## On BMS-transformations and the shock wave S -matrix

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#### Abstract

This thesis examines the relation between asymptotic BMS-symmetries and the shock wave S-matrix. After reviewing both formalisms, we see that the shock wave S-matrix in Minkowski space is invariant under supertranslations, which leads to antipodal matching of supertranslation charges between past and future null infinity. We thus identify a simple explicit scattering example of general results found by Strominger and collaborators, which has thus far gone unnoticed in the BMS-literature. For a Schwarzschild black hole, we show that the shift of the event horizon induced by a shock wave satisfies the same relation to the energy-momentum tensor in the two formalisms. This suggests that a description of particle scattering in a black hole background in terms of BMS-charges may be found. We briefly review the derivation of BMS-like symmetries acting at the event horizon of a black hole and show that the Dray-'t Hooft shock wave in tortoise coordinates generates a non-zero horizon superrotation charge. As the BMS-description of black holes is still under development, further exploration of the relation between the two formalisms is deferred to future research.


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## Notation and conventions

In this thesis, we use metric convention $(-,+,+,+)$. We use lower case Roman tensor indices $a, b$ for general tensor expressions and reserve Greek indices $\mu, \nu$ for expressions that are given in a particular coordinate system. We use upper case Roman indices $A, B$ to label the two transversal directions, as we exclusively consider $(1+3)$-dimensional space-times. Occasionally, we will use $\alpha, \beta$ to label longitudinal coordinates, namely, $t, r$ and various light-cone coordinates. However, because of possible confusion with general tensor indices $\mu, \nu$, we try to keep this to a minimum. Lastly, we use natural units, where $c, \hbar, G_{N}$, and $k_{B}$ are set to unity.

## Introduction

Hawking's discovery that Schwarzschild black holes emit thermal radiation [1] gives rise to an apparent conflict between general relativity and quantum mechanics. Namely, if a pure state collapses to form a black hole, the subsequent evaporation of this black hole would give rise to a final state consisting of thermal radiation. It thus seems as if Hawking evaporation allows pure states to evolve into mixed states via intermediate black hole formation, which would violate unitarity. On the basis of his calculation, Hawking and many others contended that unitarity is indeed violated and that information can disappear in physical systems. However, Hawking assumed in his calculation that the black hole state is unchanged under the emission of Hawking radiation, thus ignoring the gravitational backreaction at the event horizon. By taking into account this backreaction in the form of a longitudinal shift, 't Hooft has derived a unitary description for the scattering of shock waves off a Schwarzschild black hole [2] [3] [4] 5]. The unitary scattering of shock waves on a black hole background, as well other arguments such as the non-conservation of energy through disappearance of information and the unitary time evolution of the boundary CFT of an AdS-black hole space, have done much to shift the general consensus towards black hole unitarity, thus vindicating sustained resistance against the idea of information loss in black holes by 't Hooft, Susskind, and others. Indeed, Hawking himself conceded his position, admitting that he had changed his mind in favour of black hole unitarity.

Some of Hawking's last contributions to black hole physics were collaborations with Perry and Strominger [6] [7], which consider the effect of so-called BMS-transformations on various black hole spacetimes. BMS is the somewhat incomplete acronym for Bondi, van der Burg, Metzner, and Sachs, who examined the symmetries at null infinity of a broad class of asymptotically flat space-times [8] [9]. $\mathrm{B}(\mathrm{B}) \mathrm{MS}$ found that the Poincaré group of Minkowski space is extended by an infinite number of symmetries given by angle-dependent time translations. Their program was recently revisited by Barnich and Troessaert [10] [11], who tentatively extended the BMS-group to include localized generalizations of the Lorentz group, and subsequently by Strominger and collaborators, who used results due to Christodoulou and Klainermann [12] to show that BMS-transformations at future and past null infinity have to be antipodally matched for spaces in a finite parameter neighbourhood of Minkowski space [13] [14]. One of the main reasons for the recently refound interest in BMS-transformations is the fact that an infinite number of BMS-symmetries lead, via Noether's theorem, to an infinite number of conserved charges. This would allow one to store an infinite amount of (classical) information, so that it is suspected that BMS-charges could play a role in the resolution of the black hole information paradox.

In this thesis, we examine the relation between BMS-symmetries and the shock wave S-matrix. During the course of this thesis research, we encountered many allusions to the similarities between the BMSformalism and 't Hooft's S-matrix formalism [15] [16] [17] [6], but these similarities had thus far not been explored. In chapter 1 we review BMS-symmetries and how they give rise to an infinite number of antipodally matched conserved charges. We then review the derivation of the shock wave S-matrix in chapter 2. The two formalisms are then compared in chapter 3 where we will use an
alternative derivation due to the Verlindes [18 to show that shock wave scattering in Minkowski space is characterized by an infinite number of antipodally matched BMS-charges, thus giving an explicit realization of recent findings by Strominger and collaborators. As far as the we are aware, this was not previously pointed out in the literature. We then consider black holes, where the relation between the two formalisms is decidedly less clear, particularly since the BMS-description of black holes is still under development. We review Hawking, Perry, and Strominger's treatment of linearized supertranslations on a Schwarzschild black hole induced by shock waves with a non-trivial spherical profile [7]. We find that the shift of the event horizon satisfies the same constraint equation relating it to the energy-momentum tensor impinging on a black hole in the BMS-formalism [7] and the Dray-'t Hooft shock wave [2, which is used to calculate the shock wave S-matrix [3] [4]. This result, which has not been previously noted in the literature, suggests that one may describe the black hole S-matrix in terms of BMS charges at null infinity. We then briefly summarize a recent derivation by Donnay and collaborators of BMS-like symmetries acting on the event horizon of a black hole [19] [20]. We show that the Dray-'t Hooft shock wave in tortoise coordinates satisfies the required fall-off conditions and that it implants a non-zero superrotation charge at the horizon. This suggests another means of describing shock wave scattering in terms of BMS-charges, located in this case at the event horizon.

## 1 Asymptotic structure of gauge and gravitational scattering processes

We would like to analyze symmetries at the boundaries of space-time and the constraints these place on scattering events. Because of its comparative conceptual simplicity, we first consider a simple electromagnetic example, known as the Liénard-Wiechert potential. We will see that the physical data, in this case the radial component of the electric field, obeys an antipodal matching between past and future null infinity [21]. We then consider the asymptotic symmetries of a broad class of asymptotically flat space-times, where we will review the infinite-dimensional extension of the Poincaré group by BMSsymmetries. For space-times in a finite parameter neighbourhood of Minkowski space, we will see that BMS-transformations and the corresponding conserved charges obey antipodal matching between past and future null infinity.

### 1.1 A simple electromagnetic example

Let us consider the electromagnetic field strength of of massive particles labelled by $j \in\{1,2, \ldots, n\}$ with charges $Q_{j}$ and constant four-velocities $U_{j}^{\mu}=\gamma_{j}\left(1, \vec{\beta}_{j}\right)=\frac{1}{\sqrt{1-\beta_{j}^{2}}}\left(1, \vec{\beta}_{j}\right)$. The electromagnetic current is thus given by

$$
\begin{equation*}
j^{\mu}(x)=\sum_{j=1}^{n} Q_{j} \int d \tau U_{j}^{\mu} \delta^{4}\left(x_{\nu}-\left(U_{j}\right)_{\nu} \tau\right) \tag{1}
\end{equation*}
$$

The equations of motion, given by $\partial^{\nu} F_{\mu \nu}=-j_{\mu}$, are solved by the Liénard-Wiechert solution. We will focus on the radial component of the electric field, which is given by

$$
\begin{equation*}
F_{r t}(\vec{x}, t)=\frac{1}{4 \pi} \sum_{j=1}^{n} \frac{Q_{j} \gamma_{j}\left(r-t \hat{x} \cdot \vec{\beta}_{j}\right)}{\left|\gamma_{j}^{2}\left(t-r \hat{x} \cdot \vec{\beta}_{j}\right)^{2}-t^{2}+r^{2}\right|^{3 / 2}} \quad, \quad r=\sqrt{\vec{x} \cdot \vec{x}}, \quad \hat{x}=\frac{\vec{x}}{r} \tag{2}
\end{equation*}
$$

One easily sees that $F_{r t}$ vanishes at past- and future time-like infinity, denoted by $i^{-}$and $i^{+}$, respectively, by keeping $\vec{r}$ finite and taking the limit $t \rightarrow \pm \infty$. The same conclusion can be drawn for spatial infinity, denoted as $i^{0}$, by keeping $t$ finite and taking $\vec{r} \rightarrow \pm \infty$. Past and future null infinities, written as $\mathcal{I}^{-}$and $\mathcal{I}^{+}$, respectively, have a richer structure. To investigate the Liénard-Wiechert solution at $\mathcal{I}$, we introduce light-cone coordinates given by

$$
\begin{equation*}
v=t+r, \quad u=t-r . \tag{3}
\end{equation*}
$$

The topology (in the sense of homeomorphism rather than homotopy) of past and future null infinity is $\mathcal{I}^{ \pm} \simeq \mathbb{R} \otimes S^{2}$, where $u, v \in \mathbb{R}$ and $S^{2}$ is parametrized by spherical coordinates which we generally
denote by $\Theta$. To analyze $\mathcal{I}^{-}$, it is convenient to use a mixed $(v, r, \Theta)$ coordinate system, in which the Minkowski metric is given by

$$
\begin{equation*}
d s^{2}=-d v^{2}+2 d v d r+2 r^{2} \gamma_{A B} d \Theta^{A} d \Theta^{B} \tag{4}
\end{equation*}
$$

By plugging the first equality of (3) into (2), we see that the radial component of the electric field is given by

$$
\begin{equation*}
F_{r t}=F_{r v}=\frac{e^{2}}{4 \pi} \sum_{j=1}^{n} \frac{Q_{j} \gamma_{j}\left(r-(v-r) \hat{x} \cdot \vec{\beta}_{j}\right)}{\left|\gamma_{j}^{2}\left(v-r-r \hat{x} \cdot \vec{\beta}_{j}\right)^{2}-(v-r)^{2}+r^{2}\right|^{3 / 2}} \tag{5}
\end{equation*}
$$

We analyze the solution at null infinity by fixing $v$ and taking the limit $r \rightarrow \infty$. From the form of the metric in (4), we see that, by fixing $v$ and varying $r$, we traverse zero metric distance. We thus fix $v$ and take $r \rightarrow \infty$ to find the solution for $F_{r t}$ at $\mathcal{I}^{-}$

$$
\begin{equation*}
\left.F_{r t}(\hat{x})\right|_{\mathcal{I}^{-}}=\lim _{r \rightarrow \infty} F_{r v}(\vec{x}, v)=\frac{e^{2}}{4 \pi r^{2}} \sum_{j=1}^{n} \frac{Q_{j}}{\gamma_{j}^{2}\left(1+\hat{x} \cdot \vec{\beta}_{j}\right)^{2}} \tag{6}
\end{equation*}
$$

We see that if we take $\beta_{j}=0$, we get a Coulomb field for a charge $Q_{j}$ at $r=0$, as one would expect. To find the form of $F_{r t}$ at $\mathcal{I}^{+}$, we go to retarded time coordinates, in which the Minkowski metric is

$$
\begin{equation*}
d s^{2}=-d u^{2}-2 d u d r+2 r^{2} \gamma_{A B} d \Theta^{A} d \Theta^{B} \tag{7}
\end{equation*}
$$

Plugging the second equality of (3) into (2), fixing $u$, and taking the limit $r \rightarrow \infty$, gives

$$
\begin{equation*}
\left.F_{r t}(\hat{x})\right|_{\mathcal{I}^{+}}=\lim _{r \rightarrow \infty} F_{r u}=\frac{e^{2}}{4 \pi r^{2}} \sum_{j=1}^{n} \frac{Q_{j}}{\gamma_{j}^{2}\left(1-\hat{x} \cdot \vec{\beta}_{j}\right)^{2}} \tag{8}
\end{equation*}
$$

We thus see that

$$
\begin{equation*}
\left.F_{r t}(\hat{x})\right|_{\mathcal{I}^{+}}=\left.F_{r t}(-\hat{x})\right|_{\mathcal{I}^{-}} \tag{9}
\end{equation*}
$$

Reminding ourselves that $\hat{x}$ is a unit vector on the two-sphere at null infinity, we see that taking $\hat{x} \rightarrow-\hat{x}$ corresponds to the antipodal map on the $S^{2}$. Although the Liénard-Wiechert potential dates from the late $19^{\text {th }}$ century, the antipodal matching of $F_{r t}$ as in (9) was only recently pointed out in the literature [21]. One sees that the two solutions are equal only when $\vec{\beta}_{j}=0$ for all $j$. For small $\beta$, we have $\frac{1}{(1 \pm \hat{x} \cdot \vec{\beta})^{2}} \approx 1 \mp 2 \hat{x} \cdot \vec{\beta}$, which can be intuitively understood from the fact that if we accelerate towards $\mathcal{I}^{+}$, we accelerate away from $\mathcal{I}^{-}$. In this very simple example, we see that physical data at $\mathcal{I}^{+}$
and $\mathcal{I}^{-}$are antipodally matched. Although one might suspect this to be a peculiarity of the LiénardWiechert potential, this antipodal matching can be show to hold for more general electromagnetic configurations, including cases where particles scatter in the bulk. The physical reason for this is that there is a residual, local gauge transformation parametrized by a function of the spherical coordinates $\epsilon(\hat{x})$ at $\mathcal{I}^{ \pm}$, which has to satisfy $\epsilon(\hat{x})_{\mathcal{I}^{+}}=\epsilon(-\hat{x})_{\mathcal{I}^{-}}$[22].


Figure 1: In the conformal compactification of Minkowski space onto the Einstein static universe, Minkowski space is wrapped around a cylinder with topology $\mathbb{R} \times S^{3}$. The antipodal matching of electromagnetic field configurations in the Liénard-Wiechert potential, as well as more general electromagnetic configurations, is then equivalent to the continuity of physical data along the null generators of $\mathcal{I}$ even as they pass through $i^{0}$. Image taken from [23].

As this thesis is concerned mostly with gravitational scattering, we will not consider the electromagnetic case much further. We do note an intuitive illustration, due to Strominger [23], of how antipodal matching can be understood from the conformal compactification of Minkowski space. This is particularly insightful for the case of electromagnetism due to its invariance under conformal transformations. We conformally compactify Minkowski space onto the Einstein static universe, which wraps Minkowski space around a cylinder as illustrated in figure (11. Null infinity $\mathcal{I} \simeq \mathcal{I}^{+} \bigcup \mathcal{I}^{-}$is then given by the light cone of $i^{0}$. Antipodal identification corresponds to the requirement that the electromagnetic field is continuous along the null generators of $\mathcal{I}$ even when the generators pass through $i^{0}$. This is quite
surprising since $i^{0}$ is a point in the conformal compactification of Minkowski space, where physical data such as $F_{r t}$ is typically singular. We now consider the case of gravity, which we will find to be largely analogous to this simple electromagnetic example.

### 1.2 Supertranslations

We would like to analyze the asymptotic symmetries of asymptotically flat space-times. We will limit ourselves to consider only massless particles, as this significantly simplifies our calculations and suffices for comparison with the shock wave S-matrix due to 't Hooft [3] [4]. The analysis below has been generalized to include massive particles [24] [25]. There exist several notions of asymptotic flatness with corresponding fall-off conditions on metric components. One would like to pick suitable fall-off conditions to allow for 'interesting' space-times we would like to analyze but to exclude 'unphysical' space-times. Various notions of asymptotic flatness can be said to arise from various notions of what is interesting and what is physical. In this section, we follow the approach due to Bondi, van der Burg, Metzner, and Sachs, whose initials form the basis for the somewhat incomplete acronym 'BMS'. Their analysis of asymptotic structures was done in Bondi gauge, which is given by

$$
\begin{equation*}
g_{r A}=0=g_{r r} \quad, \quad \partial_{r} \operatorname{det}\left(\frac{g_{A B}}{r^{2}}\right)=0 \tag{10}
\end{equation*}
$$

This fixes all four gauge freedoms of general relativity. From $\partial_{r} \operatorname{det}\left(\frac{g_{A B}}{r^{2}}\right)=0$, it follows that $r$ is equal to the luminosity distance. The most general metric can then be written as [8 [9]

$$
\begin{equation*}
d s^{2}=-U d u^{2}-2 e^{2 \beta} d u d r+g_{A B}\left(d \Theta^{A}+\frac{1}{2} U^{A} d u\right)\left(d \Theta^{B}+\frac{1}{2} U^{B} d u\right) \tag{11}
\end{equation*}
$$

We would like to only consider metrics with finite radiative energy flux, which corresponds to the following fall-off conditions for the Weyl tensor

$$
\begin{equation*}
C_{r A r B}, C_{r u r A}=\mathcal{O}\left(1 / r^{3}\right) \tag{12}
\end{equation*}
$$

Writing the spherical metric as

$$
g_{A B}=r^{2} \gamma_{A B}+r C_{A B}+\mathcal{O}(1)
$$

the requirement in 12 leads to

$$
U^{A}=D^{B} C_{A B}
$$

where $D^{A}$ is the covariant derivative with respect to $\gamma_{A B}$, which is the metric of the unit sphere.

Spherical indices $A, B$ are henceforth raised and lowered by applying $\gamma^{A B}$ and $\gamma_{A B}$, respectively. The leading and subleading terms in the $1 / r$-expansion around the Minkowski metric at $\mathcal{I}^{+}$are given by [8] (9]

$$
\begin{align*}
d s^{2}= & \underbrace{-d u^{2}-2 d u d r+2 r^{2} \gamma_{A B} d \Theta^{A} d \Theta^{B}}_{\text {Minkowski metric }}+ \\
& +\frac{2 m_{B}}{r} d u^{2}+r C_{A B} d \Theta^{A} d \Theta^{B}+D^{A} C_{A B} d u d \Theta^{B}+ \\
& +\frac{1}{r}\left(\frac{4}{3}\left(N_{A}+u \partial_{A} m_{B}\right)-\frac{1}{8}\left(C_{A B} C^{A B}\right)\right) d u d \Theta^{A}+\ldots \tag{13}
\end{align*}
$$

In (13), $C_{A B}, m_{B}$, and $N_{A}$ depend only on $\left(u, \Theta^{A}\right)$, since we are expanding in $1 / r . m_{B}$ and $N_{A}$ are known as the Bondi mass aspect and the angular mass aspect, respectively. For BMS-symmetries known as supertranslations, to be considered in this section, only the first two lines of (13) are required. The third line plays a role for the case of superrotations, which we briefly review in the next section. Note that the class of metrics given by (13) includes the Schwarzschild metric, which is given by $m_{B}\left(u, \Theta^{A}\right)=M=$ constant and $C_{A B}=0=N_{A}$. One can easily see that the Bondi gauge choice for the transversal part of the metric entails that $C_{A B}$ is traceless. Namely, using the fact that $\ln (\mathbb{1}+B)=\sum_{n=1}^{\infty}(-1)^{n+1} \frac{B^{n}}{n}$ for $\|B\|<1$, where $\|\cdot\|$ is the matrix norm [26], we find

$$
\begin{align*}
\operatorname{det}\left(\frac{g_{A B}}{r^{2}}\right) & =\exp \operatorname{tr} \ln \left[\gamma_{A C}\left(\delta_{B}^{C}+\frac{C_{B}^{C}}{r}\right)+\mathcal{O}\left(1 / r^{2}\right)\right] \\
& =\operatorname{det} \gamma \exp \operatorname{tr}\left[\frac{C_{B}^{C}}{r}+\mathcal{O}\left(1 / r^{2}\right)\right] \\
& =\operatorname{det} \gamma\left[1+\frac{C_{A}^{A}}{r}+\mathcal{O}\left(1 / r^{2}\right)\right] \tag{14}
\end{align*}
$$

so that $\partial_{r} \operatorname{det}\left(\frac{g_{A B}}{r^{2}}\right)=0$ entails that $C_{A}^{A}=0$, as claimed. Let us consider the most general diffeomorphism $\xi$ which leaves the gauge and fall-off conditions invariant. For now, we restrict our attention to diffeomorphisms with the following asymptotic behaviour

$$
\begin{equation*}
\xi^{u}, \xi^{r} \sim \mathcal{O}(1) \quad, \quad \xi^{A} \sim \mathcal{O}(1 / r) \tag{15}
\end{equation*}
$$

These conditions entail that the vector fields $\xi$ are asymptotically $\mathcal{O}(1)$ in an orthonormal frame. Note that these conditions exclude Lorentz transformations, which asymptotically grow as $\mathcal{O}(r)$. The action on the metric of a diffeomorphism with respect to a Killing vector field $\xi$ can generally be written as

$$
\begin{equation*}
\mathcal{L}_{\xi} g_{\mu \nu}=\xi^{\lambda} \partial_{\lambda} g_{\mu \nu}+g_{\mu \lambda} \partial_{\nu} \xi^{\lambda}+g_{\nu \lambda} \partial_{\mu} \xi^{\lambda} \tag{16}
\end{equation*}
$$

The fall-off and gauge conditions of the BMS-metric in which we require to be preserved by $\xi$ are

$$
\begin{array}{ll}
g_{u u}=-1+\frac{2 m_{B}}{r}+\mathcal{O}\left(1 / r^{2}\right) & , \\
g_{A B}=g^{2} \gamma_{A B}+r C_{A B}+h_{A B}+\mathcal{O}(1 / r), & g_{r r}=0=g_{r A} \tag{17}
\end{array}
$$

where $h_{A B}$ is an additional $\mathcal{O}(1)$ term which we will need to find the form of $\xi^{r}$. Let us first consider the action of $\xi$ on the $r r$-component of the metric. This is given by

$$
\mathcal{L}_{\xi} g_{r r}=2 \partial_{r} \xi^{u} g_{u r}
$$

which is required to be zero from the Bondi gauge condition that $g_{r r}=0$. It follows that

$$
\begin{equation*}
\partial_{r} \xi^{u}=0 \quad \Rightarrow \quad \xi^{u}=\xi^{u}\left(u, \Theta^{A}\right) \tag{18}
\end{equation*}
$$

