Utrecht University, the Netherlands
Faculty of Science
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Master's Thesis

# Cosmic Inflation from Conformal Theories with Torsion

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# Abstract

In the current Master's Thesis we will be concerned with getting inflationary solutions from a conformally invariant gravity. To achieve conformality, we introduce torsion into the theory. We argue the form of a general conformally invariant scalar action for a general curved spacetime which is our starting point. We study it's properties and write it in a convenient form - this gives us interesting insights about the special features coming from the introduction of torsion. Using this action we derive the inflationary solutions. In due process we also show, that inflation is realized through radiative symmetry breaking and the mass scales of the theory are generated dynamically. After studying the properties of the inflationary solutions, we compare them to the current experimental data and examine the ranges of the free parameters of our theory for which we are getting a viable model of inflation. We discover, that this model gives us the values of the parameters that are close to Starobinsky inflation. In the end we also discuss the interesting features of our model coming from the conformal symmetry, interpret the results and make final the conclusions.

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# Notation and Abbreviations

## Notation

- We will be using Einstein summation notation:

  $R_{ab\ldots}T^{ac\ldots} = \sum_a R_{ab\ldots}T^{ac\ldots}$

- Unless specified otherwise, we work in units $\hbar = c = M_p = 1$

- We use the following sign convention: $\{-,+,+,+\}$

- Quantities without torsion:

  $\mathring{\nabla}_\mu$, $\mathring{R}_{\mu\nu}$, $\mathring{R}$

- Quantities with torsion:

  $\bar{\nabla}_\mu$, $\bar{R}_{\mu\nu}$, $\bar{R}$

## Abbreviations

Cosmic Microwave Background - CMB

Einstein Frame - EF

Jordan Frame - JF

General Relativity - GR

Standard Model - SM

Spontaneous Symmetry Breaking - SSB

Riemann Normal Coordinates - RNC

Configuration Space - CS

Field Space - FS

# Chapter 1

# Introduction

## Conformal General Relativity

It is well known, that there are four fundamental fores in nature - Electromagnetic, Weak, Strong and Gravitational. Currently the most precise way of treating them is by using the framework of Quantum Field Theory (QFT). For the first three forces - Electromagnetic, Weak and Strong we already have working QFT-s, whereas gravity (which was introduced more than 100 years ago by Einstein as the General Relativity (GR)) still has not been formulated completely and consistently as a QFT.

Though, if we look at how elegant the laws of nature usually are, it seems reasonable to expect the possibility of describing gravity as a QFT as well, giving us a universal framework, that can describe all the fundamental forces. Of course, many attempts have been made in this direction and by using the curvature parameters GR has been written in the action form so that it can be treated as a complete QFT. Since then it has been extensively studied on large length scales by using observational data from many sources such as our solar system, gravitational lensing, indirect observation of black holes etc [1]. However, at smaller length scales (or higher energies, alternatively), it turned out, that GR breaks down at quantum levels due to divergences coming from quantum corrections [2]. This means, that General Relativity is not a full QFT of gravity but rather (only) an effective theory - it works very well for low energies, but loses it's predictive power at higher energy scales. Thus, if we want to treat GR as a complete QFT, it has to be modified and during the past decades the research in this direction (called Quantum Gravity [2]) has been very active.

While it is true, that EFTs work only within a certain range of scales, they can still give us a lot of interesting and useful information about some particular energy regimes of the full theory, therefore studying the features and details of General Relativity (an EFT of full Quantum Gravity) is well worth it.

In particular, the interest in research of scale/conformally symmetric General Relativity, has been quite large [15]. To see why, let's briefly discuss conformal symmetry. Conformal transformation can be defined as:

$$\mathrm{d}s^2 \to e^{2\theta(x)}\mathrm{d}s^2 \tag{1.0.1}$$

When we do this transformation, lengths get rescaled at every point in space, but angles and dimensionless quantities remain constant. And in order for (1.0.1) to be the symmetry of the theory, it should only contain dimensionless observable quantities and should not have any intrinsic length/energy scale, because otherwise this scale would change during the transformation.

But why is conformal symmetry so interesting and attractive? It turns out, that apart from possibly being one of the fundamental symmetries of nature [3] (just like the Lorentz symmetry), conformal symmetry also significantly simplifies some aspects of a theory - for example conformal invariance allows us to calculate the 2- and 3-point functions exactly. When the theory is not conformal, though, these 2- and 3-point functions would be hard or impossible to calculate at all (especially if one has a strong coupling). So even if one is not interested in conformal field theories in the first place, one might still get a lot of information about some QFT theory or phenomenon, if it is considered in a CFT regime.

If we now look at the Standard Model (SM), that describes all the forces except gravity, we will notice, that it is nearly conformally invariant - the only dimensionful coupling constant that breaks the conformal invariance explicitly is the scalar mass term (or, alternatively, the Higgs mass term), since it has a dimensionful mass parameter, that introduces a scale into the theory:

$$V(\phi) = m^2\phi^\dagger\phi + \frac{1}{4}\lambda(\phi^\dagger\phi)^2 \tag{1.0.2}$$

So, the question is now - is it possible for the SM to be conformal in certain energy regimes? This would be very attractive due to above reasons, however, to achieve this we would need to be able to introduce a scale into the theory starting from some point, since we know, that at current energies we have this mass term. So, to answer the above question, we need to deal with a new question - how do we introduce a scale into a conformal theory?

The first idea would be to check whether it can be done using the fundamental constants such as $c$, $\hbar$ and $k_B$. Unfortunately, due to Lorentz invariance treating space and time equally, giving them the same conformal weight, these constants do not rescale under local change of lengths (1.0.1) and hence can not be used to introduce a scale into SM.

Fortunately, there is a way of doing it using matter fields. If we have some matter fields in the theory, they might acquire condensates allowing the scale to enter the theory as coupling constants of mass terms. This way, for example, a condensate of some scalar field can give us the Higgs mass that breaks the conformal invariance, or, as another example - the scalar field condensate can introduce the Planck's mass into the theory[1]. Basically, we can use the Spontaneous Symmetry Breaking (SSB) approach to introduce a scale into the theory and this is very effective, because the scales are also generated dynamically and do not simply appear in the starting theory.

If we now apply this approach to the Standard Model, then there will be no absolute intrinsic scale in it - only the local values of the field condensate $\langle\Phi^2\rangle$ play the role of the local scale. Based on this, it is reasonable to think, that in the early universe, when the temperature (and therefore the energy scale) was high, these scalar field condensates were 0 (see e.g. [1]). In such a case, the mass term will not be present in the SM Lagrangian, making it conformal. When the universe cooled down reaching lower energies, however, these fields acquired mass through the condensates (nonzero vacuum expectation value), resulting in a massive scalar field and introduction of a scale into the theory. So, basically, the local scale transformation (1.0.1) can be assumed to be a symmetry of Nature, which was spontaneously broken at some point during the evolution of the universe, but was valid in the very early history at very high energy scales.

So far, during this discussion we only considered the Standard Model, but if we now want the Grand Unification to be valid, the Standard Model has to contain gravity as well. For this, though, gravity, just like the other forces, has to be conformally invariant as well. In such a theory, gravitational observables should stay the same under (1.0.1) in all conformal frames.

Unfortunately, General Relativity does not have this property and there used to be no convenient theory of conformal General Relativity until recent times. It is true, that conformal Lagrangians can be constructed using the Weyl tensor or a non-minimally coupled scalar [4, 5], but in these theories observables, such as geodesic trajectories, depend on the conformal frame choice, and therefore we first need to significantly modify GR and then also describe the transfer from the high-energy effective theory to the low-energy effective theory (Einstein GR) for these theories to work. Additionally, stability is also compromised in models that are based on the Weyl tensor.

Though, there is an elegant way to make General Relativitiy conformal - all we have to do is drop the assumption of a symmetric connection [3] and introduce the torsion tensor (defined as the antisymmetric part of the connection) as an independent[2] geometrical notion. The effect that we get from torsion is, that during the parallel transport in a spacetime with torsion, vectors rotate and rescale (See Fig. (2.2) [3]). This might give us a geometrical intuition, why torsion is connected with conformal symmetry and how it helps

[1] It can actually also happen so, that both of these scales are introduced by the same scalar field
[2] Independent from metric/curvature

to build a scale-invariant theory of gravity [6]. This will be discussed in more detail in the following sections.

## Inflation

One of the most important ones is the theory of inflation and the interest to study it has been growing lately as well. The reason for this is, that inflation [7, 8] is a physical mechanism proposed to explain a number of major problems in cosmology - the ones like flatness and horizon problems, for example. Remarkably, it treats all these problems simultaneously and it solves them with elegance. But one of the most fascinating features of Inflation is, that it can explain the origin of large-scale structures in the universe, meaning, that without inflation making it possible for these structures to form, we wouldn't exist! Together with this, inflation allows us to look beyond the period of the universe, about which we currently don't have much information and is the only tool that gives us possibilities to study the very early universe.

According to the Inflation theory, during inflationary period small inhomogeneities are generated by the microscopic quantum fluctuations of the (expanding) early universe. These inhomogeneities are then stretched due to exponential expansion of the universe, so that their characteristic wavelength becomes larger than the causal radius.[3] After these fluctuations exit the horizon, they are not affected by the processes happening inside the causal radius anymore, therefore they are preserved in the shape that they had when escaping the causal radius. At some later time, inflation comes to an end and the universe continues expanding with much slower rate - during this time, causal radius starts increasing, allowing the stretched fluctuations to re-enter the horizon and become seeds for cosmic structures and anisotropies in the Cosmic Microwave Background (CMB) [9] we observe today [10]. This means, that CMB anisotropies were seeded by fluctuations that originated in the very early universe. To see why this model is so important, we should note, that we don't have much information about the events that happened after the end of inflation, so in principle, we would expect it to be hard/impossible to study the evolution of the universe. Luckily, as we said, the frozen-out fluctuations do not get affected by events inside the causal horizon, preserving information about the state of the universe as it was at the point of time, when these fluctuations exited the horizon. This fact gives us a possibility to get information about the very early universe, without having to know how events went during later times. To make it more intuitive, we can compare it the fossils - fossils are buried and are not affected by events that happen on the surface, then we dig them out and by studying them we have a possibility to learn more about the conditions on earth at the time when these fossils got buried, without caring about events connecting our time and epoch of the fossil burial). So, basically inflation allows us to skip some parts of the history on our way to early universe.

But how do we approach studying inflation? The most widely accepted theory is that from the particle physics point of view – we treat inflation as a hypothetical scalar field (or in some theories as a set of fields), that allows the universe to have an accelerated expansion period very soon after the Big-bang that makes it expand at least $\sim 60$ e-folds to be in accordance with observations that we have today.

There are many candidate theories for inflation, that have different features and properties. To name a few, we can have inflation with different shapes of the potential (each potential corresponds to a different model of inflation), or different ways of realizing it (Slow-Roll [11], Ultra-Slow-Roll [12], Hyperinflation [13] etc.) or with a different number of fields involved (single field vs. multi-field inflation [14]) and so on. In order to find the most realistic one among this myriad of models, precise experiments are needed. Luckily, experiments (such as Euclid) are being launched in the near future and they are expected to look at the early universe with accuracy, that has never been achieved before. As a result, we might be able get significantly better insights about inflationary models. This will be a big step towards understanding some of the most important problems of theoretical physics: the ones like the cosmological constant problem or inflation itself.

Based on all of this and considering the fact that, as we said, there is no other mechanism, that allows to look at the very early universe, we clearly see, that inflation is indeed a very important and modern topic to study today.

[3] If points in the universe are separated by distances larger than the causal radius, they can not communicate with each other now, but they still can do it in the future. This will be discussed in more detail later

### Conformal General Relativity + Inflation

As we know, all the theories that we have in theoretical physics are simply approximate models. If we now take a working model for a theory (such that it already works for many scenarios and has certain properties) and decide to extend it, this extension should ideally have 2 properties - 1) it should still work in all the scenarios, where the original model worked and 2) it should a) either expand the validity area of the theory at hand so that it extends its predictive power to some new regimes or b) give the theory some new properties.

In the current thesis, we take GR as the original theory and the new property that it acquires will be the conformaly symmetry. It has already been shown, how GR can be made conformal by introducing torsion into spacetime [3] and how important this is (we will review this in later sections). Having discussed the high importance of inflation (that can be derived from the standard GR), it would be very interesting to see, whether an extension of GR to a conformal GR **a)** is still able to give us an inflationary solution that is consistent with the current observational data [10] and **b)** is able to give us some new constraints on inflationary data.

The main goal of the thesis will be to get inflation from the conformal GR with torsion and to study its inflationary properties, compare it to the experimental data [10] and then study the dependence of the results on the parameters of our starting theory (action). Additionally, we will study whether the addition of the conformal symmetry through torsion will give this model of inflation some additional interesting features.

It should be noted, though, that our model is only classically conformal and the conformal symmetry is broken during inflation by the condensates of the scalar field and gravitational (Ricci scalar) condensates.

## Structure of the Thesis

The thesis will be structured in the following way. First of all we will get to know the basics - In Chapter 2 we review how General Relativity is made conformal using torsion. Then in Chapter 3 we explore the basics of Inflation and Cosmological perturbations. This is followed by Chapter 4, where we discuss how we choose the starting action for our model and back the claims up by calculating the quantum corrections and see, that the action can be further extended. Then we rewrite it in a convenient way, so that it looks like in the Einstein Frame (where we discover some very interesting features of the potential as a result of this step). This allows us to start deriving the inflationary solution in Chapter 5 and examine the details of the inflation model that we get. This is then followed by Chapter 6 where we discuss the results of the inflation and try to interpret them. And finally in Chapter 7 we make a conclusion.

# Chapter 2

# Making GR Conformal

***In this chapter we will review how gravity can be made conformal using Torsion tensor. We will also derive how geometrical quantities get modified by introduction of Torsion***

## 2.1 Scale, Conformal and Weyl Symmetries

First of all, to get a better idea about some subtleties of the scale symmetry and its slight variations, let's make a brief review.

In general, there are three symmetries associated with scale symmetry 1) scale symmetry itself, 2) conformal symmetry and 3) Weyl symmetry. Based on our empirical knowledge so far, every quantum theory that has a scale symmetry also has a conformal symmetry, so it is a usual practice to use these names interchangeably, therefore we will not go into the details of the comparison of scale and conformal symmetries. We will talk here about the differences between Conformal and Weyl symmetries.

Conformal transformation can be schematically written in the following way: It is a transformation $x^\mu \to x^\mu + \xi^\mu$ that results in the following changes:

$$\begin{aligned} x^\mu &\to x^\mu + \xi^\mu \\ g_{\mu\nu} &\to \Omega_\xi^2(x) g_{\mu\nu} \\ T_\mu &\to T_\mu + \partial_\mu \log \Omega_\xi(x) \\ \phi &\to \Omega_\xi^{\Delta_\phi}(x)\phi \\ \psi &\to \Omega_\xi^{\Delta_\psi}(x)\psi \\ A_\mu &\to \Omega_\xi^{\Delta_A}(x) A_\mu \end{aligned} \tag{2.1.1}$$

The main effect is that this $\Omega_\xi^2(x) g_{\mu\nu}$ is completely determined by the diffeomorphism parameter $\xi$. In case of Weyl symmetry, this does not happen, since it is not a diffeomorphism - it not a group at all. It is just a local rescaling of the metric and it changes things in the following way:

$$\begin{aligned} x^\mu &\to x^\mu \\ g_{\mu\nu} &\to \Omega^2(x) g_{\mu\nu} \\ T_\mu &\to T_\mu + \partial_\mu \log \Omega(x) \\ \phi &\to \Omega^{\Delta_\phi}(x)\phi \\ \psi &\to \Omega^{\Delta_\psi}(x)\psi \\ A_\mu &\to \Omega^{\Delta_A}(x) A_\mu \end{aligned} \tag{2.1.2}$$

For our purposes, we can use these symmetries interchangeably - this is possible only if

$$[\delta_\xi, \delta_W]Q = 0 \tag{2.1.3}$$

where $Q = x, g_{\mu\nu}, T_\mu$, $\delta_\xi$ is the conformal transformation and $\delta_W$ is the Weyl transformation.

It is trivial to see, that for $g_{\mu\nu}$ and $x$ this condition is satisfied. For $T_\mu$ (2.1.3) is also satisfied, though only at the linear level in $\mathcal{L}_\xi \partial_\mu \log(\Omega)$: $\delta_\xi \delta_W T_\mu = \mathcal{L}_\xi T_\mu + \partial_\mu \log(\Omega) + \cancel{\mathcal{L}_\xi \partial_\mu \log(\Omega)}$, where $\mathcal{L}_\xi$ is the Lie derivative.

So, for convenience, in this thesis we will be using Weyl and conformal symmetries interchangeably.

## 2.2 Introducing Torsion

In general, having a conformal theory means, that any two frames given by the (1.0.1) conformal transformation, will be physically equivalent if all physically meaningful observables are the same in both frames.

Now, in order to understand better, how gravity can be made conformal, it will be beneficial if we first have a look at Einstein's General Relativity and see how it behaves under the conformal transformation.

Considering the fact that the conformal transformation (1.0.1) is a length rescaling, it will obviously force all the quantities connected to the metric to change. For the metric itself we will have:

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\theta(x)} g_{\mu\nu}, \qquad \mathrm{d}\tau \to \mathrm{d}\tilde{\tau} = e^{\theta(x)} \mathrm{d}\tau \tag{2.2.1}$$

This change in the metric also modifies the Christoffel symbols, which serve as a connection for the General Relativity, in the following way:

$$\delta \mathring{\Gamma}^\lambda{}_{\mu\nu} = \delta^\lambda_\mu \partial_\nu \theta + \delta^\lambda_\nu \partial_\mu \theta - g_{\mu\nu} \partial^\lambda \theta \tag{2.2.2}$$

where we denote Christoffel symbols by: $\mathring{\Gamma}^\lambda{}_{\mu\nu}$.

Using these quantities we can already construct physical quantities. For example we can consider the geodesic equation - since geodesics are trajectories of the free falling bodies, they should stay invariant under (1.0.1), if one wants to claim, that two conformal frames are physically equivalent (which should be the case in a conformal theory). The geodesic equation is given by:

$$\frac{\mathrm{d}^2 x^\lambda}{\mathrm{d}\tau^2} + \mathring{\Gamma}^\lambda{}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0 \tag{2.2.3}$$

and it transforms as:

$$e^{-\theta} \frac{\mathrm{d}}{\mathrm{d}\tilde{\tau}} (e^{-\theta} \frac{\mathrm{d}x^\lambda}{\mathrm{d}\tilde{\tau}}) + e^{-2\theta} \left( \mathring{\Gamma}^\lambda{}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + 2 \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\tau}} - \dot{x}^\mu \dot{x}^\nu \partial^\lambda \theta \right) = 0 \tag{2.2.4}$$

If we look carefully now, we will see, that the third term in equation (2.2.4) can be absorbed in a parametrization of the proper time $\tau$, but the fourth one can not. So, we see, that the shape of the Geodesic equation, and therefore the geodesic trajectory itself is not the same in different conformal frames, meaning that these frames are not physically equivalent - thus, it becomes obvious, that standard General Relativity is not a conformal theory.

In general, it is still possible within this framework to construct observables that are independent of the conformal frame, but this can only be done by using the Weyl tensor [15], which is the trace-free part of the Riemann tensor. Such a quantity is indeed invariant under the rescaling (1.0.1), but it contains less information than the Riemann tensor itself and there exist some complications with quantization and the stability of the theory, that is based on this tensor.

Fortunately, as we said, there exists an elegant way of constructing the conformally invariant theories, that will make a full use of Ricci tensor, Ricci scalar and Riemann tensor. Though, for that we need to add the torsion to the space-time manifold, meaning that we drop the assumption of GR that the connection is symmetric.

Let's quickly introduce torsion. In general, when we take a partial derivative of a tensor field, it does not necessarily behave like a tensor. But in order to be able to write down covariant equations, that keep their shape in any coordinates, we need to find a way to write the derivative of a tensor field in a tensor form. This can be done by adding an additional term to the partial derivative, so that the transformation behaviour of the sum becomes like that of a tensor. This sum is called the covariant derivative and it has the following shape when acting on a contravariant vector $A^\alpha$:

$$\nabla_\beta A^\alpha = \partial_\beta A^\alpha + \Gamma^\alpha{}_{\beta\gamma} A^\gamma \tag{2.2.5}$$

The (2.2.5) behaves like a tensor only in the case when the affine connection $\Gamma^\alpha{}_{\beta\gamma}$ transforms in a non-tensor way compensating for the partial derivative transformation rule. From this statement it becomes clear, that the connection still has some freedom - if we add to $\Gamma^\alpha{}_{\beta\gamma}$ **any tensor** $C^\alpha{}_{\beta\gamma}$

$$\Gamma^\alpha{}_{\beta\gamma} \to \Gamma^\alpha{}_{\beta\gamma} + C^\alpha{}_{\beta\gamma} \tag{2.2.6}$$

In order to fix the definition of $\Gamma^\alpha{}_{\beta\gamma}$, we need to impose some constraints on it. In General Relativity these constraints look the following way:

1) The connection should be symmetric - $\Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta}$
2) We demand metric compatibility of the covariant derivative $\nabla_\alpha g_{\mu\nu} = 0$.

This results in a very specific shape of the connection - the so called Christoffel connection:

$$\Gamma^\alpha{}_{\beta\gamma} \equiv \mathring{\Gamma}^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} \Big( \partial_\beta g_{\lambda\gamma} + \partial_\gamma g_{\lambda\beta} - \partial_\lambda g_{\gamma\beta} \Big) \tag{2.2.7}$$

where we denote Christoffel connection by $\mathring{\Gamma}^\lambda{}_{\mu\nu}$. It is the simplest connection possible - it can be proved, that all the other connections can be expressed as $\mathring{\Gamma}^\lambda{}_{\mu\nu}$ + additional tensor, just like we had in (2.2.6).

Let's introduce torsion - it is now easy to see, that it corresponds to a specific choice of this $C^\alpha{}_{\beta\gamma}$ tensor. To deduce the shape of it, as we said, we drop the symmetric assumption of the connection, so that:

$$\Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta} = T^\alpha{}_{\beta\gamma} \neq 0 \tag{2.2.8}$$

where $T^\alpha{}_{\beta\gamma}$ is called the torsion.

From here we can see derive how the $C^\alpha{}_{\beta\gamma}$ tensor looks like:

$$\Gamma^\alpha{}_{\beta\gamma} = \mathring{\Gamma}^\lambda{}_{\mu\nu} + K^\alpha{}_{\beta\gamma} \tag{2.2.9}$$

where

$$K^\alpha{}_{\beta\gamma} = \frac{1}{2} \Big( T^\alpha{}_{\beta\gamma} - T_\beta{}^\alpha{}_\gamma - T_\gamma{}^\alpha{}_\beta \Big) \tag{2.2.10}$$

is called the contorsion tensor. And for the future convenience, we should also notice, that contorsion tensor is antisymmetric in the first two indices: $K_{\alpha\beta\gamma} = -K_{\beta\alpha\gamma}$, whereas the torsion is antisymmetric in the last two indices: $T_{\alpha\beta\gamma} = -T_{\alpha\gamma\beta}$.

Torsion tensor is usually divided into three irreducible components for convenience:

1) Trace vector - $T_\beta = T^\alpha{}_{\beta\alpha}$
2) Pseudotrace vector - $S^\nu = \epsilon^{\alpha\beta\mu\nu} T_{\alpha\beta\mu}$
3) The tensor $q^\alpha{}_{\beta\gamma}$ satisfying two conditions $q^\alpha{}_{\beta\alpha} = 0$ and $\epsilon^{\alpha\beta\mu\nu} q_{\alpha\beta\mu} = 0$

In terms of these quantities torsion tensor is expressed as:

$$T_{\alpha\beta\mu} = \frac{1}{3}\Big(T_\beta g_{\alpha\mu} - T_\mu g_{\alpha\beta}\Big) - \frac{1}{6}\epsilon_{\alpha\beta\mu\nu}S^\nu + q_{\alpha\beta\mu} \tag{2.2.11}$$

Let's see how all of this actually makes the theory conformal. First of all, torsion will modify the geometrical quantities - for example, the most general antisymmetric connection satisfying metric compatibility will now be given by:

$$\Gamma^\lambda{}_{\mu\nu} = T^\lambda{}_{\mu\nu} + T_{\mu\nu}{}^\lambda + T_{\nu\mu}{}^\lambda + \mathring{\Gamma}^\lambda{}_{\mu\nu} \tag{2.2.12}$$

It should be noted here, that, compared to GR, we only dropped the symmetric assumption about the connection we still demand the metricity just like it is done in GR. There is no reason why we should drop metricity as well - it would not add any benefits to our theory and would just unnecessarily complicate it. Also, up to the precision that we can measure in experiments, metric compatibility is also a feature of our universe, hence, we keep it in our model.

Now, inspired by (2.2.2) we can also postulate the transformation rule for this connection:

$$\Gamma^\mu{}_{\alpha\beta} \to \tilde{\Gamma}^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\alpha\beta} + \delta\Gamma^\mu{}_{\alpha\beta} \tag{2.2.13}$$

Now that we have a new connection, we can write down the modified geodesic equation and see how it transforms under conformal transformation (1.0.1).

$$\begin{aligned} \frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau}\nabla_\lambda\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = 0 \;\to\; \frac{\mathrm{d}x^\lambda}{\mathrm{d}\tilde{\tau}}\left(\tilde{\nabla}_\lambda\frac{\mathrm{d}x^\mu}{\mathrm{d}\tilde{\tau}}\right) &= \frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tilde{\tau}^2} + \tilde{\Gamma}^\lambda{}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tilde{\tau}}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tilde{\tau}} \\ &= e^{-\theta}\left(\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tilde{\tau}^2} + (\Gamma^\lambda{}_{\alpha\beta} + \delta\Gamma^\lambda{}_{\alpha\beta})\dot{x}^\alpha\dot{x}^\beta - \dot{\theta}\dot{x}^\mu\right) = 0 \end{aligned} \tag{2.2.14}$$

And now, by demanding that geodesics get mapped onto themselves in every conformal frame, we can deduce the transformation rules for the modified connection and symmetric and antisymmetric parts of Torsion tensor from equations (2.2.2), (2.2.12) and (2.2.14) :

$$\delta\Gamma^\lambda{}_{\alpha\beta} = \delta^\mu_\alpha\partial\beta\theta, \quad \delta T^\mu{}_{\alpha\beta} = \delta^\mu_{[\alpha}\partial_{\beta]}\theta \quad \delta T_{(\alpha\beta)}{}^\mu = g_{\alpha\beta}\partial^\mu\theta - \delta^\mu_{(\alpha}\partial_{\beta)}\theta \tag{2.2.15}$$

Notice, that $T^\mu{}_{\alpha\beta}$ transforms quite similarly to the gauge connection. If we now take a trace of it, we will have:

$$\mathrm{Tr}\Big[T^\mu{}_{\alpha\beta}\Big] = T^\mu{}_{\alpha\mu} \equiv T_\alpha \quad \text{therefore} \quad T_\alpha \to T_\alpha + \partial_\alpha \log\Omega \tag{2.2.16}$$

where $\Omega = \exp(\theta(x))$. And we see, that the torsion trace transforms exactly like a gauge connection, therefore we propose to treat it as a gauge boson of the conformal transformation. This way all the physically meaningful quantities will get mapped onto themselves in any conformal frame, making the General Relativity conformal.

It is also interesting to notice, that when we are dealing with the torsionless GR, we do not have any quantity, for which we can propose a transformation rule that will make it possible for the geodesic equation to transform covariantly, whereas in the theory with torsion, we are able to postulate the transformation rule for the Torsion tensor in such a way that it makes the equation covariant under conformal transformation.

It can be shown as well, that trace of the torsion is the only one of its irreducible components that transforms under conformal transformations and that, as we will see, if we keep only the trace part of the torsion tensor nonzero and put all its other components to zero, this will suffice to make the GR conformal. And since it is usually wise to keep the number of additionally introduced quantities into the theory minimal, we will set all the components of torsion except its trace to 0 in our calculations.

**Bi-Scalar**

The fact that torsion gives us conformality can also be easily demonstrated by using Riemann normal coordinates and looking at geodesics equation again in these coordinates:

$$\left.\frac{\mathrm{d}^2 x^\mu}{d\tau^2}\right|_{x=x_0} = 0 \Rightarrow \mathring{\Gamma}^\lambda{}_{\alpha\beta}|_{x=x_0} = 0 \tag{2.2.17}$$

And again, it is not conformally invariant, since under conformal transformation $\mathrm{d}x^\mu/\mathrm{d}\tau \to \Omega^{-1}(x)\mathrm{d}x^\mu/\mathrm{d}\tau$. Hence, we can impose (2.2.17) in one frame, but not consistently in all conformal frames. The problem can be solved by introducing torsion through the so called bi-scalar:

$$\chi(\tau,\tau_0) = \chi(x,x_0) = \exp\left(\int_{x_0;\gamma}^{x} \mathrm{d}\tau T_\mu \frac{\mathrm{d}x^\mu}{d\tau}\right)^{1} \tag{2.2.18}$$

where $T_\mu$ is torsion trace again.

