces of curves and the moduli spaces of abelian varieties, and moreover we will try to give the relationship between the two. These moduli spaces are the main objects of interest in the thesis. In a later stage, we will try to say something about their cohomology, and we would like to relate these cohomology groups (of local systems) to automorphic forms.

In this section we assume that the reader is familiar with the basic notions of algebraic geometry, such as schemes. A well known resource for this information is the book *Algebraic Geometry* by R. Hartshorne [20], which we therefore use as the prerequisite theory of this thesis. Another reference is *The Red Book* written by D. Mumford [30], but keep in mind that in this book the outdated term "prescheme" is used for what everyone now calls a scheme.

Although the use of schemes is quite abstract already, in the research of moduli spaces it is not always the case that the language of schemes is an adequate way of describing moduli spaces. Therefore moduli spaces can be defined as Deligne-Mumford stacks as well. As the theory of stacks is even more inaccessible than that of schemes, we try to avoid this way of defining moduli spaces, at least for now.

In Ch. 3 we gave the definitions as well as a few properties of curves and abelian varieties—the objects of which we will be studying the moduli spaces. Now it is time to give the definitions of the various moduli spaces of curves in Sec. 4.2. In Sec. 4.3 and 4.4 the definition of the moduli space of abelian varieties together with the connection with moduli spaces of curves are given.

In total we will give 5 definitions for different moduli spaces, which are all pretty much the same, except for the objects of the moduli space. Therefore it might be better to give just one definition of a moduli space, where we might apply the language of category theory, and then apply this definition to all the different objects that we want to consider. The same applies to the definition of a family.

4.1 Families of Curves

Definition 4.1. A family of curves of genus g is a morphism of schemes such that each fiber is a curve of genus g. We defined curves in Def. 3.1.

Definition 4.2. A morphism of schemes $f : X \to Y$ is called flat if $\mathcal{O}_X$ is a flat $\mathcal{O}_Y$-module.

We need families of curves to have the property of flatness, because it assures us that they behave nicely. For example, the genus and the degree of a curve are constant in a flat family of curves. For more information on flat morphisms and flat modules, check [30, p. 215–219].

We will also need to consider the following families.

Definition 4.3. A family $X \to T$ of n-pointed curves of genus g is a morphism of schemes such that each fiber is a curve of genus g, together with n disjoint sections $T \to X$.

A family of principally polarized abelian varieties of dimension g is a morphism of schemes such that each fiber is a principally polarized abelian varieties of dimension g.

4.2 Moduli Spaces of Curves

Definition 4.4. The fine moduli space $\mathcal{M}$ of curves of genus g is a variety that has the following two properties:

(a) the points of $\mathcal{M}$ are in one-to-one correspondence with the isomorphism classes of smooth curves of genus g;

(b) there is a flat family $\mathcal{C} \to \mathcal{M}$ of curves of genus g such that for any other flat family $X \to T$ of curves of genus g, there is a morphism $T \to \mathcal{M}$ such that X is the pull-back of $\mathcal{C}$.

The family of curves $\mathcal{C} \to \mathcal{M}$ as described in Def. 4.4 (b) is called the universal family or the universal curve of genus g. Unfortunately, there does not exist a fine moduli space of curves of genus g. Therefore we have to define a weaker version of moduli spaces: coarse moduli spaces.

If we do want to construct a fine moduli space of curves of genus g, together with a universal curve, we have to define it as a stack, which is a generalization of the notion of a scheme. It takes a lot of time to set up the theory of stacks, so we will avoid it in this thesis, even though it is a more convenient way to define moduli spaces. We refer to the book *Geometry of Algebraic Curves* [1] and the website *The Stacks Project*: `https://stacks.math.columbia.edu/`.

Definition 4.5. The (coarse) moduli space of curves of genus g, denoted by $\mathcal{M}_g$, is a variety satisfying:

(a) there is a bijection between $\mathcal{M}_g$ and the set of isomorphism classes of smooth curves of genus g;

(b) for a given flat family $X \to T$ of curves of genus g, we have a natural morphism $f : T \to \mathcal{M}_g$ such that for all $t \in T$ the curve X_t is in the isomorphism class of curves that corresponds with the point $f(t) \in \mathcal{M}_g$.

When we just speak of a moduli space in the context of schemes, we mean a coarse moduli space, and not a fine one.

It should be clear that the definition of a fine moduli space implies that it is a coarse moduli space as well. Namely let $X \to T$ be a flat family of curves of genus g. Then there is the flat family $\mathcal{C} \to \mathcal{M}$ of curves of genus g from Def. 4.4 (b) such that there exists a morphism $T \to \mathcal{M}$ that makes X the pull-back of $\mathcal{C}$ as in Def. 4.5 (b), which implies that the fiber X_t is in the isomorphism class $f(t)$.

We have already said that a fine moduli space of smooth curves of genus g does not exist, but the fact that the coarse moduli spaces do exist, is not trivial either. It is shown in [29, Thm. 5.11].

Example 4.6. Let's take a look at the coarse moduli space $\mathcal{M}_0$ of curves of genus 0. As we know every genus 0 curve is isomorphic to $\mathbb{P}^1$ and hence we must have $\mathcal{M}_0 = \{[\mathbb{P}^1]\}$ to satisfy Def. 4.5 (a). We can take the constant morphism $T \to \mathcal{M}_0$ as the morphism f required in Def. 4.5 (b), because trivially every fiber of a flat family $X \to T$ of curves of genus 0 is isomorphic to $\mathbb{P}^1$.

Example 4.7. Example when $g = 1$, the moduli space $\mathcal{M}_{1,1}$ of elliptic curves. A result in the theory of elliptic curves is that isomorphism classes can be characterized by the j-invariant: two elliptic curves are isomorphic if and only if they have the same j-invariant [20, p. 317]. As the j-invariant take values in the whole field k, we have $\mathcal{M}_{1,1} \cong \mathbb{A}^1$. We have seen another description of this moduli space in Sec. 2.1.

For higher genera the characterization of $\mathcal{M}_g$ becomes a lot harder. For $g = 2$ it turns out that the moduli space can be constructed as a quotient of $\mathbb{A}^3$ with respect to an action of $\mathbb{Z}/5\mathbb{Z}$ [30]. For $g \geq 3$ there is no explicit description of $\mathcal{M}_g$ at all. However, we do have a few properties of these moduli spaces that have been proven in the literature [20, p. 347].

—The moduli spaces $\mathcal{M}_g$ are quasi-projective and irreducible.

—The dimensions of all the moduli spaces $\mathcal{M}_g$ are known. We have already seen that $\dim \mathcal{M}_0 = 0$ and $\dim \mathcal{M}_1 = 1$. For all other genera $g \geq 2$ we have the formula $\dim \mathcal{M}_g = 3g - 3$.

Definition 4.8. The (coarse) moduli space of n-pointed curves of genus g, denoted by $\mathcal{M}_{g,n}$, is a variety satisfying:

(a) the points of $\mathcal{M}_{g,n}$ are in bijection with the isomorphism classes of smooth n-pointed curves of genus g;

(b) for a given flat family $X \to T$ of smooth n-pointed curves of genus g, there exists a morphism $f : T \to \mathcal{M}_{g,n}$ such that for all $t \in T$ the curve X_t is in the isomorphism class of n-pointed curves that corresponds to $f(t) \in \mathcal{M}_{g,n}$.

The problem with these moduli spaces $\mathcal{M}_g$ and $\mathcal{M}_{g,n}$ is that they are not projective / complete. For example we have seen in Ex. 4.7 that $\mathcal{M}_{1,1}$ is only quasi-projective. In order to solve this, we define the moduli spaces for stable curves.

Definition 4.9. The (coarse) moduli space of stable curves of genus g, denoted by $\overline{\mathcal{M}}_g$, is a variety with the property that

(a) the points of $\overline{\mathcal{M}}_g$ are in one-to-one correspondence with the isomorphism classes of stable curves of genus g;

(b) for any flat family $X \to T$ of stable curves of genus g, we have a morphism $f : T \to \overline{\mathcal{M}}_g$ such that for every $t \in T$ the fiber X_t is in the isomorphism class of stable curves determined by the point $f(t) \in \overline{\mathcal{M}}_g$.

Definition 4.10. The (coarse) moduli space of stable n-pointed curves of genus g, denoted by $\overline{\mathcal{M}}_{g,n}$, is a variety with the property that

(a) the points of $\overline{\mathcal{M}}_{g,n}$ are in one-to-one correspondence with the isomorphism classes of stable n-pointed curves of genus g;

(b) for any flat family $X \to T$ of stable n-pointed curves of genus g, we have a morphism $f : T \to \overline{\mathcal{M}}_{g,n}$ such that for every $t \in T$ the fiber X_t is in the isomorphism class of stable n-pointed curves determined by the point $f(t) \in \overline{\mathcal{M}}_{g,n}$.

Def. 4.9 gives a "compactification" of $\mathcal{M}_g$, and Def. 4.10 is in the same relation with $\mathcal{M}_{g,n}$. They are projective varieties, which is proven by Deligne, Mumford, and Knudsen [19, p. 48]. The moduli spaces $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$ are therefore called the stable compactifications of $\mathcal{M}_g$ and $\mathcal{M}_{g,n}$, respectively.

4.3 Moduli Spaces of Abelian Varieties

Definition 4.11. The (coarse) moduli space of principally polarized abelian varieties of dimension g, denoted by $\mathcal{A}_g$, is a variety satisfying

(a) the points of $\mathcal{A}_g$ are in one-to-one correspondence with the isomorphism classes of principally polarized abelian varieties of dimension g;

(b) for any flat family $X \to T$ of principally polarized abelian varieties of dimension g, there exists a natural morphism $f : T \to \mathcal{A}_g$ such that for each $t \in T$ the fiber X_t is in the isomorphism class of the abelian variety determined by the point $f(t) \in \mathcal{A}_g$.

Since principally polarized abelian varieties for $g = 1$ are the same as elliptic curves, their moduli spaces are also the same. It is given in Sec. 2.1 and Ex. 4.7.

We will see in Ch. 6 and Ch. 7 how we can construct the moduli space of principally polarized abelian varieties explicitly. It turns out that the dimension of $\mathcal{A}_g$ is given by $g(g+1)/2$.

Remark: Def. 4.11 above does not guarantee that the moduli space $\mathcal{A}_g$ is uniquely determined. For this, we should add a universal property condition. The standard proof using universal properties gives us uniqueness of $\mathcal{A}_g$ provided that $\mathcal{A}_g$ actually exists. The same remark applies to Def. 4.5, 4.8, 4.9 and 4.10.

4.4 The Torelli Map

Using the facts of Prop. 3.14 that we can construct a principally polarized abelian variety of dimension g from a smooth curve of genus g, we get the possibility to define a morphism $t : \mathcal{M}_g \to \mathcal{A}_g$ sending a curve C to its Jacobian $J(C)$ together with the principal polarization. It is called the Torelli map.

The Torelli theorem states that this map is injective, in other words that the Jacobian in combination with the principal polarization determines from which curve it came. But there is also the possibility that a principally polarized abelian variety does not come from a curve.

Theorem 4.12. (Torelli Theorem) *The Torelli map $t : \mathcal{M}_g \to \mathcal{A}_g$ is injective for all g.*

Note that for $g = 4$ the map t is not open anymore, since $\dim \mathcal{M}_4 = 3g - 3 = 9$ is not equal to $\dim \mathcal{A}_4 = g(g+1)/2 = 10$. And for higher genus we do not have equal dimensions either.

If we look at the Torelli map in the setting of stacks, then Thm. 4.12 does not hold anymore. Namely, if we consider it as a morphism of stacks for $g = 3$, the map is not an embedding, but it is 2-to-1.

5 Cohomology of Local Systems

In this section we briefly discuss some cohomological notions that we need for the thesis. One way to study schemes is with the ℓ-adic cohomology, which is étale cohomology with coefficients in $\mathbb{Q}_\ell$. One reason that étale cohomology is used in algebraic geometry comes from the fact that the Zariski topology of a variety is not fine enough. This has as a consequence for example that the sheaf cohomology of constant sheaves is zero. Grothendieck defined a new kind of "topology" on a scheme X, where the "opens" are given by étale morphisms to X. In [28] a good introduction is given to the subject of étale cohomology as well as the ℓ-adic particular case.

Using this particular theory, we can calculate the cohomology of local systems of the moduli spaces $\mathcal{M}_g$, and this is interesting for at least one reason: we can expect to find modular forms in these cohomology groups. Our eventual goal is that we find new examples of modular / automorphic forms in certain cohomology groups for higher genera such as $g = 3$, and that we try to find an explanation for why they show up.

In Sec. 5.1, we give a very brief introduction to ℓ-adic cohomology. After that, in Sec. 5.2, we discuss the specific local systems that we want to study. Finally we try to explain how to calculate the cohomology of local systems using point counts over finite fields in Sec. 5.3.

