Deformation of curves with a group action

Jeroen Huijben

Master's thesis

Supervisor: prof. dr. Gunther Cornelissen
Second reader: dr. Martijn Kool

Utrecht University

Faculty of Science
Department of Mathematics

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Abstract

In this thesis we study the deformations of curves with a finite group action over an algebraically closed field of positive characteristic. While the deformation theory of curves is well-known, there is no complete picture when a group action comes into play. When the genus of the curve is greater than 1, it is known that the deformation functor of the curve with a group action is pro-representable. For a genus 0 curve, we prove in this thesis that the deformation functor is non-pro-representable exactly when the field characteristic is 2, and the group is $\mathbb{Z}/2$, $(\mathbb{Z}/2)^2$ or the dihedral group D_n with n odd. Proving pro-representability in the other cases relies on reduction to local deformations. The non-pro-representability is proved by direct calculations. For genus 1 curves, the problem is still open. We propose an approach that might deal with elliptic curves with a small automorphism group.

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Contents

Introduction

What is deformation theory? In essence deformation theory studies the infinitesimal changes one can make to a geometric object. If X is such a geometric object (think of a curve for example), you could try to construct for some small $\varepsilon > 0$ a continuous family of spaces X_t for $t \in (-\varepsilon, \varepsilon)$, such that $X_0 \cong X$. Another way of describing this, is as a continuous map $\mathcal{X} \to (-\varepsilon, \varepsilon)$. Then the fibre above 0 is X, and the other fibres give the deformations. However, in this way I can consider the map $X \to (-\varepsilon, \varepsilon)$ sending everything to 0. The the other fibres are empty, and we do not get a sensible deformation. If X is a manifold, we could solve this by requiring that the map $\mathcal{X} \to (-\varepsilon, \varepsilon)$ is surjective on tangent spaces (a submersion), so that X must extend in the direction of the interval $(-\varepsilon, \varepsilon)$.

This might be a sensible approach to deformation theory in differential geometry, but this thesis is about deformation theory in algebraic geometry. Therefore we must find substitutes for these small intervals $(-\varepsilon, \varepsilon)$, and for the submersions. The intervals are replaced by Artinian, local rings and are introduced in Section 1.1. Submersions will be replaced by *flat maps*, and they will be used Section 1.2 to define the deformations of schemes. Last, we will put these together in a *deformation functor* and study the properties such functor has in Section 1.4.

The first form of deformation theory appeared in the work of Bernhard Riemann in 1857 [Rie57]. While the theory had not been fully developed, he essentially proved that a Riemann surface of genus g, greater than 1, has a $(3g - 3)$-dimensional space of deformations. Deformation theory in algebraic geometry was only developed much later. One of the earliest large results is by Michael Schlessinger in 1964. In his PhD thesis he gave a small set of properties a deformation functor should have in order to be *pro-representable* [Sch68].

You might wonder, why study deformation theory? This is because it is closely related to the moduli problem. An example of a moduli problem is to classify all curves of a fixed genus. An answer to this question would be a "parameter space" $\mathcal{M}$ of which the points correspond to all such curves. In other words, you want a map $\mathcal{X} \to \mathcal{M}$ of which the fibres are all such curves. You can see the similarities with the setting of deformations, except that $\mathcal{M}$ is now not only a very small interval. Determining this $\mathcal{M}$ can be difficult, but there are some ways to study it. In particular, we can find some of the local structure by doing deformation theory. For more information on such moduli problems in relation to deformation theory, see [Har10]. In my thesis I will only focus on deformation theory.

The first chapter of this thesis treats some basic results of deformation theory, which is often found in literature. The second chapter focuses on a specific deformation problem: that of curves with a group action, over an algebraically closed field of positive characteristic. When the genus of the curve is greater than 1, this deformation functor is pro-representable. The methods to prove this are already known in the literature, and are almost the same as when there is no group action.

On curves of genus 0, this thesis contains some new results. The main tools include: the classification of group actions on the projective line [VM80], translation to local deformations [BM00], and the classification of pro-representability of these local deformation functors [BC09]. Together they allow to prove pro-representability in a lot of cases. Only a few cases remain, namely the action of $\mathbb{Z}/2$, $(\mathbb{Z}/2)^2$, or the dihedral group D_n for n odd, on the projective line over a field of characteristic 2. Direct calculations on these cases show that they turn out to be non-pro-representable.

The curves of genus 1 resist these attacks. While their automorphisms are also classified, it is not always possible to translate the problem to local deformations. And [BC09] only deals with weakly ramified local deformations, so the local deformations may not be classified yet.

Chapter 1

Deformation theory

In this first chapter I give an introduction to the deformation theory of schemes. I do not cover all the basic theory and do not provide some of the more complicated proofs. However, I try to include many small arguments that other sources omit. For a more complete source on deformation theory, see e.g. [Sch68] or [Har10].

Sections 1.1 and 1.3 follow more or less [Bys09a]. Sections 1.2 and 1.4 follow [Sch68].

1.1 Artinian local rings

As said in the introduction, we have to find an algebraic analogue to small intervals $(-\varepsilon, \varepsilon)$ around 0. This is done in the following definition. For any local ring R, I will denote its maximal ideal as $\mathfrak{m}_R$.

Definition 1.1. Let k be a field, and let Λ be a complete, Noetherian, local ring with residue field k. We define the category $\mathcal{C}_\Lambda$ as follows:

- The objects are Artinian, local Λ-algebras A with residue field k, such that the structure morphism $\Lambda \to A$ induces an isomorphism $\Lambda/\mathfrak{m}_\Lambda \to A/\mathfrak{m}_A$.
- The morphisms $\varphi : A' \to A$ are Λ-algebra homomorphisms such that $\varphi(\mathfrak{m}_{A'}) \subset \mathfrak{m}_A$ (we also say that φ is *local*).

Example 1.2. The simplest example of such a category comes from taking $\Lambda = k$. All rings in the category $\mathcal{C}_k$ are finite-dimensional k-vector spaces. The ring $k[\varepsilon]/\varepsilon^2$ is 2-dimensional and dividing out the unique maximal ideal (ε) yields k, so this is indeed an object of $\mathcal{C}_k$. The ring $k[x_1, x_2]/(x_1^3, x_1^2 x_2, x_2^2)$ has a basis $1, x_1, x_1^2, x_2, x_1 x_2$, so it is 5-dimensional. Dividing out the unique maximal ideal (x_1, x_2) again yields k, so this is also an object of $\mathcal{C}_k$. A morphism $k[x_1, x_2]/(x_1^3, x_1^2 x_2, x_2^2) \to k[\varepsilon]/\varepsilon^2$ must be k-linear and must send the maximal ideal (x_1, x_2) into the maximal ideal (ε).

Another typical example comes from $k = \mathbb{F}_p$ a finite field, and $\Lambda = \mathbb{Z}_p$ the p-adic integers. Now the category $\mathcal{C}_\Lambda$ contains not only k-vector spaces, but also rings of higher characteristic $\mathbb{Z}/p^2, \mathbb{Z}/p^3, \ldots$.

Why would these categories be the correct translations of a small interval $(-\varepsilon, \varepsilon)$ to algebraic geometry? For this we have to look at the spectrum of a ring $A \in \mathcal{C}_\Lambda$. Because A is Artinian and local, it has exactly one prime ideal and therefore $\mathrm{Spec}(A)$ consists of exactly one point. We also have the residue map $A \to k$, which yields a map $\mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(A)$. In this map $\mathrm{Spec}(k)$ also consists of exactly one point, and this should correspond to 0 inside a small interval. Even though $\mathrm{Spec}(A)$ also consists of only one point, it adds some structure. Most prominently, the space $\mathrm{Spec}(k)$ has only a trivial tangent space, but for larger rings A, the tangent space of $\mathrm{Spec}(A)$ is non-trivial, as we see in the next example.

Example 1.3. The tangent space of a local ring R is the dual of the vector space $\mathfrak{m}_R/\mathfrak{m}_R^2$. We compute the tangent spaces of the previous example. For $k[\varepsilon]/\varepsilon^2$, we find that $(\varepsilon)/(\varepsilon)^2 = k\varepsilon$ is 1-dimensional, so the tangent space is also 1-dimensional (and indeed non-trivial!). For $k[x_1, x_2]/(x_1^3, x_1^2 x_2, x_2^2)$ the maximal ideal as k-vector space has a basis $x_1, x_1^2, x_2, x_1 x_2$. Its square has basis $x_1^2, x_1 x_2$, so the tangent space is 2-dimensional.

Now we have seen why we should treat these Artinian local rings as the correct parameter spaces for doing deformations. However, we haven't seen yet why these rings should be Artinian. Based on what we have seen so far, we could also expect a non-Artinian local ring like $k[x]_{(x)}$ to act as a parameter space. The reason to focus on Artinian rings, is that they are relatively small and allow for direct computations. The main tool to do this are the following *small extensions*.

Definition 1.4. Let $\varphi : A \to A'$ be a morphism in the category $\mathcal{C}_\Lambda$. We call φ a *small extension* if and only if:

- φ is surjective;
- the kernel $\ker(\varphi)$ is a principal ideal (t); and
- $\mathfrak{m}_A t = 0$.

The last property shows that (t) is an $A/\mathfrak{m}_A$-module, and thus a k-vector space.

Remark 1.5. Because $\mathfrak{m}_A$ is the unique maximal ideal of A, we see that $(t) \subset \mathfrak{m}_A$. It follows immediately that $t^2 \in \mathfrak{m}_A t = 0$.

Lemma 1.6. *Let $\varphi : A \to A'$ be a surjection in $\mathcal{C}_\Lambda$. Then there exist rings $A = B_0, B_1, \ldots, B_n = A' \in \mathcal{C}_\Lambda$ together with small extensions $\varphi_i : B_{i-1} \to B_i$ such that $\varphi = \varphi_n \circ \cdots \circ \varphi_2 \circ \varphi_1$. In other words, every surjection can be decomposed into small extensions.*

Proof. The idea is to do this inductively, but we first have to find some property of a ring that decreases in a small extension. This will be the length as Λ-module. We show that any object of $\mathcal{C}_\Lambda$ has finite length, and that the length decreases by one in a small extension. Then we can start the induction.

We will make repeated use of the fact that the length of a module M is the sum of the lengths of a submodule N and the length of the quotient M/N.

For a ring $A \in \mathcal{C}_\Lambda$ the powers of the maximal ideal $\mathfrak{m}_A^n$ will eventually be 0. So we find that

$$\text{length}(A) = \sum_{i=0}^{\infty} \text{length}(\mathfrak{m}_A^i/\mathfrak{m}_A^{i+1})$$

is a finite sum. Further, the maximal ideal $\mathfrak{m}_A$ is finitely generated and therefore each quotient $\mathfrak{m}_A^i/\mathfrak{m}_A^{i+1}$ is a finite-dimensional $A/\mathfrak{m}_A$-vector space. This dimension is also the length, so the length of A is finite.

Let $A \to A'$ be a small extension with kernel (t) a 1-dimensional k-vector space. Then

$$\text{length}(A) = \text{length}(A/(t)) + \text{length}((t)) = \text{length}(A') + 1.$$

Now we prove that statement of the lemma by induction on $\text{length}(A) - \text{length}(A')$. If this difference is 0, we are trivially done. If not, let I be the kernel of $A \to A'$ and consider the ideals $I \cap \mathfrak{m}_A^n$. For $n = 0$, this ideal is non-trivial and we know that for n large enough $\mathfrak{m}_A^n = 0$. So there is a maximal n for which $I \cap \mathfrak{m}_A^n \neq 0$. Let t be an element of this ideal. Then $\mathfrak{m}_A t \subset \mathfrak{m}_A I \cap \mathfrak{m}_A^{n+1} \subset I \cap \mathfrak{m}_A^{n+1} = 0$, so we find the decomposition $A \to A/(t) \to A'$ where $A \to A/(t)$ is small. Because

$$\text{length}(A/(t)) - \text{length}(A') = \text{length}(A) - \text{length}(A') - 1,$$

we can decompose $A/(t) \to A'$ in small extensions by induction hypothesis. □

Example 1.7. We can find this decomposition for the surjection $k[\varepsilon, \delta]/(\varepsilon^2, \delta^2) \to k$. Following the induction step, we have to find an element that is annihilated by the maximal ideal. Up to a constant factor, the only choice is $\varepsilon\delta$, so the decomposition starts as

$$k[\varepsilon, \delta]/(\varepsilon^2, \delta^2) \to k[\varepsilon, \delta]/(\varepsilon^2, \varepsilon\delta, \delta^2) \to \dots$$

One way to complete the decomposition is $k[\varepsilon, \delta]/(\varepsilon^2, \varepsilon\delta, \delta^2) \to k[\varepsilon]/\varepsilon^2 \to k$.

1.2 Deformations of schemes

In this section we will start considering deformations. To do this, we first have to introduce the notion of *flatness*.

Consider a ring R and an R-module M, then the functor $- \otimes_R M$ is right-exact. For example, given an ideal I of R, there is an exact sequence $0 \to I \to R \to R/I \to 0$. Tensoring this with M yields an exact sequence

$$I \otimes_R M \to M \to M \otimes_R R/I \to 0.$$

The image of $I \otimes_R M \to M$ is IM, so we can calculate

$$M \otimes_R R/I \cong M/IM \tag{1.1}$$

However, the map $I \otimes_R M \to IM$ need not be injective. For example, take $R = \mathbb{Z}, I = n\mathbb{Z}, M = \mathbb{Z}/n$, then $IM = n\mathbb{Z}(\mathbb{Z}/n) = 0$. However, $n\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ as $\mathbb{Z}$-module, so $I \otimes_R M \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n \cong \mathbb{Z}/n$.

Definition 1.8. Let R be a ring. We call an R-module M a *flat module* if $- \otimes_R M$ is exact. This means that for any short exact sequence $A \to B \to C \to$ of R-modules, the induced sequence $A \otimes_R M \to B \otimes_R M \to C \otimes_R M$ is also exact.

Definition 1.9. A map of rings $R \to S$ is *flat*, if S is a flat R-module.

A morphism of schemes $f : X \to Y$ is *flat*, if for every point $x \in X$ the map $f^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

It is undoable to check that a morphism of schemes is flat straight from these definition. We need to check the flatness of the map on the stalks for *all* points of X, and to do this, there is a condition on *all* exact sequences. The following proposition greatly reduces the number of checks.

Proposition 1.10. *(a) An R-module M is flat, if and only if for every ideal $I \subset R$ the map $I \otimes_R M \to R \otimes_R M \cong M$ is injective.*

(b) A morphism of schemes $f : X \to Y$ is flat, if and only if there is an affine open covering $\{V_j\}_{j \in J}$ of Y, and affine open coverings $\{U_i\}_{i \in I_j}$ of $f^{-1}(V_j)$, such that $\mathcal{O}_Y(V_j) \to \mathcal{O}_X(U_i)$ is flat for every $j \in J, i \in I_j$.

Proof. See [Sta18, Tags 00H9 & 01U2]. □

With this proposition, it is immediately clear that a flat ring map $R \to S$ induces a flat morphism $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$, which is not at all clear from the definition.

Example 1.11.

