Universiteit Utrecht

Faculteit Betawetenschappen

Equatorial waves

Bachelor thesis

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Opleiding: Natuur-en Sterrenkunde

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Januari 2017

Abstract

The common theoretical framework that describes equatorial ocean waves employs the traditional approximation. In this approximation the horizontal component $\tilde{f}$ of the Coriolis force is neglected and only the vertical component f is included. For mid to high latitudes, this approach may be justified, however at the equatorial region the vertical component f dissappears. In this work we examine equatorial waves under the traditional approximation and for a general case, i.e. under the full Coriolis force. Solutions for both theories are compared and the limitations are discussed. We find that the solutions for the traditional approximation cannot be written as the limiting case of a more general theory. Furthermore zonally symmetric solutions for the general case strongly resemble the observed $m = 2$ mode of the classical theory. Observations of $m = 2$ can thus be equally well described by a more general theory. For internal waves exlusion of $\tilde{f}$ results in markedly different expressions for the turning surface and dispersion relation. Employing the traditional approximation can be seen as oversymmetrizing the problem by aligning the rotation vector and gravity.

Contents

January 18, 2017

0.1 Introduction

The equatorial ocean is the region spanning $\pm 2.5° N, S$ from the equator. This latitudinal extent is dwarfed by the longitudinal reach of the equatorial ocean which reaches over 120° for the Pacific.

The dynamics of the equatorial region are often described by use of an approximation referred to as the *traditional* approximation. This traditional approximation neglects the horizontal component of the Earth's rotation vector. As such it is a particularly coarse approximation to make at the equatorial regions where the vertical component $2\Omega sin\phi$ of the rotation vector dissapears. For a thorough understanding of equatorial waves we may require a more robust theoretical framework.

Our aim in this work is twofold. First we examine and compare the two theoretical frameworks with which to describe ocean waves in the equatorial region. Limitations of both theoretical frameworks are discussed. Secondly we attempt to examine recent observational work in light of the two theoretical frameworks. This is a rather difficult task on account of the following: observational data on long scale ocean waves is often noisy and thus requires significant preprocessing in order to be interpretable. This preprocessing of course assumes some theoretical expectations which "shapes" as it is the resultant data.

Chapter 1

Geophysical flows: a short introduction

This chapter presents a short introduction into geophysical fluid dynamics. The governing equations are stated and some necessary background is provided.

Geophysical flows are characterised by long time and length-scales, put otherwise geophysical flows are those flows that are influenced by the rotation of the Earth and stratification of the fluid. Since the subject of this paper is ocean waves, the "fluid motion" we refer to is of course sea water and the motion considered takes place in ocean basins. However the theory describes atmospheric waves equally well with some minor modifications. Ocean waves encompass all oscillatory motions on the surface and interior of the oceans. From short scale capillary waves with periods less than a second to the low frequency Rossby waves spanning multiple longitudes. The focus in this work is on motion in the *long wave* limit, that is waves with a period larger than 10 s and wavelength greater than 150 m. Hence we do not consider capillary waves and similar short period waves.

In the sections below we derive firstly the equation of motion on a rotating sphere, after which we briefly touch upon the stratification of fluids. This allows us to state basic equations governing geophysical flows. Lastly we state some common approximations.

1.1 Motion on a rotating sphere

The dynamics of ocean waves are described by modelling the Earth as a rotating sphere with a thin stratified fluid layer. This fluid layer, i.e. the oceans, is "thin" since the average ocean depth $H \approx 4km$ is orders of magnitude smaller than the radius of the Earth at $R_\oplus = 6371$ km.

The rotation of the Earth gives rise to acceleration terms that can be interpreted as forces, referred to as pseudo-forces. The *Coriolis* force, one of the resultant pseudo-forces, plays a major role in modifying geophysical flows and is central to this work. Thus a detailed explanation of the nature of the Coriolis force is required which we present in this section. Below we examine what shape the acceleration of a particle takes in a rotating frame of reference. This will enable us to state the governing equations of motion for a fluid parcel.

Consider a right-handed orthogonal frame rotating around a fixed axis with rotation vector $\Omega = f(X_0, t).\Omega_0$ and magnitude $|\Omega| = 7,29.10^{-5} rad/s$.

For a point $\mathbf{x_r}$ moving relative to the rotating frame the velocity in the fixed or intertial frame is given by

$$\frac{d\mathbf{x_f}}{dt} = \frac{d\mathbf{x_r}}{dt} + \mathbf{\Omega} \times \mathbf{x_r} \tag{1.1}$$

The subscripts f and r refer to the fixed frame and the rotating frame respectively. The acceleration is

$$\frac{d^2\mathbf{x}_f}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{x}_r}{dt} + \mathbf{\Omega} \times \mathbf{x}_r\right) + \mathbf{\Omega} \times \left(\frac{d\mathbf{x}_r}{dt} + \mathbf{\Omega} \times \mathbf{x}_r\right) \tag{1.2}$$

which can be written as

$$\frac{d^2\mathbf{x}_f}{dt^2} = \frac{d^2\mathbf{x}_r}{dt^2} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x}_r) + 2\mathbf{\Omega} \times \frac{d\mathbf{x}_r}{dt} \tag{1.3}$$

The first term is the Euler force and arises due to changes in the angular velocity of the rotating system. These changes are of the order of $O10^{-iets}$ for the Earth and hence may be neglected.

The second term is the centrifugal force which points outwards perpendicularly to the rotation axis. This term can be rewritten by the introduction of the geopotential Φ which we define as the sum of the centrifugal potential and the gravitational potential U.

$$definitiegeopotential \tag{1.4}$$

Recall that the curl of the curl of a vector can be written as the gradient of a scalar, hence $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x_r}) = \nabla \frac{\Omega^2 x_r^2}{2}$. Employing this and the geopotential we may write (1.3) as follows

$$\frac{d^2 x_f}{dt^2} = \frac{d\mathbf{u_f}}{dt} = \frac{D\mathbf{u}}{Dt} + 2\mathbf{\Omega} \times \mathbf{u} \tag{1.5}$$

where we have introduced the convective derivative $\frac{D}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u}$ The first term of the convective derivative gives the local velocity changes $\frac{\partial \mathbf{u}}{\partial t}$ the second term gives the advective terms $\mathbf{u}.\nabla\mathbf{u}$.

The centrifugal acceleration/geopotential is subsumed into the advective terms of the convective derivative.

Two of the three resulting pseudo-forces in (1.3) are accounted for, this leaves the final term which is the Coriolis force This pseudoforce is a result of motion relative to the co-rotating frame. Note that the Coriolos force simply changes the direction of motion and does no work.

To simplify (1.5) we introduce the Coriolis parameters f and $\tilde{f}$. We start by decomposing the rotation vector $\mathbf{\Omega}$ in a local Cartesian frame with the following correspondences. The x-axis corresponds to the zonal (east-west) direction, the y-axis is the meridional (north-south) direction and the z-axis is the depth. The decomposition of Ω is

$$\mathbf{\Omega} = \begin{pmatrix} 0 \\ 2\Omega cos\phi \\ 2\Omega sin\phi \end{pmatrix} \tag{1.6}$$

where ϕ is the latitude with $\phi = 0$ at the equator. We may write the above in terms of the aforementioned Coriolis parameters, $f = 2\Omega sin\phi$ and $\tilde{f} = 2\Omega cos\phi$.

$$2\mathbf{\Omega} \times \mathbf{u} = 2\Omega \begin{pmatrix} -vsin\phi + wcos\phi \\ usin\phi \\ -ucos\phi \end{pmatrix} = \begin{pmatrix} -fv + \tilde{f}w \\ fu \\ -fu \end{pmatrix} \tag{1.7}$$

Generally the horizontal component of the Coriolis force, $\tilde{f}$, is neglected, this constitues what is often called the "traditional approximation", hereafter referred to as the TA. Employing the TA is justified in several ways. First the contribution of $f = 2\Omega cos\phi$ is rather small at mid and higher latitudes compared to the effect of f.

Secondly, by neglecting this component analytical solutions for most situations are easier to obtained. This is examined in depth in chapter 2.

1.2 Stratification

A stratified fluid is a fluid with density variations in the vertical. Both the ocean and atmosphere are stratified fluids. The basic measure of stratification is the Brunt–Väisälä frequency or buoyancy frequency N.

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dz} \tag{1.8}$$

A fluid parcel displaced vertically will oscillate with N. The larger N the more stable the fluid is, $N < 0$ signifies unstable stratification. In most models constant stratification is assumed.