The reason we introduce the bi-scalar $\chi(x,x_0)$ is, that it gives us the notion of a geometrical scalar field, as performing conformal transformation changes $T_\mu \to T_\mu + \partial_\mu \log\Omega$, such that

$$\chi(x,x_0) \to \Omega(x)\chi(x,x_0)\Omega(x)^{-1} \quad \text{and} \quad \text{in all frames} \lim_{x\to x_0}\chi(x,x_0) = 1 \tag{2.2.19}$$

meaning, that $\chi(x,x_0)$ is indeed a scalar under conformal transformation.

It might seem redundant to see how torsion makes GR conformal in two ways, but we will need the bi-scalar during the one-loop calculation in the Appendix B.

Using this bi-scalar, we can define a modified geodesic tangent vector as

$$\dot{\Gamma}^\mu = \chi(x,x_0)\dot{\Gamma}^\mu = \chi(\tau,\tau_0)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{2.2.20}$$

which transforms as $\Gamma^\mu \to \Omega(\tau_0)^{-1}\dot{\Gamma}^\mu = \Omega(x_0)^{-1}\dot{\Gamma}^\mu$ under conformal rescaling.

Using all of this together with equations (2.2.18) and (2.2.20), we can define the conformally invariant geodesics equation:

$$\frac{\mathrm{d}\dot{\Gamma}^\mu}{\mathrm{d}\tau}|x = x_0 = 0 \tag{2.2.21}$$

that will be true in any conformal frame now. So, as we see, by adding torsion to GR it acquires the conformal symmetry.

## Using Noether's Theorem

Another way to look at the issue of conformality of GR is from the point of view of Noether's Theorem.

We know, that symmetries in physical systems are accompanied with corresponding conserved charges $\nabla_\mu J_i^\mu = 0$. We can write the same condition for the Weyl symmetry in GR but in terms of the energy-momentum tensor $T_{\mu\nu}$:

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = 0 \tag{2.2.22}$$

This should hold in general, but if we calculate it, we'll see, that $g^{\mu\nu}\langle T_{\mu\nu}\rangle \neq 0$, so it looks like the symmetry is anomalous. To fix this, we can introduce the torsion into the theory. When we do that though, the condition (2.2.22) will still not be satisfied - it has to be modified as well. It will look like:

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle + \langle \nabla_\mu \Pi^\mu\rangle = 0 \tag{2.2.23}$$

[1] Locally $\chi(x,x_0)$ is well defined, as there will be only one geodesic linking $x$ and $x_0$.

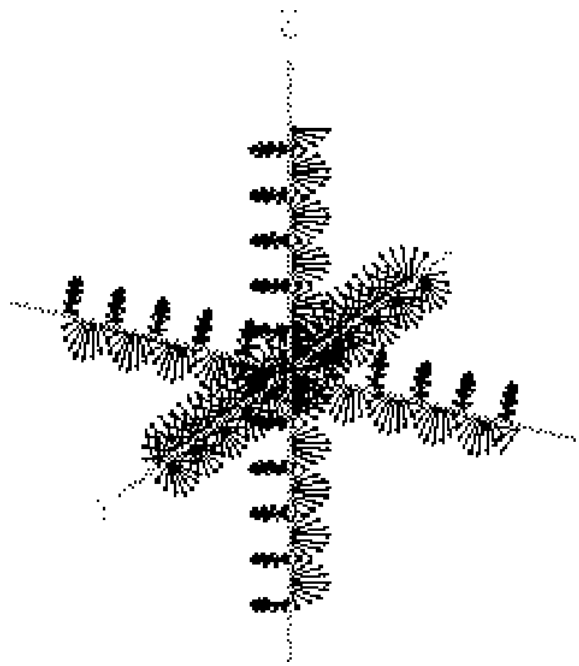


Figure 2.1: Parallel transported vectors do not change if the transport direction is along their own direction, but if the transport direction acquires an orthogonal component then the vector starts precessing along the transport direction. The so-called Cartan staircase [16] demonstrates this feature well.

But this only holds in 4 dimensions. Let's see why. We know, that

$$\langle T^{\mu}_{\mu} \rangle = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \text{Can be shown, that this can be written as} = \bar{\nabla}_{\mu}V^{\mu} \tag{2.2.24}$$

For convenience we denote $\epsilon_D \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$. So, $\langle T^{\mu}_{\mu} \rangle = \bar{\nabla}_{\mu}V^{\mu} = \overset{\circ}{\nabla}_{\mu}V^{\mu} - (D-4)T_{\mu}V^{\mu}$. The effective action can be then written as $W = \frac{D-4}{D}\epsilon_D$ where $D$ is the spacetime dimension. $\Pi^{\mu}$ is then:

$$\Pi^{\mu} = \frac{\delta W}{\delta T_{\mu}} = -(D-4)V^{\mu} \tag{2.2.25}$$

Therefore, in 4D $\bar{\nabla}_{\mu}\Pi^{\mu} = -\bar{\nabla}_{\mu}V^{\mu}$ and this means, that

$$\langle \bar{\nabla}_{\mu}\Pi^{\mu} + T^{\mu}_{\mu} \rangle = 0 \tag{2.2.26}$$

just as it is needed. So, we are convinced again, that by introducing torsion we can introduce conformal symmetry into GR.

### 2.2.1 Geometrical Features of Torsion

It is very important to mention, that it is not an accident, that conformality can be achieved by using torsion. The idea to use torsion as a quantity that gives a theory the scale invariance can be motivated by looking at the geometrical properties of the torsion. This can be demonstrated by doing a parallel transport of a vector on a manifold with torsion (see Fig.(2.1)). This figure is called the Cartan staircase [16].

If the movement direction co-linear with the vector, nothing interesting happens - the vector itself stays the same. But as soon as we start moving it in some other direction that has an orthogonal component to the direction of the vector, it starts precessing around this direction and also gets rescaled. As a consequence, if we decide to parallel-transport two vectors along each other to build a parallelogram, it will not close (see Fig.(2.2)). This rescaling property is precisely the reason why it was reasonable to try torsion as an object that would give gravity the scale symmetry.

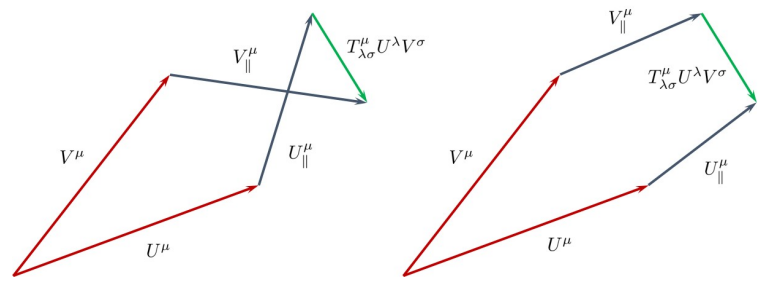



Figure 2.2: Vectors parallel transported on a manifold with torsion do not form a parallelogram - it does not close. The "difference with closing" is proportional to torsion (denoted by $T$ here). [17]

### 2.2.2 Scalar, Vector and Spinor fields

In the previous section we saw, that introducing torsion does indeed make General Relativity conformal. For future convenience (when we will be dealing with inflation/scalar fields + gravity and vector fields) we should see, how different types of quantities such as scalars, vectors and also, just to complete the picture for the Standard Model fields - spinors as well, transform under (1.0.1) and then explicitly construct the covariant derivatives for all of them.

As we said, we proposed treating torsion trace $T_\mu$ as the gauge boson of conformal transformations based on its transformation rule:

$$T_\mu \to T_\mu + \partial_\mu \theta \tag{2.2.27}$$

where $\theta$ is the conformal transformation parameter defined in (1.0.1). And referring to the discussion above, another way to motivate it is, that $T_\mu$ acting on vectors generates scale transformations, as a consequence of parallel transport.

Based on this, just like with gauge theories, we can derive a conformally covariant derivative now. Let's do this for all three types of fields in our theory.

**1) Scalars** We base it on the fact, that the scalar field transforms as:

$$\phi \to e^{-\frac{D-2}{2}\theta}\phi \tag{2.2.28}$$

So that we get for the covariant derivative:

$$\bar{\nabla}_\mu \phi = \partial_\mu \phi + \frac{D-2}{D-1} T^\lambda{}_{\lambda\mu}\phi \tag{2.2.29}$$

And we can also generalize this to some arbitrary $\Psi$ field of an arbitrary conformal weight $\omega$ as:

$$\bar{\nabla}_\mu \Psi = \nabla_\mu \Psi + (\omega_g - \omega) T_\mu \Psi \tag{2.2.30}$$

where $\nabla_\mu$ is the manifold covariant derivative (i.e. the one that guarantees the diffeomorphism invariance), and $\omega_g$ us the geometrical dimension of $\Psi$ - difference between numbers of it's up and down indices, giving us an "effective dimension" of it. E.g. if $\Psi$ is a $\binom{q}{p}$ tensor, then $\omega_g = q - p$. **Note** however, that in order to be able to construct $\overset{\omega}{\nabla}_\mu$ for a given field, we should know its scaling dimension $\omega$. This is similar to the gauge derivative of fields charged under $U(1)$: in that case, one should know the hypercharge of the representation, upon which the gauge derivative acts. $Y$, which is different for different fields - in our case, the role of the hypercharge is played by $\omega$.

In principle we can also do the same for other types of objects like Spinors and Gauge (Vector) fields:

**2) Spinors and Gauge (Vector) fields** Analogously we should look at the transformation rules for these quantities and derive the conformally covariant derivative from them:

$$\psi \to e^{-\frac{D-1}{2}\theta}\psi \tag{2.2.31}$$

$$A_\mu \to e^{-\frac{D-4}{2}\theta}A_\mu \tag{2.2.32}$$

gives us

$$\bar{\nabla}_\mu \psi = \nabla_\mu \psi + T^\lambda{}_{\lambda\mu}\psi \tag{2.2.33}$$

$$\bar{\nabla}_\mu A_\nu = \mathring{\nabla}_\mu A^\nu + T_\nu A_\mu - g_\mu \nu T_\sigma A^\sigma + \frac{D-4}{2}T_\mu A_\nu \tag{2.2.34}$$

So that now we are able to construct an invariant theory under both - gauge and conformal symmetries, as it should be (since the standard model should already be gauge invariant).

**Covariant Derivatives**

So, in the end we can classify the covariant derivatives in the following way:

1) GR covariant derivative $\mathring{\nabla}_\mu$:

$$\mathring{\nabla}_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda}V^\lambda \tag{2.2.35}$$

2) Covariant derivative with contorsion tensor $\nabla_\mu$:

$$\nabla_\mu = \mathring{\nabla}_\mu \delta^\lambda{}_\nu + K^\lambda{}_{\nu\mu} \tag{2.2.36}$$

3) Conformal covariant derivative $\bar{\nabla}_\mu$:

$$\bar{\nabla}_\mu = \mathring{\nabla}_\mu \delta^\lambda{}_\nu - K^\lambda{}_{\nu\mu} + \frac{D}{2}T_\mu \tag{2.2.37}$$

where $D$ is the dimension of the spacetime and $K^\lambda{}_{\nu\mu}$ is the contorsion tensor:

$$K^\lambda{}_{\nu\mu}V_\lambda = g_{\mu\nu}T^\lambda V_\lambda - T_\nu V_\mu \tag{2.2.38}$$

This $3^{rd}$ covariant derivative is the one that we will be using.

### 2.2.3 Modified Curvatures

And to get ready for the calculations in the following chapters, let's derive how curvatures get modified when we introduce torsion.

Luckily, Ricci tensor and scalar have already been expressed in terms of the contorsion tensor [4]. We start with Ricci tensor below and rewrite it in terms of torsion trace $T_\mu \equiv T^\lambda{}_{\mu\lambda}$ and from there we can already get the Ricci scalar as well.

$$\bar{R}_{\mu\nu} = \mathring{R}_{\mu\nu} + \underset{1}{\mathring{\nabla}_\lambda K^\lambda{}_{\mu\nu}} - \underset{2}{\mathring{\nabla}_\nu K^\lambda{}_{\mu\lambda}} + \underset{3}{K^\lambda{}_{\gamma\lambda}K^\gamma{}_{\mu\nu}} - \underset{4}{K^\lambda{}_{\mu\gamma}K^\gamma{}_{\lambda\nu}} \tag{2.2.39}$$

Where with overdots we denote the quantities that are built using Christoffel symbols and therefore do not contain torsion. To rewrite this in terms of torsion trace, we use the expression of the contorsion tensor in terms of torsion tensor (2.2.10) and torsion tensor in terms of torsion trace by using (2.2.11) and putting all the components except the torsion trace to 0:

$$K^\alpha{}_{\beta\gamma} = \frac{1}{2}\Big(T^\alpha{}_{\beta\gamma} - T_\beta{}^\alpha{}_\gamma - T_\gamma{}^\alpha{}_\beta\Big) \text{ and } T_{\alpha\beta\mu} = \Big(T_\beta g_{\alpha\mu} - T_\mu g_{\alpha\beta}\Big) \tag{2.2.40}$$

We can now express Ricci tensor with torsion trace by plugging (2.2.40) and (2.2.10) into (2.2.39). Let's do this term by term. (*For explicit details please refer to the Appendix A*)

$$\textcircled{1} = \overset{\circ}{\nabla}_\lambda K^\lambda{}_{\mu\nu} = \frac{1}{2}\Big(\underbrace{\overset{\circ}{\nabla}_\lambda T^\lambda{}_{\mu\nu}}_{\textcircled{1.I}} - \underbrace{\overset{\circ}{\nabla}_\lambda T_\mu{}^\lambda{}_\nu}_{\textcircled{1.II}} - \underbrace{\overset{\circ}{\nabla}_\lambda T_\nu{}^\lambda{}_\mu}_{\textcircled{1.III}}\Big) \tag{2.2.41}$$

$$\textcircled{1.I} = \overset{\circ}{\nabla}_\lambda T^\lambda{}_{\mu\nu} = \overset{\circ}{\nabla}_\lambda g^{\rho\lambda} T_{\rho\mu\nu} = \text{can be shown} = \Big(\overset{\circ}{\nabla}_\nu T_\mu - \overset{\circ}{\nabla}_\mu T_\nu\Big) \tag{2.2.42}$$

similarly

$$\textcircled{1.II} = \overset{\circ}{\nabla}_\lambda T_\mu{}^\lambda{}_\nu = \Big(g_{\mu\nu}\overset{\circ}{\nabla}_\lambda T^\lambda - \overset{\circ}{\nabla}_\mu T_\nu\Big) \tag{2.2.43}$$

$$\textcircled{1.III} = \text{by analogy with } \textcircled{1.II} = \Big(g_{\mu\nu}\overset{\circ}{\nabla}_\lambda T^\lambda - \overset{\circ}{\nabla}_\nu T_\mu\Big) \tag{2.2.44}$$

So that we get:

$$\boxed{\textcircled{1} = \frac{1}{2}\Big(\overset{\circ}{\nabla}_\nu T_\mu - \overset{\circ}{\nabla}_\mu T_\nu - 2g_{\mu\nu}\overset{\circ}{\nabla}_\lambda T^\lambda + \overset{\circ}{\nabla}_\mu T_\nu + \overset{\circ}{\nabla}_\nu T_\mu\Big) = \Big(\overset{\circ}{\nabla}_\nu T_\mu - g_{\mu\nu}\overset{\circ}{\nabla}_\lambda T^\lambda\Big)} \tag{2.2.45}$$

Now the second one:

$$\textcircled{2} = \overset{\circ}{\nabla}_\nu K^\lambda{}_{\mu\lambda} = \frac{1}{2}\Big(\underbrace{\overset{\circ}{\nabla}_\nu T^\lambda{}_{\mu\lambda}}_{\textcircled{2.I}} - \underbrace{\overset{\circ}{\nabla}_\nu T_\mu{}^\lambda{}_\lambda}_{\textcircled{2.II}} - \underbrace{\overset{\circ}{\nabla}_\nu T_\lambda{}^\lambda{}_\mu}_{\textcircled{2.III}}\Big) \tag{2.2.46}$$

$$\textcircled{2.I} = \text{by definition of torsion trace} = \overset{\circ}{\nabla}_\nu T_\mu \tag{2.2.47}$$

$$\textcircled{2.II} = \overset{\circ}{\nabla}_\nu T_\mu{}^\lambda{}_\lambda = ... = 0 \tag{2.2.48}$$

$$\textcircled{2.III} = \overset{\circ}{\nabla}_\nu T_\lambda{}^\lambda{}_\mu = ... = -\overset{\circ}{\nabla}_\nu T_\mu \tag{2.2.49}$$

So that

$$\boxed{\textcircled{2} = \frac{1}{2}\Big(\overset{\circ}{\nabla}_\nu T_\mu + \overset{\circ}{\nabla}_\nu T_\mu\Big) = \overset{\circ}{\nabla}_\nu T_\mu} \tag{2.2.50}$$

Now the third one:

$$\textcircled{3} = K^\lambda{}_{\gamma\lambda} K^\gamma{}_{\mu\nu} = \frac{1}{4}\Big(T^\lambda{}_{\gamma\lambda} - T_\gamma{}^\lambda{}_\lambda - T_\lambda{}^\lambda{}_\gamma\Big)\Big(T^\gamma{}_{\mu\nu} - T_\mu{}^\gamma{}_\nu - T_\nu{}^\gamma{}_\mu\Big) \tag{2.2.51}$$

Taking into account that $T^\lambda{}_{\gamma\lambda} = T_\gamma$, also that $T_\gamma{}^\lambda{}_\lambda = 0$ and $T_\lambda{}^\lambda{}_\gamma = -T^\lambda{}_{\gamma\lambda} = -T_\gamma$ we get

$$\boxed{\textcircled{3} = \frac{1}{2}T_\gamma g^{\rho\gamma}\Big(T_{\rho\mu\nu} - T_{\mu\rho\nu} - T_{\nu\rho\mu}\Big) = ... = \Big(T_\mu T_\nu - T_\lambda T^\lambda g_{\mu\nu}\Big)} \tag{2.2.52}$$

And finally:

$$\textcircled{4} = K^\lambda{}_{\mu\gamma} K^\gamma{}_{\lambda\nu} = \frac{1}{4}\Big(T^\lambda{}_{\mu\gamma} - T_\mu{}^\lambda{}_\gamma - T_\gamma{}^\lambda{}_\mu\Big)\Big(T^\gamma{}_{\lambda\nu} - T_\lambda{}^\gamma{}_\nu - T_\nu{}^\gamma{}_\lambda\Big) \tag{2.2.53}$$

After some calculations we get for it:

$$\boxed{\textcircled{4} = \left(T_\mu T_\nu - T_\lambda T^\lambda g_{\mu\nu}\right)} \tag{2.2.54}$$

And now we can finally assemble the Ricci tensor.

$$\begin{aligned}\bar{R}_{\mu\nu} &= \mathring{R}_{\mu\nu} + \textcircled{1} - \textcircled{2} + \textcircled{3} - \textcircled{4} = \\ &= \mathring{R}_{\mu\nu} - 2\mathring{\nabla}_\nu T_\mu - g_{\mu\nu}\mathring{\nabla}_\lambda T^\lambda + \left(T_\mu T_\nu - T_\lambda T^\lambda g_{\mu\nu}\right) - \left(T_\mu T_\nu - T_\lambda T^\lambda g_{\mu\nu}\right)\end{aligned} \tag{2.2.55}$$

giving us:

$$\boxed{\bar{R}_{\mu\nu} = \mathring{R}_{\mu\nu} - 2\mathring{\nabla}_\nu T_\mu - g_{\mu\nu}\mathring{\nabla}_\lambda T^\lambda + 2[T_\mu T_\nu - T_\lambda T^\lambda g_{\mu\nu}]} \tag{2.2.56}$$

and if we contract it with the metric $g^{\mu\nu}\bar{R}_{\mu\nu} = \bar{R}$ we can get the expression for the Ricci scalar:

$$\boxed{\bar{R} = \mathring{R} - 6\mathring{\nabla}_\lambda T^\lambda - 6T_\lambda T^\lambda} \tag{2.2.57}$$

This is all that we will need for our further calculations.

# Chapter 3

# Inflation - What is it?

***In this chapter we will discuss the basics of Inflation and Cosmological Perturbations***

*Note: This chapter closely follows the introduction to inflation described in [11].*

## 3.1 Basics of Inflation

Cosmology is quite special area in physics in the sense that not a lot of experiments can be conducted - we mainly rely on the observational data of the events that happened in the early universe. One of the most important sources for harvesting this kind of data is the Cosmic Microwave Background (CMB) created by quantum fluctuations in the early universe - a huge amount of statistical information is extracted from CMB (in form of 2 or 3 point functions, for example) that allows us to draw conclusions about the early universe and its structure.

Another very important source is the LSS. Certain processes (*spoiler alert* - it will turn out to be inflation) happening in the universe stretch the early universe quantum fluctuations turning them into the classical gravitational instabilities. These instabilities start growing by attracting and gathering matter around them, so that eventually they turn into the Large Scale Structures (LSS) that we observe today (like galaxies, for example). The distribution of these objects can also be studied statistically, allowing us to further constrain the theoretical models.

So, we see, that CMB does indeed play a central role in cosmology, therefore, it would be very important to know, what its origin is. Currently, the leading hypothesis that explains how these fluctuations were formed and evolved in time is *inflation* [7, 8].

This Chapter will be devoted to understanding the basics of inflation.

### 3.1.1 The Metric and Dynamics of the Universe

In order to introduce inflation, we should prepare the ground. First of all, we obviously need a metric of our universe. Based on the observational data it is reasonable to assume, that we live in a *homogeneous* and *isotropic* universe. In this case, the most general interval that we can write for an isotropic and homogeneous universe is the Friedmann-Robertson-Walker (FRW) metric:

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right). \tag{3.1.1}$$

Where $a(t)$ is the so-called the scale factor and $k$ is the curvature of the universe. The scale factor tells us how fast the universe is expanding/shrinking. Introducing the scale factor is actually very convenient, since it allows us to describe the dynamics of the universe by just a single $a(t)$ function. The shape of $a(t)$

itself also changes throughout the history of the universe depending on the type of the dominant energy[1] in the universe. Therefore, to be able to study the full evolution of the universe, we should include all of the energy content of the universe into the equation that determines $a(t)$. This can be done by using the Einstein Equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \text{ where } G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{3.1.2}$$

And based on the homogeneity and isotropy features again, we can assume, that our universe behaves like a general perfect fluid. In this case, the energy-momentum tensor will have the following shape:

$$T^{\mu}_{\nu} = (\rho + p) u^{\mu} u_{\nu} - p\ \delta^{\mu}_{\nu}. \tag{3.1.3}$$

Besides the scale factor, there is another very important parameter in Cosmology (it is actually built using the scale factor) known as the *Hubble Parameter* - $H$:

$$H \equiv \frac{\dot{a}}{a}. \tag{3.1.4}$$

Based on the sign of the Hubble Parameter we can determine, whether the universe is expanding or collapsing - it is expanding if $H > 0$ and collapsing if $H < 0$.

Employing Hubble parameter, Einstein equation (3.1.2) can be written as two coupled, non-linear ordinary differential equations:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho - \frac{k}{a^2}, \tag{3.1.5}$$

and

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p), \tag{3.1.6}$$

where overdots denote derivatives with respect to physical time $t$. These equations are called *Friedmann equations*. They explicitly connect the matter density/pressure to the scale factor.

By combining these two equations, we can derive the conservation equation for the energy-momentum tensor:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + 3H(\rho + 3p) = 0. \tag{3.1.7}$$

We also introduce the equation of state parameter $\omega$:

$$\omega \equiv \frac{p}{\rho} \tag{3.1.8}$$

This allows us to solve the Friedmann equations (3.1.5), (3.1.6) (together with (3.1.7) conservation equation) conveniently for the scale factor (i.e. evolution of the universe) in different regimes determined by $\omega$ (the energy content of the universe):

$$a(t) \propto \begin{cases} t^{2/3(1+\omega)} & \text{if } \omega \neq -1, \\ e^{Ht} & \text{if } \omega = -1. \end{cases}$$

Since, as we said, we have several different types of the energy/matter contents in the universe, when we refer to the density and pressure in the Friedmann equations, we should sum the contributions to these quantities from all the contents. To make it clear, let's say we have e.g. 2 significant energy sources in the universe, each one of them has its own energy density - in this case, the full energy density that we put in the Friedmann eqaution is given by the sum of these two densities. And the same applies to the pressure as well:

$$\rho \equiv \Sigma_{i=1}^{n} \rho_i \qquad \text{and} \qquad p \equiv \Sigma_{i=1}^{n} p_i. \tag{3.1.9}$$

[1] By the dominant energy we mean, that the universe can be dominated by either matter, radiation or a cosmological constant, for example. This becomes clear, if we think about the history of the universe - at first, when the size of the universe was extremely small and therefore the temperature was very high, we had the a lot of radiation and by far the largest amount of energy was concentrated in radiation. Though, due to the different scaling of radiation and matter energies with the scale-factor, the universe switched from the radiation dominated era to a matter dominated one some time later.

### 3.1.2 Introducing Inflation - WHY?

Now that we have enough knowledge about the metric and dynamics of the universe, we are ready to ask a question: why do we need in introduce inflation? Isn't it fine, if we just assume, that there was a "Big Bang" and then the radiation domination era, that transitioned into the matter dominated one and then the dark-matter/cosmological constant dominated one? It turns out, that this straightforward approach does not allow to solve some of the major problems in Cosmology convincingly. This served as a motivation to introduce the theory of inflation.

Let's now discuss these problems first and then we'll explain how inflation solves them.

#### Horizon Problem

In order to understand the idea of this problem better, we should introduce the notion of *comoving horizon* or, alternatively - *causal horizon*:

$$\tau \equiv \int_0^t \frac{\mathrm{d}t'}{a(t')} = \int_0^a \frac{\mathrm{d}a}{Ha^2} = \int_0^a \mathrm{d}\ln a \left(\frac{1}{aH}\right), \tag{3.1.10}$$

Intuitively the causal horizon $\tau$ is the maximum distance a light ray can travel between time 0 and time $t$. Through (3.1.10) the comoving horizon can also be connected to the so called *comoving Hubble radius* $(aH)^{-1}$, which is another important quantity. From the point of view of an observer (at a given point in time) it can be conceptually interpreted as a boundary, that connects two regions - in one of them, inside the comoving Hubble radius, the particles are moving slower than the speed of light relative to the observer, whereas in the region outside of the radius, they are moving away faster than speed of light away from the observer (important thing to note here, is that these particles are not actually moving faster than speed of light on their own - the distance between particles and the observer does indeed increase faster than the speed of light, but this happens due to the rapid expansion of the metric itself). For a universe with parameter $\omega$ we will have

$$(aH)^{-1} = H_0^{-1} a^{\frac{1}{2}(1+3\omega)}. \tag{3.1.11}$$

This allows us to see, why the comoving Hubble radius is so important for the analysis: its behaviour is determined by the sign of $(1+3\omega)$. The important point to notice here, that in case of the conventional Big Bang model, $(aH)^{-1}$ grows monotonically and we get for the comoving horizon $\tau$,

$$\tau \propto a^{\frac{1}{2}(1+3\omega)}. \tag{3.1.12}$$

In intuitive words this means, that the fraction of the universe in causal contact increases with time. This implies, that we see new patches of the universe entering our causal horizon and most importantly, that these patches have always been outside of this horizon before - so they never communicated with each other before.

If we look at the CMB structure, though, we will see, that these new patches have strikingly similar characteristics statistically, meaning, that despite the fact that they have never communicated before to come into an equilibrium with similar characteristics, they still have this feature. Of course, it can be argued, that this can still be achieved and this is true, but for this we would need to fine-tune the initial conditions very precisely and this does not look like a convincing physical theory.

#### Flatness Problem

Another issue is the flatness problem. If we look at the Einstein equation, we see, that the matter and spacetime influence each other - matter modifies the structure of the spacetime and modified spacetime backreacts on the matter. The interaction between spacetime and matter is actually dynamical, meaning that it can transport energy as gravitational waves, for example. All of this makes it very suspicious, that the spacetime is so well approximated with a flat metric.

To demonstrate, how skeptical we should be about this approximation, let's have a closer look at the Friedmann equation (3.1.5).