- Let R be a ring, S a multiplicative system, and $K \to M \to N$ an exact sequence of R-modules. Localizing in S yields again an exact sequence $S^{-1}K \to S^{-1}M \to S^{-1}N$. This can also be described as tensoring with the localization $S^{-1}R$, so $S^{-1}R$ is flat over R.
- Any free module over R is flat over R. Because the polynomial ring $R[x] = \bigoplus_{i \geq 0} Rx^i$ is a free module, it is also flat.
- Quotients are typically non-flat. For example, consider the quotient $\mathbb{Z} \to \mathbb{Z}/n$. There is an exact sequence of $\mathbb{Z}$-modules $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z}$, where the map is multiplication by n. Tensoring this with $\mathbb{Z}/n$ yields $0 \to \mathbb{Z}/n \xrightarrow{0} \mathbb{Z}/n$ with the zero map, which is not exact.
- Let M be a $k[\varepsilon]/\varepsilon^2$-module. Because $k[\varepsilon]/\varepsilon^2$ has only one non-trivial ideal (ε), checking flatness amounts to checking the injectivity of $(\varepsilon) \otimes_{k[\varepsilon]/\varepsilon^2} M \to M$. The image of this map is exactly εM, so it is equivalent to check that $(\varepsilon) \otimes_{k[\varepsilon]/\varepsilon^2} M \cong \varepsilon M$. The map $k[\varepsilon]/\varepsilon^2 \to (\varepsilon)$ which sends x to $x\varepsilon$ is surjective, and it has kernel (ε). So we can see (ε) as the quotient of $k[\varepsilon]/\varepsilon^2$ by ε. Then $(\varepsilon) \otimes_{k[\varepsilon]/\varepsilon^2} M \cong M/\varepsilon M$ by Eq. (1.1). Putting this together, we see that M is flat over $k[\varepsilon]/\varepsilon^2$, if and only if $M/\varepsilon M \cong \varepsilon M$.

It is difficult to give geometric meaning to the concept of a flat morphism, but in some cases it is possible. For example, let X, Y be regular, irreducible schemes of finite type over k, and $f : X \to Y$ be a proper morphism. Then f is flat if and only if all fibres of f have the same dimension (see [Mat86, Corollary 23.1] and [Har77, Proposition III.9.5]).

With this notion of a flat morphism, we can define a deformation. By a slight abuse of notation I will write the fibre product $Y \times_{\mathrm{Spec}(R)} \mathrm{Spec}(S)$ as $Y \otimes_R S$.

Definition 1.12. Given are a scheme X over a field k, and an Artinian local ring $A \in \mathcal{C}_\Lambda$. A deformation of X over A is a scheme $\mathcal{X}_A$ over A together with a map $\iota : X \to \mathcal{X}_A$, such that the diagram

$$\begin{array}{ccc} X & \overset{\iota}{\hookrightarrow} & \mathcal{X}_A \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(k) & \hookrightarrow & \mathrm{Spec}(A) \end{array}$$

is a fibre product square and π is a flat morphism. The condition of being a fibre product square can be restated as $\iota : X \to \mathcal{X}_A \otimes k$ being an isomorphism.

If $A' \to A$ is a morphism in $\mathcal{C}_\Lambda$, and $\iota : X \to \mathcal{X}_A$, $\iota' : X \to \mathcal{X}_{A'}$ are deformations over A and A', a morphism of deformations $\varphi : \mathcal{X}_A \to \mathcal{X}_{A'}$ over $A' \to A$ is a morphism of schemes that fits into the commutative diagram.

$$\begin{array}{ccc} X & \overset{\iota'}{\searrow} & \\ \downarrow \iota & & \\ \mathcal{X}_A & \overset{\varphi}{\longrightarrow} & \mathcal{X}_{A'} \\ \downarrow \pi & & \downarrow \pi \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A') \end{array}$$

Example 1.13.

- Consider the affine scheme $X = \mathrm{Spec}(k[x, y]/y)$, corresponding the the line $y = 0$ in the affine plane. A deformation over $k[\varepsilon]/\varepsilon^2$ is given by

$$\mathcal{X}_{k[\varepsilon]/\varepsilon^2} = \mathrm{Spec}(k[x, y, \varepsilon]/(\varepsilon^2, y - \varepsilon x^2)),$$

corresponding to an infinitesimally shallow parabola $y = \varepsilon x^2$. By the above computation, we see that

$$k[x, y, \varepsilon]/(\varepsilon^2, y - \varepsilon x^2) \otimes_{k[\varepsilon]/\varepsilon^2} k \cong k[x, y, \varepsilon]/(\varepsilon^2, y - \varepsilon x^2, \varepsilon) \cong k[x, y]/y$$

which is what we wanted.

We also have to check that $\mathcal{X}_{k[\varepsilon]/\varepsilon^2}$ is flat over $\mathrm{Spec}(k[\varepsilon]/\varepsilon^2)$, or that

$$M = k[x, y, \varepsilon]/(\varepsilon^2, y - \varepsilon x^2)$$

is a flat $k[\varepsilon]/\varepsilon^2$-module. Note that $M \cong k[x, \varepsilon]/\varepsilon^2 = k[x] \oplus \varepsilon k[x]$. By the previous example we have to check that $M/\varepsilon M \cong \varepsilon M$. However, it is clear that $\varepsilon M = \varepsilon k[x]$, and this is also the kernel of $M \to \varepsilon M$.

- If we consider instead the curve $Y = 0$ in the projective plane, the curve $YZ = \varepsilon X^2$ in $\mathbb{P}^2_{k[\varepsilon]/\varepsilon^2}$ is not a deformation. Indeed, dividing out ε yields the curve $YZ = 0$ which is not the original curve $Y = 0$.

Remark 1.14. We have now seen the notion of a deformation of schemes. However, in a similar vein you can deform many other objects. For example, the deformation theory of Galois group representations has played a prominent role in Weil's proof of Fermat's last theorem [Rib95, Section 11].

As explained in the previous section, the map $\operatorname{Spec}(k) \hookrightarrow \operatorname{Spec}(A)$ adds tangent directions to the point $\operatorname{Spec}(k)$. In the same way, a deformation $X \hookrightarrow \mathcal{X}_A$ can be seen as adding tangent directions to the scheme X, in which the scheme can be deformed. In our intuitive picture of a deformation being a family $\mathcal{X} \to (-\varepsilon, \varepsilon)$ we could really take different fibres to find a space closely resembling X. In this algebro-geometric setting this is not possible. In fact, the deformation doesn't change the topological space of the scheme X! This is the content of the following lemma.

Lemma 1.15. *Let $\iota : X \hookrightarrow \mathcal{X}_A$ be a deformation of X over A. Then ι induces a homeomorphism on the underlying toplogical spaces.*

Proof. Take an affine open covering $\{U_i\}_{i\in I}$ of $\mathcal{X}_A$. Then $\{U_i \otimes_A k\}_{i\in I}$ is an affine open covering of X. It is therefore enough to prove that U_i and $U_i \otimes_A k$ are homeomorphic. Let $U_i = \operatorname{Spec}(R)$, then $U_i \otimes_A k = \operatorname{Spec}(R \otimes_A k)$. By Eq. (1.1) we see that $R \otimes_A k \cong R/\mathfrak{m}_A R$. Because A is Artinian, there is an n such that $\mathfrak{m}_A^n = 0$. Now $(\mathfrak{m}_A R)^n = \mathfrak{m}_A^n R = 0$, so we see that $\mathfrak{m}_A R$ is a nilpotent ideal of R. It is therefore contained in every prime ideal of R. This means that $R \to R/\mathfrak{m}_A R$ induces a homeomorphism on the spectrum of prime ideals, which is exactly what we needed to show. □

The following two lemma's are rather technical, but they are very useful when proving results about deformations.

Lemma 1.16. *Let $A \in \mathcal{C}_\Lambda$ be an Artinian, local ring.*

(a) Let M, N be A-modules and $\varphi : M \to N$ a morphism, and let I be a nilpotent ideal of A. If N is flat and the induced map $\varphi : M/IM \to N/IN$ is an isomorphism, then φ is an isomorphism.

(b) Let $\mathcal{X}_A$ and $\mathcal{X}'_A$ be two deformations of X over A. If there is a morphism of deformations $\mathcal{X}_A \to \mathcal{X}'_A$, then this is an isomorphism of deformations.

(c) A flat A-module is free.

Proof. (a) Let Q be the cokernel of φ, so $M \to N \to Q \to 0$ is exact. Now tensoring with A/I is right exact. By Eq. (1.1) we get the exact sequence

$$M/IM \to N/IN \to Q/IQ \to 0.$$

The first map is an isomorphism, so $Q/IQ = 0$ or $Q = IQ$. This implies that $Q = IQ = I^2Q = \ldots$. Because I is nilpotent, at some point we find $Q = 0$. Therefore φ is surjective.

For injectivity, let K be the kernel of φ and consider the short exact sequence $0 \to \mathfrak{m}_A \to A \to k \to 0$. Because N is flat over A, this yields a short exact sequence $0 \to I \otimes_A N \to N \to N/IN \to 0$. Similar exact sequences arise for M and K, except that the first map need not be injective. Now we can form the following commutative diagram, in which all rows and columns are exact:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & I \otimes_A K & \longrightarrow & I \otimes_A M & \longrightarrow & I \otimes_A N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & K/IK & \longrightarrow & M/IM & \longrightarrow & N/IN & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Now a partial form of the snake lemma yields an exact sequence $0 \to K/IK \to M/IM \to N/IN$. The last map is an isomorphism by assumption, so $K/IK = 0$. Again this implies $K = 0$, so φ is injective and therefore an isomorphism.

(b) We prove this locally, so let U be an open of the topological space X, and let $\mathcal{B}$ and $\mathcal{B}'$ be the regular sections of $\mathcal{X}_A$ and $\mathcal{X}'_A$ over U. Then they are both flat over A, and there is a map $\varphi : \mathcal{B}' \to \mathcal{B}$ of A-modules which is an isomorphism when tensored with k. This means that $\mathcal{B}'/\mathfrak{m}_A\mathcal{B}' \cong \mathcal{B}/\mathfrak{m}_A\mathcal{B}$. Now part (a) implies that φ is an isomorphism. This holds for every open U, so $\mathcal{X}_A \cong \mathcal{X}'_A$.

(c) Let N be a flat A-module, and let S_k be a basis of the k-vector space $N/\mathfrak{m}_A A$. Now choose a set $S_A \subset N$ that maps bijectively to S_k and let M be the free A-module generated by S_A. Then there is a canonical map $M \to N$, and we know that $M/\mathfrak{m}_A M \to N/\mathfrak{m}_A N$ is an isomorphism. Hence $M \to N$ is also an isomorphism and N is free. $\square$

Lemma 1.17. *Every deformation of an affine scheme is affine.*

Proof. We will show that this reduces to [GD60, Proposition 5.1.9]. This reads: given are a scheme Y with a nilpotent, quasi-coherent sheaf of $\mathcal{O}_Y$-ideals $\mathcal{I}$. If the closed subscheme $(Y, \mathcal{O}_Y/\mathcal{I})$ is affine, then Y is affine.

Let X be an affine scheme, and $\mathcal{X}_A$ a deformation over A. Because X is quasi-compact, we can choose an affine covering of $\mathcal{X}_A$ with finitely many affine opens. For such affine open $U = \mathrm{Spec}(R)$, we know that $U \otimes_A k = \mathrm{Spec}(R/\mathfrak{m}_A R)$ (from the proof of Lemma 1.15) with $\mathfrak{m}_A R$ nilpotent. Hence $\widetilde{\mathfrak{m}_A R}$ (the sheaf associated to $\mathfrak{m}_A R$ on $\mathrm{Spec}(R)$) is a nilpotent, quasi-coherent sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-ideals, and quotienting it out gives $U \otimes_A k$. Glueing these together yields a quasi-coherent sheaf of $\mathcal{O}_{\mathcal{X}_A}$-ideals $\mathcal{I}$, such that X is obtained from quotienting out $\mathcal{I}$. Because we glue a finite number of nilpotent sheaves, the result $\mathcal{I}$ is still nilpotent. Because X is affine, we can apply the above proposition to conclude that $\mathcal{X}_A$ is affine. □

1.3 Functors on $\mathcal{C}_\Lambda$

The deformations of a scheme can be studied by putting them together in a deformation functor $\mathcal{C}_\Lambda \to \mathbf{Set}$. Before we really delve into this deformation functor, we will first look at more general functors $F : \mathcal{C}_\Lambda \to \mathbf{Set}$. We list a number of properties these functors might have, and then we look at the results we can deduce from these hypotheses. The first is:

H1 The set $F(k)$ is a singleton.

In the category $\mathcal{C}_\Lambda$ we can form fibre products. Given two maps $a : A \to C$ and $b : B \to C$, the fibre product is

$$A \times_C B = \{(x, y) \in A \times B \mid a(x) = b(y)\}.$$

This is indeed again an object of $\mathcal{C}_\Lambda$. Now applying the functor F to this fibre product, we obtain the following commutative diagram:

$$\begin{array}{ccc} F(A \times_C B) & \longrightarrow & F(B) \\ \downarrow & & \downarrow \\ F(A) & \longrightarrow & F(C). \end{array}$$

The universal property of the fibre product in **Set** now gives a map

$$\varphi_{a,b} : F(A \times_C B) \to F(A) \times_{F(C)} F(B).$$

The next hypotheses are based on this φ.

H2 The map $\varphi_{a,b}$ is an isomorphism if b is the residue map $k[\varepsilon]/\varepsilon^2 \to k$.

H3 The map $\varphi_{a,b}$ is a surjection if b is a small extension.

H4 The map $\varphi_{a,b}$ is an isomorphism if $a = b$ is a small extension.

Note that hypothesis **H3** implies a seemingly stronger one: $\varphi_{a,b}$ is surjective if b is surjective. This follows by decomposing b in small extensions. For example, if $B \to B' \to C$ are small extensions, then

$$\begin{aligned} F(A \times_C B) &\cong F((A \times_C B') \times_{B'} B) \\ &\to F(A \times_C B') \times_{F(B')} F(B) \\ &\to (F(A) \times_{F(C)} F(B')) \times_{F(B')} F(B) \\ &\cong F(A) \times_{F(C)} F(B) \end{aligned}$$

is a composition of surjective maps, hence surjective.

We can now obtain the following results from these hypotheses. First define the *tangent space of* F as $T_F = F(k[\varepsilon]/\varepsilon^2)$.

Proposition 1.18. *Let* $F : \mathcal{C}_\Lambda \to \mathbf{Set}$ *be a functor satisfying* ***H1*** *and* ***H2****.*

(a) The tangent space T_F *is a* k*-vector space.*

(b) For any small extension $A \to A'$*, the tangent space* T_F *acts on each non-empty fibre of* $F(A) \to F(A')$*.*

(c) If F *satisfies* ***H3****, this action is transitive.*

(d) If F *satisfies* ***H4****, this action is free and transitive.*

Proof. (a) By assumptions **H1** and **H2**, we see that

$$F(k[\varepsilon]/\varepsilon^2 \times_k k[\varepsilon]/\varepsilon^2) \cong F(k[\varepsilon]/\varepsilon^2) \times_{F(k)} F(k[\varepsilon]/\varepsilon^2) \cong T_F \times T_F.$$

This allows to define an addition on T_F. Consider the function

$$\begin{aligned} g : k[\varepsilon]/\varepsilon^2 \times_k k[\varepsilon]/\varepsilon^2 &\longrightarrow k[\varepsilon]/\varepsilon^2, \\ (u + v\varepsilon, u + w\varepsilon) &\longmapsto u + (v + w)\varepsilon. \end{aligned}$$

Applying F to this function and using the previous isomorphism, we obtain a map $F(g) : T_F \times T_F \to T_F$. This will be the addition on T_F.

