Utrecht University

Master thesis

Backreacting branes and fluxes in F-theory via M-theory

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June 2018

Abstract

We start this thesis by reviewing the basics of F-theory and related concepts. Then we summarize shortly the main features of AdS/CFT for D3/D7-brane systems, most importantly the relation between the RG flow and the dilaton profile, to be able to investigate whether F-theory could provide meaningful insights in this context. After these preliminaries, we construct geometries to describe Type IIB configurations via M-theory. Most notably, we found a new periodic Atiyah-Hitchin space, which describes a system of D7-branes in the presence of an O7-plane, provided we include Kaluza-Klein monopoles in the geometry. Another new result we found, is that M2-branes in such a geometry reproduce the backreaction of D3-branes on the near-horizon geometry of D7-branes. We also derived the warp factor for the backreaction of 7-brane fluxes on the periodic Atiyah-Hitchin space.

Contents

Chapter 1

Introduction

Although its name seems to imply otherwise, F-theory is actually not a fundamental theory, but rather an alternative description of a class of string vacua. To be precise, it describes 7-brane configurations in Type IIB string theory, which are quite different from their lower-dimensional variants. For instance, they source a non-trivial profile for the axio-dilaton, and backreact on the geometry even far away from the branes. This makes the relation between the dilaton and the string coupling particularly intriguing, since it suggests a varying coupling, including regimes of strong coupling. Then the idea behind F-theory is to interpret the varying axio-dilaton as the complex structure parameter of a torus, giving rise to two additional dimensions [1]. Consequently, we study a geometry instead of 7-branes, where the degenerations of the torus fiber indicate the presence of these objects, and their backreaction is incorporated as well.

However, there are some complications involved with this direct approach. First of all, the fields of Type IIB should not be allowed to depend on the fictitious dimensions. This can be fixed by assuming a limit of vanishing torus volume, that we will call the F-theory limit. The other problem is the description of this twelve-dimensional theory. Namely, a low-energy limit should give rise to supergravity, but there exists no twelve-dimensional supergravity with Lorentzian signature, although there have been alternative approaches [1, 2, 3]. A microscopic description is also not viable, due to the non-perturbative nature of the vacua.

Therefore an alternative way to realize this geometric approach is needed, which can be achieved via M-theory. This prescription utilizes the duality between M-theory on a torus with Type IIB on a circle. Via one circle we can relate the axio-dilaton and metric of Type IIB through T-duality to fields in Type IIA, and via the other circle we can lift these fields to geometry in M-theory. Therefore, the geometric approach at 7-brane systems is encoded naturally in the dual M-theory formulation. Furthermore the F-theory limit results in a spacetime dimension on the Type IIB side, due to the inverse relation between circle lengths in T-duality.

Thus formulating F-theory is already quite involved, but this trouble will be worthwhile. First of all, F-theory inherits many features of Type IIB string theory, such as gravitational physics and localized gauge degrees of freedom at the D-branes. However, in perturbative Type IIB only the gauge groups SU(k), SO($2k$) and Sp($2k$) can be realized, whereas F-theory allows for exceptional gauge groups E_6, E_7 and E_8 as well. Especially for particle physics phenomenology this aspect is useful, because it allows model building for grand unified theories. Namely, E_6 is one of the viable candidates to be the gauge group of a GUT, together with SU(5) and SO(10).

Fluxes are essential for the picture sketched in the above paragraph. For instance, they are used for moduli stabilization, to create effective potentials for the moduli fields of the internal manifold. For consistency they can also be needed to cancel D3-brane tadpoles. And last but not least, fluxes for the 7-brane gauge fields have been used to create a chiral spectrum, as done in for instance [4]. This application is very useful for the model building of GUTs, since our current Standard Model has a chiral spectrum.

To complete the picture, we should include the backreaction of these D3-branes and fluxes as well. From the Type IIB perspective this aspect has been studied extensively, for example in [5]. We will take a different point of view in this thesis, by considering the backreaction via a dual M-theory setup. The D3-branes can be dualized to M2-branes, and the 7-brane fluxes are encoded in a flux for the 4-form field strength of M-theory. Then one can study the backreaction as considered in [6]. It results in a warp factor that depends on the torus fiber, and we are not entirely sure what this should map to in F-theory.

In this thesis, we will use the same approach as [7], by considering an appropriate local description of the Calabi-Yau fourfold. They motivated their geometry from a system of k D7-branes, and we will include an O7-plane in this setup, such that we can consider SO($2k$) gauge enhancement instead of SU(k) gauge enhancement. This leads us to consider a new periodified Atiyah-Hitchin space instead of periodified Taub-NUT space, both of which we will construct in Chapter 4.

After these constructions, we will consider a stack of M2-branes in such a geometry. This configuration dualizes to a D3/D7-brane system, which has actually been studied extensively in the AdS/CFT literature [8, 9, 10, 11, 12]. In this context, the D7-branes add flavor fields to the gauge theory living on the stack of D3-branes, which makes it a QCD-like model. Here the AdS/CFT correspondence can be used to map the backreaction of the D7-branes on the supergravity side to the running of the gauge coupling on the field theory side. They use a logarithmic approximation on the field theory side that breaks down at large length scales, which is dual to the Landau pole which causes the gauge theory to break down at large energy scales. We should note that our M-theory setup reproduces precisely this logarithmic behavior of the supergravity side, which has not been considered previously in this way.

1.1 Outline of the thesis

We start this thesis with a review of F-theory and related concepts in Chapter 2. Then Chapter 3 gives a short summary on the most important aspects of the D3/D7-brane system for the AdS/CFT correspondence. Thereafter Chapter 4 discusses the geometries that we will need in the remaining part of this thesis. In Chapter 5 these spaces describe part of the space transverse to a stack of M2-branes, and we consider the backreaction of these M2-branes on them. In Chapter 6 we consider the backreaction of 7-brane fluxes on these geometries.