If we define

$$\Omega(a) \equiv \frac{\rho(a)}{\rho_{crit}(a)}, \qquad \rho_{crit}(a) \equiv 3H(a)^2, \tag{3.1.13}$$

(where $\rho_{crit}$ is the (critical) energy density for the universe to be flat), then the Friedmann equation can be written as

$$1 - \Omega = -\frac{k}{(aH)^2}. \tag{3.1.14}$$

It is important to note, that $\Omega(a)$ depends on time and since in the standard Big-Bang model the inverse of the comoving Hubble radius $(aH)$ decreases with time (meaning that the comoving radius itself grows), from (3.1.14) it becomes clear, that $|\Omega - 1|$ diverges with time. This implies, that $\Omega = 1$ is an unstable condition. But $\Omega(a_0) \sim 1$ corresponds to a flat universe, therefore the observed flatness of the universe today requires us to fine-tune $\Omega$ extremely close to 1 in the early universe.

Again, as with the Horizon problem, it is important to note, that both of these problems can be addressed within the framework of the standard Big-bang model, but since this requires very finely tuned initial conditions, it is highly improbable, that Big-Bang model is a complete theory.

On the other hand, as we will now demonstrate, if we introduce a period of inflation into this model, then both of these problems get solved simultaneously and very elegantly.

### 3.1.3 Inflation

It is very interesting to note, that if we pay a closer attention to the above problems, we will see, that they have one thing in common - in both cases, the issues arise because of the fact, that the comoving Hubble radius is strictly increasing in case of the standard Big-Bang model. The main idea of Inflation in this context is simple, yet beautiful - we simply assume, that there was a period in the history of the universe, where the behaviour of the comoving Hubble radius was decreasing.

#### General Idea

To better grasp the intuition behind inflation, let us recall how the comoving horizon we defined as a logarithmic integral of the comoving Hubble radius in (3.1.10):

$$\tau = \int_0^a \mathrm{d}\ln a \left(\frac{1}{aH}\right). \tag{3.1.15}$$

It is important to notice, though, that there is a subtle difference between the notions of the comoving horizon $\tau$ and the comoving Hubble radius $(aH)^{-1}$.

$\tau$ - If particles are separated by distances greater than $\tau$, they ***never*** could have communicated with one another.

$(aH)^{-1}$ - If they are separated by distances larger than $(aH)^{-1}$, then they can not communicate with each other ***now***.

The fact that there is a difference between these notions is what makes it possible to solve the horizon problem - it allows us to assume, that $\tau$ is indeed much larger than $(aH)^{-1}$ today ( meaning that the patches that entered the horizon can not communicate now), but this has not always been the case, therefore the patches **could** have communicated in the past, allowing them to reach a shared equilibrium, therefore giving them the same statistical properties. (See Fig. (3.1))

This can be clearly seen from the (3.1.15) expression, if we assume, that the comoving Hubble radius was much larger in the early universe than it is today (in this case $\tau$ would get the largest contribution

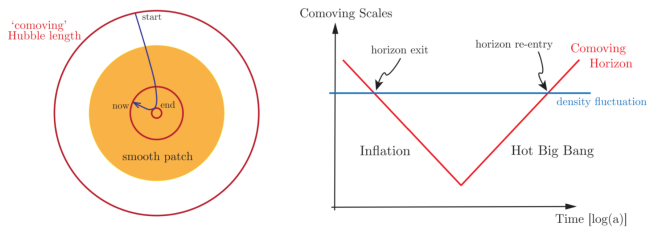



Figure 3.1: [11]
**Left**: Evolution of the comoving Hubble radius in an inflationary universe. " During inflation, the comoving Hubble sphere decreases in size and starts expanding after inflation. Inflation is therefore a mechanism to 'zoom-in' on a smooth sub-horizon patch" [11].
**Right**: Timeline (evolution) of scales.

from it in the past). Basically, we resolve both of the issues by demanding a period when the Hubble radius was decreasing and Inflation gives us that, since during Inflation $a(t)$ grows exponentially, whereas $H$ stays approximately constant, meaning that during inflationary period, the Hubble radius does indeed decrease.

Now that we know the general idea of Inflation, let's actually check whether the problems can indeed be solved conveniently:

### Flatness Problem + Inflation

To address this problem, we should look at the Friedmann equation (3.1.5). We clearly see, that if we decrease the comoving Hubble radius, this will result in *"a flatter universe"* rather than the other way around. This will obviously solve the problem, since the $\Omega = 1$ basically turns out to be an attractor point during inflation.

### Horizon Problem + Inflation

Let's think what would be the results of a decreasing comoving horizon - based on the intuitive definition of the comoving horizon it becomes clear, that parts of the present universe, that are coming into the horizon today and some that are still outside of it, were inside the horizon before inflation (See Fig.(3.1)). This directly implies, that these patches were in a causal contact before inflation hence, as we said above, this would result in an information exchange that allows these patches to acquire similar statistical properties back then. Though, later, due to the presence of inflation, they were pushed out of the horizon and, of course, lost the causal contact with other parts.

So, we clearly see, that assuming an inflationary period in the standard Big-Bang model, the uniformity of the Cosmic Microwave Background gets demystified and elegantly explained.

### Conditions for Inflation

In general, there are numerous ways to write an equivalent mathematical definitions for Inflation. We will only present two of them below.

1) **Accelerated Expansion**

From the relation

$$\frac{d}{dt}(aH)^{-1} = \frac{-\ddot{a}}{(aH)^2}, \tag{3.1.16}$$

it is evident, that a decreasing comoving Hubble radius implies the accelerated expansion, where the acceleration is given by:

$$\frac{d^2a}{dt^2} > 0. \tag{3.1.17}$$

2) **Negative pressure**

Accelerated expansion condition can also be translated in terms of conditions on the energy-momentum tensor. For this we put $\ddot{a} > 0$ in equation (3.1.6) to get:

$$p < -\frac{1}{3}\rho \tag{3.1.18}$$

Implying, that accelerated expansion of the universe is equivalent to the condition of having a negative pressure.

And for further convenience (we will exploit this in the end of the thesis), we can also rewrite the first definition (3.1.17) in terms of Hubble flow parameter derivatives:

$$\frac{\ddot{a}}{a} = H^2(1-\epsilon_1), \text{ where } \epsilon_1 \equiv -\frac{\dot{H}}{H^2} \tag{3.1.19}$$

This gives us another definition for inflation - now in terms of $\epsilon$:

$$\boxed{\epsilon_1 = -\frac{\dot{H}}{H^2} = -\frac{\mathrm{d}\ln(H)}{\mathrm{d}N} < 1} \tag{3.1.20}$$

where we introduced N - the number of e-folds: When inflation was introduced, as we said, the intention was to cure cosmological problems and in order to do so, one of the requirements for inflation was to expand the universe at least 50-60 e-folds, meaning that the universe should have expanded $\sim e^{60}$ times for the theoretical predictions to be in accordance with the experimental data. The number of e-folds is defined as:

$$\mathrm{d}N = \frac{\mathrm{d}a}{\mathrm{d}t}\frac{1}{a}\mathrm{d}t = \frac{\mathrm{d}a}{a} = \mathrm{d}(\ln(a)) \implies N = \ln(a) \implies a = e^N \tag{3.1.21}$$

### 3.1.4 Inflation as a scalar field

The most common way of treating inflation is from the the particle physics point of view - in the simplest models of inflation we treat it as a single scalar field $\phi$ - the inflaton, which, as we could have guessed from the name, drives inflation. The choice of scalar field is not accidental - according to measurements of CMB fluctuations, they are isotropic, therefore there should not be a preferred direction and if we had treated inflaton as a vector or tensor field, we would obviously have introduced a ceratin direction into the problem, which would contradict the experimental data.

We can write the action for this scalar field (in this case it's minimally coupled to gravity):

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] = S_{EH} + S_\phi. \tag{3.1.22}$$

The potential $V(\phi)$ describes the self-interactions of the scalar field. So, scalar fields serve as matter fields in this action and we should therefore derive the expression for the energy-momentum tensor for it:

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + V(\phi)\right) \tag{3.1.23}$$

And also, the EoM for the field itself is:

$$\frac{\delta S_\phi}{\delta\phi} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi) + V_{,\phi} = 0 \tag{3.1.24}$$

If we now choose FLRW as the metric and use the cosmological limit, where we treat the field as homogeneous (so it becomes only time dependent $\phi(t,\mathbf{x}) = \phi(t)$), then the energy-momentum tensor will look like that of a perfect fluid

$$T_\mu{}^\nu = \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} \tag{3.1.25}$$

where

$$\rho = \frac{1}{2}\dot{\phi} + V(\phi) \tag{3.1.26}$$

$$p = \frac{1}{2}\dot{\phi} - V(\phi) \tag{3.1.27}$$

From here it can be shown, that the equations that govern the dynamics of the homogeneous scalar field in and the spacetime have the following shape:

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \qquad \text{and} \qquad H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \tag{3.1.28}$$

From these equations it becomes clear, that the main parameter defining the dynamics of inflation is the potential, so each potential corresponds to a different model of inflation.

### 3.1.5 Slow-Roll Inflation

In cases when we treat inflation as a scalar field, apart from having different models defined by different potentials (as we just said above), within each model we can also have different styles of inflation such as slow-roll, ultra slow-roll, etc. The style can be defined by parameters such as friction, single-field or multi-field inflation (where instead of having only 1 field we can have 2 fields, for example and one of them can be responsible for making the other field to roll down the potential slowly, even when the potential itself can be steep (e.g. Hyperinflation, where centrifugal force coming from one field can slow down the other field to achieve a slow-roll inflation in case of a steep potential)) or some other parameters. In this section we discuss the most common and successful (at least so far) type of inflation - the slow-roll inflation.

As we saw in (3.1.19), when we have a homogeneous scalar field, it will accelerate the universe in the following way:

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p) = H^2(1 - \epsilon_1), \text{ where } \epsilon_1 = \frac{3}{2}(\omega + 1) = \frac{1}{2}\frac{\dot{\phi}^2}{H^2} \tag{3.1.29}$$

We will now call $\epsilon_1$ the slow-roll parameter. The reason for this name is that since we need an exponential expansion during inflation, it happens only in the limit where $\epsilon_1 \to 0$, which implies from (3.1.28) that

$$\dot{\phi}^2 \ll V(\phi) \tag{3.1.30}$$

meaning, that the velocity of the field rolling down the potential is small - for this reason, we call this kind of inflation a slow-roll inflation. Though, it should be noted, that the accelerated expansion can be kept for a long enough time only if the second derivative of $\phi$ is small enough

$$\ddot{\phi}^2 \ll |3H\dot{\phi}|, |\partial_\phi V(\phi)| \tag{3.1.31}$$

And this means, that the second slow-roll parameter $\epsilon_2$ should be small as well

$$\epsilon_2 = -\frac{\ddot{\phi}}{H\dot{\phi}} = \epsilon - \frac{1}{2\epsilon}\frac{\mathrm{d}\epsilon}{\mathrm{d}N} \tag{3.1.32}$$

and the condition is $|\epsilon_2| < 1$, so that $\epsilon$ does not change too fast with each e-fold.

Using number of e-folds we can also write a universal formula for any slow-roll parameter:

$$\epsilon_{n+1} \equiv \frac{\mathrm{d}\ln(|\epsilon_n|)}{\mathrm{d}N}, \quad n \geq 0 \text{ and } \epsilon_0 \equiv \frac{1}{H} \tag{3.1.33}$$

This will give expressions for any slow-roll parameter through Hubble constant. For the first two it will be:

$$\epsilon_1 = -\frac{\dot{H}}{H^2} \text{ and } \epsilon_2 = -\frac{\ddot{H}}{\dot{H}} \tag{3.1.34}$$

So, to have a slow-roll inflation, the following conditions have to be satisfied:

$$\epsilon_1, |\epsilon_2| < 1 \tag{3.1.35}$$

So, the usual slow-roll potential looks the following way: (See Fig.(3.2a)) and we see that it is very flat for long enough time to make the universe expand enough number of e-folds and then, when the field starts rolling down the potential, $\epsilon_1$ becomes of order 1 and inflation ends. After this, since the field dynamics has the Hubble friction term, it looses energy and this energy is transfered into the environment, making the universe heat up - this period of oscillations around the minimum of the potential until the whole energy of inflation is given away is called reheating.

An alternative name for the slow-roll parameters written in terms of Hubble parameter $H$ are geometrical slow-roll parameters (since as it is clear from this name, they are defined using solely geometrical quantities) or Hubble flow parameters.

### Small- and Large-Field Inflation

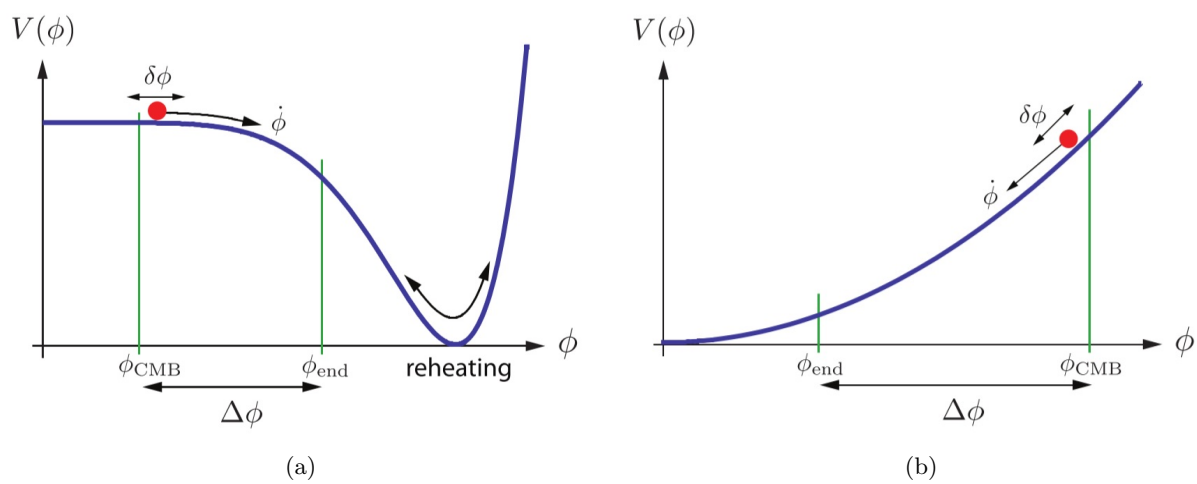
During the single field Slow-Roll inflation, depending on the initial conditions for the field we can also have 2 cases - the big and small-field inflations.

*Small-Field Inflation*

In this case the field starts at $|\phi| < M_p$ values in the instable equilibrium and due to quantum fluctuations gets displaced from it and starts rolling down to the minimum of the potential - non-zero vacuum (see.Fig 2 [11]). This kind of scenario is usually seen in models involving Spontaneous Symmetry Breaking phenomenon (like we will have in our model). (See Fig. (3.2a))

*Large-Field Inflation*

In large-field inflation the field starts rolling from $|\phi| > M_p$ towards the minimum of the potential, which is usually situated around $\phi = 0$, from the large initial values. (See Fig. (3.2b))




(a) (b)

Figure 3.2: Small- and Large-Field Inflation [11]

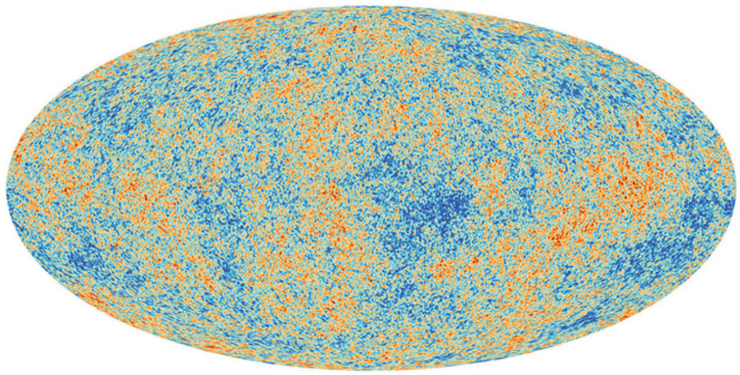


Figure 3.3: Cosmic Microwave Background [10]. The key observation is, that the inhomogeneities were small - this allows us to treat them as linear perturbations around the homogeneous FLRW background

## 3.2 Cosmological Perturbations

The treatment that we used the previous section was completely classical. In this section we will already include quantum effects that arise during inflation, since this is the actual source of slight inhomogeneities in the Cosmic Microwave Background that we observe.

If we simply find the solution of (3.1.28), we will see, that the whole universe will expand in the same way, therefore we will get a perfectly homogeneous universe if we start with FLRW metric. This is of course not realistic - in real universe we have inhomogeneities in form of Large scale structures and CMB (see Fig.(3.3)), for example. It becomes possible to predict these inhomogeneities if we include quantum fluctuations into our problem: quantum fluctuations are local and might slow down or accelerate the rolling down of the field on the slope of the potential (Fig.(3.2a)) and (Fig.(3.2b)), resulting in different expansions of different local regions of the universe - some regions stay dominated by the potential for longer time than others. This can be seen in the CMB (see Fig.(3.3)).

Additionally, these inhomogeneities induce relative density fluctuations in the universe and overdense regions will start attracting matter from the regions around them and hence growing. This can explain the Large Scale Structure existence in the universe.

### 3.2.1 Linear Perturbations

In order to study the structure of the universe properly, we have to include these effects of quantum fluctuations in our model. Luckily, based on the observation we can assume, that these inhomogeneities are so small, that we can treat them as linear perturbations around the homogeneous FLRW background. This simplifies their study significantly. If we denote all the main quantities (metric $g_{\mu\nu}$ and matter fields $T_{\mu\nu}$ such as $\phi, \rho, p$ and so on) collectively as $X(t, \mathbf{x})$, then we can separate the background and fluctuation parts. The background part will obviously be only time dependent, whereas the fluctuations part will also depend on the position in space:

$$X(t, \mathbf{x}) \equiv \bar{X}(t) + \delta X(t, \mathbf{x}) \tag{3.2.1}$$

where $\bar{X(t)}$ is the background value of the field and $\delta X(t, \mathbf{x})$ is the fluctuation.

### 3.2.2 Gauge freedom

We should be very cautios here, though - this split (3.2.1) of the main quantities into background and perturbative parts is not unique - it depends on the coordinate choice. If we are not careful, we might introduce fictitious perturbations into our problem, which enter the problem only due to the choice of the coordinates and are therefore not physical.

To avoid these issues, we should consider a full set perturbations that includes both - metric and matter perturbations. In this case, by using gauge transformations like we did above with time, we will be able to trade matter perturbations to metric ones and vice versa.

Another, more convenient, way of dealing with these subtleties is to use the gauge-invariant quantities, that we can build using specific combinations of perturbations.

### 3.2.3 SVT Decomposition

Another very convenient method to simplify the treatment of the cosmological perturbations is the SVT (Scalar-Vector-Tensor) decomposition. In some sense we are lucky that our universe can be assumed to be spatially flat, homogeneous and isotropic with a very good precision - without the symmetries associated with these properties (such as spatial or time translations and rotations) it would not be allowed to decompose the perturbations of matric and stress-energy tensor into independent scalar, vector and tensor parts.

This decomposition becomes especially convenient in the Fourier space:

$$\delta X_{\mathbf{k}}(t) = \int \mathrm{d}^3 x \delta X(t, \mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{3.2.2}$$

and it can be proved [11], that at linear order, the translational symmetry of perturbation EoM-s allows us to treat each Fourier mode individually - this is incredibly convenient, since we can just take a single $\mathbf{k}$ Fourier mode and study only its evolution without having to take into account any other modes. For example, we can see when it leaves the Hubble horizon (the condition for that is $k = aH$) and when it re-enters it again.

A similar argument applies to the rotational symmetry. We can consider rotations around a single mode $\mathbf{k}$:

$$\delta X_{\mathbf{k}}(t) = e^{im\psi} \delta X_{\mathbf{k}}(t) \tag{3.2.3}$$

$m$ is called the helicity of the perturbation. For scalars it is equal to $m = 0$, for vectors $m = \pm 1$ and for tensors $m = \pm 2$. (3.2.3) allows us to treat scalar, vector and tensor perturbations independently as well (again, at the linear order and again, for the proof please refer to [11]).

So, all in all, (3.2.2) and (3.2.3) significantly simplify the treatment of cosmological perturbations by allowing to treat each Fourier mode $\mathbf{k}$ and each type of the perturbations separately (at linear order).

Let's actually have a look at specific FLRW examples of this treatment in the sections below.

### 3.2.4 Metric Perturbations

We split the metric and the inflaton into homogeneous background and quantum perturbation parts:

$$\phi(t, \mathbf{x}) = \bar{\phi}(t) + \delta\phi(t, \mathbf{x}) \tag{3.2.4}$$

$$g_{\mu\nu}(t, \mathbf{x}) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \mathbf{x}) \tag{3.2.5}$$

We can now employ the SVT decomposition, so that we get for the invariant interval:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\mu = -(1 + 2\Phi)\mathrm{d}t^2 + 2aB_i \mathrm{d}x^i \mathrm{d}t + a^2[(1 - 2\Psi)\delta_{i,j} + E_{ij}]\mathrm{d}x^i \mathrm{d}x^j \tag{3.2.6}$$

where

$$B_i \equiv \partial_i B - S_i \;\text{ where }\; \partial_i S^i = 0 \tag{3.2.7}$$

$$E_{ij} \equiv 2\partial_{ij} E + 2\partial_{(i} F_{j)} + h_{ij} \;\text{ where }\; \partial_i F^i = 0 \;\text{ and }\; h^i_i = \partial^i h_{ij} = 0 \tag{3.2.8}$$

and $\Phi$ and $\Psi$ are the scalar perturbations. Things also get simplified by the fact that the vector perturbations decay, so we can simply ignore them.

Now we should have a look at gauge transformations to see, how these components of cosmological perturbations transform under them and then using these rules we'll construct the gauge invariant quantities later.

Tensor fluctuations turn out to be gauge-invariant, but the scalar ones transform under the gauge transformations

$$\begin{aligned} t &\to t + \alpha \\ x^i &\to x^i + \delta^{ij}\beta_{,j} \end{aligned} \tag{3.2.9}$$

in the following way:

$$\begin{aligned} &\Phi \to \Phi - \dot{\alpha} \\ &B \to B + a^{-1}\alpha - a\dot{\beta} \\ &E \to E - \beta \\ &\Psi \to \Psi + H\alpha \end{aligned} \tag{3.2.10}$$

*Note: Under the* (3.2.9) *transformations the* $\mathrm{d}s^2$ *interval stays invariant.*

### 3.2.5 Matter Perturbations

During inflation the main contribution to the energy-momentum tensor comes from the inflation itself. Therefore, after inflation the perturbations to $T_{\mu\nu}$ will look like

$$\begin{aligned} T^0_0 &= -(\bar{\rho} + \delta\rho) \\ T^0_i &= (\bar{\rho} + \bar{p}) a v_i \\ T^i_0 &= -(\bar{\rho} + \bar{p})(v^i - B^i)/a \\ T^i_j &= \delta^i_j(\bar{\rho} + \delta\rho) + \Sigma^i_j \end{aligned} \tag{3.2.11}$$

and again, the tensor $\Sigma^i_j$ is gauge-invariant, whereas the scalar quantities like the density, pressure and momentum density ($(\delta q)_{,i} \equiv (\bar{\rho} + \bar{p}) v_i$) transform like

$$\begin{aligned} &\delta\rho \to \delta\rho - \dot{\bar{\rho}}\alpha \\ &\delta p \to \delta p - \dot{\bar{\rho}}\alpha \\ &\delta q \to \delta q + (\bar{\rho} + \bar{p})\alpha \end{aligned} \tag{3.2.12}$$

### 3.2.6 Constructing Gauge-Invariant Quantities

Now we have enough information to start constructing gauge-invariant quantities. Using (3.2.10) and (3.2.12), we can combine some quantities to get a gauge-invariant combination. The first one is the *curvature perturbation of uniform-density hypersurfaces* [18]

$$-\zeta \equiv \Psi + \frac{H}{\dot{\bar{\rho}}}\delta\rho \tag{3.2.13}$$

This quantity can be interpreted geometrically as a spatial curvature of the constant-density hypersurfaces. We can also define the adiabatic matter perturbations (another gauge-invariant quantity) - they satisfy:

$$\delta p_{en} \equiv \delta p - \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}}\delta\rho \tag{3.2.14}$$

In the single-field inflation, that we will be treating in our model (3.2.14) condition is always satisfied. When this condition is satisfied, $\zeta_{\mathbf{k}}$ has a very important feature - it does not evolve outside of the Hubble horizon (i.e when $k \ll aH$ is satisfied). In a spatially flat gauge $\zeta$ perturbation is the dimensionless density perturbation $\frac{1}{3}\delta\rho/(\bar{\rho}+\bar{p})$. Therefore $\zeta$ can be connected to the CMB and LSS fluctuations and hence, as we promised in the introduction, these fluctuations will be conserved outside of the causal Horizon, preserving the information about the primordial spectra.

Another very important gauge-invariant scalar is defined in the following way:

$$\mathcal{R} \equiv \Psi + \frac{H}{\bar{\rho}+\bar{p}}\delta q \tag{3.2.15}$$

*where $\delta q$ is the scalar part of the 3-momentum density $T^0_i = \partial_i \delta q$.* $\mathcal{R}$ is called the *comoving curvature perturbation* and can also be interpreted geometrically - it shows the spatial curvature of comoving hypersurfaces (constant $\phi$).

Interestingly enough, $\zeta$ and $\mathcal{R}$ are equal on superhorizon scales ($k \ll aH$) [11]. Following the common trend, we will use the power spectrum of $\mathcal{R}$ to compare theoretical predictions of our model to the experimental data.

### 3.2.7 Power Spectrum and Spectral Tilt

Obviously, the data that satellites harvest about CMB and LSS is statistical, therefore we need to employ statistical methods to study it. The most important statistical property of the scalar perturbations is the so-called power-spectrum of $\mathcal{R}$ - it basically measures the importance of each $\mathbf{k}$ mode in the perturbations. It can be expressed in the following way:

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k}+\mathbf{k}') P_{\mathcal{R}}(k) \tag{3.2.16}$$

where the expectation value means the ensemble average of the fluctuations. To better see the clumpiness of cosmological perturbations around $k$, we also define a dimensionless power spectrum:

$$\Delta_s^2 \equiv \Delta_{\mathcal{R}}^2 = \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k) \tag{3.2.17}$$

It is also often written in the following way for convenience:

$$\Delta_s^2(\mathbf{k}, t) = A_\star \left(\frac{k}{k_\star}\right)^{n_s - 1} \tag{3.2.18}$$

This makes it easier to study the scale-dependence of the power spectrum. We call the parameter that controls the scale dependence the scalar spectral tilt $n_s$:

$$n_s - 1 \equiv \frac{\mathrm{d}\Delta_s^2}{\mathrm{d}\ln(k)} \tag{3.2.19}$$

where the scale-invariant power spectrum corresponds to $n_s = 1$. We can also define the scale-dependence of the $n_s$ itself:

$$\gamma_s \equiv \frac{\mathrm{d}n_s}{\mathrm{d}\ln(k)} \tag{3.2.20}$$

# Chapter 4

# Our Model

***In this chapter we will discuss how we choose the starting action for our model and how we make it more convenient for inflationary studies by switching to the Einstein Frame***

## 4.1 Determining the starting action

Now we are ready to start building our model of inflation. First of all, we should determine the starting action. For that we need to put some requirements on it - it should

1) Be general for any curved spacetime
2) Have conformal symmetry
3) Be a gravitational action with scalar fields
4) Should be renormalizeable (perturbatively)
5) Contain all the possible terms that satisfy the above conditions

Each one of these conditions is achieved as follows:

1) This is done in a standard way by keeping the terms Lorentz covariant and adding $\sqrt{-g}$ to the measure

2) We modify quantities with torsion as explained in Chapter 2 and make the scalar couple conformally to gravity

3) We should include a conformal term $\sim \bar{R}$, so that it can later be turned into the Hilbert-Einstein term $\frac{1}{2}M_p\bar{R}$. We also add the kinetic term for scalar fields $\sim (\bar{\nabla}\phi)^2$ and finally the self-interaction of the scalar fields, which should be proportional to $\sim \phi^4$, since otherwise it will not be conformal - it will have a dimensionful coupling constant that violates the scale symmetry

4) To achieve renormalizeability, the action should contain the terms that will be able to absorb all the quantum corrections coming from the perturbative expansion of the action. This is done in Appendix B and we discover, that other conformal terms like $\sim \bar{R}^2$ should also be included in the action [1]

All of this leads us to the following starting action:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\alpha\bar{R}^2 + \frac{1}{2}(\bar{\nabla}\phi)^2 + \beta\phi^2\bar{R} - \frac{\lambda}{4}\phi^4\right) \tag{4.1.1}$$

[1] In general, we should also include the Weyl terms $\sim \bar{R}^2_{\mu\nu}$ into the action, but since they violate the unitarity and there is currently no consistent method of dealing with this, we choose the coefficient in front of this term to be 0

## 4.2 Switching to Einstein Frame

### 4.2.1 Einstein and Jordan Frames

In theories of gravity, depending on the task that one wants to accomplish, the action can be written in different conformal frames for convenience. Switching from one frame to another one is done by performing a conformal transformation. All of them, except one specific frame, are called Jordan frames. That one special frame is usually called Einstein frame [20].