For the multiplication, we consider for each $\lambda \in k$ the map $g_\lambda : k[\varepsilon]/\varepsilon^2 \to k[\varepsilon]/\varepsilon^2$ given by $u + v\varepsilon \mapsto u + \lambda v\varepsilon$. Applying F gives $F(g_\lambda) : T_F \to T_F$ which will be scalar multiplication by λ.

Now it remains to show that this structure satisfies the axioms of a vector space, for example that $\lambda\mathbf{v} + \mu\mathbf{v} = (\lambda + \mu)\mathbf{v}$ for $\lambda, \mu \in k$, $\mathbf{v} \in T_F$. This follows by noting that $g \circ (g_\lambda, g_\mu) = g_{\lambda+\mu}$, and applying F. The other axioms can be checked in a similar fashion.

(b) We will first prove the isomorphism

$$\Theta : A \times_k k[\varepsilon]/\varepsilon^2 \cong A \times_{A'} A.$$

Write (t) for the kernel of $A \to A'$, which is a k-vector space. Hence every element of $A \times_{A'} A$ can be written as $(a, a + \lambda t)$ with $a \in A$, $\lambda \in k$. Write $\bar{a}$ for the image of a under the residue map $A \to k$, then we see that every element of $A \times_k k[\varepsilon]/\varepsilon^2$ can be given by $(a, \bar{a} + \lambda\varepsilon)$. Now it is clear that $(a, \bar{a} + \lambda\varepsilon) \overset{\sim}{\longleftrightarrow} (a, a + \lambda t)$ gives a bijection $A \times_k k[\varepsilon]/\varepsilon^2 \leftrightarrow A \times_{A'} A$. Because it is multiplicative, the isomorphism follows.

Applying F to this isomorphism, and again using **H1** and **H2**, we obtain

$$F(A) \times T_F \cong F(A \times_k k[\varepsilon]/\varepsilon^2) \cong F(A \times_{A'} A) \to F(A) \times_{F(A')} F(A).$$

All these map are compatible with the projection map to the first factor $F(A)$, so we obtain a map $(x, \eta) \mapsto (x, y)$. Now the action of η on x is defined to be y, and we denote it by $x+\eta$. To check that this is an action, note that the following two maps are equal:

$$\begin{aligned}
& & A \times_k k[\varepsilon]/\varepsilon^2 \times_k k[\varepsilon]/\varepsilon^2 &\longrightarrow A \times_{A'} A, \\
(\pi_1, \pi_3) \circ (\mathrm{id}, \Theta) \circ (\Theta, \mathrm{id}) : & & (a, \bar{a} + \lambda\varepsilon, \bar{a} + \mu\varepsilon) &\longmapsto (a, a + \lambda t, \bar{a} + \mu\varepsilon) \\
& & &\longmapsto (a, a + \lambda t, a + \lambda t + \mu t) \\
& & &\longmapsto (a, a + \lambda t + \mu t), \\
\Theta \circ (\mathrm{id}, g) : & & (a, \bar{a} + \lambda\varepsilon, \bar{a} + \mu\varepsilon) &\longmapsto (a, \bar{a} + (\lambda + \mu)\varepsilon) \\
& & &\longmapsto (a, a + (\lambda + \mu)t).
\end{aligned}$$

Applying F yields that $(x+\eta)+\xi = x+(\eta+\xi)$ for $x \in F(A)$, $\eta, \xi \in T_F$.

(c) It follows that $F(A) \times T_F \cong F(A \times_{A'} A) \to F(A) \times_{F(A')} F(A)$ is surjective. Now take $x, y \in F(A)$ that lie in the same fibre, then we have $(x, y) \in F(A) \times_{F(A')} F(A)$, so it is reached by a pair (x, η). Then $y = x + \eta$. We conclude that the action of T_F on every *non-empty* fibre is transitive.

(d) In this case we even have $F(A) \times T_F \cong F(A \times_{A'} A) \cong F(A) \times_{F(A')} F(A)$, so the η of part (c) is unique, and we find that the action on the *non-empty* fibres is free and transitive. □

Actually, more can be deduced from the stated assumptions, but for this we first introduce some terminology.

Definition 1.19. Let $F : \mathcal{C}_\Lambda \to \mathbf{Set}$ be a functor. We say F *is pro-representable by* R, if there is a complete, Noetherian, local Λ-algebra R, such that

$$F \cong \mathrm{Hom}_{\Lambda,\mathrm{loc}}(R, -).$$

We understand Hom-functors very well, so we also understand pro-representable functors very well. However, not all deformation functors we will encounter are pro-representable. Instead they might satisfy a slightly weaker notion, that we will define now.

Definition 1.20. Let $F, G : \mathcal{C}_\Lambda \to \mathbf{Set}$ be two functors, and let $\eta : F \to G$ be a natural transformation. We say that η is *smooth*, if for every surjection $A \to A'$ in $\mathcal{C}_\Lambda$, the map

$$F(A) \to F(A') \times_{G(A')} G(A)$$

is surjective. We say η is *étale* if η is smooth and $\eta : T_F \to T_G$ is an isomorphism.

Lemma 1.21. *Let F, G be two functors satisfying **H1**, and let $\eta : F \to G$ be a smooth natural transformation. Then $\eta : F(A) \to G(A)$ is surjective for all $A \in \mathcal{C}_\Lambda$.*

Proof. Because we have a surjection $A \to k$, it follows from the definition that $F(A) \to F(k) \times_{G(k)} G(A)$ is surjective. But $F(k)$ and $G(k)$ are singletons, hence $F(k) \times_{G(k)} G(A) \cong G(A)$. □

Definition 1.22. For a complete, Noetherian, local Λ-algebra R we define

$$h_R = \mathrm{Hom}_{\Lambda,\mathrm{loc}}(R, -).$$

An étale natural transformation $h_R \to F$ is called a *versal hull of* F.

A versal hull can often give a good enough description of F. In particular, because $h_R(A) \to F(A)$ is always surjective, all deformations over A can be found from this versal hull. The following theorem by Schlessinger shows how the notions of pro-representability and a versal hull, correspond with the hypotheses introduced before. See [Sch68] for the proof.

Theorem 1.23 (Schlessinger's criterion)**.** *Let F be a functor $\mathcal{C}_\Lambda \to$ **Set** satisfying **H1**.*

*(a) The functor F has a versal hull, if and only if it satisfies **H2**, **H3** and if T_F is finite-dimensional.*

*(b) The functor F is pro-representable, if and only if it satisfies **H2**, **H3**, **H4** and if T_F is finite-dimensional.*

Remark 1.24. The assumption that T_F is finite-dimensional ensures that the ring R is Noetherian. Without this assumption, F will also have a versal hull/be pro-representable, but the ring R need not be Noetherian.

Why would we expect such a result to follow from only those four assumptions? First of all, the universal property of a fibre product shows that

$$\mathrm{Hom}(R, A \times_C B) \cong \mathrm{Hom}(R, A) \times_{\mathrm{Hom}(R,C)} \mathrm{Hom}(R, B).$$

So if F is pro-representable, it is clear that $\varphi_{a,b}$ is always an isomorphism. Schlessinger's criterion shows that the converse is true, without even needing all these isomorphisms.

1.4 Deformation functors and pro-representability

Now that we have seen the deformations of a scheme, we can gather all information of these deformations into a functor. See the following definition.

Definition 1.25. Let X be a scheme over k. Define the *deformation functor* $D_X : \mathcal{C}_\Lambda \to$ **Set** of X as follows:

- $D_X(A)$ are all isomorphisms classes of deformations of X over A.
- $D_X(A \to A')$ sends a deformation $\mathcal{X}_A$ of X over A, to the deformation $\mathcal{X}_A \otimes_A A'$ over A'.

It is not right away clear that the image of a morphism $D_X(A \to A')$ is well-defined, because we have to check that $\mathcal{X}_A \otimes_A A'$ is indeed a deformation over A'. The fibre product condition is fulfilled, because $(\mathcal{X}_A \otimes_A A') \otimes_{A'} k \cong \mathcal{X}_A \otimes_A (A' \otimes_{A'} k) = \mathcal{X}_A \otimes_A k \cong X$. Second, we need that $\mathcal{X}_A \otimes_A A' \to \mathrm{Spec}(A')$ is flat. We can check this on affine opens of

$\mathcal{X}_A$. Let $\mathrm{Spec}(R)$ be an affine open of $\mathcal{X}_A$, then R is a flat A-module. This determines an open $\mathrm{Spec}(R \otimes_A A')$ of $\mathcal{X}_A \otimes_A A'$. Now we have the isomorphisms $- \otimes_{A'} (R \otimes_A A') \cong (- \otimes_{A'} A') \otimes_A R = - \otimes_A R$, and we know this functor is left-exact. Therefore $R \otimes_A A'$ is flat over A', and we see that $\mathcal{X}_A \otimes_A A'$ is flat over A'. Hence it is indeed a deformation.

With the results of the previous section in mind, we might hope that D_X is pro-representable. This is not always the case, but we will prove that D_X has a hull if X is smooth and proper. We will also show that $D_{\mathbb{P}^1_k}$ and $D_{\mathrm{Spec}(k[\![t]\!])}$ are actually pro-representable.

Checking property **H1** is simple. If $\mathcal{X}_k$ is a deformation of X over k, then $X \cong \mathcal{X}_k \otimes_k k = \mathcal{X}_k$, so X is the only deformation over k.

Proposition 1.26. *The deformation functor D_X has the properties **H3** and **H2**.*

Proof. **(H3)** We let $A \to C, B \to C$ be maps in $\mathcal{C}_\Lambda$, such that $B \to C$ is a small extension. Let $E = A \times_C B$ be the fibre product. Consider deformations $\mathcal{X}_A$, $\mathcal{X}_B$ over resp. A, B that restrict to the same deformation $\mathcal{X}_C$ over C, as shown in the following diagram:

$$\begin{array}{ccccc} \mathcal{X}_A & & & & \mathcal{X}_B \\ \uparrow & & & & \uparrow \\ \mathcal{X}_A \otimes_A C & \overset{\sim}{\longleftrightarrow} & \mathcal{X}_C & \overset{\sim}{\longleftrightarrow} & \mathcal{X}_B \otimes_B C \end{array}.$$

Note that by Lemma 1.15, all the deformations $\mathcal{X}_A$, $\mathcal{X}_B$, $\mathcal{X}_C$ have homeomorphic topological spaces. Therefore, we can consider them as one topological space $|X|$, with the three structure sheaves $\mathcal{O}_A$, $\mathcal{O}_B$, $\mathcal{O}_C$. Now I claim that there is a sheaf $\mathcal{O}_E = \mathcal{O}_A \times_{\mathcal{O}_C} \mathcal{O}_B$ on $|X|$, with sections $\mathcal{O}_A(U) \times_{\mathcal{O}_C(U)} \mathcal{O}_B(U)$ over U. And furthermore, $\mathcal{X}_E = (|X|, \mathcal{O}_A \times_{\mathcal{O}_C} \mathcal{O}_B)$ is a deformation of X over E, which restricts to $\mathcal{X}_A$ and $\mathcal{X}_B$ over A and B respectively.

First we show that $U \mapsto \mathcal{O}_A(U) \times_{\mathcal{O}_C(U)} \mathcal{O}_B(U)$ is indeed a sheaf on $|X|$. We can easily see that it is a presheaf, and it is even a fibre product in the category of presheaves on $|X|$. Further, the inclusion of the category of sheaves on $|X|$ in the category of presheaves on $|X|$ preserves limits, and therefore also fibre products. This follows because the sheafification functor is left adjoint to the inclusion. A more down-to-earth proof is possible by checking directly from the definition that $U \mapsto \mathcal{O}_A(U) \times_{\mathcal{O}_C(U)} \mathcal{O}_B(U)$ is a sheaf. The category-theoretic approach has the benefits that it is shorter, and it shows immediately that this sheaf is actually a fibre product of sheaves on $|X|$.

Next we want that $\mathcal{X}_E$ is actually a scheme. To do this, choose an affine open $(U, \mathcal{O}_C|_U) \cong \mathrm{Spec}(R_C)$ of $\mathcal{X}_C$. By Lemma 1.17 its deformations $(U, \mathcal{O}_A|_U)$, $(U, \mathcal{O}_B|_U)$ are affine, say they are isomorphic to $\mathrm{Spec}(R_A)$, $\mathrm{Spec}(R_B)$. Then $(U, \mathcal{O}_E|_U)$ is isomorphic to $\mathrm{Spec}(R_A \times_{R_C} R_B)$, so it is also affine. Doing this for all U in an affine open cover of $\mathcal{X}_C$, yields an affine open cover of $\mathcal{X}_E$, hence it is a scheme.

To show that $\mathcal{X}_E$ is actually a deformation, it needs to be flat over E. In the notation of the previous paragraph, it is enough to show that $R_E = R_A \times_{R_C} R_B$ is flat over $E = A \times_C B$. Because R_A is flat over A, it follows from Lemma 1.16 that R_A is a free A-module. Let S_A be an A-basis, and define S_C to be its image in R_C. Because

$R_C \cong R_A \otimes_A C$, the module R_C is also free with basis S_C. Because $B \to C$ is surjective, the map $R_B \to R_C$ is also surjective. Therefore we can choose a subset S_B of R_B that maps bijectively to R_C. Now there is a canonical map $\bigoplus_{s\in S_B} sB \to R_B$, and tensoring with C gives $\bigoplus_{s\in S_C} sC \to R_C$ which is an isomorphism. Because $C \cong B/I$ with I nilpotent and R_B is flat, we use Lemma 1.16 to conclude that $\bigoplus_{s\in S_B} sB \to R_B$ is also an isomorphism. Now it easily follows that R_E is also a free E-module with E-basis $S_A \times_{S_C} S_B$. Hence it is indeed a flat E-module. We also see that tensoring with A and B yields R_A and R_B respectively, and therefore tensoring with k yields $\mathcal{O}_X(U)$. So $\mathcal{X}_E$ is indeed a deformation of X over E, and it restricts to $\mathcal{X}_A$ and $\mathcal{X}_B$.

(H2) From the previous part it follows that the map

$$D_X(A \times_k k[\varepsilon]/\varepsilon^2) \to D_X(A) \times_{D_X(k)} D_X(k[\varepsilon]/\varepsilon^2)$$

is surjective, because we constructed a 'fibre sum of deformations'. We will now show injectivity. Let $\mathcal{X}_A$, $\mathcal{X}_{k[\varepsilon]/\varepsilon^2}$ be deformations of X over respectively A, $k[\varepsilon]/\varepsilon^2$. Then there is the fibred sum deformation $\mathcal{X}_{A'}$ over $A' = A \times_k k[\varepsilon]/\varepsilon^2$ restricting to $\mathcal{X}_A$ and $\mathcal{X}_{k[\varepsilon]/\varepsilon^2}$:

$$\mathcal{X}_A \longrightarrow \mathcal{X}_{A'} \longleftarrow \mathcal{X}_{k[\varepsilon]/\varepsilon^2} \ .$$

Let $\mathcal{Z}$ be another deformation of X over A' such that the restrictions to A and $k[\varepsilon]/\varepsilon^2$ are isomorphic to $\mathcal{X}_A$ and $\mathcal{X}_k[\varepsilon]/\varepsilon^2$. This gives a diagram

$$\mathcal{X}_A \overset{\sim}{\longleftrightarrow} \mathcal{Z} \otimes_{A'} A \longrightarrow \mathcal{Z} \longleftarrow \mathcal{Z} \otimes_{A'} k[\varepsilon]/\varepsilon^2 \overset{\sim}{\longleftrightarrow} \mathcal{X}_{k[\varepsilon]/\varepsilon^2} \ .$$

Now the universal property of the fibred sum deformation yields a morphism of deformations $\mathcal{X}_{A'} \to \mathcal{Z}$ and by Lemma 1.16 this is an isomorphism. We conclude that $D_X(A \times_k k[\varepsilon]/\varepsilon^2) \to D_X(A) \times_{D_X(k)} D_X(k[\varepsilon]/\varepsilon^2)$ is indeed an isomorphism. □

We can almost apply Schlessinger's criterion in order to prove that D_X has a versal hull, we only need to know that the tangent space T_{D_X} is finite-dimensional. For this, we need the following result from [Har10, Theorem 5.3], which we will not prove.