1.3 The basic equations

In the previous section we derived the equation of motion for a particle in a rotating frame. To fully describe wave dynamics we require several other formulas for the fundamental physical laws of conservation of momentum, mass and energy. Conservation of momentum follows from including the pressure and buoyancy forces to (1.5). Conservation of mass for a fluid parcel, also know as the continuity equation is

$$\frac{\partial \rho}{\partial t} + \rho \nabla . \mathbf{u} = 0 \tag{1.9}$$

for an incompressible fluid $\nabla . \mathbf{u} = 0$.

As this work is focused primarily on dynamics we will not consider the salinity equation and other equations which describe the thermodynamic state of the fluid.

The full set of governing equations for a parcel of fluid with density ρ and velocity $\mathbf{u} = (u, v, w)$ on a rotating sphere is

$$\frac{D\mathbf{u}}{dt} + 2\Omega \times \mathbf{u} = -\frac{1}{\rho}\nabla p + \nabla \Phi + \mathcal{F} \tag{1.10}$$

$$\frac{\partial \rho}{\partial t} + \rho \nabla . \mathbf{u} = 0 \tag{1.11}$$

where $\mathcal{F}$ is a forcing term and p is the pressure. When written out equation (1.9) gives three momentum equations for the u, v and w components of the velocity vector $\mathbf{u}$. We have included a forcing term for the sake of giving a complete picture however most of the solutions we will examine in this work are unforced.

The gradient of the geopotential ∇g = Simplification of the full set of governing equations (1.9)-(1.10) is necessary to obtain analytical and numerical results.

The study of geophysical flows is abound with approximation methods and it is beyond the scope of this work to summarize most of them. In the section below only those approximation methods most relevant for this paper are stated. Other methods such as non-dimensionalization/scale analysis and linearization assumptions are explained when used.

1.3.1 The Boussinesq approximation and the hydrostatic approximation

In the ocean the relative fluid density difference is of the order of $O\ 10^{-3}$ This justifies the assumption that the density fluctuations around an average value are very small. Under the Boussinesq approximation the density ρ is seperated in a mean reference value ρ_0 that is constant in time and a variation $\rho'(x, y, z, t)$

$$\rho = \rho_0 + \rho'(x, y, z, t)\text{with}|\rho'| << \rho_0 \tag{1.12}$$

which we subtitute in the continuity equation (1.8). Note that the terms including ρ' are much smaller than those involving ρ_0, hence we may drop these much smaller terms. Since the mean density is assumed constant $\frac{\partial \rho_0}{\partial t}$ vanishes and obtain the following simplified form of the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial x}{\partial z} = 0 \tag{1.13}$$

With regards to the pressure we employ the hydrostatic approximation, i.e. we consider the fluid in hydrostatic balance. As the horizontal scale of the motion under consideration is much greater than the vertical scale this assumption is justified in most cases. For a fluid in hydrostatic balance the vertical pressure gradient is $\frac{dp}{dz} = -\rho g$.

1.3.2 The β-plane approximation

The β-plane approximation consists of the linearization of the trigonometric terms of the Coriolis parameters such that f and $\tilde{f}$ vary linearly with latitude. For details on this approximation and the justification hereof the reader is referred to "Blablbala" (BLABLBLA).

To obtain a Coriolis parameter that varies linearly with latitude we perform a Taylor expansion of the Coriolis parameters around a central latitude $\phi_0 = 0$ at the equator.

$$f = 2\Omega(\phi + O(\phi^3)) = y\beta + O(y^3) \tag{1.14}$$

$$\tilde{f} = 2\Omega(1 + O(\phi^2)) = \tilde{f}_0 \tag{1.15}$$

where y is the meridional distance from the equator ϕ_0 and $\beta = \frac{2\Omega}{a}$ with a the radius of the Earth. β is the meridional gradient of f at the equator. On the "traditional" β-plane $f = \beta y$, the "nontraditional" β-plane refers to $f = \beta y + \tilde{f}_0$.

1.4 Summary

To summarize we state the full set of governing equations for a Boussinesq approximated fluid with the complete Coriolis force on a non-traditional β-plane. In future sections we will not state the variable dependence of the velocity vector, assumed throughout is $\mathbf{u}(x, y, z, t) = \begin{pmatrix} u(x, y, z, t) \\ v(x, y, z, t) \\ z(x, y, z, t) \end{pmatrix}$

$$\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} - fv + \tilde{f}w = -\frac{1}{\rho_0} \tag{1.16}$$

$$\frac{\partial v}{\partial t} - v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho_0} \tag{1.17}$$

$$\frac{\partial w}{\partial t} - w\frac{\partial w}{\partial z} - fu = -\frac{1}{\rho_0} \tag{1.18}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1.19}$$

A final remark on linearization. The equations of motion (1.15)-(1.17) are oft linearized. Linearization, i.e. neglecting the advective terms, is justified as the wave amplitudes under consideration are small disturbances of a ground state.

Chapter 2

Equatorial waves

Equatorial wave dynamics

2.1 Classical equatorial wave theory

In this section we examine the classical equatorial wave theory for surface and interfacial waves which neglects the horizontal component of the Coriolis force. The derivation of the governing equations for equatorial waves largely follows the canonical paper by Matsuno (1966).
However as Matsuno's work is for the atmospheric case we introduce some adjustments to make it more in line with the oceanographic case.

2.1.1 Governing equations

Consider a divergent barotropic model on the equatorial β-plane, i.e. a 1-layer incompressible and homogenous $N = 0$ fluid with a free surface at hydrostatic balance.
The horizontal component of the Coriolis force/parameter $\tilde{f}$ = is neglected. Denote the mean value of the surface by H and the small deviations of this mean by h. The boundary conditions at the surface are

$$p|_{z=0} = gh \tag{2.1}$$

$$w|_{z=0} = \frac{\partial h}{\partial t} \tag{2.2}$$

Employing all the above we obtain the following linearized *shallow water* equations on an equatorial β-plane

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial h}{\partial x} = 0 \tag{2.3}$$

$$\frac{\partial v}{\partial t} + fu + g\frac{\partial h}{\partial y} = 0 \tag{2.4}$$

$$H\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + \frac{\partial h}{\partial t} = 0 \tag{2.5}$$

where g is the gravitational acceleration and $f = \beta y$ the Coriolis parameter.

While the shallow water equations given above for a 1-layer model are adequate for the atmosphere they are too simplistic a model for ocean dynamics. Hence we modify the model slightly by considering a *reduced gravity* model, also known as a $1\frac{1}{2}$ layer model. In such a model a thin layer of warm water lies atop an infinitely deep cold motionless layer and is seperated by a sharp thermocline. The interface between the two layers has a neutral position denoted by D' and the position at a given time is given by a field $\xi(x, y, t)$. The free surface of the top layer is $\eta(x, y, t)$ and the variable thickness h of this layers is $h(x, y, t) = \eta(x, y, t) + D' - \xi(x, y, t)$. Denote the density in layer one by ρ_* and in layer 2 as $\rho_* + \Delta\rho$. The hydrostatic equation for layer two with pressure p_2 reads

$$\frac{\partial p_2}{\partial z} = -(\rho_* + \Delta\rho)g \tag{2.6}$$

Clearly at the interface the following boundary conditions must hold

$$p_1|_{z=D'+\xi} = p_2|_{z=-D'+\xi} \tag{2.7}$$

$$\frac{D(z + D' - \xi)}{Dt}|_{z=-D'+\xi} = 0 \tag{2.8}$$

If we evaluate this condition and insert zero velocity for the second layer we see that the horizontal gradient ∇_H of p_2 is zero

$$p_2 = -g(\rho_* + \Delta\rho)(z + D' - \xi) + \rho_* g h + p_{atm} \tag{2.9}$$

$$\nabla_H \xi = -\frac{\rho_*}{\Delta\rho}(\nabla_H \eta) \tag{2.10}$$

Thus variations in the free surface level correspond to variations in the thermocline level that are much larger.

The equations for the reduced gravity model are

$$\frac{\partial u}{\partial t} - fv + g'\frac{\partial h}{\partial x} = 0 \tag{2.11}$$

$$\frac{\partial v}{\partial t} + fu + g'\frac{\partial h}{\partial y} = 0 \tag{2.12}$$

$$D'\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big) + \frac{\partial h}{\partial t} = 0 \tag{2.13}$$

Note that these equations are similar to the shallow water equations (2.3)-(2.5) and differ only in the replacement of the gravity g by the reduced gravity $g' = \frac{\Delta\rho}{\rho_*}g$, $h \to \eta$ and $H \to D'$. This simple modification to the shallow water equations results in velocity values that are more realistic.