The action that we motivated above, is written in the Jordan frame:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\alpha\bar{R}^2 - \frac{1}{2}(\bar{\nabla}\phi)^2 + \beta\phi^2\bar{R} - \frac{\lambda}{4}\phi^4\right) \tag{4.2.1}$$

And we would like to rewrite it in the following way:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}M_p^2\bar{R} - \frac{1}{2}(\bar{\nabla}\phi)^2 - V(\phi)\right) \tag{4.2.2}$$

This is the shape that the action has in the Einstein frame. Each of these two frames have their respective advantages and disadvantages. Let's briefly discuss them.

First of all, our starting action (4.1.1) is written in the Jordan frame. It is clear, that motivating the action shape is much easier in the Jordan frame - one of the reasons is, that, in general, we have no idea how the $V(\phi_E)$ term will look like, therefore we can not impose the above conditions on the action in the Einstein Frame (EF).

Another advantage of Jordan Frame (JF) can again be traced back to those conditions: in JF the conformal symmetry of the action is explicit. This is not the case in the EF, since we fixed the gauge, so the symmetry is now hidden.

And finally, we can calculate the quantum corrections in the JF much easier than in EF. And the results of this calculation (see Appendix B) turn out to be very important, as we saw, to actually complete the shape of the starting action. Besides, at the end of this section we will see another interesting result coming from quantum corrections.

As to the Einstein Frame, it is way more convenient for studying the inflationary properties of the model - it allows us to separate the canonical kinetic terms, Hilbert-Einstein part of the action and the potential of the scalar field explicitly. Studying the shape of the potential will reveal the inflationary properties of the model (this will be discussed in Chapter 5).

### 4.2.2 Einstein Frame

Let's actually bring the Jordan action to the EF shape. Since our action is conformal, we can not apply the usual strategies of going from JF to EF (in non-conformal actions we simply transform them conformally in such a way to make the shape look like the EF shape). Our strategy will be to use the Lagrange multiplier method, that allows us to expose all the dynamical degrees of freedom and by **fixing the gauge** appropriately we can achieve the shape of the action like (4.2.2), which will be the on-shell (clasically) equivalent action to our initial one.

To get a better idea of this method, we can briefly discuss a simple example:

$$S = \int \mathrm{d}t(\dot{x}^2 + \dot{y}^2) \tag{4.2.3}$$

In this case, the equations of motion for these fields are given by $\ddot{x} = 0$ and $\ddot{y} = 0$. But if we now add the Lagrange multiplier, we can force the solutions to behave in a certain way. For example, we can force them to stay on a circle by adding the following multiplier:

$$S = \int \mathrm{d}t(\dot{x}^2 + \dot{y}^2 + \lambda(x^2 + y^2 - R^2)) \tag{4.2.4}$$

where $R = const$ is the radius of a circle. From here we can derive the EoM for $\lambda$ and it will be a constraint equation:

$$\frac{\delta S}{\delta \lambda} = 0 \Rightarrow x^2 + y^2 = R^2 \tag{4.2.5}$$

which will force $x$ and $y$ to stay on a $R$ radius circle, since $x = R\cos\theta$ and $y = R\sin\theta$. And then we will have to plug this back into the action to rewrite it as a function of $\theta$ and then by deriving the $\theta$ EoM we will get the final solution for $\theta$ and therefore for $x$ and $y$.

The main result of this multiplier is that it allows us to add a constraint equation to the system. Let's see how this can be used in our case now.

We employ this trick in such a way, that we will get rid of the explicit $R^2$ term by hiding it in the Lagrange constraint part. Though, for this purpose we have to introduce an additional scalar field and this will be the cost of using this trick - instead of 1 inflaton $\phi$ field we will have 2 fields now - $\phi$ and $\zeta$, but we will not loose any dynamical information anymore by going to the Einstein frame, just as we need.

The action will now look the following way:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\alpha\zeta^2 - \frac{1}{2}(\bar{\nabla}\phi)^2 + \beta\phi^2\zeta - \frac{\lambda}{4}\phi^4 + \omega^2(\bar{R} - \zeta)\right) \tag{4.2.6}$$

where we have introduced $\zeta$ field and $\omega$ is the Lagrange multiplier that will force $\zeta$ to be equal to $\bar{R}$ with the constraint equation, so that (4.2.6) is equivalent to (4.2.1). It might look surprising, that we have moved from the action with 4 derivatives acting on metric to an equivalent action with only 2 derivatives acting on metric. The reasons for this is, that in some cases, when we rewrite an action into an equivalent but more convenient form for some purposes, some properties of it are not so explicit anymore, despite the fact, that they are still contained in it.

We see now, that we have 3 fields $\phi$, $\zeta$ and $\omega$, instead of 2 that we were anticipating. The reason is, that to make the constraint equation work, we need a temporary field $\zeta$, for which we will now get the EoM, then solve it and plug the solution back into the action, so that we are left with 2 fields that we want to have - $\phi$ and $\omega$.

Let's get rid of $\zeta$. The EoM will be:

$$\frac{\delta S}{\delta \zeta} = 0 \quad \Rightarrow \quad 2\alpha\zeta + \beta\phi^2 - \omega^2 = 0 \quad \Rightarrow \quad \zeta = -\frac{\beta\phi^2 - \omega^2}{2\alpha} \tag{4.2.7}$$

and now we plug this back into the action to ged rid of $\zeta$ presence there. We get:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{(\beta\phi^2 - \omega^2)^2}{4\alpha} - \frac{1}{2}(\bar{\nabla}\phi)^2 - \beta\phi^2\frac{\beta\phi^2 - \omega^2}{2\alpha} - \frac{\lambda}{4}\phi^4 + \omega^2\left(\frac{\beta\phi^2 - \omega^2}{2\alpha} + \bar{R}\right)\right) \tag{4.2.8}$$

And demanding the invariance of the action with respect to a conformal transformation, we can deduce the transformation rule for $\omega$: $\omega \to \Omega^{-1}\omega$.

For convenience we could also group some terms in the action:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{\beta\phi^2 - \omega^2}{2\alpha}\left(\frac{\beta\phi^2 - \omega^2}{2} - \beta\phi^2 + \omega^2\right) - \frac{1}{2}(\bar{\nabla}\phi)^2 - \frac{\lambda}{4}\phi^4 + \omega^2\bar{R}\right] \tag{4.2.9}$$

or simplified:

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[ -\frac{1}{2}(\bar{\nabla}\phi)^2 - \frac{\lambda}{4}\phi^4 - \frac{(\beta\phi^2-\omega^2)^2}{4\alpha} + \omega^2\bar{R}\Big] \tag{4.2.10}$$

If we now split the torsion trace into the longitudinal and transverse parts, it turns out, that due to conformal symmetry, the latter is not dynamical (it is simply not sourced), hence we can put it to 0 and retain only the longitudinal part of the torsion trace. $T_\mu \to \partial_\mu\sigma(x)$. Taking into this account and writing out the covariant derivatives explicitly, we arrive at:

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[ -\frac{\lambda}{4}\phi^4 - \frac{(\beta\phi^2-\omega^2)^2}{4\alpha} + \omega^2\bar{R} - \frac{1}{2}g^{\mu\nu}\Big(\partial_\mu\phi + \phi\partial_\mu\sigma\Big)\Big(\partial_\nu\phi + \phi\partial_\nu\sigma\Big)\Big] \tag{4.2.11}$$

And we are now ready to bring this action to the Einstein form by fixing the gauge - choosing a specific form for $\omega$, such that $\omega^2\bar{R}_E$ term turns into $\frac{1}{2}M_p^2\bar{R}_E$. This is very simple to do and we get:

$$\omega^2\bar{R} = \frac{2\omega^2M_p^2}{2M_p^2}\bar{R} = \text{should be} = \frac{1}{2}M_p^2\bar{R} \qquad \Rightarrow \qquad \omega^2 = \frac{M_p^2}{2} \tag{4.2.12}$$

Under such a gauge fix the action becomes:

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[ -\frac{\lambda}{4}\phi^4 - \frac{\Big(\beta\phi^2 - \frac{M_p^2}{2}\Big)^2}{4\alpha} + \frac{M_p^2}{2}\bar{R} - \frac{1}{2}g^{\mu\nu}\Big(\partial_\mu\phi + \phi\partial_\mu\sigma\Big)\Big(\partial_\nu\phi + \phi\partial_\nu\sigma\Big)\Big] \tag{4.2.13}$$

So, we have successfully arrived at the Einstein frame action by using the Lagrange multiplier trick, that allowed us not to get rid of any derivatives of the metric, hence, preserving all the dynamical information of the system. The price, however, was the introduction of a second scalar field, that manifests itself in terms, that contain $\partial_\nu\sigma$.

Now in order to be able to judge the inflationary properties of the system, we should bring this action to the canonical form, meaning, that the mass term and both of the kinetic terms should be brought to the canonical form. Luckily, one of them, for $\phi$ is already in the canonical form, so we should take care of the other one now.

The kinetic term for $\sigma$ comes from the curvature term - to have the de Sitter background, we need to have $M_p^2\mathring{R}$ and when we write out $\bar{R}$ explicitly, we see, that it contains this $\mathring{R}$ and torsion parts as well that can be written as $\partial_\nu\sigma$, giving us the second kinetic term:

$$M_p^2\bar{R} = M_p^2\Big[\mathring{R} - 6\mathring{\nabla}_\lambda T^\lambda - 6T_\lambda T^\lambda\Big] \qquad \to \qquad M_p^2\bar{R} = M_p^2\Big[\mathring{R} - 6\partial_\lambda\sigma\partial^\lambda\sigma\Big]$$

where we dropped the $6\mathring{\nabla}_\lambda T^\lambda$ term from the action, because it is a full derivative.

Thus we get for the other kinetic term $-6\frac{1}{2}M_p^2\partial_\mu\sigma\partial^\mu\sigma$. To normalize it canonically, we introduce a new field $\tilde{\chi} \equiv \sqrt{6}M_p\sigma$ and $\partial_\mu\tilde{\chi} = \sqrt{6}M_p\partial_\mu\sigma \;\Rightarrow\; \partial_\mu\sigma = \frac{\partial_\mu\tilde{\chi}}{\sqrt{6}M_p}$. This gives us

$$-6\frac{1}{2}M_p^2\partial_\mu\sigma\partial^\mu\sigma = -\frac{1}{2}\partial_\mu\tilde{\chi}\partial^\mu\tilde{\chi} \tag{4.2.14}$$

Let's put this back into the action and see what we get:

$$\begin{aligned} S = \int \mathrm{d}^4x\sqrt{-g}\Big[\frac{M_p^2}{2}\mathring{R} - \frac{1}{2}(\partial_\mu\tilde{\chi})^2 - \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4}\phi^4 - \\ - \frac{\Big(2\beta\phi^2 - M_p^2\Big)^2}{16\alpha} - \frac{1}{2}\frac{(\partial_\mu\tilde{\chi})^2\phi^2}{6M_p^2} - g^{\mu\nu}\phi\frac{\partial_\mu\tilde{\chi}\partial_\nu\phi}{\sqrt{6}M_p}\Big] \end{aligned} \tag{4.2.15}$$

This turns out to be an important result. To understand this better, we should have a look at the field-space metric of this action (4.2.15).

$$G_{IJ} = \begin{bmatrix} 1 & \frac{\phi}{\sqrt{6}M_p} \\ \frac{\phi}{\sqrt{6}M_p} & 1 + \frac{\phi^2}{(6M_p)^2} \end{bmatrix} \tag{4.2.16}$$

where $\phi_I \equiv (\phi, \tilde{\chi})$ and $I = 1, 2$.

## 4.3 Diagonalizing the Field-Space Metric

Now we would like to study the inflationary properties of the model. For this we need to simplify the action to the standard form, to which standard approaches of finding an inflationary solution can be applied. For this purpose we will now diagonalize the field space metric.

$$G_{IJ} = \mathrm{d}\phi^2 + \mathrm{d}\tilde{\chi}^2\left(1 + \frac{\phi^2}{6M_p^2}\right) + \frac{2\phi}{\sqrt{6}M_p}\mathrm{d}\tilde{\chi}\mathrm{d}\phi \tag{4.3.1}$$

To diagonalize this metric, let's define a new variable:

$$\chi = \tilde{\chi} + f(\phi) \implies \mathrm{d}\chi = \mathrm{d}\tilde{\chi} + f'(\phi)\mathrm{d}\phi \tag{4.3.2}$$

From here we get:

$$\begin{aligned} \mathrm{d}\tilde{\chi} &= \mathrm{d}\chi - f'(\phi)\mathrm{d}\phi \\ \mathrm{d}\tilde{\chi}^2 &= \mathrm{d}\chi^2 - 2f'(\phi)\mathrm{d}\chi\mathrm{d}\phi + f'(\phi)^2\mathrm{d}\phi^2 \end{aligned} \tag{4.3.3}$$

Plugging these back into (4.3.1), we get:

$$\begin{aligned} G_{IJ} = \mathrm{d}\phi^2 + \mathrm{d}\chi^2\left(1 + \frac{\phi^2}{6M_p^2}\right) - 2f'(\phi)\mathrm{d}\chi\mathrm{d}\phi\left(1 + \frac{\phi^2}{6M_p^2}\right) + \\ + f'(\phi)^2\mathrm{d}\phi^2\left(1 + \frac{\phi^2}{6M_p^2}\right) + \frac{2\phi}{\sqrt{6}M_p}\mathrm{d}\chi\mathrm{d}\phi - \frac{2\phi}{\sqrt{6}M_p}f'(\phi)\mathrm{d}\phi^2 \end{aligned} \tag{4.3.4}$$

And now we want to choose such a form for $f(\phi)$ that it kills the mixed term:

$$\frac{2\phi}{\sqrt{6}M_p} - 2f'(\phi)\left(1 + \frac{\phi^2}{6M_p^2}\right) = 0 \implies f'(\phi) = \frac{\phi}{\sqrt{6}M_p\left(1 + \frac{\phi^2}{6M_p^2}\right)} \tag{4.3.5}$$

This will give us:

$$\begin{aligned} G_{IJ} &= \mathrm{d}\phi^2 + \mathrm{d}\chi^2\left(1 + \frac{\phi^2}{6M_p^2}\right) + \frac{\phi^2\mathrm{d}\phi^2}{\left(\sqrt{6}M_p\left(1 + \frac{\phi^2}{6M_p^2}\right)\right)^2}\left(1 + \frac{\phi^2}{6M_p^2}\right) - \frac{2\phi^2\mathrm{d}\phi^2}{6M_p^2\left(1 + \frac{\phi^2}{6M_p^2}\right)} = \\ &= \frac{\mathrm{d}\phi^2}{\left(1 + \frac{\phi^2}{6M_p^2}\right)} + \left(1 + \frac{\phi^2}{6M_p^2}\right)\mathrm{d}\chi^2 \end{aligned} \tag{4.3.6}$$

So that we have a diagonal field-space metric:

$$G_{IJ} = \begin{bmatrix} \frac{1}{1+\frac{\phi^2}{(6M_p)^2}} & 0 \\ 0 & 1 + \frac{\phi^2}{(6M_p)^2} \end{bmatrix} \tag{4.3.7}$$

For an even more convenience, we should also canonically normalize the kinetic term of $\phi$. For this we employ a field-redefinition:

$$-\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,\underbrace{(\partial_\mu\phi)^2\left(\frac{1}{1+\frac{\phi^2}{(6M_p)^2}}\right)}_{\equiv(\partial_\mu\psi)^2} \tag{4.3.8}$$

so that

$$(\partial_\mu\psi) = (\partial_\mu\phi)\sqrt{\frac{1}{1+\frac{\phi^2}{(6M_p)^2}}} \tag{4.3.9}$$

getting rid of derivatives, we get

$$\psi = \int \mathrm{d}\phi\sqrt{\frac{1}{1+\frac{\phi^2}{(6M_p)^2}}} \tag{4.3.10}$$

which gives

$$\psi = \sqrt{6}M_p \operatorname{arcsinh}\left(\frac{\phi}{\sqrt{6}M_p}\right) \tag{4.3.11}$$

and

$$\phi = \sqrt{6}M_p \sinh\left(\frac{\psi}{\sqrt{6}M_p}\right) \tag{4.3.12}$$

We can plug this back into (4.3.6) to finally get the canonically normalized metric for the scalar field:

$$G_{IJ} = \mathrm{d}\psi^2 + \cosh^2\left(\tilde{\psi}\right)\mathrm{d}\chi^2 \tag{4.3.13}$$

Where we used that $\tilde{\psi} \equiv \frac{\psi}{\sqrt{6}M_p}$ and $1+\sinh^2\left(\frac{\psi}{\sqrt{6}M_p}\right) = \cosh^2\left(\tilde{\psi}\right)$.

All of this now gives us the canonically normalized action that will be our main working action from now on:

$$\boxed{S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_p^2}{2}\mathring{R} - \frac{1}{2}(\partial_\mu\psi)^2 - \frac{1}{2}\cosh^2\left(\tilde{\psi}\right)(\partial_\mu\chi)^2 - V(\psi)\right]} \tag{4.3.14}$$

Where the potential has the following shape:

$$\boxed{V(\psi) = \frac{1}{\alpha}\left[\frac{\tilde{\lambda}}{4}(6M_p^2)^2\sinh^4\left(\tilde{\psi}\right) + \frac{M_p^4}{16}\left(12\beta\sinh^2\left(\tilde{\psi}\right)-1\right)^2\right]} \tag{4.3.15}$$

*where for further convenience we pulled $\alpha$ out of the brackets and defined $\tilde{\lambda} \equiv \alpha\lambda$.*

## 4.4 Analyzing the Potential

Let's take a moment here to understand, what we derived.

We started with an action with 2 scalar fields - inflaton $\phi$ and torsion $\chi$. We worked on this action to significantly simplify it and in the end we arrived at the (4.3.14) action, that can serve as a model of a two-field inflation.

The interesting part about this simplified (i.e. canonically normalized (except the kinetic term of $\chi$)) action is, that the presence of the torsion $\chi$ field in the initial action (4.1.1) does not "go unnoticed" and its

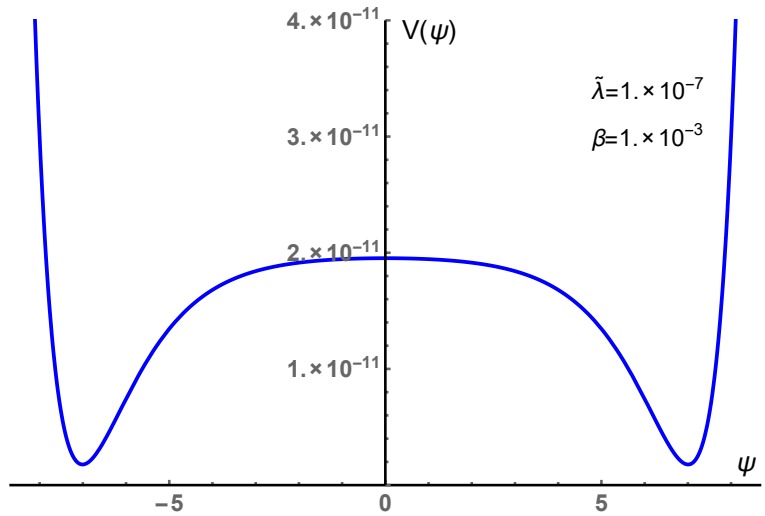



Figure 4.1: By properly choosing the values of the coupling constants, the potential can be made flat around the origin. This allows us to get a long enough period of slow-roll regime for the field, so that it makes the universe expand at least 60 e-folds (satisfying one of the necessary conditions of inflation)

effect is now reflected in both - a) the potential part as well as b) the kinetic part.

a) In order to get a better idea about the the potential energy part, we can calculate the Ricci scalar for the Configuration Space (CS) metric (4.3.7):

$$R_{CS} = -\frac{1}{3M^2} \tag{4.4.1}$$

This might correspond either to the Anti-de Sitter field space or the Hyperbolic one. Since we have the positive coefficients in front of both of the diagonal elements (unlike opposite signs in the case of AdS), we get a Hyperbolic space (negatively curved and maximally symmetric):

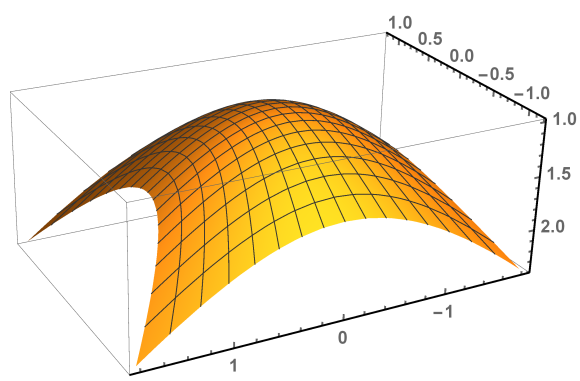



Figure 4.2: Negatively curved and maximally symmetric Configuration Space

This is the reason, why we arrived at the potential that contains hyperbolic functions and this is very important, since the hyperbolical potential is flat near $\phi \sim 0$ area (See Fig.(4.1)), therefore in order to get a slow-roll inflation, we can use the small-field initial conditions $\phi \sim 0$, which corresponds to initially being close to the conformal point and then rolling down the potential to the non-zero vacuum to acquire a condensate. This is exactly in accordance with the scenario, where the Standard Model is conformal at high energies (or, alternatively, conformal in the beginning of the history of the universe) and this symmetry is broken at later times by the condensate acquired by the scalar field.

So, we realize inflation using Radiative Symmetry Breaking of the conformal symmetry.

In general, conformal inflations have been studied before [21], but the crucial difference with our model is, that these models either employ the Large field inflation (meaning that they do not start at a conformal point) or start with conformal symmetry being explicitly broken. Our model, on the other hand, realizes inflation by employing the mechanism of spontaneous symmetry breaking starting close to the conformal point of matter, so that the field breaks the conformal symmetry spontaneously.

All of this is very interesting, since it comes out naturally from introducting torsion into the model without any further fine-tuning. So, it is ensured by the torsion, that a) it makes the GR conformal and b) it allows inflaton to start rolling from the conformal point and c) it breaks the conformal symmetry spontaneously, allowing the masses (like Planck's Mass and the field mass itself) to be generated dynamically using condensates.

b) As to the kinetic part of the action, to understand the role of $\chi$ in it, we should have a look at the hyperbolic space metric in the polar coordinates for a 2D manifold:

$$\mathrm{d}s^2_{emb} = -\mathrm{d}X_1^2 + \mathrm{d}X_2^2 + \mathrm{d}X_3^2 \tag{4.4.2}$$

where we used that the standard coordinates are expressed in the following way:

$$\begin{aligned} X_1 &= \cosh\left(\tilde{\psi}\right)\cosh(\chi) \\ X_2 &= \cosh\left(\tilde{\psi}\right)\sinh(\chi) \\ X_3 &= \sinh\left(\tilde{\psi}\right) \end{aligned} \tag{4.4.3}$$

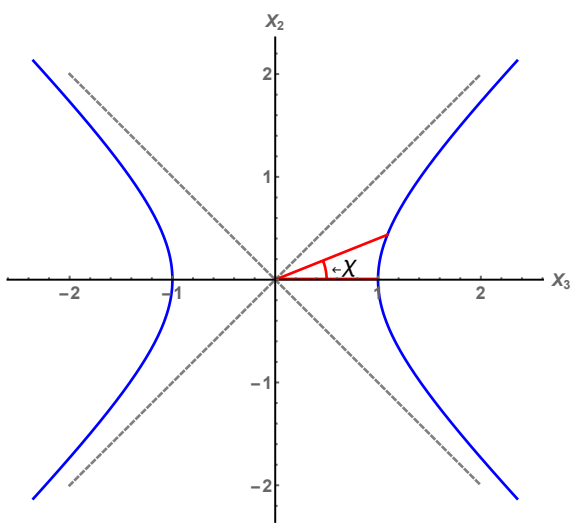



Figure 4.3: Plot of the manifold for a fixed $X_1$ coordinate

If we now compare (4.4.2) to what we have in (4.3.13), we clearly see, that $\chi$ plays the role of an angle $\tau$.

It is important to note, that $\chi$ is not a circular angle - it is a hyperbolic angle (Note that this is in the Field Space) (see Fig.(4.3)) and it can change from $-\infty$ to $\infty$ (and not $0 \to 2\pi$), therefore when we call the expression involving its derivative an angular momentum (in the Field Space), it is not the usual angular momentum that we have in a circular motion. Though, since they are similar, we will still keep calling it an angular momentum of the Field Space.

So, in the end we can say, that $\chi$ basically gives the $\psi$ field an angular momentum (and we will also see what kind of effect does this have on the final dynamics of $\psi$).

It also interesting to notice, that since $\chi$ enters action only as a derivative:

1) $\chi$ has a shift symmetry:

$$\chi \to \chi' = \chi + \theta \quad \Longrightarrow \quad S(\psi, \chi) = S(\psi, \chi') \tag{4.4.4}$$

meaning that we have a flat direction in the potential. It is also noteworthy, that under this transformation is not covariant (non-linearly realized by $\chi$).

2) $\chi$ is massless, since if we go to the Fourier space, $\partial_\mu \to k_\mu$ and by going to the 0 momentum limit, $\chi$ terms vanish in the action (decouple from it).

If we now recall, how Goldstone Bosons behave in a theory - 1) they give the flat direction, 2) they realize the symmetry non-linearly, 3) they're massless, and that 4) just like $\chi$ does in our case, they come from the symmetry transformation, we clearly see that $\chi$ is the Goldstone Boson in our case.

## 4.5 Remainder Cosmological Constant and Dynamically Generated Masses

It is also very interesting to see, what is the potential energy of $\phi$ field after it looses all of its kinetic energy - i.e. when it goes to the vacuum. To calculate this, we take the first derivative of the potential (4.3.15) and see, for what value of $\phi$ does it achieve the minimum and then by substituting into (4.3.15) again, we will find the value of the $V(\phi)$ that is left after reheating has finished.

For the derivative we have

$$\lambda\phi^3 + \frac{\beta\phi\left(2\beta\phi^2 - M_p^2\right)}{2\alpha} = 0 \tag{4.5.1}$$

from where it's trivial to find, that

$$\phi_0 = \sqrt{\frac{\beta M_p^2}{2(\lambda\alpha + \beta^2)}} \tag{4.5.2}$$

and consequently:

$$\boxed{V(\phi_0) = \frac{\lambda M_p^4}{16(\lambda\alpha + \beta^2)}} \tag{4.5.3}$$

Let's stop here and analyze this result, because we can draw interesting conclusions from it.

First of all, we should notice, that to get the flat inflationary potential like in Fig.(4.1), we take the coupling constants to be positive. This means that not only will this remaining (constant) potential energy be positive, but we can also use the results of one-loop quantum corrections to make some implications about the value of this potential. Let's discuss this in more details.

We have 3 free coupling constants in our model: $\alpha$, $\beta$ and $\lambda$. If we now look at how these coupling constants are modified by quantum corrections (See Appendix A), we will discover, that we can build a very interesting hierarchy for their values at one-loop level.

$$\begin{aligned} &1)\ \delta\lambda \propto \lambda^2, \beta^2 \\ &2)\ \delta\beta \propto \lambda, \beta \\ &3)\ \delta\alpha \propto \beta, \alpha \end{aligned} \tag{4.5.4}$$

where $\delta\alpha$, $\delta\beta$ and $\delta\lambda$ are quantum corrected coupling constants.

The object of main interest for us is the $\delta\lambda$ term. We see, that the quantum corrections to $\lambda$ are proportional to both - $\lambda^2$ and $\beta^2$. However, the coefficient of the $\beta^2$ term is proportional to $R^2$, which in turn, as we already derived, is proportional to $H^4$ (since $R \propto H^2$). And as discussed in the resutls section, $H \sim 10^{-6}$ and the maximum working value of $\beta$ is $\beta = 10^{-3}$, meaning that the corrections to $\lambda$ coupling constant proportional to $\beta^2$ will be of order $\beta^2 H^4 \sim 10^{-30}$ (or even smaller, if we decrease $\beta$). This argument implies, that until $\lambda$ reaches $10^{-30}$ values (or even smaller for smaller $\beta$), we can say, that $\delta\lambda$ is proportional only to $\lambda$ itself, so that making $\lambda$ smaller will not violate the perturbative hierarchy - the quantum corrections $\delta\lambda$ will decrease together with decrease of $\lambda$ and hence will not become of order $\lambda$ itself (until we reach the point where $\beta^2$ terms become dominant - this is the lower bound for $\lambda$).