Theorem 1.27. *For a smooth scheme X over k, the tangent space of D_X is given by $\mathrm{H}^1(X, \mathcal{T}_X)$. Here $\mathcal{T}_X$ is the tangent sheaf of X.*

If the scheme X is not only smooth, but proper as well, the cohomology group $\mathrm{H}^1(X, \mathcal{T}_X)$ is a finite-dimensional k-vector space, and this is exactly what we need to apply Schlessinger's criterion. For the sake of completeness, we record this result.

Corollary 1.28. *When X is a smooth, proper scheme over k, the deformation functor D_X has a versal hull.*

As conclusion to this chapter, we will give two schemes with a pro-representable deformation functor: the projective line $\mathbb{P}^1_k$ over k, and the spectrum of the power series ring $\mathrm{Spec}(k[\![t]\!])$. This can be done by first computing the versal hull that comes out of Schlessinger's criterion and then showing that this hull is an isomorphism. I will give a proof without Schlessinger's criterion, but relying on some more basic results of this chapter.

Theorem 1.29. *The deformation functors of the projective line* $\mathbb{P}^1_k$ *and the spectrum of the power series ring* $\operatorname{Spec}(k[\![t]\!])$ *are isomorphic to the constant functor* $\mathcal{C}_\Lambda \to \mathbf{Set}$*, sending every object to a singleton. In particular this means that* $D_{\mathbb{P}^1}$ *is pro-represented by* Λ*.*

Proof. Let $A \in \mathcal{C}_\Lambda$, then we first show that $\mathbb{P}^1_k$ and $\operatorname{Spec}(k[\![t]\!])$ both have at least one deformation over A. The map $\operatorname{Spec}(k) \hookrightarrow \operatorname{Spec}(A)$ yields a map

$$\mathbb{P}^1_k = \mathbb{P}^1_{\mathbb{Z}} \times \operatorname{Spec}(k) \to \mathbb{P}^1_{\mathbb{Z}} \times \operatorname{Spec}(A) = \mathbb{P}^1_A.$$

It is clear that this induces an isomorphism $\mathbb{P}^1_k \cong \mathbb{P}^1_A \times_{\operatorname{Spec}(A)} \operatorname{Spec}(k)$. Further, $\mathbb{P}^1_A$ is covered by the affine patches $\operatorname{Spec}(A[t])$ and $\operatorname{Spec}(A[t^{-1}])$ which are flat over A (see Example 1.11). We see that $\mathbb{P}^1$ is indeed a deformation of $\mathbb{P}^1$ over A.

A deformation of $\operatorname{Spec}(k[\![t]\!])$ is given by $\operatorname{Spec}(k[\![t]\!]) \to \operatorname{Spec}(A[\![t]\!])$ which comes from the quotient map $A \to k$. Again it is clear that $A[\![t]\!] \otimes_A k \cong k[\![t]\!]$, and because $A[\![t]\!]$ is a free A-module it is also flat. Hence $A[\![t]\!]$ is a deformation of $k[\![t]\!]$ over A.

Next we show that the deformation functors of $\mathbb{P}^1_k$ and $\operatorname{Spec}(k[\![t]\!])$ both have trivial tangent space. Both are smooth schemes over k, so we can use Theorem 1.27 to compute their tangent space. First, $\operatorname{Spec}(k[\![t]\!])$ is an affine scheme, so the first cohomology of the quasi-coherent sheaf $\mathcal{T}_{\operatorname{Spec}(k[\![t]\!])}$ vanishes. Hence the tangent space of $D_{\operatorname{Spec}(k[\![t]\!])}$ is indeed trivial. Second, the tangent sheaf $\mathcal{T}_{\mathbb{P}^1_k}$ has degree 2, so it is isomorphic to $\mathcal{O}_{\mathbb{P}^1_k}(2)$. We know that $\mathrm{H}^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(2))$ is trivial, so the tangent space of $D_{\mathbb{P}^1_k}$ is also trivial.

Now both deformation functors $D_{\mathbb{P}^1_k}$ and $D_{\operatorname{Spec}(k[\![t]\!])}$ has the following properties: they satisfy **H1**, **H2**, **H3**, the tangent space is trivial and the image of each $A \in \mathcal{C}_\Lambda$ is non-empty. We will show that any functor F having these properties is isomorphic to the constant functor mapping each $A \in \mathcal{C}_\Lambda$ to a singleton.

We will prove by induction on the length of $A \in \mathcal{C}_\Lambda$ that $F(A)$ is a singleton. The base case is $A = k$, and then it is true because F satisfies **H1**.

For the induction step, decompose the residue map $A \to k$ as $A \to A' \to k$, where $A \to A'$ is a small extension. By Lemma 1.6, the length of A' is smaller than that of A, so by the induction hypothesis we know that $F(A')$ is a singleton. Then the map $F(A) \to F(A')$ has exactly one fibre, which is non-empty because $F(A)$ is non-empty. Because the functor F satisfies **H1**, **H2** and **H3**, Proposition 1.18 implies that the tangent space T_F acts transitively on this one fibre. Because the tangent space is trivial, this means that the fibre contains exactly one point, which proves that $D_{\mathbb{P}^1}(A)$ also consists of only one point. □

Chapter 2

Group actions on curves

In this chapter we will focus on curves. A curve is defined as a projective, regular, integral schemes of dimension 1 over an algebraically closed field k. The deformation of curves over $\mathbb{C}$ was already studied by Riemann, and it is well known that these deformations are all pro-representable. The same holds in positive characteristic (see e.g. [Ser06, Corollary 2.6.6]). The main result in this chapter is on curves in positive characteristic with a finite group of automorphisms. If the genus of the curve is at least 2, the deformation of a curve with this automorphism group is pro-representable, which can be proved in almost the same way as for curves without a group action. Curves of genus 0 on the other hand, have lots of automorphisms. In this chapter I present a complete classification of the pro-representable deformations of genus 0 curves with a group action.

The case of genus 1 remains, and it is yet unknown whether these curves with group action have a pro-representable deformation functor. The most complicated cases arise when an elliptic curve in characteristic 2 or 3 has a large automorphism group.

Section 2.2 are all minor adaptations of [Ser06]. Sections 2.3 to 2.5 take as starting point an unfinished draft [Bys09b] of Jakub Byszewski.

2.1 Deformation problem

We start by introducing the deformation problem for this case.

Definition 2.1. Given is a curve X and a finite group G that acts faithfully on X. A deformation of this pair (X, G) over $A \in \mathcal{C}_\Lambda$, consists of a deformation $\mathcal{X}_A$ of X over A and a group action of G on $\mathcal{X}_A$, such that the map $\iota : X \to \mathcal{X}_A$ is G-equivariant.

A morphism of deformations, is a morphisms as defined in Definition 1.12 that is also G-equivariant.

I will not always include the group G in the notation of deformations, but the action of G will always be present. Now it is simple to give the definition of the deformation functor of this pair (X, G). We will call it the global deformation functor, because later we will encounter a local deformation functor.

Definition 2.2. The *global deformation functor of the pair* (X, G), notation $D_{X,G}$: $\mathcal{C}_\Lambda \to \mathbf{Set}$, is defined as follows:

- $D_{X,G}(A)$ are all isomorphism classes of deformations of (X, G) over A.
- $D_{X,G}(A \to A')$ sends a deformation $\mathcal{X}_A$ of (X, G) over A, to the deformation $\mathcal{X}_A \otimes_A A'$ over A'.

Note that the G-action on $\mathcal{X}_A$ indeed induces a G-action on $\mathcal{X}_A \otimes_A A'$. An element $g \in G$ with an action $g : \mathcal{X}_A \to \mathcal{X}_A$ yields a map $\mathcal{X}_A \otimes_A A' \to \mathcal{X}_A \xrightarrow{g} \mathcal{X}_A$. There is also the projection map $\mathcal{X}_A \otimes_A A' \to \mathrm{Spec}(A')$, so the universal property of $\mathcal{X} \otimes_A A'$ yields $g : \mathcal{X}_A \otimes_A A' \to \mathcal{X} \otimes_A A'$.

To state the theorem, we have to introduce one last concept. The field k has positive characteristic, but we might be interested in deformations to characteristic 0. Therefore we need the *ring of Witt vectors* $W(k)$. This is a complete discrete valuation ring, with residue field k (see [Haz12, Section 17]). It then follows from [Nag75, Theorem 30.3] that $W(k)$ is a Henselian ring. This means that a polynomial over $W(k)$ which factors into coprime factors over k, also has a factorisation over $W(k)$. The prime example for a ring of Witt vectors, is that over the prime field $\mathbb{F}_p$. Then $W(\mathbb{F}_p)$ is the ring of p-adic integers $\mathbb{Z}_p$. This is indeed a ring of characteristic 0, and the unique maximal ideal generated by p induces a complete valuation. And of course Hensel's lemma applies to $\mathbb{Z}_p$.

From now on we will always consider $\Lambda = W(k)$, and we will write $\mathcal{C}$ instead of $\mathcal{C}_{W(k)}$ and W instead of $W(k)$. With this definition, we can state the main result.

Theorem 2.3. *Let a curve X over an algebraically closed field k of positive characteristic be given, together with a faithful group action of the finite group G on X. The pro-representability of $D_{X,G}$ depends on the genus of X, the characteristic of k and the group G. It is non-pro-representable in the case when $g(X) = 0$, and $\mathrm{char}(k) = 2$ and G is isomorphic to $\mathbb{Z}/2$, $(\mathbb{Z}/2)^2$, or D_n for n odd. In all other cases where the genus of X is not 1, the deformation functor is pro-representable.*

As might be expected, the proof distinguishes several cases. If the genus of X is greater than 1 (in Section 2.2), we can easily compute that $\mathrm{H}^0(X, \mathcal{T}_X)^G = 0$ and this will imply the pro-representability of $D_{X,G}$. The case of genus 0 (in Section 2.3) requires a completely different approach. Because our base field k is algebraically closed, we know that $X \cong \mathbb{P}^1$ and there is a complete classification in the literature [VM80] of the finite groups acting on the projective line in positive characteristic. Luckily, we do not have to check all these cases by hand. For many we will see that the global deformation functor is isomorphic to a local deformation functor that is pro-representable. The few groups that remain turn out to be non-pro-representable.

When the genus is 1 (in Section 2.5), we can try to use the same strategies. If the cohomology group $\mathrm{H}^0(X, \mathcal{T}_X)^G$ is trivial, we can take the approach of the $g > 1$ case. We can also classify the group actions that do not fall in this case, but we cannot go to the local deformation functor as in the $g = 0$ case. Instead we attempted to prove an isomorphism to another deformation functor that is pro-representable. This might deal with cases that G is a translation group of an elliptic curve. Even if this works, there are still some elliptic curves with large automorphism groups that seem impervious to either attack.

2.2 Curves with genus greater than 1

In this case we will prove pro-representability of the global deformation functor by considering automorphisms of the deformations. For this we introduce the following notation.

Definition 2.4. Let $A \to A'$ be a morphism in $\mathcal{C}$. Let $\mathcal{X}_A$ be a deformation of a curve X over A that induces the deformation $\mathcal{X}_{A'}$ over A'. Now we define the automorphism group $\mathrm{Aut}_A(\mathcal{X}_A/\mathcal{X}_{A'})$ as the A-automorphisms of $\mathcal{X}_A$ that restrict to the identity on $\mathcal{X}_{A'}$.

For deformations of X with a group action of G, we denote this automorphism group as $\mathrm{Aut}_{A,G}(\mathcal{X}_A/\mathcal{X}_{A'})$.

The following lemma relates these automorphism groups to the cohomology group $\mathrm{H}^0(X, \mathcal{T}_X)^G$, which we will call the *group of infinitesimal automorphisms of* (X, G). The lemma after that gives a criterion to prove pro-representability based on this group of infinitesimal automorphisms.

Lemma 2.5. *Given are a curve X and a finite group G acting faithfully on X. Let $A' \to A$ be a small extension in $\mathcal{C}$ and let $\mathcal{X}_{A'}$ be a deformation over A'. If $\mathcal{X}_A$ is the fibre above* $\mathrm{Spec}(A)$*, then* $\mathrm{Aut}_{A',G}(\mathcal{X}_{A'}/\mathcal{X}_A) \cong \mathrm{H}^0(X, \mathcal{T}_X)^G$.

Proof. We first prove that $\mathrm{Aut}_{A'}(\mathcal{X}_{A'}/\mathcal{X}_A) \cong \mathrm{H}^0(X, \mathcal{T}_X)$. Then we show that this isomorphism is compatible with the action of G on both spaces, from which we conclude that $\mathrm{Aut}_{A',G}(\mathcal{X}_{A'}/\mathcal{X}_A) \cong \mathrm{H}^0(X, \mathcal{T}_X)^G$.

By Lemma 1.15, we know that a deformation of X has the same underlying topological space. An automorphism of a deformation is therefore the identity on the underlying topological space and it is only an automorphism of the structure sheaf. Now take an affine open covering $\{U_i\}_i$ of X, such that these opens are affine in $\mathcal{X}_{A'}$. Restricting this affine open covering to A and k yields an affine open covering of $\mathcal{X}_A$ and X. For each of these opens U, we will prove that $\mathrm{Aut}_{A'}(\mathcal{O}_{\mathcal{X}_{A'}}(U)/\mathcal{O}_{\mathcal{X}_A}(U)) \cong \mathcal{T}_X(U)$. Gluing these isomorphisms together gives $\mathrm{Aut}_{A'}(\mathcal{X}_{A'}/\mathcal{X}_A) \cong \mathrm{H}^0(X, \mathcal{T}_X)$.