The equations (2.11)-(2.13) require some algebraic manipulation such that solutions may be found.

First let us crossdifferentiate (2.11) and (2.12) to obtain the time-derivative of the vertical component of $\nabla \times \mathbf{u}$

$$\frac{\partial}{\partial t}\Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{f}{D'}h\Big) = 0 \tag{2.14}$$

We may eliminate u and h resulting in

$$\frac{\partial}{\partial t}\Big(\frac{\partial^2 v}{\partial t^2} - c^2\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\Big) + \beta^2 y^2 v\Big) - c^2\beta\frac{\partial v}{\partial x} = 0 \tag{2.15}$$

with $c^2 = D'g'$ the squared velocity of pure gravity waves in a reduced gravity model. The above can be rewritten as a single equation for the meridional structure $V(y)$ for plane-wave solutions of the form $v = V(y)e^{i(kx-\omega t)}$

$$\frac{d^2V}{dy^2} + (\frac{\omega}{c}^2 - k^2 - \frac{k\beta}{\omega} - \frac{\beta}{c}^2 y^2)V = 0 \tag{2.16}$$

The above equation is made nondimensional by scaling with

$$\hat{y} = \sqrt{\frac{\beta}{c}y} \qquad \hat{\omega} = \frac{\omega}{\sqrt{\beta c}} \qquad \hat{k} = \sqrt{\frac{c}{\beta}}k \tag{2.17}$$

For the remainder of this chapter we only consider non-dimensional variables so we immediatly drop the hat notation. The nondimensional equation reads

$$\frac{d^2V}{dy^2} + \left(\omega^2 - k^2 + \frac{k}{\omega} - y^2\right)V = 0 \tag{2.18}$$

As we are only interested in the equatorial region we require that the wave motion approaches zero for poleward latitudes, i.e. $v \to 0$ for $y \to \pm\infty$. Equation (14) satisfies this boundary condition only when

$$\omega^2 - k^2 + \frac{k}{\omega} = 2m + 1 \tag{2.19}$$

where m is an integer, we will refer to m as the meridional mode of the solution. The reader may recognize the preceding as the Schrodinger equation for the simple harmonic oscillator which has the solution

$$V(y) = Ce^{\frac{-y^2}{2}} H_m(y) \tag{2.20}$$

This is the parabolic cylinder function where $\mathcal{H}_m(y)$ are the Hermite polynomials of order m. The first four parabolic cylinder functions which correspond to the meridional structure $V_m(y)$ are shown in figure 2.1. Note that the waves are confined to some $R_{eq} =$ $c\beta$ from the equator.

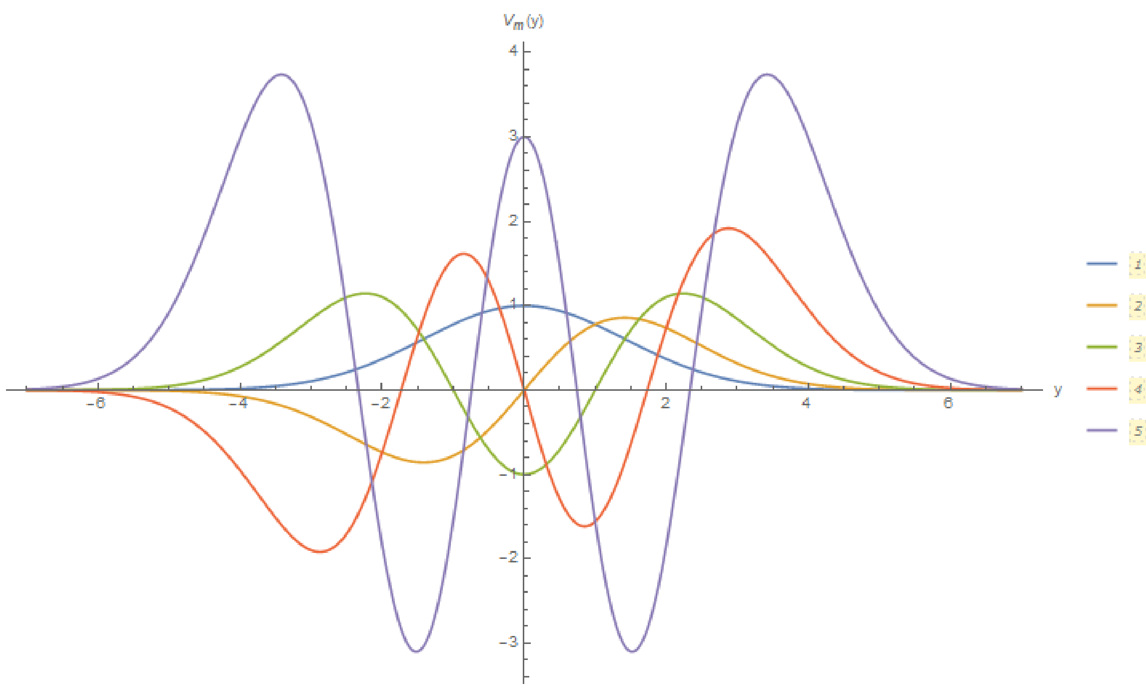

Figure 2.1: The parabolic cylinder functions governing the meridional structure $V_m(y) = e^{\frac{-y^2}{2}}\mathcal{H}_m(y)$ for the first five modes. For even m the functions are even and maxima occur at the equator. For odd m the functions are odd and maxima occur off the equator.

2.1.2 Eigenfunctions

One can obtain eigensolutions for the zonal velocity u and the height h by rewriting (2.11) and (2.13) with the assumption that the time dependence is $e^{-i\omega t}$.

$$u = \frac{1}{i(\omega^2 - k^2)}\left(\omega y v + k\frac{\partial v}{\partial y}\right) \tag{2.21}$$

$$h = \frac{1}{-i(\omega^2 - k^2)}\left(k y v + \omega\frac{\partial v}{\partial y}\right) \tag{2.22}$$

The meridional velocity is $v = e^{i(kx-\omega t)}V_m(y)$. To obtain the derivative of v one can use the recurrence relation for Hermite polynomials.

$$\frac{dH_m(y)}{dy} = 2mH_{m-1}(y) \tag{2.23}$$

$$H_{m+1}(y) = 2yH_m(y) - 2mH_{m-1}(y) \tag{2.24}$$

Combining the above leads to the following eigenfuntions for the meridional velocity v, zonal velocity u and height h

$$\begin{pmatrix} v \\ u \\ h \end{pmatrix}_{ml} = \begin{pmatrix} i(\omega_{ml}^2 - k^2)\psi_m \\ \frac{1}{2}(\omega_{ml} - k)\psi_{m+1} + m(\omega_{ml} + k)\psi_{m-1} \\ \frac{1}{2}(\omega_{ml} - k)\psi_{m+1} - m(\omega_{ml} + k)\psi_{m+1} \end{pmatrix} \tag{2.25}$$

with $\psi_m = e^{\frac{-y^2}{2}}H_m(y)$. The subscript m denotes the mode while the subscript l denotes the root of the dispersion relation.

Note that when m is odd v is odd and u and h are even. For m is even v is even and u and h are odd. These symmetry properties of the velocity fields and pressure field are often invoked when analyzing observational data.

The eigenfunctions in (2.25) form an orthogonal complete system. To see this rewrite (2.25) as $\Omega\xi + i\omega\xi = 0$ where Ω and ξ are

$$\Omega = \begin{pmatrix} 0 & -y & ik \\ y & 0 & \frac{d}{dy} \\ ik & \frac{d}{dy} & 0 \end{pmatrix} \tag{2.26}$$

$$\xi = \begin{pmatrix} u \\ v \\ h \end{pmatrix} \tag{2.27}$$

The matrix Ω is skew-Hermitian. The eigenfunctions of a skew-Hermitian matrix are orthogonal. To prove completeness simply recall that the Hermite funtions are complete.

2.1.3 Equatorial Kelvin waves

Initially we have demanded that $m > 0$ to obtain (2.18) since we have assumed that $v_m(y) \neq 0$.