5.1 ℓ-Adic Cohomology

In what follows, let X be a scheme over a field k, and let ℓ be a prime number not equal to the characteristic of k. We can define the ℓ-adic cohomology of X as follows. Let $H^i(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})$ be the étale cohomology of the (constant) sheaf associated to the presheaf $U \mapsto \mathbb{Z}/\ell^n\mathbb{Z}$. We can take the inverse limit of these cohomology groups over $n \in \mathbb{N}$:

$$H^i(X_{\text{ét}}, \mathbb{Z}_\ell) := \varprojlim H^i(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z}),$$

and obtain the étale cohomology of X with coefficients in $\mathbb{Z}_\ell$. Then the ℓ-adic cohomology is defined as the tensor product $H^i(X_{\text{ét}}, \mathbb{Q}_\ell) := H^i(X_{\text{ét}}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

As $\mathbb{Q}_\ell$ is a field, the resulting cohomology groups are $\mathbb{Q}_\ell$-vector spaces. Moreover, we have the property that the only non-zero cohomology groups H^i are those with $0 \leq i \leq 2n$, with n the dimension of X.

We can do this construction for more general sheaves. For this we need some definitions.

Definition 5.1. A sheaf of $\mathbb{Z}_\ell$-modules on X (or an ℓ-adic sheaf) is a collection $\mathcal{F}_n$ of sheaves and also morphisms $f_{n+1} : \mathcal{F}_{n+1} \to \mathcal{F}_n$ with the following properties:

(a) $\mathcal{F}_n$ is a constructible sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules, for all n, and

(b) the map $f_{n+1} : \mathcal{F}_{n+1} \to \mathcal{F}_n$ gives us an isomorphism $\mathcal{F}_{n+1}/\ell^n\mathcal{F}_{n+1} \to \mathcal{F}_n$, also for all n.

A sheaf of $\mathbb{Q}_\ell$-vector spaces is also a $\mathbb{Z}_\ell$-sheaf $\mathcal{F} = (\mathcal{F}_n)$, except that we define the cohomology groups differently:

$$H^i(X_{\text{ét}}, \mathcal{F}) = \left(\varprojlim H^i(X_{\text{ét}}, \mathcal{F}_n)\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

It is clear that the construction of $H^i(X_{\text{ét}}, \mathbb{Q}_\ell)$ above is an example of this.

For the definition above we needed to know what a constructible sheaf is. So for completeness we give the definition below. The collection of sheaves $\mathbb{Z}/\ell^n\mathbb{Z}$ together with quotient maps $f_{n+1} : \mathbb{Z}/\ell^{n+1}\mathbb{Z} \to \mathbb{Z}/\ell^n\mathbb{Z}$ satisfies Def. 5.1 and therefore gives us a first example of a sheaf of $\mathbb{Z}_\ell$-modules.

Definition 5.2. A sheaf $\mathcal{F}$ on $X_{\text{ét}}$ is constructible if

(a) for every closed immersion $i : Z \to X$ with Z irreducible, there exists a non-empty open subset $U \subseteq Z$ such that $(i_*\mathcal{F})|U$ is locally constant and

(b) the stalks of $\mathcal{F}$ are finitely generated.

The cohomology groups for sheaves of $\mathbb{Q}_\ell$-vector spaces are also $\mathbb{Q}_\ell$-vector spaces. They have the property that they are finite dimensional if X is proper over k.

Finally, we also give the notions of compactly supported cohomology and inner cohomology.

Definition 5.3. Let $\mathcal{F}$ be a torsion sheaf on a variety X. We then define the compactly supported cohomology as $H^i_c(X_{\text{ét}}, \mathcal{F}) := H^i(Z_{\text{ét}}, j_!\mathcal{F})$ where Z is any complete variety containing X as a dense open subvariety and j is the inclusion map.

A sheaf $\mathcal{F}$ is called a torsion sheaf if all the $\mathcal{F}(U)$ are torsion groups, in other words if all the sections have finite order. The compactly supported cohomology can be defined for the sheaves $\mathbb{Z}/\ell^n\mathbb{Z}$, and taking the inverse limit and tensoring with $\mathbb{Q}_\ell$ we also obtain the compactly supported ℓ-adic cohomology.

One thing needed for the definition of compactly supported cohomology of a variety X is that there exists a complete variety containing X as a dense open subvariety. We have already seen that it is possible for the moduli spaces $\mathcal{M}_g$ to define a completion $\overline{\mathcal{M}}_g$. It is not obvious that every variety does admit a completion, but it turns out that it is possible. However, there might be more than one completion, and we actually have to show that the definition above is independent of the use of the completion Z. See for this [28, p. 119].

The image of the inclusion $H^i_c(X_{\text{ét}}, \mathcal{F}) \to H^i(X_{\text{ét}}, \mathcal{F})$ is denoted by $H^i_!(X_{\text{ét}}, \mathcal{F})$ and is called the inner cohomology.

One thing that makes compactly supported cohomology particularly interesting is that it allows us to generalize Poincaré duality. The standard result is the following. If X is smooth and proper over k and has dimension n, then $H^{2n}(X, \mathbb{Q}_\ell)$ is one dimensional, and

$$H^i(X, \mathbb{Q}_\ell) \times H^{2n-i}(X, \mathbb{Q}_\ell) \to H^{2n}(X, \mathbb{Q}_\ell)$$

is a perfect pairing for all $0 \leq i \leq 2n$. More generally, we can give a similar result for finite locally constant sheaf $\mathcal{F}$ in terms of compactly supported cohomology.

5.2 Local Systems

Let $\pi : Y \to X$ be a morphism of schemes. Given a sheaf $\mathcal{F}$ on $Y_{\text{ét}}$, we define $\pi_*\mathcal{F}$ as the sheaf on $X_{\text{ét}}$ with $\Gamma(U, \pi_*\mathcal{F}) = \Gamma(U \times_X Y, \mathcal{F})$. The functor π_* takes sheaves on $Y_{\text{ét}}$ to sheaves on $X_{\text{ét}}$. It is left exact, so we can take its right derived functors, $R^i\pi_*$. So we can take a sheaf $\mathcal{F}$ on $Y_{\text{ét}}$ and construct the sheaf $R^i\pi_*\mathcal{F}$ on $X_{\text{ét}}$. These are called the higher direct images of $\mathcal{F}$.

To do so, construct for the sheaf $\mathcal{F}$ on $Y_{\text{ét}}$ an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \cdots .$$

This is possible as the category of sheaves on $Y_{\text{ét}}$ has enough injectives. We apply the functor π_* to this exact sequence and we remove $\pi_*\mathcal{F}$ to obtain the cochain complex

$$0 \longrightarrow \pi_*\mathcal{I}^0 \xrightarrow{d^0} \pi_*\mathcal{I}^1 \xrightarrow{d^1} \cdots$$

of which we will take the cohomology objects as $R^i\pi_*\mathcal{F}$. In particular $R^1\pi_*\mathcal{F} = \ker d^1/\operatorname{im} d^0$. The sequence above is really a complex since π_* is left exact.

Proposition 5.4. *The higher direct image $R^i\pi_*\mathcal{F}$ is the sheaf on $X_{\text{ét}}$ associated with the presheaf $U \mapsto H^i(U \times_X Y, \mathcal{F})$.*

If π_* is exact, then $R^i\pi_*\mathcal{F} = 0$ for all $i > 0$. This holds for $\pi : Y \to X$ a closed immersion, or more generally for a finite map.

Definition 5.5. A local system is the term used for a sheaf of modules $\mathcal{L}$ that is locally constant on the space. In other words for each point of the space there exists a neighborhood such that $\mathcal{L}$ restricted to that neighborhood is a constant sheaf of modules.

We use $\mathbb{V} = R^1\pi_*\mathbb{Q}$ or $\mathbb{V} = R^1\pi_*\mathbb{Q}_\ell$ and symmetric powers $\mathbb{V}_k := \mathrm{Sym}^k(\mathbb{V})$ of those. Here π denotes the map from the universal curve to the moduli space for genus 1. For $\pi_* : \mathcal{C} \to \mathcal{M}_g$ with $g > 1$ we define

$$\mathbb{V}_\lambda := \mathrm{Sym}^{\lambda_1-\lambda_2}(\wedge^1\mathbb{V}) \otimes \mathrm{Sym}^{\lambda_2-\lambda_3}(\wedge^2\mathbb{V}) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_g}(\wedge^g\mathbb{V}),$$

where Sym denotes symmetric powers, $\wedge$ denotes exterior powers and $\lambda = (\lambda_1, \ldots, \lambda_g)$ such that $\lambda_1 \geq \cdots \geq \lambda_g \geq 0$. If we take $\lambda = (k, 0, \ldots, 0)$ then we get the same $\mathbb{V}_k$ as defined above. The $\mathbb{V}$, $\mathbb{V}_k$ and $\mathbb{V}_\lambda$ are all referred to as local systems.

Theorem 5.6. *(Leray spectral sequence.) Let $\pi : Y \to X$ be a morphism of schemes. For any sheaf $\mathcal{F}$ on $Y_{\text{ét}}$ we have a spectral sequence*

$$H^r(X_{\text{ét}}, R^i\pi_*\mathcal{F}) \Rightarrow H^{r+i}(Y_{\text{ét}}, \mathcal{F}).$$

Probably it would be interesting to explain what spectral sequences are. For now we refer to the book [27]. Spectral sequences are used to relate cohomology groups to each other. For example, the cohomology of $\mathcal{M}_{1,n}$ can be found by looking at the cohomology for local systems.

Lefschetz Fixed-point Formula

The following will be useful for our calculation of the cohomology of local systems. Assume that X is a smooth and proper scheme over k, and let $f : X \to X$ be an automorphism with fixed points that are isolated. The morphism f must have the property that each fixed point has multiplicity 1. The most important example of such a morphism f will be the Frobenius, denoted by F_q or Frob_q, where q is a prime power. The "standard" Lefschetz fixed-point formula is then given by:

$$L(f, X) = \sum_i (-1)^i \mathrm{tr}(f^* \mid H^i(X, \mathbb{Q}_\ell)),$$

where $L(f, X)$ is the number of points of X fixed by f, and f^* is the induced map on the cohomology group [20, p. 453].

One problem of the standard Lefschetz fixed-point formula is that moduli spaces should be viewed as stacks, and that we also want to study the cohomology of local systems instead of $\mathbb{Q}_\ell$. Reference [3] comes up with the following generalized formula

$$\sum_{x \in X(\mathbb{F}_q)} \frac{\operatorname{tr}(\operatorname{Frob}_q \mid \mathcal{F}_x)}{\operatorname{Aut}_{\mathbb{F}_q}(x)} = \sum_i (-1)^i \operatorname{tr}(\operatorname{Frob}_q \mid H^i_{\mathrm{c}}(X, \mathcal{F})),$$

where $\mathcal{F}$ is any locally constant sheaf on X. So in particular the local systems that we want to study can be applied. To explain the formula, informally we have to count each fixed point with a factor one over the number of automorphisms of that point, and this is due to the fact that we work with a Deligne-Mumford stack now instead of a scheme.

5.3 Experimental Calculations

We will review the method used by Bergström, Faber and Van der Geer to calculate the cohomology of local systems of moduli spaces of curves in order to find automorphic representations (in particular Siegel modular forms). It is explained in a nice way in an article by Faber and Pandharipande [13].

We want to calculate the number of automorphism classes of elliptic curves over finite fields, so $|\mathcal{M}_{1,1}(\mathbb{F}_p)|$. Here we have to keep in mind that we have to count elliptic curves C over $\mathbb{F}_p$ with weight factor $1/|\operatorname{Aut}_{\mathbb{F}_p}(C)|$. The result is $|\mathcal{M}_{1,1}(\mathbb{F}_p)| = p$. If we consider curves of genus 1 with up to 10 marked points, we all get point counts that satisfy polynomial formulas in p. However, if we consider 11 marked points, then suddenly $|\mathcal{M}_{1,11}(\mathbb{F}_p)| = f_{11}(p) - \tau(p)$, where f_{11} is a polynomial of degree 11 and τ is the function associated to the cusp form of weight 12 discussed in Ex. 2.10.

Part III
Towards Automorphic Representations

In this part we will first define Siegel modular forms, and the try to view them as automorphic forms and as automorphic representations. The article [2] is used for this purpose.

6 Modular Forms

The use of the term modular forms for different definitions might be a bit confusing. So to make it clear from the start we use the terms classical modular form for the ones defined in Def. 2.5, scalar-valued Siegel modular forms for Def. 6.5 and vector-valued Siegel modular forms for Def. 6.8. If we use the term Siegel modular forms we mean one of the last two definitions and the term modular forms can refer to any of the three.

As the theory of modular forms has become very successful and a vital part in the mathematical research in many areas in the 20th century, the natural thing to do was to generalize the definition of modular forms to functions over a higher dimensional analogue of the upper half plane, but the way to do this is not entirely obvious. However, Carl Ludwig Siegel was able to do this, and this was the birth of Siegel modular forms.