To simplify notation, we denote B, $\mathcal{B}_A$, $\mathcal{B}_{A'}$ for the regular functions of X, $\mathcal{X}_A$, $\mathcal{X}_{A'}$ over U. Then we obtain the following diagram where all squares are pushouts:

$$\begin{array}{ccccc} B & \longleftarrow & \mathcal{B}_A & \longleftarrow & \mathcal{B}_{A'} \\ \uparrow & & \uparrow & & \uparrow \\ k & \longleftarrow & A & \longleftarrow & A' \end{array} .$$

Let (t) be the kernel of $A' \to A$, which is a 1-dimensional k-vector space. This can be written as a short exact sequence of A'-modules

$$0 \to k \cdot t \to A' \to A \to 0.$$

Because $\mathcal{B}_{A'}$ is a deformation of B, we know that $\mathcal{B}_{A'} \otimes_{A'} -$ is an exact functor so applying it to the above exact sequence yields

$$0 \to B \cdot t \to \mathcal{B}_{A'} \to \mathcal{B}_A \to 0.$$

Let φ be an automorphism of $\mathrm{Aut}_{A'}(\mathcal{B}_{A'}/\mathcal{B}_A)$. The function $x \mapsto \varphi(x) - x$ becomes the zero map on $\mathcal{B}_A$, so on $\mathcal{B}_{A'}$ the image lies in $B \cdot t$. Hence there is a function $d : \mathcal{B}_{A'} \to B$ such that $\varphi(x) = x + d(x) \cdot t$. This function is actually an A'-derivation: it is A'-linear and we see that

$$\begin{aligned} d(b_1 b_2) \cdot t &= \varphi(b_1 b_2) - b_1 b_2 \\ &= \varphi(b_1)\varphi(b_2) - b_1 b_2 \\ &= (b_1 + d(b_1) \cdot t)(b_2 + d(b_2) \cdot t) - b_1 b_2 \\ &= (b_1 d(b_2) + d(b_1) b_2) \cdot t. \end{aligned}$$

On the other hand, given an A'-derivation $d : \mathcal{B}_{A'} \to B$, we can construct the morphism $x \mapsto x + d(x) \cdot t$. It is indeed an automorphism, because $x \mapsto x - d(x) \cdot t$ is its inverse. Hence we obtain that $\mathrm{Aut}_{A'}(\mathcal{B}_{A'}/\mathcal{B}_A) \cong \mathrm{Der}_{A'}(\mathcal{B}_{A'}, B) \cong \mathrm{Der}_k(B, B) \cong \mathcal{T}_X(U)$. All these isomorphisms are G-equivariant, hence $\mathrm{Aut}_{A',G}(\mathcal{B}_{A'}/\mathcal{B}_A) \cong \mathrm{H}^0(X, \mathcal{T}_X)^G$. □

Lemma 2.6. *Given are a curve X and a finite group G acting faithfully on X. Assume that the group of infinitesimal automorphisms of the pair (X, G) is trivial. Then we have the following results.*

(a) For every deformation $\mathcal{X}$ of (X, G) over A, we have that $\mathrm{Aut}_{A,G}(\mathcal{X}_A/X)$ is trivial.

(b) The global deformation functor $D_{X,G}$ is pro-representable.

Proof. (a) We do this by induction on the length of A as $W(k)$-module. The length 0 case is trivial.

For A of positive length, we can factor the residue map $A \to k$ as $A \to A' \to k$, where $A \to A'$ is small and $\mathrm{length}(A') = \mathrm{length}(A) - 1$. Let $\mathcal{X}_{A'}$ be the induced deformation over A'. By induction we know that $\mathrm{Aut}_{A',G}(\mathcal{X}_{A'}/X)$ is trivial, and hence that an automorphism of $\mathrm{Aut}_{A,G}(\mathcal{X}_A/X)$ restricts to the identity on $\mathcal{X}_{A'}$. But $\mathrm{Aut}_{A,G}(\mathcal{X}_A/\mathcal{X}_{A'})$ is isomorphic to the group of infinitesimal automorphisms, hence is trivial. We conclude that $\mathrm{Aut}_{A,G}(\mathcal{X}_A/X)$ also consists of only the identity.

(b) To prove that $D_{X,G}$ has a hull, we can apply the same proof as in Proposition 1.26 to obtain **H2** and **H3**. We only have to make sure that the fibre sum deformation has a G-action. In the notation in that proof, there is a G-action on the sheaves $\mathcal{O}_A, \mathcal{O}_B, \mathcal{O}_C$. This yields a G-action on the fibre product $\mathcal{O}_A \times_{\mathcal{O}_C} \mathcal{O}_B$ as well, which is the sheaf of the fibre sum deformation.

Because a curve is proper and smooth, Theorem 1.27 applies and the tangent space of D_X is finite-dimensional. For a single deformation $\mathcal{X}_{k[\varepsilon]/\varepsilon^2}$ of X over $k[\varepsilon]/\varepsilon^2$, the space of automorphisms is also finite-dimensional over k by Lemma 2.5. For any two group actions of G on $\mathcal{X}_{k[\varepsilon]/\varepsilon^2}$, the two actions of an element $g \in G$ must differ by an element of $\mathrm{Aut}_{k[\varepsilon]/\varepsilon^2}(\mathcal{X}_{k[\varepsilon]/\varepsilon^2}/X)$. Therefore the space of possible group actions on $\mathcal{X}_{k[\varepsilon]/\varepsilon^2}$ is finite-dimensional, and hence the tangent space of $D_{X,G}$ is also finite-dimensional. We conclude that $D_{X,G}$ has a hull.

In order to prove pro-representability, we only need to show that condition **H4** is satisfied. Consider two small extensions $A', A'' \to A$ and create the fibre product $\bar{A} = A' \times_A A''$. Consider deformations $\mathcal{X}_{A'}$, $\mathcal{X}_{A''}$ over resp. A', A'' such that they restrict to the same deformation $\mathcal{X}_A$ over A, as shown in the following diagram:

$$\begin{array}{ccccc} \mathcal{X}_{A'} & & & & \mathcal{X}_{A''} \\ \uparrow & & & & \uparrow \\ \mathcal{X}_{A'} \otimes_{A'} A & \xleftrightarrow{\sim} & \mathcal{X}_A & \xleftrightarrow{\sim} & \mathcal{X}_{A''} \otimes_{A''} A \end{array} .$$

Again we have the fibre sum deformation $\mathcal{X}_{\bar{A}}$ that produces the pushout digram

$$\begin{array}{ccc} & \mathcal{X}_{\bar{A}} & \\ \nearrow & & \nwarrow \\ \mathcal{X}_{A'} & & \mathcal{X}_{A''} \\ \nwarrow & & \nearrow \\ & \mathcal{X}_A & \end{array} .$$

To prove that this is the unique deformation extending $\mathcal{X}_{A'}$ and $\mathcal{X}_{A''}$, let $\mathcal{Z}$ be another deformation of X over $\bar{A}$ such that the restrictions to A' and A'' are isomorphic to $\mathcal{X}_{A'}$ and $\mathcal{X}_{A''}$. We can restrict these isomorphisms to A, and this produces the following diagram:

$$\begin{array}{ccccccc} & & & \mathcal{Z} & & & \\ & & \nearrow & & \nwarrow & & \\ \mathcal{X}_{A'} & \xleftrightarrow{\sim} & \mathcal{Z} \otimes_{\bar{A}} A' & & \mathcal{Z} \otimes_{\bar{A}} A'' & \xleftrightarrow{\sim} & \mathcal{X}_{A''} \\ \nwarrow & & \nwarrow & & \nearrow & & \nearrow \\ & \mathcal{X}_{A'} \otimes_{A'} A & \xleftrightarrow{\sim} & \mathcal{Z} \otimes_{\bar{A}} A & \xleftrightarrow{\sim} & \mathcal{X}_{A''} \otimes_{A''} A & \\ & \sim\updownarrow & & & & \sim\updownarrow & \\ & \mathcal{X}_A & & & & \mathcal{X}_A & \end{array} .$$

In this diagram, the maps $\mathcal{X}_A \to \mathcal{X}_{A'}, \mathcal{X}_{A''}$ are the same as above. Furthermore, the induced isomorphism between the $\mathcal{X}_A$'s on the bottom is an isomorphism of deformations. It therefore induced the identity on X, and now we make crucial use of the fact that $\mathrm{Aut}_A(\mathcal{X}_A/X)$ is trivial. Because of this, the isomorphism between the $\mathcal{X}_A$'s is actually the identity and we obtain the commuting diagram

$$\begin{array}{ccc} & \mathcal{Z} & \\ \nearrow & & \nwarrow \\ \mathcal{X}_{A'} & & \mathcal{X}_{A''} \\ \nwarrow & & \nearrow \\ & \mathcal{X}_{A} & \end{array} .$$

The universal property of $\mathcal{X}_{\bar{A}}$ induces a morphism of deformations $\mathcal{X}_{\bar{A}} \to \mathcal{Z}$. By Lemma 1.16 this morphism is an isomorphism, so we have found that there is indeed exactly one deformation over $\bar{A}$ restricting to $\mathcal{X}_{A'}$ and $\mathcal{X}_{A''}$. □

With these two lemmas, we can easily prove Theorem 2.3 in the case $g > 1$.

Proof of Theorem 2.3, case $g > 1$. The tangent bundle $\mathcal{T}_X$ has degree $2 - 2g$, so in case $g > 1$ this degree is negative. Therefore $\mathrm{H}^0(X, \mathcal{T}_X)$ is trivial, and the group of automorphisms $\mathrm{H}^0(X, \mathcal{T}_X)^G$ is also trivial. Now Lemma 2.6 implies that $D_{X,G}$ is pro-representable. □

2.3 Pro-representable deformations of genus 0 curves

We have arrived at the case where X has genus 0. Because we work over an algebraically closed field, we can assume that X has at least one k-point, and therefore $X \cong \mathbb{P}^1_k$. We have seen in Theorem 1.29 that all deformations of $\mathbb{P}^1$ are trivial. Hence studying the deformations of $(\mathbb{P}^1, G)$ is mainly about deforming the group action.

Our main tool in getting a grip on these deformations is a classification of the group actions on $\mathbb{P}^1$ in positive characteristic by [VM80]. While it would be possible to check all these group actions for pro-representability, I present my proof which does not need to determine all deformation functors one by one. We prove that the deformation functor is often isomorphic to a *local* deformation functor, for which pro-representability is determined in [BC09]. We now introduce these local deformation functors.

Definition 2.7. We denote Γ_A for the A-automorphisms of the ring of formal power series $A[\![t]\!]$. A *local G-action* is an injective homomorphism $G \to \Gamma_A$. A deformation of a given local G-action $\rho : G \to \Gamma_k$ to $A \in \mathcal{C}$ is a local G-action $\rho_A : G \to \Gamma_A$, such that it is equal to ρ after composing with the $\mod \mathfrak{m}_A$ map $\Gamma_A \to \Gamma_k$. Two such deformations are isomorphic if they are related by an isomorphism of Γ_A that is the identity on Γ_k.

The deformation functor $D_{G,\mathrm{loc}} : \mathcal{C} \to \mathbf{Set}$ assigns to every $A \in \mathcal{C}$ the isomorphism classes of deformations of ρ to A.

There is another description of this deformation functor. In Theorem 1.29 we have seen that the deformations of $\mathrm{Spec}(k[\![t]\!])$ are trivial. A local G-action $G \to \Gamma_k$ is equivalent to a faithful action of G on $\mathrm{Spec}(k[\![t]\!])$, so we can consider the deformations of the pair $(\mathrm{Spec}(k[\![t]\!]), G)$. But a deformation of $\mathrm{Spec}(k[\![t]\!])$ over A is isomorphic to $\mathrm{Spec}(A[\![t]\!])$, so we find exactly the lifted local G-actions $G \to \Gamma_A$.

Definition 2.8. Given a curve X with a faithful action of the finite group G. Consider the set of points of X having non-trivial stabilizer group under this action. For every G-orbit in this set, we choose one point, call these points $p_1, \dots, p_\ell$. For every point p_i its stabilizer group G_{p_i} acts on the completion of $\mathcal{O}_{X,p_i}$. Because the curve X is regular, this completion is isomorphic to $k[\![t]\!]$, so there is a local G_{p_i}-action. We define the *local deformation functor* $D_{X,G,\mathrm{loc}}$ of the pair (X, G) as the product $\prod_{i=1}^{\ell} D_{G_{p_i},\mathrm{loc}}$.

In the next lemma we see why this local deformation functor is so useful.

Lemma 2.9. *(a) There is formally smooth morphism $D_{X,G} \to D_{X,G,\mathrm{loc}}$.*

(b) The morphism $D_{\mathbb{P}^1,G} \to D_{\mathbb{P}^1,G,\mathrm{loc}}$ is formally étale.

Proof. *(a)* Let $\mathcal{X}_A$ be a deformation of X over A. For a ramification point $p_i \in X$ the ramification group G_i acts on the local ring of $p_i \in X$. By Lemma 1.15 the topological space of $\mathcal{X}_A$ is homeomorphic to that of X, so we can also consider $p_i \in \mathcal{X}_A$ and look at its local ring. Because G has a lifted action on $\mathcal{X}_A$, and G_i fixes p_i, we obtain an action of G_i on the local ring $\mathcal{O}_{\mathcal{X}_A,p_i}$. We can easily lift this to an action of G_i on the completion $\widehat{\mathcal{O}}_{\mathcal{X}_A,p_i}$. Now it is easy to see that $\mathrm{Spec}(\widehat{\mathcal{O}}_{\mathcal{X}_A,p_i})$ is a deformation of $\mathrm{Spec}(\widehat{\mathcal{O}}_{X,p_i})$, so we obtain a deformation of $(\mathrm{Spec}(\widehat{\mathcal{O}}_{X,p_i}), G_i)$. As we have seen that $D_{G_i,\mathrm{loc}} \cong D_{\mathrm{Spec}(k[\![t]\!]),G_i}$, we obtain the map to $D_{X,G,\mathrm{loc}}$ by varying over all ramification points. Smoothness of this morphism is proven in [BM00, Theorem 3.3.4].

(b) It remains to show that the tangent space of the global and local deformation functors have the same dimension. The difference in dimension is given in [BM00, Corollary 3.3.5] by $\dim \mathrm{H}^1(\mathbb{P}^1/G, \pi_*^G(\mathcal{T}_{\mathbb{P}^1}))$ and we want to show that this is trivial. Here $\pi : \mathbb{P}^1 \to \mathbb{P}^1/G$ is the quotient curve of the action of G on $\mathbb{P}^1$, and π_*^G is the *G-equivariant pushforward*, which sends a sheaf $\mathcal{F}$ on $\mathbb{P}^1$ to the sheaf $U \mapsto \mathcal{F}(\pi^{-1}(U))^G$ (the G-invariant sections over $\pi^{-1}(U)$). The sheaf $\pi_*^G(\mathcal{T}_{\mathbb{P}^1})$ is computed in [CK03] for ordinary curves X. As the projective line is indeed ordinary, we can use their result. They define a *ramification divisor* Δ on $\mathbb{P}^1/G$. For this, consider the branch points of π. The ramification group of such point P is always of the form $(\mathbb{Z}/p)^t \rtimes \mathbb{Z}/n$. We put all points for which $t = 0$, or $t = 1$ and $p = 2$, in a set T. All other branch points form a set W. Then the divisor Δ is

$$\Delta = \sum_{P \in T} P + \sum_{Q \in W} 2Q.$$

Now the result is that $\pi_*^G(\mathcal{T}_{\mathbb{P}^1}) \cong \mathcal{T}_{\mathbb{P}^1/G}(-\Delta)$.

We can simplify this as follows. In a non-constant map of curves $\pi : \mathbb{P}^1 \to \mathbb{P}^1/G$ the genus cannot increase (by the Riemann-Hurwitz theorem), so we see that $\mathbb{P}^1/G$ also has genus 0, hence $\mathbb{P}^1/G \cong \mathbb{P}^1$. Under this isomorphism we have that $\mathcal{T}_{\mathbb{P}^1/G}(-\Delta) \cong \mathcal{T}_{\mathbb{P}^1}(-\Delta) \cong \mathcal{O}_{\mathbb{P}^1}(2 - \Delta)$. We are interested in $\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2 - \Delta))$, which is by Serre duality isomorphic to the dual of $\mathrm{H}^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(\Delta - 2)) \cong \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\Delta - 4))$. We see that this is trivial when $\deg \Delta \leq 3$, so we are left to prove this.