However $m = -1$ also gives rise to an acceptable solution which corresponds to an eastward propagating Kelvin wave described by

$$V_m(y) = 0 \tag{2.28}$$

$$u_m = V_m(y)e^{i(kx-\omega t)} \tag{2.29}$$

We may also obtain the Kelvin wave by directly input $v(x, y) \equiv 0$ in the governing equations

$$i\omega u + ikh = 0 \tag{2.30}$$

$$yu + \frac{\partial h}{\partial y} = 0 \tag{2.31}$$

$$i\omega u + iku = 0 \tag{2.32}$$

This set of equations have solutions only when $(\omega - k)(\omega + k) = 0$ For $\omega = -k$ and $\omega = k$ the solutions are respectively

$$u = h = Ce^{\frac{y^2}{2}} \tag{2.33}$$

$$-u = h = Ce^{\frac{y^2}{2}} \tag{2.34}$$

Note that (2.32) does not satisfy the boundary condition hence it must be discarded.

2.1.4 Dispersion relation

In the sections above, analytical solutions for "classical" equatorial waves were obtained. We have seen that the amplitude for equatorial waves is confined to some R_{eq} from the equator. In this section we examine the disprsion relation to gain more insight into the properties of the solutions in (2.25) To that end we examine the frequency equation (2.19) and the resultant dispersion relations which are plotted in figure 2.3.

Each mode $m > 0$ has three distinct roots for the frequency $\omega_i(k)$, $i = 1, 2, 3$ which corresponds to two gravity waves and one Rossby wave as can be seen in figure 2.2.

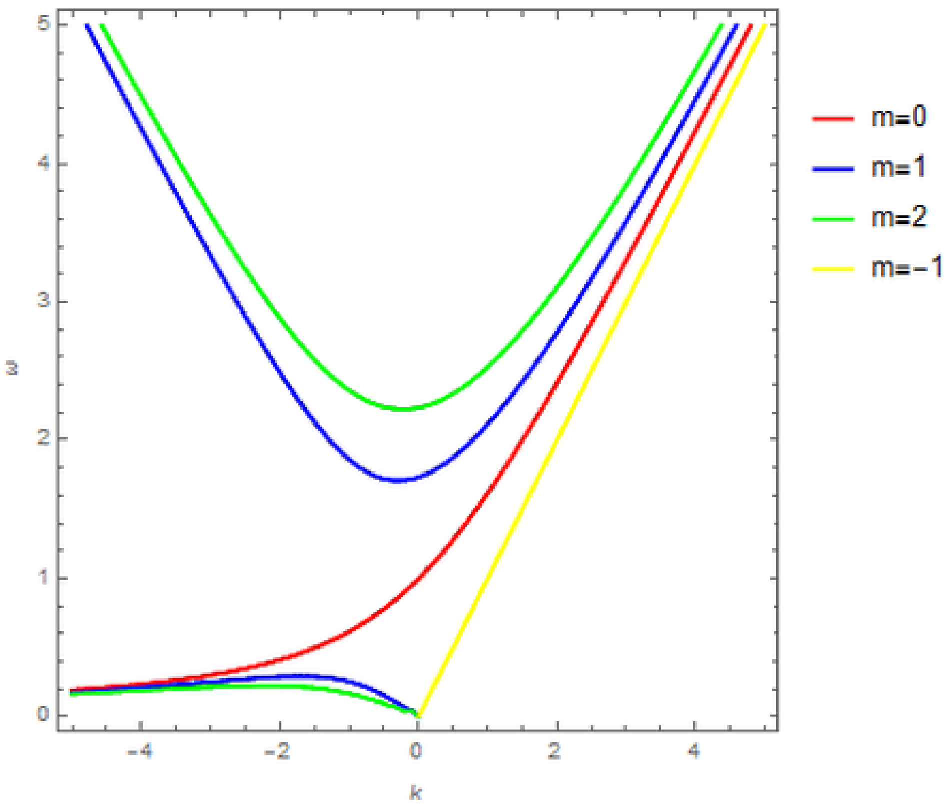

Figure 2.2: Dispersion relations for $m = -1, 0, 1, 2$

Of primary interest to us are the dispersion relations for the first three meridional modes which shown in figure 2.2

For the lowest meridional mode $m = 0$ equation (2.25) can be factored into:

$$(\omega - k)(\omega^2 + k\omega - 1) = 0 \tag{2.35}$$

Solutions are obtained by considering m as a continous parameter

$\omega_i(k, 0) = \lim_{m\to 0)\omega_i(k,m)}$, where $i = 1, 2, 3$ denotes the solution to the cubic frequency equation. Employing the above these roots of (2.19) are

$$\omega_1 = -\frac{k}{2} - \sqrt{\frac{k}{2}^2 + 1} \tag{2.36}$$

$$\omega_2 = \begin{cases} \sqrt{(\frac{k}{2})^2 + 1} - \frac{k}{2} & (\text{for} \quad k \leq \frac{1}{\sqrt{2}}) \\ k & (\text{for} \quad k \geq \frac{1}{\sqrt{2}}) \end{cases} \tag{2.37}$$

$$\omega_3 = \begin{cases} k & (\text{for} \quad k \leq \frac{1}{\sqrt{2}}) \\ \sqrt{(\frac{k}{2})^2 + 1} - \frac{k}{2} & (\text{for} \quad k \geq \frac{1}{\sqrt{2}}) \end{cases} \tag{2.38}$$

The root $\omega = -k$ gives rise to a velocity component that is unbounded as $\left|y\right| \rightarrow \infty$ hence it is inadmissable, Matsuno (1966), LeBlond and Mysak (1978)

The other root for $m = 0$ corresponds to the mixed gravity-planetary wave which is often reffered to as the Yanai wave. For large wavenumber $k > 0$ it behaves as a high frequency gravity wave whereas for large negative wavenumber it is a low frequency Rossby wave.

$\omega_2 = \omega_3$ at $k = \frac{1}{\sqrt{2}}$

For $m = 1$ and $m = 2$ the root of the dispersion relations correspons to three waves, two gravity waves and one Rossby wave. The two gravity waves have both $k < 0$ and $k > 0$ and thus propagate both west and eastward whereas the Rossby wave only propagates westward. We note that for small frequencies and wavenumber, $\omega << 1, k << 1$ equatorial Rossby waves are nondispersive.

The frequency of the gravity waves for a given m is

$$\omega < \sqrt{\frac{1}{2} + m - \sqrt{m(m+1)}} \tag{2.39}$$

or

$$\omega > \sqrt{\frac{1}{2} + m + \sqrt{m(m+1)}} \tag{2.40}$$

2.2 Equatorial waves under the full Coriolis force

In this section we attempt to generalise Matsuno's work by including the complete Coriolis force. The limitations of the traditional approximation have been noted by many authors, see LeBlond and Mysak (1978); Gerkema et al (2008). However as the equations for the velocities are not seperable under inclusion of $\tilde{f}$ finding satisfactory analytical solutions is not an easy feat.

Ideally we would have some system of equations for a non-hydrostatic non-traditional β-plane that approaches the work of Matsuno (1966) for some suitable limit such as $\tilde{f} \to 0$. Several authors, Kasahara (2003), Fruman (2009), Roundy and Janiga (2012), have attempted such a generalizations however their analytical models suffer from defects. See Rabitti (2016) section 2.3 for a more in depth critical view of these generalizations. Additionally most generalizations of Matsuno's work have so far focused on the atmospheric case.

The inclusion of the full Coriolis force will affect not only surface and internal waves but also Ekman spirals, deep convection and equatorial jets. However examining these phenomena is beyond the scope of this text, for a concise summary of non-tradititional geophysical fluid dynamics the reader is refered to Gerkema *et. al* (2008)

The derivations in this chapter largely follows Schmidt (2013) and Rabitti (2016). We employ a non-hydrostatic model on a non traditional β-plane and make no assumption regarding the seperability of the solutions is made.

First we examine non-traditional equatorial waves with $v \equiv 0$, i.e. Kelvin waves, as this is the most simple to model. In the subsequent sections we consider other types of non-traditional equatorial waves. Lastly a comparison is made between the solutions of the TA and the solutions under the full Coriolis force. The reader may skip the extensive derivation of the results in sections 2.2.1 and 2.2.3 and immediatly move on to the interpretation of these results in 2.2.4

2.2.1 Non traditional equatorial Kelvin waves

Consider a non-hydrostatic linearly stratified model, i.e. $N > 0$ constant, on a non-traditional equatorial β plane with $v \equiv 0$. The Coriolis parameters are approximated as $\tilde{f} = 2\Omega cos(\phi) \approx 2\Omega$ and $f = 2\Omega sin(\phi) \approx 2\Omega\phi \approx \frac{2\Omega y}{a} = \beta y$ where a is the radius of the Earth. Note that $b = g\frac{\rho'}{\rho_*}$ is the buoyancy acceleration

The equations are

$$\frac{\partial u}{\partial t} + 2\Omega w = -- \frac{\partial p}{\partial x} \tag{2.41}$$

$$\beta y u = -\frac{\partial p}{\partial y} \tag{2.42}$$

$$\frac{\partial w}{\partial t} - 2\Omega u = -\frac{\partial p}{\partial z} + b \tag{2.43}$$

$$\frac{\partial b}{\partial t} + N^2 w = 0 \tag{2.44}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{2.45}$$

Assume that the time evolution of the buoyancy b is $e^{-i\omega t}$, now we can write b in terms of w and eliminate it from (2.43). We can also eliminate p from the equations by taking the curl.