Siegel modular forms are very interesting in the theory of moduli spaces as well. As we have seen there is a natural connection between classical modular forms and elliptic curves, which explains why classical modular forms come up in the cohomology of $\mathcal{M}_{1,1}$. We would like to have something similar happening in $\mathcal{M}_g$ for higher genus, and this turns out be, to a certain extent, in the form of Siegel modular forms.

In the research of moduli spaces of higher genus, Siegel modular forms have for example been found in the cohomology for $g = 2$ in [12] and have been conjectured to exist in the cohomology for $g = 3$ in [4]. These Siegel modular forms occurring in the cohomology are actually vector-valued. They can be defined quite easily and naturally as well and fortunately they can also be seen as automorphic forms. It might be tempting to also expect Siegel modular forms in the cases $g > 3$, but for these genera a complication arises.

In modern research, modular forms as well as Siegel modular forms are all viewed in the language of automorphic forms. It might be interesting to put the Siegel modular forms which appear in the cohomology of moduli spaces in the context of automorphic forms and see if this point of view can partly fill the gaps that are existing in the current understanding of these cohomology groups.

The chapter is structured as follows. First, we will look in Sec. 6.1 at the symplectic group and how it acts on the Siegel upper half space. In Sec. 6.2, we define Siegel modular forms in the classical way, that is to say as functions with values in $\mathbb{C}$, and observe that they are a convincing generalization of the original modular forms. In Sec. 6.3 the notion of vector-valued Siegel forms is explained. We will see in Ch. 7 that we can associate automorphic forms to them.

6.1 The Symplectic Group

Let us explain some notation. Fix a natural number $g \in \mathbb{N}$. For R any commutative ring, in our case it will be $\mathbb{Z}, \mathbb{R}$ or $\mathbb{C}$, let $\mathrm{Sp}_{2g}(R)$ be the group of matrices $\gamma \in \mathrm{GL}_{2g}(R)$ with the property that $\gamma^{\mathrm{t}} J_{2g} \gamma = J_{2g}$, where J_{2g} denotes the $2g \times 2g$-matrix of the form $(0, I_g; -I_g, 0)$ and I_g denotes the $g \times g$ identity matrix. We call this group the symplectic group.

Proposition 6.1. *If $\gamma = (A, B; C, D) \in \mathrm{Sp}_{2g}(R)$, with A, B, C, D all $g \times g$-matrices, the defining condition above is equivalent to the relations $A^{\mathrm{t}} D - C^{\mathrm{t}} B = I_g$ and $A^{\mathrm{t}} C = C^{\mathrm{t}} A$ and $B^{\mathrm{t}} D = D^{\mathrm{t}} B$.*

Proof. This follows immediately if we carry out the multiplication $\gamma^t J_{2g} \gamma$ with this γ and compare it to J_{2g}. For the calculation we note that $\gamma^{\mathrm{t}} = (A^{\mathrm{t}}, C^{\mathrm{t}}; B^{\mathrm{t}}, D^{\mathrm{t}})$. □

Definition 6.2. We denote by $\mathbb{H}_g$ the Siegel upper half space. It is defined as the set consisting of symmetric complex $g \times g$-matrices with positive definite imaginary part.

It is clearly a generalization of the upper half plane as $\mathbb{H}_1 = \mathbb{H}$.

Proposition 6.3. *We can give an action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on the Siegel upper half space by $\gamma Z = (AZ + B)(CZ + D)^{-1}$, where $\gamma = (A, B; C, D) \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and $Z \in \mathbb{H}_g$. This action is well-defined.*

Proof. First we have to check that $CZ + D$ is invertible, which holds if and only if it has non-zero determinant (as the matrix has coefficients in $\mathbb{C}$). We write $Z = X + iY$, where X and Y have real entries and Y is positive definite. As a trick, we can calculate

$$\begin{aligned}
(C\overline{Z} + D)^{\mathrm{t}}(AZ + B) &- (A\overline{Z} + B)^{\mathrm{t}}(CZ + D) \\
&= \overline{Z}(C^{\mathrm{t}} A - A^{\mathrm{t}} C)Z + \overline{Z}(C^{\mathrm{t}} B - A^{\mathrm{t}} D) + \\
&\quad (D^{\mathrm{t}} A - B^{\mathrm{t}} C)Z + D^{\mathrm{t}} B - B^{\mathrm{t}} D \\
&= Z - \overline{Z} = 2iY, \qquad (1)
\end{aligned}$$

where we use the relations of Prop. 6.1 for the second equality.

Now, suppose that $(CZ + D)\xi = 0$ has a solution. Then using Eq. (1) we find that $\overline{\xi}^{\mathrm{t}} Y \xi = 0$. As Y is positive definite, we must have that $\xi = 0$. We conclude that $\det(CZ + D) \neq 0$ as $(CZ + D)\xi = 0$ has only the trivial solution.

Next we show that γZ lies in $\mathbb{H}_g$, in other words that it is symmetric and that its imaginary part is positive definite. The first follows from the calculation

$$\begin{aligned}(CZ+D)^{\mathrm{t}}&(\gamma Z-(\gamma Z)^{\mathrm{t}})(CZ+D)\\ &=(CZ+D)^{\mathrm{t}}(AZ+B)-(AZ+B)^{\mathrm{t}}(CZ+D)\\ &=Z-Z^{\mathrm{t}}=0, \qquad (2)\end{aligned}$$

where for the third equality we first expand and then use the same relations from Prop. 6.1 as for the previous equation. We see that $Z=Z^{\mathrm{t}}$ now implies that gZ is symmetric as well. Combining Eq. (1) and Eq. (2), we find that the imaginary part of γZ is positive definite:

$$\begin{aligned}(C\overline{Z}+D)^{\mathrm{t}}&(\gamma Z-(\overline{\gamma Z})^{\mathrm{t}})(CZ+D)\\ &=(C\overline{Z}+D)^{\mathrm{t}}(AZ+B)-(A\overline{Z}+B)^{\mathrm{t}}(CZ+D)\\ &=2iY.\end{aligned}$$

We now know that the action is well-defined. □

Constructing $\mathcal{A}_g$

One thing which makes the action of Sp_{2g} on the Siegel upper half space $\mathbb{H}_g$ so interesting is that we can use it to construct the moduli space of principally polarized abelian varieties of dimension g. This is essentially the same as what we did in Ch. 2. There we found that $\mathcal{M}_{1,1}\cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ in Sec. 2.1 which translates in this setting to $\mathcal{A}_1\cong \mathrm{Sp}_2(\mathbb{Z})\backslash\mathbb{H}_1$. We can generalize this to the following claim.

Proposition 6.4. *Let $g\in\mathbb{N}$. The moduli space $\mathcal{A}_g$ of principally polarized abelian varieties can be constructed as*

$$\mathcal{A}_g\cong \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g.$$

Proof. We use an element $T\in\mathbb{H}_g$ to define the lattice $\mathbb{Z}^g\oplus T\mathbb{Z}^g\subset\mathbb{C}^g$. Using this lattice we can construct a principally polarized abelian variety $\mathbb{C}^g/L$. It turns out that all isomorphism classes of principally polarized abelian varieties can be represented using such a lattice $\mathbb{Z}^g\oplus T\mathbb{Z}^g$. So the Siegel upper half space $\mathbb{H}_g$ parametrizes those abelian varieties.

The isomorphism class of $\mathbb{C}^g/L$ is independent of the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathbb{H}_g$. This follows from a similar calculation as done on p. 25. Conclusion: $\mathcal{A}_g\cong \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$. □

Prop. 6.4 gives us the suspicion that Siegel modular forms, which are defined below using the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathbb{H}_g$, have everything to do with $\mathcal{A}_g$.

6.2 Siegel Modular Forms

Definition 6.5. We mean with a scalar-valued Siegel modular form of weight k and degree g a function $f : \mathbb{H}_g \to \mathbb{C}$ with the following properties:

(a) the function satisfies the transformation condition

$$f(\gamma Z) = \det(CZ + D)^k f(Z)$$

for all $\gamma = (A, B; C, D) \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and all $Z \in \mathbb{H}_g$.

(b) the function f is holomorphic and

(c) it is holomorphic at infinity, as in the definition of classical modular forms.

Condition (c) of Def. 6.5 is actually only needed for degree $g = 1$. This follows from the so-called Koecher principle, which is explained in [17, p. 191].

Of course a silly modular form for any weight here is the zero function. Also for weight 0 all constant functions $\mathbb{H}_g \to \mathbb{C}$ are modular forms. As we have seen in the definition of classical forms and as we will see in the future definitions of vector-valued Siegel modular forms and automorphic forms, the constant functions will always occur as trivial examples.

Example 6.6. Let us take a look at the case $g = 1$. In this case $\gamma = (A, B; C, D) \in \mathrm{Sp}_2(\mathbb{Z})$, with $A, B, C, D \in \mathbb{Z}$, will have to satisfy the relation $AD - BC = 1$, which shows that $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, and the transpose relations become trivial. It tells us that $\mathrm{Sp}_2(\mathbb{Z})$ is actually equal to $\mathrm{SL}_2(\mathbb{Z})$. Also the definition of $\mathbb{H}_1$ reduces to that of $\mathbb{H}$. We conclude that Siegel modular forms of dimension 1 coincide with classical modular forms for $\mathrm{SL}_2(\mathbb{Z})$.

Example 6.7. There also exists a notion of Eisenstein series for scalar-valued Siegel modular forms, called Klingen Eisenstein series, which is much more involved compared to the Eisenstein series for classical modular forms. However, a relative simple example of these for degree g is given by $E_{g,0,k} = \sum \det(CZ + D)^{-k}$, where the sum is over C, D coming from a full set of representatives $\gamma = (A, B; C, D)$ for the cosets $\mathrm{GL}_g(\mathbb{Z})\backslash\mathrm{GSp}_{2g}(\mathbb{Z})$. It turns out that these are scalar-valued Siegel modular forms of weight k for each even weight $k > 2$, see [17, p. 193]. For the definition of $\mathrm{GSp}_{2g}(\mathbb{Z})$ we point forward to p. 60.

In our definition of any modular form we required the transformation property to hold under the action of the full modular group, so in our case $\mathrm{Sp}_{2g}(\mathbb{Z})$. We could instead take any so-called congruence subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$, and this gives rise to modular forms of level > 1. For this thesis we will focus on modular forms of level 1.

6.3 Vector-valued Siegel Modular Forms

A representation of a group G on a vector space V over a field k is a group homomorphism from G to $\mathrm{GL}(V)$ (the group of all automorphisms of V).

Definition 6.8. Let $\rho : \mathrm{GL}_g(\mathbb{C}) \to \mathrm{GL}(V)$ be a representation with V a finite dimensional $\mathbb{C}$-vector space. We mean with a Siegel modular form of weight ρ and degree g a function $f : \mathbb{H}_g \to \mathbb{C}$ with the following properties:

(a) the function satisfies the transformation condition

$$f(\gamma Z) = \rho(CZ + D) f(Z)$$

for all $\gamma = (A, B; C, D) \in \mathrm{Sp}_{2g}(\mathbb{Z})$ and all $Z \in \mathbb{H}_g$.

(b) the function f is holomorphic and

(c) it is holomorphic at infinity.

Example 6.9. If we take the representation $\rho = \det^k : \mathrm{GL}_g(\mathbb{C}) \to \mathbb{C}$ we return to the original definition of scalar-valued Siegel modular forms.

We can restrict ourselves to irreducible representations of $\mathrm{GL}_g(\mathbb{C})$. To see this, we consider two vector-valued Siegel modular forms f_1 and f_2 of degree g, one of weight (ρ_1, V_1), the other of weight (ρ_2, V_2). We can now construct a vector-valued Siegel modular form of weight $(\rho, V_1 \oplus V_2)$, where $\rho : G \to GL(V_1 \oplus V_2)$ is defined by $\gamma \mapsto \mathrm{diag}(\rho_1(\gamma), \rho_2(\gamma))$. This Siegel modular form is given by $f : \mathbb{H}_g \to V_1 \oplus V_2, Z \mapsto (f_1(Z), f_2(Z))$. We can immediately see that this is in fact a Siegel modular form if we try to write down the transformation condition (a) of Def. 6.8 for f. It is therefore sufficient to study only the case that we have irreducible representations as weights.