For $g_{u r}$, we find

$$
\begin{aligned}
\mathcal{L}_{\xi} g_{u r} & =\xi^{\mu} \partial_{\mu} g_{u r}+g_{u \mu} \partial_{r} \xi^{\mu}+g_{r \mu} \partial_{u} \xi^{\mu} \\
& =g_{u r}\left(\partial_{r} \xi^{r}+\partial_{u} \xi^{u}\right)+\mathcal{O}\left(1 / r^{2}\right) \\
& =-\left(\partial_{r} \xi^{r}+\partial_{u} \xi^{u}\right)+\mathcal{O}\left(1 / r^{2}\right)
\end{aligned}
$$

From (15), $\partial_{u} \xi^{u}=\mathcal{O}(1)$ and $\partial_{r} \xi^{r}=\mathcal{O}(1 / r)$ at $\mathcal{I}^{+}$, so that

$$
\begin{equation*}
\xi^{u}=f\left(\Theta^{A}\right)+\mathcal{O}\left(r^{-2}\right) \tag{19}
\end{equation*}
$$

Similarly, we require that $\partial_{r} \xi^{r}=\mathcal{O}\left(r^{-2}\right)$ so that, to leading order, $\xi^{r}$ is independent of $r$. We now expand $\xi^{\mu}$ in $1 / r$

$$
\begin{equation*}
\xi=f \partial_{u}+\sum_{n=0}^{\infty} \frac{\xi^{(n) r}}{r^{n}} \partial_{r}+\sum_{n=1}^{\infty} \frac{\xi^{(n) A}}{r^{n}} \partial_{A} \tag{20}
\end{equation*}
$$

i.e. $\xi^{(n) \mu}$ is the $n^{\text {th }}$ order term in the $1 / r$-expansion of $\xi^{\mu}$. We then have

$$
\begin{align*}
\mathcal{L}_{\xi} g_{u u} & =f \partial_{u} g_{u u}+\xi^{r} \partial_{r} g_{u u}+\xi^{A} \partial_{A} g_{u u}+2 g_{u r} \partial_{u} \xi^{r}+2 g_{u A} \partial_{u} \xi^{A} \\
& =-2 \partial_{u} \xi^{(0) r}+\frac{2 f \partial_{u} m_{B}-2 \partial_{u} \xi^{(1) r}+\left(\partial_{u} \xi^{(1) A}\right) D^{B} C_{A B}}{r}+\mathcal{O}\left(1 / r^{2}\right) \tag{21}
\end{align*}
$$

We will return this expression in due time to read off the action of a supertranslation on the Bondi mass aspect. The action of a supertranslation on $g_{r A}$ is given by

$$
\begin{align*}
\mathcal{L}_{\xi} g_{r A} & =\partial_{r} \xi^{B} g_{B A}+g_{u r} \partial_{A} f \\
& =-\gamma_{A B}\left(\xi^{(1) B}+D^{B} f\right)-\frac{2 \gamma_{A B} \xi^{(2) B}+C_{A B} \xi^{(1) B}}{r}+\mathcal{O}\left(1 / r^{2}\right) \tag{22}
\end{align*}
$$

$\mathcal{L}_{\xi} g_{r A}$ has to equal zero to respect Bondi gauge, from which we find that

$$
\begin{equation*}
\xi^{(1) A}=-D^{A} f \quad, \quad \xi^{(2) A}=\frac{1}{2} C^{A B} D_{B} f \tag{23}
\end{equation*}
$$

For $g_{A B}$, we have

$$
\begin{align*}
\mathcal{L}_{\xi} g_{A B} & =f \partial_{u} g_{A B}+\xi^{r} \partial_{r} g_{A B}+\xi^{C} \partial_{C} g_{A B}+g_{u B} \partial_{A} f+\partial_{A} \xi^{C} g_{C B}+g_{u A} \partial_{B} f+\partial_{B} \xi^{C} g_{C A} \\
& =r\left(f \partial_{u} C_{A B}+2 \gamma_{A B} \xi^{(0) r}+D_{A} \xi_{B}^{(1)}+D_{B} \xi_{A}^{(1)}\right) \\
& +f \partial_{u} h_{A B}+C_{A B} \xi^{(0) r}+2 \gamma_{A B} \xi^{(0) r}+2 \gamma_{A B} \xi^{(1) r}+\frac{1}{2}\left(D^{C} C_{C A}\right) D_{B} f+\frac{1}{2}\left(D^{C} C_{C B}\right) D_{A} f \\
& +\xi^{(1) C} D_{C} C_{A B}+D_{A} \xi_{B}^{(2)}+D_{B} \xi_{A}^{(2)}+C_{A C} D_{B} \xi^{(1) C}+C_{B C} D_{A} \xi^{(1) C}+\mathcal{O}(1 / r) \tag{24}
\end{align*}
$$

From the condition that the $\mathcal{O}(r)$-term is traceless, as found in 14 , it follows that

$$
\begin{equation*}
\xi^{(0) r}=\frac{1}{2} D^{A} D_{A} f \tag{25}
\end{equation*}
$$

For $g_{A B}$, given in 17 , we have

$$
\begin{aligned}
\operatorname{det}\left[\frac{g_{A B}}{r^{2}}\right] & =\exp \operatorname{tr} \log \left[\gamma_{A B}\left(\delta_{A}^{B}+\frac{1}{r} C_{A}^{B}+\frac{1}{r^{2}} h_{A}^{B}+\mathcal{O}\left(1 / r^{3}\right)\right)\right] \\
& =\operatorname{det} \gamma \exp \operatorname{tr} \log \left(\delta_{A}^{B}+\frac{1}{r} C_{A}^{B}+\frac{1}{r^{2}} h_{A}^{B}+\mathcal{O}\left(1 / r^{3}\right)\right) \\
& =\operatorname{det} \gamma\left[1+\frac{C_{A}^{A}}{r}+\frac{1}{r^{2}}\left(h_{A}^{A}-\frac{C_{A}^{B} C_{B}^{A}}{2}\right)+\mathcal{O}\left(1 / r^{3}\right)\right] .
\end{aligned}
$$

Hence, $\partial_{r} \operatorname{det}\left[\frac{g_{A B}}{r^{2}}\right]=0$ leads to

$$
\begin{equation*}
h_{A}^{A}=\frac{C_{A}^{B} C_{B}^{A}}{2} . \tag{26}
\end{equation*}
$$

To find the $\mathcal{O}(1 / r)$-term of $\xi^{r}$, we look at the action of a supertranslation on (26), i.e. we require that $\delta h_{A}^{A}=C^{A B} \delta C_{A B}$. Using (23) as well as the tracelessness of $C_{A B}$, we find

$$
\begin{align*}
\delta h_{A}^{A} & =C^{A B} \delta C_{A B} \\
f \partial_{u} h_{A}^{A}+4 \xi^{(1) r}+2\left(D^{A} f\right) D^{B} C_{A B}-C_{A B} D^{A} D^{B} f & =f C^{A B} \partial_{u} C_{A B}-2 C_{A B} D^{A} D^{B} f, \tag{27}
\end{align*}
$$

so that

$$
\begin{equation*}
\xi^{(1) r}=-\frac{1}{4}\left[2\left(D^{A} f\right) D^{B} C_{A B}-C_{A B} D^{A} D^{B} f\right] . \tag{28}
\end{equation*}
$$

Combining (19), 23), (25), and 27), the final result for $\xi$ is given by

$$
\begin{align*}
\xi^{\mu} \partial_{\mu}= & f \partial_{u}+\left(-\frac{D^{A} f}{r}+\frac{\frac{1}{2} C^{A B} D_{B} f}{r^{2}}+\mathcal{O}\left(1 / r^{3}\right)\right) \partial_{A} \\
& +\left(\frac{1}{2} D_{A} D^{A} f-\frac{\frac{1}{2}\left(D^{A} f\right) D^{B} C_{A B}+\frac{1}{4} C_{A B} D^{A} D^{B} f}{r}+\mathcal{O}\left(1 / r^{2}\right)\right) \partial_{r} . \tag{29}
\end{align*}
$$

Note, from the first term on the right hand side of (29), that the $\ell=0$ and $\ell=1$-modes in the expansion of $f=f(z, \bar{z})$ in spherical harmonics correspond to global time- and spatial translations, respectively. Defining the Bondi news tensor as

$$
\begin{equation*}
N_{A B}:=\partial_{u} C_{A B}, \tag{30}
\end{equation*}
$$

the action of a supertranslation $g_{u u}$ and $g_{A B}$ is given by

$$
\begin{align*}
\mathcal{L}_{\xi} g_{u u} & =\frac{1}{r}\left[2 f \partial_{u} m_{B}+\left(D^{A} f\right) D^{B} N_{A B}+\frac{1}{2}\left(\partial_{u} C_{A B}\right) D^{A} D^{B} f\right]+\mathcal{O}\left(1 / r^{2}\right), \\
\mathcal{L}_{\xi} g_{A B} & =r\left[f \partial_{u} C_{A B}+\gamma_{A B} D_{C} D^{C} f-2 D_{A} D_{B} f\right]+\mathcal{O}(1) . \tag{31}
\end{align*}
$$

From (13), (30), and (31), we read off

$$
\begin{align*}
\mathcal{L}_{\xi} C_{A B} & =f \partial_{u} C_{A B}+\gamma_{A B} D_{C} D^{C} f-2 D_{A} D_{B} f \\
\mathcal{L}_{\xi} N_{A B} & =f \partial_{u} N_{A B} \\
\mathcal{L}_{\xi} m_{B} & =f \partial_{u} m_{B}+\frac{1}{2}\left(D^{A} f\right) D^{B} N_{A B}+\frac{1}{4} N_{A B} D^{A} D^{B} f . \tag{32}
\end{align*}
$$

For convenience, we now parametrize $S^{2}$ using stereographic coordinates. That is, we write $\Theta^{A} \in\{z, \bar{z}\}$, where $\bar{z}$ is the complex conjugate of $z$. Stereographic coordinates at $\mathcal{I}^{+}$are defined as

$$
\begin{equation*}
\left.z\right|_{\mathcal{I}^{-}}=\frac{x_{1}+i x_{2}}{x_{3}+r} \quad,\left.\quad \bar{z}\right|_{\mathcal{I}^{-}}=\frac{x_{1}-i x_{2}}{x_{3}+r} \quad ; \quad r^{2}=\vec{x} \cdot \vec{x} . \tag{33}
\end{equation*}
$$

The Minkowski metric at $\mathcal{I}^{+}$is then given by

$$
d s^{2}=-d u^{2}-2 d u d r+2 r^{2} \gamma z \bar{z} d z d \bar{z} \quad, \quad \gamma_{z \bar{z}}=\frac{2}{(1+z \bar{z})^{2}}
$$

For convenience, we choose the spherical metrics at $\mathcal{I}^{+}$and $\mathcal{I}^{-}$to be related by the antipodal map, which is given by $z \rightarrow-1 / \bar{z}$. This will simplify a lot of expressions throughout the remainder of this thesis. Thus, at $\mathcal{I}^{+}$, we choose

$$
\begin{equation*}
\left.z\right|_{\mathcal{I}^{+}}=-\frac{r+x^{3}}{x^{1}-i x^{2}} . \tag{34}
\end{equation*}
$$

One can easily show that $\gamma_{z \bar{z}} d z d \bar{z}$ is invariant under the antipodal map, as one would expect. In these coordinates, supertranslations at $\mathcal{I}^{+}$are given by

$$
\begin{equation*}
\xi=\xi^{\mu} \partial_{\mu}=f \partial_{u}-\frac{1}{r}\left(D^{z} f \partial_{z}+D^{\bar{z}} f \partial_{\bar{z}}\right)+D^{z} D_{z} f \partial_{r} \tag{35}
\end{equation*}
$$

and their effect on $C_{z z}$ and $m_{B}$ is [23]

$$
\begin{align*}
\mathcal{L}_{\xi} C_{z z} & =f \partial_{u} C_{z z}-2 D_{z}^{2} f \\
\mathcal{L}_{\xi} m_{B} & =f \partial_{u} m_{B}+\frac{1}{4}\left[2\left(D_{z} N^{z z}\right) D_{z} f+N^{z z} D_{z}^{2} f+\text { c.c. }\right] \tag{36}
\end{align*}
$$

One immediately sees that $\mathcal{L}_{\xi} N_{z z}=f \partial_{u} C_{z z}$ from the fact that $\partial_{u} f=0$. The first term on the right hand side of both equalities in 36 is simply a translation along light-cone coordinate $u$ which is modulated by some angle-dependent function $f=f(z, \bar{z})$. Note that flat Minkowski space is given by $m_{B}=0=C_{z z}$, which immediately gives $N_{z z}=0$. If we act on such a space-time with a supertranslation, we see from (36) that the resulting space-time will still have $m_{B}=0=N_{z z}$, so that the corresponding Riemann tensor will still be zero. We thus see that there is an entire class of flat space-times characterized by a function $C=C(z, \bar{z})$, where

$$
C_{z z}=-2 D_{z}^{2} C \quad, \quad \mathcal{L}_{\xi} C=C+f
$$

We would like to see how $m_{B}$ changes when we insert energy in our space-time in the form of a matter stress energy source $T_{\mu \nu}^{M}$. To do so, we consider the leading term in the $1 / r$-expansion of the uu-component of Einstein's equations. A tedious but straightforward calculation reveals

$$
\begin{align*}
8 \pi T_{u u}^{M} & =R_{u u}-\frac{1}{2} g_{u u} R \\
& =\frac{1}{r^{2}}\left[-2 \partial_{u} m_{B}+\frac{1}{2}\left(D_{z}^{2} N^{z z}+D_{\bar{z}}^{2} N^{\bar{z} \bar{z}}-N_{z z} N^{z z}\right)\right]+\mathcal{O}\left(1 / r^{3}\right), \tag{37}
\end{align*}
$$

which we rewrite to give

$$
\begin{equation*}
\partial_{u} m_{B}=-4 \pi G r^{2} T_{u u}^{M}+\frac{1}{4}[\underbrace{D_{z}^{2} N^{z z}+D_{\bar{z}}^{2} N^{\bar{z} \bar{z}}}_{\sim \partial \Gamma}-\underbrace{N_{z z} N^{z z}}_{\sim \Gamma^{2}}]+\mathcal{O}(1 / r) . \tag{38}
\end{equation*}
$$

One can easily convince oneself that terms in (38) which are linear in $N_{z z}$ arise from Riemann tensor components involving the derivative of the Christoffel connection, while those quadratic in $N_{z z}$ arise from terms involving the square of the connection. Note that the latter terms correspond to gravitational wave contributions to the energy-momentum tensor. Following Strominger [13, we define

$$
\begin{equation*}
T_{u u}:=4 \pi r^{2} T_{u u}^{M}+\frac{1}{4} N_{z z} N^{z z} \tag{39}
\end{equation*}
$$

There is, of course, an analogous story at $\mathcal{I}^{-}$to the one described above for $\mathcal{I}^{+}$. At $\mathcal{I}^{-}$, we write the metric as

$$
\begin{equation*}
d s^{2}=-d v^{2}+2 d v d r+2 r^{2} \gamma_{z \bar{z}} d z d \bar{z}+\frac{m_{B}^{-}}{r} d v^{2}+r D_{z z} d z^{2}+r D_{\bar{z} \bar{z}} d \bar{z}^{2}+\ldots \tag{40}
\end{equation*}
$$

Repeating the above analysis at $\mathcal{I}^{-}$reveals that the supertranslations here are given by

$$
\begin{equation*}
\zeta=g \partial_{v}+\frac{1}{r}\left(D^{z} g \partial_{z}+D^{\bar{z}} g \partial_{\bar{z}}\right)-D^{z} D_{z} g \partial_{r} \quad, \quad g=g(z, \bar{z}) \tag{41}
\end{equation*}
$$

Note that the radial component of the asymptotic Killing vector at $\mathcal{I}^{-}$has the opposite sign of that at $\mathcal{I}^{+}$by comparing with equation (35). Writing the Bondi news at $\mathcal{I}^{+}$as $M_{z z}=\partial_{v} D_{z z}$, we find that

$$
\begin{equation*}
\mathcal{L}_{\zeta} D_{z z}=g \partial_{v} D_{z z}+2 D_{z}^{2} g \quad, \quad \mathcal{L} M_{z z}=g \partial_{v} M_{z z} \tag{42}
\end{equation*}
$$

and 23
$\partial_{v} m_{B}^{-}=4 \pi G r^{2} T_{v v}^{M}+\frac{1}{4}\left[D_{z}^{2} M^{z z}+D_{\bar{z}}^{2} M^{\bar{z} \bar{z}}+M_{z z} M^{z z}\right]+\mathcal{O}(1 / r)=: T_{v v}+\frac{1}{4}\left(D_{z}^{2} M^{z z}+D_{\bar{z}}^{2} M^{\bar{z} \bar{z}}\right)+\mathcal{O}(1 / r)$.

The sign difference between the matter stress energy tensors in (38) and 43) can be easily understood from the fact that energy carried by massless particles enters space-time at $\mathcal{I}^{-}$and exits it at $\mathcal{I}^{+}$.

Hence, assuming the null energy condition, $T_{v v}\left(T_{u u}\right)$ gives a positive (negative) contribution to the Bondi mass.

### 1.3 Superrotations

The fall-off conditions on $\xi$ given by (15) exclude Lorentz transformations. B(B)MS [8] analyzed diffeomorphisms which need not satisfy condition (15); we briefly review these here. At $\mathcal{I}^{+}$, global Lorentz transformations are given by

$$
\begin{equation*}
\left.\zeta_{z}\right|_{\mathcal{I}^{+}}=Y^{z} \partial_{z}+\frac{u}{2} D_{z} Y^{z} \partial_{u}+\text { c.c. }, \quad Y^{z} \in\left\{1, z, z^{2}, i, i z, i z^{2}\right\} . \tag{44}
\end{equation*}
$$

The action on the metric of general $Y^{z}$ is given by

$$
\begin{equation*}
\mathcal{L}_{\zeta} g_{z z}=2 r^{2} \gamma_{z \bar{z}} \partial_{\bar{z}} Y^{z}+\mathcal{O}(r), \tag{45}
\end{equation*}
$$

so that we only have globally well-defined $Y^{z}$ which satisfy the fall-off conditions for the spherical metric in (17) in case of the Lorentz transformations indicated in (44). All the other metric components respect the fall-off conditions under the action of $\zeta$, so that 45) is the only offending term. The form of equation (45) led $\mathrm{B}(\mathrm{B}) \mathrm{MS}$ to ignore extensions to the Lorentz group [8] [9]. However such extensions recently received new-found attention due to the fact that expression (45) is the same equation found by Belavin, Polyakov, and Zamolodchikov in their analysis of 2D CFT's [27], where such singular transformations are used to great effect. This led various authors to suggest that one should allow for transformations parametrized by $Y^{z}=z^{n}$ for general $n \in \mathbb{Z}$ [28] [29] [10] [11]. These (pseudo)symmetries are known as superrotations. Due to their violating the fall-off conditions, the interpretation of superrotations is decidedly less clear than that of the supertranslations we encountered in the previous section.

### 1.4 The antipodal matching condition

In the section 1.2 we saw that the fall-off and Bondi gauge conditions in (17) at $\mathcal{I}$ are left invariant by an asymptotic Killing vector which translates the time coordinate in an angle-dependent manner. Note that $\mathcal{I}=\mathcal{I}^{-} \bigcup \mathcal{I}^{+}$and that we have an asymptotic Killing vector at both $\mathcal{I}^{+}$and $\mathcal{I}^{-}$, given by (35) and (41], respectively. It is easy to see that we cannot transform $\mathcal{I}^{+}$and $\mathcal{I}^{-}$independently, for example, a global time translation on Minkowski space corresponds to taking $f(z, \bar{z})=$ constant $=g(z, \bar{z})$. In general, $f$ and $g$ are allowed to depend non-trivially on spherical coordinates, so that these functions have to matched in some way for the combined transformation to be a symmetry of space-time. The need for a matching condition arises from the fact that supertranslations have a non-trivial effect on the physical data as they act at the boundary of space-time. For example, a supertranslation at $\mathcal{I}^{-}$ $\left(\mathcal{I}^{+}\right)$can change the time at which a particles enters (exits) space-time. Let us consider the simplest conceivable such example, which is a radially propagating massless particle entering Minkowski space
at the south pole at $v=0$ on $\mathcal{I}^{-}$and exiting at the north pole at $u=0$ on $\mathcal{I}^{+}$. If we now act on $\mathcal{I}^{-}$with a supertranslation that shifts $v$ at the south pole by some amount $\Delta$, we clearly have to act on $\mathcal{I}^{+}$with a supertranslation which shifts the north pole by the same amount, so that the particle exits at $u=\Delta$. One can of course consider more complicated situations with multiple particles that interact in the interior to make the non-trivial effect of a supertranslation more acute. Yet this simple example illustrates an important point, namely, that supertranslations at $\mathcal{I}^{-}$and $\mathcal{I}^{+}$have to be antipodally matched for the combined transformation to be a symmetry of space-time [13]. As the supertranslations have a non-trivial effect on the physical data, as expressed in (36), these have to be antipodally matched as well.


Figure 2: The action of a supertranslation, illustrated here on $\mathcal{I}^{+}$, is to translate the time coordinate in an angle-dependent manner. The null generators of $\mathcal{I}^{+}$and $\mathcal{I}^{-}$meet in the limit where they approach $i^{0}$; these limits are indicated as $\mathcal{I}_{-}^{+}$and $\mathcal{I}_{+}^{-}$, respectively. Antipodal matching of supertranslations leads to antipodal matching of physical data at $\mathcal{I}_{-}^{+}$and $\mathcal{I}_{+}^{-}$. Image adapted from [23].