For this we consider all possible faithful actions of finite groups G on $\mathbb{P}^1$ in positive characteristic. These are determined by [VM80] and are summarized in the following table. In this table, p is the characteristic of the base field k. The column 'char' indicates

Table 2.1: Finite groups acting on $\mathbb{P}^1$ in positive characteristic p, with ramification behaviour of the quotient $\mathbb{P}^1 \to \mathbb{P}^1/G$.

G	char	G_{P_1}	Δ_{P_1}	G_{P_2}	Δ_{P_2}	G_{P_3}	Δ_{P_3}	$\deg\Delta$
$\mathbb{Z}/n$	$(p,n)=1$	$\mathbb{Z}/n$	1	$\mathbb{Z}/n$	1	–	–	2
$(\mathbb{Z}/p)^t$		$(\mathbb{Z}/p)^t$	1, 2	–	–	–	–	1, 2
D_n	$p=2$, $(p,n)=1$	$\mathbb{Z}/2$	1	$\mathbb{Z}/n$	1	–	–	2
	$p\neq 2$, $(p,n)=1$	$\mathbb{Z}/2$	1	$\mathbb{Z}/2$	1	$\mathbb{Z}/n$	1	3
A_4	$p\neq 2,3$	$\mathbb{Z}/2$	1	$\mathbb{Z}/3$	1	$\mathbb{Z}/3$	1	3
S_4	$p\neq 2,3$	$\mathbb{Z}/2$	1	$\mathbb{Z}/3$	1	$\mathbb{Z}/4$	1	3
A_5	$p=3$	$\mathbb{Z}/6$	2	$\mathbb{Z}/5$	1	–	–	3
	$p\neq 2,3,5$	$\mathbb{Z}/2$	1	$\mathbb{Z}/3$	1	$\mathbb{Z}/5$	1	3
$(\mathbb{Z}/p)^t \rtimes \mathbb{Z}/n$	$n \mid p^t-1$	$(\mathbb{Z}/p)^t \rtimes \mathbb{Z}/n$	1, 2	$\mathbb{Z}/n$	1	–	–	2, 3
$\mathrm{PSL}_2(\mathbb{F}_{p^t})$	$p\neq 2$	$(\mathbb{Z}/p)^t \rtimes \mathbb{Z}/\frac{p^t-1}{2}$	2	$\mathbb{Z}/\frac{p^t+1}{2}$	1	–	–	3
$\mathrm{PGL}_2(\mathbb{F}_{p^t})$		$(\mathbb{Z}/p)^t \rtimes \mathbb{Z}/(p^t-1)$	1, 2	$\mathbb{Z}/(p^t+1)$	1	–	–	2, 3

any restrictions on this characteristic in order to have the group action. Each column G_{P_i} indicates the ramification group of a branch point of $\mathbb{P}^1/G$, and Δ_{P_i} is the degree of the divisor Δ in this point P_i. For every group action there are at most three branch points. The total degree of Δ is in the last column, and in this way we have verified that $\deg\Delta$ is indeed always at most 3.

We conclude that the tangent spaces in $D_{\mathbb{P}^1,G} \to D_{\mathbb{P}^1,G,\mathrm{loc}}$ have the same dimension. Because the map is smooth, the map on tangent spaces is surjective and hence it is an isomorphism. This combines to give a formally étale map. □

Before we start the proof of the $g=0$ case, we need one small fact from [Bys09a]:

Lemma 2.10. *Let $D, E : \mathcal{C} \to$ **Set** be two functors, and let $\eta : D \to E$ be a formally étale morphism. If E is pro-representable, and D has a versal hull, then η is an isomorphism. In particular is D also pro-representable.*

Proof. Let $\theta : h_R \to D$ be a hull of D. Then composing this with $D \to E$ gives a hull $\eta\theta : h_R \to D \to E$ of E. Because E is pro-representable, there is also a trivial hull $\mathrm{id} : E \to E$. Because the hull of a pro-representable functor is unique up to a unique isomorphism, we find that $\eta\theta$ is an isomorphism, and in particular θ is injective. As θ is a hull, it is also surjective, hence it is an isomorphism. This already implies that D is pro-representable. Because we knew that $\eta\theta$ is an isomorphism, it also follows that η is an isomorphism. □

Now we have all we need to start the proof of the genus 0 case.

Proof of Theorem 2.3, case $g = 0$. We start by combining the results of Lemma 2.9 and Lemma 2.10. The functor $D_{\mathbb{P}^1,G}$ has a versal hull, which follows from the proof of Lemma 2.6. The morphism $D_{\mathbb{P}^1,G} \to D_{\mathbb{P}^1,G,\mathrm{loc}}$ is étale, so if $D_{\mathbb{P}^1,G,\mathrm{loc}}$ is pro-representable, we know immediately that $D_{\mathbb{P}^1,G}$ is pro-representable!

Let's now describe what happens with the local deformation functor $D_{X,G,\mathrm{loc}}$, as given in [BC09]. They considered the local deformation functor over a perfect field of positive characteristic, hence it applies to our situation. They showed that a local deformation functor $D_{k[\![t]\!],G}$ is pro-representable, except when $\mathrm{char}(k) = 2$ and $G \cong \mathbb{Z}/2, (\mathbb{Z}/2)^2$ in which case the deformation functor is non-pro-representable.

This means that the local deformation functor $D_{X,G,\mathrm{loc}}$ is also pro-representable, except when $\mathrm{char}(k) = 2$ and one of the ramification groups is $\mathbb{Z}/2$ or $(\mathbb{Z}/2)^2$. Looking in Table 2.1, this means that the only groups for which the local deformation is non-pro-representable are $\mathbb{Z}/2$, $(\mathbb{Z}/2)^2$, D_n for n odd, and $\mathrm{PGL}_2(\mathbb{F}_2)$. For all other groups, we can immediately conclude that the deformation functor $D_{X,G}$ is pro-representable.

Note that $\mathrm{PGL}_2(\mathbb{F}_2) \cong D_3$, so the only groups left to consider are $\mathbb{Z}/2$, D_n for n odd and $(\mathbb{Z}/2)^2$, all with $\mathrm{char}(k) = 2$. These are never pro-representable, as will be shown in the next section. □

2.4 Non-pro-representable deformations

In this section we only prove the following theorem, which requires a lot of computations. The counter-examples, showing that a versal hull is not an isomorphism are the same as in the local case in [BC09].

Theorem 2.11. *When $\mathrm{char}(k) = 2$ and $G \cong \mathbb{Z}/2$, $G \cong (\mathbb{Z}/2)^2$, $G \cong D_n$ for $n \geq 3$ odd, the deformation functor $D_{\mathbb{P}^1,G}$ is never pro-representable.*

We first state the following lemma, which will be needed in the calculations.

Lemma 2.12. *Consider the matrix $X = \begin{pmatrix} 0 & 1 \\ C & 1 \end{pmatrix}$ for a variable C. Then there exist polynomials $p_n \in \mathbb{Z}[C]$ such that $X^n = \begin{pmatrix} Cp_{n-1} & p_n \\ Cp_n & p_{n+1} \end{pmatrix}$ for all positive integers n. These polynomials satisfy $p_0 = 0$, $p_1 = 1$, and $p_{n+1} = p_n + Cp_{n-1}$ for all positive integers n. For positive n, the degree of the polynomial p_n is given by $\lfloor \frac{n-1}{2} \rfloor$.*

A closed form over $\mathbb{Z}[\zeta_n]$ (where ζ_n is a primitive n-th root of unity) is given by

$$p_n(C) = \prod_{k=1}^{\frac{n-1}{2}} (1 + (\zeta_n^k + \zeta_n^{-k} + 2)C).$$

Proof. Note that we can easily define the polynomials by the recurrent relation $p_0 = 0$, $p_1 = 1$ and $p_{n+1} = p_n + Cp_{n-1}$ for all $n > 0$. It is clear that all coefficients are positive in these polynomials. The first two statements in the lemma can now both be proven by induction on n.

We see that $p_2 = 1$, so it is clear that $X = \begin{pmatrix} Cp_0 & p_1 \\ Cp_1 & p_2 \end{pmatrix}$. For the induction step, we assume that $X^n = \begin{pmatrix} Cp_{n-1} & p_n \\ Cp_n & p_{n+1} \end{pmatrix}$ for some positive n. Then we see that

$$X^{n+1} = \begin{pmatrix} Cp_{n-1} & p_n \\ Cp_n & p_{n+1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ C & 1 \end{pmatrix} = \begin{pmatrix} Cp_n & Cp_{n-1} + p_n \\ Cp_{n+1} & Cp_n + p_{n+1} \end{pmatrix} = \begin{pmatrix} Cp_n & p_{n+1} \\ Cp_{n+1} & p_{n+2} \end{pmatrix},$$

which prove the induction step.

Next we look at the degrees. For $n = 1, 2$ both p_1, p_2 have degree 0. For the induction step, we assume that p_{n-1} and p_n have degree respectively $\lfloor \frac{n-2}{2} \rfloor$ and $\lfloor \frac{n-1}{2} \rfloor$. Because both have positive coefficients, we compute the degree of $p_{n+1} = p_n + Cp_{n-1}$ as the maximum of $\lfloor \frac{n-1}{2} \rfloor$ and $\lfloor \frac{n-2}{2} \rfloor + 1 = \lfloor \frac{n}{2} \rfloor$. But this maximum is clearly $\lfloor \frac{n}{2} \rfloor$, as we wanted.

It remains to show the closed form. First, note that the constant term of p_n is always 1. It is therefore enough to show that the zeroes of p_n are $\frac{-1}{\zeta_n^k + \zeta_n^{-k} + 2}$. We see that C is a zero of p_n, exactly if X^n is a multiple of the identity matrix. Note that the characteristic polynomial of $\begin{pmatrix} 0 & 1 \\ C & 1 \end{pmatrix}$ is $\lambda^2 - \lambda - C$. Assuming for the moment that $C \neq -1/4$, we see that its eigenvalues λ_1, λ_2 are different and it has an eigenbasis. The n-th power of this matrix has the same eigenbasis, with eigenvectors λ_1^n, λ_2^n. As this n-th power must be a multiple of the identity matrix, we find that $\lambda_1^n = \lambda_2^n$. We see that λ_1/λ_2 is a power of ζ_n. Then

$$\lambda^2 - \lambda - C = (\lambda - \lambda_2\zeta_n^k)(\lambda - \lambda_2) = \lambda^2 - (1 + \zeta_n^k)\lambda_2\lambda + \zeta_n^k\lambda_2^2.$$

Equating the coefficients yields $C = -\zeta_n^k\lambda_2^2 = \frac{-\zeta_n^k}{(1+\zeta_n^k)^2} = \frac{-1}{\zeta_n^k + \zeta_n^{-k} + 2}$ for $k \neq n/2$. If we assume that $k \neq 0$, these zeroes are not equal to $-1/4$, so this produces $\lfloor \frac{n-1}{2} \rfloor$ different roots of p_n. This matches the degree of p_n and therefore we find that closed form. □

Proof of Theorem 2.11. First two remarks that simplify the computations. We have seen in Theorem 1.29 that the deformations of $\mathbb{P}^1$ are trivial. Hence a deformation of $(\mathbb{P}^1_k, G)$ over A is isomorphic to $(\mathbb{P}^1_A, G)$ for some G-action on $\mathbb{P}^1_A$. Furthermore we know that all automorphisms of $\mathbb{P}^1_A$ are given by Möbius transformations [MFK94, Section 0.5]. Hence we can write all automorphisms as 2×2-matrices in PGL_2. In the calculations we will use the letters a, b, c, d, x, y, z, w for elements in k.

Case $G \cong \mathbb{Z}/2 = \langle s \mid s^2 = e \rangle$. The proof goes in four steps: we put the action of $\mathbb{Z}/2$ in a standard form, then we calculate the tangent space, next we determine a versal hull, and last we check that this hull is not universal.

Step 1. Let $\rho : G \to \mathrm{PGL}_2(k)$ be the action of G and write $\rho(s) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we have that $\rho(s)^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{pmatrix}$. This needs to be a multiple of the identity matrix, so we get $a^2 = d^2$. Because the field k has characteristic 2, this implies $d = -a$. We see right away that the top-right and bottom-left entry are now 0, so indeed $\rho(s)^2 = \mathrm{id}$. Note that the determinant of $\rho(s)$ is $a^2 + bc$, which is therefore non-zero. Because we work over an algebraically closed field, we can also extract its square root $r = \sqrt{a^2 + bc}$. If $b = c = 0$, we see that $\rho(s) = \mathrm{id}$ while the action was faithful. Hence either b or c is non-zero. This means that one of the matrices $A_1 = \begin{pmatrix} a+r & b \\ a & b \end{pmatrix}$, $A_2 = \begin{pmatrix} c & a+r \\ c & a \end{pmatrix}$ has non-zero determinant. Their inverses (if they exist) are respectively given by $A_1^{-1} = \begin{pmatrix} b & b \\ a & a+r \end{pmatrix}$ and $A_2^{-1} = \begin{pmatrix} a & a+r \\ c & c \end{pmatrix}$. Now we easily compute

$$A_1 \rho(s) A_1^{-1} = \begin{pmatrix} br^2 & 0 \\ br^2 & br^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{if } b \neq 0,$$
$$A_2 \rho(s) A_2^{-1} = \begin{pmatrix} cr^2 & 0 \\ cr^2 & cr^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{if } c \neq 0.$$

In either case, we see that $\rho(s)$ is conjugated to $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$. From now one we will assume that $\rho(s) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$. (Note that we could leave out the minus sign, but we keep it because this form lifts to characteristic 0.)

Step 2. Now we calculate the tangent space. Consider a lift $\tilde{\rho} : G \to \mathrm{PGL}_2(k[\varepsilon]/\varepsilon^2)$, so $\tilde{\rho}(s) = \begin{pmatrix} 1 + a\varepsilon & b\varepsilon \\ 1 + c\varepsilon & -1 + d\varepsilon \end{pmatrix}$. This lift still must have order 2, so we compute

$$\tilde{\rho}(s^2) = \begin{pmatrix} 1 + b\varepsilon & 0 \\ (a+d)\varepsilon & 1 + b\varepsilon \end{pmatrix}.$$

We conclude that $a = -d$. Equivalent lifts are conjugated by an element of the form $\begin{pmatrix} 1 + x\varepsilon & y\varepsilon \\ z\varepsilon & 1 + w\varepsilon \end{pmatrix}$. Computing the conjugate of $\tilde{\rho}(s)$ we find

$$\begin{pmatrix} 1 + (a+y)\varepsilon & b\varepsilon \\ 1 + (c + w - x)\varepsilon & -1 - (a+y)\varepsilon \end{pmatrix}.$$

From this we see that two lifts of $\rho(s)$ are conjugate if and only if the top-right entry is the same. And we also see that all lifts of ρ to $k[\varepsilon]/\varepsilon^2$ are given by $\left\{ \tilde{\rho} : s \mapsto \begin{pmatrix} 1 & b\varepsilon \\ 1 & -1 \end{pmatrix} \mid b \in k \right\}$.