We can for example give the irreducible representations for $g = 2$, as these are given by $\mathrm{Sym}^j \otimes \det^k$ with $j, k \in \mathbb{Z}$ and $j \geq 0$, with Sym^j the j-th symmetric power of the standard representation. More generally, for any $g \in \mathbb{N}$, the irreducible finite dimensional representations of $\mathrm{GL}_g(\mathbb{C})$ correspond one to one with tuples $(\lambda_1, \ldots, \lambda_g)$ with $\lambda_1 \geq \ldots \geq \lambda_g \geq 0$ decreasing non-negative integers. This tuple specifies the representation

$$\mathrm{Sym}^{\lambda_1 - \lambda_2}(\wedge^1 V) \otimes \mathrm{Sym}^{\lambda_2 - \lambda_3}(\wedge^2 V) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_g}(\wedge^g V),$$

where V is the standard representation of $\mathrm{GL}_g(\mathbb{C})$. The wedge symbol $\wedge^i$ denotes the i-th exterior power of V.

Siegel Operator and Siegel Cusp Forms

We finally want to mention that we can also define when Siegel modular forms are called cusp forms, as a generalization of the cusp forms that we defined for classical modular forms in Def. 2.9. This is important for us, because it are the cusp forms that occur in the cohomology of moduli spaces of curves. In order to give the definition, we first need to introduce the Siegel operator.

Definition 6.10. Let $g > 1$. We define the Siegel operator Φ on the space $M_\rho(\mathrm{Sp}_{2g}(\mathbb{Z}))$ by

$$\Phi f = \lim_{t \to \infty} f \begin{pmatrix} Z & 0 \\ 0 & it \end{pmatrix},$$

where $f \in M_\rho(\mathrm{Sp}_{2g}(\mathbb{Z}))$, $t \in \mathbb{R}$ and $Z \in \mathbb{H}_{g-1}$.

We can take this limit as we require Siegel modular forms to be holomorphic at infinity (and they automatically are for $g > 1$). The resulting function $\Phi f : \mathbb{H}_{g-1} \to \mathbb{C}$ turns out to be a Siegel modular form of degree $g-1$. So the Siegel operator is a useful tool to construct Siegel modular forms of a lower degree. But we can also use it to give the following definition.

Definition 6.11. A Siegel cusp form is a Siegel modular form f such that $\Phi f = 0$. The space of cusp forms is denoted by $S_\rho(\mathrm{Sp}_{2g}(\mathbb{Z})) \subseteq M_\rho(\mathrm{Sp}_{2g}(\mathbb{Z}))$ or simply S_ρ. We write $S_k(\mathrm{Sp}_{2g}(\mathbb{Z})$ for the space of scalar-values cusp forms of weight k.

7 Automorphic Forms

7.1 Definition

Before we can give our main definition of automorphic forms, we first have to give an introduction to the adele ring.

Adeles

We start with the field of rational numbers $\mathbb{Q}$. The rational numbers admit several absolute values: the standard absolute value, but also the p-adic absolute values with p prime. Taking the completion of $\mathbb{Q}$ with respect to these different absolute values, gives us the field of real numbers $\mathbb{R}$ and the fields of p-adic numbers $\mathbb{Q}_p$, respectively.

Taking the restricted product of all the $\mathbb{Q}_p$, with compact opens $\mathbb{Z}_p$ consisting of p-adic integers, i.e. p-adic numbers with p-adic absolute value at most 1, we obtain the finite adeles $\mathbb{A}_\mathrm{f}$. For the definition of the restricted product and more on adeles, you can consult [9, Ch. 5].

We can define the ring of finite adelic integers in another way as the direct product $\widehat{\mathbb{Z}} = \prod_{p \text{ prime}} \mathbb{Z}_p$, where $\mathbb{Z}_p$ is again the ring of p-adic integers. Then we can define the ring of finite adeles as the tensor product $\mathbb{A}_\mathrm{f} := \mathbb{Q} \otimes_\mathbb{Z} \widehat{\mathbb{Z}}$.

The ring of adeles is then the product $\mathbb{A} := \mathbb{R} \times \mathbb{A}_\mathrm{f}$.

In the definition of the adelic ring and actually in the whole definition of automorphic forms we can use any global field instead of $\mathbb{Q}$. However, we do not need this generality at this point and stick to the rationals.

Groups used for the Definition

A group over which we will define automorphic forms will need to have the following properties.

—Such a group has to be a locally compact group, which means that it is a topological group with an underlying space that is locally compact and Hausdorff. This is to be able to define a measure on the space.

—The group has to be able to be defined over both $\mathbb{Q}$ and over $\mathbb{A}$. This happens in practice when the group consists of matrices, because we can then just change the entries to lie in $\mathbb{Q}$ or $\mathbb{A}$.

There are many ways to give a satisfying definition for the group G. Many authors require G to be reductive over $\mathbb{Q}$ as an algebraic group. Others require it to be an affine group scheme of finite type over $\mathbb{Z}$. But for now we will just consider subgroups of GL_n that are locally compact.

Example 7.1. Examples of the groups that we will consider are $G = \mathrm{GL}_n$, SL_n, GSp_{2n}, Sp_{2n}, O_n or SO_n. These are in practice the only groups for which automorphic forms are defined.

The general symplectic group GSp_{2n} mentioned in the example above is defined as the group consisting of elements $\gamma \in \mathrm{GL}_{2n}(R)$, for some commutative ring R, such that $\gamma^{\mathrm{t}} J_{2n} \gamma = \mu(\gamma) J_{2n}$ for some $\mu(\gamma) \in \mathrm{GL}_1(R)$. The function $\mu : \mathrm{GSp}_{2n}(R) \to \mathrm{GL}_1(R)$ that is consequently defined is called the multiplier homomorphism. It has $\mathrm{Sp}_{2n}(R)$ as its kernel.

We can come across projective versions of these $G(R)$, denoted by $\mathrm{P}G(R)$. This is defined as the quotient of the group $G(R)$ by its center. The elements of $\mathrm{P}G(R)$ therefore correspond to the equivalence classes of the equivalence relation $\gamma \sim \lambda\gamma$, with $\gamma \in G(R)$ and $\lambda \in R^*$.

Proposition 7.2. *Let G be a locally compact group over $\mathbb{Q}$. Then we know that on the space $G(\mathbb{Q})\backslash G(\mathbb{A})$ there exists a unique positive $G(\mathbb{A})$-invariant Radon measure μ (up to scaling).*

Proof. Due to the construction of $\mathbb{A}$ as a restricted product, the group $G(\mathbb{A})$ will be a locally compact group as well. Using the diagonal embedding $\mathbb{Q} \to \mathbb{A}$, we can show that $\mathbb{Q}$ is discrete in $\mathbb{A}$, so we can use [9, Thm. 3.15], which gives us the required measure μ. □

Definition of Automorphic Forms

We are now ready for the definition of automorphic forms.

Definition 7.3. Let G be a locally compact (linear algebraic) group over $\mathbb{Q}$. Then we define an automorphic form φ for G as a function $\varphi : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ such that

(a) φ is square integrable with respect to the measure that we found in Prop. 7.2 and

(b) φ is invariant under the right action of $G(\widehat{\mathbb{Z}})$.

With square integrable we mean that

$$\int_{G(\mathbb{Q})\backslash G(\mathbb{A})} \varphi(\gamma)^2 \,\mathrm{d}\mu$$

is finite with respect to the measure found in Prop. 7.2. If we define a norm on $G(\mathbb{Q})\backslash G(\mathbb{A})$ as the square root of this integral, then this space becomes a Hilbert space. The inner product is defined as

$$\langle f, g \rangle := \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} f(\gamma)\overline{g(\gamma)} \,\mathrm{d}\mu.$$

Instead of giving the condition that an automorphic forms φ has to be square integrable, we can impose the restriction that φ has to be slowly increasing [6, p. 190 & 194]. However, this has its own drawback, as this requires a norm on the domain of φ.

The definition above for automorphic forms in the adelic setting. However, as an intermediate step, we could define "real" automorphic forms as functions over $G(\mathbb{Z})\backslash G(\mathbb{R})$ that are square integrable with respect to the unique positive $G(\mathbb{R})$-invariant Radon measure [9, Ch. 3].

We must note that automorphic forms are defined in many different ways in the literature. Some definitions are equivalent, some are not, and some definitions are in greater generality than others. For example, we could also reformulate condition (b) of Def. 7.3 by defining automorphic forms as functions $G(\mathbb{Q})\backslash G(\mathbb{A})/G(\widehat{\mathbb{Z}}) \to \mathbb{C}$. We choose the definition as used in [8, p. 117], as this is an article that contains some nice results that we may want to study during the rest of the thesis.

Another Description of $\mathcal{A}_g$

We can give another description of $\mathcal{A}_g$ than found before. This makes it more clear why $\mathcal{A}_g$ and its associated Siegel modular forms have so much to do with automorphic forms.

We have already seen that $\mathcal{A}_g \cong \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$ in Prop. 6.4. We first need to rewrite $\mathbb{H}_g$. Note that $\mathrm{Sp}_{2g}(\mathbb{R})$ acts on $\mathbb{H}_g$ as well, exactly in the same way as $\mathrm{Sp}_{2g}(\mathbb{Z})$. Moreover, this action is transitive, so for all $Z \in \mathbb{H}_g$ there exists a $\gamma \in \mathrm{Sp}_{2g}(\mathbb{R})$ such that $\gamma \cdot I = Z$, where I is an $n \times n$ diagonal matrix with all diagonal entries equal to the imaginary unit i. However, there are elements that stabilize I, and denote this stabilizer subgroup by K. We then conclude that $\mathbb{H}_g = \mathrm{Sp}_{2g}(\mathbb{R})/K$, so $\mathcal{A}_g \cong \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathrm{Sp}_{2g}(\mathbb{R})/K$. For the last step of translating this to adelic coefficients, we can use [7, Prop. 3.5]. Eventually we get:

$$\mathcal{A}_g \cong \mathrm{Sp}_{2g}(\mathbb{Q})\backslash\mathrm{Sp}_{2g}(\mathbb{A})/\mathrm{Sp}_{2g}(\widehat{\mathbb{Z}})K.$$

7.2 Modular Forms as Automorphic Forms

We have to be able to decompose $G(\mathbb{A})$ as $G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})G(\widehat{\mathbb{Z}})$ in order to define automorphic forms. If we use the so-called principle of strong approximation on these groups G, we can establish this decomposition.

Example 7.4. Classical modular forms are automorphic forms for $G = \mathrm{GL}_2$. To be more precise, let $f : \mathbb{H} \to \mathbb{C}$ be a classical cusp form of weight k. We will first map f to a function φ_f on $G(\mathbb{R})$, namely by defining $\varphi_f(g) := (ci+d)^{-k}f(gi)$. Here c and d are the bottom entries of g and $i \in \mathbb{H}$ is the imaginary unit. This function will be an automorphic form in the real setting.

In order to show that this φ_f is actually an automorphic form, we have to show that it is invariant under the left action of $G(\mathbb{Z})$ and that it is square integrable. The first follows from the transformation property of f.

$$\begin{aligned}\varphi_f(\gamma g) &= (c_{\gamma g}i + d_{\gamma g})^{-k}f(\gamma gi) = (c_{\gamma g}i + d_{\gamma g})^{-k}(c_\gamma gi + d_\gamma)^k f(gi)\\ &= (c_\gamma gi + d_\gamma)^{-k}(c_g i + d_g)^{-k}(c_\gamma gi + d_\gamma)^k f(gi) = \varphi_f(g).\end{aligned}$$

Here we write c_M and d_M for the bottom entries of the matrix M denoted in the subscript. We see that we can actually view φ_f as a function on $G(\mathbb{Z})\backslash G(\mathbb{R})$. The square integrability is shown in [9, p. 96].

The final step is to view this as an adelic automorphic form as in Def. 7.3. This can be achieved by finding an injection of $L^2(G(\mathbb{Z})\backslash G(\mathbb{R}))$ into $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$, as is done in [9, p. 171].

Example 7.5. Scalar-valued Siegel modular forms of degree n can be mapped to functions on $G(\mathbb{A})$ if we take $G = \mathrm{GSp}_{2n}$. Write an element of $G(\mathbb{A})$ as $g = g_{\mathbb{Q}}g_{\mathbb{R}}g_{\hat{\mathbb{Z}}}$. If we have a scalar-valued Siegel modular form of weight k for degree n, we send it to φ_f, defined by $\varphi_f(g) = (CI+D)^{-k}f(g_{\mathbb{R}}I)$. In this case C and D are the bottom $n \times n$-matrices of $g_{\mathbb{R}}$, and I is the diagonal $n \times n$-matrix with all diagonal entries equal to i. More information can be found in [2].

Let f be a vector-valued Siegel modular form of weight ρ and degree g and assume that ρ is irreducible. We can then associate to f a function on the adelic group $\mathrm{GSp}_{2g}(\mathbb{A})$. Define $\tilde{\varphi}(\gamma) = \rho(CI+D)^{-1}f(\gamma_{\mathbb{R}}I)$. The notation is the same as in Ex. 7.5. The problem now is that this defines a vector-valued function on G($\mathbb{A}$), and in our definition of automorphic forms, they are defined to be scalar-valued. To solve this issue, we take a non-zero linear form L on V (which is just a linear map $V \to \mathbb{C}$) and define $\varphi(\gamma) = L(\tilde{\varphi}(\gamma))$ to obtain a scalar-valued function. This definition is dependent on the choice of L, but if we look further to the automorphic representation associated to φ (to be defined later), then the choice of L does not matter anymore. For more information on the construction of an automorphic form from a vector-valued Siegel modular form, we refer to [2, p. 194], where the whole method is explained.