To see how antipodal matching of supertranslations comes about, we first define the future and past limits of null infinity

$$
\begin{equation*}
\mathcal{I}_{ \pm}^{+}:=\lim _{u \rightarrow \pm \infty} \mathcal{I}^{+}, \quad \mathcal{I}_{ \pm}^{-}:=\lim _{v \rightarrow \pm \infty} \mathcal{I}^{-} \tag{46}
\end{equation*}
$$

as indicated in figure 2 Following Strominger, we then apply a result from Christodoulou and Klainermann's seminal work on asymptotically flat spaces, which proved the existence of geodesically complete asymptotically Minkowski solutions [12]. For these space-times, one has

$$
\begin{equation*}
\left.m_{B}\right|_{\mathcal{I}_{+}^{+}}=0=\left.m_{B}^{-}\right|_{\mathcal{I}_{-}^{-}} \tag{47}
\end{equation*}
$$

Hence, if black holes are allowed to form in the interior, they have to radiate away in finite time. Further, in Christodoulou-Klainermann spaces, the Bondi news $N_{z z}$ decays like $|v|^{3 / 2}$ or faster for $|v| \rightarrow \infty$, with the same condition in terms of $|u|$ for the $|u| \rightarrow \infty$. For such spaces, one can show that 13]

$$
\left.C_{z z}\right|_{\mathcal{I}_{-}^{+}}=-2 D_{z}^{2} C(z, \bar{z})
$$

where $C$ is a function on the two-sphere at $\mathcal{I}^{-}$. One can then integrate $N_{z z}=\partial_{v} C_{z z}$ with integration constant $\left.C_{z z}\right|_{\mathcal{I}_{+}^{-}}=-D_{z}^{2} C$ to give $C_{z z}$ for all of $\mathcal{I}^{+}$. Similarly, we have $\left.D_{z z}\right|_{\mathcal{I}_{+}^{-}}=-2 D_{z}^{2} D$, from which we find $M_{z z}$ for all of $\mathcal{I}^{-}$. The physical data at $\mathcal{I}^{+}$and $\mathcal{I}^{-}$are therefore specified by

$$
\left\{\left.m_{B}^{-}\right|_{\mathcal{I}_{+}^{-}},\left.D\right|_{\mathcal{I}_{+}^{-}}, M_{z z}\right\} \quad, \quad\left\{\left.m_{B}\right|_{\mathcal{I}_{-}^{+}},\left.C\right|_{\mathcal{I}_{-}^{+}}, N_{z z}\right\}
$$

Let us then consider the action of a global Lorentz transformation on $\mathcal{I}$. These act on $m_{B}$ and $m_{B}^{-}$as

$$
\begin{align*}
\mathcal{L}_{\chi_{+}} m_{B} & =\left(\chi_{+}^{z} \partial_{z}+\frac{u}{2} D_{z} \chi_{+}^{z} \partial_{u}+\frac{3 D_{z} \chi_{+}^{z}}{2}\right) m_{B}-\frac{u}{2} \partial_{u}\left[U_{z}\left(1-D_{z}^{2}\right) \chi_{+}^{z}\right] \\
\mathcal{L}_{\chi_{-}} m_{B}^{-} & =\left(\chi_{-}^{-z} \partial_{z}+\frac{u}{2} D_{z} \chi_{-}^{-z} \partial_{u}+\frac{3 D_{z} \chi_{-}^{-z}}{2}\right) m_{B}+\frac{u}{2} \partial_{u}\left[V_{z}\left(1-D_{z}^{2}\right) \chi_{-}^{-z}\right], \tag{48}
\end{align*}
$$

where $U_{z}=\frac{1}{2} D^{z} C_{z z}$ and $V_{z}=-\frac{1}{2} D^{z} D_{z z}$, as in 11) and (13), and $\chi_{ \pm} \in\left\{1, i, z, i z, z^{2}, i z^{2}\right\}$. Global Lorentz transformations are then given by

$$
\begin{equation*}
\chi_{+}^{z}=\chi_{-}^{-z} . \tag{49}
\end{equation*}
$$

We see that these leave invariant the following matching condition [13]

$$
\begin{equation*}
\left.m_{B}(z, \bar{z})\right|_{\mathcal{I}_{-}^{+}}=\left.m_{B}^{-}(z, \bar{z})\right|_{\mathcal{I}_{+}^{-}} \quad,\left.\quad C(z, \bar{z})\right|_{\mathcal{I}_{-}^{+}}=-\left.D(z, \bar{z})\right|_{\mathcal{I}_{+}^{-}} \tag{50}
\end{equation*}
$$

Remember that spherical coordinates $z$ and $\bar{z}$ at $\mathcal{I}^{+}$and $\mathcal{I}^{-}$are related by the antipodal map $z \mapsto-1 / \bar{z}$ as per our definitions in (33) and (34). Further, note that our conventions for $C$ and $D$, which were chosen to resemble the form of a supertranslation with $f=C, g=D$, are different from those used by Strominger in [23] and [13. In addition to Lorentz transformations, the conditions in (50) are invariant under CPT inversions. This can be seen from the fact that a parity inversion corresponds to the antipodal map on the two-sphere, and time inversion takes us from $\mathcal{I}_{-}^{+}$to $\mathcal{I}_{+}^{-}$. The charge
inversion then switches the sign of the energy, akin to the sign switch between (38) and (43), so that ingoing (outgoing) energy flux gives a strictly positive (negative) contribution to the energy content of space-time. The combined supertranslations which preserve (50) are given by

$$
\begin{equation*}
\left.f(z, \bar{z})\right|_{\mathcal{I}^{+}}=\left.g(z, \bar{z})\right|_{\mathcal{I}^{-}} . \tag{51}
\end{equation*}
$$

We thus find that $f$ and $g$ have to antipodally matched to preserve (50). Equation (51) holds for all of $\mathcal{I}^{+}$and $\mathcal{I}^{-}$since $f$ and $g$ do not depend on time, so that fixing them at $\mathcal{I}_{-}^{+}$and $\mathcal{I}_{+}^{-}$fixes them for all of $\mathcal{I}^{+}$and $\mathcal{I}^{-}$, respectively. Writing a point on $\Omega=(\theta, \varphi)$ and its antipode as $-\Omega=(\pi-\theta, \pi+\varphi)$, we note that

$$
\begin{equation*}
Y_{\ell, m}(-\Omega)=(-1)^{\ell} Y_{\ell, m}(\Omega) \tag{52}
\end{equation*}
$$

Expanding the supertranslations at $\mathcal{I}^{+}$and $\mathcal{I}^{-}$as $f=\sum_{\ell, m} f_{\ell, m} Y_{\ell, m}$ and $g=\sum_{\ell, m} g_{\ell, m} Y_{\ell, m}$, respectively, we find that the antipodal matching of $f$ and $g$ is expressed in spherical harmonics as

$$
\begin{equation*}
f_{\ell, m}=(-1)^{\ell} g_{\ell, m} \tag{53}
\end{equation*}
$$

One can easily see that the antipodal matching in (50) implies that $N_{z}$ should be antipodally matched between $\mathcal{I}^{-}$and $\mathcal{I}^{+}$as well. Namely the $u z$-component of Einstein equations for (13) is given by

$$
\begin{equation*}
\partial_{u} N_{z}=-u \partial_{u} \partial_{z} m_{B}+\frac{1}{4} \partial_{z}\left(D_{z}^{2} C^{z z}-D_{\bar{z}}^{2} C^{\bar{z} \bar{z}}\right)-T_{u z} \tag{54}
\end{equation*}
$$

where $T_{u z}$ is a convenient momentum density expression defined as

$$
\begin{equation*}
T_{u z}:=8 \pi \lim _{r \rightarrow \infty}\left[r^{2} T_{u z}^{M}\right]-\frac{1}{4} \partial_{z}\left(C_{z z} N^{z z}\right)-\frac{1}{2} C_{z z} D_{z} N^{z z} \tag{55}
\end{equation*}
$$

One thus sees that (50) fixes $N_{z}$ up to an integration constant in the limit where $\mathcal{I}$ approaches $i^{0}$. In [30] [31, it is shown that this integration constant has to be antipodally matched as

$$
\begin{equation*}
\left.N_{z}(z, \bar{z})\right|_{\mathcal{I}_{-}^{+}}=\left.N_{z}(z, \bar{z})\right|_{\mathcal{I}_{+}^{-}} \tag{56}
\end{equation*}
$$

by showing that this is equivalent to a subleading soft graviton theorem.
The upshot is that we have an infinite number of symmetries at $\mathcal{I}^{+}$and $\mathcal{I}^{-}$, known as supertranslations, which are angle-dependent translations of the retarded and advanced light-cone coordinates, respectively. By considering simple examples, such as global time translations or the action of supertranslation with compact support on a massless particle, it is easy to see that supertranslations at $\mathcal{I}^{+}$and $\mathcal{I}^{-}$are not independent but have to be matched in some non-trivial manner. By using
results from the work of Christodoulou and Klainermann, Strominger has argued that the appropriate matching condition is antipodal matching. This in turn fixes the angular momentum aspect appearing in the subleading term of the BMS-metric up to an integration constant. One can then show that this integration constant has to similarly obey antipodal matching as well by showing its equivalence to a recently found subleading soft graviton theorem.

### 1.5 BMS charges

Via Noether's theorem, the infinite number of symmetry transformations at $\mathcal{I}$ lead to an infinite number of conserved charges, which we consider here. As noted before, a global time transformation corresponds to a supertranslation given by

$$
f(z, \bar{z})=\text { constant }=g(z, \bar{z})
$$

The corresponding conserved charge is given by global energy conservation. If we allow $f$ and $g$ to depend non-trivially on angular coordinates $z$ and $\bar{z}$, the corresponding conserved charge is a 'localized' version of energy conservation. The rigorous treatment of conserved charges arising from diffeomorphisms in general relativity is known as the covariant canonical formalism [32] 33]. The reason that this formalism is needed is because more familiar treatments, such as the one due to Komar, do not take into account the conserved charge that is radiated away. As the covariant canonical formalism is rather involved, we follow Strominger [23] by using an analogy with electromagnetism and some of the same simplifying assumptions which we use throughout this chapter. In particular, we assume the null energy condition holds, that the Bondi mass aspect goes to zero at $i^{ \pm}$, and that the radiative energy goes to zero at the boundaries of $\mathcal{I}$. These conditions hold for any Christodoulou-Klainermann space-time; we used these in section 1.4 to derive antipodal matching for such space-times. These conditions ensure that the conserved charge at $\mathcal{I}^{+}\left(\mathcal{I}^{-}\right)$can be found from the physical data at $\mathcal{I}_{-}^{+}$ $\left(\mathcal{I}_{+}^{-}\right)$.

We would thus like to calculate the conserved charge corresponding to a Killing vector $\xi^{a}$. Killing vectors are is implicitly defined by Killing's equation

$$
\begin{equation*}
\nabla^{(a} \xi^{b)}=0 \tag{57}
\end{equation*}
$$

which entails that $\nabla^{a} \xi^{b}$ is antisymmetric. The role of the electromagnetic field strength tensor $F^{a b}$ will be played by $\nabla^{a} \xi^{b}$ in the electromagnetic-gravitational analogy. Namely, the electric charge inside a volume $V$ is given by

$$
\begin{equation*}
Q_{V}=\int_{V} d * F=\frac{1}{e^{2}} \int_{\partial V} * F=\int_{\partial V} F^{a b} \epsilon_{a b c d} d x^{c} d x^{d} \tag{58}
\end{equation*}
$$

where Stokes' theorem was used in the second equality. The corresponding expression for a Killing vector $\xi^{a}$ is given by

$$
\begin{equation*}
Q_{\xi}=-\frac{1}{8 \pi} \int_{\partial V} \nabla^{a} \xi^{b} \epsilon_{a b c d} d x^{c} d x^{d} \tag{59}
\end{equation*}
$$

If $\xi$ is a constant time-like vector, $Q_{\xi}$ is simply (proportional to) the Komar mass. We apply (59p to (13), where we take $S^{2}$ to be the sphere at $\mathcal{I}_{-}^{+}$and $\xi$ to be a supertranslation, given by (35). That is, we calculate

$$
\begin{equation*}
Q_{\xi}=-\frac{1}{8 \pi} \int_{\mathcal{I}_{-}^{+}} d^{2} z \gamma_{z \bar{z}} r^{2} \nabla^{[u} \xi^{r]} \tag{60}
\end{equation*}
$$

Plugging equation 35 into $\sqrt[60]{ }$ and using the fact that, to leading order in $1 / r$

$$
\nabla^{[u} \xi^{r]}=-\nabla_{r} \xi^{r}-\nabla_{u} \xi^{u}+\nabla_{r} \xi^{u}=-\frac{2}{r^{2}} f m_{B}
$$

we find

$$
\begin{align*}
& Q_{f}^{+}=\frac{1}{4 \pi} \int_{\mathcal{I}_{-}^{+}} d^{2} z \gamma_{z \bar{z}} f m_{B} \\
& Q_{g}^{-}=\frac{1}{4 \pi} \int_{\mathcal{I}_{-}^{+}} d^{2} z \gamma_{z \bar{z}} f m_{B}^{-} \tag{61}
\end{align*}
$$

where the bottom equality is found by repeating the calculation at $\mathcal{I}^{-}$.
which are the supertranslation charges found in [34]. Using (50) and (51), we see that

$$
\begin{equation*}
Q_{f}^{+}=Q_{g}^{-} \tag{62}
\end{equation*}
$$

We will use (51) to parametrize supertranslations at $\mathcal{I}^{+}$and $\mathcal{I}^{-}$by the same function $f$, keeping in mind that these functions are related between $\mathcal{I}^{+}$and $\mathcal{I}^{-}$by the antipodal map. Integrating by parts and using the fact that $\left.m_{B}\right|_{\mathcal{I}_{+}^{+}}=0=\left.m_{B}^{-}\right|_{\mathcal{I}_{-}^{-}}$and $\left.N_{z z}\right|_{\mathcal{I}_{+}^{+}}=0=\left.N_{z z}\right|_{\mathcal{I}_{-}^{-}}$for Christodoulou-Klainermann spaces, we find 13

$$
\begin{align*}
& Q_{f}^{+}=\frac{1}{4 \pi G} \int_{\mathcal{I}^{+}} d u d^{2} z \gamma_{z \bar{z}} f\left[T_{u u}-\frac{1}{4}\left(D_{z}^{2} N^{z z}+D_{\bar{z}}^{2} N^{\bar{z} \bar{z}}\right)\right] \\
& Q_{f}^{-}=\frac{1}{4 \pi G} \int_{\mathcal{I}^{-}} d v d^{2} z \gamma_{z \bar{z}} f\left[T_{v v}+\frac{1}{4}\left(D_{z}^{2} M^{z z}+D_{\bar{z}}^{2} M^{\bar{z} \bar{z}}\right)\right]
\end{align*}
$$

We see that, in the absence of gravitational waves, the conserved quantity is the energy flux at $\mathcal{I}$ modulated by some function $f$. In addition to the familiar term coming from the energy-momentum tensor, we have a term which is linear in the Bondi news. We can rewrite this as the following limit

$$
\begin{align*}
\int_{\mathcal{I}^{+}} d u d^{2} z \gamma_{z \bar{z}} f\left(D_{z}^{2} N^{z z}+D_{\bar{z}}^{2} N^{\bar{z} \bar{z}}\right) & =\lim _{\omega \rightarrow 0} \frac{1}{2} \int_{\mathcal{I}^{+}} d u d^{2} z \gamma_{z \bar{z}}\left[\left(e^{i \omega u}+e^{-i \omega u}\right)\left(N^{z z} D_{z}^{2} f+N^{\bar{z} \bar{z}} D_{\bar{z}}^{2} f\right)\right] \\
& =\lim _{\omega \rightarrow 0} \int_{\mathcal{I}_{-}^{+}} d^{2} z \gamma_{z \bar{z}}\left(C^{z z} D_{z}^{2} f+C^{\bar{z} \bar{z}} D_{\bar{z}}^{2} f\right) \tag{64}
\end{align*}
$$

We see that the term in 64 is the soft $(\omega \rightarrow 0)$ limit of a metric perturbation with a polarization tensor $\sim D_{z}^{2} f[13]$. Let us consider a supertranslation given by $f=\delta^{2}(z-w)$, which gives

$$
\begin{equation*}
\int_{\mathcal{I}^{+}} d u \gamma_{w \bar{w}}\left[T_{u u}-\frac{1}{4}\left(D_{w}^{2} N^{w w}+D_{\bar{w}}^{2} N^{\bar{w} \bar{w}}\right)\right]=\int_{\mathcal{I}^{-}} d v \gamma_{w \bar{w}}\left[T_{v v}+\frac{1}{4}\left(D_{w}^{2} M^{w w}+D_{\bar{w}}^{2} M^{\bar{w} \bar{w}}\right)\right] \tag{65}
\end{equation*}
$$

In the words of Strominger, the conserved charge is "energy [...] at every angle" (page 17 of [13]). We see that, in the absence of gravitational waves, this is indeed the case. However, it seems a bit imprecise to refer to the terms linear in the Bondi news as "energy", hence we will refrain from doing so and instead refer to $Q_{f}^{ \pm}$as supertranslation charge. Let us consider scattering events with non-zero transversal momentum transfer, so that the sources of the energy-momentum tensors change direction. For such cases, it is not at all intuitively clear that $Q_{f}^{-}$and $Q_{f}^{+}$should be antipodally matched. This is where the soft graviton terms in come into play. Note that the covariant derivatives kill the $\ell=0$ - terms in the expansion of $N_{z z}$ and $M_{z z}$ in spherical waves, so that integrating them over the twosphere with constant $f$ gives zero. However, they can have local non-zero contributions on $S^{2}$ which may be positive or negative. These soft terms ensure that 65 holds at all angles on $S^{2}$. Strominger and collaborators have shown that the Ward identity corresponding to supertranslation invariance is equivalent to Weinberg's soft graviton theorem [14], which provides further physical underpinning for the above analysis.

One can derive a similar set of infinite conserved charges corresponding to the superrotations. Namely, from (56), one can show that there is an infinite number of antipodally matched superrotation charges given by

$$
\begin{align*}
& Q_{Y}^{+}=\frac{1}{8 \pi} \int_{\mathcal{I}_{-}^{+}} d^{2} z \gamma_{z \bar{z}}\left(Y^{z} N_{z}+Y^{\bar{z}} N_{\bar{z}}\right) \\
& \| \\
& Q_{Y}^{-}=\frac{1}{8 \pi} \int_{\mathcal{I}_{+}^{-}} d^{2} z \gamma_{z \bar{z}}\left(Y^{z} N_{z}+Y^{\bar{z}} N_{\bar{z}}\right) \tag{66}
\end{align*}
$$

It has recently been shown that the Ward identities corresponding to antipodally matched superrotation charge conservation are equivalent to a subleading soft graviton theorem [35] [36, [37, which indicates that they have physical significance. However, we saw that superrotations locally violate the BMS fall-off conditions, hence their precise interpretation is as of yet unclear.

To summarize, by considering the metric at $\mathcal{I}$ of asymptotically flat space-times, one can derive an infinite-dimensional extension of the Poincare symmetries of Minkowski space, given by angledependent translations of the advanced and retarded time coordinates. These transformations, known as supertranslations, have a non-trivial effect on the physical data. This requires a matching condition between $\mathcal{I}^{-}$and $\mathcal{I}^{+}$for the combined transformation to be a symmetry of space-time. The appropriate matching condition is found to be antipodal matching. From this, we find antipodally matched supertranslation charge conservation. A similar analysis holds for localized versions of Lorentz transformations, known as superrotations, for which the corresponding Ward identity is equivalent to a subleading soft graviton theorem. However, as supertranslations locally violate the asymptotic falloff conditions, their interpretation is less clear. The BMS-formalism is rather general, in particular, it holds for a large class of physical situations, including situations where one has highly non-linear effects including black hole formation and evaporation in the interior. In the following sections, we will consider an examples of gravitational scattering and examine how they are acted on by BMS-like transformations.

## 2 The shock wave scattering matrix

Some of the most explicit examples of gravitational scattering events are provided by the shock wave S-matrix due to 't Hooft. This S-matrix takes into account the longitudinal drag induced by the shock wave of highly energetic particles while ignoring transverse momentum transfer. This approximation is appropriate for scattering events where the transverse distance between particles is typically much larger than their longitudinal wavelengths, as is the case at the event horizon of a black hole. We will briefly consider the construction of gravitational shock waves after which we treat the derivation of the shock wave S-matrix in Minkowski space and at the event horizon of a Schwarzschild black hole.

### 2.1 Constructing gravitational shock waves

There are multiple methods for constructing shock waves in general relativity. In particular the "cut-and-paste" approach due to Penrose [38] [39] and the boosted black hole construction due to Aichelburg and Sexl 40, which were then generalized by Dray and 't Hooft to include backgrounds of non-constant curvature [2]. We will focus here on the cut-and-paste method. We take a background space-time, denoted by $\mathcal{M}$, which we divide along a null hypersurface into two patches, denoted by $\mathcal{M}^{+}$and $\mathcal{M}^{-}$. We then apply a 'warp', given by a translation of one of the light-cone coordinates parametrized by a function of the transversal coordinates, to one patch of the space-time, say $\mathcal{M}^{+}$. The two patches are then glued back together, which introduces non-zero energy-momentum at the hyperplane. The cut-and-paste method is illustrated in figure 3. Let us consider this construction in Minkowski space. We write the Minkowski metric in Cartesian light-cone coordinates as

$$
d s^{2}=-d \bar{u} d \bar{v}+d \zeta d \bar{\zeta}
$$

where

$$
\bar{v}=t+z, \quad \bar{u}=t-z, \quad \zeta=x+i y
$$

We cover the two patches $\mathcal{M}^{+}$and $\mathcal{M}^{-}$by coordinate charts given by ( $u, v, \zeta, \bar{\zeta}$ ) and $\left(u^{\prime}, v^{\prime}, \zeta^{\prime}, \bar{\zeta}^{\prime}\right)$, respectively. Let us consider a situation where $u^{\prime}=u$ and $\zeta^{\prime}=\zeta$ but $v^{\prime}=v+F(\zeta, \bar{\zeta})$, where $F(\zeta, \bar{\zeta})$ is known as the the warp function, which is equivalent to the shift function we will encounter in the Dray-'t Hooft construction to be considered below. This function tells us how the advanced time coordinate $v$ is shifted as we go from $\mathcal{M}^{-}$to $\mathcal{M}^{+}$. We place the boundary between $\mathcal{M}^{+}$and $\mathcal{M}^{-}$at $u=0$. The Penrose junction condition is the condition that the coordinates are continuous at the boundary, which can be written as

$$
\begin{equation*}
(u=0, v, \zeta, \bar{\zeta})_{\mathcal{M}^{-}}=(u=0, v+F(\zeta, \bar{\zeta}), \zeta, \bar{\zeta})_{\mathcal{M}^{+}} \tag{67}
\end{equation*}
$$

The metric for the full space is then given by

$$
\begin{equation*}
d s_{0}^{2}=-d v d u-d u \theta(u)\left(\partial_{\zeta} F d \zeta+\partial_{\bar{\zeta}} F d \bar{\zeta}\right)+d \zeta d \bar{\zeta} \tag{68}
\end{equation*}
$$

We perform the following coordinate transformation

$$
v \rightarrow v-\theta(u) F
$$

which gives

$$
\begin{equation*}
d s^{2}=d s_{0}^{2}-\delta(u) F d u^{2} \tag{69}
\end{equation*}
$$

For the case of a single particle located at $u=0=\zeta$, we have an energy-momentum tensor of the form

$$
T_{\mu \nu}=\mu \delta(u) \delta^{(2)}(\zeta)\left(\partial_{u}\right)_{\mu}\left(\partial_{u}\right)_{\nu}
$$

By using the fact that Einstein's equations are given by

$$
\begin{equation*}
\Delta_{h} F=2 \partial_{\zeta} \partial_{\bar{\zeta}} F=8 \pi T_{u u} \tag{70}
\end{equation*}
$$

where $\Delta_{h}$ is the Laplacian of the transversal part of the metric, as well as the fact that $\partial_{\zeta} \partial_{\bar{\zeta}} \ln (\zeta \bar{\zeta})=$ $4 \pi \delta^{(2)}(\zeta)^{1}$ we find that the corresponding shift function is given by

$$
\begin{equation*}
F=2 \mu \ln (\zeta \bar{\zeta}) \delta(u) \tag{71}
\end{equation*}
$$

The resulting metric is know as the AichelBurg-Sexl metric [40]. Aichelburg and Sexl arrived at this metric via a different approach than the cut-and-paste method, namely, by boosting a black hole to the speed of light $(v \rightarrow \infty)$ whilst simultaneously taking its mass to zero $(m \rightarrow 0)$ in such a way that its energy with respect to a stationary observer, $\mu=m\left(1-v^{2}\right)^{-1 / 2}$, is kept constant. We will not consider this approach further since it does not lead to conceptual clarification for the purposes of this thesis.