Step 3. Given this form of the tangent space, we guess the following versal hull:

$$\eta : h_{W[\![\alpha]\!]} \longrightarrow D_{\mathbb{P}^1, \mathbb{Z}/2},$$
$$(f : W[\![\alpha]\!] \to A) \longmapsto \left(\tilde{\rho} : s \mapsto \begin{pmatrix} 1 & f(\alpha) \\ 1 & -1 \end{pmatrix} \right).$$

An easy calculation shows that this lift maps s^2 to the identity, hence it is a deformation. A W-morphism $f : W[\![\alpha]\!] \to A$ maps α into $\mathfrak{m}_A$, and is completely determined by the image of α. This already shows that this map is an isomorphism on tangent spaces. To prove that it is a hull, we only need that it is formally smooth. For this, consider a small extension $A' \to A$, and let t generate the kernel of this map. We have to show that the map

$$h_{W[\![\alpha]\!]}(A') \to h_{W[\![\alpha]\!]}(A) \times_{D_{\mathbb{P}^1,\mathbb{Z}/2}(A)} D_{\mathbb{P}^1,\mathbb{Z}/2}(A')$$

is surjective. This means that we have a map $f : W[\![\alpha]\!] \to A$, which gives a deformation $s \mapsto \begin{pmatrix} 1 & f(\alpha) \\ 1 & -1 \end{pmatrix}$. We also have a lift of this to A', write this as $\begin{pmatrix} 1+at & m+bt \\ 1+ct & -1+dt \end{pmatrix}$, with $m \in \mathfrak{m}_A$ mapping to $f(\alpha) \in \mathfrak{m}_{A'}$. This deformation must again square to the identity. Using that $t^2 = mt = 0$, we compute the square $\begin{pmatrix} 1+m+bt & 0 \\ (a+d)t & 1+m+bt \end{pmatrix}$ and therefore $a + d = 0$. Now we conjugate by $\begin{pmatrix} 1 & ct \\ 0 & 1+at \end{pmatrix}$, this gives

$$\begin{aligned}\begin{pmatrix} 1 & ct \\ 0 & 1+at \end{pmatrix}\begin{pmatrix} 1+at & m+bt \\ 1+ct & -1-at \end{pmatrix}\begin{pmatrix} 1+at & -ct \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1+ct & m+bt \\ 1+ct & -1-ct \end{pmatrix} \\ &= \begin{pmatrix} 1 & m+bt-ct \\ 1 & -1 \end{pmatrix}.\end{aligned}$$

Hence we obtain this deformation from $f : W[\![\alpha]\!] \to A'$, sending $\alpha \mapsto m + bt - ct$. This shows the surjectivity we wanted.

Step 4. This shows that $\eta : h_{W[\![\alpha]\!]} \to D_{\mathbb{P}^1,\mathbb{Z}/2}$ is indeed a versal hull. If the deformation functor is pro-representable, η is an isomorphism by Lemma 2.10. We will show that this is not the case, by showing that $\eta(k[\varepsilon]/\varepsilon^3)$ is not injective. There are two different maps $f : W[\![\alpha]\!] \to k[\varepsilon]/\varepsilon^3$ yielding the deformations $\begin{pmatrix} 1 & \varepsilon \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \varepsilon+\varepsilon^2 \\ 1 & -1 \end{pmatrix}$. However, a quick calculation shows that these are conjugated by $\begin{pmatrix} 1+\varepsilon & \varepsilon \\ 0 & 1 \end{pmatrix}$. Therefore η is not an isomorphism and $D_{\mathbb{P}^1,\mathbb{Z}/2}$ is indeed non-pro-representable.

Case $G \cong D_n = \langle r, s \mid r^n = e, s^2 = e, (rs)^2 = e \rangle$. We follow the same steps as in the previous case. We can also recycle some of the calculations performed for s.

Step 1. We again write $\rho : G \to \mathrm{PGL}_2(k)$ for the action of G, and we again assume that $\rho(s) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$. Next we look at $\rho(r) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\rho(rs) = \begin{pmatrix} a+b & -b \\ c+d & -d \end{pmatrix}$. Because rs has order two, we find analogous to the previous case that $d = a + b$. Now assume that $b = 0$, then $\rho(r) = \begin{pmatrix} a & 0 \\ c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix}$. In this form we easily compute that $\rho(r^n) = \begin{pmatrix} 1 & 0 \\ n \cdot c/a & 1 \end{pmatrix}$. This must be the identity, so because n is odd, this means that $c = 0$. But then $\rho(r)$ is the identity, while the action of G is faithful. This means that $b \neq 0$, and by rescaling we may assume that $b = 1$. Now consider the matrix $A = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, then we easily see that $A\rho(s)A^{-1} = \rho(s)$. When we conjugate $\rho(r)$ by A, we see that

$$A\rho(r)A^{-1} = \begin{pmatrix} 0 & 1 \\ c - a - a^2 & 1 \end{pmatrix}.$$

Hence we may take $\rho(r) = \begin{pmatrix} 0 & 1 \\ C & 1 \end{pmatrix}$ for some $C \in k$. According to Lemma 2.12, C must satisfy $p_n(C) = 0$.

Step 2. Let $\tilde{\rho}$ be a lift of ρ to $k[\varepsilon]/\varepsilon^2$. Then $\tilde{\rho}(s) = \begin{pmatrix} 1 + a_1\varepsilon & b_1\varepsilon \\ 1 + c_1\varepsilon & -1 - a_1\varepsilon \end{pmatrix}$ (because it has order 2) and $\tilde{\rho}(r) = \begin{pmatrix} a_2\varepsilon & 1 + b_2\varepsilon \\ C + c_2\varepsilon & 1 + d_2\varepsilon \end{pmatrix}$. Now we can compute

$$\tilde{\rho}(rs) = \begin{pmatrix} 1 + (c_1 + a_2 + b_2)\varepsilon & -1 - (a_1 + b_2)\varepsilon \\ 1 + C + (a_1C + c_1 + c_2 + d_2)\varepsilon & -1 + (-a_1 + b_1C - d_2)\varepsilon \end{pmatrix}.$$

This must have order 2, so as we have seen before this implies that the top-left and bottom-right entry add up to 0, or $c_1 + a_2 + b_2 = a_1 - b_1C + d_2$.

We now first conjugate $\tilde{\rho}(s)$ and $\tilde{\rho}(r)$ by $A = \begin{pmatrix} 1 + (b_1C + c_1)\varepsilon & 0 \\ a_2\varepsilon & 1 + a_1\varepsilon \end{pmatrix}$. This yields

$$A\tilde{\rho}(s)A^{-1} = \begin{pmatrix} 1 & b_1\varepsilon \\ 1 - b_1C\varepsilon & -1 \end{pmatrix}, \quad A\tilde{\rho}(r)A^{-1} = \begin{pmatrix} 0 & 1 \\ C + (a_2 + b_2C + c_2)\varepsilon & 1 \end{pmatrix}.$$

The n-th power of this conjugate of $\tilde{\rho}(r)$ must also be the identity. Writing $t = a_2 + b_2C + c_2$, Lemma 2.12 shows that C and $C + t\varepsilon$ are both zeroes of the polynomial p_n, so we have

$$0 = p_n(C + t\varepsilon) = p_n(C) + p_n'(C)t\varepsilon = p_n'(C)t\varepsilon.$$

However, this polynomial has $\lfloor \frac{n-1}{2} \rfloor$ distinct zeroes, so $p_n'(C) \neq 0$ and we conclude that $t = 0$. Hence $A\tilde{\rho}A^{-1} = \begin{pmatrix} 0 & 1 \\ C & 1 \end{pmatrix}$ is just the same as $\rho(r)$.

This shows that all lifts to $k[\varepsilon]/\varepsilon^2$ are conjugated to $s \mapsto \begin{pmatrix} 1 & b\varepsilon \\ 1 - bC\varepsilon & -1 \end{pmatrix}$, $r \mapsto \begin{pmatrix} 0 & 1 \\ C & 1 \end{pmatrix}$. Furthermore, these are all not conjugated amongst themselves, because the lifts of s are not conjugated (as in the previous case). Hence again we have a 1-dimensional tangent space.

Step 3. Based on this form of the tangent space, we again guess a versal hull:

$$\eta : h_{W[\![\alpha]\!]} \longrightarrow D_{\mathbb{P}^1, D_n},$$
$$(f : W[\![\alpha]\!] \to A) \longmapsto \left(\tilde{\rho} : s \mapsto \begin{pmatrix} 1 & f(\alpha) \\ 1 - f(\alpha)C & -1 \end{pmatrix}, r \mapsto \begin{pmatrix} 0 & 1 \\ C & 1 \end{pmatrix}\right).$$

Note that C must be lifted to A to still satisfy $p_n(C) = 0$. This is possible because C lifts to W by the Henselian property of W.

A quick check again shows that $s^2, (sr)^2, r^n$ are mapped to the identity matrix, so these are indeed deformations. It is also an isomorphism on tangent spaces, hence the only thing left is to show formal smoothness. Again let $A' \to A$ be a small extension with kernel generated by t, let $f : W[\![\alpha]\!] \to A$ be a morphism, and let

$$s \mapsto \begin{pmatrix} 1+at & m+bt \\ 1-mC+ct & -1-at \end{pmatrix}, \quad r \mapsto \begin{pmatrix} xt & 1+yt \\ C+zt & 1+wt \end{pmatrix}$$

be a lift of the induced deformation (so $A' \ni m \mapsto f(\alpha) \in A$). Now sr maps to

$$\begin{pmatrix} 1-Cm+ct+xt+yt & -1-at-yt \\ a+C-Cm+ct+aCt+wt+zt & -1+Cm-at+bCt-wt \end{pmatrix},$$

which must have order 2, hence $c+x+y = a-bC+w$. Now after conjugating by $\begin{pmatrix} 1+(c+bC)t & 0 \\ xt & 1+at \end{pmatrix}$, we see that

$$s \mapsto \begin{pmatrix} 1 & m+bt-2qt \\ 1-C(m+bt-2qt) & -1 \end{pmatrix}, \quad r \mapsto \begin{pmatrix} 0 & 1 \\ C+(\ldots)t & 1 \end{pmatrix}.$$

Because r must have order n, we see again that the the extra term in the bottom-left must vanish, hence $r \mapsto \begin{pmatrix} 0 & 1 \\ C & 1 \end{pmatrix}$. Hence this deformation arises from the map $W[\![\alpha]\!] \to A'$ sending α to $m+bt-2qt$.

Step 4. It remains to show that η is not injective, so we consider the deformations over $k[\varepsilon]/\varepsilon^3$ sending s to $\begin{pmatrix} 1 & \varepsilon \\ 1-C\varepsilon & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \varepsilon+\varepsilon^2 \\ 1-C(\varepsilon+\varepsilon^2) & -1 \end{pmatrix}$, and in both cases r to $\begin{pmatrix} 0 & 1 \\ C & 1 \end{pmatrix}$. These are conjugated by $\begin{pmatrix} 1+\varepsilon & \varepsilon \\ C\varepsilon & 1 \end{pmatrix}$, and this show indeed that η is not an isomorphism. Hence $D_{\mathbb{P}^1, D_n}$ is non-pro-representable.

Case $G \cong (\mathbb{Z}/2)^2 = \langle a, b \mid a^2 = e, b^2 = e, (ab)^2 = e \rangle$. Again, we follow the same 4 steps.

Step 1. Again write $\rho : G \to \mathrm{PGL}_2(k)$ for the action of G, and again assume that $\rho(a) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$. Because b also has order 2, we know that $\rho(b) = \begin{pmatrix} x & y \\ z & -x \end{pmatrix}$. Now computing $\rho(ab)$ yields $\begin{pmatrix} x & y \\ x-z & x+y \end{pmatrix}$ which must have order 2. This means that $y = -2x$, and we find that $\rho(b) = \begin{pmatrix} x & -2x \\ z & -x \end{pmatrix}$. If $x = 0$, this matrix is not invertible, so we can scale such that $x = 1$, and then we have $\rho(b) = \begin{pmatrix} 1 & -2 \\ z & -1 \end{pmatrix}$. This always gives a faithful representation of G, as long as z is not 0 or 1.

Step 2. We compute the tangent space, so let $\tilde{\rho}$ be a lift to $k[\varepsilon]/\varepsilon^2$. Then

$$\tilde{\rho}(a) = \begin{pmatrix} 1+p\varepsilon & q\varepsilon \\ 1+r\varepsilon & -1-p\varepsilon \end{pmatrix}, \quad \tilde{\rho}(b) = \begin{pmatrix} 1+s\varepsilon & -2+t\varepsilon \\ z+u\varepsilon & -1-s\varepsilon \end{pmatrix}.$$

Now $\tilde{\rho}(ab)$ must also have order 2, which boils down to $p+s+qz = 2r-p-s-t$. Now we conjugate by $\begin{pmatrix} 1+(r-p)\varepsilon & \frac{u+(p-r-s)z}{z(z-1)}\varepsilon \\ 0 & 1+\frac{u+(p-r-s)z}{z(z-1)}\varepsilon \end{pmatrix}$, note that this is well-defined, as z is not 0 or 1. This results in

$$a \mapsto \begin{pmatrix} 1 & c\varepsilon \\ 1 & -1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & -2-zc\varepsilon \\ z & -1 \end{pmatrix}.$$

Again, all these deformations are not conjugated, because the a's are not conjugated. Hence there is a 1-dimensional tangent space.

Step 3. The versal deformation is now given by

$$\begin{aligned} \eta : h_{W[\![\alpha]\!]} &\longrightarrow D_{\mathbb{P}^1,(\mathbb{Z}/2)^2}, \\ (f : W[\![\alpha]\!] \to A) &\longmapsto \left(\tilde{\rho} : a \mapsto \begin{pmatrix} 1 & f(\alpha) \\ 1 & -1 \end{pmatrix}, b \mapsto \begin{pmatrix} 1 & -2-zf(\alpha) \\ z & -1 \end{pmatrix}\right). \end{aligned}$$

In this versal deformation z is arbitrarily lifted to W to give a value in A.

It's again an easy check to see that $a^2, b^2, (ab)^2$ are sent to the identity and hence that it is indeed a deformation. And it is also clearly an isomorphism on tangent spaces. So last we have to check smoothness. For this, we let $A' \to A$ be a small extension with kernel generated by t, let $f : W[\![\alpha]\!] \to A$ be a morphism, and let

$$a \mapsto \begin{pmatrix} 1+ct & m+dt \\ 1+et & -1-ct \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1+ft & -2-zf(\alpha)+gt \\ z+ht & -1-ft \end{pmatrix}$$

be a lift of the induced deformation (with $A' \ni m \mapsto f(\alpha) \in A$). We see that ab maps to

$$\begin{pmatrix} 1+mz+(c+f+dz)t & -2-m-mz+(-2c-d+g)t \\ 1-z+(e+f-h-cz)t & -1-mz+(c-2e+f+g)t \end{pmatrix},$$

which must have order 2. Therefore we find that $c+f+dz = 2e-c-f-g$. Now we conjugate by $\begin{pmatrix} 1+(e-c)t & \frac{h+(c-e-f)z}{z(z-1)}t \\ 0 & 1+\frac{h+(c-e-f)z}{z(z-1)}t \end{pmatrix}$. Under this conjugation we find that

$$a \mapsto \begin{pmatrix} 1 & m+ut \\ 1 & -1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 1 & -2-zut \\ z & -1 \end{pmatrix},$$

for some $u \in k$. We see this comes from the morphism $f' : W[\![\alpha]\!] \to A'$, sending $\alpha \mapsto m+ut$. We conclude that η is étale.

Step 4. It again remains to exhibit two equivalent deformations that come from different maps in $\mathrm{Hom}_W(W[\![\alpha]\!], -)$. For this we take the ring $A = W/16$ of characteristic 16, and we consider the deformations coming from the maps $W[\![\alpha]\!] \to A$ sending $\alpha \mapsto -2$ and $\alpha \mapsto 6$. These deformation are conjugated by $\begin{pmatrix} 5-4z(1-z) & -4-4z(1-z) \\ 2z(1-z) & 1 \end{pmatrix}$.