For a reader with insufficient string theory knowledge, we refer to books such as [13, 14, 15, 16, 17], and we point out [18] for readers with a lacking mathematical background.

1.2 Units and conventions

We will work with metric signature $(-,+,\ldots,+)$ throughout this thesis.

The gravitational constant κ of a d-dimensional theory can be related to the characteristic length scale ℓ of the theory. Similar relations hold for the parameters T_p and μ_p of Dp-branes and Mp-branes. Our conventions can be summarized in the following way

$$\frac{1}{2\kappa^2} = \frac{2\pi}{\ell^{d-2}}, \quad T_p = \mu_p = \frac{2\pi}{\ell^{p+1}}. \tag{1.1}$$

Note that have length $\ell_s = 2\pi\sqrt{\alpha'}$ for Type II string theories, and ℓ_M for M-theory (which is related by $\ell_M = 2\pi\ell_{11}$ to the conventions in [13]).

We will denote the rank of a differential form by a subscript, i.e. a p-form C_p, and it can be written as

$$C_p = \frac{1}{p!}(C_p)_{\mu_1\cdots\mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}. \tag{1.2}$$

The Hodge star operator $*$ is defined by

$$*C_p = \frac{\sqrt{\pm g}}{p!(d-p)!} C_{\mu_1\cdots\mu_p} \varepsilon^{\mu_1\cdots\mu_p}{}_{\nu_{p+1}\ldots\nu_d} dx^{\nu_{p+1}} \wedge \ldots \wedge dx^{\nu_d}, \tag{1.3}$$

where ε with lower indices is the antisymmetric Levi-Civita tensor, and the $\pm$-sign depends on if we are dealing with a Euclidean/Lorentzian manifold. Note that, using these conventions, we have

$$F_p \wedge *F_p = d^d x \sqrt{\pm g} \frac{1}{p!}(F_p)_{\mu_1\cdots\mu_p}(F_p)^{\mu_1\cdots\mu_p}. \tag{1.4}$$

We typically use (r,ϕ,z) as cylindrical coordinates on the 3-dimensional base of the geometries that we will consider. Then the vielbeins $e^r = dr$, $e^\phi = r d\phi$, $e^z = dz$ such that $e_r \wedge e_\phi \wedge e_z$ is the volume form. For calculating Hodge duals, it is useful to point out that

$$*(e^{i_1} \wedge \ldots \wedge e^{i_k}) = e^{i_{k+1}} \wedge \ldots \wedge e^{i_n}, \tag{1.5}$$

where $(i_1, \ldots, i_n)$ must be an even permutation of $(1, \ldots, n)$. For the periodified case, one must use $e_z = r_B dz$ instead. If we include the circle fibration, we must define instead

$$\begin{aligned} \tilde{e}^0 &= \frac{1}{\sqrt{V}}(dt + U), & \tilde{e}^r &= \sqrt{V} e^r, \\ \tilde{e}^\phi &= \sqrt{V} e^\phi, & \tilde{e}^z &= \sqrt{V} e^z, \end{aligned} \tag{1.6}$$

with $\tilde{e}^0 \wedge \tilde{e}^r \wedge \tilde{e}^\phi \wedge \tilde{e}^z$ as volume element, which specifies the orientation we use as well.

Chapter 2

F-theory and M-theory Preliminaries

F-theory describes a class of non-perturbative string vacua, namely vacua involving 7-branes. Ordinarily, one would try to describe these vacua through perturbative Type IIB string theory. However, due to the fact that 7-branes source a non-trivial profile for the string coupling, another description might be more appropriate. From the SL(2,$\mathbb{Z}$)-symmetry that Type IIB possesses, one can motivate the existence of two additional dimensions, in the form of a torus with vanishing area. The symmetry would be realized through the modular group of the torus, and the string coupling would be part of its complex structure parameter. The fact that this approach embeds the non-trivial profile of the string coupling into the geometry, makes it promising for describing the 7-brane physics, but it turns out that there exists no twelve-dimensional supergravity with Lorentzian signature.

Luckily, M-theory provides us with a viable alternative. By compactification on a torus, we can relate it via Type IIA string theory and T-duality to Type IIB string theory. Then by allowing the torus parameters to vary over the remaining nine-dimensional space, or part thereof, it turns out that we can realize the non-trivial profile due to 7-branes in this M-theory setup as well.

This latter approach via M-theory, is the approach that we will use in this thesis, so it is important that we discuss it thoroughly first. We start with the basics of Type IIB string theory, focusing on the SL(2,$\mathbb{Z}$)-symmetry and 7-branes. Then we explain the relevant aspects of M-theory, such as the aforementioned duality with Type IIB. Once we have covered these basics, we are ready to move on to the approach of F-theory via M-theory. Especially important will be the discussion on warped metrics, since this is the main object of study in this thesis. Furthermore, we will try to highlight all concepts that will be needed in later chapters.

For the interested reader, more extensive reviews on F-theory are [19, 20, 21], upon which we draw heavily.