Following Dray and 't Hooft [2], the cut-and-paste method can be generalized to background metrics with non-constant curvature. Starting from the following metric ansatz

[^0]

Figure 3: In the cut-and-paste method of constructing shock waves, the space-time $\mathcal{M}$ is divided into two patches $\mathcal{M}^{-}$and $\mathcal{M}^{+}$along a null hypersurface. One then applies a warp to one of the two patches, after which the patches are glued back together. This induces a non-zero energy-momentum distribution on the null hypersurface along which the cut was made. Image adapted from 41.

$$
\begin{equation*}
d s_{0}^{2}=2 A(u, v) d u d v+2 g(u, v) h(\zeta, \bar{\zeta}) d \zeta d \bar{\zeta} \tag{72}
\end{equation*}
$$

and following analogous steps to those outlined above, we arrive at

$$
\begin{equation*}
d s^{2}=d s_{0}^{2}-A(u, v) F(\zeta, \bar{\zeta}) \delta(\tilde{u}) d u^{2} \tag{73}
\end{equation*}
$$

where $\tilde{u}$ is the retarded time coordinate of the null hypersurface along which the space-time was cut, which is where the source of the shock wave is located. The metric components and the shift function are required to satisfy

$$
\begin{equation*}
\delta(\tilde{u})\left(\frac{A}{g} \Delta_{h}-\frac{\partial_{u} \partial_{v} g}{g}\right) F=8 \pi T_{u u} \quad,\left.\quad \partial_{v} A\right|_{u=\tilde{u}}=0=\left.\partial_{v} g\right|_{u=\tilde{u}} \tag{74}
\end{equation*}
$$

where $\Delta_{h}$ is again the transversal Laplacian, now no longer necessarily Cartesian. For the case of a shock wave on a Schwarzschild black hole background, the last equality forces the shock wave to propagate along the event horizon. We will return to this point in due time. For the case of Minkowski space, we have

$$
A=-\frac{1}{2}, \quad g=1
$$

so that we recover the shift function of the Aichelburg-Sexl metric 70 .

### 2.2 Derivation of the shock wave S-matrix

We now consider the effect a of a shock wave on massless geodesics and use it to derive a gravitational S-matrix [4]. Labelling the two transversal coordinates by $\Theta_{A} \in(\zeta, \bar{\zeta})$, the massless geodesics of the Aichelburg-Sexl metric in (69) and (70), are given by

$$
\begin{align*}
v(u) & =v_{0}+\theta(u)\left(F+u \partial_{u} \Theta_{A}(u) \partial_{A} F\right) \\
\Theta_{A}(u) & =\Theta_{A}^{0}-\frac{1}{2} u \theta(u) \partial_{A} F \tag{75}
\end{align*}
$$

We would like to consider gravitational scattering for which the transverse distance between interacting particles, which is known as the impact parameter, is very large compared to the longitudinal wavelengths of interacting particles. In the example of an Aichelburg-Sexl metric sourced by a particle at $\zeta=0$, the impact parameter is given by $b:=\sqrt{\zeta \bar{\zeta}}$. For situations where, in natural units, $\frac{\mu}{b} \ll 1$, we can ignore terms in (75) proportional to $\partial_{A} F \sim \frac{1}{b}$. That is, we only take into account the shift that the shock wave induces on other massless particles.

We now consider a gravitational scattering process given by some in-state $\left|\mathrm{in}_{0}\right\rangle$, a scattering matrix $S$, and an out-state given by $\mid$ out $\left._{0}\right\rangle=S\left|\mathrm{in}_{0}\right\rangle$. The amplitude is written as

$$
\begin{equation*}
\left.\mathcal{N}=\left\langle\text { in }_{0}\right| \text { out }_{0}\right\rangle, \tag{76}
\end{equation*}
$$

where $\mathcal{N}$ is some numerical factor of which the absolute value is fixed by requiring unitarity but which has a phase that is in principle arbitrarty. We restrict to situations where the Hilbert space is spanned by the momentum distributions, so that we can write $\left|\mathrm{in}_{0}\right\rangle=\left|p_{\mathrm{in}_{0}}\right\rangle$ and $\mid$ out $\left._{0}\right\rangle=\left|p_{\text {out }_{0}}\right\rangle$. We now use the Aichelburg-Sexl metric to express the effect of momentum perturbations around our previous inand out-states. Namely, if we add a particle with some null-like longitudinal four-momentum $\delta p_{\text {in }}$ at some transverse distance $\zeta$, the position of outgoing particles in our space-time are shifted as

$$
\begin{equation*}
\delta u_{\text {out }}\left(\zeta^{\prime}\right)=F\left(\zeta-\zeta^{\prime}\right) \delta p_{\text {in }}(\zeta) \tag{77}
\end{equation*}
$$

The effect on the out-state is then given by

$$
\begin{equation*}
S\left|p_{\mathrm{in}_{0}}+\delta p_{\text {in }}\right\rangle=\exp \left[i p_{\text {out }} \delta v\right]\left|p_{\text {out }_{0}}\right\rangle . \tag{78}
\end{equation*}
$$

If we now consider adding momenta given by a distribution in the form of $p_{\mathrm{in}}(\zeta)$ to $\left|\mathrm{in}_{0}\right\rangle$, the effect on the scattering process is [4]

$$
\begin{equation*}
\langle\text { out }| S \mid \text { in }\rangle=\left\langle\text { out }_{0}\right| S\left|\mathrm{in}_{0}\right\rangle \exp \left[4 i \int d^{2} \zeta^{\prime} p_{\text {out }}\left(\zeta^{\prime}\right) \ln \left|\zeta^{\prime}-\zeta\right| p_{\text {in }}(\zeta)\right] \tag{79}
\end{equation*}
$$

Since we assume that the Hilbert space is completely spanned by the momentum distributions $p_{\text {in }}(\zeta)$ and $p_{\text {out }}(\zeta)$, we can write a unitary scattering matrix as

$$
\begin{equation*}
\left\langle p_{\text {out }}\left(\zeta^{\prime}\right)\right| S\left|p_{\text {in }}(\zeta)\right\rangle=\mathcal{N} \exp \left[4 i \int d^{2} \zeta^{\prime} p_{\text {out }}\left(\zeta^{\prime}\right) \ln \left|\zeta^{\prime}-\zeta\right| p_{\text {in }}(\zeta)\right] \tag{80}
\end{equation*}
$$

$u_{\text {in }}$ and $u_{\text {out }}$ are the positions canonically conjugate to $p_{\text {in }}$ and $p_{\text {out }}$, respectively. We set these to zero prior to the insertion of in- and out-momenta i.e. $u_{\mathrm{in}}^{0}=0=u_{\text {out }}^{0}$. $u_{\mathrm{in}}$ and $u_{\text {out }}$ are then given in terms of $p_{\text {out }}$ and $p_{\text {in }}$ by

$$
\begin{align*}
u_{\text {out }}(\zeta) & =-4 \int d^{2} \zeta^{\prime} p_{\text {in }}\left(\zeta^{\prime}\right) \ln \left|\zeta-\zeta^{\prime}\right| \\
u_{\text {in }}(\zeta) & =4 \int d^{2} \zeta^{\prime} p_{\text {out }}\left(\zeta^{\prime}\right) \ln \left|\zeta-\zeta^{\prime}\right| \tag{81}
\end{align*}
$$

From the canonical commutation relations, given by

$$
\left[u_{\text {in }}(\zeta), p_{\text {in }}\left(\zeta^{\prime}\right)\right]=i \delta^{(2)}\left(\zeta-\zeta^{\prime}\right)=\left[u_{\text {out }}(\zeta), p_{\text {out }}\left(\zeta^{\prime}\right)\right]
$$

we find that the shift effect induced by the shock wave gives rise to additional non-trivial commutation relations, given by

$$
\begin{align*}
{\left[u_{\text {in }}(\zeta), u_{\text {out }}\left(\zeta^{\prime}\right)\right] } & =-4 i \ln \left|\zeta-\zeta^{\prime}\right| \\
{\left[p_{\text {in }}(\zeta), p_{\text {out }}\left(\zeta^{\prime}\right)\right] } & =-\frac{i}{8 \pi} \partial_{\zeta} \partial_{\bar{\zeta}} \delta^{(2)}\left(\zeta-\zeta^{\prime}\right) \tag{82}
\end{align*}
$$

For notational convenience, we write $u_{\mathrm{in}}=x^{+}$and $u_{\mathrm{out}}=x^{-}$. The in- and out-momenta are then given in terms of in- and out-going energy-momentum tensors as

$$
\begin{align*}
p_{\text {in }}(\zeta) & =\left.\int T_{++}\left(x^{+}, x^{-}, \zeta\right) d x^{+}\right|_{x^{-}=0} \\
p_{\text {out }}(\zeta) & =-\left.\int T_{--}\left(x^{+}, x^{-}, \zeta\right) d x^{-}\right|_{x^{+}=0} \tag{83}
\end{align*}
$$

Using the fact that the shock wave metric in 73 is in Kerr-Schild form, Aichelburg and Balasin 42 ] [43] have shown that $p_{\text {in }}$ and $p_{\text {out }}$ are the Bondi momenta of the flat space shock wave, the future and
past limits of which are equal to the supertranslation charge at $\mathcal{I}^{-}$and $\mathcal{I}^{+}$, respectively. We review the Kerr-Schild decomposition and the computation of the Bondi momentum of the Aichelburg-Sexl shock wave in appendix A.

Let us now consider the metric of the Schwarzschild black hole, which, in tortoise coordinates is given by

$$
\begin{equation*}
d s^{2}=\left(1-\frac{2 M}{r}\right)\left(-d t^{2}+\left(d r^{*}\right)^{2}\right) r^{2}\left(d \theta^{2}+\sin ^{2} \theta d \varphi^{2}\right) \tag{84}
\end{equation*}
$$

where

$$
r^{*}:=r+2 M \ln \left(\frac{r}{2 M}-1\right)
$$

The equation of motion of a massless scalar field $\phi$ is given in these coordinates by

$$
\begin{equation*}
\partial_{t}^{2} \phi=\left[\frac{1}{4 M}\left(\partial_{r^{*}}\right)^{2}+\frac{2 M}{r^{2}} \partial_{r^{*}}+\frac{1}{r^{2}}\left(1-\frac{2 M}{r}\right) \ell(\ell+1)\right] \phi . \tag{85}
\end{equation*}
$$

We thus see that the term coming from angular momentum goes to zero as we approach the event horizon at $r=2 M$. We thus get a separation between longitudinal and transversal length scales as we approach the horizon, so that we can ignore angular momentum as we approach the horizon and instead focus on the longitudinal shift that is induced by the shock waves of highly energetic particles, as we did for the case of Minkowski space. We now go to Kruskal coordinates, which are defined as

$$
\begin{align*}
V & :=e^{\left(r^{*}+t\right) / 4 M} \\
U & :=-e^{\left(r^{*}-t\right) / 4 M} \tag{86}
\end{align*}
$$

$r$ and $t$ are implicitly given in terms of $U$ and $V$ by

$$
\begin{align*}
U V & =-\left(\frac{r}{2 M}-1\right) e^{r / 2 M} \\
V / U & =e^{t / 2 M} \tag{87}
\end{align*}
$$

The Schwarzschild metric is given in these coordinates by

$$
\begin{equation*}
d s^{2}=-\frac{32 M^{3}}{r} e^{-r / 2 M} d V d U+r^{2}\left(d \theta^{2}+\sin ^{2} \theta d \varphi^{2}\right) \tag{88}
\end{equation*}
$$

i.e. $A=-\frac{16 M^{3}}{r} e^{-r / 2 M}$ and $g=r^{2}$ in metric ansatz 74 . From the first equality of 87) we see that

$$
\partial_{V} r=-\frac{4 M^{2} U e^{-r / 2 M}}{r}=\frac{U}{V} \partial_{U} r,
$$

so that $\partial_{V} r \propto U$ and $\partial_{U} r \propto V$. From the last equality of (74), we have

$$
\begin{equation*}
\partial_{V} g=2 r \partial_{V} r=-8 M^{2} U e^{-r / 2 M}, \tag{89}
\end{equation*}
$$

which is required to equal zero at the $U$-coordinate of the shock wave. We thus see that a shock wave travelling along the $V$-direction is required to lie at $U=0$, the converse holds for a shock wave travelling along the $U$-direction. The future (past) event horizon of the Schwarzschild black hole is located at $U=0(V=0)$, so that the shock wave travels along the horizon. As opposed to the Aichelburg-Sexl metric, one therefore cannot take a meaningful limit to $\mathcal{I}$, which complicates comparison with the BMS-formalism. We compute

$$
\partial_{U} \partial_{V} g=-8 M^{2}\left(1-\frac{U}{2 M} \partial_{U} r\right) e^{-r / 2 M}=-\frac{16 M^{3}}{r} e^{-r / 2 M}=A,
$$

where we used the first equality in (87). From the first equality of (74), the shift function is then required to satisfy

$$
\begin{align*}
\left(D^{2}-1\right) F\left(\Omega, \Omega^{\prime}\right) & =\left.32 \pi P(g A)\right|_{U=0} \delta^{(d-2)}\left(\Omega, \Omega^{\prime}\right) \\
& =:-\tilde{\kappa} \delta^{(d-2)}\left(\Omega, \Omega^{\prime}\right), \quad \tilde{\kappa}:=2^{10} \pi e^{-1} M^{4} P, \tag{90}
\end{align*}
$$

where we use spherical coordinates $\Omega=(\theta, \varphi)$ and with corresponding Laplacian $\Delta_{\Omega}=D^{2}$. We then define

$$
\begin{array}{ll}
U^{+}:=\alpha R V, & U^{-}:=\beta R U, \\
P^{+}:=\gamma \frac{P_{\text {out }}}{R^{3}}, & P^{-}:=\delta \frac{P_{\text {in }}}{R^{3}} . \tag{91}
\end{array}
$$

Here, $\alpha, \beta, \gamma, \delta \in\{+,-\}$, so that these coordinates cover both patches of our maximally extended spacetime, indicated in figure 4 Namely, for positive $\alpha$ and negative $\beta, U^{ \pm}$cover region $I$, while, for inverted signs, $U \pm$ cover region $I I$. As quantum operators, the positions and momenta satisfy

$$
\begin{equation*}
\left[U^{ \pm}(\Omega), P^{\mp}\left(\Omega^{\prime}\right)\right]=i \delta^{(2)}\left(\Omega-\Omega^{\prime}\right), \tag{92}
\end{equation*}
$$

where $U^{ \pm}$and $P^{ \pm}$are to be interpreted as their corresponding operators henceforth. Defining
$\tilde{F}\left(\Omega, \Omega^{\prime}\right)=\frac{F\left(\Omega, \Omega^{\prime}\right)}{\tilde{\kappa}}$ and writing the Schwarzschild radius as $R M$, the effect of the shift is given by $[4$

$$
\begin{align*}
& V(\Omega)=-8 \pi R^{2} \int d^{d-2} \Omega^{\prime} \tilde{F}\left(\Omega, \Omega^{\prime}\right) P_{\mathrm{out}}\left(\Omega^{\prime}\right), \\
& U(\Omega)=8 \pi R^{2} \int d^{d-2} \Omega^{\prime} \tilde{F}\left(\Omega, \Omega^{\prime}\right) P_{\mathrm{in}}\left(\Omega^{\prime}\right), \tag{93}
\end{align*}
$$

where we distinguish between particles going into and coming out of the event horizon, with corresponding momenta $P_{\text {in }}$ and $P_{\text {out }}$, respectively. As in the case of Minkowski space, the fact that the insertion of momenta induces a shift in the positions leads to an additional set of commutation relations

$$
\begin{equation*}
\left[U^{+}(\Omega), U^{-}\left(\Omega^{\prime}\right)\right]=i \tilde{F}\left(\Omega-\Omega^{\prime}\right) . \tag{94}
\end{equation*}
$$

We expand $U^{ \pm}$and $P^{ \pm}$in spherical harmonics:

$$
\begin{equation*}
U^{ \pm}(\Omega)=\sum_{\ell, m} U_{\ell m}^{ \pm} Y_{\ell m}(\Omega), \quad P^{ \pm}(\Omega)=\sum_{\ell, m} P_{\ell m}^{ \pm} Y_{\ell m}(\Omega) \tag{95}
\end{equation*}
$$

The coefficients in the above expansion satisfy

$$
\begin{equation*}
\left[U_{\ell, m}^{ \pm}, P_{\ell^{\prime}, m^{\prime}}^{\mp}\right]=i \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}}, \tag{96}
\end{equation*}
$$

which follows immediately plugging (95) into (94). Expanding both sides of (93) and using (90) gives

$$
\begin{aligned}
\left(D^{2}-1\right) \sum_{\ell, m} U_{\ell m}^{ \pm} Y_{\ell m}(\Omega) & = \pm 8 \pi R^{-2} \int d^{d-2} \Omega^{\prime} \delta\left(\Omega, \Omega^{\prime}\right) \sum_{\ell^{\prime}, m^{\prime}} P_{\ell^{\prime} m^{\prime}}^{ \pm}, Y_{\ell^{\prime} m^{\prime}}\left(\Omega^{\prime}\right) \\
-\sum_{\ell, m}\left(\ell^{2}+\ell+1\right) U_{\ell m}^{ \pm} Y_{\ell m}(\Omega) & = \pm 8 \pi R^{-2} \sum_{\ell^{\prime}, m^{\prime}} P_{\ell^{\prime} m^{\prime}}^{ \pm} Y_{\ell^{\prime} m^{\prime}}(\Omega)
\end{aligned}
$$

We thus arrive at a very elegant expression relating $U^{ \pm}$to $P^{ \pm}[17]$

$$
\begin{equation*}
U_{\ell m}^{ \pm}=\mp \frac{8 \pi}{R^{2}\left(\ell^{2}+\ell+1\right)} P_{\ell m}^{ \pm}=: \mp \lambda P_{\ell m}^{ \pm} \tag{97}
\end{equation*}
$$

The additional commutation relations can then be succinctly expressed as

$$
\begin{align*}
{\left[U_{\ell, m}^{+}, U_{\ell^{\prime}, m^{\prime}}^{-}\right] } & =i \lambda \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}} \\
{\left[P_{\ell, m}^{+}, P_{\ell^{\prime}, m^{\prime}}^{-}\right] } & =-\frac{i}{\lambda} \delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}} \tag{98}
\end{align*}
$$

We see that the different $\ell$ - and $m$ - modes completely decouple. We now consider a single $(\ell, m)$-pair and drop the subscripts on $U^{ \pm}$and $P^{ \pm}$for notational convenience. From (98), we see that we can write the position operators as $U^{ \pm}= \pm i \lambda \partial_{U \mp}$. Accordingly, the inner products of the eigenstates of $U^{ \pm}$and $P^{ \pm}$are given by

$$
\begin{align*}
\left\langle U^{ \pm} \mid P^{\mp}\right\rangle & =\frac{1}{\sqrt{2 \pi}} \exp \left(i U^{ \pm} P^{ \pm}\right) \\
\left\langle U^{+} \mid U^{-}\right\rangle & =\frac{1}{\sqrt{2 \pi \lambda}} \exp \left(\frac{i U^{+} U^{-}}{\lambda}\right) \\
\left\langle P^{+} \mid P^{-}\right\rangle & =\sqrt{\frac{\lambda}{2 \pi}} \exp \left(i \lambda P^{+} P^{-}\right) \tag{99}
\end{align*}
$$

From the second of the above equalities, we find, for an arbitrary state $|\psi\rangle$

$$
\begin{equation*}
\left\langle U^{-} \mid \psi\right\rangle=\psi^{\text {out }}\left(U^{-}\right)=\int_{-\infty}^{\infty} d U^{+}\left[\left\langle U^{-} \mid U^{+}\right\rangle\left\langle U^{+} \mid \psi\right\rangle\right]=\int_{-\infty}^{\infty} \frac{d U^{+}}{\sqrt{2 \pi \lambda}} \exp \left(-i U^{+} U^{-} / \lambda\right) \psi^{\text {in }}\left(U^{+}\right) \tag{100}
\end{equation*}
$$