Introduce a streamfunction $\psi(x, y, z)$ and a substitution for N and ω

$$u = -\frac{\partial \psi}{\partial z} e^{-i\omega t} \tag{2.46}$$

$$w = \frac{\partial \psi}{\partial x} e^{-i\omega t} \tag{2.47}$$

$$\alpha^2 = \frac{N^2}{\omega^2} - 1 \tag{2.48}$$

$$\tilde{\omega} = \omega - \frac{N^2}{\omega} = -\alpha^2\omega \tag{2.49}$$

The system of equations now reads

$$\frac{\partial^2\psi}{\partial z^2} - \alpha^2\frac{\partial^2\psi}{\partial x^2} = 0 \tag{2.50}$$

$$2\Omega\frac{\partial}{\partial z}\left(\frac{y}{a}\frac{\partial\psi}{\partial z} + \frac{\partial\psi}{\partial y}\right) = i\tilde{\omega}\frac{\partial\psi}{\partial xy} \tag{2.51}$$

$$-2\Omega\frac{\partial}{\partial x}\left(\frac{y}{a}\frac{\partial\psi}{\partial z} + \frac{\partial\psi}{\partial y}\right) = i\omega\frac{\partial\psi}{\partial zy} \tag{2.52}$$

This system can be solved by the method of characteristic curves, also known as the method of lines. Briefly stated this method entails finding curves, called *characteristics*, along which the PDE becomes an ODE. This ODE can be solved along the curves and this solution can be transformed into a solution for the orginal PDE.

Hence we introduce new coordinates $\chi_+ = x + \alpha z$, $\chi_- = x - \alpha z$ and $\xi = \frac{y^2}{2a}$. The solution to (2.50) can be written as the superposition of two functions of either χ_+ or χ_-

$$\psi = G^+(\chi_+, \xi_)) + G^-(\chi_-, \xi) \tag{2.53}$$

Now (2.51) and (2.52) can be decoupled

$$\partial_{\chi_+}[(\frac{i\omega}{2\Omega} + \frac{1}{\alpha})\partial_\xi + \partial_{\chi_+}]G^+(\chi_+, \xi) = 0 \tag{2.54}$$

$$\partial_{\chi_-}[(\frac{i\omega}{2\Omega} - \frac{1}{\alpha})\partial_\xi + \partial_{\chi_-}]G^-(\chi_-, \xi) = 0 \tag{2.55}$$

Substituting $\gamma^\pm = \frac{i\omega}{2\Omega} \pm \frac{1}{\alpha}$ in the above and integrating results in

$$(\gamma^\pm\partial_\xi + \partial_{\chi_\pm})G^\pm(\chi_\pm, \xi) = C^\pm(\xi) \tag{2.56}$$

with $C^\pm$ is an arbitrary function only of ξ and thus of y in the old coordinates. However we may set $C^\pm$ to zero as the velocities u, w are defined as the x, z derivatives of the stream function (2.46) and (2.47). The desired stream function is given below in the new and old coordinates respectively

$$\psi = G^+(\xi - \gamma^+\chi_+) + G^-(\xi - \gamma^-\chi_-) \tag{2.57}$$

$$\psi = G^+\left(\frac{y^2}{2a} - i\left(\frac{\omega}{2\Omega} - \frac{i}{\alpha}\right)(x + \alpha z)\right) + G^-\left(\frac{y^2}{2a} - i\left(\frac{\omega}{2\Omega} + \frac{i}{\alpha}\right)(x - \alpha z)\right) \tag{2.58}$$

Below we consider solutions of (2.50)-(2.52) for a fixed boundary and solutions that decay exponentially.

2.2.2 Solutions for non-traditional equatorial Kelvin waves

We examine whether solutions of (2.49) exist for a fixed boundary where the normal velocity is zero.

$$\psi_x|_{z=0} = w|_{z=0} = 0 \tag{2.59}$$

$$\partial_{\chi_+}G^+|_{\chi_+=\chi} + \partial_{\chi_-}G^-|_{\chi_-=\chi} = 0 \tag{2.60}$$

with $\chi_+ = \chi_- = \chi$ at $z = 0$

$$-\gamma^+(G_H^+)'(\xi - \gamma^+\chi) = \gamma^-(G_H^-)'(\xi - \gamma^-\chi)\forall\chi, \xi \tag{2.61}$$

This requirement cannot be satisfied ever hence there are no non-traditional Kelvin waves for a fixed boundary.

Solutions of (2.50)-(2.52) that decay exponentially however do exist and have been studied for the atmospheric case. As these waves decay rapidly with depth they are trapped near the surface. The asymptotic condition for the exponentially decaying solution is

$$\lim_{z\to\infty} \psi = 0 \tag{2.62}$$

The pressure can be written in terms of ψ by introducing $p = e^{-i\omega t}(P^+G^+ + P^-G^-)$ for some constant $P^\pm$.

$$p = -2\Omega\alpha e^{-i\omega t}(\gamma^+G^+ - \gamma^-G^-) \tag{2.63}$$

We define $G^\pm$ for a given wavenumber k as

$$G_k^+(X) = A_k e^{-Xk/(\frac{\omega}{2\Omega}+1)} \tag{2.64}$$

$$G_k^- = A_k e^{-\frac{y^2k\beta}{2\omega+4\Omega}} e^{k(ix+z)} \tag{2.65}$$

which becomes zero as $z \to -\infty$ so $G_k^+ \equiv 0$ is required. We use the surface boundary condition $\frac{\partial p}{\partial t} = gw$ to obtain the dispersion relation

$$k = \frac{\omega^2}{g}\Big(\frac{2\Omega}{\omega} - \sqrt{1 - \Big(\frac{N^2}{\omega^2}\Big)}\Big) \tag{2.66}$$

which becomes complex for $N > \omega$. The derived wavenumber can only satisfy (2.62) for deep water waves.

We now examine whether the solution for the non-traditional Kelvin wave approaches some known result for the limit $\gamma^\pm = 0$.

For $\gamma^\pm = 0$ and $\omega^2 > N^2$ the substitution (2.48) becomes

$$\alpha' = \sqrt{1 - \frac{N}{\omega}} \tag{2.67}$$

so that the frequency bound is $\omega^2 >> 4\Omega^2 + N^2$. Employing the same choice of G_k as for the exponentially decaying solutions $\sum_k G_k = \sum_k A_k e^{-k\frac{2\Omega}{\omega}X}$. For a given k the streamfunction reads

$$\psi_k = e^{-\frac{y^2k\beta}{2\omega}} e^{ikx}(A_k^+ e^{-\alpha' kz} + A_k^- e^{\alpha' kz}) \tag{2.68}$$

which is rewritten as

$$\psi_k = A_k e^{-\frac{y^2k\beta}{2\omega}} e^{ikx} sinh(\alpha' k(z+H)) \tag{2.69}$$

The frequency follows from applying the bounary condition at $z = 0$

$$\omega^2 = -\frac{gk(G^+ + G^-)}{\alpha'(G^+ - G^-)}\Big|_{z=0} = \frac{gk}{\alpha'} tanh(\alpha' kH) \tag{2.70}$$

This dispersion relation bears resemblance to the dispersion relation of surface gravity waves $\omega^2 = gktanh(kH)$, however the inclusion of α' in (2.70) complicates the matter.

2.3 Equatorial waves

In this section we examine solutions for wave motion on a non-hydrostatic non-traditional equatorial β-plane for a homogenous fluid. We aim to find general free solutions for meridionally bounded waves under the full Coriolis force.

The governing equations are

$$\frac{\partial \mathbf{u}}{\partial t} + 2\mathbf{\Omega} \times \mathbf{u} = -\nabla p \tag{2.71}$$

where $\mathbf{\Omega} = \Omega(\hat{e}_y + \frac{y}{a}\hat{e}_z)$. Take the curl of (2.71) to obtain the following vorticity balance.