2.1 Type IIB supergravity

The spectrum of oriented closed strings can be divided in several sectors. Because we can relate the fermionic modes to the bosonic modes by supersymmetry, it is sufficient to state only the bosonic sectors. Furthermore, since we are interested in low-energy behavior, we will restrict our discussion to the massless modes. The sector that is identical for Type IIA and Type IIB is the NS-NS sector, which contains the dilaton ϕ, the metric g and the Kalb-Ramond potential B_2. The other sector is the R-R sector, and contains the potentials C_1, C_3 for Type IIA, and C_0, C_2, C_4 for Type IIB. Due to our main interest in Type IIB, this will be our focus here. The field strengths of the Type IIB potentials are given by

$$\begin{aligned} H_3 &= dB_2, \quad F_1 = dC_0, \quad F_3 = dC_2 - C_0 dB_2, \\ F_5 &= dC_4 - \frac{1}{2}C_2 \wedge dB_2 + \frac{1}{2}B_2 \wedge dC_2. \end{aligned} \tag{2.1}$$

Now, since Type IIB string theory is a chiral theory of supersymmetry and gravity, it follows that the massless modes should be described by the unique chiral $\mathcal{N} = 2$ supergravity. Indeed, computation of the tree level string scattering amplitudes results in the associated action. This action can be expressed in the string frame as

$$\begin{aligned} S^{\text{IIB}}_{\text{str}} =& \frac{1}{2\kappa_{10}^2}\int_{\mathcal{M}_{10}} e^{-2\phi}\left(R * 1 - \frac{1}{2}H_3 \wedge *H_3 + 4d\phi \wedge *d\phi\right) \\ &- \frac{1}{4\kappa_{10}^2}\int_{\mathcal{M}_{10}}\left(F_1 \wedge *F_1 + F_3 \wedge *F_3 + \frac{1}{2}F_5 \wedge *F_5 + C_4 \wedge H_3 \wedge F_3\right). \end{aligned} \tag{2.2}$$

It turns out that the equations of motion following from the action are not sufficient. Namely, each C_p has a magnetic dual C_{8-p}. And since we did not include these duals in our formulation, i.e. integrated them out, we must impose the Bianchi identity $dF_p = 0$ by hand. However, for C_4 this is a bit more subtle, since it is its own dual. Hence we need to impose the self-duality condition on its field strength

$$F_5 = *F_5. \tag{2.3}$$

The equations of motion following from the action, together with this self-duality condition and the Bianchi identities, yield a complete description of Type IIB supergravity.

2.1.1 SL$(2,\mathbb{Z})$-symmetry

At supergravity level, we can show that the theory possesses an SL$(2,\mathbb{R})$-symmetry. Then we will argue that, by semi-classical arguments, it is broken down to SL$(2,\mathbb{Z})$ for the full string theory. Let us also mention that, for historical reasons, this symmetry is often called S-duality. And because the SL$(2,\mathbb{Z})$ group appears as the modular group of the torus as well, the associated transformations are typically called modular transformations.

For verifying this symmetry property, the string frame action is apparently not the most convenient expression. Namely, the string frame metric depends on the dilaton, and S-duality transformations act on the dilaton. Hence it is convenient to switch to the Einstein frame, in which the metric is independent of the dilaton. We can transform to this frame by applying a Weyl transformation $g \to \Omega^{-2} g$ on the metric, if we choose $\Omega = e^{-\phi/4}$. Furthermore, it is useful to redefine our fields in the axio-dilaton τ and G_3 as

$$\tau = C_0 + ie^{-\phi}, \quad G_3 = F_3 - ie^{-\phi} H_3. \tag{2.4}$$

In combination with the Weyl transformation, this results in the following expression for the Einstein frame action

$$\begin{aligned} S_{\mathrm{E}}^{\mathrm{IIB}} =& \frac{1}{2\kappa_{10}^2} \int \left(R * 1 - \frac{d\tau \wedge *d\bar{\tau}}{2\tau_2} - \frac{G_3 \wedge *\bar{G}_3}{2\tau_2} \right) \\ &+ \frac{1}{2\kappa_{10}^2} \int \left(-\frac{1}{4} F_5 \wedge *F_5 + \frac{C_4 \wedge G_3 \wedge \bar{G}_3}{4i\tau_2} \right). \end{aligned} \tag{2.5}$$

Now consider an element of the symmetry group, given by

$$\begin{pmatrix} r & q \\ s & p \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}). \tag{2.6}$$

Then it acts on our redefined field content as

$$\tau \to \frac{r\tau + q}{s\tau + p}, \quad G_3 \to \frac{G_3}{s\tau + p}. \tag{2.7}$$

Alternatively, it acts on the 2-form potentials as

$$\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to \begin{pmatrix} r & q \\ s & p \end{pmatrix} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix}. \tag{2.8}$$

Using the second transformation rule, one can quickly verify that F_5 is invariant. Hence its kinetic term and the self-duality condition are both invariant. Then, using the first transformation rule, one can verify that the rest of the action is invariant as well.

Thus we have established that Type IIB supergravity has an $\mathrm{SL}(2, \mathbb{R})$-symmetry, and we are ready to argue that the full string theory has an $\mathrm{SL}(2, \mathbb{Z})$ symmetry at most. The first argument for a reduced symmetry group at the quantum level, involves D(-1)-instantons. Namely, using the D-brane solutions, which we will discuss in subsection 2.2.1, one can compute that $S_{\mathrm{inst}} = 2\pi\tau$. Then, through the path integral formalism, we need quantized shifts in τ, such that $\exp(iS_{\mathrm{inst}})$ keeps the same value. As an example, we can map $\tau \to \tau + s$ by $r = p = 1$ and $s = 0$, and this would fix $q \in \mathbb{Z}$.

Another argument involves the fact that F1-strings and D1-strings are charged under our supergravity fields. For instance, the F1-string is coupled to the NS-NS 2-form B_2 via

$$S_{\mathrm{F1}} \supset \frac{1}{2\pi\alpha'} \int_{\mathcal{W}_{1,1}} B_2, \tag{2.9}$$

where $\mathcal{W}_{1,1}$ denotes the string worldsheet. From this term, it follows that B_2 is sourced electrically by F1-strings. Similarly, D1-strings are charged electrically under the R-R 2-form C_2, via the Chern-Simons term of D1-strings.