Where the 'in' and 'out' superscripts for $\psi$ indicate that the in- and outgoing states are defined in terms of $U^{+}$and $U^{-}$, respectively. We thus see from 100 that the in- and outgoing wavefunctions are related by a simple Fourier transformation. From the fact that the map between $\psi^{\text {out }}$ and $\psi^{\text {in }}$ is a Fourier transformation, which is an invertible operation, it follows that the scattering matrix that relates $\psi^{\text {out }}$ to $\psi^{\text {in }}$ is unitary. We now derive a more explicit form of the scattering matrix to further clarify this point. To do so, we introduce tortoise coordinates $\rho^{ \pm}$as

$$
\begin{equation*}
U^{+}=: \alpha e^{\rho^{+}}, \quad U^{-}=: \beta e^{\rho^{-}} \tag{101}
\end{equation*}
$$

so that $\rho^{ \pm}$is given by

$$
\begin{equation*}
\rho^{ \pm}=\frac{1}{4 M}\left(r^{*} \pm t\right)+\ln (2 M) \tag{102}
\end{equation*}
$$

Demanding that $\psi\left(U^{+}\right)$is normalized to unity gives

$$
\begin{align*}
1 & =\int_{-\infty}^{\infty}\left|\psi\left(U^{+}\right)\right|^{2} d U^{+}=\int_{-\infty}^{0}\left|\psi\left(U^{+}\right)\right|^{2} d U^{+}+\int_{0}^{\infty}\left|\psi\left(U^{+}\right)\right|^{2} d U^{+} \\
& =\sum_{\alpha \in\{+,-\}} \int_{-\infty}^{\infty}\left|\psi\left(\alpha e^{\rho^{+}}\right)\right|^{2} e^{\rho^{+}} d \rho^{+} . \tag{103}
\end{align*}
$$

The same result holds for $\psi\left(U^{-}\right)$i.e. we simply replace $U^{+}, \rho^{+}$, and $\alpha$ in 103 by $U^{-}, \rho^{-}$, and $\beta$. We then define convenient new field variables as

$$
\begin{equation*}
\phi\left(\alpha, \rho^{+}\right)=e^{\rho^{+} / 2} \psi\left(\alpha e^{\rho^{+}}\right), \quad \phi\left(\beta, \rho^{-}\right)=e^{\rho^{-} / 2} \psi\left(\alpha e^{\rho^{-}}\right) . \tag{104}
\end{equation*}
$$

Plugging this into 100 gives

$$
\begin{aligned}
\phi^{\text {out }}\left(\beta, \rho^{-}\right) & =\frac{1}{\sqrt{2 \pi \lambda}} \int_{-\infty}^{\infty} d U^{+} e^{\frac{\rho^{+}+\rho^{-}}{2}} \exp \left(-i U^{+} U^{-} / \lambda\right) \phi^{\text {in }}\left(\alpha, \rho^{+}\right), \\
& =\sum_{\alpha \in\{+,-\}} \int_{-\infty}^{\infty} \frac{d x}{\sqrt{2 \pi}} \exp \left(\frac{x}{2}-i \alpha \beta e^{x}\right) \phi^{\mathrm{in}}\left(\alpha, x+\log \lambda-\rho^{-}\right), \quad x:=\rho^{+}+\rho^{-}-\log \lambda .
\end{aligned}
$$

Hence

$$
\binom{\phi^{\text {out }}\left(+, \rho^{-}\right)}{\phi^{\text {out }}\left(-, \rho^{-}\right)}=\int_{-\infty}^{\infty} d x\left(\begin{array}{ll}
A_{+}(x) & A_{-}(x)  \tag{105}\\
A_{-}(x) & A_{+}(x)
\end{array}\right)\binom{\phi^{\text {in }}\left(+, \rho^{+}\right)}{\phi^{\text {in }}\left(-, \rho^{+}\right)} \quad, \quad A_{ \pm}(x):=\frac{1}{\sqrt{2 \pi}} \exp \left(\frac{x}{2} \mp i e^{x}\right)
$$

We now go to momentum space by writing

$$
\begin{align*}
\phi^{\mathrm{in}}\left( \pm, \rho^{+}\right) & =\int_{-\infty}^{\infty} \frac{d k}{\sqrt{2 \pi}} \tilde{\phi}^{\mathrm{in}}( \pm, k) e^{-i k \rho^{+}} \\
\phi^{\mathrm{out}}\left( \pm, \rho^{-}\right) & =\int_{-\infty}^{\infty} \frac{d k}{\sqrt{2 \pi}} \tilde{\phi}^{\text {out }}( \pm, k) e^{-i k \rho^{-}} . \tag{106}
\end{align*}
$$

Defining $X:=e^{x}$ and using the fact that

$$
\begin{equation*}
\int_{0}^{\infty} d X X^{-\left(\frac{1}{2}+i k\right)} e^{ \pm i X}=\Gamma\left(\frac{1}{2}-i k\right) e^{ \pm \frac{\pi}{4}(2 k+1)} \tag{107}
\end{equation*}
$$

we find that the S -matrix in momentum space for a single $(\ell, m)$-pair is given by [17]

$$
\binom{\tilde{\phi}^{\text {out }}(+, k)}{\tilde{\phi}^{\text {out }}(-, k)}=\underbrace{\frac{e^{-i \pi / 4}}{\sqrt{2 \pi}} \Gamma\left(\frac{1}{2}-i k\right)\left(\frac{8 \pi}{R^{2}\left(\ell^{2}+\ell+1\right)}\right)^{-i k}\left(\begin{array}{cc}
e^{-\frac{1}{2} \pi k} & i e^{\frac{1}{2} \pi k}  \tag{108}\\
i e^{\frac{1}{2} \pi k} & e^{-\frac{1}{2} \pi k}
\end{array}\right)}_{=: S}\binom{\tilde{\phi}^{\mathrm{in}}(+, k)}{\tilde{\phi}^{\text {out }}(-, k)}
$$

By using

$$
\left|\Gamma\left(\frac{1}{2}-i k\right)\right|^{2}=\frac{\pi}{\cosh \pi k}
$$

we see that

$$
\begin{equation*}
S^{\dagger} S=\mathbb{1} \tag{109}
\end{equation*}
$$

that is, the S-matrix is manifestly unitary. Note that both signs of $\alpha$ and $\beta$ contribute to the S-matrix. That is, the S-matrix in 108 has off-diagonal elements which map between external regions $I$ and $I I$ in figure 4 even though regions $I$ and $I I$ are not in causal contact. Hence, a unitary scattering requires demands that we have access to physical data in both external regions. One way to do so is to identify regions $I$ and $I I$ [5] we will consider this point in the next section.

Before we consider the identification of regions $I$ and $I I$, let us look at the symmetries of the black hole S-matrix. These include, of course, the ten Poincaré symmetries. If we fix the center of mass position of the black hole, the only remaining Poincaré symmetries are global rotations and time translations, the latter of which are given by boosts in the radial coordinate. Rotation and time translation symmetry give rise to the conservation of mass and angular momentum, respectively. The statement that classical black holes are, in the absence of electromagnetism, fully characterized by their mass and angular momentum is famously known as the no-hair theorem. The black hole S-matrix, however, has an infinite symmetry group. Namely, for every pair of angular quantum numbers $(\ell, m)$, we can freely perform the following transformation

$$
\begin{align*}
& U_{\ell, m}^{+} \rightarrow a_{\ell, m} U_{\ell, m}^{+} \\
& U_{\ell, m}^{-} \rightarrow \frac{1}{a_{\ell, m}} U_{\ell, m}^{-}, \quad a_{\ell, m} \in \mathbb{R} \tag{110}
\end{align*}
$$

which, in tortoise coordinates, corresponds to

$$
\begin{align*}
\rho_{\ell, m}^{+} & \rightarrow \rho_{\ell, m}^{+}+\ln \left(a_{\ell, m}\right) \\
\rho_{\ell, m}^{-} & \rightarrow \rho_{\ell, m}^{-}-\ln \left(a_{\ell, m}\right) \tag{111}
\end{align*}
$$



Figure 4: The maximally extended conformal (Penrose) diagram of the Schwarzschild black hole, with regions $I$ and $I I$ and $\mathcal{I}^{ \pm}$indicated. As the direction of time is reversed between regions $I$ and $I I$, one could justifiably invert $\mathcal{I}^{-}$and $\mathcal{I}^{+}$in region $I I$; we choose them as such to simplify comparison with Minkowski space. Additionally illustrated is the shift induced by the shock wave of a highly energetic particle close to the horizon, given by the black arrow. We see that outgoing (Hawking) particles, indiciated by colored arrows, are dragged in the direction of the energetic ingoing particle. Particles which enter region $I$ close to the horizon, such as the blue particle above, can be dragged across the horizon and enter region $I I$. This is the physical interpretation of the fact that the black hole S-matrix transfers signals between ragions $I$ and $I I$. Image adapted from [5].

By looking at 105 , we immediately see that the transformation in 111) leaves the S-matrix invariant. In momentum space, this corresponds to

$$
\begin{align*}
& P_{\ell, m}^{+} \rightarrow \frac{1}{\left(a_{\ell, m}\right)} P_{\ell, m}^{+}, \\
& P_{\ell, m}^{-} \rightarrow\left(a_{\ell, m}\right) P_{\ell, m}^{-} . \tag{112}
\end{align*}
$$

The on-shell condition for the momentum reads 4]

$$
\begin{equation*}
2 P_{\ell, m}^{+} P_{\ell, m}^{-}=\tilde{P}_{\ell, m}^{2} \tag{113}
\end{equation*}
$$

where $\tilde{P}_{\ell, m}$ is the transversal momentum corresponding to $(\ell, m)$. We thus see that $\tilde{P}_{\ell, m}$ is conserved under the symmetry transformation (112).

### 2.3 Antipodal identification and particle dragging

As can be seen from equations 105 and 108 , the shock wave $S$-matrix takes particles from region $I$ to $I I$ (and back), hence we need to understand the meaning of region $I I$ in order to interpret shock wave scattering on Schwarzschild black holes. If we assume that we can scatter shock waves from a Schwarzschild black hole unitarily, then all the physical data should, in principle, be accessible to us. We are thus led to identify regions $I$ and $I I$ with a single copy of the external space-time of a black hole. Let us consider the requirements such an identification would need to satisfy. First of all, the metric should be left invariant under this identification, and the identification itself should to be invariant under Lorentz transformation and CPT-inversions. Further, we demand that this identification has no fixed points i.e. no points that are mapped to themselves. This requirement follows from the fact that if we quotient a manifold by an equivalence relation which has a fixed point, the resulting quotient manifold has (naked) conical singularities 44, which would violate the cosmic censorship conjecture [45]. Lastly, the identification has to be an involution, which means that the square of this map has to equal the identity map. The full group of transformations which leaves the Schwarzschild metric invariant is $O(1,3)$, the $(1+3)$-dimensional Lorentz group, which has four connected components. We can then apply a famous result due to Brouwer, which states that every map on an even-dimensional sphere that has no fixed points is homotopy equivalent to the antipodal map 44. We thus see that the action of the identification on the bifurcation sphere has to be the antipodal map, which is given by

$$
\begin{equation*}
(\theta, \varphi) \sim(\pi-\theta, \pi+\varphi) \tag{114}
\end{equation*}
$$

Spherical symmetry precludes any identification that is related to the antipodal map by some nontrivial continuous map. That is, if we were to identify $(\theta, \varphi) \sim(\pi-\theta, \varphi+\pi+\alpha)$ with $\alpha \in \mathbb{R}, \alpha \neq 2 \pi n$, $n \in \mathbb{Z}$, we would break spherical symmetry along the azimuthal angle. The bifurcation sphere is located at $U^{+}=0=U^{-}$; for finite values of $U^{ \pm}$, the requirement that the identification is an involution and that it is invariant under Lorentz transformations leads us to identify $U^{ \pm} \sim-U^{ \pm}$, so that the full identification is given by 5

$$
\begin{equation*}
\left(U^{+}, U^{-}, \theta, \varphi\right) \sim\left(-U^{+},-U^{-}, \pi-\theta, \pi+\varphi\right) \tag{115}
\end{equation*}
$$

In terms of the canonically conjugate momenta, this gives

$$
\begin{equation*}
P^{ \pm}(\theta, \varphi)=-P^{ \pm}(\pi-\theta, \pi+\varphi) \tag{116}
\end{equation*}
$$

We noted in equation 52 that

$$
Y_{\ell, m}(\pi-\theta, \pi+\varphi)=(-1)^{\ell} Y_{\ell, m}(\theta, \varphi)
$$

we see that only the odd $\ell$-modes contribute to expansions of $U_{\ell, m}^{ \pm}$and $P_{\ell, m}^{ \pm}$in spherical harmonics [5]. This entails that the symmetries in (110) and (112) have to antipodally matched as well. This gives rise to a very similar picture to that of the supertranslations at $\mathcal{I}$. Namely, we have an infinite number of symmetries at the event horizon given by independent translations of each $(\ell, m)$-pair of the Kruskal coordinates as in 110 . The event horizon of the Schwarzschild black hole is the light-cone of the bifurcation sphere, where the transformations in (111) have to be antipodally matched for the combined transformation to be a symmetry of the S-matrix. This entails that the physical data is continuous along the null generators of the event horizon even as they pas through the bifurcation sphere. Similarly, at $\mathcal{I}$, we can independently translate each $(\ell, m)$-pair of advanced and retarded time coordinates. $\mathcal{I}$ is the light-cone of $i^{0}$; it is at $i^{0}$ that the translations of advanced and retarded time coordinates have to be antipodally matched for the combined transformation to be a diffeomorphism. This is equivalent to the fact that the physical data is continuous along the generators of $\mathcal{I}$ even as they pass through $i^{0}$. However, these situations are not completely analogous, as the antipodal matching at the event horizon is between $U^{a}>0$ and $U^{a}<0, a \in\{+,-\}$. Nonetheless, this similarity warrants further comparison between the two formalisms, which is the purpose of the remainder of this thesis. It is interesting to note here that the degrees of freedom of the black hole S-Matrix can be represented by an infinite number of inverted harmonic oscillators, one for each partial wave 46]. This gives a map from the boosts at infinity to a unique Hamiltonian for the dilatation operator on the null-horizon. One might wonder whether all the asymptotic symmetries, in particular BMS-symmetries, have such a near-horizon map, as this would provide a very direct link between the two formalisms.

To summarize, we considered the gravitational scattering of massless particles whose longitudinal wavelengths are much smaller than their mutual impact parameter, such as is typically the case close to the event horizon of a black hole. By considering the longitudinal shift induced by the shock wave of an ingoing particle on the positions of outgoing particles, we follow 't Hooft and derive a unitary gravitational S-matrix. In the case of an eternal black hole, the S-matrix has non-zero elements which map between between external regions $I$ and $I I$. Assuming that the physical data in a single external region of a black hole provides a unitary description thus leads us to identify regions $I$ and $I I$. By demanding that this identification leaves the metric invariant, that it is an involution with no fixed points, and that it is invariant under Lorentz- and CPT-transformations, we arrive at antipodal identification of regions $I$ and $I I$.

## 3 Relating the two formalisms

We finished chapter 1 by noting that the BMS-formalism is very general, as it describes a large class of cases including those where one has highly non-linear effects, such as black hole formation and evaporation, in the interior. The price one pays for this is that the description is limited to the boundary of space-time, where gravity is weakly coupled and well-behaved. It is, in general, unclear how to extend the action of BMS-transformations from null infinity into the interior. We would like to see how BMS-transformations act on explicit examples of gravitational scattering, in particular, the shock wave S-matrix described in the previous chapter. The reasons for applying the BMS-formalism to shock wave scattering go beyond the fact that this provides an explicit example, as there are many similarities between the two formalisms. Indeed, in the process of writing this thesis, we found that several allusions to analogies between the two formalisms had previously appeared in the literature. For example, Arcioni and Dappiaggi stated on the BMS-formalism that "The final picture one gets is quite similar to the scenario proposed by 't Hooft [...] in the context of black holes" (page 36 of [15]), citing in particular the fact that the relevant degrees of freedom reside on the boundary of space-time. In another paper, Arcioni points out a similarity between the small angle behaviour of the membrane paradigm Green's function and the shift function at the boundary of a black hole [16]. More recently, Penna has shown that the conserved charges of the membrane paradigm are the same as those arising from BMS-invariance 47, hence Arcioni pointed out another analogy between the BMS-formalism and the shock wave S-matrix, albeit likely unknowingly. Similarly, 't Hooft mentions the apparent close relation between the BMS approach and the black hole S-matrix [17. Lastly, Hawking, Perry, and Strominger (HPS) point out in a footnote of one of their papers [6] the similarity between expressions appearing in the shock wave scattering formalism and the (electromagnetic) BMS-formalism, where they cite discussions on the matter with Polchinski. They state that "Perhaps future work will relate these effects" (footnote 9 of [6]); this is the aim of the remainder of this thesis.

Using an alternative derivation due to the Verlindes [18], we will find that the shock wave S-matrix in Minkowski space is invariant under translations of the asymptotic time coordinate that are allowed to depend on transversal coordinates. This will lead to antipodal matching of in- and outgoing momentum flux between $\mathcal{I}^{+}$and $\mathcal{I}^{-}$, which, in the absence of gravitational wave, corresponds to antipodally matched supertranslation charge. Shock wave scattering in Minkowski space thus provides an an explicit example of recent findings by Strominger and collaborators, which had thus far gone unnoticed in the literature. We then consider the case of an eternal black hole, where there are clear similarities but the overall analogy is much less apparent, in particular since the BMS-formalism is still under development here. We consider the effect of a linearized supertranslation induced by a shock wave as derived by HPS [7] and show that the resulting shift of the event horizon satisfies the same expression as the Dray-'t Hooft shock wave [2]. We then briefly review the derivation of BMS-like symmetries acting on the event horizon of a black hole [19] [20]. We show that the Dray-'t Hooft shock wave in tortoise coordinates satisfies the required fall-off conditions and that it implants a non-zero superrotation charge at the horizon. However, as the black hole S-matrix is based on the shock wave in Kruskal coordinates,
this result is as yet of limited relevance to the S -matrix formalism. We consider how these results could provide a description of shock wave scattering off black holes in terms of BMS-charges and, conversely, what the S-matrix formalism could tell us about the BMS-matching condition in the context of black holes.

### 3.1 Lagrangian approach to shock wave scattering

One can derive the shock wave scattering matrix in Minkowski space using a Lagrangian approach, which is better suited to comparison with the BMS group at null infinity. Due to the fact that the final result is the same as that of section 2.2 , we do not repeat all steps of the derivation here; more details can be found in 18 . We again consider a situation with a hierarchy between longitudinal and transversal scales. If the coordinates are kept dimensionless, the dimension of the metric is (length) ${ }^{2}$. For a system with typical length scale $L$, the corresponding metric scales as

$$
\begin{equation*}
G_{\mu \nu}=L^{2} \hat{G}_{\mu \nu}, \tag{117}
\end{equation*}
$$

where $\hat{G}_{\mu \nu}$ is dimensionless. The dimensionality of the Einstein-Hilbert action $S_{E H}[G]=\int \sqrt{G} R$ is also given by (length) ${ }^{2}$, so that it satisfies

$$
\begin{equation*}
S_{E H}[G]=S_{E H}\left[L^{2} \hat{G}\right]=L^{2} S_{E H}[\hat{G}] . \tag{118}
\end{equation*}
$$

Although, in quantum gravity, we need to integrate over all metrics, the main contribution arises from metrics with a typical size that is comparable to the size of the system. It is therefore natural to expect that $\hat{g}_{\mu \nu}=\mathcal{O}(1)$, so that the coupling constant multiplying $S_{E H}$ is

$$
\begin{equation*}
g(L)=\frac{\ell_{\mathrm{Pl}}}{L} \tag{119}
\end{equation*}
$$

where $\ell_{\mathrm{PI}}$ is the Planck length. This line of reasoning is often used to argue that, when $L \sim \ell_{\mathrm{PI}}$, the system becomes strongly coupled. Consider now a situation where particles with Planckian energies scatter at impact parameters much larges than Planck length. This situation is thus characterized by a longitudinal length scale $\ell_{/ /} \sim \ell_{\mathrm{Pl}}$ and a transversal length scale $\ell_{\perp} \gg \ell_{\mathrm{Pl}}$. We write longitudinal coordinates as $x^{\alpha}=:(t, x)$, the transversal coordinates are written as $\tilde{x}^{i}=(y, z)$. Following [18], we then make the following gauge choice

$$
G_{\mu \nu}=\left(\begin{array}{cc}
g_{\alpha \beta} & 0  \tag{120}\\
0 & h_{i j}
\end{array}\right)
$$

Following the previous discussion, we extract the typical length scales from $g_{\alpha \beta}$ and $h_{i j}$ as follows

$$
\begin{align*}
g_{\alpha \beta} & =\ell_{/ j}^{2} \hat{g}_{\alpha \beta} \\
h_{i j} & =\ell_{\perp}^{2} \hat{h}_{i j} \tag{121}
\end{align*}
$$

A straightforward calculation, done in the appendix of [18], reveals that the Einstein-Hilbert action splits up into a longitudinal and a transversal part, i.e. $\left.S_{E H}[G]=S_{/} / g, h\right]+S_{\perp}[g, h]$, in the following manner

$$
\begin{equation*}
S_{E H}[G]=\underbrace{\int d^{2} x \sqrt{g h}\left(R_{h}+\frac{1}{4} h^{i j} \partial_{i} g_{\alpha \beta} \partial_{j} g_{\gamma \delta} \epsilon^{\alpha \gamma} \epsilon^{\beta \delta}\right)}_{=: S_{l}}+\underbrace{\int d^{2} y \sqrt{g h}\left(R_{g}+\frac{1}{4} g^{\alpha \beta} \partial_{\alpha} h_{i j} \partial_{\beta} h_{k l} \epsilon^{i k} \epsilon^{j l}\right)}_{=: S_{\perp}} . \tag{122}
\end{equation*}
$$

Here, $R_{g}$ and $R_{h}$ are the Ricci scalars corresponding to $g_{\alpha \beta}$ and $h_{i j}$, respectively, and the Levi-Civita symbols $\epsilon^{\alpha \beta}$ and $\epsilon^{i j}$ include appropriate factors of metric determinants so that they transform as tensors. The components of the action are referred to as longitudinal and transversal from their dependence on the corresponding length scales, which is given by

$$
\begin{align*}
S_{/ /}\left[\ell_{/ /}^{2} \hat{g}, \ell_{\perp}^{2} \hat{h}\right] & =\ell_{/}^{2} S_{/ /}[\hat{g}, \hat{h}] \\
S_{\perp}\left[\ell_{/ /}^{2} \hat{g}, \ell_{\perp}^{2} \hat{h}\right] & =\ell_{\perp}^{2} S_{\perp}[\hat{g}, \hat{h}] . \tag{123}
\end{align*}
$$