Using this argument for $\lambda$ in our case, we can argue, that the remainder potential energy (4.5.3) can be made very small. But the question is - why would we want to make this value small?

The answer can be revealed by looking at the effect, that this potential energy has on the expansion of the universe. Since it is constant and what's important - positive, it corresponds to an accelerated expansion - just like we have today in the era of the cosmological constant. Thus, it will contribute positively to the overall cosmological constant left after the symmetry breakings during the evolution of the universe.

Therefore, it can in general be interpreted as a remnant cosmological constant after inflation. In the usual models of inflation, the remnant potential energy after inflation is put to 0 by hand, since having a positive energy still corresponds to an accelerated expansion. In our model, however, we can achieve the smallness (i.e. being almost 0) of this positive energy by simply choosing the smaller values for $\lambda$.

Additionally, it can be argued [22], that due to (4.5.4) dependence, we can set some specific small (nonzero) value for this potential energy that will help us achieve the energy corresponding to the remainder cosmological constant that we observe today with *less* fine-tuning.

Additionally this nonzero positive value gives us another important result - the mass scales of the problem like Planck's scale $M_p$ and scalar field mass get dynamically generated through the mechanism of SSB [23].

# Chapter 5

# Inflation from Conformal Action

***In this chapter we will derive the equations of motion for the inflaton and will solve them to get the inflationary solution***

## 5.1 Studying Inflationary Properties

Now we can start studying the inflationary properties of our model using this action, since we brought it to the following shape:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_p^2}{2}\mathring{R} - \frac{1}{2}(\partial_\mu\psi)^2 - \frac{1}{2}\cosh^2\left(\tilde{\psi}\right)(\partial_\mu\chi)^2 - V(\psi)\right] \tag{5.1.1}$$

where:

$$V(\psi) = \frac{1}{\alpha}\left[\frac{\tilde{\lambda}}{4}(6M_p^2)^2\sinh^4\left(\tilde{\psi}\right) + \frac{M_p^4}{16}\left(12\beta\sinh^2\left(\tilde{\psi}\right) - 1\right)^2\right] \tag{5.1.2}$$

### 5.1.1 Friedmann Equations and Slow-Roll parameter

First of all, we need to derive the Einstein Equation and resultant Friedmann equations. Therefore, again, we should vary the action with respect to metric

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = 0$$

We can schematically write $\delta S$ as:

$$\delta S = \delta S_1 + \delta S_2 + \delta S_3 + \delta S_4 + \delta S_5 \tag{5.1.3}$$

where:

$$\begin{aligned}
\delta S_1 &= \int \mathrm{d}^4x\ \mathcal{L}\ \delta\sqrt{-g} = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\ \mathcal{L}g_{\mu\nu}\delta g^{\mu\nu} \\
\delta S_2 &= -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\cosh^2\left(\tilde{\psi}\right)\partial_\mu\chi\partial_\nu\chi\delta g^{\mu\nu} \\
\delta S_3 &= -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\ \partial_\mu\psi\partial_\nu\psi\delta g^{\mu\nu} \\
\delta S_4 &= \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\ M_p^2\mathring{R}_{\mu\nu}\delta g^{\mu\nu} \\
\delta S_5 &= \int \mathrm{d}^4x\sqrt{-g}\ M_p^2 g^{\mu\nu}\delta\mathring{R}_{\mu\nu}
\end{aligned} \tag{5.1.4}$$

and

$$\mathcal{L} = \frac{M_p^2}{2}\mathring{R} - \frac{1}{2}(\partial_\mu\psi)^2 - \frac{1}{2}\cosh^2\Big(\tilde{\psi}\Big)(\partial_\mu\chi)^2 - \frac{1}{\alpha}\left[\frac{\tilde{\lambda}}{4}(6M_p^2)^2\sinh^4\Big(\tilde{\psi}\Big) + \frac{M_p^4}{16}\left(12\beta\sinh^2\Big(\tilde{\psi}\Big) - 1\right)^2\right] \tag{5.1.5}$$

Notice, that $\delta S_5$ contains only $\delta\mathring{R}_{\mu\nu}$. If we look at it's multiplier in $\delta S_5$, we'll see that there are only different powers of metric and constants. Considering this, together with the fact, that $\mathring{R}_{\mu\nu}$ contains metric compatible covariant derivatives, when we do integration by parts, all these derivatives will act only on metric, giving us 0. So, we can simply drop $\delta S_5$, so that we get:

$$\delta S = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\Big\{\mathcal{L}g_{\mu\nu} - \ M_p^2\mathring{R}_{\mu\nu} + \partial_\mu\psi\partial_\nu\psi + \cosh^2\Big(\tilde{\psi}\Big)\partial_\mu\chi\partial_\nu\chi\Big\}\delta g^{\mu\nu} \tag{5.1.6}$$

This gives us Einstein Equation:

$$-\mathcal{L}g_{\mu\nu} + \ M_p^2\mathring{R}_{\mu\nu} - \partial_\mu\psi\partial_\nu\psi - \cosh^2\Big(\tilde{\psi}\Big)\partial_\mu\chi\partial_\nu\chi = 0 \tag{5.1.7}$$

Employing $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, we get:

$$\boxed{M_p^2G_{\mu\nu} + \frac{1}{2}g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi g_{\mu\nu} + \frac{1}{2}\cosh^2\Big(\tilde{\psi}\Big)g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi g_{\mu\nu} - \partial_\mu\psi\partial_\nu\psi - \cosh^2\Big(\tilde{\psi}\Big)\partial_\mu\chi\partial_\nu\chi + V(\psi)g_{\mu\nu} = 0} \tag{5.1.8}$$

Again, since we use the cosmological limit, we will have (using normal and not conformal time):

$$R_{\mu\nu} = \begin{bmatrix} -\frac{3\ddot{a}}{a} & 0 & 0 & 0 \\ 0 & a\ddot{a} + 2\dot{a}^2 & 0 & 0 \\ 0 & 0 & a\ddot{a} + 2\dot{a}^2 & 0 \\ 0 & 0 & 0 & a\ddot{a} + 2\dot{a}^2 \end{bmatrix} \tag{5.1.9}$$

and the corresponding Ricci scalar will be of the following shape :

$$R = \frac{6\big(a\ddot{a} + \dot{a}^2\big)}{a^2} \tag{5.1.10}$$

Using the same argument, as before for Friedmann equation derivation in cosmological limit, we see, that only the 00 and $ii$ components of (5.1.8) survive. This gives us Friedmann equations. Let's see how they look after plugging (5.1.9) and (5.1.10).

**1) 00 - component of** (5.1.7)

$$\begin{aligned} &\left(R_{00} - \frac{1}{2}Rg_{00}\right)M_p^2 = \rho \implies \left(-3\frac{\ddot{a}}{a} + \frac{1}{2}\frac{6}{a^2}(a\ddot{a} + \dot{a}^2\right)M_p^2 = \rho \implies \\ &\left(-3\frac{\ddot{a}}{a} + 3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2\right)M_p^2 = \rho \implies 3\left(\frac{\dot{a}}{a}\right)^2M_p^2 = \rho \end{aligned} \tag{5.1.11}$$

So that by putting $M_p = 1$ we get for the first Friedmann equation:

$$\boxed{H^2 = \frac{1}{3}\rho} \tag{5.1.12}$$

**2) ij - component of** (5.1.7)

$$\begin{aligned} &\left(R_{ii} - \frac{1}{2}Rg_{ii}\right)M_p^2 = pa^2 \implies a\ddot{a} + 2\dot{a}^2 - \frac{1}{2}6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]a^2 = pa^2 \implies \\ &a\ddot{a} + 2\dot{a}^2 - 3a\ddot{a} - 3\dot{a}^2 = pa^2 \implies -2a\ddot{a} - \dot{a}^2 = pa^2 \implies 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -p \end{aligned} \tag{5.1.13}$$

So that we get for the second Friedmann equation:

$$\boxed{2\frac{\ddot{a}}{a} + H^2 = -p} \tag{5.1.14}$$

Now we can combine these two (5.1.12) and (5.1.14) to get:

$$\dot{H} + H^2 = -\frac{1}{6}\Big(\rho + 3p\Big) \tag{5.1.15}$$

Now we need the expressions for $\rho$ and p, to be able to use this equation. For this we should look at $T_{\mu\nu}$ for our case. Considering, that we are using FLRW metric, $T_{\mu\nu}$ will have the shape of energy-momentum tensor of an ideal fluid

$$T_{\mu}{}^{\nu} = \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} \tag{5.1.16}$$

we will get for our case:

$$T_{\mu\nu} = \partial_\mu\psi\partial_\nu\psi + \cosh^2\Big(\tilde{\psi}\Big)\partial_\mu\chi\partial_\nu\chi - \frac{1}{2}\Big(\partial_\alpha\psi\partial^\alpha\psi + \cosh^2\Big(\tilde{\psi}\Big)\partial_\alpha\chi\partial^\alpha\chi + 2V(\psi)\Big)g_{\mu\nu} \tag{5.1.17}$$

so that we get:

$$\begin{aligned} T_{00} &= \dot{\psi}^2 - \frac{\dot{\psi}^2}{2} + \cosh^2\Big(\tilde{\psi}\Big)\dot{\psi}^2 - \frac{\cosh^2\Big(\tilde{\psi}\Big)\dot{\chi}^2}{2} + V(\psi) \equiv \rho \ \text{ and} \\ T_{ii} &= \frac{\dot{\psi}^2}{2}a^2 + \frac{\cosh^2\Big(\tilde{\psi}\Big)\dot{\chi}^2}{2}a^2 - V(\psi)a^2 \equiv pa^2 \end{aligned} \tag{5.1.18}$$

therefore:

$$\begin{aligned} \rho &= \frac{1}{2}\dot{\psi}^2 + \frac{1}{2}\cosh^2\Big(\tilde{\psi}\Big)\dot{\chi}^2 + V(\psi) \\ p &= \frac{1}{2}\dot{\psi}^2 + \frac{1}{2}\cosh^2\Big(\tilde{\psi}\Big)\dot{\chi}^2 - V(\psi) \end{aligned} \tag{5.1.19}$$

So, now we simply plug these back into (5.1.12) and to get

$$\boxed{H^2 = \frac{1}{3}\left[\frac{1}{2}\dot{\psi}^2 + \frac{1}{2}\cosh^2\Big(\tilde{\psi}\Big)\dot{\chi}^2 + V(\psi)\right]} \tag{5.1.20}$$

Besides, recalling that $\epsilon_1 = -\frac{\dot{H}}{H^2}$, also that the following relation is true: $\frac{\ddot{a}}{a} = H^2(1-\epsilon_1)$ and that $\dot{H} + H^2 = \frac{\ddot{a}}{a}$, using (5.1.15) we can get the following relation:

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\Big(\rho + 3p\Big) = H^2(1-\epsilon_1) \tag{5.1.21}$$

which in turn (combined with (5.1.12)) gives:

$$\boxed{\epsilon_1 = \frac{3}{2}(1+\omega) = \frac{3}{2}\frac{\dot{\psi}^2 + \cosh^2\Big(\tilde{\psi}\Big)\dot{\chi}^2}{\frac{1}{2}\dot{\psi}^2 + \frac{1}{2}\cosh^2\Big(\tilde{\psi}\Big)\dot{\chi}^2 + V(\psi)}} \tag{5.1.22}$$

### 5.1.2 Deriving Equations of Motion for the Scalar Fields

Now that we derived Friedmann equations, we can also derive the equation of motion for the scalar field that we will have to solve with help of Friedmann equations. This should give us the inflationary solution.

From (5.1.1) it follows directly:

$$\sqrt{-g}\frac{\partial V(\psi)}{\partial \psi} + \partial_0\left(\sqrt{-g}\partial_0\psi\right) - \frac{\cosh\left(\tilde{\psi}\right)\sinh\left(\tilde{\psi}\right)}{\sqrt{6}M_p}\dot{\chi}^2\sqrt{-g} = 0 \tag{5.1.23}$$

using the product rule and dividing everything by $-\sqrt{-g}(= -a^3(t))$ we get for the EoM:

$$\boxed{\ddot{\psi} + 3H\dot{\psi} + \partial_\psi V(\psi) - \frac{\cosh\left(\tilde{\psi}\right)\sinh\left(\tilde{\psi}\right)}{\sqrt{6}M_p}\dot{\chi}^2 = 0} \tag{5.1.24}$$

Besides, we, we should also derive the EoM for the $\chi$ field as well:

$$\partial_0\left[\frac{1}{2}2\sqrt{-g}g^{00}\partial_0\chi\cosh^2\left(\tilde{\psi}\right)\right] = 0 \tag{5.1.25}$$

So that:

$$\boxed{\dot{\chi} = \frac{-c}{\sqrt{-g}\cosh^2\left(\tilde{\psi}\right)}} \tag{5.1.26}$$

### 5.1.3 Switching to number of e-folds

As we said, one of the requirements for inflation was to make the universe expand at least 50-60 e-folds. Therefore, since the restriction is put on number of e-folds (N), it makes sense to use it as a measure of time instead of the time itself. So, we need to rewrite our equations in terms of N. By definition (3.1.21) we have:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\mathrm{d}a}{\mathrm{d}t}\frac{1}{a} \implies \frac{\mathrm{d}N}{\mathrm{d}t} = H$$

so that we will have for the slow-roll parameters:

$$\epsilon_0 = \frac{1}{H} \implies \epsilon_1 = \frac{\mathrm{d}\ln(\epsilon_0)}{\mathrm{d}N} = \frac{\dot{a}}{a}\frac{\mathrm{d}\frac{1}{H}}{\mathrm{d}N} = -\frac{1}{H}\frac{\mathrm{d}H}{\mathrm{d}N} \tag{5.1.27}$$

and

$$\dot{\psi} = \frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{\mathrm{d}\psi}{\mathrm{d}N}\frac{\mathrm{d}N}{\mathrm{d}t} = H\frac{\mathrm{d}\psi}{\mathrm{d}N} \tag{5.1.28}$$

therefore:

$$\ddot{\psi} = \frac{\mathrm{d}}{\mathrm{d}t}\left(H\frac{\mathrm{d}\psi}{\mathrm{d}N}\right) = H\frac{\mathrm{d}}{\mathrm{d}N}\left(H\frac{\mathrm{d}\psi}{\mathrm{d}N}\right) = H^2\frac{\mathrm{d}\psi}{\mathrm{d}N}^2 - H^2\epsilon_1\frac{\mathrm{d}\psi}{\mathrm{d}N} \tag{5.1.29}$$

Now we can use these expressions to rewrite 1)$H$, 2)$\epsilon_1$, 3)EoM for $\chi$ and finally 4) EoM for $\psi$ in terms of number of e-folds.

Let's start with $H$:

$$H^2 = \frac{1}{3}\left[\frac{1}{2}H^2\left(\frac{\mathrm{d}\psi}{\mathrm{d}N}\right)^2 + \frac{1}{2}\cosh^2\left(\tilde{\psi}\right)H^2\left(\frac{\mathrm{d}\chi}{\mathrm{d}N}\right)^2 + V(\psi)\right] \tag{5.1.30}$$

giving us:

$$H^2 = \frac{V(\psi)}{3 - \frac{1}{2}\left[\left(\frac{\mathrm{d}\psi}{\mathrm{d}N}\right)^2 + \cosh^2\left(\tilde{\psi}\right)\left(\frac{\mathrm{d}\chi}{\mathrm{d}N}\right)^2\right]} \tag{5.1.31}$$

Now let's change (5.1.22) for $\epsilon_1$ into e-folds:

$$\epsilon_1 = \frac{3}{2}\frac{\left(\frac{\mathrm{d}\psi}{\mathrm{d}N}\right)^2 + \cosh^2\left(\tilde{\psi}\right)\left(\frac{\mathrm{d}\chi}{\mathrm{d}N}\right)^2}{\left(\frac{\mathrm{d}\psi}{\mathrm{d}N}\right)^2 + \cosh^2\left(\tilde{\psi}\right)\left(\frac{\mathrm{d}\chi}{\mathrm{d}N}\right)^2 + \frac{V(\psi)}{H^2}} \tag{5.1.32}$$

and by plugging (5.1.31) here we get the final expression for $\epsilon_1$:

$$\boxed{\epsilon_1 = \frac{1}{2}\left[\left(\frac{\mathrm{d}\psi}{\mathrm{d}N}\right)^2 + \cosh^2\left(\tilde{\psi}\right)\left(\frac{\mathrm{d}\chi}{\mathrm{d}N}\right)^2\right]} \tag{5.1.33}$$

We can now also rewrite $\chi$ in terms of e-folds:

$$\boxed{\frac{\mathrm{d}\chi}{\mathrm{d}N} = -\frac{1}{H}\frac{c}{a^3\cosh^2\left(\tilde{\psi}\right)}} \tag{5.1.34}$$

and we see, that $H$ is expressed through $\chi$, whereas $\chi$ itself is expressed using $H$. To break this loop-dependence, we plug (5.1.34) into (5.1.31)

$$H^2 = \frac{V(\psi)}{3 - \frac{1}{2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}N}\right)^2 - \frac{1}{2}\frac{c^2}{a^6 H^2\cosh^2\left(\tilde{\psi}\right)}} \tag{5.1.35}$$

and express $H$ from the result to get the shape of it:

$$\boxed{H^2 = \frac{V(\psi) + \frac{1}{2}\frac{c^2}{a^6\cosh^2\left(\tilde{\psi}\right)}}{3 - \frac{1}{2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}N}\right)^2}} \tag{5.1.36}$$

and now let's finally also rewrite the (3.1.28) in terms of e-folds:

$$H^2\frac{\mathrm{d}^2\psi}{\mathrm{d}N^2} + H^2(3-\epsilon_1)\frac{\mathrm{d}\psi}{\mathrm{d}N} + \partial_\psi V(\psi) - \frac{\sinh\left(\tilde{\psi}\right)}{\sqrt{6}M_p}\frac{c^2}{a^6\cosh^3\left(\tilde{\psi}\right)} = 0 \tag{5.1.37}$$

dividing this by $H^2$ and plugging in (5.1.36) we finally get:

$$\boxed{\frac{\mathrm{d}^2\psi}{\mathrm{d}N^2} + (3-\epsilon_1)\frac{\mathrm{d}\psi}{\mathrm{d}N} + \left(\partial_\psi V(\psi) - \frac{\sinh\left(\tilde{\psi}\right)}{\sqrt{6}M_p}\frac{c^2}{a^6\cosh^3\left(\tilde{\psi}\right)}\right)\frac{3 - \frac{1}{2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}N}\right)^2}{V(\psi) + \frac{1}{2}\frac{c^2}{a^6\cosh^2\left(\tilde{\psi}\right)}} = 0} \tag{5.1.38}$$

This is the equation that we have to solve and try to get the inflationary solution. In order to achieve this, the initial conditions have to be chosen appropriately. Besides, in order for this model of inflation to be valid, the inflationary solution should give us predictions that are in accordance with experimental measurements. In order to be able to compare the predictions with the data in the next chapter, let's briefly discuss, what do experiments actually measure and how should we compare our results with their measurements.

## 5.2 How to compare Model Predictions vs. Experimental data

The main parameter that is being measured is, as we said, the Power-spectrum. (3.2.17). It is often written in the following way:

$$\Delta_s^2(\mathbf{k}, t) = A_\star \left(\frac{k}{k_\star}\right)^{n_s - 1} \tag{5.2.1}$$

where $A_\star$ is the amplitude of the scalar power spectrum at some chosen pivot scale $k_\star$ (we will discuss this below).

So, the main parameters that can be measured in the experiment are the power spectrum amplitude at the pivot scale $A_\star$ and the spectral tilt $n_s$. Hence, if we want to compare theoretical predictions to the experimental data, we should compare the predictions for these two quantities. Though, there is another important quantity, which we have not mentioned so far, but it is also being measured - the tensor to scalar ratio.

Obviously, in a similar way as we treated scalar components of cosmological perturbations, we can also calculate the power spectrum for tensor components of cosmological perturbations. Experiments can also measure the ratio of the tensor power spectrum to the scalar power spectrum, therefore we should compare this quantity from our predictions to the experimental results as well.

Conveniently enough, many of these quantities can be written using the Hubble flow functions (we called them slow-roll parameters in the 3rd chapter, as it is usually the case, but we actually didn't use the slow-roll approximation when we defined them using geometrical quantities).

$$\begin{aligned} \Delta^2_{\omega_\xi}(k^\star) \equiv A_{\omega_{\xi^\star}} &= \left(\frac{H}{2\pi}\right)^2 \frac{1}{2M_p^2 \epsilon_1} \\ n_s - 1 &= -2\epsilon_1 - \epsilon_2 \\ r &\equiv \frac{\Delta_t^2}{\Delta_s^2} = 16\epsilon_1 \end{aligned} \tag{5.2.2}$$

And now discuss why we introduce the pivot scale and how it is used. Let's consider CMB - in the CMB we measure the correlation functions of the temperature anisotropies. It should be noted, though, that we can do this only up to a certain resolution and not for all the possible values for $\mathbf{k}$. To understand this better, imagine that we take the $\mathbf{k}$ too small, so that the corresponding wavelength becomes comparable to the Hubble (causal) horizon in size. If we now calculate the correlation functions at these scales, we will not have enough points within the horizon, between which we can calculate the correlation function, meaning that we will not be able to draw reliable statistical conclusions from these results. The range of values of $\mathbf{k}$, that can give us reliable results corrseponds to $10^{-4} Mpc \lesssim k \lesssim 0.1 Mpc$.

As we said, the interesting part about these correlations is, that they get conserved on super-Hubble scales - first they exit the Hubble radius horizon, get "frozen" there and later, at some point they re-enter the Hubble radius in the shape that they had when they exited the Hubble radius - this is the shape that we observe now. Depending on the inflationary models and reheating temperature ([19], [10]) it can be assumed, that the modes that we observe today were created around 50-60 e-folds before the end of inflation. Hence, to be able to compare the theoretical predictions of our model with the experimental data, we should look at these modes.

Obviously, the value of the power spectrum will slightly differ for each mode, since each mode exits the Hubble radius at different times. Therefore, we usually take some reference/pivot scale somewhere between 50 and 60 e-folds (say N=55 e-folds) and compute the 2 important parameters - the amplitude of the power spectrum and the spectral tilt. In CMB we usually choose a scale $k_\star = 0.005 Mpc^{-1}$ [10] and compute these two parameters for it. According to the latest measurements, $A_{k_\star} = 2.1955 \cdot 10^{-9} \pm 0.103 \cdot 10^{-9}$ and $n_s = 0.9655 \pm 0.0062$ [10].

To compare the predictions of our model to these values, we should therefore:

1) Identify the point at which inflation ends
2) Determine the number of e-folds that corresponds to $N \in (50, 60)$ e-folds before the end of inflation.
3) Calculate the value of the power spectrum amplitude, $n_s$ and tensor to scalar ratio at those values (r).
4) Plot the graph of $n_s$ vs. $r$.
5) Compare with the results of [10].

This was the general strategy that we used to get the results presented in the following chapter.

# Chapter 6

# Results

***In this chapter we are going to discuss how we choose parameters for our model and what kind of results for what ranges of free parameters of the theory do we get. Then we are going to compare these predicted results of our model to the experimental measurements from [10].*** Additionally, we are going to discuss some interesting features of our model coming from the conformal symmetry in the theory.

## 6.1 Choosing Parameters

Before we can demonstrate the results, we should first discuss the initial conditions and then what free parameters do we have in the theory and how do we approach choosing their values.

As initial conditions we decided to choose $\psi(0) = 10^{-3}M_p$ and $\psi'(0) = 0$, where $'$ denotes the derivative with respect to e-folds. The reason for the choice of $\psi(0)$ was simple - we wanted to start very close to the conformal point, but not too close, since then the initial condition would be less than the quantum uncertainty $\delta\psi \sim 10^{-6}M_p$ and this would introduce some complications in explanations, therefore we went for the safe option. As to the velocity of the field - we have the same constraint involving quantum uncertainty, but since in this case the uncertainty is of order $\langle\dot{\psi}\rangle \sim H^2 = \left(\frac{H^2}{M_p}\right)^2 M_p^2 \sim (10^{-11} - 10^{-12})M_p^2$, i.e. very small, we can safely put it to 0 in our code. As to the variation of these quantities, the effect is clear - higher initial velocity, as well as the larger initial value for the field would simply result in a smaller number of e-folds in the expansion.

Now let's discuss the free parameters. In our model we have 4 free parameters - 3 coupling constants $\alpha$, $\beta$ and $\lambda$ (or $\tilde{\lambda}$) and 1 integration constant $c$ coming from the equation of motion of $\chi$.

First of all, let's clear things about $\alpha$. As we mentioned early, we pulled $\alpha$ out of the brackets in the potential and defined $\tilde{\lambda} \equiv \alpha\lambda$ for convenience

$$V(\psi) = \frac{1}{\alpha}\left[\frac{\tilde{\lambda}}{4}(6M_p^2)^2\sinh^4\left(\tilde{\psi}\right) + \frac{M_p^4}{16}\left(12\beta\sinh^2\left(\tilde{\psi}\right) - 1\right)^2\right] \tag{6.1.1}$$

We should also define $c \to \tilde{c} = c\sqrt{\alpha}$. If we now carefully look at the equation of motion (5.1.38):

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}N^2} + (3-\epsilon_1)\frac{\mathrm{d}\psi}{\mathrm{d}N} + \left(\partial_\psi V(\psi) - \frac{\sinh\left(\tilde{\psi}\right)}{\sqrt{6}M_p\alpha}\frac{\tilde{c}^2}{a^6\cosh^3\left(\tilde{\psi}\right)}\right)\frac{3 - \frac{1}{2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}N}\right)^2}{V(\psi) + \frac{1}{2}\frac{\tilde{c}^2}{\alpha a^6\cosh^2\left(\tilde{\psi}\right)}} = 0 \tag{6.1.2}$$

taking into account, that

$$\frac{\mathrm{d}\chi}{\mathrm{d}N} = -\frac{1}{H}\frac{\tilde{c}}{\sqrt{\alpha}a^3\cosh^2\left(\tilde{\psi}\right)} \quad \text{and} \quad H^2 = \frac{V(\psi) + \frac{1}{2}\frac{\tilde{c}^2}{\alpha a^6\cosh^2\left(\tilde{\psi}\right)}}{3 - \frac{1}{2}\left(\frac{\mathrm{d}\psi}{\mathrm{d}N}\right)^2}$$

together with:

$$\epsilon_1 = \frac{1}{2}\left[\left(\frac{\mathrm{d}\psi}{\mathrm{d}N}\right)^2 + \cosh^2\left(\tilde{\psi}\right)\left(\frac{\mathrm{d}\chi}{\mathrm{d}N}\right)^2\right]$$

we see, that $\alpha$ simply drops out of the equation of motion and it does not affect the evolution of the inflation anymore. This is the reason, why it was convenient to pull it out of the potential.

It is important to note, that $\alpha$ still plays an important role in the model by controlling the amplitude of the power spectrum:

$$\Delta^2_{\omega_\xi}(k^\star) \equiv A_{\omega_{\xi^\star}} = \left(\frac{H}{2\pi}\right)^2 \frac{1}{2M_p^2\epsilon_1} \quad \text{where} \quad H(0) = \frac{1}{48\alpha} \tag{6.1.3}$$

so $\alpha$ determines the initial value of the Hubble parameter allowing us to tweak the amplitude of the power spectrum to the proper values of [10] without affecting any dynamics of the inflaton and consequently the $n_s$ and $r$ as well. This is clearly very convenient as well.

Now let's have a look at the other coupling constants - $\tilde{\lambda}$ and $\beta$. Recall the requirements for the slow-roll inflation:

1) The potential needs to be flat, so that the slow-roll condition applies

2) Since inflation has to end at some point, we need to violate $\epsilon_1 \ll 1$ condition - this is only achieved by having a steep slope away from the origin

Keeping these requirements in mind, we should look at the shape of the potential:

$$V(\psi) = \frac{1}{\alpha}\left[\underbrace{\frac{\tilde{\lambda}}{4}(6M_p^2)^2\sinh^4\left(\tilde{\psi}\right)}_{\text{I}} + \frac{M_p^4}{16}\left(\underbrace{144\beta^2\sinh^4\left(\tilde{\psi}\right)}_{\text{II}} - \underbrace{24\beta\sinh^2\left(\tilde{\psi}\right)}_{\text{III}} + 1\right)\right] \tag{6.1.4}$$

We clearly see, that there is a negative parabola coming from the III term - this term will be dominant near the origin, compared to the terms proportional to $\sinh^4\left(\tilde{\psi}\right)$. Though, if we start increasing $\lambda$, we can make the I term become significant closer to the origin as well, destroying the clear minimum structure, meaning that we will not get a steep slope anymore and we won't be able to stop inflation (See Fig. (6.1)).