Then, if we perform an S-duality transformation, we map an F1-string into an object with p units of F1-string charge and q units of D1-string charge. We will encounter this object in subsection 2.2.4 as a (p,q)-string, which is a generalization of the ordinary F1-strings and D1-strings through this symmetry. However, since we can not consider fractional strings, this restricts once again part of our symmetry group to integer coefficients. And if we combine the restrictions we obtained so far, it turns out that we have only an $\mathrm{SL}(2,\mathbb{Z})$ symmetry group for the full string theory at most.

As a last remark for this section, let us point out that we have already found evidence for the geometrical approach at the modular group. Consider a twelve-dimensional theory on a torus. Then its complex structure parameter would appear precisely as τ in the kinetic term of the Einstein frame action (Eq. (2.5)). As second argument, consider a 3-form potential in this 12-dimensional theory. Then its Kaluza-Klein reduction along one of the torus direction would result in one of the 2-forms C_2 or B_2, with exactly the $\mathrm{SL}(2,\mathbb{Z})$-symmetry that acts on them. However, this is were the evidence stops pointing in our favor as well, since we do not have a 1-form, corresponding to the reduction of this 3-form along both directions. Another problem is the fact that there exists no twelve-dimensional supergravity with a Lorentzian signature, although there have been attempts through supergravities with different signatures [1, 2, 3]. For now, we will leave this geometric approach for what it is, and return to the idea in section 2.4.

2.2 D-branes and O-planes

Since we have covered the basics of closed strings at supergravity level, we are ready to include open strings to our theory. This feature can be achieved through the presence of D-branes, on which the open strings can end. First we review the basics of these objects, such as the terms they contribute to the action. Then we discuss orientifold projections, and how these lead to an $\mathrm{SO}(2k)$ gauge enhancement instead of an $\mathrm{SU}(k)$ gauge enhancement. We will conclude with arguments that stress the importance of the $\mathrm{SL}(2,\mathbb{Z})$ symmetry group, most importantly the backreaction of 7-branes.

2.2.1 Basics of D-branes

As suggested above, the presence of D-branes requires additional terms for our action. Similar to the terms for the Type IIB supergravity, they can be calculated from string scatterings. It is important to note, however, that these terms are localized to the world-volume of the D-brane. The first term that we consider is the Dirac-Born-Infeld (DBI) action. Most importantly, it captures the physics of the gauge field $A_{\mathrm{D}p}$, that arises from the open strings that end on the D-brane. It is given below

$$S_{\mathrm{DBI}} = -T_p \int_{\mathcal{W}_{1,p}} d^{p+1}\xi e^{-\phi} \sqrt{-\det(\pi^*(g) + \pi^*(B_2) + 2\pi\alpha' F_{\mathrm{D}p})}, \tag{2.10}$$

where π^* denotes the pull-back to the D-brane worldvolume $\mathcal{W}_{1,p}$, and $F_{\mathrm{D}p}$ the field strength of the gauge field. Note that this term only describes a single D-brane, since the complete non-linear generalization to a stack of k D-branes is not yet known. We do, however, know, that it should be described by an SU(k) gauge theory at lowest order in coupling.

The other term that arises due to the D-branes, is a topological term, the Chern-Simons (CS) action. Because its full expression is rather involved, we only include the part of its content that is relevant for this thesis

$$S_{\mathrm{CS}} \supset \mu_p \int_{\mathcal{W}_{1,p}} e^{-\pi^*(B)-2\pi\alpha' F_{\mathrm{D}p}} \sum_q \pi^*(C_q), \tag{2.11}$$

where the exponential should be calculated through its Taylor series with the wedge product. Through the term $\mu_p \pi^*(C_{p+1})$ from this CS action, it follows that the R-R forms are sourced by D-branes, as mentioned before in the context of the D1-string.

Another important aspect of D-branes for this thesis, is their backreaction on the geometry, which results in a warp factor. From general considerations, one would expect this backreaction to preserve Poincaré symmetry in the directions parallel to the D-brane, but break it to a rotational symmetry around the object in the other directions. This was precisely the reasoning used in [22], where they found a class of brane solutions to supergravity. The Dp-brane interpretation for these solutions, i.e. that open strings end on this object and that it sources the R-R $(p+1)$-form, came later in [23].

Either way, it results in a warped metric, where the warping will be described by a function A, which depends on the transverse coordinates $x_\perp$. This solution can be summarized by the equations below

$$\begin{aligned} &ds^2 = e^{-A/2} ds_\parallel^2 + e^{A/2} ds_\perp^2, \\ &C_{p+1} = g_s^{-1} e^{-A} \mathrm{dvol}_\parallel, \qquad\qquad e^\phi = g_s e^{(3-p)A/4}, \end{aligned} \tag{2.12}$$

where we split the metric in a parallel and a transverse part, with $\mathrm{dvol}_\parallel$ denoting the volume form of the parallel space.

The behavior of the warp factor can be most easily analyzed through the equations of motion of C_{p+1}, from which we obtain for a stack of N Dp-branes in $\mathbb{R}^{1,9}$

$$\Delta_\perp e^A = 2N\kappa_{10}^2 \mu_p \delta(x_\perp). \tag{2.13}$$

Such an harmonic equation can be solved by the function $Q_{\mathrm{D}p} r_\perp^{-(p-7)}$ (for $p < 7$), where $r_\perp$ denotes the transverse radial distance. One can identify the charge $Q_{\mathrm{D}p}$ of this solution by integrating the equation over a sphere in the transverse space.[1] Furthermore, we need to add 1 to our solution, to recover the background geometry $\mathbb{R}^{1,9}$ asymptotically far away. Then we obtain

$$e^A = 1 + \frac{Q_{\mathrm{D}p}}{r_\perp^{7-p}}, \tag{2.14}$$

[1] Charge $Q_{\mathrm{D}p} = (4\pi)^{(5-p)/2}\Gamma((7-p)/2)(\alpha')^{(7-p)/2} g_s N$, following [13].