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{u}) + \beta v \hat{e}_z - 2\Omega(\partial_y + \frac{y}{a}\partial_z) \tag{2.72}$$

As usual we assume the time dependence is $e^{-i\omega t}$, we employ this and the continuity condition $\nabla.\mathbf{u} = 0$ to rewrite (2.72)

$$i\omega\Delta\mathbf{u} + \beta\begin{pmatrix}\partial_y v - \partial_y w\\ -\partial_x v\\ \partial_z u\end{pmatrix} - \frac{2\Omega}{i\omega}L\begin{pmatrix}0\\0\\ \beta v\end{pmatrix} + \frac{4\Omega^2}{i\omega}L^2\begin{pmatrix}u\\v\\w\end{pmatrix} = \mathbf{0} \tag{2.73}$$

where $L = \partial_y + \frac{y}{a}{}_z$ and L^2 is computed using Leibniz's rule. Assume that the velocities are proportional to e^{ikx}, this is possible since there are no mixed derivatives involving x. The normalized frequency is introduced $\omega' = \frac{\omega}{2\Omega}$ and we immediatly drop the prime notation. This leads to

$$[-(\omega^2k^2 + \frac{\omega k}{a}) + (\omega^2 - 1)\partial_y^2 + (\omega^2 - \frac{y^2}{a^2})\partial_z^2 - 2\frac{y}{a}\partial_y\partial_z - \frac{1}{a}\partial_z]v = 0 \tag{2.74}$$

This equation can be analysed by studying its characteristic curves. The characteristics will provide suitable coordinates with which to rewrite the equation. Before employing the method of characteristics we set all factors in front of ∂_y^2 equal to one and make use of the scaling

$$(k,l) = \frac{1}{a}(k',l') \qquad y = a\sqrt{1-\omega^2}y' \qquad (x,z) = a(x',z') \qquad t = \frac{1}{2\Omega}t' \tag{2.75}$$

after which (2.74) becomes

$$[\mu^2 - \partial_y^2 + (\Omega^2 - (1-\omega^2)y^2)\partial_z^2 - 2y\partial_y\partial_z - \partial_z]v = 0 \tag{2.76}$$

with $\mu^2 = \omega k(1 + \omega k)$.

Note that (2.76) only has oscillatory solutions if it is of the hyperbolic type. Rewrite (2.76) for the usual PDE notation

$$Ad_{yy} + 2Bd_{yz} + Cd_{zz} + lowerorderterms = 0 \tag{2.77}$$

with $A = -1, \quad B = -y$ and $C = \omega^2 - (1-\omega^2)y^2$. Recall that for a hyperbolic PDE the determinant of the coefficient matrix Z must be less then zero. In this case $\delta = det(Z) = AC - B^2 = -\omega^2(1 + y^2) < 0$ for all values of y if $\omega < 1$.

The two characteristics of the equation are related as

$$\frac{dz}{dy} = \frac{B \pm \sqrt{-\delta}}{A} = y \pm \omega\sqrt{y^2+1} \tag{2.78}$$

Hence if we introduce coordinates $z - \int \frac{B\pm\sqrt{-\delta}}{A}dy = \chi_\pm$ the PDE can be written for only mixed derivatives

$$\partial_{\chi_+}\partial_{\chi_-}v + ... = 0 \tag{2.79}$$

To eliminate the mixed derivatives we again introduce new coordinates $\chi = z - \frac{y^2}{2}$ and $\eta = y$ such that (2.76) reads

$$[\mu^2 - \partial_\eta^2 - \omega^2(1+\eta^2)\partial_\xi^2]v = 0 \tag{2.80}$$

Finally we combine all the manipulations and insert them into (2.73) to obtain the following system of equations where the continuity equation is used to replace $\frac{\partial w}{\partial z}$

$$(-\omega^2k^2 + \omega^2(1+\eta^2)\partial_\xi^2 - \partial_\eta^2)\begin{pmatrix}u\\v\\w\end{pmatrix} - i\omega\begin{pmatrix}(\frac{2}{\sqrt{1-\omega^2}})(v_\eta - \eta v_\xi) + iku\\ -ikv\\ u_\xi\end{pmatrix} + \begin{pmatrix}0\\0\\(\frac{1}{\sqrt{1-\omega^2}})(v_\eta - \omega^2\eta v_\xi)\end{pmatrix} = \mathbf{0} \tag{2.81}$$

A solution for v can be obtained as follows. First write v as seperable $v = e^{l\xi}V(\eta)$ which gives

$$[\mu^2 - \partial_\eta^2 - \omega^2(1+\eta^2)l^2]V(\eta) = 0 \tag{2.82}$$

This becomes Hermite's differential equation if we take $V = e^{\frac{-\omega l \eta^2}{2}}\mathcal{H}(\sqrt{\omega l}\eta)$

$$\mathcal{H}_m'' - 2\sqrt{\omega l}\eta\mathcal{H}_m' + 2m\mathcal{H} = 0 \tag{2.83}$$

where

$$2m+1 = \frac{\omega}{l}(k^2 - l^2 + \frac{k}{\omega}) \tag{2.84}$$

for m a positive integer as the other solutions diverge for $y \to \pm\infty$.The solutions are the Hermite polynomials $\mathcal{H}_m$ which we already encountered in section 2.1. The dispersion relation in (2.84) can be written for dimensional quantities

$$\omega_m = -\beta\frac{k-(2m+1)l}{k^2-l^2} \tag{2.85}$$

with $k \neq l$ since $k = l$ gives $H_0 = 1$. The dispersion relation for a fixed l is shown in figure 2.3.

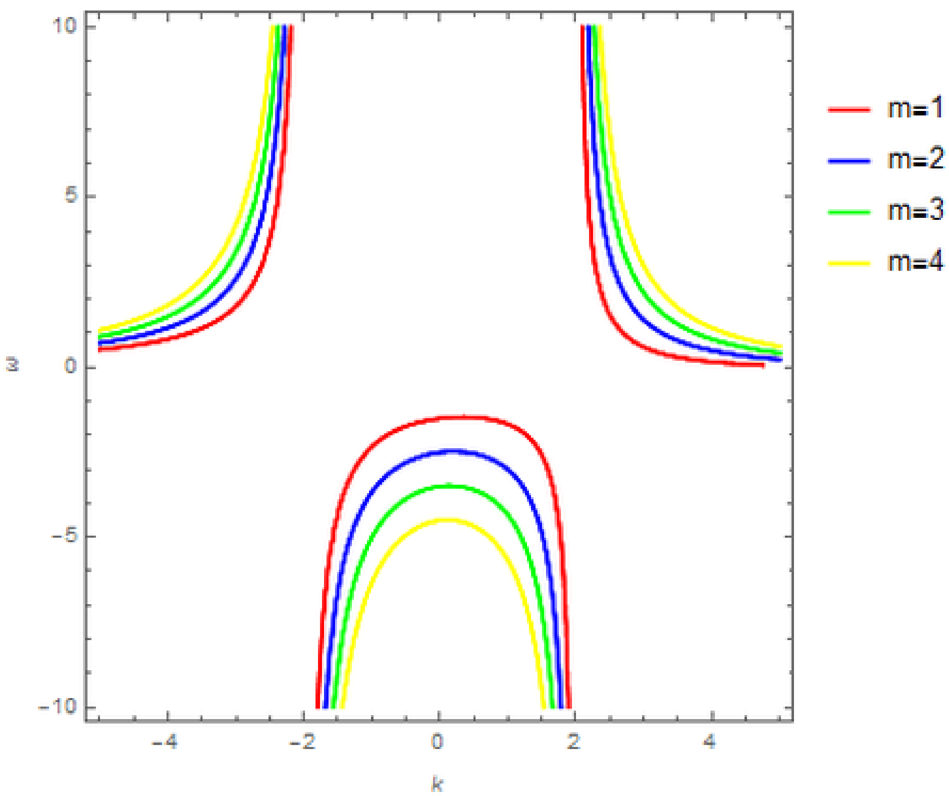

Figure 2.3: Disperion relations for the first four modes for $\omega = -\frac{1}{2}$ and $l = 2$

For a single component of the meridional velocity with wavenumbers k, l and mode m we can write

$$v_{k,l,m} = \sqrt{1-\omega_m^2}e^{i(kx-\omega_m t)+l(z-(1+\omega_m)\frac{y^2}{2})}\mathcal{H}_m(y\sqrt{\omega_m l}) \tag{2.86}$$