This is a clear indication that to get a working model, we should keep the $\lambda \ll \beta$ hierarchy.

This is also convenient, since if we follow this hierarchy now, $\beta$ becomes dominant throughout the whole range of $\psi$ evolution and $\lambda$ does not play a significant role anymore (See Fig.(6.5b) ), hence $\beta$ will be our main controlling parameter, making it easier to tweak the results.

So, in the end we can roughly assign the properties to each of the coupling constants:

1) $\alpha$ - allows us to easily control the amplitude of the power spectrum

2) $\lambda$ - the main parameter that controls the steepness of the slope of reheating (we need the slope to be steep enough to be able to end inflation)

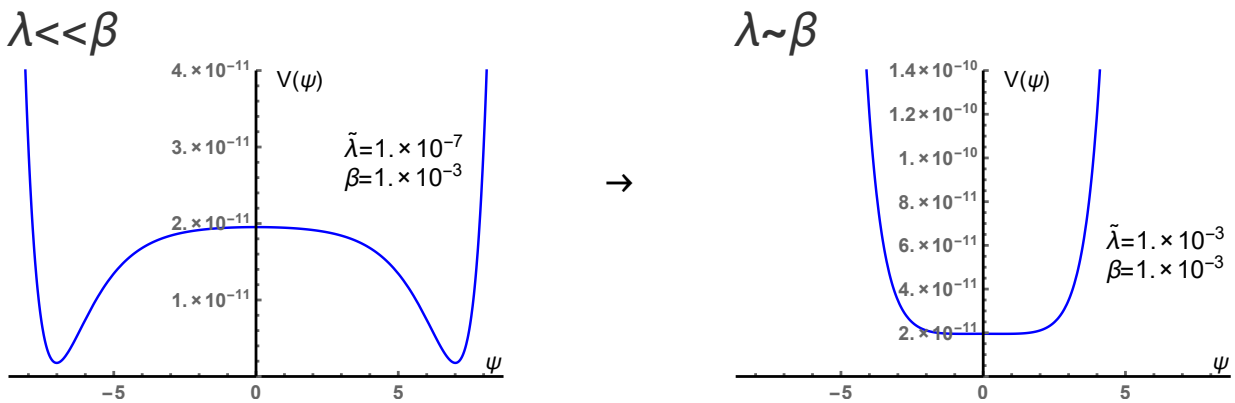



Figure 6.1: $\lambda$ and $\beta$ hierarchy effect on the potential. On the first plot we have the clear minima, whereas on the second plot minima are completely abscent now (away from the origin)

3) $\beta$ - the main parameter that controls the position of the minimum in the potential - i.e. how far does the flatness stretch from the origin

Since $\alpha$ does not affect the inflationary results, in our simulations we varied $\beta$ and $\lambda$. And, as mentioned, we also kept the needed $\lambda \ll \beta$ hierarchy in check.

As to the $c$ term, we will discuss 2 different cases - 1) $c = 0$ and 2) $c \neq 0$, since the second case introduces some interesting effects into the model. Let's discuss what kind of results we acquired in each of these cases.

## 6.2 Results

**1)** $\alpha$

First of all let's discuss $\alpha$ to get it quickly out of the way. As we said, it controls the amplitude of the power spectrum and it does not affect any other quantities other than this amplitude, therefore we simply choose the value of $\alpha$ such that we get an agreement with the measurements of [10]. For example, in the case with $\beta = 1.5 \cdot 10^{-3}$ and $\tilde{\lambda} = 1 \cdot 10^{-8}$ we need $\alpha = 4.9 \cdot 10^{8}$ so that for the amplitude we get $\Delta = 2.17 \cdot 10^{-9}$. That's all that there is to $\alpha$.

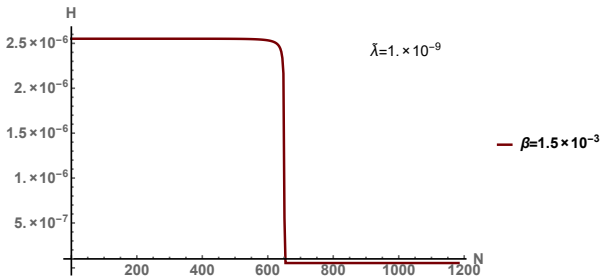
### 6.2.1 c = 0

**2)** $\beta$

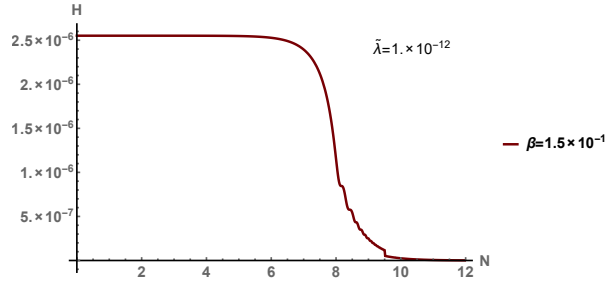
Now we should obviously have a look at the main parameter of our problem - $\beta$ and see how it should be chosen in the first place. As our main constraint on the choice of $\beta$ value we use the requirement of having at least $\sim 50 - 60$ e-folds of expansion.

To see how much the universe expands in our model for a chosen set of parameters, we look at the evolution of the Hubble parameter $H$. As we remember, $H = const \implies$ inflationary period, so inflation

will last until the $H$ vs $N$ plot stays flat. For example for $\beta = 1.5 \cdot 10^{-3}$ we get (See Fig.(6.2a)).




(a) For $\beta = 1.5 \cdot 10^{-3}$ inflation lasts for enough time - more than $N > 600$ e-folds




(b) For $\beta = 1.5 \cdot 10^{-1}$ inflation does not last enough anymore as $N < 50$ e-folds

Figure 6.2: Good vs. Bad cases

This value of $\beta$ apparently gives us enough number of e-folds. If we increase $\beta$ though, we might not get enough N (See Fig.(6.2b)) anymore. This means, that $\beta$ is bounded from above and it should at least be $\beta < 10^{-2}$.

We can see the dependence of $N(\beta)$ (i.e. how $\beta$ affects our results) well on the following plot (See Fig. (6.3)).

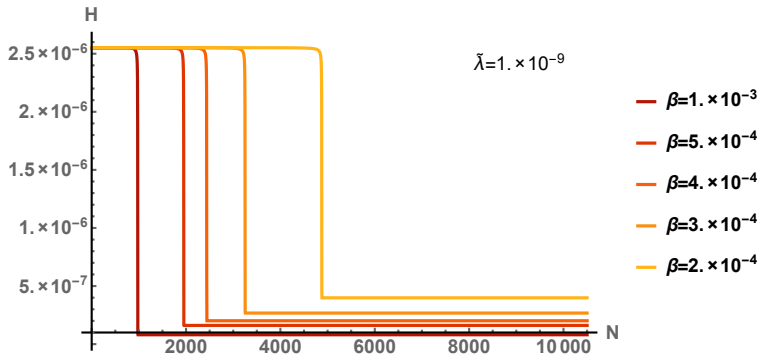



Figure 6.3: Dependence of $N$ on $\beta$

Now that we have enough number of e-folds and controlled amplitude of the power spectrum, we should have a look at other parameters measured by the experiment - 1) the spectral tilt and the 2) tensor to scalar ratio.

To see, what results we get for these two quantities, as we discussed above, when we solve the main equations, we should choose the number of e-folds $N_\star$ (at which we measure $n_s$ and $r$) such that there is still 50-60 e-folds left before the $\epsilon_1$ becomes of order 1. After this we look at the expressions for $n_s$ and $r$ and evaluate them at this $N_\star$ time and then compare with the experimental results.

To make this procedure clearer, let's take (6.2a) as an example and plot the corresponding graph of $\epsilon_1$ (See Fig.(6.4)):

By doing this for varying values of $\beta$ we can plot the $r$ vs. $n_s$ graph showing the dependence of the results on $\beta$ . We plot the results for every integer $N$ between 50 and 60 e-folds (11 points in total for each value of $\beta$) (See Fig.(6.5a)). The blue shaded area is the range of allowed values for $r$ and $n_s$ according to [10].

As we see, there is only a limited range of results for $r$ vs. $n_s$ - if we increase the $\beta$ values too much, it goes out of the measurement bounds to the left. As we decrease it, though, the dependence of $r$ on $n_s$ seems

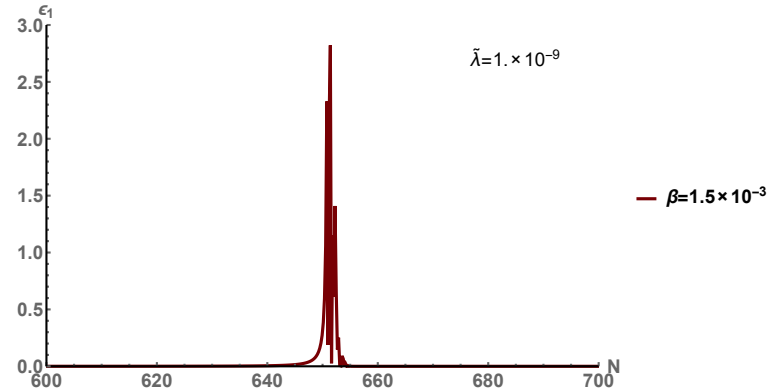
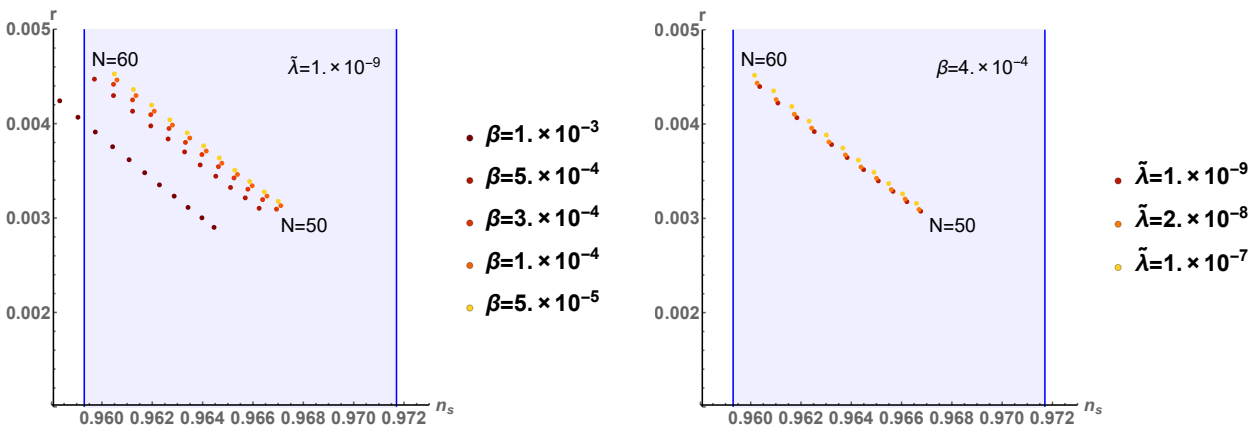



Figure 6.4: $\epsilon_1$ blows up when the field starts rolling down the steep potential and inflation ends




(a) Dependence of the results on various values of $\beta$. As we see, with lowering $\beta$ the line starts asymptoting some value and not going too far to the right (For a fixed $\lambda$)).

(b) Dependence of the results on various values of $\lambda$. In this case, the line barely moves even with change of $\lambda$ by two orders of magnitude, just as we anticipated.

Figure 6.5: $\beta$ and $\lambda$ dependence

to be assymptoting to some slope and not going too far to the right.

The main point here is, though, that it's clear from the plot that we can indeed get a viable inflationary model for certain range of values of the coupling constants! We are both - getting the correct amplitude for the power spectrum, as well as getting the proper $n_s$ and $r$ values.

**3)** $\lambda$

And finally let's have a look at the effect of $\lambda$. As we said, we should maintain the needed hierarchy $\lambda \ll \beta$. In this case $\lambda$ can mainly control the steepness of the slope (See Fig.(6.6)) and not much more, as it is clear from Fig.(6.5b)). Even the change by 2 orders of magnitude does not introduce much shift in the plot, proving our conclusions about the role of $\lambda$.

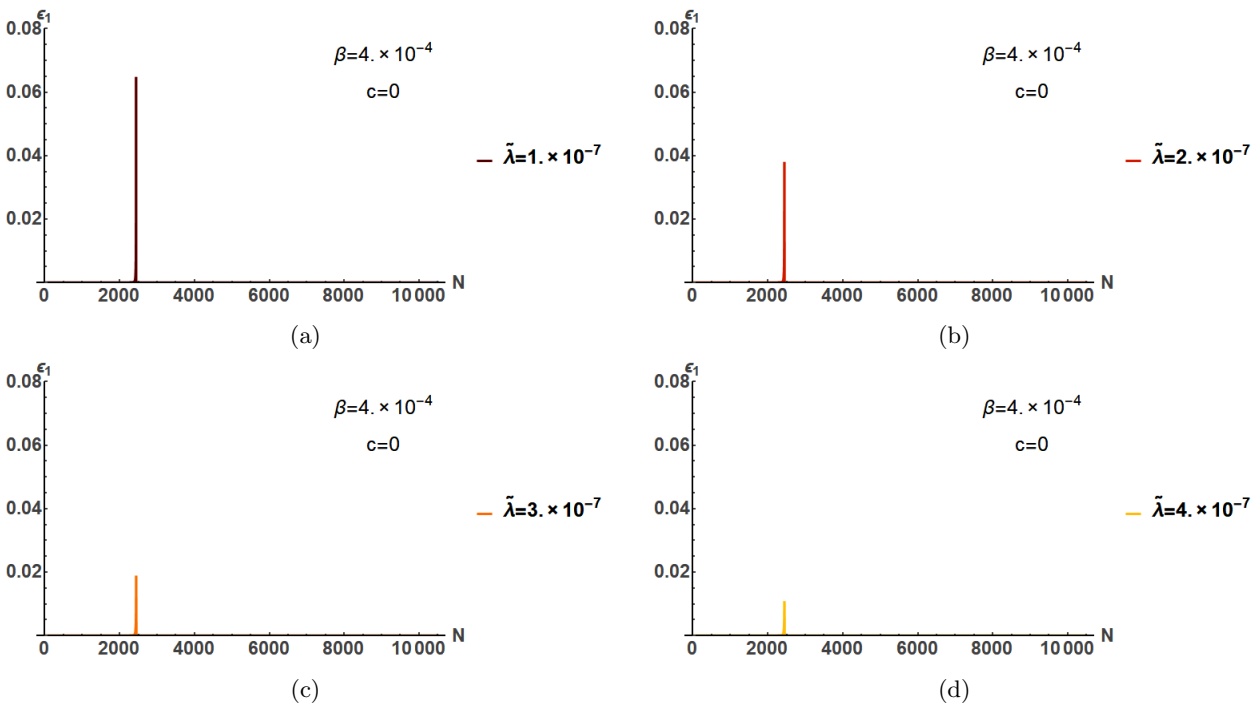



Figure 6.6: Dependence of the slope steepness on $\lambda$ - a higher value of $\epsilon_1$ means a steeper slope

So, as we see, in case of $c = 0$ we get a range of values for the coupling constants, for which our model of inflation gives predictions that are within the measurement constrains.

### 6.2.2 $c \neq 0$

Now let's "turn on" the effects coming from $c$ and see what we get.

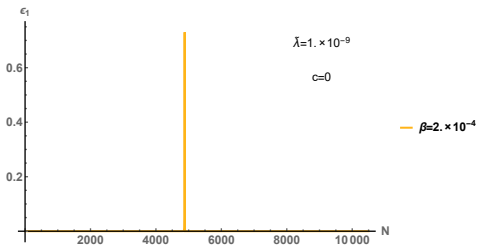
First of all, we should notice, that terms containing $c$ in (6.1.2) are suppressed by the $a^6$ factor, meaning that they will only be important in the very beginning of inflation, since $a = e^N$.

Before actually plugging $c$ into the equation to see the results, we can already make some intuitive predictions about its behaviour - recall, that $c$ term comes from the equation of motion of the torsion field $\chi$, meaning that $c$ controls the amount of the angular momentum present initially. And since we are starting near the origin, considering the fact, that the angular momentum would obviously be accompanied by the centrifugal force, we can expect this centrifugal force to speed up the inflation and make it roll down the potential faster than it would have done in $c = 0$ case. Besides, this initial period of the so-called kination (when kinetic energy dominates the potential energy) will obviously decrease the value of $H$, at which the field will start slowly rolling, meaning that we should also expect a less steep slope for $H$ i.e. smaller $\epsilon_1$ value at the blowup point.

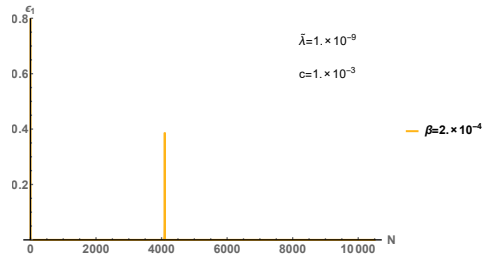
Let's now have a look at the actual results (See Fig.(6.7)).

If we look closely, we do indeed see, that both effects appeared - 1) the blowup point moved to the left and 2) the slope became less steep.

To understand this even better, it is interesting to look at the initial part of H and how it evolves (See Fig.(6.8)).

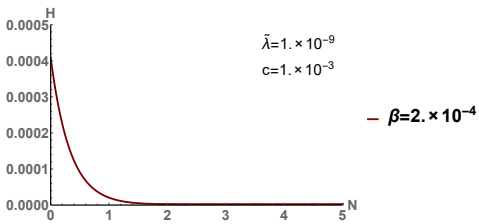



(a)

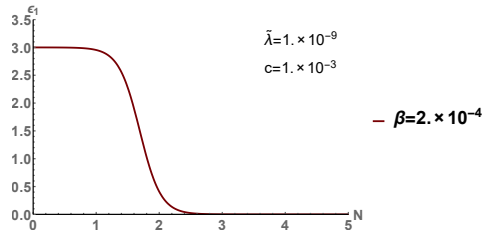



(b)

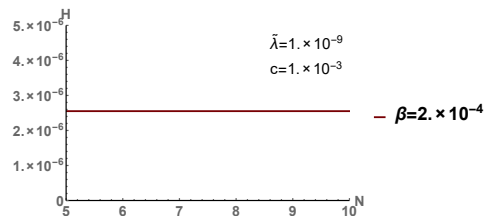
Figure 6.7: With $c$ present we do indeed see the $\epsilon_1$ blowup point move to the left, meaning that inflation ended sooner than with $c = 0$




(a)




(b)




(c)

Figure 6.8: $c$ introduces an initial period of the so-called kination

We see, that initially $H$ starts decreasing rapidly because of the huge centrifugal force acting on the inflaton. $\epsilon_1$ is also, obviously, very large during this period and then flattens together with $H$ (See Fig.(6.8c)). After this, the evolution continues exactly like we had in the case with $c = 0$.

The interesting part about $c$ is precisely the kination effect - due to the kinetic energy domination in the beginning, we can not use the following approximation expression for the power spectrum anymore:

$$\Delta_s^2(\mathbf{k}, t) = A_\star \left(\frac{k}{k_\star}\right)^{n_s - 1}$$

meaning, that at the largest length scales(the earlier the events happen in terms of e-folds, the larger the length scales affected by them are), we should be able to observe the effects of kination in the power spectrum. Therefore, we could be able to distinguish our model from other models of inflation using the effect coming from $c$, which is the same as torsion, or even more generally - the effects of the initial conformal symmetry in our model. We are not going to discuss this in this thesis, though - this will be the next step in this project.

**Large $c$**

There is a bound from a above for $c$ though - if we keep increasing it, we will be getting less and less number of e-folds, therefore at some point we will not have enough expansion, to coincide with the experimental measurements.

### 6.2.3 Scale of Inflation

And finally let's also quickly calculate the scale of the inflation that we get. As a scale of inflation we take the value of $H$ when it is flat in the beginning. Taking into account, that all the plotted values are in units of the Planck's mass, we get for the scale:

$$A_s = \frac{H^2 M_p^2}{16\epsilon_1} \quad \text{and} \quad \frac{H^2}{M_p^2} = \frac{\pi^2}{2} r A_s \tag{6.2.1}$$

giving us the following scale:

$$H = \sqrt{\frac{\pi^2}{2} r A_s} M_p \approx 3 \cdot 10^{-6} M_p \approx 10^{13} \text{ GeV} \tag{6.2.2}$$

### 6.2.4 Difference from $\alpha$-attractors

As we said, there are various models of inflation and one of the most important type of them is called the $\alpha$-attractor.

One of the main differences with our case, that comes from the conformal symmetry, is that we can not vary the curvature of the configuration space, since adding any otherwise allowed coefficients to the kinetic terms, that give us the possibility to modify the $R_{CS}$, will violate the conformal symmetry explicitly. In $\alpha$-attractors, on the other hand, such terms can be added and this allows one to change the value of the tensor to scalar ratio $r$ within the allowed boundaries ranging from the Starobinsky [8] model to the so-called $m^2\phi^2$ models.

So in our case we end up with values of $r$ that are close to those of the Starobinsky model, namely $r \sim 10^{-3}$.

# Chapter 7

# Conclusions and Outlook

Now it's finally time to make the final conclusions and discuss the future possibilities within this project.

First of all, we motivated why it was interesting to study this project and discussed why and how can the General Relativity be made (in an elegant way) conformal . We also motivated why it was important to check, whether we could get a working inflationary model within this framework.

After this, starting with a general conformal action for a scalar field $\phi$ expanded up to $R^2$ terms, we derived an action, from which we could actually start going towards getting an inflationary solution. It turned out, that this action acquired some very interesting features such as the hyperbolic field-space, consequently resulting in a hyperbolic potential, that allowed us to make it flat near the origin and also flat for long enough, so that we could get a viable slow-roll model. This feature was also very important, since it realized inflation through SSB, allowing us to start the evolution of the field at (or close to) the conformal point and then by acquiring a condensate to have dynamically generated scales in the theory. It should be mentioned, though, that there is a difference compared to the "conventional" SSB (with a $U(1)$ symmetry breaking, for example) in the sense that in our case the range of $\chi$ Goldstone boson $\in (-\infty, \infty))$ unlike $\in (0, 2\pi)$. This was also caused by the hyperbolic field-space.

Another interesting feature was also the remnant cosmological constant (to properly discuss this last one, we also had to calculate the one-loop quantum corrections to it (see Appendix B)). And finally, we discovered, that torsion gave the inflaton an angular momentum, that resulted in a period of kination in the beginning of the evolution.

What is noteworthy, though, is, that all of these interesting properties came out naturally (without having to fine-tune any parameters or make any assumptions) from the fact that we introduced torsion into the theory.

After this we studied the free parameters of our theory to find their ranges, for which we could get a viable inflationary model that agreed with the measurement data given in [10]. We discussed, how we approached choosing them and what kind of results we should have expected in different cases. And indeed, in the end we managed to find the ranges, for which the model is valid and discussed how the dynamics of the inflation behaves under different conditions.

As an outlook for the future we are planning to calculate the power-spectrum for the matter perturbations explicitly to see, whether we can in principle observe the kink in it (as it is predicted by the theory). If there is a kink, it should be at the largest scales (i.e. smallest $k$ values) and if these scales can in principle be observed today (they might still be outside of the causal horizon for us), then this would be a good test for the current model and would also give us a possibility to differentiate it from the other models.

# Acknowledgements and Bibliography

First of all, I would like to thank my supervisor Tomislav for his support and guidance - especially in those moments, when it seemed like there was no solution to the problem. I definitely earned a lot of experience and knowledge from him, together with learning the new ways of approaching a problem and thinking in general, which I am sure will be of a great benefit for me in the future, so I am very happy about that. What I also really liked while working with Tomislav was his attitude towards the students and willingness to help us even if it took him 2 hours of an unplanned meeting. And finally, his big interest in the topic itself made the working process way more exciting and interesting as well.

As to the technical side, I liked his friendly working style rather than a formal one, and it was also always easy to set an appointment when there was a problem and I appreciate this a lot.

Aside from the academic interaction, I also enjoyed the normal life interactions very much - discussing topics ranging from football (involving actual interactive explanations (using little kicks in my shoulder) to demonstrate the fouls conceded on the football players) to fighter jets and their performance. This actually made the working atmosphere more relaxed and joyful and I am very happy about this as well.

I would also like to say a very big thank you to my daily supervisor Stefano. I am very thankful for his time and his willingnes to help whenever there was a problem (even if he could not come up with a good solution at the moment, he would always take his time find the explanation later and discuss it with me). I was also very impressed by the amount of knowledge he has - he would always come up with an intuitive explanation from a physical point of view to all of my questions from various areas of physics, therefore he is a great example for me what being a good young scientist really means and this inspired me to work more, expand my views in physics and be more excited about the beauty of the nature in general. I also really appreciate his patience in answering my questions, even if they were not particularly good at times. So, I must say that I feel very fortunate to have worked with him as my daily supervisor.