This solution will be useful throughout this thesis. Namely, we will try to solve the harmonic equation for more complicated background geometries. However, these geometries typically have limits in which they look like $\mathbb{R}^{1,9}$, or other variants of Minkowski spacetime. Then we can solve the complicated harmonic equation, and select the solution (up to a pre-factor) that is singular at the position of the D-brane. This pre-factor can be fixed by taking the limit to Minkowski spacetime, and match our solution with Equation (2.14). It is also the reason that we can often neglect factors of ℓ_s in our equations, since we will fix these in this manner. Note that this procedure applies for M-branes as well, as we will see in Chapter 5.

2.2.2 Basics of O-planes

In addition to D-branes, we can also add so-called O-planes to our setups. These objects are the result of the orientifold projection involving the string reversal operator Ω and a $\mathbb{Z}_2$ involution σ on the geometry. Besides these operations, one might also need an additional minus sign $(-1)^{F_L}$ for the left-moving worldsheet fermions, depending on the number of coordinates that σ changes. This minus sign is needed for O7-planes for instance, but we will neglect this aspect here. The inclusion of unoriented string worldsheets in the calculation of string amplitudes results in this object. Our discussion below is due to [24, 25].

The string reversal operator Ω acts on the worldsheet coordinates of the string, which results in an exchange of its left- and right-moving modes. For our purposes, it is useful to point out that it maps the left-moving massless vector mode of the open string into minus the right-moving mode, and vice versa. Of course, we still have a choice of sign for the way Ω acts on the vacuum, and hence a positive (negative) sign would send the left-moving massless vector state into minus (plus) the right-moving state.

As an example, we consider an involution σ that reflects the coordinates $x_{p+2}, \ldots, x_{10}$ of $\mathbb{R}^{1,9}$, together with k Dp-branes and k image Dp-branes. Then, if we let σ act on an open string state between a D-brane and an image D-brane, it interchanges the Chan-Paton factors of the state.

Now we are interested in the combined operation $\Omega\sigma$, especially in the context $2k$ Dp-branes at $x_{p+2} = \ldots = x_{10} = 0$. Normally, stacking $2k$ D-branes results in an $\mathrm{SU}(2k)$ gauge enhancement, but the operation $\Omega\sigma$ projects out part of its massless vectors. If Ω acts with a positive sign on the string vacuum, combined with σ interchanging the CP factors, this results in the anti-symmetrization of the generators, and hence an $\mathrm{SO}(2k)$ gauge enhancement instead. Similarly, a negative sign will result in the generators of a symplectic gauge group.

As mentioned, it turns out that the orientifold projection $\Omega\sigma$ gives rise to an object in our theory, called an O-plane. This object is located at the fixed locus of σ, or if this locus contains several disconnected components, there is an O-plane for each of these components. In our example, it corresponds precisely to the subspace specified by $x_{p+2} = \ldots = x_{10} = 0$. This explains the phrase that stacking D-branes on top of an O-plane results in orthogonal or symplectic gauge enhancements. Note as well that, if the reflection acts on the coordinate of a circle, we have two fixed points instead of one, and hence two O-planes.

The sign of Ω acting on the vacuum is directly related to the charge of the O-plane, and it is negative if Ω acts with a plus sign, and vice versa. More precisely, its parameters can be expressed in D-brane parameters as $T_{\mathrm{O}p} = \pm 2^{p-4} T_p$ and $\mu_{\mathrm{O}p} = \pm 2^{p-4}\mu_p$. Similarly, we can also write down a DBI-term and CS-term for this object. However, since it is a non-dynamical object, it does not have an associated gauge field, and it also does not couple to Kalb-Ramond field B_2. However, this does mean that it contributes to the equation of motion of C_{p+1}. If the O-plane and D-branes are located in a compact manifold without boundary, integration of ΔC_p over the transverse space gives zero. Hence we find the following condition

$$\sum_i \mu_p D_i + \mu_{\mathrm{O}p} D_{\mathrm{O}p} = 0 \tag{2.15}$$

where D_i and $D_{\mathrm{O}p}$ denote the divisors of the manifold that the D-branes and O-plane wrap respectively. For historical reasons, this equation is often called the tadpole condition, due to its origin in tadpole diagrams. For the tadpoles to be cancelled, we need an O-plane with negative charge, and 2^{p-5} brane/image-brane pairs ($p > 4$). Thus, as an example, if we stack 32 D9-branes and an O9-plane together, this gives rise to an SO(32) gauge group, and this Type IIB orientifold is indeed dual to Type I string theory, which has exactly this gauge group.

2.2.3 Backreaction of 7-branes

Following the backreaction of Dp-branes we considered previously (for $p < 7$), we are ready to discuss the more complicated case for $p = 7$. It is typical of codimension two objects to source logarithmic dependence on the transverse coordinates, and these 7-branes are no exceptions to this rule. Here we will analyze this behavior thoroughly, which is due to [26, 13].

Instead of parametrizing the transverse space by real coordinates x, y, it is useful to define a complex coordinate $z = x + iy$. Then supersymmetry conditions, following from the dilatini (see for instance [9]), imply that τ must be a holomorphic or anti-holomorphic function in z.[2] Without loss of generality, we will consider τ to be a holomorphic function.

First let us consider the case of a stack of k D7-branes at the origin $z = 0$. We know that these objects couples to the R-R form C_8 via the CS-term, which is the magnetic dual of C_0. Hence we obtain as equation of motion

$$d * F_9 = dF_1 = k\delta(z). \tag{2.16}$$

[2]Meromorphic or anti-meromorphic in the presence of singularities due to 7-branes.