□

2.5 Genus 1 curves

In this case, the approaches of the previous sections don't work. It might be that the curve with the group action has infinitesimal automorphisms, in which case we cannot apply Lemma 2.6.

The idea is that curves of genus 1 have the structure of an elliptic curve (once we have chosen a point as identity) and therefore admit a group structure. This group structure allows one to construct infinitesimal automorphisms by "adding an infinitesimal point". These automorphisms will disappear if we consider deformations of triples (X, G, p) where p is a fixed k-point of X. Therefore, if we can prove that (X, G) and (X, G, p) have the same deformations, we might still obtain pro-representability. Let us now first look at the deformation functor of such triple.

Definition 2.13. Given a curve X, a group action of G on X, and a k-point p of X. A deformation of the triple (X, G, p) over $A \in \mathcal{C}$ is a deformation $\mathcal{X}_A$ of X over A together with a group action of G such that $\iota : X \to \mathcal{X}_A$ is G-equivariant, and an A-point p_A that lies over the point p of X.

An isomorphism of deformations is an A-automorphism that is compatible with the the inclusion of X, the group action, and the chosen points p_A. This defines the deformation functor $D_{X,G,p}$.

For these deformation we can define automorphism groups $\mathrm{Aut}_{A,G,p_A}(\mathcal{X}_A/\mathcal{X}_{A'})$ just as in Definition 2.4 such that they also fix the point p_A. In this case, we have the following characterisation for these automorphism groups over small extensions.

Lemma 2.14. *Let X be a curve with a group action of G, and let p be a k-point. Let $A' \to A$ be a small extension in $\mathcal{C}$ and $\mathcal{X}_{A'}$, $\mathcal{X}_A$ be deformations of the triple (X, G, p). Then we have $\mathrm{Aut}_{A',G,p_{A'}}(\mathcal{X}_{A'}/\mathcal{X}_A) \cong \mathrm{H}^0(X, \mathcal{T}_X(-p))^G$. Again, we call this the group of infinitesimal automorphisms for the triple (X, G, p).*

Proof. Just as in the proof for Lemma 2.5, we prove the isomorphism by considering affine opens of $\mathcal{X}_{A'}$. For the opens that do not contain the point p_A, the same proof holds. For an open U that contains the point p_A, we will prove that the automorphisms of this open correspond to sections of $\mathcal{T}_X(-p)$ over this open.

We recycle the notation of the proof for Lemma 2.5, using B, $\mathcal{B}_{A'}$ and $\mathcal{B}_A$ for the regular functions of X, $\mathcal{X}_{A'}$ and $\mathcal{X}_A$ over U. We have already seen that $\mathrm{Aut}_{A'}(\mathcal{B}_{A'}/\mathcal{B}_A) \cong \mathcal{T}_X(U)$ holds. Now we look what happens when we want to preserve the point $p_{A'}$. This point corresponds to a map $p_{A'} : \mathcal{B}_{A'} \to A'$ that commutes with the map $p : B \to k$.

Let θ be an automorphism of $\mathrm{Aut}_{A',p_{A'}}(\mathcal{B}_{A'}/\mathcal{B}_A)$, then it can be written as $\theta(x) = x + t \cdot d(x)$ where $d : \mathcal{B}_{A'} \to B$ is an A'-derivation. This automorphism must be compatible with the point $p_{A'}$, so we must have that $p_{A'}(x) = p_{A'}(\theta(x))$. Writing out gives $p_{A'}(x) = p_{A'}(x) + p_{A'}(t \cdot d(x)) = p_{A'}(x) + t \cdot p(d(x))$, which means that $p(d(x)) = 0$ for every $x \in \mathcal{B}_{A'}$. In other words, the derivation must vanish at the point p, which corresponds with a section of $\mathcal{T}_X(-p)$. Hence we find that $\mathrm{Aut}_{A',p_{A'}}(\mathcal{B}_{A'}/\mathcal{B}_A) \cong \mathcal{T}_X(-p)(U)$, and gluing these isomorphisms yields $\mathrm{Aut}_{A',p_{A'}}(\mathcal{X}_{A'}/\mathcal{X}_A) \cong \mathrm{H}^0(X, \mathcal{T}_X(-p))$. This isomorphism is G-equivariant, hence we obtain that $\mathrm{Aut}_{A',G,p_{A'}}(\mathcal{X}_{A'}/\mathcal{X}_A) \cong \mathrm{H}^0(X, \mathcal{T}_X(-p))^G$. ☐

Proposition 2.15. *Let X be a curve of genus 1, G a group which acts faithfully on X, and p a point on X. Then the deformation functor $D_{X,G,p}$ is pro-representable.*

Proof. Because X has genus 1, the sheaf $\mathcal{T}_X(-p)$ has degree -1 and therefore the cohomology $\mathrm{H}^0(X, \mathcal{T}_X(-p))^G$ is trivial. This implies by Lemma 2.14 that $\mathrm{Aut}_{A',G,p_{A'}}(\mathcal{X}_{A'}/\mathcal{X}_A)$ is trivial.

We are now almost in the situation to apply Lemma 2.6, but our deformations also include a point on the curve X. However, the proof can easily be adapted by finding a point on the fibre sum deformation, and checking that the proof of unicity goes the same when all deformations are pointed. With these adaptations we conclude that $D_{X,G,p}$ is pro-representable. □

Now follows an attempt at proving $D_{X,G}$ is pro-representable.

Attempted proof at pro-representability of $D_{X,G}$. In the case at hand, the tangent sheaf $\mathcal{T}_X$ is isomorphic to the structure sheaf, so $\mathrm{H}^0(X, \mathcal{T}_X)$ has dimension 1. Now we consider the following two cases: G acts trivially on $\mathrm{H}^0(X, \mathcal{T}_X)$, or not. In the second case, we see that $\mathrm{H}^0(X, \mathcal{T}_X)^G$ is a strict subspace of the 1-dimensional $\mathrm{H}^0(X, \mathcal{T}_X)$, so it is trivial. Hence Lemma 2.6 applies and $D_{X,G}$ is pro-representable.

Now we may assume that G acts trivially on $\mathrm{H}^0(X, \mathcal{T}_X)$. Because we work over an algebraically closed field k, we know that X has a k-point p. Then we will prove that $D_{X,G,p}$ and $D_{X,G}$ are isomorphic. There is an obvious natural transformation $\eta : D_{X,G,p} \to D_{X,G}$, which maps a deformation of (X, G, p) to a deformation of (X, G) by forgetting the lift of p. We prove that it is surjective and attempt to prove that it is injective, from which the isomorphism would follow.

First we prove surjectivity, so take a deformation $\mathcal{X}_A$ of the pair (X, G). First of all, the morphism $\mathcal{X}_A \to \mathrm{Spec}(A)$ is flat and the only geometric fibre above $\mathrm{Spec}(k)$ is X, which is regular. Therefore the morphism is smooth [Har77, Theorem III.10.2], and also formally smooth. Now the point p gives a morphism $\mathrm{Spec}(k) \xrightarrow{p} X \hookrightarrow \mathcal{X}_A$ that fits in the following diagram of solid arrows:

$$\begin{array}{ccc} \mathrm{Spec}(k) & \longrightarrow & \mathcal{X}_A \\ \big\downarrow & \nearrow & \big\downarrow \\ \mathrm{Spec}(A) & = & \mathrm{Spec}(A) \end{array} .$$

The map $\mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(A)$ factors into small extension where each kernel has square zero. Then the formal smoothness of $\mathcal{X}_A \to \mathrm{Spec}(A)$ implies that there is a lift $p_A : \mathrm{Spec}(A) \to \mathcal{X}_A$ (dotted arrow in the diagram). Now $\mathcal{X}_A, p_A$ is a deformation of the triple (X, G, p) and $\eta(\mathcal{X}_A, p_A) = \mathcal{X}_A$. Hence η is indeed surjective.

For injectivity, suppose we have a deformation $\mathcal{X}_A$ of X over A, with two A-points p_A, p'_A lifting p. Because the map $\mathcal{X}_A \to \mathrm{Spec}(A)$ is smooth, and the geometric fibre X has genus 1, the scheme $\mathcal{X}_A$ also has a group structure over A with p_A as the identity [KM85]. With this group structure, there is an A-automorphism of the scheme $\mathcal{X}_A$ that translates p'_A to p_A. Call this translation τ, then we have that $\tau p'_A = p_A$. To check that

τ is also an automorphism of the deformation $\mathcal{X}_A$, we have to check that it is the identity on X, and that it is compatible with the group action of G on $\mathcal{X}_A$.

For the first, consider the induced automorphism $\tilde{\tau}$ of X. If we denote the inclusions $\iota_k : \mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(A), \iota_X : X \hookrightarrow \mathcal{X}_A$, then we see that

$$\iota_X p = p_A \iota_k = \tau p'_A \iota_k = \tau \iota_X p = \iota_X \tilde{\tau} p.$$

Because ι_X is mono, we see that $p = \tilde{\tau} p$. Now we see that $\tilde{\tau}$ is a translation of X that sends p to itself, hence it must be an isomorphism.

To complete the proof of injectivity, one still needs to check that τ commutes with the group action of G on the deformation $\mathcal{X}_A$. □

The last step, showing that this translation τ commutes with the lifted group action of G seems impossible to prove. If G consists of only translations, and the lifted group actions are also translation, they will commute. So we are reduced to the following question:

Question 2.16. Let $\tau : X \to X$ be a translation of an elliptic curve X, and let $\tau_A : \mathcal{X}_A \to \mathcal{X}_A$ be an automorphism of the deformation $\mathcal{X}_A$ that restricts to τ. Must τ_A be a translation of the elliptic curve $\mathcal{X}_A$?

If this is true, the above approach deals with all cases where G contains only translations. But if G has some other automorphisms, its lifts will not be translations and there should be no reason that the lifted group action commutes with the translation τ.

This means that the approach will fail if the curve X has an automorphism that is not a translation, but which acts trivially on $\mathrm{H}^0(X, \mathcal{T}_X)$. We will now determine the curves having such an automorphism. Because we can always choose a point O on the curve X, we may assume that (X, O) is an elliptic curve. Now if φ is such an automorphism of X, we can consider the morphism $P \mapsto \varphi(P) - P$ from X to X. Because it is a map between smooth projective curves, it is either constant or surjective. If it is constant, φ is a translation. If it is surjective, there will be a point Q such that $\varphi(Q) - Q = O$, i.e. $\varphi(Q) = Q$. Hence choosing Q is the identity makes φ an automorphism of the elliptic curve (X, Q). We are interested in these automorphisms.

Lemma 2.17. *Let (X, O) be an elliptic curve over an algebraically closed field k. If (X, O) has a non-trivial automorphism acting trivially on $\mathrm{H}^0(X, \mathcal{T}_X)$, we are in one of the following cases:*

- $\mathrm{char}(k) = 3$, *X is isomorphic to the curve $y^2 = x^3 - x$, and the automorphism is given by $(x, y) \mapsto (x+1, y)$ or $(x, y) \mapsto (x+2, y)$. The corresponding automorphism group is $\mathbb{Z}/3$.*
- $\mathrm{char}(k) = 2$, *X is isomorphic to the curve $y^2 + xy = x^3 + c$ for some $c \in k^\times$, and the automorphism is given by $(x, y) \mapsto (x, y+x)$. The corresponding automorphism group is $\mathbb{Z}/2$.*
- $\mathrm{char}(k) = 2$, *X is isomorphic to the curve $y^2 + y = x^3$, and the automorphism is given by $(x, y) \mapsto (x + s^2, y + sx + t)$ for $s, t \in k$ satisfying $s^4 = s$ and $t^2 + t = s^3$. The corresponding automorphism group is the quaternion group Q_8.*

NB: we only give the equation on an affine part. The actual elliptic curve is its closure in $\mathbb{P}^2$*, with the unique point* $(0:1:0)$ *on the line at infinity being the identity.*

Proof. We write the elliptic curve (X, O) in Weierstrass form. When $\mathrm{char}(k) \neq 2, 3$ we can even take the form $y^2 = x^3 + Ax + B$. By the isomorphism $\mathcal{T}_X \cong \Omega_X$, the global section of the tangent sheaf corresponds to the global 1-form dx/y (called the invariant differential in [Sil09]). So we are looking for automorphisms that preserve this 1-form. Any automorphism is of the form $(x, y) \mapsto (u^2x, u^3y)$ (see [Sil09, Proposition III.3.1]). However, this maps $dx/y \mapsto u^{-1}dx/y$ and this action is trivial if and only if $u = 1$. So in this cases, only the identity acts trivially on $\mathrm{H}^0(X, \mathcal{T}_X)$.

For the $\mathrm{char}(k) = 2, 3$ cases, we use the normalized forms of [Sil09, Proposition A.1.1] and we consider the cases one by one. (Note there is a small error: in the case $\mathrm{char}\, K = 2$ and $j = 0$, the discriminant must be a_3^4.) Proposition A.1.2 even gives all automorphisms. For $\mathrm{char}(k) = 3$ with j-invariant $j(X, O) \neq 0$, we have the Weierstrass form $y^2 = x^3 + a_2x^2 + a_6$. Every automorphism is of the form $(x, y) \mapsto (u^2x, u^3y)$ and this does not preserve the 1-form dx/y unless $u = 1$. For $\mathrm{char}(k) = 3, j(X, O) = 0$ we see that (X, O) is isomorphic to $y^2 = x^3 - x$ because they are elliptic curves over an algebraically closed field with the same j-invariant. The automorphisms of this curve are given by $(x, y) \mapsto (u^2x + r, u^3y)$. Such an automorphism acts in the global 1-form as $dx/y \mapsto u^{-1}dx/y$, so we must have $u = 1$. Further the r must satisfy $r^3 = r$, so $r = 0, 1, 2$. This gives the first curve with automorphism group $\mathbb{Z}/3$.

For $\mathrm{char}(k) = 2$, $j(X, O) \neq 0$, we have the Weierstrass form $y^2 + xy = x^3 + a_2x^2 + a_6$ with j-invariant $1/a_6$. Because the j-invariant determines the isomorphism class of an elliptic curve over an algebraically closed field, we may assume that $a_2 = 0$. The only non-trivial automorphism is $(x, y) \mapsto (x, y + x)$, and this preserves the global 1-form dx/x. These yield the curves of the second case with automorphism group $\mathbb{Z}/2$.

For $\mathrm{char}(k) = 2$, $j(X, O) = 0$, we see that (X, O) is isomorphic to $y^2 + y = x^3$ because both have j-invariant 0. Every automorphism is of the form $(x, y) \mapsto (u^2x + s^2, u^3y + u^2sx + t)$. The global 1-form is in this case given by dx, and the automorphism acts trivial if and only if $u = 1$. Further the automorphism must satisfy $s^4 = s$ and $t^2 + t = s^6 = s^3$. This gives the curve of the third case and an automorphism group of order 8. All automorphisms with $s \neq 0$ have order 4, so the automorphism group is the quaternion group Q_8. □

Even if the answer to Question 2.16 is yes, we cannot deal with this curves by the 'fixing a point-trick'. Going to a local deformation as in Section 2.3 also seems unfruitful, because the method assumed a weakly ramified local group action. For the first and third case this might not be the case, as these curves are not ordinary [CK03]. For the second case, going to the local group action is useless as we know that it is non-pro-representable. It seems therefore that a new approach is needed to determine the pro-representability of these deformation functors.

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