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# Appendix

# Appendix A

# Einstein and Jordan Frame equivalence

***In this appendix we are checking the consistency of Einstein frame results that we derived in the main part of the thesis. We do this by calculating the minimal value of the potential energy in Jordan frame and then comparing it to that coming from the Einstein frame. For this we first derive the Einstein equation, then Friedmann equations and finally we find the de Sitter solution at the minimum of the potential and calculate the value of the potential at that point and compare it to the results coming from the Einstein frame calculations, presented in the main part of the thesis.***

## A.1 Physical Quantity Modification by Torsion

We can now express Ricci tensor with torsion trace by plugging (2.2.40) and (2.2.10) into (2.2.39). Let's do this term by term.

$$\textcircled{1} = \mathring{\nabla}_\lambda K^\lambda{}_{\mu\nu} = \frac{1}{2}\Big(\underbrace{\mathring{\nabla}_\lambda T^\lambda{}_{\mu\nu}}_{\textcircled{1.\mathrm{I}}} - \underbrace{\mathring{\nabla}_\lambda T_\mu{}^\lambda{}_\nu}_{\textcircled{1.\mathrm{II}}} - \underbrace{\mathring{\nabla}_\lambda T_\nu{}^\lambda{}_\mu}_{\textcircled{1.\mathrm{III}}}\Big) \tag{A.1.1}$$

$$\begin{aligned}\textcircled{1.\mathrm{I}} &= \mathring{\nabla}_\lambda T^\lambda{}_{\mu\nu} = \mathring{\nabla}_\lambda g^{\rho\lambda} T_{\rho\mu\nu} = \text{using metric compatibility we can take the metric out} = g^{\rho\lambda}\mathring{\nabla}_\lambda T_{\rho\mu\nu} = \\ &= \frac{1}{3} g^{\rho\lambda}\Big(\mathring{\nabla}_\lambda T_\mu g_{\rho\nu} - \mathring{\nabla}_\lambda T_\lambda g_{\rho\mu}\Big) = \text{using metricity again} = \frac{1}{3} g^{\rho\lambda}\Big(g_{\rho\nu}\mathring{\nabla}_\lambda T_\mu - g_{\rho\mu}\mathring{\nabla}_\lambda T_\lambda\Big) = \\ &= \frac{1}{3}\Big(\delta^\lambda_\nu \mathring{\nabla}_\lambda T_\mu - \delta^\lambda_\mu \mathring{\nabla}_\lambda T_\nu\Big) = \frac{1}{3}\Big(\mathring{\nabla}_\nu T_\mu - \mathring{\nabla}_\mu T_\nu\Big)\end{aligned} \tag{A.1.2}$$

$$\textcircled{1.\mathrm{II}} = \mathring{\nabla}_\lambda T_\mu{}^\lambda{}_\nu = g^{\rho\lambda}\mathring{\nabla}_\lambda T_{\mu\rho\nu} = \frac{1}{3} g^{\rho\lambda}\Big(g_{\mu\nu}\mathring{\nabla}_\lambda T_\rho - g_{\mu\rho}\mathring{\nabla}_\lambda T_\nu\Big) = \frac{1}{3}\Big(g_{\mu\nu}\mathring{\nabla}_\lambda T^\lambda - \mathring{\nabla}_\mu T_\nu\Big) \tag{A.1.3}$$

$$\textcircled{1.\mathrm{III}} = \text{by analogy with } \textcircled{1.\mathrm{II}} = \frac{1}{3}\Big(g_{\mu\nu}\mathring{\nabla}_\lambda T^\lambda - \mathring{\nabla}_\nu T_\mu\Big) \tag{A.1.4}$$

So that we get:

$$\boxed{\begin{aligned}\textcircled{1} &= \frac{1}{6}\Big(\mathring{\nabla}_\nu T_\mu - \mathring{\nabla}_\mu T_\nu - 2g_{\mu\nu}\mathring{\nabla}_\lambda T^\lambda + \mathring{\nabla}_\mu T_\nu + \mathring{\nabla}_\nu T_\mu\Big) = \frac{1}{6}\Big(2\mathring{\nabla}_\nu T_\mu - 2g_{\mu\nu}\mathring{\nabla}_\lambda T^\lambda\Big) = \\ &= \frac{1}{3}\Big(\mathring{\nabla}_\nu T_\mu - g_{\mu\nu}\mathring{\nabla}_\lambda T^\lambda\Big)\end{aligned}} \tag{A.1.5}$$

Now the second one:

$$\textcircled{2} = \mathring{\nabla}_\nu K^\lambda{}_{\mu\lambda} = \frac{1}{2}\Big(\underbrace{\mathring{\nabla}_\nu T^\lambda{}_{\mu\lambda}}_{\textcircled{2.\mathrm{I}}} - \underbrace{\mathring{\nabla}_\nu T_\mu{}^\lambda{}_\lambda}_{\textcircled{2.\mathrm{II}}} - \underbrace{\mathring{\nabla}_\nu T_\lambda{}^\lambda{}_\mu}_{\textcircled{2.\mathrm{III}}}\Big) \tag{A.1.6}$$

$$\textcircled{2.\mathrm{I}} = \text{by definition of torsion trace} = \mathring{\nabla}_\nu T_\mu \tag{A.1.7}$$

$$\begin{aligned}\textcircled{2.\mathrm{II}} &= \mathring{\nabla}_\nu T_\mu{}^\lambda{}_\lambda = \mathring{\nabla}_\nu g^{\rho\lambda} T_{\mu\rho\lambda} = \text{since some indices are dummy here, we can relabel them} = \mathring{\nabla}_\nu g^{\lambda\rho} T_{\mu\lambda\rho} = \\ &= \text{using symmetry of metric and antisymmetry of torsion in last two indices} = -\mathring{\nabla}_\nu g^{\rho\lambda} T_{\mu\rho\lambda} \\ &\implies T_\mu{}^\lambda{}_\lambda = 0\end{aligned} \tag{A.1.8}$$

$$\begin{aligned}\textcircled{2.\mathrm{III}} &= \mathring{\nabla}_\nu T_\lambda{}^\lambda{}_\mu = \mathring{\nabla}_\nu g^{\rho\lambda} T_{\lambda\rho\mu} = \text{using metricity} = g^{\rho\lambda}\mathring{\nabla}_\nu T_{\lambda\rho\mu} = \frac{1}{3} g^{\rho\lambda}\Big(g_{\lambda\mu}\mathring{\nabla}_\nu T_\rho - g_{\lambda\rho}\mathring{\nabla}_\nu T_\mu\Big) = \\ &= \Big(\delta^\rho_\mu \mathring{\nabla}_\nu T_\rho - 4\mathring{\nabla}_\nu T_\mu\Big) = -\mathring{\nabla}_\nu T_\mu\end{aligned} \tag{A.1.9}$$

So that

$$\boxed{\textcircled{2} = \frac{1}{2}\Big(\mathring{\nabla}_\nu T_\mu + \mathring{\nabla}_\nu T_\mu\Big) = \mathring{\nabla}_\nu T_\mu} \tag{A.1.10}$$

Now the third one:

$$\textcircled{3} = K^\lambda{}_{\gamma\lambda} K^\gamma{}_{\mu\nu} = \frac{1}{4}\Big(T^\lambda{}_{\gamma\lambda} - T_\gamma{}^\lambda{}_\lambda - T_\lambda{}^\lambda{}_\gamma\Big)\Big(T^\gamma{}_{\mu\nu} - T_\mu{}^\gamma{}_\nu - T_\nu{}^\gamma{}_\mu\Big) \tag{A.1.11}$$

Taking into account that $T^\lambda{}_{\gamma\lambda} = T_\gamma$, also that $T_\gamma{}^\lambda{}_\lambda = 0$ and $T_\lambda{}^\lambda{}_\gamma = -T^\lambda{}_{\gamma\lambda} = -T_\gamma$ we get

$$\begin{aligned}\textcircled{3} &= \frac{1}{2} T_\gamma g^{\rho\gamma}\Big(T_{\rho\mu\nu} - T_{\mu\rho\nu} - T_{\nu\rho\mu}\Big) = \frac{1}{6} T_\gamma g^{\rho\gamma}\Big(T_\mu g_{\rho\nu} - T_\nu g_{\rho\mu} - T_\rho g_{\mu\nu} + T_\nu g_{\mu\rho} - T_\rho g_{\nu\mu} + T_\mu g_{\nu\rho}\Big) = \\ &= \frac{1}{6}\Big(T_\gamma T_\mu \delta^\gamma_\nu - T_\gamma T_\nu \delta^\gamma_\mu - T_\gamma T^\gamma g_{\mu\nu} + T_\gamma T_\nu \delta^\gamma_\mu - T_\gamma T^\gamma g_{\mu\nu} + T_\gamma T_\mu \delta^\gamma_\nu\Big) = \\ &= \frac{1}{6}\Big(T_\nu T_\mu - T_\mu T_\nu - T_\lambda T^\lambda g_{\mu\nu} + T_\mu T_\nu - T_\lambda T^\lambda g_{\mu\nu} + T_\mu T_\nu\Big) = \frac{2}{6}\Big(T_\mu T_\nu - T_\lambda T^\lambda g_{\mu\nu}\Big)\end{aligned} \tag{A.1.12}$$

So, we get:

$$\boxed{\textcircled{3} = \frac{1}{3}\Big(T_\mu T_\nu - T_\lambda T^\lambda g_{\mu\nu}\Big)} \tag{A.1.13}$$

And finally:

$$\textcircled{4} = K^\lambda{}_{\mu\gamma} K^\gamma{}_{\lambda\nu} = \frac{1}{4}\Big(T^\lambda{}_{\mu\gamma} - T_\mu{}^\lambda{}_\gamma - T_\gamma{}^\lambda{}_\mu\Big)\Big(T^\gamma{}_{\lambda\nu} - T_\lambda{}^\gamma{}_\nu - T_\nu{}^\gamma{}_\lambda\Big) \tag{A.1.14}$$

And taking into account

$$T^{\lambda}{}_{\mu\gamma} = \frac{1}{3}\Big(T_\mu \delta^\lambda_\gamma - T_\gamma \delta^\lambda_\mu\Big) \tag{A.1.15}$$

$$T_\mu{}^\lambda{}_\gamma = \frac{1}{3}\Big(T^\lambda g_{\mu\gamma} - T_\gamma \delta^\lambda_\mu\Big) \tag{A.1.16}$$

$$T_\gamma{}^\lambda{}_\mu = \frac{1}{3}\Big(T^\lambda g_{\mu\gamma} - T_\mu \delta^\lambda_\gamma\Big) \tag{A.1.17}$$

$$T^{\gamma}{}_{\lambda\nu} = \frac{1}{3}\Big(T_\lambda \delta^\gamma_\nu - T_\nu \delta^\gamma_\lambda\Big) \tag{A.1.18}$$

$$T_\lambda{}^\gamma{}_\nu = \frac{1}{3}\Big(T^\gamma g_{\lambda\nu} - T_\nu \delta^\gamma_\lambda\Big) \tag{A.1.19}$$

$$T_\nu{}^\gamma{}_\lambda = \frac{1}{3}\Big(T^\gamma g_{\lambda\nu} - T_\lambda \delta^\gamma_\nu\Big) \tag{A.1.20}$$

we get

$$\begin{aligned} 4 &= \frac{1}{36}\Big(T_\mu \delta^\lambda_\gamma - T_\gamma \delta^\lambda_\mu - T^\lambda g_{\mu\gamma} + T_\gamma \delta^\lambda_\mu - T^\lambda g_{\mu\gamma} + T_\mu \delta^\lambda_\gamma\Big)\Big(T_\lambda \delta^\gamma_\nu - T_\nu \delta^\gamma_\lambda - T^\gamma g_{\lambda\nu} + T_\nu \delta^\gamma_\lambda - T^\gamma g_{\lambda\nu} + T_\lambda \delta^\gamma_\nu\Big) = \\ &= \frac{4}{36}\Big(T_\mu \delta^\lambda_\gamma - T^\lambda g_{\mu\gamma}\Big)\Big(T_\lambda \delta^\gamma_\nu - T^\gamma g_{\lambda\nu}\Big) = \frac{1}{9}\Big(T_\mu T_\nu - T_\mu T_\nu - T_\lambda T^\lambda g_{\mu\nu} + T_\mu T_\nu\Big) \end{aligned} \tag{A.1.21}$$

So, we get

$$\boxed{4 = \frac{1}{9}\Big(T_\mu T_\nu - T_\lambda T^\lambda g_{\mu\nu}\Big)} \tag{A.1.22}$$

And now we can finally assemble the Ricci tensor.

$$\begin{aligned} \bar{R}_{\mu\nu} &= \mathring{R}_{\mu\nu} + 1 - 2 + 3 - 4 = \\ &= \mathring{R}_{\mu\nu} - \frac{2}{3}\mathring{\nabla}_\nu T_\mu - \frac{1}{3} g_{\mu\nu} \mathring{\nabla}_\lambda T^\lambda + \frac{1}{3}\Big(T_\mu T_\nu - T_\lambda T^\lambda g_{\mu\nu}\Big) - \frac{1}{9}\Big(T_\mu T_\nu - T_\lambda T^\lambda g_{\mu\nu}\Big) \end{aligned} \tag{A.1.23}$$

giving us:

$$\boxed{\bar{R}_{\mu\nu} = \mathring{R}_{\mu\nu} - \frac{2}{3}\mathring{\nabla}_\nu T_\mu - \frac{1}{3} g_{\mu\nu} \mathring{\nabla}_\lambda T^\lambda + \frac{2}{9}[T_\mu T_\nu - T_\lambda T^\lambda g_{\mu\nu}]} \tag{A.1.24}$$

## A.2 Einstein Equation

Now that we have expressed the modified quantities in terms of torsion trace, we can start the actual derivation of Einstein's equation.

We start with an action:

$$S = \int \mathrm{d}^4x \sqrt{-g}\Big(\bar{R}^2 + \frac{1}{2}(\bar{\nabla}\phi)^2 + \beta\phi^2 \bar{R} + \gamma\phi^4\Big) = \int \mathrm{d}^4x \sqrt{-g}\mathcal{L} \tag{A.2.1}$$

where $\mathcal{L} \equiv \bar{R}^2 + \frac{1}{2}(\bar{\nabla}\phi)^2 + \beta\phi^2\bar{R} + \gamma\phi^4$.

In order to derive the Einstein's equation, we should vary the action with respect to metric

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = 0 \tag{A.2.2}$$

We can schematically write $\delta S$ as:

$$\delta S = \delta S_1 + \delta S_2 + \delta S_3 + \delta S_4 + \delta S_5 + \delta S_6 \tag{A.2.3}$$

where:

$$\delta S_1 = \int \mathrm{d}^4x \, \mathcal{L} \, \delta\sqrt{-g} \tag{A.2.4}$$

$$\delta S_2 = \int \mathrm{d}^4x \sqrt{-g}\beta\phi^2 \bar{R}_{\mu\nu}\delta g^{\mu\nu} \tag{A.2.5}$$

$$\delta S_3 = \int \mathrm{d}^4x \sqrt{-g}\beta\phi^2 g^{\mu\nu}\delta\bar{R}_{\mu\nu} \tag{A.2.6}$$

$$\delta S_4 = 2\int \mathrm{d}^4x \sqrt{-g}\bar{R} g^{\mu\nu}\delta\bar{R}_{\mu\nu} \tag{A.2.7}$$

$$\delta S_5 = 2\int \mathrm{d}^4x \sqrt{-g}\bar{R}\bar{R}_{\mu\nu}\delta g^{\mu\nu} \tag{A.2.8}$$

$$\delta S_6 = \frac{1}{2}\int \mathrm{d}^4x \sqrt{-g} \; \bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi\delta g^{\mu\nu} \tag{A.2.9}$$

In order to be able to do these variations, we first need to see what the variation of $\bar{R}_{\mu\nu}$ is and that's what we are going to calculate now. From (2.2.56) we get

$$\delta\bar{R}_{\mu\nu} = \delta\mathring{R}_{\mu\nu} - \frac{2}{3}\delta(\partial_\nu T_\mu - \mathring{\Gamma}^\sigma{}_{\nu\mu}T_\sigma) - \frac{1}{3}\delta[g_{\mu\nu}\mathring{\nabla}_\lambda(g^{\lambda\sigma}T_\sigma)] + \frac{2}{9}[0 - \delta(T_\lambda g^{\lambda\gamma}T_\gamma g_{\mu\nu})] \tag{A.2.10}$$

where we used the fact, that a manifold with torsion is defined in terms of $g_{\mu\nu}$ and torsion tensor, which are independent, therefore varying $T_\mu$ with respect to $g_{\mu\nu}$ is 0. Besides, $T_\mu$ is defined with lower index, so whenever we see $T^\mu$ we should first lower it with metric and only then perform the variation, therefore $\delta T^\mu \neq 0$.

$$\begin{aligned}\delta\bar{R}_{\mu\nu} = \delta\mathring{R}_{\mu\nu} + \frac{2}{3}T_\sigma\delta\mathring{\Gamma}^\sigma{}_{\nu\mu} - \frac{1}{3}g_{\mu\nu}(\mathring{\nabla}_\lambda T_\sigma)\delta g^{\lambda\sigma} - \frac{1}{3}g^{\lambda\sigma}(\mathring{\nabla}_\lambda T_\sigma)\delta g_{\mu\nu} - \frac{1}{3}g^{\lambda\sigma}g_{\mu\nu}\delta(\mathring{\nabla}_\lambda T_\sigma) - \\ \frac{2}{9}T_\lambda T_\gamma g^{\lambda\gamma}\delta g_{\mu\nu} - \frac{2}{9}T_\lambda T_\gamma g_{\mu\nu}\delta g^{\lambda\gamma} = \delta\mathring{R}_{\mu\nu} + \frac{2}{3}T_\sigma\delta\mathring{\Gamma}^\sigma{}_{\nu\mu} - \frac{1}{3}g_{\mu\nu}(\mathring{\nabla}_\lambda T_\sigma)\delta g^{\lambda\sigma} - \frac{1}{3}g^{\lambda\sigma}(\mathring{\nabla}_\lambda T_\sigma)\delta g_{\mu\nu} + \\ + \frac{1}{3}g^{\lambda\sigma}g_{\mu\nu}T_\rho\delta\mathring{\Gamma}^\rho{}_{\lambda\sigma} - \frac{2}{9}T_\lambda T_\gamma g^{\lambda\gamma}\delta g_{\mu\nu} - \frac{2}{9}T_\lambda T_\gamma g_{\mu\nu}\delta g^{\lambda\gamma}\end{aligned} \tag{A.2.11}$$

where we used that for the covariant vector field $T_\sigma$ the covariant derivative is $\mathring{\nabla}_\lambda T_\sigma = \partial_\lambda T_\sigma - \mathring{\Gamma}^\rho{}_{\lambda\sigma}T_\rho$.

We would like to write everything in terms of $\delta g^{\mu\nu}$ now. For this purpose, we need to first vary the Christoffel symbols and write this variation in terms of metric.

$$\delta\mathring{\Gamma}^\sigma{}_{\nu\mu} = -\frac{1}{2}\left[g_{\lambda\mu}\mathring{\nabla}_\nu(\delta g^{\lambda\sigma}) + g_{\lambda\nu}\mathring{\nabla}_\mu(\delta g^{\lambda\sigma}) - g_{\mu\alpha}g_{\nu\beta}\mathring{\nabla}^\sigma(\delta g^{\alpha\beta})\right] \tag{A.2.12}$$

$$\delta\mathring{\Gamma}^\rho{}_{\lambda\sigma} = -\frac{1}{2}\left[g_{\gamma\lambda}\mathring{\nabla}_\sigma(\delta g^{\gamma\rho}) + g_{\gamma\sigma}\mathring{\nabla}_\lambda(\delta g^{\gamma\rho}) - g_{\lambda\alpha}g_{\sigma\beta}\mathring{\nabla}^\rho(\delta g^{\alpha\beta})\right] \tag{A.2.13}$$

We should integrate now these by parts to be able to rewrite everything in terms of $\delta g^{\mu\nu}$. All of this is part of $\delta S_4 = 2\int \mathrm{d}^4x\sqrt{-g}\bar{R}g^{\mu\nu}\delta\bar{R}_{\mu\nu}$ and $\delta S_3 = \int \mathrm{d}^4x\sqrt{-g}\beta\phi^2 g^{\mu\nu}\delta\bar{R}_{\mu\nu}$. Additionally, each term with $\delta\Gamma$ also has a torsion trace in front, so that when we integrate $\delta\Gamma$ terms by parts, the derivatives will move to all these terms that are in front of $\delta\Gamma$. For simplicity we will only have a look at integration by parts of one term and all the others will just be analogous in derivation.

$$-2\cdot\frac{1}{2}\cdot\frac{2}{3}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}\bar{R}T_\sigma g_{\lambda\mu}\mathring{\nabla}_\nu(\delta g^{\lambda\sigma}) = \text{after integrating by parts we get a boundary term which}$$
$$= \text{can be set to 0 + using metricity} = \frac{2}{3}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}g_{\lambda\mu}\mathring{\nabla}_\nu(\bar{R}T_\sigma)\delta g^{\lambda\sigma} \tag{A.2.14}$$

We will get similar results for the other 5 terms coming from Christoffel variation. Let's write them all down.

$$\frac{2}{3}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}g_{\lambda\mu}\mathring{\nabla}_\nu(\bar{R}T_\sigma)\delta g^{\lambda\sigma} \tag{A.2.15}$$
$$\frac{2}{3}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}g_{\lambda\nu}\mathring{\nabla}_\mu(\bar{R}T_\sigma)\delta g^{\lambda\sigma} \tag{A.2.16}$$
$$-\frac{2}{3}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}g_{\mu\alpha}g_{\nu\beta}\mathring{\nabla}^\sigma(\bar{R}T_\sigma)\delta g^{\alpha\beta} \tag{A.2.17}$$
$$\frac{1}{3}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}g_{\gamma\lambda}g_{\mu\nu}g^{\lambda\sigma}\mathring{\nabla}_\sigma(\bar{R}T_\rho)\delta g^{\gamma\rho} \tag{A.2.18}$$
$$\frac{1}{3}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}g_{\gamma\sigma}g_{\mu\nu}g^{\lambda\sigma}\mathring{\nabla}_\lambda(\bar{R}T_\rho)\delta g^{\gamma\rho} \tag{A.2.19}$$
$$-\frac{1}{3}\int \mathrm{d}^4x\sqrt{-g}g^{\mu\nu}g_{\lambda\alpha}g_{\sigma\beta}g_{\mu\nu}g^{\lambda\sigma}\mathring{\nabla}^\rho(\bar{R}T_\rho)\delta g^{\alpha\beta} \tag{A.2.20}$$

Contracting all the indices and using common relations with metrics, we get:

$$\frac{2}{3}\int \mathrm{d}^4x\sqrt{-g}\mathring{\nabla}_\lambda(\bar{R}T_\sigma)\delta g^{\lambda\sigma} \tag{A.2.21}$$
$$\frac{2}{3}\int \mathrm{d}^4x\sqrt{-g}\mathring{\nabla}_\lambda(\bar{R}T_\sigma)\delta g^{\lambda\sigma} \tag{A.2.22}$$
$$-\frac{2}{3}\int \mathrm{d}^4x\sqrt{-g}g_{\alpha\beta}\mathring{\nabla}^\sigma(\bar{R}T_\sigma)\delta g^{\alpha\beta} \tag{A.2.23}$$
$$\frac{4}{3}\int \mathrm{d}^4x\sqrt{-g}\mathring{\nabla}_\gamma(\bar{R}T_\rho)\delta g^{\gamma\rho} \tag{A.2.24}$$
$$\frac{4}{3}\int \mathrm{d}^4x\sqrt{-g}\mathring{\nabla}_\gamma(\bar{R}T_\rho)\delta g^{\gamma\rho} \tag{A.2.25}$$
$$-\frac{4}{3}\int \mathrm{d}^4x\sqrt{-g}g_{\alpha\beta}\mathring{\nabla}^\rho(\bar{R}T_\rho)\delta g^{\alpha\beta} \tag{A.2.26}$$

Since all of these indices are dummy here, we can relabel them in order to get $\delta g^{\mu\nu}$ with every term. We can also group (A.2.21) with (A.2.22) and also (A.2.24) with (A.2.25). We get

$$\frac{4}{3}\int \mathrm{d}^4x\sqrt{-g}\mathring{\nabla}_\mu(\bar{R}T_\nu)\delta g^{\mu\nu} \tag{A.2.27}$$
$$-\frac{2}{3}\int \mathrm{d}^4x\sqrt{-g}g_{\mu\nu}\mathring{\nabla}^\lambda(\bar{R}T_\lambda)\delta g^{\mu\nu} \tag{A.2.28}$$
$$\frac{8}{3}\int \mathrm{d}^4x\sqrt{-g}\mathring{\nabla}_\mu(\bar{R}T_\nu)\delta g^{\mu\nu} \tag{A.2.29}$$
$$-\frac{4}{3}\int \mathrm{d}^4x\sqrt{-g}g_{\mu\nu}\mathring{\nabla}^\lambda(\bar{R}T_\lambda)\delta g^{\mu\nu} \tag{A.2.30}$$

And we notice now, that (A.2.27) and (A.2.29) and also (A.2.28) and (A.2.30) are the same, so that we can sum them to get:

$$\frac{12}{3}\int \mathrm{d}^4x\sqrt{-g}\mathring{\nabla}_\mu(\bar{R}T_\nu)\delta g^{\mu\nu} \tag{A.2.31}$$

$$-\frac{6}{3}\int \mathrm{d}^4x\sqrt{-g}g_{\mu\nu}\mathring{\nabla}^\lambda(\bar{R}T_\lambda)\delta g^{\mu\nu} \tag{A.2.32}$$

So, the only remaining problem in (A.2.11) is $\delta g_{\mu\nu}$. This is fixed by the following trick:

$$\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\eta}\delta g^{\rho\eta} \tag{A.2.33}$$

and then relabeling the indices in the problematic terms to get $\delta g^{\mu\nu}$:

$$\begin{aligned} &g^{\mu\nu}g^{\lambda\sigma}\mathring{\nabla}_\lambda(T_\sigma)\delta g_{\mu\nu} = -g^{\lambda\sigma}\mathring{\nabla}_\lambda(T_\sigma)g_{\mu\rho}g_{\nu\eta}\delta g^{\rho\eta}g^{\mu\nu} = \text{relabeling indices} = -g^{\lambda\sigma}g^{\rho\eta}g_{\mu\rho}g_{\nu\eta}\mathring{\nabla}_\lambda(T_\sigma)\delta g^{\mu\nu} \\ &= -g^{\rho\eta}g_{\mu\rho}g_{\nu\eta}\mathring{\nabla}_\lambda(T^\lambda)\delta g^{\mu\nu} = -g_{\mu\nu}\mathring{\nabla}_\lambda(T^\lambda)\delta g^{\mu\nu} \end{aligned} \tag{A.2.34}$$

$$g^{\mu\nu}T_\gamma T_\lambda g^{\lambda\gamma}\delta g_{\mu\nu} = -T_\lambda T^\lambda g_{\mu\nu}\delta g^{\mu\nu} \tag{A.2.35}$$

So that we can finally write the variation in terms of $\delta g^{\mu\nu}$:

$$\begin{aligned} \delta S_4 = 2\int \mathrm{d}^4x\sqrt{-g}\bar{R}g^{\mu\nu}\delta\bar{R}_{\mu\nu} = 2\int \mathrm{d}^4x\sqrt{-g}\bar{R}g^{\mu\nu}\delta\mathring{R}_{\mu\nu} + \int \mathrm{d}^4x\sqrt{-g}\Big[\bar{R}\Big( -\frac{4}{3}\mathring{\nabla}_\nu T_\mu + \frac{1}{3}g_{\mu\nu}\mathring{\nabla}_\lambda T^\lambda + \\ + \frac{2}{9}T_\lambda T^\lambda g_{\mu\nu} - \frac{8}{9}T_\mu T_\nu\Big) + 4\mathring{\nabla}^\mu(\bar{R}T_\nu) - 2\mathring{\nabla}^\lambda(\bar{R}T_\lambda)g_{\mu\nu}\Big]\delta g^{\mu\nu} \end{aligned} \tag{A.2.36}$$

But as we said, we also have $\delta\bar{R}_{\mu\nu}$ in $\delta S_3$ and by summing $\delta S_3 + \delta S_4$ we should simply replace $\bar{R}$ with $\bar{R} + \frac{\beta\phi^2}{2}$.