After dividing the action by $\ell_{\mathrm{Pl}}^{2}$, we see that this scattering regime is characterized by two distinct coupling constants

$$
\begin{gather*}
g_{/ /}:=\frac{\ell_{\mathrm{Pl}}}{\ell_{/ /}} \sim 1 \\
g_{\perp}:=\frac{\ell_{\mathrm{Pl}}}{\ell_{\perp}} \ll 1 . \tag{124}
\end{gather*}
$$

From the fact that $g_{\perp} \ll 1$, we see that $S_{\perp}$ is very weakly coupled, so that we can limit ourselves to consider only configurations with

$$
\begin{equation*}
S_{\perp}=0 \tag{125}
\end{equation*}
$$

We write the metric in a convenient parametrization given by

$$
\begin{align*}
g_{\alpha \beta} & =e^{\phi(X, Y)} \eta_{a b} \partial_{\alpha} X^{a} \partial_{\beta} X^{b} \\
h_{i j} & =e^{\chi(X, Y)} \delta_{p q} \partial_{i} Y^{p} \partial_{j} Y^{q} \tag{126}
\end{align*}
$$

$X^{a}$ can thus be seen as maps of the two-dimensional $x^{\alpha}$ - plane which are allowed to vary in the $y$-direction; a similar interpretation holds for $Y^{p}$. Demanding that equation 125 is satisfied leads to

$$
\begin{align*}
\partial_{\alpha} h_{i j} & =0 \\
R_{g} & =0 . \tag{127}
\end{align*}
$$

The metric which satisfies these conditions is of the form

$$
\begin{align*}
h_{i j} & =h_{i j}(\tilde{x}) \\
g_{\alpha \beta} & =\eta_{a b} \partial_{\alpha} X^{a} \partial_{\beta} X^{b} \tag{128}
\end{align*}
$$

We see from $\partial_{\alpha} h_{i j}=0$ that there are no gravitational waves, as these are time-dependent fluctuations of the transverse traceless part of the metric. The authors of [18] then apply this formalism to Minkowski space, which we denote by $\mathcal{M}$. They thus consider the scattering of Aichelburg-Sexl shock waves, as we considered at the start of section 2.2. One can then show that $S_{/ /}$is equal to a total derivative, so that it reduces to an action at the boundary of $\mathcal{M}$, which we denote by $\partial \mathcal{M}$. For the case of Minkowski space, $\partial \mathcal{M}=\mathcal{I}$. We write the boundary values of $X^{a}$ as

$$
\begin{equation*}
\bar{X}^{a}:=\left.X^{a}\right|_{\partial \mathcal{M}} \tag{129}
\end{equation*}
$$

The boundary action is then given by

$$
\begin{equation*}
S_{\partial \mathcal{M}}[\bar{X}]=\int d x^{\alpha} \int \sqrt{h} \epsilon_{a b}\left(\partial_{i} \bar{X}^{a} \partial_{\alpha} \partial^{i} \bar{X}^{b}+R_{h} \bar{X}^{a} \partial_{\alpha} \bar{X}^{b}\right) \tag{130}
\end{equation*}
$$

The authors of 18 then introduce a 'time' variable $\tau$ to parametrize $x^{\alpha}(\tau)$, which are given by advanced and retarded time at $\mathcal{I}^{-}$and $\mathcal{I}^{+}$, respectively. The boundary action is then given by

$$
\begin{equation*}
S_{\partial M}[\bar{X}]=\int d \tau \int \sqrt{h} \epsilon_{a b} \partial_{\tau} \bar{X}^{a}\left(\Delta_{h}-R_{h}\right) \bar{X}^{b} \tag{131}
\end{equation*}
$$

so that the canonically conjugate momentum of $\bar{X}^{a}$ is

$$
\begin{equation*}
\epsilon^{a b} P_{b}:=\left(\Delta_{h}-R_{h}\right) \partial_{\tau} \bar{X}^{a} \tag{132}
\end{equation*}
$$

Canonical quantization then to the following commutation relation

$$
\begin{equation*}
\left[\bar{X}^{a}(\tilde{x}), \bar{X}^{b}\left(\tilde{x}^{\prime}\right)\right]=i \epsilon^{a b} F\left(\tilde{x}-\tilde{x}^{\prime}\right) \tag{133}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(\Delta_{h}-R_{h}\right) F\left(\tilde{x}-\tilde{x}^{\prime}\right)=\delta^{(2)}\left(\tilde{x}-\tilde{x}^{\prime}\right) \tag{134}
\end{equation*}
$$

By comparing with equation $(74)$ and 81 , we see that these results are the same as those derived previously by 't Hooft [3] [4]. Note that the boundary action is invariant under the transformation

$$
\begin{equation*}
\bar{X}^{a}(\tilde{x}, \tau) \rightarrow \bar{X}^{a}(\tilde{x}, \tau)+f(\tilde{x}) \tag{135}
\end{equation*}
$$

where $f(\tilde{x})$ is an arbitrary function of the transversal coordinates. The fact that this is a symmetry of the theory could already be seen from the fact that it leaves $g_{\alpha \beta}$ in 128 invariant. Note that the symmetry transformation given in 135 is very similar to a supertranslation, as it is a translation of the time coordinate parametrized by a function which is allowed to depend on transversal coordinates. To make this analogy more precise, we consider the conserved charge corresponding to this symmetry. By plugging 135 into 131 and using 132 , we find 18

$$
\begin{equation*}
\int d \tau d^{2} \tilde{x} f(\tilde{x}) P_{a}(\tau, \tilde{x})=0 \tag{136}
\end{equation*}
$$

By choosing $f(\tilde{x})$ to be proportional to $\delta^{(2)}\left(\tilde{x}_{0}\right)$ for some $\tilde{x}_{0}$, this gives

$$
\begin{equation*}
\int d \tau P_{a}^{\text {in }}\left(\tau, \tilde{x}_{0}\right)=-\int d \tau P_{a}^{\text {out }}\left(\tau, \tilde{x}_{0}\right) \tag{137}
\end{equation*}
$$

We now look at how equation (137) gives antipodal matching of supertranslation charge as in 62). The limits $t \rightarrow+\infty$ and $t \rightarrow-\infty$ take us to $\mathcal{I}^{+}$and $\mathcal{I}^{-}$, respectively. We write $x^{ \pm}=t \pm x$, where $x$ is the Cartesian coordinate, as below (120). Then, for a particular value $x_{0}^{-}=$constant, the limits to $\mathcal{I}^{+}$and $\mathcal{I}^{-}$correspond to $x \rightarrow+\infty$ and $x \rightarrow-\infty$, respectively. Choosing our coordinate system such that $x=r \cos \theta, \tilde{x}=0$ corresponds to $\theta=\pi$ for $x<0$ and to $\theta=0$ for $x>0$. Hence, $\int d \tau P_{a}^{\text {in }}(\tau, \tilde{x}=0)=-\int d \tau P_{a}^{\text {out }}(\tau, \tilde{x}=0)$ for $a=+$ gives antipodal matching of ingoing momentum flux at $\theta=\pi$ and outgoing momentum flux at $\theta=\pi$, the converse is found for $a=-$. As there are no gravitational waves generated by shock wave scattering in Minkowski space, as we noted below equation (128), it seems from equations (63) and (83) that the conserved charge in 136 is the supertranslation


Figure 5: The type of scattering considered above concerns the map between in- and outgoing momentum distributions $P_{ \pm}^{\text {in }}$ and $P_{ \pm}^{\text {out }}$. From time-independent reparametrization invariance of the boundary coordinates $\bar{X}^{a}$, it follows that $\int d \tau P_{a}^{\text {in }}(\tau, \tilde{x})=\int d \tau P_{a}^{\text {out }}(\tau, \tilde{x}), \forall \tilde{x}, a \in\{+,-\}$. Image taken from 18.
charge of the Aichelburg-Sexl shock waves which we scatter here. Aichelburg and Balasin have shown that this is indeed the case [42] [43. This calculation, which is somewhat non-trivial due to the distribution-valued and plane-fronted nature of the Aichelburg-Sexl shock wave, is summarized in appendix B

To summarize, we have shown that the antipodal supertranslation charge conservation, found recently by Strominger and collaborators [13] [14, is explicitly realized in the older work of 't Hooft [2] 4], as is evident in later work by the Verlindes [18]. As far as we are aware, this fact has thus far gone unnoticed in the literature.

### 3.2 BMS analysis of Schwarzschild metrics

We would like to extend the above analysis to the case of Schwarzschild black holes. As stated at the start of this chapter, the BMS-formalism is much less understood in this context than in the case of Minkowski space. The main reason for this is that supertranslations act at null infinity; it is generally very complicated to extend their non-linear action into the interior of space-time. Indeed, as emphasised before, the reason for considering the symmetries at null infinity is that gravity becomes weakly coupled here, so that non-linear effects can be ignored. Further, the antipodal matching of BMStransformations has been shown for Christodoulou-Klainermann spaces which, from the requirement that the Bondi mass aspect goes to zero at past and future time-like infinity (47), do not include Schwarzschild black holes. Indeed, as classical massless particles entering space-time at $\mathcal{I}^{-}$will simply disappear into the black hole without leaving a trace at $\mathcal{I}^{+}$, it is not clear how supertranslations should
be matched between $\mathcal{I}^{-}$and $\mathcal{I}^{+}$in the presence of an eternal black hole.
We therefore consider a linearized treatment of BMS-transformations on a Schwarzschild black hole due to Hawking, Perry, and Strominger (HPS) [7. We will see how a shock wave with a non-trivial spherical profile induces a supertranslation and compare its effect on the event horizon to that of a Dray-'t Hooft shock wave. The Schwarzschild metric is written in advanced Bondi coordinates as

$$
\begin{equation*}
d s^{2}=-V d v^{2}+2 d v d r+2 r^{2} \gamma_{A B} d \Theta^{A} d \Theta^{B} \quad, \quad V:=1-\frac{2 M}{r} \tag{138}
\end{equation*}
$$

HPS assume that the Bondi news falls as $\frac{1}{|v|^{3 / 2}}$ or faster, so that $C_{A B}$ remains finite at the future and past boundaries of $\mathcal{I}^{-}$and $\mathcal{I}^{+}$, respectively. They then assume antipodal matching as in equation (50), which would lead to antipodal matching of supertranslations as well. However, no further physical for this antipodal matching is given by HPS, nor, as far as we are aware, anywhere else in the literature. We return to this point at the end of this section. Let us consider an infinetesimal supertranslation parametrized by a vector field $\xi$ which preserves Bondi gauge. This result was derived in section 1.2 and is repeated here for convenience

$$
\begin{equation*}
\xi=\xi^{\mu} \partial_{\mu}=f \partial_{v}+\frac{1}{r} D^{A} f \partial_{A}-\frac{1}{2} D^{2} f \partial_{r}, \quad f=f(\Theta) \tag{139}
\end{equation*}
$$

The linearized action of this supertranslation on the metric components is given by

$$
\begin{align*}
\mathcal{L}_{\xi} g_{v v} & =\frac{M D^{2} f}{r^{2}} \\
\mathcal{L}_{\xi} g_{v A} & =-\frac{1}{2} D_{A}\left(2 V f+D^{2} f\right) \\
\mathcal{L}_{\xi} g_{A B} & =r\left(2 D_{A} D_{B} f-\gamma_{A B} D^{2} f\right) \tag{140}
\end{align*}
$$

so that the supertranslated Schwarzschild metric is

$$
\begin{align*}
d s^{2}+\mathcal{L}_{\xi} d s^{2}= & -\left(V-\frac{M D^{2} f}{r^{2}}\right) d v^{2}+2 d v d r-\left(2 V f+D^{2} f\right) d v d \Theta^{A}+ \\
& +\left(r^{2} \gamma_{A B}+2 r D_{A} D_{B} f-r \gamma_{A B} D^{2} f\right) d \Theta^{A} d \Theta^{B} \tag{141}
\end{align*}
$$

After a supertranslation, the event horizon of a Schwarzschild black hole is thus located at

$$
\begin{equation*}
R+\mathcal{L}_{\xi} R=2 M+\frac{1}{2} D^{2} f \tag{142}
\end{equation*}
$$

Note that a supertranslation does not implant linearized supertranslation charge on a Schwarzschild
black hole, as we still have $m_{B}=M$ in 140 . This is similar to the fact that a regular translation or does not impart momentum on a black hole. However, by comparing with (13), we see that the angular momentum aspect of a supertranslated Schwarzschild black hole is

$$
\begin{equation*}
\mathcal{L}_{f} N_{A}=-3 M \partial_{A} f, \tag{143}
\end{equation*}
$$

so that its linearized superrotation charge, given by expression 66), is [7]

$$
\begin{equation*}
Q_{Y}^{-}=-\frac{3}{8 \pi} \int_{\mathcal{I}_{+}^{-}} d^{2} \Theta \sqrt{\gamma} M Y^{A} \partial_{A} f \tag{144}
\end{equation*}
$$

Therefore, supertranslations implant superrotation charges on a Schwarzschild background. Following HPS [7, we show that a Schwarzschild black hole can be supertranslated by sending in a shock wave with a non-trivial spherical profile. As we saw in the review of the cut-and-paste method, a shock wave can be seen as a hyperplane along which geodesics undergo an instantaneous supertranslation, namely, geodesics are instantaneously shifted along a function of the transversal coordinates. Let us see what the form is of a supertranslation induced by a linearized shock wave on a Schwarzschild black hole background. We consider a shock wave of the following form

$$
\begin{equation*}
T_{v v}=\frac{\mu+\hat{T}(\Theta)}{4 \pi r^{2}} \delta\left(v-v_{0}\right) \tag{145}
\end{equation*}
$$

In order to satisfy $\nabla^{a} T_{a b}=0$, we require $1 / r$-corrections to the energy momentum tensor as

$$
\begin{align*}
T_{v v} & =\frac{1}{4 \pi r^{2}}\left(\mu+\hat{T}+\frac{1}{r} \hat{T}^{(1)}\right) \delta\left(v-v_{0}\right) \\
T_{v A} & =\frac{\hat{T}_{A}}{4 \pi r^{2}} \delta\left(v-v_{0}\right) \tag{146}
\end{align*}
$$

We assume that $\hat{T}$ only has $\ell \geqslant 2$ - components in the expansion in spherical waves, so that its momentum is equal to zero. The $\ell=0$ - term is of course given by $\mu . \hat{T}^{(1)}$ and $\hat{T}_{A}$ are then determined by

$$
\left(D^{2}+2\right) \hat{T}^{(1)}=-6 M \hat{T} \quad, \quad D^{A} \hat{T}_{A}=\hat{T}^{(1)}
$$

We introduce a Green's function implicitly defined as

$$
\begin{equation*}
\frac{1}{4} D^{2}\left(D^{2}+2\right) G(\Theta ; \tilde{\Theta})=\frac{1}{\sqrt{\gamma}} \delta^{(2)}(\Delta \Theta) \tag{147}
\end{equation*}
$$

where $\Delta \Theta$ is the angle between $\Theta^{A}$ and $\tilde{\Theta}^{A}$. This is solved by

$$
\begin{equation*}
G(\Theta ; \tilde{\Theta})=\frac{1}{\pi} \sin ^{2} \frac{\Delta \Theta}{2} \ln \sin ^{2} \frac{\Delta \Theta}{2} \tag{148}
\end{equation*}
$$

We then define a function $\hat{C}(\Theta)$ as

$$
\begin{equation*}
\hat{C}(\Theta):=\int d \tilde{\Theta} G(\Theta ; \tilde{\Theta}) \hat{T}(\Theta) \tag{149}
\end{equation*}
$$

in which the components of the energy momentum tensor are expressed as

$$
\begin{align*}
\hat{T} & =\frac{1}{4} D^{2}\left(D^{2}+2\right) \hat{C} \\
\hat{T}^{(1)} & =-\frac{3 M}{2} D^{2} \hat{C} \\
\hat{T}_{A}^{(1)} & =-\frac{3 M}{2} \partial_{A} \hat{C}, \tag{150}
\end{align*}
$$

so that the final result is given by

$$
\begin{align*}
T_{v v} & =\frac{1}{4 \pi r^{2}}\left[\mu+\frac{1}{4} D^{2}\left(D^{2}+2\right) \hat{C}\right] \delta\left(v-v_{0}\right)-\frac{3 M}{8 \pi r^{3}} D^{2} \hat{C} \delta\left(v-v_{0}\right) \\
T_{v A} & =-\frac{3 M}{8 \pi r^{2}} D_{A} \hat{C} \delta\left(v-v_{0}\right) \tag{151}
\end{align*}
$$

By evaluating $T_{v v}$ at $r=2 M$, we find

$$
\left.T_{v v}\right|_{r=2 M}=\frac{1}{16 \pi M^{2}}\left[\mu+\frac{1}{4} D^{2}\left(D^{2}-1\right) \hat{C}\right] \delta\left(v-v_{0}\right)
$$

which has a similar form to the defining function of the shift equation of the Dray - 't Hooft shock wave in 90. . We will further consider this point below. Solving Einstein's equations at $\mathcal{I}^{-}$then gives, to leading order in $1 / r$

$$
\begin{equation*}
\partial_{v} m=\frac{1}{4} D_{A} D_{B} N^{A B}+(\mu+\hat{T}(\Theta)) \delta\left(v-v_{0}\right) \tag{152}
\end{equation*}
$$

which is the result found in (43). HPS assume that $\partial_{A} m=0$ at all times. They then integrate 152 ) over $S^{2}$ to find

$$
\begin{equation*}
m=M+\mu \theta\left(v-v_{0}\right) . \tag{153}
\end{equation*}
$$

We thus have to solve

$$
\begin{equation*}
D_{A} D_{B} C^{A B}=-4 \hat{T}\left(\Theta^{A}\right) \theta\left(v-v_{0}\right)=-D^{2}\left(D^{2}+2\right) \hat{C} \theta\left(v-v_{0}\right) \tag{154}
\end{equation*}
$$

We use the fact that covariant derivatives on the unit sphere act as

$$
\begin{equation*}
\left[D^{B}, D_{A}\right] V_{B}=V_{A} \tag{155}
\end{equation*}
$$

for general vector $V_{B}$ on $S^{2}$. The solution is then given by

$$
C_{A B}=-2 \theta\left(v-v_{0}\right)\left(D_{A} D_{B} \hat{C}-\frac{1}{2} \gamma_{A B} D^{2} \hat{C}\right)
$$

From the tracelessness of $C_{A B}$, we easily see that

$$
\begin{align*}
\frac{1}{8} N_{A B} N^{A B} & =\frac{1}{2}\left(D_{A} D_{B} \hat{C}-\frac{1}{2} \gamma_{A B} D^{2} \hat{C}\right)\left(D^{A} D^{B} \hat{C}-\frac{1}{2} \gamma^{A B} D^{2} \hat{C}\right)\left(\delta\left(v-v_{0}\right)\right)^{2} \\
& =\frac{1}{2}\left(\left(D_{A} D_{B} \hat{C}\right)\left(D^{A} D^{B} \hat{C}\right)\right)\left(\delta\left(v-v_{0}\right)\right)^{2} \tag{156}
\end{align*}
$$

Hence the square of the Bondi news gives a square of $\delta\left(v-v_{0}\right)$, which is not well-defined even in a distributional sense. Interestingly, $\delta^{2}$-terms also appear in the calculation of the shock wave metric by Dray and 't Hooft [2], where one can show by replacing them with the limit of a Gaussian function that their effect on the metric is trivial. This point is not considered by HPS, we will not consider it here either but defer it to future work. We write the linearized metric variations as $\delta g_{a b}=h_{a b}$. For (151), the solutions of the linearized Einstein equations give

$$
\begin{align*}
h_{v v} & =\theta\left(v-v_{0}\right)\left(\frac{2 \mu}{r}-\frac{M D^{2} \hat{C}}{r^{2}}\right) \\
h_{v A} & =\theta\left(v-v_{0}\right) \partial_{A}\left(1-\frac{2 M}{r}+\frac{1}{2} D^{2}\right) \hat{C} \\
h_{A B} & =-2 r \theta\left(v-v_{0}\right)\left(D_{A} D_{B} \hat{C}-\frac{1}{2} \gamma_{A B} D^{2} \hat{C}\right) \tag{157}
\end{align*}
$$

we consider the general treatment of linearized Einstein equations for linearized perturbations below, but let us first consider the form of these particular solutions. Namely, by comparing (157) with the supertranslated Schwarzschild metric, given by equation (140), HPS show that the effect of a linearized shock wave in 151 is a change of the black hole mass by $\mu$ and a supertranslation parametrized by a function $f=-C$. That is, send in a spherical shock wave given by

$$
T_{v v}=\frac{\mu \delta\left(v-v_{0}\right)}{4 \pi r^{2}}
$$

which brings the Schwarzschild metric to a metric of the following form

$$
\begin{equation*}
d s^{2}=-\left(1-\frac{2 M+\theta\left(v-v_{0}\right) \mu}{r}\right) d v^{2}+2 d v d r+2 r^{2} \gamma_{A B} d \Theta^{A} d \Theta^{B} \tag{158}
\end{equation*}
$$

We act on this space with a linearized supertranslation, of which the radial component is given by $-\frac{1}{2} D^{2} f$, hence a supertranslation radially translates the event horizon, as well as the shock wave profile, by said term. Note that $D^{2}$ projects out the $\ell=0$ - components of the expansion of $f$ in spherical harmonics. As before, we assume that $f$ has no $\ell=1$ - component. Plugging in the solution for the action of a supertranslation on the Schwarzschild metric, given by equation (140), but now with total mass $M+\mu$, we see that the final result is the same as (157). That is, the effect of 151 ) on the Schwarzchild metric can be be succinctly expressed as

$$
\begin{equation*}
h_{a b}=\theta\left(v-v_{0}\right)\left(\mathcal{L}_{f=-\hat{C}}+\frac{2 \mu}{r} \delta^{v}{ }_{a} \delta^{v}{ }_{b}\right) . \tag{159}
\end{equation*}
$$