We can analyze the behaviour of C_0 by integrating this equation over a disc D in the transverse space, centered at $z = 0$. Then we can rewrite this via Stokes' theorem into an integral over a circle S, and we find

$$\int_D dF_1 = \oint_S F_1 = \oint_S dC_0 = k. \tag{2.17}$$

Thus, by circling around the D7-branes, we find that the R-R 0-form shifts by $C_0 \to C_0 + k$. This means that the presence of 7-branes induces a monodromy for τ, and that this field is not single-valued. Normally this would be a problem for a theory, but in this case it corresponds precisely to the $\mathrm{SL}(2,\mathbb{Z})$ of Type IIB, namely the modular transformation with $r = p = 1$, $q = k$ and $s = 0$. Similar behaviour is showed by the complex logarithm $\log z$. Actually, we can even solve our equation of motion with this function, although it will turn out to be a naive solution. This solution is given by

$$\tau = \tau^0 + \frac{k}{2\pi i}\log z, \tag{2.18}$$

where τ^0 is the integration constant of the solution. Close to the origin we find that τ_2, the imaginary part of τ, diverges to ∞. This means that the string coupling becomes very small, and hence that the solution is reliable. However, if we move further away from the origin, τ_2 becomes negative, and therefore the string coupling as well, which implies that our solution breaks down at large distances. Later, we will find that we can solve this problem by using the SL(2,$\mathbb{Z}$) symmetry of our theory.

First, now that we have established that τ has a non-trivial profile, we will analyze the implications this makes for the energy of our solutions, due to its kinetic term. Of course, we must take the backreaction of the 7-branes on the geometry into account as well. This gives us as ansatz for the metric in the Einstein frame

$$ds^2 = ds_{\parallel}^2 + e^{B(z,\bar{z})}dz d\bar{z}, \tag{2.19}$$

where B describes the warping due to the 7-branes. Actually, we find that this warping does not alter the equation for τ, since its equation of motion is given by

$$\partial\bar{\partial}\tau + \frac{2\partial\tau\bar{\partial}\tau}{\bar{\tau} - \tau} = 0. \tag{2.20}$$

We can derive the kinetic energy for τ from the Einstein frame action (Eq. (2.5)). Since the non-trivial dependence of τ is restricted to the transverse space, we only need to integrate the kinetic term over the complex z-plane. Then we obtain as energy density

$$\mathcal{E} = -\frac{i}{\kappa_{10}^2}\int d^2 z \frac{\partial\tau\bar{\partial}\bar{\tau}}{(\tau - \bar{\tau})^2}. \tag{2.21}$$

By a change of coordinates, we can be rewrite this expression as an integral over τ, with the image of τ as integration domain. Furthermore, we can apply Stokes' theorem, such that we only need to integrate over the boundary of this domain. This yields

$$\begin{aligned}\mathcal{E} &= -\frac{i}{\kappa_{10}^2}\int d^2\tau \partial_\tau \bar{\partial}_{\bar{\tau}} \log(\tau - \bar{\tau}) = \frac{i}{\kappa_{10}^2}\int d\tau \partial_\tau \log(\tau - \bar{\tau}) \\ &= \frac{i}{\kappa_{10}^2}\int d\tau \frac{1}{\tau - \bar{\tau}}.\end{aligned} \tag{2.22}$$

Clearly this result is not finite, if the image of τ is the entire complex plane. Hence we need our $\mathrm{SL}(2,\mathbb{Z})$ symmetry once again for consistency. Using this symmetry, we can restrict the integral to the fundamental domain F of the modular group.[3] Then we find

$$\mathcal{E} = \frac{i}{\kappa_{10}^2}\int_{\partial F} d\tau \frac{1}{\tau - \bar{\tau}} = \frac{\pi}{6\kappa_{10}^2}. \tag{2.23}$$

Motivated by this need for the SL(2,$\mathbb{Z}$)-symmetry, we can use the modular invariant j-function, which uniquely maps the fundamental domain τ to the complex plane.[4] Since the explicit expression for this function is rather complicated, let us only state its expansion in $q = \exp(2\pi i\tau)$, which turns out to be sufficient for our purposes anyway. It is given by

$$j(\tau) = \frac{1}{q} + 744 + 196884q + \dots. \tag{2.24}$$

We can use this expansion to determine the leading order behavior of τ in certain limits. First, we will analyze the region around the pole $q = 0$, where we should recover the weak-coupling limit we found previously (Eq. (2.18)). Therefore, we suggest that the j-function equals the following meromorphic function

$$j(\tau(z)) = c + (\lambda/z)^k, \tag{2.25}$$

where c corresponds to the asymptotic value of τ, and λ corresponds to the length scale associated with the weak-coupling region. Indeed, we retrieve in the limit $z/\lambda \ll 1$ as leading order behavior

$$e^{-2\pi i\tau} \simeq (\lambda/z)^n \quad \Longrightarrow \quad \tau \simeq \frac{k}{2\pi i}\log(z/\lambda), \tag{2.26}$$

whereas we find $\tau \simeq j^{-1}(c)$ for $z/d \gg 1$, instead of the breakdown of our solution. This Equation (2.25), specifies the complete profile for τ.

Now we can use this solution to construct the warp factor. From the Einstein equations we find that

$$\partial\bar{\partial}B = \partial\bar{\partial}\log(\tau_2). \tag{2.27}$$

It means that we can identify e^B and τ_2 up to multiplication by a holomorphic and an anti-holomorphic function. Hence we suggest the following ansatz

$$e^{B(z,\bar{z})} = f(z)\bar{f}(\bar{z})\tau_2 \tag{2.28}$$

We need the Einstein frame metric to be modular invariant, so e^B needs to be invariant under modular transformations. Since τ_2 transforms under the modular group, we must counter this by an appropriate choice of f. This can be achieved by setting

$$f(z) = \eta^2(\tau)g(z). \tag{2.29}$$

[3]This domain is specified by the conditions $|\tau| > 1$, $|\tau_1| < 1/2$.