The last thing to do, before we can write down Einstein's equation is to vary the normal Ricci tensor $\delta\mathring{R}_{\mu\nu}$ - when we were dealing with Hilbert-Einstein action, we only had powers of metric in front of $\delta\mathring{R}_{\mu\nu}$, therefore we simply dropped this term there, since integration by parts doesn't give us any nonzero results due to metricity [24]. In the current action, though, we have other terms in front of it as well, so we should do integration by parts of $\delta\mathring{R}_{\mu\nu}$ properly. The expression for it can be read off from [24]:

$$\int \mathrm{d}^4x\sqrt{-g}\Big(2\bar{R} + \beta\phi^2\Big)\mathring{\nabla}_\sigma\Big[g_{\mu\nu}\mathring{\nabla}^\sigma\delta g^{\mu\nu} - \mathring{\nabla}_\lambda\delta g^{\sigma\lambda}\Big] \tag{A.2.37}$$

to get rid of the derivatives acting on $\delta g^{\mu\nu}$ we integrate by parts:

$$\begin{aligned} &\int \mathrm{d}^4x\sqrt{-g}\Big[-\mathring{\nabla}_\sigma\Big(2\bar{R} + \beta\phi^2\Big)g_{\mu\nu}\mathring{\nabla}^\sigma\delta g^{\mu\nu} + \mathring{\nabla}_\sigma\Big(2\bar{R} + \beta\phi^2\Big)\mathring{\nabla}_\lambda\delta g^{\sigma\lambda}\Big] = \text{doing it again} = \\ &= \int \mathrm{d}^4x\sqrt{-g}\Big[\mathring{\nabla}^\sigma\mathring{\nabla}_\sigma\Big(2\bar{R} + \beta\phi^2\Big)g_{\mu\nu}\delta g^{\mu\nu} - \mathring{\nabla}_\lambda\mathring{\nabla}_\sigma\Big(2\bar{R} + \beta\phi^2\Big)\delta g^{\sigma\lambda}\Big] = \text{relabeling indices} = \\ &= \int \mathrm{d}^4x\sqrt{-g}\Big[\mathring{\nabla}_\lambda\mathring{\nabla}^\lambda\Big(2\bar{R} + \beta\phi^2\Big)g_{\mu\nu} - \mathring{\nabla}_\nu\mathring{\nabla}_\mu\Big(2\bar{R} + \beta\phi^2\Big)\Big]\delta g^{\mu\nu} \end{aligned} \tag{A.2.38}$$

Now we finally have all the terms with $\delta g^{\mu\nu}$ and we can write down the action variation properly:

$$\delta S = \delta S_1 + \delta S_2 + \delta S_3 + \delta S_4 + \delta S_5 + \delta S_6 = \int \mathrm{d}^4x\sqrt{-g}\Bigg\{ -\underbrace{\frac{1}{2}\mathcal{L}g_{\mu\nu}}_{\delta S_1} + \underbrace{\beta\phi^2\bar{R}_{\mu\nu}}_{\delta S_2} + \underbrace{2\bar{R}\bar{R}_{\mu\nu}}_{\delta S_5} +$$
$$+ \underbrace{\left[g_{\mu\nu}\mathring{\nabla}^2\left(2\bar{R}+\beta\phi^2\right) - \mathring{\nabla}_\nu\mathring{\nabla}_\mu\left(2\bar{R}+\beta\phi^2\right)\right]}_{\delta\mathring{R}_{\mu\nu}} + \underbrace{\frac{1}{2}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi}_{\delta S_6} +$$
$$+ \underbrace{\left[\left(\bar{R}+\frac{\beta\phi^2}{2}\right)\left(\frac{1}{3}g_{\mu\nu}\mathring{\nabla}_\lambda T^\lambda + \frac{2}{9}T_\lambda T^\lambda g_{\mu\nu} - \frac{8}{9}T_\mu T_\nu\right) + 4\mathring{\nabla}^\mu\left(\left(\bar{R}+\frac{\beta\phi^2}{2}\right)T_\nu\right) - 2\mathring{\nabla}^\lambda\left(\left(\bar{R}+\frac{\beta\phi^2}{2}\right)T_\lambda\right)g_{\mu\nu}\right]}_{\text{Torsion Part}}\Bigg\}\delta g^{\mu\nu} \tag{A.2.39}$$

And finally we can already get Einstein's Equation by using (A.2.2):

$$\boxed{\begin{aligned}&\left(2\bar{R}+\beta\phi^2\right)\bar{R}_{\mu\nu} - \frac{1}{2}\mathcal{L}g_{\mu\nu} + \left[g_{\mu\nu}\mathring{\nabla}^2\left(2\bar{R}+\beta\phi^2\right) - \mathring{\nabla}_\nu\mathring{\nabla}_\mu\left(2\bar{R}+\beta\phi^2\right)\right] + \frac{1}{2}\bar{\nabla}_\mu\phi\bar{\nabla}_\nu\phi + \\ &+ \left[\left(\bar{R}+\frac{\beta\phi^2}{2}\right)\left(\frac{1}{3}g_{\mu\nu}\mathring{\nabla}_\lambda T^\lambda + \frac{2}{9}T_\lambda T^\lambda g_{\mu\nu} - \frac{8}{9}T_\mu T_\nu\right) + 4\mathring{\nabla}^\mu\left(\left(\bar{R}+\frac{\beta\phi^2}{2}\right)T_\nu\right) - 2\mathring{\nabla}^\lambda\left(\left(\bar{R}+\frac{\beta\phi^2}{2}\right)T_\lambda\right)g_{\mu\nu}\right] = T_{\mu\nu}\end{aligned}} \tag{A.2.40}$$

where

$$\mathcal{L} \equiv \bar{R}^2 + \frac{1}{2}(\bar{\nabla}\phi)^2 + \beta\phi^2\bar{R} + \gamma\phi^4$$

## A.3 Equations of Motion

Now we would like to derive the equations of motion for the other two fields - $T_\mu$ and $\phi$. We start with the same action again

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\alpha\bar{R}^2 + \frac{1}{2}(\bar{\nabla}\phi)^2 + \beta\phi^2\bar{R} + \gamma\phi^4\right) \tag{A.3.1}$$

Let's start with $\phi$. We write the Euler-Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \bar{\nabla}_\mu\frac{\partial\mathcal{L}}{\partial(\bar{\nabla}_\mu\phi)} = 0 \tag{A.3.2}$$

from where we immediately get

$$\boxed{2\beta\phi\bar{R} + 4\gamma\phi^3 - \bar{\nabla}_\mu\bar{\nabla}^\mu\phi = 0} \tag{A.3.3}$$

Now let's derive the equation of motion for the torsion trace. For this purpose we should write out some of the terms containing torsion explicitly. Using the expression that we derived for the Ricci scalar (generated by $g^{\mu\nu}\bar{R}_{\mu\nu}$) and also writing out $(\bar{\nabla}\phi)^2$ explicitly:

$$\bar{R}_{\mu\nu} = \mathring{R}_{\mu\nu} - \frac{2}{3}\mathring{\nabla}_\nu T_\mu - \frac{1}{3}g_{\mu\nu}\mathring{\nabla}_\lambda T^\lambda + \frac{2}{9}[T_\mu T_\nu - T_\lambda T^\lambda g_{\mu\nu}] \tag{A.3.4}$$

And

$$\bar{R} = \bar{R}_{\mu\nu}g^{\mu\nu} = \mathring{R} - \frac{2}{3}\mathring{\nabla}_\nu T^\nu - \frac{4}{3}\mathring{\nabla}_\lambda T^\lambda + \frac{2}{9}[T_\nu T_\nu - 4T_\lambda T^\lambda] = \mathring{R} - \frac{6}{3}\mathring{\nabla}_\nu T^\nu - \frac{6}{9}T_\lambda T^\lambda \tag{A.3.5}$$

So

$$\boxed{\bar{R} = \mathring{R} - 2\mathring{\nabla}_\lambda T^\lambda - \frac{2}{3}T_\lambda T^\lambda} \tag{A.3.6}$$

Now let's have a look at the derivative:

$$\frac{1}{2}\bar{\nabla}_\mu\phi\bar{\nabla}^\mu\phi = \frac{1}{2}\Big((\partial_\mu + T_\mu)\phi\Big)\Big((\partial^\mu + T^\mu)\phi\Big) = \frac{1}{2}\mathring{\nabla}_\mu\phi\mathring{\nabla}^\mu\phi + \mathring{\nabla}^\mu\phi T^\mu\phi + \frac{1}{2}T_\mu T^\mu\phi^2 \tag{A.3.7}$$

So that we can write from (A.3.7) and (A.3.6):

$$\frac{\delta\frac{1}{2}(\bar{\nabla}\phi)^2}{\delta T_\mu} = \phi\mathring{\nabla}^\mu\phi + T^\mu\phi^2 \tag{A.3.8}$$

And

$$\frac{\delta\bar{R}}{\delta T_\mu} = -2\mathring{\nabla}^\mu - \frac{4}{3}T^\mu \tag{A.3.9}$$

Now we're ready to vary the action with respect to $T_\mu$. For simplicity we only look at torsion dependent terms: $\mathcal{L}_T = \alpha\bar{R}^2 + \frac{1}{2}(\bar{\nabla}\phi)^2 + \beta\phi^2\bar{R}$

$$\begin{aligned}
\frac{\delta S_T}{\delta T_\mu} &= \int \mathrm{d}^4x\sqrt{-g}\Big(2\alpha\bar{R}\frac{\delta\bar{R}}{\delta T_\mu} + \frac{\delta\frac{1}{2}(\bar{\nabla}\phi)^2}{\delta T_\mu} + \beta\phi^2\frac{\delta\bar{R}}{\delta T_\mu}\Big)\delta T_\mu = \\
&= \int \mathrm{d}^4x\sqrt{-g}\Big((2\alpha\bar{R} + \beta\phi^2)(-2\mathring{\nabla}^\mu - \frac{4}{3}T^\mu) + \phi\mathring{\nabla}^\mu\phi + T^\mu\phi^2\Big)\delta T_\mu = \text{integrating by parts} = \\
&= \int \mathrm{d}^4x\sqrt{-g}\Big(-2\mathring{\nabla}^\mu(2\alpha\bar{R} + \beta\phi^2) - \frac{4}{3}(2\alpha\bar{R} + \beta\phi^2)T^\mu + \phi\mathring{\nabla}^\mu\phi + T^\mu\phi^2\Big)\delta T_\mu = \text{demanding that this} = 0
\end{aligned} \tag{A.3.10}$$

So that we get for the equation of motion:

$$\boxed{2\mathring{\nabla}^\mu(2\alpha\bar{R} + \beta\phi^2) + \frac{4}{3}(2\alpha\bar{R} + \beta\phi^2)T^\mu + \phi\mathring{\nabla}^\mu\phi + T^\mu\phi^2 = 0} \tag{A.3.11}$$

## A.4 Friedmann Equations

Let's now derive the Friedmann equations. These are all the non-trivial components of Einstein's equation (A.2.40). Besides, since we are working in Cosmology, we usually consider the following approximation:

$$\begin{aligned}
\phi &= \phi(t) \\
g_{\mu\nu} &= a(t)^2\eta_{\mu\nu}
\end{aligned} \tag{A.4.1}$$

and the only non-trivial component of torsion trace vector will be $T_0(t)$. This approximation, obviously, changes the geometric quantities.

$$R_{\mu\nu} = \begin{bmatrix} \frac{3(\dot{a}^2 - a\ddot{a})}{a^2} & 0 & 0 & 0 \\ 0 & \frac{\dot{a}+a\ddot{a}}{a^2} & 0 & 0 \\ 0 & 0 & \frac{\dot{a}+a\ddot{a}}{a^2} & 0 \\ 0 & 0 & 0 & \frac{\dot{a}+a\ddot{a}}{a^2} \end{bmatrix} \tag{A.4.2}$$

and the corresponding Ricci scalar will be of the following shape:

$$R = \frac{6\ddot{a}}{a^3} \tag{A.4.3}$$

Additionally, since we can perform the Weyl tarnsformation, we can simply set $T_0 = 0$ using $-\partial_0\theta = T_0$. So that the whole torsion part of the EE will be 0 now.

If we also pay a closer attention to the fact, that both - the metric $g_{\mu\nu}$ and $R_{\mu\nu}$ are diagonal in this approximation and that $\phi(t)$ and $R(t)$ are only time-dependent, then we'll see, that since all the terms in EE 1) either contain $g_{\mu\nu}$ or $R_{\mu\nu}$ or alternatively 2) two derivatives of $\phi(t)^2$ or $R(t)^2$ with $\mu$ and $\nu$, both of which are only time-dependent, all the $ij$ $(i, j = 1, 2, 3)$ components of EE with $i \neq j$ will vanish. Thus, the only non-trivial components of EE that we are left with are 1) 00 and 2) $ij$ with $i = j$ components. Let's write them out explicitly.

$$\begin{aligned}&(2\mathring{R} + \beta\phi^2)\mathring{R}_{00} - \frac{1}{2}g_{00}\left(\mathring{R}^2 + \frac{1}{2}\mathring{\nabla}_0\phi\mathring{\nabla}^0\phi + \beta\phi^2\mathring{R} + \gamma\phi^4\right) + \\ &+ \left[g_{00}\mathring{\nabla}_0\mathring{\nabla}^0\left(2\mathring{R} + \beta\phi^2\right) - \mathring{\nabla}_0\mathring{\nabla}^0\left(2\mathring{R} + \beta\phi^2\right)\right] + \frac{1}{2}\bar{\nabla}_0\phi\bar{\nabla}^0\phi = \rho(t)\end{aligned} \tag{A.4.4}$$

And if we plug $R_{00}$ and $R$ explicitly, we will get the **I Friedmann equation**:

$$\boxed{\begin{aligned}&\left(2\frac{6\ddot{a}}{a^3} + \beta\phi^2\right)\frac{3(\dot{a}^2 - a\ddot{a})}{a^2} + \frac{1}{2}a^2\left(\left(\frac{6\ddot{a}}{a^3}\right)^2 + \frac{1}{2}\mathring{\nabla}_0\phi\mathring{\nabla}^0\phi + \beta\phi^2\frac{6\ddot{a}}{a^3} + \gamma\phi^4\right) - \\ &- \left[a^2\mathring{\nabla}_0\mathring{\nabla}^0\left(2\frac{6\ddot{a}}{a^3} + \beta\phi^2\right) + \mathring{\nabla}_0\mathring{\nabla}^0\left(2\frac{6\ddot{a}}{a^3} + \beta\phi^2\right)\right] + \frac{1}{2}\bar{\nabla}_0\phi\bar{\nabla}^0\phi = \rho(t)\end{aligned}} \tag{A.4.5}$$

If we now consider $ij$ component of EE, we'll see, that all the derivatives of $\phi$ and $\mathring{R}$ will vanish, since these objects are only time-dependent. And for convenience we can also write:

$$R_{ij} = \frac{\dot{a} + a\ddot{a}}{a^2}g_{ij} \tag{A.4.6}$$

That will give us:

$$\left(2\frac{6\ddot{a}}{a^3} + \beta\phi^2\right)\frac{\dot{a} + a\ddot{a}}{a^4}g_{ij} - \frac{1}{2}g_{ij}\left(\left(\frac{6\ddot{a}}{a^3}\right)^2 + \beta\phi^2\frac{6\ddot{a}}{a^3} + \gamma\phi^4\right) + g_{ij}\mathring{\nabla}_0\mathring{\nabla}^0\left(2\frac{6\ddot{a}}{a^3} + \beta\phi^2\right) = p(t)g_{ij} \tag{A.4.7}$$

and now we can cancel the metric and get the final expression for $ij$ component of EE giving us the **II Friedmann equation**:

$$\boxed{\left(2\frac{6\ddot{a}}{a^3} + \beta\phi^2\right)\frac{\dot{a} + a\ddot{a}}{a^4} - \frac{1}{2}\left(\left(\frac{6\ddot{a}}{a^3}\right)^2 + \beta\phi^2\frac{6\ddot{a}}{a^3} + \gamma\phi^4\right) + \mathring{\nabla}_0\mathring{\nabla}^0\left(2\frac{6\ddot{a}}{a^3} + \beta\phi^2\right) = p(t)} \tag{A.4.8}$$

## A.5 Finding a Solution in the minimum of the potential

In order to understand what we will be doing, recall, that Friedmann equations govern the expansion of the universe - they tell us, how the scale factor behaves with time, due to the matter content of the universe. As we discussed, if we take the Hubble parameter as a constant $\frac{\dot{a}}{a} = H = const$, meaning that we have a constant potential energy coming from the matter content of the universe, we will get inflationary solution for the scale factor - $a \sim e^{Ht}$. In order to check the consistency of the Einstein Frame against the Jordan frame, we will discuss the solutions of the equations at the minimum of the potential, meaning that it is constant there. For that, we should impose $H = const$ condition

$$\frac{\dot{a}}{a} = H = const = \Lambda \tag{A.5.1}$$

(we will use $\Lambda \equiv H$) in Friedmann equations and see, if this is one of the solutions of it - basically, whether Friedmann equations derived for our theory with torsion allow us to have such a solution.

Since we have 3 variables - $\phi$, $a(\eta)$ and $H$ and only 2 equations, we should express $\phi$ and $a$ in terms of $\eta$ conformal time and then plug it into EE or Friedmann equations and see, for what $H$-s do they get satisfied - we will get a family of solutions.

Expression for $a(\eta)$ can be derived from (A.5.1), whereas $\phi(\eta)$ can be derived from the equations of motion. Let's derive these.

$a(\eta)$ first: since our metric is written in terms of conformal time, we should also solve this equation in terms of $\eta$, where $\mathrm{d}\eta = \frac{\mathrm{d}t}{a}$. Plugging this into (A.5.1) and taking into account, that we have $a(\eta_{initial} = 0) = 0$, we get:

$$\frac{\mathrm{d}a}{a^2} = H\mathrm{d}\eta \Rightarrow -\frac{1}{a} = H\eta \tag{A.5.2}$$

$$\boxed{a = -\frac{1}{\eta H}} \tag{A.5.3}$$

Now let's have a look at the equation of motion for $\phi$.

$$\mathring{\Box}\phi - 2\beta\mathring{R}\phi - 4\gamma\phi^3 = 0 \tag{A.5.4}$$

First of all, we rewrite

$$g_{\mu\nu} = a^2\eta_{\mu\nu} = (H\eta)^{-2}\eta_{\mu\nu} \tag{A.5.5}$$
$$g^{\mu\nu} = a^{-2}\eta^{\mu\nu} = (H\eta)^{2}\eta^{\mu\nu} \tag{A.5.6}$$
$$\sqrt{-g} = \sqrt{(H\eta)^{-8}} = (H\eta)^{-4} \tag{A.5.7}$$

Then

$$\partial_\eta\partial_\eta a = \frac{-2}{H\eta^3}$$
$$R = \frac{6\ddot{a}}{a^3} = 12H^2$$

From here we can deduce (keeping in mind, that D=4 dimensions):

$$\mathring{\Box}\phi = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu\phi = (H\eta)^4\partial_\mu\Big((H\eta)^{-2}\partial_\nu\phi\Big) \tag{A.5.8}$$

Since $\phi$ is only time-dependent and since, due to our choice of metric, we are already working with conformal time, we get:

$$\mathring{\Box}\phi = (H\eta)^4\partial_\eta\Big((H\eta)^{-2}\partial_\eta\phi\Big) = (H\eta)^4\Big((\partial_\eta\partial_\eta\phi)(H\eta)^{-2} - 2H^{-2}\eta^{-3}\partial_\eta\phi\Big) = (H\eta)^2\partial_\eta^2\phi - 2H^2\eta\partial_\eta\phi \tag{A.5.9}$$

So that we get the equation of motion written in the cosmological approximation

$$\partial_\eta^2\phi - \frac{2}{\eta}\partial_\eta\phi - \frac{24\beta\phi H^2 + 4\gamma\phi^3}{H^2\eta^2} = 0 \tag{A.5.10}$$

For the convenience, though, we would like to get rid of the single-derivative term. To achieve this, we perform the field redefinition:

$$\phi = a^{\frac{D-2}{2}}\varphi = H\eta\varphi \tag{A.5.11}$$

Let's calculate the terms of (A.5.10) in terms of the new field:

$$\partial_\eta^2\phi = \partial_\eta^2(H\varphi + H\eta\partial_\eta\varphi) = H\partial_\eta\varphi + H\partial_\eta\varphi + H\eta\partial_\eta^2\varphi \tag{A.5.12}$$

$$-\frac{2}{\eta}\partial_\eta\phi = -\frac{2}{\eta}(H\varphi + H\eta\partial_\eta\varphi) = -\frac{2}{\eta}H\varphi - \frac{2}{\eta}H\eta\partial_\eta\varphi \tag{A.5.13}$$

Summing (A.5.12) and (A.5.13):

$$H\partial_\eta\varphi + H\partial_\eta\varphi + H\eta\partial_\eta^2\varphi - \frac{2}{\eta}H\varphi - \frac{2}{\eta}H\eta\partial_\eta\varphi = H\eta\partial_\eta^2\varphi - \frac{2}{\eta}H\varphi \tag{A.5.14}$$

So, indeed, we managed to kill the single derivative through this redefinition and now we have

$$H\eta\partial_\eta^2\varphi - \frac{2}{\eta}H\varphi - \frac{24\beta\phi H^2 + 4\gamma\phi^3}{H^2\eta^2} = 0 \tag{A.5.15}$$

And after rewriting it in a neater way, we finally get the EoM:

$$\boxed{\partial_\eta^2\varphi - \frac{(2+24\beta)H^2 + 4\gamma(H\eta)^2\varphi^2}{(H\eta)^2}\varphi = 0} \tag{A.5.16}$$

This is the final version of the $\varphi$ equation of motion and this has to be solved now for $\varphi(\eta)$, which we will then have to plug into Friedmann equation to get a family of solutions in terms of $H$.

To solve this equation, we use the following Ansatz:

$$\boxed{\varphi = \varphi_0\eta^\alpha} \tag{A.5.17}$$

where $\varphi_0$ is the value of of $\varphi$ at the starting point. We get (denoting $\beta' \equiv 2 + 24\beta$):

$$\alpha(\alpha-1)\varphi_0\eta^{\alpha-2} - \frac{\beta' H^2 + 4\gamma(H\eta)^2\varphi_0^2\eta^{2\alpha}}{(H\eta)^2}\varphi_0\eta^\alpha = 0 \tag{A.5.18}$$

we can find one solution, when $3\alpha = \alpha - 2 \;\Rightarrow\; \alpha = -1$ (the point is to find at least one solution and that will be enough, since that will allow there to be an inflation). And now we can derive a constraint for $\varphi_0$:

$$(2-\beta')\varphi_0\eta^{-3} - 4\gamma\varphi_0^3\eta^{-3} = 0 \;\Rightarrow\; \varphi_0\Big((2-\beta') - 4\gamma\varphi_0^2\Big) = 0 \;\Rightarrow\; \varphi_0 = \pm\sqrt{\frac{2-\beta'}{4\gamma}} \tag{A.5.19}$$

So, we get the solution:

$$\boxed{\varphi = \pm\sqrt{\frac{2-\beta'}{4\gamma}}\eta^{-1}} \tag{A.5.20}$$

Now, as a final step, we have to see, whether this solution satisfies the Friedmann equations and for what values of $H$. This will give us a family of solutions, that we will have to compare to the solutions coming from the Einstein Frame.

$$\varphi = \pm\sqrt{\frac{2-\beta'}{4\gamma}}\eta^{-1} \equiv \frac{A}{\eta} \;\text{ so that }\; \phi = H\eta\varphi = HA = const. \tag{A.5.21}$$

Therefore, all the derivatives in the Friedmann equations will vanish, since $R = 12H^2$ as well. So, we get:

$$2\alpha\mathring{R}\left(\mathring{R}_{\mu\nu} - \frac{1}{4}\mathring{R}g_{\mu\nu}\right) + \beta\phi^2\left(\mathring{R}_{\mu\nu} - \frac{1}{2}\mathring{R}g_{\mu\nu}\right) - \frac{1}{2}\gamma\phi^4 g_{\mu\nu} = 0 \tag{A.5.22}$$

Now, since $\mathring{R} = g^{\mu\nu}\mathring{R}_{\mu\nu}$, by multiplying the both sides of the equation by $g_{\mu\nu}$ we will get:

$$\mathring{R}g_{\mu\nu} = 4\mathring{R}_{\mu\nu} \quad\Longrightarrow\quad \mathring{R}_{\mu\nu} = \frac{\mathring{R}g_{\mu\nu}}{4} \;\text{ and thus }\; \mathring{R}_{\mu\nu} - \frac{\mathring{R}g_{\mu\nu}}{4} = 0 \tag{A.5.23}$$

This gives us for the equation:

$$\beta\phi^2\Big(G_{\mu\nu} - \frac{1}{2}\frac{\gamma}{\beta}\phi^2\Big)g_{\mu\nu} = 0 \quad \Longrightarrow \quad \text{switching to } \lambda \quad \Longrightarrow \quad \frac{1}{2}\beta\phi^2\Big(-\frac{1}{4}R + \frac{\lambda}{\beta}\phi^2\Big)g_{\mu\nu} = 0 \tag{A.5.24}$$

And recalling that $R = 12H^2$ we get:

$$\phi \propto H^2 \tag{A.5.25}$$

meaning, that there is no restriction on $H$ and that the (A.5.17) Ansatz works for any $H$.

## A.6 Comparing Frames

Now we can already compare the results for the minimum in both frames. To be able to do that, we should take some physical objects and compare them in both frames. In general, such an object would look rather different in Jordan and Einstein frames, but our case is a bit different. In general, when one transforms to the Einstein frame, a conformal transformation is employed. In our case, though, we have a conformal action, so a conformal transformation leaves it invariant. What we do to go to the action, that will look like that in the Einstein frame, is that we fix the gauge. This will, obviously, keep the physical objects invariant and that's what we are going to check now.

As a physical object (i.e. the one that is invariant under the conformal transformation) we choose the following ratio:

$$\frac{\bar{R}}{\omega^2} \tag{A.6.1}$$

$$\text{since } \bar{R} \to \Omega^{-2}\bar{R} \text{ and } \omega^2 \to \Omega^{-2}\omega^2$$

And to check, whether this ratio stays the same in both frames, we should calculate $\omega^2$ and $\bar{R}$ in each frame separately and then compare their ratio in both frames.

**1) Let's start with the Einstein Frame:**

In this frame we calculate $\bar{R}$ value from the Einstein equation derived in this frame. The value for $(\omega^2)^{EF}$ is simply obtained from the gauge-fixing condition: $\Omega^2 = -\frac{2\omega^2}{M_p^2} \quad \Longrightarrow \quad \omega^2 = -\frac{\Omega^2 M_p^2}{2}$

For $\bar{R}^{EF}_{\mu\nu}$ we get:

$$\bar{R}^{EF}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\bar{R}^{EF} + \frac{V(\phi_0)g_{\mu\nu}}{M_p^2} = 0 \tag{A.6.2}$$

and expressing $\bar{R}^{EF}_{\mu\nu}$ through $\bar{R}^{EF}$:

$$-\frac{1}{4}g_{\mu\nu}\bar{R}^{EF} + \frac{V(\phi_0)g_{\mu\nu}}{M_p^2} = 0 \tag{A.6.3}$$

giving us:

$$\bar{R}^{EF} = \frac{4V(\phi_0)}{M_p^2} \tag{A.6.4}$$

Therefore for the ratio we get:

$$\frac{\bar{R}^{EF}}{(\omega^2)^{EF}} = \frac{\lambda}{4(\beta^2+\alpha\lambda)M_p^2}\cdot\frac{1}{\omega^2\Omega^{-2}} = \frac{\lambda}{4(\beta^2+\alpha\lambda)M_p^2}\cdot\frac{-2}{\Omega^2\Omega^{-2}M_p^2} = -\frac{\lambda}{2(\beta^2+\alpha\lambda)} \tag{A.6.5}$$

**2) In the Jordan frame:**

We obtain the value of $\bar{R}$ simply by calculating it from the FLRW metric and then plugging the expression for H into it, which gives us $\bar{R} = 12H^2$. Then we express $\omega^2$ through $\phi^2$ and $\bar{R}$ using the EoM for $\zeta$ (see the main part of the thesis, Chapter 4). And we also find the expression for $\phi$ by solving the EoM (we did this in the previous section).

The EoM for $\omega$ gives us the constraint that $\Psi = \bar{R}$, and the EoM for $\Psi$ gives us the value for $\Psi = -\frac{\beta\phi^2+\omega^2}{2\alpha}$, from where we can actually derive the needed ratio.

$$\bar{R} = -\frac{\beta\phi^2 + \omega^2}{2\alpha} \tag{A.6.6}$$

where

$$\omega^2 = -2\alpha\bar{R} - \beta\phi^2 \;\text{ with }\; \phi^2 = \frac{24\beta H^2}{\lambda} \;\text{ and }\; \bar{R} = 12H^2 \tag{A.6.7}$$

So that:

$$\frac{\bar{R}^{JF}}{(\omega^2)^{JF}} = -\frac{\bar{R}}{2\alpha\bar{R} + \beta\phi^2} = -\frac{12H^2}{24\alpha H^2 + \frac{24\beta^2 H^2}{\lambda}} = -\frac{\lambda}{2(\beta^2 + \alpha\lambda)} \tag{A.6.8}$$

Meaning, that we just proved the equivalence and the consistency of the results in both frames, just as we wanted.

# Appendix B

# One-loop Quantum Corrections

## B.1 One-loop Quantum Corrections

In order to calculate the one-loop corrections in this model, we used the Jordan frame action, since it simplified performing this calculation significantly.

$$S = \int \mathrm{d}^4x\sqrt{-g}\bigg( -\frac{1}{2}(\bar{\nabla}\phi)^2 + \beta\phi^2\bar{R} + \frac{\lambda}{4}\phi^4\bigg) \tag{B.1.1}$$

In order to be able to use the familiar regularization and then renormalization schemes, we switch to the Riemann Normal Coordinates (RNC), since the spacetime is locally flat in these coordinates, allowing us to use the standard method of Dimensional regularization.

In these coordinates (after quite some calculations) we arrive at the following perturbative solution for the Green's function:

$$\mathcal{G}_F(k) = \frac{1}{k^2 + M(x')} - \frac{iM_\alpha^{(1)}(x')}{k^2 + M(x')}\partial^\alpha \frac{1}{k^2 + M(x')} + \frac{A_{\alpha\beta}(x')}{k^2 + M(x')}\partial^\alpha\partial^\beta \frac{1}{k^2 + M(x')} + \mathcal{O}\bigg(\frac{1}{k^8}\bigg) \tag{B.1.2}$$

And then using the method of the heat-kernel we can write:

$$\mathcal{G}_F(y) = -i\int_0^\infty \mathrm{d}s K(x;x';s) \tag{B.1.3}$$

and after quite some calculations again, we finally arrive at the result for the one-loop quantum corrections [25]:

$$\begin{aligned} W_{div} = &-\frac{i\,\mathrm{Tr}\,\mathbb{1}}{2} i\int \frac{\mathrm{d}^Dx}{(4\pi)^{D/2}}\sqrt{-g}\bigg[\bigg(\bigg(\beta - \frac{1}{6}\bigg)\bar{R} + \lambda\phi_0^2\bigg)^{\frac{D}{2}}\Gamma\bigg(-\frac{D}{2}\bigg) - \\ &- \frac{1}{3}\bigg(\frac{1}{180}\bar{R}_{\alpha|(\beta\gamma)|\delta}\bar{R}^{\alpha\beta\gamma\delta} - \frac{1}{180}\bar{R}_{\alpha\beta}\bar{R}^{\alpha\beta} + \frac{(D-2)^2}{48}\bigg(F_{\alpha\beta}F^{\alpha\beta} + \mathrm{Tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu})\bigg)\bigg)\cdot \\ &\cdot\bigg(\bigg(\beta - \frac{1}{6}\bigg)\bar{R} + \lambda\phi_0^2\bigg)^{\frac{D-4}{2}}\Gamma\bigg(2 - \frac{D}{2}\bigg)\bigg] \end{aligned} \tag{B.1.4}$$