8 Automorphic Representations

We have already seen modular forms and automorphic forms and we have convinced ourselves why these are interesting things to study. It turns out that there is another way of looking at these, and that is as representations. There has been a lot of research on group representations, but on the other hand the results have been for finite groups or compact groups for example, and the groups that we want to study are a lot harder to tackle. Still though it might give some interesting insights in the cohomology of moduli spaces, and it at least gives us a different viewpoint rather than just considering Siegel modular forms.

First in Sec. 8.1 we will give some basic definitions from representation theory that we will need. In Sec. 8.2 we will introduce automorphic representations, and we will try to associate these to modular and automorphic forms in Sec. 8.3.

8.1 Basics of Group Representations

A representation of a group G on a vector space V is a pair (π, V) with π a group homomorphism $G \to \mathrm{GL}(V)$. With $\mathrm{GL}(V)$ we mean the group of automorphisms on V. If G is a topological group and V a Banach space, then if $G \times V \to V, (g, v) \mapsto \pi(g)v$ is continuous we speak of a continuous representation.

A subspace W of V that is invariant under the group action is called a subrepresentation. It is of course a representation itself and can be written as $(\pi|_{\mathrm{GL}(W)}, W)$. If V has exactly two subrepresentations, namely the zero-dimensional subspace and V itself, then the representation is said to be irreducible.

A unitary representation of a group G is a representation π of G on a Hilbert space V, with inner product $\langle *, * \rangle$, such that $\pi(g)$ is a unitary operator for every $g \in G$. This means that $\pi(g)$ preserves the inner product of the Hilbert space or in other words that $\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$ for all $v, w \in V$.

The dimension of a representation $G \to \mathrm{GL}(V)$ is the dimension of V.

The multiplicity of an irreducible subrepresentation $W \subseteq V$ is defined as the dimension of $\mathrm{Hom}_G(W, V)$, which is the vector space of homomorphisms $W \to V$ that commute with the action of G.

8.2 Defining Automorphic Representations

For the theory of automorphic representations, just as for the theory of automorphic forms, we can choose various topological groups G for which we give the definition. Some authors require G to be a reductive group. Others require it to be a linear algebraic group, and for example Chenevier and Lannes [8] give the definition for so called $\mathbb{Z}$-groups, which are affine group schemes of finite type over $\mathbb{Z}$.

Choosing one of these definitions, we will mostly see the example $G = \mathrm{Sp}_{2g}$ for now. In this situation we look at $G = G(\mathbb{A})$ and $V = L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z)$ with Z the neutral component of the center of $G(\mathbb{R})$. As representation we take the right representation π defined by $\pi(\gamma)\varphi(*) = \varphi(*\gamma)$ for all $\gamma \in G(\mathbb{A})$ and $\varphi \in V$. The inner product (p. 60) is invariant under the right action of $G(\mathbb{A})$ by Prop 7.2, so we see that π is a unitary representation.

In this case we will only consider

$$\mathcal{A}^2(G) := L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/Z \cdot G(\widehat{\mathbb{Z}}))$$

consisting of automorphic forms of level 1. As stated above, there is a representation of $G(\mathbb{R})$ on this space. Moreover, for every prime number p, there is also the action of the group ring

$$H_p(G) := \mathbb{Z}[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)].$$

The elements of this ring are called Hecke operators at p. Recall that the elements of a convolution / group ring $\mathbb{Z}[\Gamma]$ of a group Γ over $\mathbb{Z}$ are mappings $\Gamma \to \mathbb{Z}$ of finite support, with multiplication of two mappings f and g defined by convolution: $f * g : x \mapsto \sum_{\gamma\in\Gamma} f(\gamma)g(\gamma^{-1}x)$. We want to study $\mathcal{A}^2(G)$ by considering the commuting actions of $G(\mathbb{R})$ and the Hecke algebra $H(G) := \otimes_p H_p(G)$.

We denote $\Pi(G)$ the set of isomorphism classes of the form $\pi_\infty \otimes \pi_{\mathrm{f}}$, with π_∞ a unitary irreducible representation of $G(\mathbb{R})$ and π_{f} a one dimensional representation of the ring $H(G)$.

We write $m(\pi)$ for the multiplicity of π as a subrepresentation of $\mathcal{A}^2(G)$. It is finite according to a theorem by Harish-Chandra [18, Thm. 1.1].

Definition 8.1. A discrete automorphic representation of G is an element of $\Pi(G)$ with $m(\pi) \neq 0$. We denote by $\Pi_{\mathrm{disc}}(G) \subseteq \Pi(G)$ the subset of these representations.

Write:

$$\mathcal{A}^2(G) = \mathcal{A}^2_{\text{disc}}(G) \oplus \mathcal{A}^2_{\text{cont}}(G) \quad \text{with} \quad \mathcal{A}^2_{\text{disc}}(G) = \bigoplus_{\pi \in \Pi_{\text{disc}}(G)} m(\pi)\pi.$$

The space $\mathcal{A}^2_{\text{disc}}(G)$ contains the subgroup $\mathcal{A}^2_{\text{cusp}}(G)$ of cusp forms. We denote by $\Pi_{\text{cusp}}(G) \subseteq \Pi_{\text{disc}}(G)$ the subset of elements that are subrepresentations in $\mathcal{A}^2_{\text{cusp}}(G)$.

8.3 Automorphic Forms as Representations

The above follows the definition of automorphic representations that has been given in [8]. In the article [2] the definition is for $G = \text{PGSp}_{2g}$, which is therefore slightly different from what we have seen before. Since it is nicely explained how we need to see Siegel modular forms as automorphic representations in this article, and since we want to follow it as correctly as possible, we will work in this section with PGSp_{2g}.

We will associate an automorphic representation of PGSp_{2n} with a Hecke eigenform of degree g. A Hecke eigenform is a Siegel modular form that is an eigenvector under the action of all the Hecke operators for $\text{Sp}_{2g}(\mathbb{Z})$.

Let f be a cuspidal Hecke eigenform of degree n and weight k and let φ_f be the corresponding automorphic form on $G(\mathbb{A})$ defined as we did in Sec. 7.2 on modular forms and automorphic forms. So φ_f lies in $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$, and moreover we can show that it will lie in the cuspidal subspace $L^2_0(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$. We can use the right action on this space to get all right translates of φ_f. Denote with V_f the subspace that is spanned by all these right translates.

Let π be an irreducible subrepresentation for V_f of the unitary representation as defined on p. 64. We know that we can find such a π as it is a fact that $L^2_0(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$ decomposes discretely into representations that are irreducible. Then this π will be an automorphic representation of $G(\mathbb{A})$ which is trivial on $Z(\mathbb{A})$. Therefore we can view π as an automorphic representation of $\text{PGSp}_{2n}(\mathbb{A})$.

We can do the same for vector-valued Siegel modular forms. Recall that to define a (scalar valued) automorphic form from a vector valued Siegel modular form (values in V), we had to compose it with a linear form $V \to \mathbb{C}$. The choice of linear form $L : V \to \mathbb{C}$ does not matter as we take right translates to form the representation.

We can also immediately construct the automorphic representation belonging to $f \in S_k(\mathrm{Sp}_{2g}(\mathbb{Z}))$ as follows: for π_∞ we take the lowest weight representation π_k constructed in [2, 3.5] and for π_p we take the spherical principal series representation of $\mathrm{PGSp}_{2n}(\mathbb{Q}_p)$ with certain Satake parameters. The precise formulation requires of course a lot of work, but it is nice to see that we can also directly connect modular forms with automorphic representations, instead of using automorphic forms as an intermediate step.

We can construct the representation associated to vector-valued Siegel modular forms just as in the scalar-valued case, except that we have to be careful with π_∞. The representation of $\mathrm{GL}_n(\mathbb{C})$ used for the vector valued Siegel modular form plays an important role in choosing π_∞ here.

We have now seen how we can connect modular forms with automorphic representations. This correspondence is quite strong. For example, the L-functions for a modular form and its representation coincide, and the lifting of modular forms to automorphic forms is invariant under the Hecke action.

Part IV
Lattices

Besides using modular forms to construct automorphic forms (for Sp_{2g}), as we have seen in Sec. 7.2, we can also use lattices. Using this second approach, we obtain examples of automorphic representations for groups such as O_n, SO_n and $\mathrm{SO}_{p,q}$. It might be the case that these automorphic representations are occurring in the cohomology of moduli spaces as well, and maybe these representations can explain parts for which the occurrence is not understood. In fact, Mégarbané has found an automorphic representation for $\mathrm{SO}_{4,3}$ which he calls $\Delta_{23,13,5}$ [26]. This representation has presented itself in the calculations of Faber and others as well, looking at local systems with weight $\lambda = (11, 3, 3)$ for genus $g = 3$ [4].

On the other hand we can construct Siegel modular forms with the help of the θ-series of a lattice, so this makes lattices interesting as well. So all in all the concept of lattices is a very important one in the theory of automorphic forms and represenations.

The following chapters can be found in this part:

—Ch. 9, Lattices (p. 69).

—Ch. 10, Orthogonal Groups (p. 73). These groups are defined using the lattices defined in the previous chapter, and they are used to define certain automorphic representations in the next chapter.

—Ch. 11, Automorphic Forms for Orthogonal Groups (p. 75).

9 Lattices

Lattices are part of a more general definition, namely quadratic groups. We will follow the article of Chenevier and Renard [7] in the definitions that follow.

9.1 Quadratic Groups

Let L be an abelian group. Recall that a quadratic form $q : L \to \mathbb{Z}$ is a map satisfying

—$q(n \cdot x) = n^2 \cdot q(x)$ for all $n \in \mathbb{Z}$ and $x \in L$ and

—the map $L \times L \to \mathbb{Z}$ defined by $(x, y) \mapsto x \cdot y = q(x + y) - q(x) - q(y)$ is a bilinear form.

Definition 9.1. The term quadratic groups is used for abelian groups that are free of finite rank and are equipped with a quadratic form.

A quadratic group L with quadratic form q has a determinant $\det(L) \in \mathbb{Z}$, which is by definition the determinant of the symmetric bilinear form $x \cdot y$ associated to q. We say that L is non-degenerate if $\det(L) = \pm 1$ or ± 2, but this is by no means standard terminology. Notice that $x \cdot x = q(2x) - 2q(x) = 2q(x)$ and therefore $x \cdot x$ is even, which implies that the map $(x, y) \mapsto x \cdot y$ is alternating on $L/2L$. This has as consequence that $\det(L) = \pm 1$ only if n even and $\det(L) = \pm 2$ only if n odd.

We will finally define the signature of a quadratic group as the signature (p, q) of its quadratic form. It turns out that for non-degenerate quadratic groups $p - q \equiv -1, 0, 1 \pmod 8$, so there are no quadratic forms with a signature that does not satisfy the congruence.

Positive Definite Case

In the positive definite case, that is to say $(p, q) = (n, 0)$, we can view the quadratic group L as a lattice in euclidean space $L \otimes \mathbb{R}$. Consider for this the standard euclidean space $\mathbb{R}^n$ with inner product $(x_i) \cdot (y_i) := \sum_i x_i y_i$ and denote $\mathcal{L}_n$ the set of lattices $L \subseteq \mathbb{R}^n$ coming from a non-degenerate quadratic group. In other words: if we restrict the map $x \mapsto x \cdot x/2$ to L and use it as quadratic form then L becomes a non-degenerate quadratic group.

Root Systems

We can study these lattices using root systems. Let $R \subset \mathbb{R}^n$ be a root system of rank n with the property that $x \cdot x = 2$ for all $x \in R$.

Definition 9.2. We use the following notion of a root system $R \subset \mathbb{R}^n$:
—R is a finite set that generates $\mathbb{R}^n$, and $0 \notin R$;
—$x \cdot x = 2$ for all $x \in R$ (as we have already stated);
—$x \cdot y \in \mathbb{Z}$ for all $x, y \in \mathbb{Z}$;
—for all $x \in R$, reflections through the hyperplane of x leave R stable.