[4]More precisely, it maps to the Riemann sphere $\mathbb{C} \cup \{\infty\}$, and the cusp, $\tau = i\infty$, maps to ∞.

Furthermore, we do not want the metric to be degenerate. Therefore, the warp factor e^B must be non-zero everywhere. Near the origin we find that the warp factor behaves as $\tau_2|(z/\lambda)^{k/12}|^2|g(z)|^2$. Accordingly, we must fix g to be

$$g(z) = (z/\lambda)^{-k/12}. \tag{2.30}$$

In addition to solving the Einstein equations, this choice turns out to solve the singular part of our equations due to the 7-branes as well. The final result is given by

$$e^{B(z,\bar{z})} = \tau_2 \frac{\eta^2(\tau)\bar{\eta}^2(\bar{\tau})}{|z/\lambda|^{k/6}}. \tag{2.31}$$

The behavior of this solution near the origin has been considered, but we have not discussed the how the metric behaves for large z/λ yet. In this limit, we find that τ behaves as $j^{-1}(c)$, and hence $\eta(\tau)$ is constant as well. Then the metric behaves at leading order as

$$ds_\perp^2 \sim |(z/\lambda)^{-k/12}dz|^2 = (r/\lambda)^{-k/6}(dr^2 + r^2 d\phi^2), \tag{2.32}$$

where we switched to polar coordinates via $z = re^{i\phi}$. Such an expression for the metric means that the space has a deficit angle. Let us make this explicit by defining a set of alternative coordinates

$$\begin{aligned} \rho &= \frac{1}{1-k/12}(r/\lambda)^{1-k/12}, \\ \theta &= (1-k/12)\phi. \end{aligned} \tag{2.33}$$

In this coordinate system, the transverse metric looks like flat space

$$ds_\perp^2 \sim d\rho^2 + \rho^2 d\theta^2, \tag{2.34}$$

but we have $\theta \in [0, 2\pi(1-k/12)]$ instead. From this restricted domain for the angle, we find that each D7-brane contributes $\pi/6$ to a deficit angle. And for $12 \leq n \leq 24$, this leads us to rather non-trivial spaces. Namely, for $n = 12$, we find that the transverse space is a cylinder. For $12 < n < 24$, we have a transverse space that is not smooth. And last, for $n = 24$, we find that the space even becomes compact. Topologically it is equal to $\mathbb{P}^1$, the complex projective space. And if we combine it with τ into a torus fibration over $\mathbb{P}^1$, we obtain a K3 surface.

Especially the last case is interesting, since it motivates us to study the transverse space via a Calabi-Yau 2-fold. This would give us a way to study the 7-branes beyond the weak-coupling regime. We can generalize the idea to a higher-dimensional Kähler base, instead of $\mathbb{P}^1$, as well, which would lead us to study Calabi-Yau n-folds. Through these complex manifolds, we would also be able to allow for $\mathrm{SL}(2,\mathbb{Z})$ monodromies different from those due to D7-branes. These monodromies correspond to the 7-branes that we discuss in the next subsection.

Another important notion about this solution, especially for the AdS/CFT context, is that it does not possess spherical symmetry in the complex z-plane. Although the metric does have this feature close to the D7-branes, this is no longer the case for larger length scales. For instance, in the relation for τ in Eq. (2.25), only $z \to e^{2\pi i n/k}z$ (for $n \in \mathbb{Z}$) is a symmetry, instead of generic rotations.

2.2.4 Monodromies and (p, q) 7-branes

So far, we have established that dealing with D7-branes needs the SL(2,$\mathbb{Z}$)-symmetry for various consistency conditions. We also found that this symmetry group relates ordinary F1-strings to objects with p units of F1-string charge and q units of D1-string charge. It inspires us to consider these new objects, which we shall call (p, q)-strings, as well. We can even extend this idea by considering (p, q) 7-branes, i.e. 8-dimensional hypersurfaces on which open (p, q)-strings end. These objects were originally proposed in [27], upon which this subsection draws heavily, together with [28].

We know that the monodromy matrix due to circling around a D7-brane is given by

$$M_{1,0} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.35}$$

We also know that we can map an F1-string into a (p, q)-string as

$$(q, p) = (0, 1) \begin{pmatrix} r & s \\ q & p \end{pmatrix}. \tag{2.36}$$

Note that we must require $pr - qs = 1$ such that we our (p, q)-string can not decompose into a multiple-string solution. Combining this map with the monodromy matrix of a D7-brane, it follows that the monodromy matrix for a (p, q) 7-brane is given by

$$M_{p,q} = \begin{pmatrix} r & s \\ q & p \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r & s \\ q & p \end{pmatrix} = \begin{pmatrix} 1 + pq & p^2 \\ -q^2 & 1 - pq \end{pmatrix}. \tag{2.37}$$

Let us mention that we can argue the monodromy matrix of the O7-plane as well, from the tadpole condition. First consider an O7-plane with 4 coincident D7-branes in the quotient space. Then moving around this stack in the quotient space corresponds to moving to the opposite point in the full space. Hence the monodromy matrix for this system can be argued from $\Omega(-1)^{F_L}$ acting on $(C_2, B_2)^{\mathrm{T}}$, since it should correspond to our orientifold projection. It is known that B_2 has negative Ω-parity, whereas C_2 is in the R-R sector, and hence gets a minus sign under $(-1)^{F_L}$. Thus we find minus the identity matrix as associated monodromy matrix for this stack. Then if we invert the monodromy due to the D7-branes, we obtain for the O7-plane

$$M_{\mathrm{O7}} = -M_{1,0}^{-4} = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}. \tag{2.38}$$

As a last remark, we should point out that a geometric approach can be very useful in the context of these (p, q) 7-branes. For instance, if one knows the full profile of τ, one can identify an object simply by the monodromy around it. Furthermore, via Type IIB it is not immediately clear how to determine which setups of 7-branes lead to consistent theories, and which do not. But via the geometric approach, these consistency conditions will be embedded in the geometry itself, which we discuss explicitly in section 2.4.1.