HPS then derive the presymplectic form corresponding to the supertranslation charge by fixing the residual gauge transformations which preserve the Bondi gauge. For the case of constructing the presymplectic form of the supertranslation charge, the energy momentum tensors on the right hand side of Einstein's equations can be set to zero, as is done by HPS. We will instead include these terms and the shift they induce on the event horizon further below. The linearized Einstein tensor is given by

$$
\begin{equation*}
\square h_{a b}+2 R_{a c b d} h^{c d}-2 R_{c(a} h_{b)}^{c}-\nabla_{a} \nabla_{b} h_{b}^{c}-\nabla_{b} \nabla_{c} h_{a}^{c}+\nabla_{a} \nabla_{b} h=0 . \tag{160}
\end{equation*}
$$

This leads to the following constraint equations

$$
\begin{array}{r}
\frac{1}{2 M}\left(\partial_{v}-\frac{1}{4 M} D^{2}\right) h_{v v}+\frac{1}{4 M^{2}}\left(\partial_{v}-\frac{1}{4 M}\right) D^{A} h_{A v}=8 \pi T_{v v}^{M} \\
\frac{1}{2} D_{A} \partial_{r} h_{v v}+\left(\frac{1}{4 M}-\frac{1}{2} \partial_{v}\right) D_{A} h_{v r}+\left(\frac{1}{2 M} \partial_{v}-\partial_{v} \partial_{r}-\frac{1}{4 M^{2}}\left(D^{2}+1\right)\right) h_{v A}=8 \pi T_{v A}^{M} \tag{161}
\end{array}
$$

HPS additionally demand that $\left.\partial_{v} h_{a b}\right|_{v= \pm \infty}=0$, so that the linearized metric perturbations at the boundaries of the event horizon, given by $\mathcal{H}_{ \pm}^{+}$, are independent of advanced time. The Dray-'t Hooft shock wave is exactly located at the past boundary of the future event horizon, so that this may seem like an important difference. However, HPS impose this restriction by hand to simplify calculations, and state that this can be relaxed. By plugging $\left.\partial_{v} h_{a b}\right|_{v= \pm \infty}=0$ into 161 , we find that

$$
\begin{array}{r}
{\left.\left[\frac{1}{2 M} D^{A} h_{v A}+D^{2} h_{v v}\right]\right|_{\mathcal{H}_{ \pm}^{+}}=0,} \\
{\left.\left[\partial_{r} D_{A} h_{v v}+\frac{1}{2 M} D_{A} h_{v r}+\frac{1}{4 M^{2}}\left(D^{2}+1\right) h_{v A}+\frac{1}{4 M^{2}} D_{A} D^{B} h_{v B}\right]\right|_{\mathcal{H}_{ \pm}^{+}}} \tag{162}
\end{array}=0 .
$$

HPS then consider the most general diffeomorphism which leaves the Schwarschild metric in Bondi coordinates invariant, which is given by

$$
\begin{equation*}
\zeta=X \partial_{v}-\frac{1}{2}\left(r D_{A} X^{A}+D^{2} X\right) \partial_{r}+X^{A} \partial_{A}+\frac{1}{r} D^{A} X \partial_{A}, \quad X=X(v, \Theta), \quad X^{A}=X^{A}(v, \Theta) \tag{163}
\end{equation*}
$$

Note that $X$ and $X^{A}$ are allowed to depend on advanced time, as we do not impose fall-off conditions at $\mathcal{I}$. The effect on the linearized variations of the metric is then

$$
\begin{align*}
\mathcal{L}_{\zeta} h_{v v} & =\frac{M}{r^{2}} D^{2} X-\left(2 V+D^{2}\right) \partial_{v} X+\left(\frac{M}{r}-r\right) D_{A} X^{A} \\
\mathcal{L}_{\zeta} h_{v r} & =\partial_{v} X-\frac{1}{2} D_{A} X^{A} \\
\mathcal{L}_{\zeta} h_{v A} & =\left(r \partial_{v}-V\right) \partial_{A} X-\frac{1}{2} D_{A} D^{2} X+r^{2} \partial_{v} X_{A}-\frac{r}{2} D_{A} D_{B} X^{B} \\
\mathcal{L}_{\zeta} h_{A B} & =r\left(2 D_{A} D_{B} X-\gamma_{A B} D^{2} X\right)+r^{2}\left(D_{(A} X_{B)}-\gamma_{A B} D_{C} X^{C}\right) \tag{164}
\end{align*}
$$

We then use the residual diffeomorphisms to set to zero the following combinations of linearized metric variations

$$
\begin{equation*}
2\left(1+M \partial_{r}\right) h_{v v}+h_{v r}=0=h_{v A}+2 M D_{A} h_{v v} \tag{165}
\end{equation*}
$$

This fixes all diffeomorphisms in (163) except for those with $\partial_{v} X=0=\partial_{v} X^{A}$, that is, the diffeomorphisms which satisfy the asymptotic fall-off conditions. Plugging 165 into 161 , the linearized constraints reduce to [7]

$$
\begin{align*}
-\frac{1}{2 M}\left(D^{2}-1\right) \partial_{v} h_{v v} & =8 \pi T_{v v}^{M}, \\
\partial_{v}\left(M D_{A} \partial_{r} h_{v v}-\frac{1}{2} \partial_{r} h_{v A}+\frac{1}{8 M^{2}} D^{B} h_{A B}\right) & =8 \pi T_{v A}^{M} \tag{166}
\end{align*}
$$

The first equation in 166 is very reminiscent of the defining equation of the shift function in Kruskal
coordinates, given by (90); we previously noted such a similarity below equation (151). Let us further consider this for the case of a shock wave impinging on the event horizon. We plug $h_{v v}$ from 157 ) into the left hand side of the first equation of 166 and evaluate the resulting expression at the event horizon to find

$$
\begin{equation*}
\left(D^{2}-1\right) \partial_{v} h_{v v}=-\frac{\mu}{M}-\frac{1}{4 M}\left(D^{2}-1\right) D^{2} \hat{C} \delta\left(v-v_{0}\right) \tag{167}
\end{equation*}
$$

Comparing with (151), we see that the $\ell=0$ - term drops out. Let us therefore focus on $\ell \geqslant 2$ components, given by the second term on the right hand side of 167 . We saw that the effect of a supertranslation parametrized by some function $f$ is to translate the radial coordinate as $r \rightarrow r-\frac{1}{2} D^{2} f$. The shock wave in (151) induces a supertranslation given by $f=-\hat{C}$. The corresponding shift in $r$ is therefore given by

$$
\begin{equation*}
\delta r(\Theta)=\frac{1}{2} D^{2} \hat{C} \tag{168}
\end{equation*}
$$

We plug this into 166 and 167 to find

$$
\begin{equation*}
\left.\frac{\delta\left(v-v_{0}\right)}{4 M^{2}}\left(D^{2}-1\right) \delta r\right|_{r=2 M}=8 \pi T_{v v}^{M}-\frac{\mu}{2 M^{2}} \tag{169}
\end{equation*}
$$

Let us compare this with the Dray-'t Hooft shock wave in Kruskal coordinates $U=-e^{-u / 4 M}, V=$ $e^{v / 4 M}$, located now at $V=0$. Using the fact that

$$
U V=-\left(\frac{r}{2 M}-1\right) e^{r / 2 M}
$$

we have

$$
\partial_{U} r=-\frac{4 M^{2} V e^{-r / 2 M}}{r}
$$

so that

$$
\begin{equation*}
\delta r=\delta U \partial_{U} r=-\frac{4 M^{2} V e^{-r / 2 M}}{r} F \tag{170}
\end{equation*}
$$

Inverting the above expression gives $F=-\frac{r e^{r / 2 M}}{4 M^{2} V} \delta r$. We evaluate this expression at the event horizon of the unperturbed black hole at $r=R=2 M$ and plug it into the constraint equation 74 , now for a shock wave at $V=0$ rather than $U=0$, to find

$$
\begin{equation*}
\left.\frac{1}{M U} \delta(V)\left(D^{2}-1\right) \delta r\right|_{r=2 M}=8 \pi T_{V V}^{M} \tag{171}
\end{equation*}
$$

where the energy-momentum tensor has been given a superscript ' $M$ ' as it consists only of matter contributions. We transform this expression to Bondi coordinates. We have $\delta(V)=\frac{4 M \delta(\tilde{v})}{V}$, where $\tilde{v}$ is again the $v$-coordinate of the shock wave, in this case given by taking $v$ to minus infinity. This slightly obfuscates the interpretation in these coordinates, but it seems to be simply a coordinate artefact. In (171), we do not evaluate $\partial_{v} V=-(4 M)^{-1} V$ at $v=\tilde{v}$ since we only get a non-zero contribution at $\tilde{v}$ due to the presence of $\delta(\tilde{v})$. Simple tensor transformation gives $T_{V V}^{M}=\frac{16 M^{2}}{V^{2}} T_{v v}^{M}$, which we plug in to find

$$
\left.\frac{1}{M V} \frac{4 M}{V} \delta(\tilde{v})\left(D^{2}-1\right) \delta r\right|_{r=2 M}=8 \pi \frac{16 M^{2}}{V^{2}} T_{v v}^{M}
$$

We rewrite this to give

$$
\begin{equation*}
\left.\frac{1}{4 M^{2}} \delta(\tilde{v})\left(D^{2}-1\right) \delta r\right|_{r=2 M}=8 \pi T_{v v}^{M} \tag{172}
\end{equation*}
$$

We thus see that the dependence of the shift of the event horizon on the $\ell \geqslant 2$-components of the energy-momentum tensor is the same in HPS [7] as it is for the older formalism of 't Hooft [3] [4] based on the Dray-'t Hooft shock wave [2].

The same expression can be shown to hold for the Dray-'t Hooft shock wave in tortoise coordinates, in which the Schwarzschild metric is written as

$$
\begin{equation*}
d s^{2}=-\left(1-\frac{2 M}{r}\right) d u d v+r^{2} \gamma_{A B} d \Theta^{A} d \Theta^{B} \tag{173}
\end{equation*}
$$

which also satisfies the metric ansatz in 72 . Denoting the shift function in this coordinate system by $\tilde{f}$ and using

$$
\frac{1}{2}(v-u)=r+2 M \ln \left(\frac{r}{2 M}-1\right)
$$

we find that the radial coordinate is shifted as

$$
\begin{equation*}
\delta r(\Theta)=\frac{\partial r}{\partial u} \delta u=\frac{1}{2}\left(1-\frac{2 M}{r}\right) \tilde{f} . \tag{174}
\end{equation*}
$$

Using the fact that metric component $A=-\frac{1}{2}\left(1-\frac{2 M}{r}\right)$ in metric ansatz 72$)$, we have

$$
\tilde{f}=\frac{\delta r}{A} .
$$

Pluggin this into constraint equation (74) gives

$$
\begin{equation*}
\left.\frac{1}{4 M^{2}} \delta(\tilde{v})\left(D^{2}-1\right) \delta r\right|_{r=2 M}=8 \pi T_{v v}^{M} \tag{175}
\end{equation*}
$$

By comparing with 167 and 168 , we see that the shift of the radial coordinate at the event horizon satisfies the same constraint equation relating it to the energy-momentum tensor impinging on the black hole. However, from $\sqrt[174]{ }$, we see that $\tilde{f}$ blows up for any non-zero $\delta r$. This is related to the fact that, in tortoise coordinates, the event horizon is located at infinite coordinate distance, so that one has to shift geodesics by an infinite amount in order to drag them across the event horizon. This point obfuscates the interpretation of the shock wave in tortoise coordinates, which is why Kruskal coordinates are more convenient for this purpose. However, the fact that the shift of the event horizon calculated by HPS satisfies the same equation as the one found by Dray and 't Hooft in both Kruskal and tortoise coordinates suggests that this is not merely a coincidence. The fact that the shifts of the event horizon satisfy the same equation in HPS [7] as in the black hole S-matrix formalism [3] [4] based on the Dray-'t Hooft shock wave [2] was not pointed out by HPS, nor by other authors.

HPS have thus found the shift of the event horizon of a black hole due to a supertranslation induced by a shock wave with a non-trivial spherical profile. We find that this shift satisfies the same relation to the energy-momentum tensor at the event horizon as the Dray-'t Hooft shock wave [2] used by 't Hooft to derive a unitary black hole S-matrix [3] 4. This suggests that one might be able to describe the black hole S-matrix in terms of BMS charges at $\mathcal{I}$. In particular, we saw in equation (144) that a supertranslation implants superrotation charge on the black hole, so that the scattering of shock waves from Schwarzschild black hole may be described in terms of the superrotation charges implanted by these shock waves. We leave further exploration of this point to future work, as it is beyond the scope of this thesis.

In spite of clear similarities, there are obvious and important differences between the two approaches as well. First and foremost, the calculations by HPS and Dray \& 't Hooft are done in different gauges. Further, there is no mass change in the black hole S-matrix, whereas the black hole mass in [7] changes by $\mu$ in the expressions derived by HPS. However, as we saw from equation (159) and the expressions that followed, the effect of the shock wave 'splits up' into a mass change and an induced supertranslation, so that we could consider them separately and find that the action of the induced shift in $r$ at the event horizon is the same in the two formalisms. There are no gravitational waves in the black hole S-matrix, as it describes the longitudinal shift induced by a shock wave while ignoring transverse momentum transfer, whereas the fact that $N_{A B} \neq 0$ in the calculation by HPS signals the presence of gravitational waves. Lastly, HPS assume that the time dependence of the metric variations goes to zero at the boundaries of the event horizon, while the Dray-'t Hooft shock wave is required to propagate along the horizon. However, this last point seems to be merely a simplifying assumption on the part of HPS rather than a true physical difference between the two formalisms.

As noted previously, HPS assume antipodal matching to hold in a single external region in the context of a black hole, but no justification for this is given in their work [7, nor, as far as we are aware, is this justified elsewhere in the literature, for example by a soft theorem in the context of a black hole.

Radially propagating classical particles entering space-time at $\mathcal{I}^{-}$simply disappear into the black hole without leaving a trace at $\mathcal{I}^{+}$, so there seems to be no reason why we would not be allowed to act on $\mathcal{I}^{-}$with a supertranslation without acting at $\mathcal{I}^{+}$with an antipodally matched supertranslation as well. However, the black hole S-matrix tells us that this classical picture is not correct, as the information of infalling particles can be retrieved if one has access to both external regions of the maximally extended Schwarzschild black hole. We therefore suspect that the S-matrix can clarify the matching of BMS-transformations in the context of black holes, which we hope to further explore in future work.

### 3.3 BMS transformations at the event horizon

It has recently been shown that a structure similar to the BMS group at $\mathcal{I}$ can be found at the event horizon of a $(1+3)$-dimensional black hole [19] 20 . We briefly review this here and show that the Dray-'t Hooft shock wave in tortoise coordinates satisfies the gauge and fall-off conditions and generates a non-zero horizon superrotation charge. On the event horizon of a black hole, tortoise coordinates satisfy

$$
g^{\mu \nu} \partial_{\mu} v \partial_{\nu} v=0
$$

where, for a Schwarzschild black hole, $v=t+r^{*}=t+r+2 M \ln \left(\frac{r}{2 M}-1\right)$, as we will consider below. From the fact that the product of the tangent and normal spaces of the two-sphere looks locally like $\mathbb{R}^{3}$, one can always choose a vector $\partial_{\rho}$ that is normal to the event horizon. The spherical coordinates $\Theta^{A}$ are chosen such that they are constant along $\rho$, i.e. [20]

$$
\begin{equation*}
g^{\alpha \beta} \partial_{\alpha} v \partial_{\beta} \Theta^{A}=0 \tag{176}
\end{equation*}
$$

This gives the following gauge conditions on the metric components

$$
\begin{array}{ll}
g^{v \rho}=1 & , \quad g^{\rho \rho}=0=g^{v v} \\
g_{v \rho}=1 & , \quad g_{\rho \rho}=0=g_{\rho A} \tag{177}
\end{array}
$$

Setting the horizon at $\rho=0$ and assuming it is non-expanding, the other metric components are given by 48

$$
\begin{align*}
g_{v v} & =-2 \kappa \rho+\mathcal{O}\left(\rho^{2}\right) \\
g_{v A} & =\rho \theta_{A}+\mathcal{O}\left(\rho^{2}\right) \\
g_{A B} & =\Omega_{A B}+\rho \lambda_{A B}+\mathcal{O}\left(\rho^{2}\right) \tag{178}
\end{align*}
$$

where $\kappa$ is the surface gravity which is at this point, allowed to depend on $v$ and $\Theta^{A}$, as are $\theta_{A}, \Omega_{A B}$, and $\lambda_{A B}$. The class of near-horizon metrics to be considered can thus be written as [19]

$$
\begin{equation*}
d s^{2}=-2 \kappa \rho d v^{2}+2 d \rho d v+2 \theta_{A} \rho d v d \Theta^{A}+\left(\Omega_{A B}+\rho \lambda_{A B}\right) d \Theta^{A} d \Theta^{B}+\mathcal{O}\left(\rho^{2}\right) \tag{179}
\end{equation*}
$$

Note that the metric for a linearized shock wave on a Schwarzschild black hole derived by HPS, written in 157), does not respect these fall-off conditions. In particular, $g_{v A}$ contains an $\mathcal{O}(1)$ - term, whereas $g_{v A}$ in 179 has to be $\mathcal{O}(\rho)$. A map between the BMS-formalisms has, as far as we are aware, not appeared in the literature, and Donnay herself stated some months ago that it is not known to her (if and) how the two are related [49].

We would like to find vector fields $\chi=\chi^{\mu} \partial_{\mu}$ along which (177) and (178) are preserved. One then finds the following fall-off conditions for $\chi[20$

$$
\begin{align*}
& \chi^{v}=f\left(v, \Theta^{A}\right) \\
& \chi^{\rho}=-\rho \partial_{v} f+\rho^{2} \frac{1}{2} \Omega^{A B} \Theta_{A} \partial_{B} f+\mathcal{O}\left(\rho^{3}\right) \\
& \chi^{A}=Y^{A}\left(\Theta^{B}\right)+\rho \Omega^{A C} \partial_{C} f+\frac{1}{2} \rho^{2} \Omega^{A D} \Omega^{B C} \lambda_{D B} \partial_{C} f+\mathcal{O}\left(\rho^{3}\right) \tag{180}
\end{align*}
$$

Assuming that $\delta_{\chi} \kappa=0=\partial_{i} \kappa, i \in\{v, A\}$, i.e. that $\kappa$ is constant both along the horizon as well as along $\chi$, we find

$$
\begin{equation*}
\kappa \partial_{v} f+\partial_{v}^{2} f=0 \tag{181}
\end{equation*}
$$

so that $f$ is of the form

$$
\begin{equation*}
f\left(\Theta^{A}, v\right)=T\left(\Theta^{A}\right)+e^{-\kappa v} S\left(\Theta^{A}\right) \tag{182}
\end{equation*}
$$

so that any dependence of $f$ on $v$ decays exponentially. One can then construct the charge variation at the horizon [50 [51], which, for constant $\kappa$ and $\partial_{v} \Omega_{A B}=0$, can be integrated to find [19] [20]

$$
\begin{equation*}
Q_{f, Y}=\frac{1}{16 \pi} \int d^{2} \Theta \Omega_{A B}\left(2 \kappa T-Y^{C} \theta_{C}\right) \tag{183}
\end{equation*}
$$

where we leave out an integration constant.
The Schwarzschild metric, of course, satisfies the conditions in 178). In particular, $v$ is simply given by

$$
v=t+r^{*}
$$

For convenience, we repeat here the Schwarzschild metric in tortoise coordinates

$$
\begin{equation*}
d s^{2}=-\left(1-\frac{2 M}{r}\right) d v^{2}+2 d v d r+r^{2} \gamma_{z \bar{z}} d z d \bar{z} \tag{184}
\end{equation*}
$$

Rewriting $\kappa=\frac{1}{4 M}$ and performing the following coordinate change

$$
\begin{equation*}
r=\frac{1}{2 \kappa}(1+2 \kappa \rho), \tag{185}
\end{equation*}
$$

we find that

$$
\begin{equation*}
d s^{2}=-2 \kappa \rho d v^{2}+2 d \rho d v+\left(\frac{1}{4 \kappa^{2}}+\frac{\rho}{\kappa}\right) \gamma_{z \bar{z}} d z d \bar{z}+\mathcal{O}\left(\rho^{2}\right) . \tag{186}
\end{equation*}
$$

One easily sees that this metric belongs to the class given by 178 with

$$
\begin{array}{ll}
\Theta_{z}=0, & \Omega_{z \bar{z}}=\frac{\gamma_{z \bar{z}}}{4 \kappa^{2}}, \quad \Omega_{z z}=0 \\
\lambda_{z \bar{z}}=\frac{1}{\kappa}, & \lambda_{z z}=0 \tag{187}
\end{array}
$$

Let us consider the Dray-'t Hooft shock wave (73) in tortoise coordinates. We start from the shock wave metric given by

$$
\begin{equation*}
d s^{2}=2 \hat{A} d u(d \hat{v}-\delta(u) \tilde{f} d u)+\hat{g} \gamma_{A B} d \Theta^{A} d \Theta^{B} \tag{188}
\end{equation*}
$$

where

$$
v=t+r^{*} \quad, \quad u=t-r^{*}
$$

In (188), $\hat{v}, \hat{A}$, and $\hat{g}$ are defined as

$$
\begin{equation*}
\hat{v}=v+\theta(\tilde{u}) \tilde{f}, \quad \hat{A}=A(u, \hat{v}), \quad \hat{g}=g(u, \hat{v}) \tag{189}
\end{equation*}
$$

where $\tilde{u}$ is given by the limit of $u$ to $-\infty$, which is the location of the event horizon. We have

$$
A(u, v)=-\frac{1}{2}\left(1-\frac{2 M}{r}\right)=-\frac{1}{2 \partial_{r} r^{*}}
$$