By making use of the recurrence relation for the Hermite polynomials u and w for wavenumbers k, l and mode m are.

$$u_{k,l,m} = -i\sqrt{\frac{l}{\omega_n}} e^{i(kx-\omega t)+l(z-(1+\omega_m)\frac{y^2}{2})}(\frac{\omega_m - 1}{l-k} m\mathcal{H}_{m-1} + \frac{1}{2}\frac{\omega_m + 1}{l+k}\mathcal{H}_{m+1}) \tag{2.87}$$

$$w_{k,l,m} = \sqrt{\frac{l}{\omega_m}} e^{i(kx=\omega t)+l(z-(1+\omega_m)\frac{y^2}{2})}(-\frac{\omega_m - 1}{l-k} m\mathcal{H}_{m-1} + \frac{1}{2}\frac{\omega_m + 1}{l+k}\mathcal{H}_{m+1}) \tag{2.88}$$

here the argument of the Hermite polynomials is $y\sqrt{\omega_n l}$.
Given the above we can also find the pressure field which is

$$p = \frac{\tilde{p}}{\rho_* 2\Omega a} = i\frac{1-\omega^2}{l-k}\frac{1}{\sqrt{\omega l}} e^{i(kx-\omega_m t)+l(z-(1+\omega_m)\frac{y^2}{2})}(m\mathcal{H}_{m-1} - \frac{1}{2}\mathcal{H}_{m+1}) \tag{2.89}$$

2.3.1 Interpretation of results

In the previous section we derived solutions for oscillatory motio in the equatorial ocean under full Coriolis force. Now we will attempt an interpretation of these results and examine the properties of the theoretically predicted waves. First of we remark that the solutions in (2.86)-(2.89) clearly do not transform to Matsuno's solution for any limit. The velocities u, v and w in (W)-(w) are all functions of z whereas for Matsuno's solutions u and v are not functions of z.

The disperion relations in figure 3 and (40), (41) are not as easily interpreted as those in section 2.1. For a given mode m the frequency ω is a function of two wavenumbers k, l. From figure 2.3 we can glean that the wave propagation is both $k < 0$ westward and $k > 0$ eastward for a fixed wavenumber l.

An illuminating case is that for low frequencies $\frac{\omega}{2\Omega} << 1$ and zonally symmetric waves $k = 0$. The dispersion relation (41) then becomes

$$\omega_m == \beta\frac{2m+1}{l} \tag{2.90}$$

this results in an imaginary argument $\sqrt{\omega_m l}$ in the Hermite polynomials. These Hermite polynomials with complex argument are plotted in figure 2.4. The modii shown on the left plot of figure 2.4 are all similar differing only in the amplitude. Note that the modii shown in the left plot of figure 2.4 are similar in form to the $m = 2$ mode found under the TA, see figure 2.1 This suggests that observational results interpreted as a Matsuno $m = 2$ mode may equally well be explained otherwise.

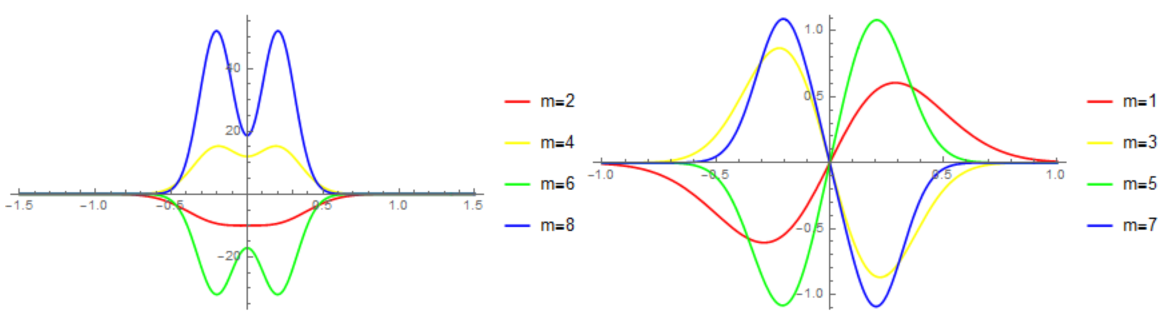

Figure 2.4: Plot of $e^{\frac{2m+1}{2\omega}y^2}H_m(i\sqrt{2m+1}y)$ for $\omega = -\frac{1}{4}$ for the first 4 modii. The left plot shows the real part of the function for even m. The right plot shows the imaginary part of the function for odd m. The amplitude differences between modes are large therefore the amplitudes as scaled so that they may all fit in one plot. Note that these are solutions for zonal symmetry $k = 0$

2.4 Summary

In the sections above we derived solutions for equatorial waves for the classical case and for the 'non-traditional' case.

The solutions presented in 2.1 form an orthogonal and complete set of eigenfunctions which clearly makes for a good result.

However we must consider the model employed to derive the solutions. The shallow water equation originaly adopted by Matsuno (1966) gives values for the velocity and spatial scales that do not correspond to observation. To correct for this we instead used the reduced gravity model. The reduced gravity model itself is also a strongly simplifid model. By employing the motionless bottom and sharp thermocline all dynamics within the layer are lost.

In sections 2.2 and 2.3 solutions were derived for equatorial waves under the full Coriolis force. Deriving these solutions requires more work as the problem is not seperable. We have seen that these non-traditional solutions,for waves with non-zero meridional velocity, do not approach those of Matsuno for any limit. This is due to the dependence of u, w on z. However as can be seen in figure 2.3 zonally symmetric solutions bear strong resemblance to the $m = 2$ of the classical solutions. Observations of this mode in for example Wakata (2007), Farrar (2008) may equally well be explained by the 'non-traditional' approach.

Both theoretical frameworks have their appeal and limitations. There is also the question of which framework best explains observational resuts. This is however not as simple as it sounds.

The long waves under consideration have long time scales and data is often very noisy on such scales. Disturbances of shorter scale processes abound and strong preprocessing is required to interpret the data. This preprocessing presumes that the observations align with theoretical predictions. Thus the data might already biased prior to interpretation.

Consider for example the symmetry properties of the meridional structures derived in section 2.1. These symmetry properties are often invoked in order to justify the symmetrization and antisymmetrization of data signals see Farrar (2008).

Due to such inherent data bias' it has proven difficult for us to interpret the work of other authors in a new light.

Chapter 3

Internal waves

In the previous section we derived solutions for surface and interfacial waves at the equator. Surface waves are familiar to most people as their effects are easily observed. However the rotating stratified shell that is the Earth also gives rise to another type of waves, namely *internal* waves, these waves reach their maximum amplitude in the interior of the fluid. Hence these type of waves are not as easily observable as their surface counterparts. However the presence of internal waves can be deduced from their effect on surface waves. Internal waves also play an important role in the diapycnal mixing of the deep ocean as they can propagate both vertically and horizontally.

The restoring forces giving rise to internal waves are buoyancy (gravity) and the Coriolis force. For gravity to act as a restoring force the fluid must the stably stratified, that is $N^2 > 0$. Internal gravity waves arise when gravity is the sole restoring force. Gyroscopic (inertial) waves result from rotation as the restoring force. If the restoring force is a combination of gravity and rotation one speaks of internal inertio-gravity waves. It is this type of internal wave which will be our focus.

Below we give a short theoretical discription of these waves under the full Coriolis force. We have seen that the solutions for surface waves under inclusion of $\tilde{f}$ resemble those of the TA for zonally symmetric cases. When it comes to internal waves however neglecting $\tilde{f}$ dramatically changes the solutions.

The solutions for internal waves are also closely linked to the geometry of the domain/container which we briefly touch upon.

A complete description of internal waves is outside the scope of this text. In the section below we merely wish to highlight the role of $\tilde{f}$ in internal wave trapping and the resultant effect on deep water mixing. For a more in depth treatment of internal waves the reader is referred to Gerkema (2008), LeBlond and Mysak (1978), Swart (2007) and Maas (2005).