An important result in the theory of root systems is that irreducible components of a root system R can be classified: they are of type A, D, or E. If we denote by $e_i \in \mathbb{R}^n$ the standard basis elements of $\mathbb{R}^n$, then these types are given by

$$\begin{aligned}
\mathrm{A}_n &= \{\pm(e_i - e_j) : 1 \leq i < j \leq n+1\} \subset \mathbb{R}^{n+1}, \\
\mathrm{D}_n &= \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \subset \mathbb{R}^n, \\
\mathrm{E}_8 &= \mathrm{D}_8 \cup \{(x_i) = (\pm 1/2, \ldots, \pm 1/2) : \textstyle\prod_i x_i > 0\} \subset \mathbb{R}^8, \\
\mathrm{E}_7 &= \mathrm{A}_7 \cup \{(x_i) = (\pm 1/2, \ldots, \pm 1/2) : \textstyle\sum_i x_i = 0\} \subset \mathbb{R}^8.
\end{aligned}$$

Here we must note that the $\pm$-signs are independent from each other. So for example D_n contains $2n(n-1)$ elements. In the list above, the root system E_6 is omitted because we do not need it in this thesis. It should be clear that the root system A_n is a root system of rank n, even though we defined it embedded in $\mathbb{R}^{n+1}$. The same holds for E_7.

Root Lattices

The set R generates a lattice in $\mathbb{R}^n$ that we call $L(R)$, and we will see this is a quadratic group through the quadratic form $x \mapsto x \cdot x/2$. This is called the root lattice that is associated with R. It contains exactly the same information as R because of the property that $R = \{x \in L(R) : x \cdot x = 2\}$. It turns out that the root lattices A_1, E_7, E_8 and $E_8 \oplus A_1$, respectively belonging to the root systems A_1, E_7, E_8 and $\mathrm{E}_8 \sqcup \mathrm{A}_1$, are non-degenerate, and they are of rank 1, 7, 8 and 9. These are also the only ones of rank < 15. Note that confusion can arise due to the notation. We choose to denote the root systems with upright letters and their associated lattices with italics.

9.2 Theta Series

We will later show how we can use the lattices as defined above to define automorphic forms for SO_n. But we can also use them to construct Siegel modular forms, automorphic forms for Sp_{2g}. They can be realized as the θ-series of a lattice. This has for example been done by Chenevier and Lannes to make automorphic forms out of even unimodular lattices [8, Ch. V].

We will quickly explain how this works. Pick an even unimodular lattice $L \subseteq \mathbb{R}^n$ and choose a degree g. Then the θ-series of L for degree g is defined as

$$\theta_g(L) = \sum_{v \in L^g} q^{v \cdot v/2}.$$

Here $v = (v_i)$ is a g-tuple of lattice points of L, and $q^{v \cdot v/2} = e^{\pi i \operatorname{tr}((v \cdot v)\tau)}$ with $\tau \in \mathbb{H}_g$ and $v \cdot v = (v_i \cdot v_j)_{i,j}$ a $g \times g$ Gramian matrix. In this case $v_i \cdot v_j$ is given by the inner product on $\mathbb{R}^n$. It turns out that $\theta_g(L)$ is a scalar-valued Siegel-modular form of degree g and of weight $n/2$ (for the full modular group $\mathrm{Sp}_{2g}(\mathbb{Z})$).

We have the following classification of even unimodular lattices for $n < 32$. There is a unique one for $n = 8$, up to rotations and reflections of $\mathrm{O}_n(\mathbb{R})$. There are 2 for $n = 16$ and finally 24 in dimension $n = 24$. Even unimodular lattices exist only if n is a multiple of 8, and the number of them explodes as n grows. For example for $n = 32$ it is known that there are already millions. These lattices will all give examples of modular forms (but of course these modular forms are not all distinct: for example for $g = 1$ we already know that there are not that many modular forms).

10 Orthogonal Groups

Define for each commutative ring R the subgroup $\mathrm{O}_L(R)$ of the general linear group $\mathrm{Aut}(L \otimes R)$ that consists of elements g such that $q_R \circ g = q_R$. Here the notation q_R has the following meaning: the quadratic form q can be extended to R, and we denote this by $q_R : L \otimes R \to R$. In this way we get the orthogonal group scheme $\mathrm{O}_L \subseteq \mathrm{Aut}_L$ over $\mathbb{Z}$ associated with L.

Now assume that L is non-degenerate. If n is even, then O_L is a smooth group scheme over $\mathbb{Z}$. It has exactly two connected components, and we shall write $\mathrm{SO}_L \subseteq \mathrm{O}_L$ for the neutral component. If n is odd, we simply define SO_L as the kernel of the determinant. In each case it turns out that SO_L is reductive over $\mathbb{Z}$, and even semi-simple if $n \neq 2$, which are properties that allow us to define automorphic forms for them. This is a reason for working with SO_L rather than O_L.

10.1 Positive Definite Case

In general dimension $n = 8k + s$, with $s = -1$ or 0 or 1 we get an example of a positive definite non-degenerate quadratic group L_n by looking at the direct sums $E_7 \oplus E_8^{k-1}$ or E_8^k or $E_8^k \oplus A_1$. We will simply write O_n and SO_n for O_{L_n} and SO_{L_n} if we have to deal with these lattices. Keep an eye on the fact that for higher rank these are not anymore the unique lattices with that particular rank.

10.2 Indefinite Case

More generally, if $p \geq q > 0$ are integers, and if $p - q \equiv_8 -1$ or 0 or 1, then it turns out that non-degenerate quadratic groups with signature (p, q) are unique, so we can simply write $\mathrm{SO}_{p,q}$ for their special orthogonal group schemes. We can also give a concrete description. For example, if we take $p = q + 1$, then we can define $\mathrm{O}_{q+1,q}$ as follows.

Definition 10.1. Let R be a commutative ring. The group $\mathrm{O}_{q+1,q}(R)$ consists of elements $\gamma \in \mathrm{GL}_{2q+1}(R)$ that satisfy

$$\gamma^{\mathrm{t}} J \gamma = J, \qquad \text{with } J = \begin{pmatrix} 0_q & I_q & \\ I_q & 0_q & \\ & & 2 \end{pmatrix},$$

where 0_q is the $q \times q$-matrix with all entries equal to zero, and I_q is the $q \times q$ identity matrix. In this way the quadratic form is preserved under the action of $\mathrm{O}_{q+1,q}$. For γ to be in $\mathrm{SO}_{q+1,q}$ we furthermore require that $\det \gamma = 1$.

In low dimensions these groups coincide with groups that we already know. We then have the following isomorphisms over $\mathbb{Z}$:

$$\mathrm{SO}_{1,1} \cong \mathbb{G}_{\mathrm{m}}, \quad \mathrm{SO}_{2,1} \cong \mathrm{PGL}_2 \quad \text{and} \quad \mathrm{SO}_{3,2} \cong \mathrm{PGSp}_4.$$

Let us show the last isomorphism as an example. Consider the conjugation action of $\mathrm{PGSp}_4(R)$ on matrices of the form

$$\begin{pmatrix} x_3 & x_2 & 0 & -x_1 \\ x_5 & -x_3 & x_1 & 0 \\ 0 & x_4 & x_3 & x_5 \\ -x_4 & 0 & x_2 & -x_3 \end{pmatrix},$$

with $x_1, \ldots, x_5 \in R$. These matrices have the property that $AJ_4 = J_4^{\mathrm{t}} A$, with J_4 as defined on p. 54. We define a quadratic form on this 5-dimensional space by $\mathrm{tr}(A^2)/2 = 2x_1x_4 + 2x_2x_5 + 2x_3^2$. Then there are subgroups of finite rank of this space of matrices with addition as group operation that form quadratic groups of signature $(3, 2)$. The conjugation action of $\mathrm{PGSp}_4(R)$ corresponds precisely to the action of the special orthogonal group of these quadratic groups, because it leaves the quadratic form invariant.

11 Automorphic Forms for SO-groups

We start with a description of the space of automorphic forms for SO_n. We use the definition as given in the thesis of Mégarbané and his articles (for example [26]).

11.1 Definition

Definition 11.1. Let (W, ρ) be a finite dimensional representation of $\mathrm{SO}_n(\mathbb{R})$ over $\mathbb{C}$. Then we can define automorphic forms of weight ρ for SO_n as functions $f : \mathcal{L}_n \to W$ for which we have $f(\gamma \cdot L) = \rho(\gamma) \cdot f(L)$ for all $\gamma \in \mathrm{SO}_n(\mathbb{R})$. The vector space of such automorphic forms is denoted by $M_\rho(\mathrm{SO}_n)$.

It is possible to show that $M_\rho(\mathrm{SO}_n)$ is a finite dimensional vector space for all n and all weights ρ. It is property that we also have seen for $M_k(\mathrm{SL}_2)$ and $M_\rho(\mathrm{Sp}_{2g})$.

We have got the isomorphism

$$\mathcal{L}_n \cong \mathrm{SO}_n(\mathbb{Q})\backslash\mathrm{SO}_n(\mathbb{A})/\mathrm{SO}_n(\widehat{\mathbb{Z}})$$

which allows us to view the functions f as described above as automorphic forms for $G = \mathrm{SO}_n$ [8, p. v].

As we have seen, Mégarbané uses the term automorphic form for the definition above, even though it does not satisfy Def. 7.3 straight away. I think the definition also resembles Def. 6.8 of vector-valued Siegel modular forms, as this is also defined in terms of a representation.

Of course we can also define automorphic forms for the groups $\mathrm{SO}_{p,q}$ and the associated lattices of signature (p, q). In this section the choice was made to give the definition only for the case SO_n for the time being.

11.2 Kneser Neighbours and Hecke Operators

With regard to lattices, we can define so-called Kneser neighbours of a lattice L. These are lattices that are related to L in a certain way. We use the Kneser neighbours to describe the Hecke operators acting on $\mathcal{L}_n$.

Definition 11.2. Let A be a finite abelian group. Two lattices $L, L' \in \mathcal{L}_n$ are A-Kneser neigbours, or A-neighbours for short, if $L/(L \cap L') \cong L'/(L \cap L') \cong A$. If $A = \mathbb{Z}/p\mathbb{Z}$, then we speak of p-neighbours.

Definition 11.3. Let A be a finite abelian group. For a lattice $L \in \mathcal{L}_n$ we define $T_A(L)$ as the formal sum over all A-neighbours of L. In this way we obtain a Hecke oparator T_A that we can view as an endomorphism of $\mathbb{Z}[\mathcal{L}_n]$. If $A = \mathbb{Z}/p\mathbb{Z}$ then we denote the Hecke operator by T_p.

The elements $f \in M_\rho(\mathrm{SO}_n)$ can be seen as functions $\mathbb{Z}[\mathcal{L}_n] \to W$. The Hecke operators act on $\mathcal{L}_n$ and thus we can define an action on $M_\rho(\mathrm{SO}_n)$ as well. In formulas this action is given by

$$T_A(f)(L) := \sum_{L' \in V_A(L)} f(L'),$$

where $V_A(L)$ is the set of A-neighbours of L.

It is a fact that the operators T_A that we just defined over the vector space $M_\rho(\mathrm{SO}_n)$ have the following property: there exists a basis of $M_\rho(\mathrm{SO}_n)$ of elements that are eigenvectors for all T_A at the same time.

The Hecke operators give us a way to study (the properties of) automorphis forms. Moreover, it should hold that the trace of the Hecke operators agrees with the trace of the Frobenius morphisms on the ℓ-adic cohomology groups of local systems on moduli spaces. So Hecke operators give us a way to link automorphic forms with the cohomology of moduli spaces.

11.3 Calculating Hecke Eigenvalues

In this paragraph we will shortly explain how Mégarbané calculated the Hecke traces of a number of automorphic forms for special orthogonal groups. It are these traces that have been compared with calculations done by Bergström, Faber and Van der Geer.

Let $n \in \mathbb{N}$ such that $n \equiv -1, 0, 1 \pmod 8$, and let $L \in \mathcal{L}_n$ be a lattice. We write $\mathrm{SO}(L) = \{\gamma \in \mathrm{SO}_n(\mathbb{R}) : \gamma L = L\}$.

We are particularly interested for the method for $n = 7$, because those Hecke traces seem to correspond to certain automorphic representations in the cohomology of $\mathcal{M}_3$. So we can restrict ourselves to the case $n = 7, 8, 9$. Let $L_0 \in \mathcal{L}_n$ (for example $L_0 = E_7$ for $n = 7$). The group $\mathrm{SO}(L_0)$ acts naturally on the set of A-neighbours of L_0 denoted by $V_A(L_0)$. Write $\mathcal{V}_i$ for the orbits of this action, and take for each i an element $g_i \in \mathrm{SO}_n(\mathbb{R})$ such that we have $g_i L_0 \in \mathcal{V}_i$. Using this notation, we can write down a formula for the traces of Hecke operators acting on spaces of automorphic forms for SO_n.