We now transform to coordinates $\left(u, r, \Theta^{A}\right)$. We use

$$
\begin{array}{cc}
\partial_{u} \hat{v}=1+\delta(\tilde{u}) f, & \partial_{u} u=1 \\
\partial_{r} \hat{v}=\partial_{r} v=-A^{-1}, & \partial_{r} u=0 \\
\partial_{A} \hat{v}=\theta(\tilde{u}) \partial_{A} \tilde{f}, & \partial_{A} u=0 \tag{190}
\end{array}
$$

as well as

$$
\hat{A}=\frac{-1}{\partial_{r} \hat{v}}=\frac{-1}{\partial_{r} v}=A, \quad \hat{g}=(r+\theta(\tilde{u}) \delta r)^{2}=(r-\theta(\tilde{u}) A \tilde{f})^{2}=r^{2}-\theta(\tilde{u}) 2 A r \tilde{f}-\theta(\tilde{u})(A \tilde{f})^{2}
$$

This gives

$$
\begin{align*}
& g_{r r}=0=g_{r A} \\
& g_{r u}=-1 \\
& g_{u u}=2 \partial_{u} \hat{v} A-2 A \tilde{f} \delta=2 A=-\left(1-\frac{2 M}{r}\right) \\
& g_{u A}=A \theta(\tilde{u}) \partial_{A} \tilde{f}=-\frac{1}{2}\left(1-\frac{2 M}{r}\right) \theta(\tilde{u}) \partial_{A} \tilde{f} \\
& g_{A B}=\hat{g} \gamma_{A B}=(r-\theta(\tilde{u}) A \tilde{f})^{2}=\left(r^{2}-\theta(\tilde{u}) 2 A r \tilde{f}-\theta(\tilde{u})(A \tilde{f})^{2}\right) \gamma_{A B} \tag{191}
\end{align*}
$$

Plugging in 185, this gives

$$
\begin{align*}
g_{\rho \rho} & =0=g_{\rho A} \\
g_{\rho u} & =-1 \\
g_{u u} & =-\frac{\rho}{2 M}+\mathcal{O}\left(\rho^{2}\right) \\
g_{u A} & =-\frac{\rho}{4 M} \theta(\tilde{u}) \partial_{A} \tilde{f}+\mathcal{O}\left(\rho^{2}\right) \\
g_{A B} & =4 M^{2}+(4 M+\theta(\tilde{u}) \tilde{f}) \rho+\mathcal{O}\left(\rho^{2}\right) \tag{192}
\end{align*}
$$

Using $\kappa=\frac{1}{4 M}$, we see that 191 belongs to the class of metrics given by (179). Further, we see from $g_{u A}$ that

$$
\begin{equation*}
\theta_{A}=-\frac{1}{4 M} \theta(\tilde{u}) \partial_{A} \tilde{f} \tag{193}
\end{equation*}
$$

By comparing with 183, we thus find that the Dray-'t Hooft shock wave gives rise to a non-zero superrotation charge, given by

$$
\begin{equation*}
Q_{Y}=\frac{\theta(\tilde{u})}{64 \pi M} \int d^{2} \Theta \Omega_{A B} Y^{C} \partial_{C} \tilde{f} \tag{194}
\end{equation*}
$$

When transforming to coordinates $\left(v, r, \Theta^{A}\right)$, using

$$
\partial_{r} u=-2\left(1-\frac{2 M}{r}\right)^{-1}=A^{-1} \quad, \quad \partial_{A} \hat{v}=\theta(u) \partial_{A} \tilde{f}
$$

we have

$$
\begin{equation*}
g_{r A}=\partial_{r} u \partial_{A} \hat{v} A=\theta \partial_{A} \tilde{f} \tag{195}
\end{equation*}
$$

so that the gauge condition that $g_{r A}=0$ is violated in coordinate system $\left(v, r, \Theta^{A}\right)$. Similarly, if we consider the Dray-'t Hooft shock wave $(74)$ in Kruskal coordinates, we have

$$
\hat{V}=V+\theta(U) F(\Theta)=e^{\left(u+2 r^{*}\right) / 4 M}+\theta(U) F(\Theta)
$$

so that, in coordinates $\left(u, r, \Theta^{A}\right)$

$$
\begin{equation*}
g_{u A}=\partial_{u} U \partial_{A} \hat{V}\left(-\frac{16 M^{3}}{r} e^{-r / 2 M}\right)=-\frac{4 M^{2}}{r} e^{-(u+2 r) / 4 M} \theta(U) \partial_{A} F \tag{196}
\end{equation*}
$$

We see that 196 is finite at $r=2 M$, so that the Dray-'t Hooft shock wave in Kruskal coordinates violates the fall-off conditions for horizon-BMS. The same can easily be shown for $g_{v A}$ in coordinates $\left(v, r, \Theta^{A}\right)$.

We thus find that the Dray-'t Hooft shock wave in tortoise coordinates implants a non-zero horizonBMS charge in the form of $(194)$. However, the S-matrix is derived from Dray-'t Hooft shock wave in Kruskal coordinates, which does not respect the fall-off conditions. Further, no map between horizonBMS and BMS at $\mathcal{I}$ currently exists in the literature. It would be interesting to see if horizon-BMS can be applied to the gravitational S-matrix, in particular, to see if shock waves leave a 'footprint' at the horizon in the form of superrotation charges. We hope to further explore this in future work.

## Conclusion

In this thesis, we compared the BMS- and the shock wave scattering formalisms. We first reviewed BMS-symmetries, which are an infinite-dimensional extension of the Poincare group acting at the null infinities of asymptotically flat space-times. We focused on angle-dependent time-translations at $\mathcal{I}$ known as supertranslations, as these are relatively well understood. Following Strominger and collaborators, we saw that supertranslations have to be antipodally matched between $\mathcal{I}^{-}$and $\mathcal{I}^{+} . \mathcal{I}$ is the light-cone of $i^{0}$, from which we see that antipodal matching is equivalent to the continuity of physical data along the null generators of $\mathcal{I}$ even as they pass through $i^{0}$. In the absence of gravitational waves, antipodal matching of supertranslations corresponds to antipodal matching of in- and outgoing energy flux between $\mathcal{I}^{-}$and $\mathcal{I}^{+}$. We then reviewed the construction of gravitational shock waves and their unitary scattering as described by the shock wave S-matrix. We saw that the longitudinal shift induced by a shock wave in the vicinity of an event horizon can transfer signals between the two external regions of the maximally extended Schwarzschild black hole. By demanding unitary scattering in a single 'universe', that is, a single external region of the maximally extended Schewarzschild black hole, we are led to identify the two external regions via antipodal identification at the event horizon. The black hole S-matrix is characterized by an infinite number of symmetries at the event horizon, given by independent boosts for each spherical wave in the expansion of Kruskal coordinates, which then have to be antipodally matched at the bifurcation sphere. As the event horizon is the light-cone of the bifurcation sphere, we see that this picture is very reminiscent of supertranslations at $\mathcal{I}$. In particular, physical data is continuous along the generators of the horizon as they pass through the bifurcation sphere, as was the case for physical data along the null generators of $\mathcal{I}$ as they pass through $i^{0}$.

Further comparison of the two formalisms in the context of black holes is complicated by the fact that the BMS-formalism here is still under development. We are thus led to compare the two formalisms in Minkowski space. By employing a Lagrangian approach, we saw that the S-matrix in Minkowski space is invariant under translations of the time coordinate that are allowed to depend non-trivially on transversal coordinates, which is very reminiscent of supertranslation invariance. The corresponding conserved charge equals the supertranslation charge of the scattering shock waves. We thus find that the antipodally matched supertranslation charge conservation found recently by Strominger and collaborators is explicitly realized for the shock wave S-matrix in Minkowski space which appeared in the literature more than twenty-five years earlier. As far as we are aware, this fact has not previously been pointed out in the literature. We then considered the action of BMS-transformations on black hole space-times, where the analogy is decidedly less clear. Following Hawking, Perry, and Strominger, we considered the action of a linearized supertranslation on a Schwarzschild black hole and show that it can be induced by a linearized shock wave with a non-trivial spherical profile. By comparing with the defining equation of the Dray-'t Hooft shock wave, we see that the relation of the shift of the event horizon to the energy-momentum tensor impinging on the black hole is the same in the two formalisms. This suggests that one might be able to describe the black hole S-matrix in terms of

BMS-charges at $\mathcal{I}$. However, different gauge choices and the absence of mass change and gravitational waves in the S-matrix formalism complicates this relation, so that we cannot but defer its further exploration to future work. Hawking, Perry, and Strominger tacitly assume that BMS-transformations are antipodally matched even for Schwarzschild space-times as long as the Bondi news falls off fast enough, although antipodal matching has only been shown to hold for Christodoulou-Klainermann spaces, which do not include Schwarzschild black holes. We suggest that the S-matrix formalism may be used to further clarify the matching conditions for BMS-transformations on black hole space-times. We then briefly review the recent derivation of BMS-like transformations acting on the event horizon of a black hole. We see that the Dray-'t Hooft shock wave in tortoise coordinates respects the gauge and fall-off conditions of horizon-BMS and that it generates a non-zero superrotation charge. However, the S-matrix is based on the Dray-'t Hooft shock wave in Kruskal coordinates, and no map between the BMS-structures at $\mathcal{I}$ and the event horizon currently exists, so that we do not explore this point much further. We have thus identified a number of results relating the two formalisms in the context of black holes, which we hope will add to their mutual extension and clarification.

## Appendices

## A BMS analysis in Kerr-Schild decomposition

We now calculate the Bondi four-momentum of the Aichelburg-Sexl (AS) metric, of which the past (future) limit at $\mathcal{I}^{+}\left(\mathcal{I}^{-}\right)$equals the supertranslation charge. We follow the analysis by Aichelburg and Balasin in [42, [43]. The authors make use of the Kerr-Schild metric decomposition [52], where the metric is decomposed into a flat part and a part that is proportional to the square of a null geodesic vector field. Explicitly, this is written as

$$
\begin{equation*}
g_{a b}=\eta_{a b}+S k_{a} k_{b} \tag{197}
\end{equation*}
$$

where $S$ is a scalar function. $k^{a}=\eta^{a b} k_{b}$ is a null vector with respect to $\eta_{a b}$, such that it is null with respect to $g_{a b}$ as well, that is

$$
g_{a b} k^{a} k^{b}=0=\eta_{a b} k^{a} k^{b}
$$

By demanding that $g_{a c} g^{c b}=\delta^{b}{ }_{c}$, we easily see that that the inverse metric is given by

$$
\begin{equation*}
g^{a b}=\eta^{a b}-S k^{a} k^{b} \tag{198}
\end{equation*}
$$

We choose $k^{a}$ to be affinely parametrised, which gives

$$
\begin{equation*}
k^{a} \nabla_{a} k^{b}=0=k^{a} \partial_{a} k^{b} . \tag{199}
\end{equation*}
$$

The Christoffel connection is then given by

$$
\begin{align*}
\Gamma_{b c}^{a} & =\frac{1}{2} g^{a d}\left(\partial_{b} g_{c d}+\partial_{c} g_{b d}-\partial_{d} g_{b c}\right) \\
& =\frac{1}{2}\left(\partial_{b}\left(S k^{a} k_{c}\right)+\partial_{c}\left(S k^{a} k_{b}\right)-\partial^{a}\left(S k_{b} k_{c}\right)+S k^{a}(k \cdot \partial)\left(S k_{b} k_{c}\right)\right) \tag{200}
\end{align*}
$$

where the last term arises from $g^{a b}-\eta^{a b}=-S k^{a} k^{b}$. For a vector $\xi^{b}$, we then have

$$
\begin{equation*}
\nabla^{(a} \xi^{b)}=\partial^{(a} \xi^{b)}+\frac{1}{2}\left((\xi \cdot \partial)\left(S k^{a} k^{b}\right)-S(k \cdot \partial) S(\xi \cdot k) k^{a} k^{b}\right) \tag{201}
\end{equation*}
$$

If the right hand side of (201) is (asymptotically) zero, $\xi^{b}$ is a(n asymptotic) Killing vector. In the absence of gravitational waves, the conserved charge corresponding to a Killing vector field $\xi$ is given
by the Komar expression

$$
\begin{equation*}
Q_{\xi}=-\frac{1}{4 \pi} \int_{\mathcal{I}} \nabla^{a} \xi^{b} \epsilon_{a b c d} d x^{c} d x^{d}=-\frac{1}{4 \pi} \int_{\mathcal{I}} \nabla^{a} \xi^{b} \epsilon_{a b A B} d \Theta^{A} d \Theta^{B} \tag{202}
\end{equation*}
$$

where $\Theta^{A}$ and $\Theta^{B}$ are the spherical coordinates parametrizing the two-sphere at $\mathcal{I}$. Due to the antisymmetry of the Levi-Civita tensor, we see that the lower case indices $a, b$ necessarily label longitudinal coordinates, namely, $t, r$ and functions thereof. Labeling longitudinal coordinates by $\alpha$, $\beta$, the condition for $\xi$ to be an asymptotic Killing vector is therefore

$$
\begin{equation*}
\nabla^{(\alpha} \xi^{\beta)}=\partial^{(\alpha} \xi^{\beta)}+\frac{1}{2}\left((\xi \cdot \partial)\left(S k^{\alpha} k^{\beta}\right)-S(k \cdot \partial) S(\xi \cdot k) k^{\alpha} k^{\beta}\right)=\mathcal{O}\left(1 / r^{2}\right) \tag{203}
\end{equation*}
$$

Hence, the longitudinal components of $\xi$ are only allowed to depend on transversal coordinates, as we found in the BMS analysis at $\mathcal{I}$ which we treated in section 1.2 in particular expression 29 .

## B Supertranslation charges for (boosted) Schwarzschild black hole

We now calculate the BMS-charges for the Schwarzschild black hole and the Aichelburg-Sexl metric. For a Schwarzschild black hole of mass $M$, we have

$$
\begin{equation*}
d s^{2}=-\left(1-\frac{2 M}{r}\right) d t^{2}+\left(1-\frac{2 M}{r}\right)^{-1} d r^{2}+r^{2} d \Omega_{(2)}^{2} \tag{204}
\end{equation*}
$$

We then go to coordinates $(\bar{t}, r, \theta, \phi)$, where

$$
\begin{equation*}
\bar{t}:=t-2 M \ln (r-2 M) \tag{205}
\end{equation*}
$$

so that

$$
\begin{equation*}
d s^{2}=-d \bar{t}^{2}+d r^{2}+r^{2} d \Omega_{(2)}^{2}+\frac{2 M}{r}(d \bar{t}-d r)^{2} \tag{206}
\end{equation*}
$$

The Schwarzschild metric is now expressed in Kerr-Schild decomposition, given by equation (197), with

$$
\begin{equation*}
S=\frac{2 M}{r} \quad, \quad k^{a}=\left(1, e_{r}^{i}\right) \tag{207}
\end{equation*}
$$

where $e_{r}^{i}$ is the unit radial vector with respect to $\eta_{a b}$. Let us calculate the Komar integral for an asymptotic Killing vector $\xi$, which is necessarily independent of $t$ and $r$. That is, we calculate

$$
\begin{equation*}
Q_{\xi}=-\frac{1}{8 \pi} \int_{S^{2}} d^{2} z \gamma_{z \bar{z}} r^{2} \nabla^{[t} \xi^{r]} \tag{208}
\end{equation*}
$$

where we take $S^{2}$ to be a sphere at $\mathcal{I}^{+}$, not necessarily located at $\mathcal{I}_{-}^{+}$. We use the fact that

$$
\nabla^{[t} \xi^{r]}=-\left(\Gamma_{t \mu}^{r} \xi^{\mu}+\Gamma_{r \mu}^{t} \xi^{\mu}\right)
$$

as well as the fact that the only non-zero Christoffel connections are given by

$$
\begin{equation*}
\Gamma_{r t}^{t}=\frac{M}{r^{2}}=\Gamma_{t t}^{r}, \quad \Gamma_{r r}^{t}=\frac{2 M}{r^{2}}, \tag{209}
\end{equation*}
$$

where we only write the first term in the $1 / r$ - expansion. We then find that

$$
\begin{equation*}
Q_{\xi}=-\frac{1}{8 \pi} \int_{S^{2}} d^{2} z \gamma_{z \bar{z}} r^{2} \nabla^{[t} \xi^{r]}=-\frac{1}{8 \pi} \int_{S^{2}} d^{2} z \gamma_{z \bar{z}}(-2 M)\left(\xi^{t}+\xi^{r}\right)=M \xi^{t} \tag{210}
\end{equation*}
$$

where we used the fact that the term $\xi^{r}$ integrates to zero. We see that, if we choose $S^{2}$ to lie at the past (future) limit of $\mathcal{I}^{+}\left(\mathcal{I}^{-}\right), 210$ gives the supertranslation charge of a Schwarzschild black hole found by replacing $m_{B} \rightarrow M$ in (63).

Let us now consider the Aichelburg-Sexl metrix, given by (69) and 71, which corresponds to a Schwarzschild black hole boosted to its light-like limit whilst keeping its total energy constant. As opposed to the Schwarzschild black hole, the energy of an Aichelburg-Sexl shock wave exits space-time after finite retarded time. The Bondi momentum will therefore have some non-trivial time dependence. The Bondi momentum at $\mathcal{I}^{+}$at some retarded time $u_{0}$ is given by the top equality of 63 with lower integral bound at $u_{0}$. We write the Aichelburg-Sexl metric in Cartesian coordinates

$$
\begin{align*}
d s^{2} & =-\left(1-F \delta\left(t-x_{3}\right)\right) d t^{2}+\left(1+F \delta\left(t-x_{3}\right)\right) d x_{3}^{2}+2 F \delta\left(t-x_{3}\right) d t d x_{3}+d x_{1}^{2}+d x_{2}^{2} \\
& =\left(\eta_{\mu \nu}+\bar{h}_{\mu \nu}\right) d x^{\mu} d x^{\nu} \quad, \quad F=-4 p_{\text {in }} \ln \left(x_{1}^{2}+x_{2}^{2}\right) . \tag{211}
\end{align*}
$$

We see that this metric is in Kerr-Schild form, namely

$$
\begin{equation*}
g_{\mu \nu}=\eta_{\mu \nu}+S k_{\mu} k_{\nu}, \quad k^{\mu}=\left(\partial_{t}\right)^{\mu}+\left(\partial_{3}\right)^{\mu} \quad, \quad S=F \delta\left(t-x_{3}\right) \tag{212}
\end{equation*}
$$

where we write vectors as their corresponding partial derivatives. We use light-cone coordinates $v=$ $t+r$ and $u=t-r$ as well as spherical coordinates $(\theta, \phi)$, we have

$$
F=-4 p_{\text {in }} \delta(t-z) \ln \left(\sqrt{x_{1}^{2}+x_{2}^{2}}\right)=\mu \delta\left(t-\frac{v-u}{2} \cos \theta\right) \ln \left(\frac{v-u}{2} \sin \theta\right)
$$

As a distribution, the Dirac delta 'function' satisfies

$$
x \partial_{x} \delta(x)=-\delta(x)
$$

From 200, we find that, for a general Kerr-Schild metric

$$
\begin{equation*}
\nabla^{[a} \xi^{b]}=(k \cdot \xi) \partial^{[a} S k^{b]} \tag{213}
\end{equation*}
$$

Combining these expressions gives, for the Komar expression of the Bondi momentum $P_{\xi}^{\mathrm{Bondi}}(u) 42$

$$
\begin{align*}
P_{\xi}^{\text {Bondi }}(u) & =\lim _{v \rightarrow \infty}-\frac{1}{8 \pi} \int_{S^{2}} \nabla^{[a} \xi^{b]} \epsilon_{a b c d} \\
& =(k \cdot \xi) \lim _{v \rightarrow \infty} \int p_{\text {in }} \delta(u(1+\cos \theta)+v(1-\cos \theta))(v-u) \sin \theta d \theta d \phi \tag{214}
\end{align*}
$$

From the fact that

$$
\lim _{v \rightarrow \infty} \delta(u(1+\cos \theta)+v(1-\cos \theta))(v-u) \sin \theta=\lim _{v \rightarrow \infty} \delta(v(1-\cos \theta)) v \sin \theta=\delta(\theta)
$$

we find

$$
\begin{equation*}
P_{\xi}^{\mathrm{Bondi}}(u)=p_{\mathrm{in}} \theta(-u) p_{a} \xi^{a} \tag{215}
\end{equation*}
$$

In terms of the expression for the charge of a supertranslation, now parametrized by a function $f(\theta, \varphi)$, we find for an Aichelburg-Sexl shock wave at $\theta=0$ that

$$
\begin{equation*}
Q_{f}=p_{\mathrm{in}} f(\theta=0, \varphi) \tag{216}
\end{equation*}
$$

Note that the ADM-momentum is given by [53] 42 ]

$$
\begin{align*}
P^{A D M} & =\lim _{r \rightarrow \infty} \frac{1}{8 \pi} \int_{S^{2}} \nabla^{[a} \xi^{b]} \epsilon_{a b c d} \\
& \left.=(k \cdot \xi) \lim _{r \rightarrow \infty} \int \mu \delta(t-r \cos \theta) r\right] \sin \theta d \theta d \phi \\
& =-p_{\text {in }} k_{a} \xi^{a} \tag{217}
\end{align*}
$$

so that we see that the Bondi momentum equals the ADM momentum up until the retarded time at which the energy exits space-time, in accordance with the findings of [54] and 55]. However, we also see that the momentum is null-like, seemingly violating a result due to Ashtekar and Horowitz which states that neither the ADM nor the Bondi four-momentum of an asymptotically flat system can be null-like [56]. However, Ashtekar and Horowitz assume that the physical data is non-singular, which is not the case for the Aichelburg-Sexl metric.

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[^0]:    ${ }^{1}$ To show that $\partial_{\zeta} \partial_{\bar{\zeta}} \ln (\zeta \bar{\zeta})=4 \pi \delta^{(2)}(\zeta)$, we revert back to Cartesian coordinates by writing $\zeta=x+i y$ and integrate over $d^{2} \zeta$. The left hand side gives $\int d^{2} \zeta \partial \ln (\zeta \bar{\zeta})=\oint \frac{x d y-y d x}{x^{2}+y^{2}}$, where we used Stokes' theorem. Defining $r e^{i \theta}=x+i y$, the left hand side reduces to $\int d \theta=4 \pi$. Dividing this expression by two and taking the positive root of the argument of the logarithm so that it remains real-valued, we find that $\partial_{\zeta} \partial_{\bar{\zeta}} \ln (\zeta \bar{\zeta})=4 \pi \delta^{(2)}(\zeta)$.