3.1 Internal inertia-gravity waves

We now give an overview of the linear internal wave theory under inclusion of $\tilde{f}$ after Gerkema et al. (2008) and LeBlond and Mysak (1979). For the sake of simplicity we consider motion on the $f, \tilde{f}$-plane so $\beta = 0$ and $\tilde{f}, f$ are constants. As customary we start with the momentum and continuity equations, now on the $f, \tilde{f}$-plane . Wave amplitudes are assumed to be infinitesimal so we linearize the equations by dropping the advective terms. The time-dependence of the velocities is assumed to be $e^{-i\omega t}$ and N is constant.

$$-i\omega u + \tilde{f}w - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{3.1}$$

$$-i\omega v + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{3.2}$$

$$-i\omega w(1 - \frac{N^2}{\omega^2}) - \tilde{f}u = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{3.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3.4}$$

The velocities u, v and w are written in terms of the pressure so that one single equation for the pressure results.

$$A\frac{\partial^2 p}{\partial y^2} - 2B\frac{\partial p}{\partial yz} + C\frac{\partial^2 p}{\partial z^2} + D\frac{\partial^2 p}{\partial x^2} = 0 \tag{3.5}$$

where $A = (\omega^2 - N^2 - \tilde{f}^2), B = (f\tilde{f}), C = (\omega^2 - f^2), D = (\omega^2 - N^2)$.
Equation (3.5) is also known as the Poincare equation. Note that due to the inclusion of $\tilde{f}$ the terms A and D are unequal implying anistropy in the horizontal plane. Under the TA $A = D$ and the mixed derivative dissapears. To obtain dispersion relations we make the ansatz of $p \propto e^{i\mathbf{k}.\mathbf{x}}$ with $\mathbf{k} = (k, l, m)$ in (3.5). This gives

$$\omega^2 = N^2\frac{k^2 + l^2}{k^2 + l^2 + m^2} + 4\Omega^2\frac{(lcos\phi + msin\phi)^2}{k^2 + l^2 + m^2} \tag{3.6}$$

The above is rewritten for θ, the angle between $\mathbf{k}$ and the plane perpendicular to the local gravity vector, and α the angle between $\mathbf{k}$ and the local Coriolis parameter vector.

$$\omega^2 = N^2cos^2\theta + 4\Omega^2cos^2\alpha \tag{3.7}$$

This highlights an interesting property of internal waves namely their *monoclinic* nature. The dispersion relation fixes θ, α but not $|\mathbf{k}|$.

The dispersion relation (3.7) implies that the frequency range for internal waves is $4\Omega^2 < \omega^2 < N^2$. In most parts of the ocean $N^2 > 4\Omega^2$ however for the deep ocean however stratification is very low and $N^2 > 4\Omega^2$ may not hold. From (3.7) we can see that the group velocity vector $c_{gr} = \nabla_k\omega(\mathbf{k})$ is perpendicular to the wave vector $\mathbf{k}$. Recall that the wave energy travels along the group velocity vector, thus for internal waves the energy travels parallel to the phase lines.

Under the TA the dispersion relation reads

$$\omega^2 = N^2\frac{k^2 + l^2}{k^2 + l^2 + m^2} \tag{3.8}$$

The exclusion of $\tilde{f}$ results in a markedly different dispersion relation.

Another noteworthy property of internal waves is their limitation to a *critical* latitude. We will see that the expression for this critical latitude differs greatly when neglecting $\tilde{f}$. To derive an expression for the critical latitude we start by analyzing (3.5) for the zonally symmetric case

$$A\frac{\partial^2 p}{\partial y^2} - 2B\frac{\partial p}{\partial yz} + C\frac{\partial^2 p}{\partial z^2} = 0 \tag{3.9}$$

For oscillatory solutions to exist (3.9) must be a hyperbolic thus $AC - B^2 < 0$. This provides the bounds of the frequency domain

$$\omega^2(\omega^2 - N^2) + 4\Omega^2) + 4\Omega^2N^2sin^2(\phi) < 0 \tag{3.10}$$

We can solve (3.10) for the latitude ϕ for a given frequency ω to obtain the critical latitude ϕ_c, LeBlond and Mysak (1978), Gerkema et al. (2008).

$$\phi_c = \pm\arcsin\Big[\frac{\omega^2}{4\Omega^2} - \frac{\omega^2}{N^2}\Big(\frac{\omega^2}{4\Omega^2} - 1\Big)\Big]^{\frac{1}{2}} \tag{3.11}$$

with + for the Northern hemisphere and − for the Southern hemisphere. Thus internal waves of a frequency ω propagating poleward cannot exist beyond the critical latitude ϕ_c. The imaginary surface spanned by ϕ_c is called the *turningsurface*. The turning surface delineates the penetrable region where (3.5) is hyperbolic, and the impenetrable region where (3.5) is elliptic. Beyond the turning surface only exponentially decaying solutions exist.

The expression derived for the critical latitude (3.11) is for the full Coriolis force. Under the traditional approximation, $N \to \infty$, (3.11) reverts to

$$\phi_{c,TA} = \pm\arcsin\Big(\frac{w}{2\Omega}\Big) = \phi_i \tag{3.12}$$

Under the TA the critical latitude approaches the inertial latitude, i.e. the latitude at which the wave frequency ω is equal to the intertial frequency $|f|$.

3.1.1 Reflection of internal waves and wave attractors

The Poincaré pressure equation (3.5) in three-dimensions for a bounded domain is generally ill-posed and underdetermined. Smooth solutions do exist for some geometries such as a full sphere or cylinder, Rieutord *et. al*, 2000, however solutions for most cases will exhibit singularities.

Consider the two-dimensional Poincare equation in a bounded domain. The solutions hereof are constant along the characteristic lines of the equation which coincide with the ray paths of the internal wave.

The characteristics reflect of the boundaries and the trajectories of these lines converge towards a periodic orbit called the *waveattractor*. Small scale wave-attractors may be suppressed by the effects of viscosity however the properties of wave attractor are robust enough to have been observed in experimental settings, see Maas and Lam (1997).

In section 3.1 we found that internal waves are monoclinic, upon reflection focusing or defocusing may occur under certain conditions. An internal wave propagating along a ray,with angle θ to the vertical, incident upon a sloping bottom with angleγ, will upon reflection maintain θ to the vertical. The sloping bottom then acts as a lens resulting in focusing of the internal wave localized around the wave attractor. The higher wave energy at the bottom results in intense mixing of deep ocean waters.

The focusing or defocusing of internal waves is a consequence of symmetry breaking as seen in the example above where the sloping bottom 'breaks' the symmetry. Focusing of internal waves wil not occur for "over-symmetrized" geometries, that is for boundaries that are parallel or perpendicular to the restoring direction. In essence adopting the TA means oversymmetrizing the setting by aligning the rotation axis to gravity Gerkema *et.al*. Under the TA symmetry breaking will occur for sloping bottoms and other topographic boundary features. Inclusion of $\tilde{f}$ results in symmetry breaking as the rotation axis and gravity are not aligned so that focusing of internal waves may possibly even occur on smooth topography, Holmes *et. al* (2016), Shrira and Townsend (2013).

3.2 Summary

In the previous sections expressions for the dispersion relation and critical latitude for internal waves was derived. We have seen that employing the traditional approximation for internal waves results in markedly different dispersion relations and turning surfaces.

Upon reflection of a sloping bottom internal waves become focused. More generally, upon reflection at a symmetry breaking boundary focusing of internal waves will occur. The reflected waves will converge towards a wave attractor. Deep ocean mixing will be localized around the wave attractor.

Chapter 4

Conclusion

In this paper we examined two different theoretical frameworks for the description of equatorial ocean dynamics. The traditional approximation was shown to have limitations. That this theory is well established should not mean that we can not improve upon our understanding of equatorial wave dynamics.

It is remarkeable that the classical solutions can not be derived as the limiting case of a more general theory. This hints at a fundamental flaw in the formulation of traditional equatorial waves.

A 'non-traditional' or more general framework can offer an alternative. However the solutions for this theory do not provide a complete and orthogonal set. Thus deriving solutions under forcing and other disturbances is not as straight forward.

It remains to be seen whether the general solutions correspond adequatly to observational results. However as the general theory is by its nature more realistic due to the inclusion of $\tilde{f}$ one would expect observations to align more to the general theory than to the classical one.

We have seen that for the zonally symmetric case the solutions for the general theory are similar to the $m =$ mode of the classical solutions. Observations of this $m = 2$ mode can thus also be explained by the more general theory.

In the case of internal waves the differences between the theoretical results of both framework is pronounced. The critical latitudes for inclusion or exclusion of $\tilde{f}$ differ significantly. Thus in describing internal waves one must keep in mind the result of neglecting $\tilde{f}$ on the solutions.

The role of $\tilde{f}$ may be pronounced for deep ocean mixing as it provides further symmetry breaking necessary for the focusing of internal waves.

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