Theorem 11.4. *Let $n = 7, 8$ or 9, let A be a finite abelian group and let (ρ, W) be a finite dimensional representation of* $\mathrm{SO}_n(\mathbb{R})$. *Using the above* *notation, we have the following formula*

$$\operatorname{tr}(T_A \mid M_\rho(\mathrm{SO}_n)) = \frac{1}{|\mathrm{SO}(L_0)|} \sum_i \left(|\mathcal{V}_i| \sum_{\gamma \in \mathrm{SO}(L_0)} \operatorname{tr}(\gamma g_i \mid W) \right)$$

for the traces of the Hecke operators T_A *acting on the space* $M_\rho(\mathrm{SO}_n)$ *of automorphic forms for* SO_n*.*

This formula is very useful: we only need to have a good description of the representations of $\mathrm{SO}_n(\mathbb{R})$ and of the lattices of $\mathcal{L}_n$ to get to know everything we want to know about $M_\rho(\mathrm{SO}_n)$. The only problem may be that the calculations can become difficult to do by hand. Mégarbané made use of a computer program to do the calculations for $n = 7, 8, 9$ as presented in [26]. The results of all his calculations can be found on his web page `http://megarban.perso.math.cnrs.fr/tracehecke.html`.

Part V

Making the Connection

In Pt. II we have looked at the moduli space of curves of genus g. In Pt. III we have looked at Siegel modular forms, automorphic forms, and automorphic representations. Now it is time to try and relate these parts to each other.

—Ch. 12, Teichmüller Modular Forms (p. 81). Here we introduce Teichmüller modular forms, the analogue of Siegel modular forms, as certain global sections on $\mathcal{M}_g$. Beware that these are not automatically automorphic forms, but we will try to construct them as lift of automorphic forms.

—Ch. 13, To Higher Cohomology Groups (p. 83). With Teichmüller modular modular forms we study automorphic forms occurring in H^0-groups of $\mathcal{M}_g$. However, we can also find them in higher cohomology groups.

—Ch. 14, Using a Map Towards Lattices (p. 85). We are especially interested in automorphic forms for SO-groups found in the cohomology of $\mathcal{M}_3$. One way of understanding this connection is by trying to associate lattices to curves of genus 3.

12 Teichmüller Modular Forms

12.1 Introducing Teichmüller Modular Forms

Teichmüller modular forms have been introduced by Ichikawa in [21]. In short, he defined them as global sections of line bundles on the moduli space of curves of a certain genus. The notion of Teichmüller modular forms is an analogy of Siegel modular forms, which are global sections of line bundles (or vector bundles) on the moduli space of principally polarized abelian varieties of a certain dimension. The name comes from the Teichmüller space.

Definition 12.1. Ichikawa introduces Teichmüller modular forms in [21] as sections in

$$T_{g,h}(R) := \Gamma(\mathcal{M}_g \otimes R, \lambda^{\otimes h}),$$

with $h \geq 1$ the weight, R a commutative ring (we will predominantly take $R = \mathbb{C}$ here), and λ an invertible sheaf on $\mathcal{M}_g$ that is defined as

$$\lambda := \wedge^g \pi_*(\Omega_{\mathcal{C}/\mathcal{M}_g}),$$

where $\pi : \mathcal{C} \to \mathcal{M}_g$ is the universal curve and $\Omega_{\mathcal{C}/\mathcal{M}_g}$ the relative sheaf of differentials. This means that we should view $\mathcal{M}_g$ as a stack in order to give this definition. The space of Teichmüller modular forms for genus g, of weight h and over R is thus denoted by $T_{g,h}(R)$.

Ichikawa subsequently points out that $T_{g,h}(k)$ is finite dimensional for every field k with characteristic $\neq 2$, a property that we are used to in the context of "regular" modular forms.

Furthermore, it is obvious that we can generalize the definition so that we can also speak of vector-valued Teichmüller modular forms. Vector-valued Siegel-modular forms will also fit in via t.

The result of the second article of Ichikawa [22] is that the graded ring of scalar-valued Teichmüller modular forms for $g = 3$ is generated by pullbacks of scalar-valued Siegel modular forms of degree $g = 3$ together with χ_9, a Teichmüller modular form of weight 9. The proof relies on the properties of $t : \mathcal{M}_3 \to \mathcal{A}_3$ and is quite straightforward.

An interesting question might be: are there Teichmüller modular forms that do not come from automorphic forms. The form ξ_9 might be a candidate.

12.2 Vector-valued Teichmüller Modular Forms

As said before, Ichikawa introduced Teichmüller modular forms in [21]. In all the articles he wrote on Teichmüller modular forms, he only considered and defined scalar-valued ones, just as we did in Def. 12.1. However, vector-valued Teichmüller modular forms are more useful since vector-valued Siegel modular forms are found in the cohomology of $\mathcal{A}_g$ and therefore also in the cohomology of $\mathcal{M}_g$ (see our discussion above).

But before we can define those, we first need to introduce the Hodge bundle $\mathbb{E} = \pi_*(\Omega_{\mathcal{C}/\mathcal{M}_g})$. Here $\pi : \mathcal{C} \to \mathcal{M}_g$ is the universal curve and the vector bundle $\mathbb{E}$ is called the Hodge bundle. Notice that for the scalar-valued definition we already used the Hodge bundle as follows: $\lambda = \wedge^g \mathbb{E}$. Now, let $\rho = (a_1, \ldots, a_g)$ with $a_i \in \mathbb{Z}_{\geq 0}$. Then we define $\mathbb{E}_\rho$ to be the vector bundle obtained from $\mathbb{E}$ by applying the Schur functor associated to the irreducible representation of $\mathrm{GL}_g(\mathbb{C})$ with highest weight

$$(a_1 + \cdots + a_g, a_2 + \cdots + a_g, \ldots, a_{g-1} + a_g, a_g).$$

Definition 12.2. Let $g > 1$, and let $\mathcal{M}_g$ denote the moduli stack of smooth curves of genus g. Then we define the space of Teichmüller modular forms of weight ρ and genus g as

$$T_{g,\rho} = H^0(\mathcal{M}_g \otimes R, \mathbb{E}_\rho).$$

We can compare this with the definition of scalar-valued Teichmüller modular forms by taking $\rho = (0, \ldots, 0, h)$.

Pullbacks of Siegel modular forms always yield Teichmüller modular forms. It turns out that each Teichmüller modular forms for $g = 2$ equals a pullback of a Siegel modular form of degree 2 up to an *integral* power of χ_{10}, a Siegel modular form of weight 10.

We also find Teichmüller modular forms for $g = 3$ with help of the Torelli map, but not all Teichmüller modular forms are pullbacks of Siegel modular forms anymore, since $t : \mathcal{M}_3 \to \mathcal{A}_3$ is 2-to-1 as a morphism of stacks. One such Teichmüller modular form not coming from a Siegel modular form is χ_9 of weight 9 for genus 3. Its square is equal to χ_{18}, (a pullback of) a certain Siegel modular form of weight 18.

To explain these other Teichmüller modular forms, it may be useful to find maps similar to the Torelli map t, in order to lift other automorphic forms. We are already quite certain that automorphic forms for SO_7 or $\mathrm{SO}_{4,3}$ occur in the higher cohomology of groups of $\mathcal{M}_3$, so they may be found as Teichmüller modular forms (which are elements of H^0-groups) as well.

13 To Higher Cohomology Groups

In Ch. 12 we talked about ways to find automorphic forms in H^0 of the moduli spaces of curves. However, originally, in the work of Bergström, Faber and Van der Geer, the automorphic forms occur in the higher cohomology groups of local systems.

In my opinion, there are two ways to understand these higher cohomology groups. The first way is to first understand the higher cohomology groups of local systems of moduli spaces of abelian varieties, and then identify them with higher cohomology groups of moduli spaces of curves via the Torelli map.

The second way is to first understand the Teichmüller modular forms, that is, the H^0-groups of moduli spaces of curves. After that, we can try to relate the higher cohomology groups with these H^0-groups. For this we may use tools such as the Hodge decomposition and the Leray spectral sequence.

14 Using a Map Towards Lattices

The automorphic forms for $G = \mathrm{SO}_7$ found by Mégarbané may be explained in the cohomology of $\mathcal{M}_3$ by constructing a map $\mathcal{M}_3 \to \mathcal{L}_7$ with the right properties. This follows from the fact that we have the identification $\mathcal{L}_7 = \mathrm{SO}_7(\mathbb{Q})\backslash\mathrm{SO}_7(\mathbb{A})/\mathrm{SO}_7(\widehat{\mathbb{Z}})$. Similarly, we may want to look at $G = \mathrm{SO}_{4,3}$ and associated lattices of signature $(4,3)$.

It may be a bit naive to think that this is the way to give the explanation. However, there are three arguments to give that make this method seem reasonable.

(1) We have the isomorphisms $\mathrm{PGL}_2 \cong \mathrm{PGSp}_2 \cong \mathrm{SO}_{2,1}$ and $\mathrm{PGSp}_4 \cong \mathrm{SO}_{3,2}$, see p. 74. So these special orthogonal groups really have something to do with $\mathcal{M}_g$.

(2) We have to find some structure in curves of genus 3 and higher. One way to do this could be to associate lattices (which have a group structure) to these curves.

(3) Where else would the automorphic forms that are found in the cohomology of $\mathcal{M}_3$ come from? We know that some are in fact Siegel modular forms coming from the Torelli map $t : \mathcal{M}_3 \to \mathcal{A}_3$, but there are also automorphic forms in the cohomology that are not coming from Siegel modular forms.

There are a few articles in recent research that explain a construction for curves of genus $g = 3$ that might be useful in our search of such a map.

—*A complex hyperbolic structure for the moduli space of curves of genus three* by Shigeyuki Kondō [23].

—*Moduli of plane quartics, Göpel invariants and Borcherds products* by Shigeyuki Kondō [24].

—*On occult period maps* by Stephen Kudla and Michael Rapoport [25].

14.1 Constructing Lattices of Rank 7

Before we give the construction of a lattice isomorphic to E_7 from a curve of genus 3 that is nonhyperelliptic, we have to know what hyperelliptic and nonhyperelliptic curves are. For completeness, we explain this here.

Hyperelliptic and Nonhyperelliptic Curves

A curves C of genus 1, or an elliptic curve, has the special property that there is a morphism from C to $\mathbb{P}^1$ of degree 2. We want to consider curves of higher genus with the same property.

Let C be a curve of genus $g \geq 2$. If there exists a finite morphism $f : C \to \mathbb{P}^1$ of degree 2, then C is called hyperelliptic. Needless to say, if such a morphism does not exist, then C is called nonhyperelliptic.

For genus 3 there exist both hyperelliptic curves and nonhyperelliptic curves. The subvariety of $\mathcal{M}_3$ of hyperelliptic curves of genus 3 is 5 dimensional [20, Ex. IV.5.5.6]. As $\mathcal{M}_3$ has dimension 6, we conclude that curves of genus 3 are in general nonhyperelliptic curves, so it is not a big problem if we limit ourselves to these nonhyperelliptic curves. They are represented by smooth curves in $\mathbb{P}^2$ of degree 4.

A Construction for Curves of Degree 4

We will first restrict our study of genus 3 curves to those that are not hyperelliptic. The construction below comes from [23].

As stated before, a nonhyperelliptic curve of genus 3 is represented by a smooth plane curve C of degree 4. We can give a double cover S of $\mathbb{P}^2$ that is branched along C. Namely, if C is defined by the degree 4 polynomial $f(X, Y, Z)$, then we take the variety defined by $f(X, Y, Z) = T^2$ in $\mathbb{P}(1, 1, 1, 2)$. (So for the weighted projective space we take homogeneous coordinates $(X : Y : Z : T)$.)

Apparently, S can be obtained from $\mathbb{P}^2$ by blowing up at 7 general points. Let e_0 be the pull-back of the class of a line on $\mathbb{P}^2$ and for $1 \leq i \leq 7$ let e_i be the classes of exceptional curves of these blow-ups. Then $e_0^2 = 1$, $e_i^2 = -1$ for all $1 \leq i \leq 7$, and $e_i \cdot e_j = 0$ for all $i \neq j$. Put

$$\alpha_0 = e_0 - e_1 - e_2 - e_3 \quad \text{and} \quad \alpha_i = e_i - e_{i+1} \quad \text{for } 1 \leq i \leq 6.$$

Then the $\{\alpha_i\}$ generate a lattice that is isometric to E_7, where E_7 is the root lattice associated to the root system E_7 as introduced on p. 70. The term isometric means that the quadratic form defined on them is identical. So as quadratic groups these lattices are equal. The quadratic form in this case is defined by using the intersection pairing as bilinear form.

Alternative Construction

Alternatively, consider the covering transformation σ of S over $\mathbb{P}^2$. Apparently the α_i are constructed in such a way that $\sigma(\alpha_i) = -\alpha_i$ for $0 \leq i \leq 6$. So put

$$H_- := \{x \in H^2(S, \mathbb{Z}) : \sigma^*(x) = -x\},$$

then also $H_- \cong E_7$. Of course we have to check that H_- is of rank 7, and that the bilinear form defined by the cup product will give the right quadratic form.

It might be interesting to note that in [24], which is a subsequent article of [23] by Kondō, automorphic forms are mentioned as well, associated to a similar construction as given above. However, those automorphic forms are meromorphic and of level > 1, so they are not directly useful in our context.

14.2 Abelian Varieties of Dimension 7

Finally, I want to talk about an interesting fact from the article by Kudla and Rapoport [25]. For this they use a construction of another lattice coming from [23].

Let C be a smooth non-hyperelliptic curve of genus $g = 3$, so we can assume that C is a smooth curve in $\mathbb{P}^2$ of degree 4. Let X be the four cyclic cover of $\mathbb{P}^2$ branched along C (in contrast to the surface S we considered in Sec. 14.1 that was a twofold cover of $\mathbb{P}^2$). Let τ be a covering transformation of X of order 2. Then we define

$$L_- = \{x \in H^2(X, \mathbb{Z}) : \tau^*(x) = -x\}$$

This turns out to be a lattice of rank 14. We must note that this lattice has determinant 2^8, so it is degenerate according to Sec. 9.1 and therefore the research of Chenevier-Lannes, Chenevier-Renard, and Mégarbané does not apply.

However, it is still quite interesting, as we can use this lattice to construct an abelian variety. This abelian variety is of dimension 7, and has a polarization of degree 2^6. So these abelian varieties are different from the principally polarized ones that we have studied in Pt. II. If we denote by $\mathcal{B}_7$ the moduli space of these abelian varieties constructed above, and by $\mathcal{N}_3$ the moduli space of smooth non-hyperelliptic curves of genus 3, then we get a map $\mathcal{N}_3 \to \mathcal{B}_7$. Moreover, as a morphism of schemes this is an open embedding [25, Thm. 7.1]. So if we can give a description of modular / automorphic forms living on $\mathcal{B}_7$, we can immediately lift these to obtain Teichmüller modular forms on $\mathcal{M}_3$!

Part VI

End Matter

We end this thesis with a conclusion (p. 91), where we also describe some applications and give suggestions for further research. After the conclusion the references (p. 93) and the index (p. 97) can be found.

Conclusion

The goal of the thesis is to examine the role of automorphic forms and automorphic representations in relation to the cohomology of moduli spaces of curves and abelian varieties. We can summarize our findings as follows.

—Siegel modular forms are defined in a natural way as global sections of certain vector bundles over the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g. Due to the Torelli map $t : \mathcal{M}_g \to \mathcal{A}_g$ these Siegel modular forms can also be pulled back to $\mathcal{M}_g$. However, for $g \geq 4$ we have $\dim \mathcal{M}_g \neq \dim \mathcal{A}_g$, so such a pullback could vanish and in any case the cohomology on $\mathcal{A}_g$ associated to a Siegel modular form will be very different from the cohomology on $\mathcal{M}_g$ associated to the pullback. And also for $g = 3$ we cannot explain all automorphic representations in the cohomology by pulling back Siegel modular forms. We therefore need alternatives to the Torelli map in order to study the automorphic forms that are found in its cohomology.

—Siegel modular forms can be viewed as automorphic forms or representations. They are automorphic forms for $G = \mathrm{Sp}_{2g}$. But we can also define automorphic forms for special orthogonal groups for example.

—The automorphic forms for $G = \mathrm{SO}_7$ found by Mégarbané may be explained in the cohomology of $\mathcal{M}_3$ by constructing a map $\mathcal{M}_3 \to \mathcal{L}_7$ with the right properties such that we can lift the automorphic froms to $\mathcal{M}_3$. This follows from the fact that we have the identification

$$\mathcal{L}_7 \cong \mathrm{SO}_7(\mathbb{Q})\backslash\mathrm{SO}_7(\mathbb{A})/\mathrm{SO}_7(\widehat{\mathbb{Z}})$$

and the automorphic forms for SO_7 are defined on the space on the right hand side. Similarly, we may want to look at $G = \mathrm{SO}_{4,3}$ and associated lattices of signature $(4, 3)$.

Applications

Of course with any form of scientific research it is tempting to ask for the purpose of this research. This is not always easy to answer, especially when it comes to fundamental mathematical research. Still though it can be useful to think about it, and in my opinion, there are 3 particular uses for the things that have been described in this thesis, at least in other mathematical research.

(1) It gives us a way to find examples of automorphic representations. One thing that makes automorphic representations so inaccessible in comparison to classical modular forms is that there are very few concrete examples of

automorphic representations, and moreover it might be unclear in what way we can obtain these representations. We have seen that we can find automorphic representations in the cohomology of moduli spaces. In fact, Bergström, Faber and Van der Geer used their calculations to predict the existence of Siegel modular forms of certain weights for degree 2 and 3.

(2) It gives us more insight in the moduli spaces of curves. Moduli spaces of curves are notoriously difficult to study, as we generally do not have explicit constructions for them and know little about them, especially for higher genera. Calculating the cohomology of these spaces is one way of trying to get a better understanding.

(3) It is an extensive example of the Langlands correspondence between automorphic representations and Galois representations. If we consider ℓ-adic cohomology (of moduli spaces of curves for example), then we can connect ℓ-adic representations to it. With the things that we have done in this thesis, we can interpret these as automorphic representations as well.

These kinds of correspondences between representations are conjectures that are part of the Langlangds program. The Langlands program is one of the most important topics in modern mathematical research.

Further Research

A number of suggestions for further research are listed below.

(1) Trying to find more automorphic representations. This can for example be done by extending the work of Chenevier-Renard [7], or by trying to calculate the cohomology of local systems for curves of genus 4.

(2) We have seen that the Torelli map has been very useful in explaining automorphic representations for Sp_{2g} in the cohomology of local systems of moduli spaces of curves of genus g. However, it does not explain everything, so it will be useful to find new maps that can explain the automorphic representation in the cohomology of moduli spaces, especially for $g \geq 3$. In particular, it might be interesting to elaborate on the beginning that was made in Ch. 14.

(3) Studying automorphic representations in the cohomology of other moduli spaces. For example, the moduli space of curves with level structure will have something to do with automorphic representations of level > 1.

References

[1] E. Arbarello, M. Cornalba, P.A. Griffiths, *Geometry of Algebraic Curves Volume II*, Springer, http://www.springer.com/br/book/9783540426882, 2011.

[2] M. Asgari, R. Schmidt, *Siegel Modular Forms and Representations*, manuscripta mat. 104, p. 173–200, Springer, https://link.springer.com/article/10.1007/PL00005869, 2001.

[3] K.A. Behrend, *The Lefschetz Trace Formula for Algebraic stacks*, Invent. math. 112, p. 127–149, https://link.springer.com/content/pdf/10.1007/BF01232427.pdf, 1993.

[4] J. Bergström, C. Faber, G. van der Geer, *Siegel modular forms of degree three and the cohomology of local systems*, Selecta Math. 20, p. 83–124, https://arxiv.org/abs/1108.3731, 2014.

[5] J. Bergström, N.P. Dummigan, T. Mégarbané, *Eisenstein congruences for* $\mathrm{SO}(4,3)$*,* $\mathrm{SO}(4,4)$*, spinor and triple product L-values*, Experimental Mathematics, https://arxiv.org/abs/1605.00819, 2016.

[6] A. Borel, H, Jacquet, *Automorphic Forms and Automorphic Representations*, Proceedings of Symposia in Pure Mathematics Vol. 33 Part 1, p. 189–202, American Mathematical Society, 1979.

[7] G. Chenevier, D. Renard, *Level one algebraic cuspforms of classical groups of small rank*, Memoirs of the A.M.S. 1121, vol. 237, http://gaetan.chenevier.perso.math.cnrs.fr/articles/dimform.pdf, 2015.

[8] G. Chenevier, J. Lannes, *Formes automorphes et réseaux unimodulaires pairs*, to appear in Ergebnisse der Mathematik und ihrer Grenzgebiete under the title *Automorphic forms and even unimodular lattices* (in english), http://gaetan.chenevier.perso.math.cnrs.fr/articles/CL_main.pdf, 2016.

[9] A. Deitmar, *Automorphic Forms*, Springer, http://www.springer.com/gp/book/9781447144342, 2010.

[10] P. Deligne *Formes modulaires et représentations ℓ-adiques*, Séminaire N. Bourbaki, 1968–1969, exp. nº 355, p. 139–172, http://www.numdam.org/article/SB_1968-1969__11__139_0.pdf, 1969.

[11] P. Deligne, D. Mumford, *The irreducibility of the space of curves of a certain genus*, Publications Mathématiques de l'IHÉS, Volume 36 (1969), p. 75–109, http://www.numdam.org/item/PMIHES_1969__36__75_0, 1969.

[12] C. Faber, G. van der Geer, *Sur la cohomologie des systèmes locaux sur les espaces de modules des courbes de genre 2 et des surfaces abéliennes*, part I and II, C. R. Acad. Sci. Paris, Sér. I 338, p. 381–384, https://arxiv.org/pdf/math/0305094.pdf, 2004.

[13] C. Faber, P. Pandharipande, *Tautological and non-tautological cohomology of the moduli space of curves*, Handbook of Moduli (G. Farkas and I. Morrison, eds.), Volume I, p. 293–330, Advanced Lectures in Mathematics, International Press, Boston, https://arxiv.org/abs/1101.5489, 2013.

[14] G. Faltings, C.-L. Chai, *Degeneration of Abelian Varieties*, Springer, 1990.

[15] O. Forster, *Lectures on Riemann Surfaces*, Springer, http://www.springer.com/gp/book/9780387906171, 1981.

[16] W. Fulton, *Algebraic Curves*, http://www.math.lsa.umich.edu/~wfulton/CurveBook.pdf, 2008.

[17] G. van der Geer, *Siegel Modular Forms and their Applications*, In J.H. Bruinier, G. van der Geer, G. Harder, & D. Zagier (Eds.), *The 1-2-3 of modular forms: Lectures at a summer school in Nordfjordeid, Norway*, p. 181–246), Springer, http://www.springer.com/la/book/9783540741176, 2008.

[18] Harish-Chandra, *Automorphic forms on semisimple Lie groups*, Lecture Notes in Math. 62, Springer, 1968.

[19] J. Harris, I. Morrison, *Moduli of Curves*, Springer, http://www.springer.com/gp/book/9780387984384, 1998.

[20] R. Hartshorne, *Algebraic Geometry*, Springer, http://www.springer.com/us/book/9780387902449, 1977.

[21] T. Ichikawa, *On Teichmüller Modular Forms*, Math. Ann. 299, p. 731–740, Springer, 1994.

[22] T. Ichikawa, *Teichmüller Modular Forms of Degree 3*, American Journal of Mathematics, Vol. 117, No. 4, p. 1057–1061, http://www.jstor.org/stable/2374959, 1995.

[23] S. Kondō, *A complex hyperbolic structure for the moduli space of curves of genus three*, J. Reine Angew. Math., Vol. 525, p. 219–32, 2000.

[24] S. Kondō, *Moduli of plane quartics, Göpel invariants and Borcherds products*, Int. Math. Res. Not., no. 12, p. 2825–2860, 2011.

[25] S. Kudla, M. Rapoport, *On occult period maps*, Pacific J. Math., Vol. 260, no. 2, p. 565–581, 2012.

[26] T. Mégarbané, *Traces des operateurs de Hecke sur les espaces de formes automorphes de* SO_7*,* SO_8 *ou* SO_9 *en niveau* 1 *et poids arbitraire*, to appear in Journal de Théorie des Nombres de Bordeaux, https://arxiv.org/abs/1604.01914, 2016.

[27] J.S. Milne, *Étale Cohomology*, Princeton University Press, 1980.

[28] J.S. Milne *Elliptic Curves*, http://www.jmilne.org/math/Books/ectext5.pdf, 2006.

[29] D. Mumford, J. Fogarty, K. Kirwan, *Geometric Invariant Theory*, Third Enlarged Edition, Springer, http://www.springer.com/la/book/9783540569633, 1994.

[30] D. Mumford, *The Red Book of Varieties and Schemes*, Second, Expanded Edition, Springer, http://www.springer.com/la/book/9783540632931, 1999.

[31] C.L. Siegel, *Einführung in die Theorie der Modulfunktionen n-ten Grades*, Math. Ann. Volume 116, p. 617–657, https://doi.org/10.1007/BF01597381, 1939.

[32] J.H. Silverman, *The Arithmetic of Elliptic Curves*, Second Edition, Springer, http://www.springer.com/la/book/9780387094939, 1994.

[33] J.-L. Verdier, *Sur les intégrales attachées aux formes automorphes*, Séminaire N. Bourbaki, 1960–1961, exp. n° 216, p. 149–175, http://www.numdam.org/article/SB_1960-1961__6__149_0.pdf, 1961.

[34] D. Zagier, *Elliptic Modular Forms and Their Applications*, In J.H. Bruinier, G. van der Geer, G. Harder, & D. Zagier (Eds.), *The 1-2-3*

of modular forms: Lectures at a summer school in Nordfjordeid, Norway, p. 1–104), http://www.springer.com/la/book/9783540741176 Springer, 2008.

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