2.3 M-theory and eleven-dimensional supergravity

Now that we have stressed the most important features of the Type IIB perspective on F-theory, we can start with the M-theory perspective on F-theory. First, for completeness, we shall discuss the basics of M-theory in this section, and highlight its relation to Type IIB. This discussion below is due to [13], and we use the procedure as outlined in e.g. [19] for duality with Type IIB.

Since M-theory is a theory of supersymmetry and gravity in eleven dimensions, it can be argued that, at long wavelengths, it should be described by the eleven-dimensional supergravity. The action for its bosonic content is given by

$$S_{\mathrm{M}} = \frac{1}{2\kappa_{11}^2}\int\left(R*1-\frac{1}{2}G_4\wedge *G_4-\frac{1}{6}A_3\wedge G_4\wedge G_4+\ell_M^6 A_3\wedge X_8\right), \tag{2.39}$$

which contains a 4-form field strength $G_4 = dA_3$, and a polynomial of fourth order in the Riemann curvature X_8, besides the metric. Similar to the supergravity fields in Type II string theories, this 3-form potential A_3 can be sourced by objects. Its electric sources are the M2-branes, and its magnetic sources are the M5-branes. The coupling to the M2-brane can be described by

$$S_{\mathrm{M2}} = -T_{\mathrm{M2}}\int_{\mathcal{W}_{1,2}} A_3. \tag{2.40}$$

It turns out that this eleven-dimensional supergravity can be related to Type IIA string theory. Namely, in the strong coupling limit of this theory ($g_{\mathrm{IIA}} \to \infty$), all D-branes become light objects. Then the lightest D-brane is the D0-brane, with mass $2\pi/(g_{\mathrm{IIA}}\ell_s)$, which follows from their tension T_0, similar to string tension and their mass. Bound BPS states of N D0-branes have N times this mass, and can be identified with the Kaluza-Klein states of a circle compactification with the following length

$$L = 2\pi R = g_{\mathrm{IIA}}\ell_s. \tag{2.41}$$

Hence Type IIA at strong coupling gives rise to an eleven-dimensional theory of supersymmetry and gravity, and if we consider only massless modes, this suggests that it should be described by the eleven-dimensional supergravity. Therefore we have argued that, at supergravity level, Type IIA at strong coupling is dual to M-theory on a circle.

The gravitational constant of this eleven-dimensional theory must be related to the ten-dimensional gravitational constant through a simple multiplication with the circle length

$$\kappa_M^2 = L\kappa_{10}^2 = \frac{1}{4\pi}(g_{\mathrm{IIA}})^3\ell_s^9. \tag{2.42}$$

Combining this with $4\pi\kappa_M^2 = \ell_M^9$, we find

$$\ell_M^3 = g_{\mathrm{IIA}}\ell_s^3. \tag{2.43}$$

Consistency with these parameters, since we can relate g_{IIA} and the background value of ϕ^{IIA}, suggests that we can relate the metrics in the following manner

$$\begin{aligned} \ell_M^{-2} ds_{\mathrm{M}}^2 &= e^{\frac{4}{3}\phi^{\mathrm{IIA}}}(dx+\ell_s^{-1}C_1^{\mathrm{IIA}})^2 + e^{-\frac{2}{3}\phi^{\mathrm{IIA}}}, \ell_s^{-2} ds_{\mathrm{IIA}}^2, \\ \ell_M^{-3}A_3 &= \ell_s^{-3}C_3^{\mathrm{IIA}} + \ell_s^{-2}B_2^{\mathrm{IIA}}\wedge dx. \end{aligned} \tag{2.44}$$

The first equation suggests that the metric, dilaton and the RR 1-form of Type IIA lift to the geometry of M-theory, as we have mentioned before. One can verify that compactification of the eleven-dimensional supergravity in this manner reproduces Type IIA supergravity, but this verification is rather involved and detailed, and will not be included here.

2.3.1 Duality with Type IIB

Next, we want to use this duality with Type IIA to find a duality with Type IIB. We know that we can relate Type IIA and Type IIB through a T-duality, and that the lengths of the respective circles are related by

$$L_B = \frac{\ell_s^2}{L_A} \tag{2.45}$$

Hence a small circle on the Type IIA side is dual to a large circle on the Type IIB side. With this identification, we can motivate that a spacetime direction in Type IIB can be related to Type IIA compactified on a circle. Using this reasoning, we want to compactify M-theory on a torus with vanishing area v, which we already mentioned as the F-theory limit. It also turns out that the SL(2,$\mathbb{Z}$) symmetry of type IIB can be related to the modular group of this torus.

Our starting point is the following expression for the metric

$$ds^2_{\mathrm{M}} = ds^2_{1,8} + \frac{v}{\tau_2}((dx_A + \tau_1 dx_B)^2 + \tau_2^2 dx_B^2), \tag{2.46}$$

where we will use the x_A-circle for duality with Type IIA, and the x_B-circle for duality with Type IIB. Then reduction along the x_A-circle gives as Type IIA content

$$\begin{aligned} ds^2_{\mathrm{IIA}} &= \frac{\ell_s^2}{\ell_M^2} e^{\frac{2}{3}\phi^{\mathrm{IIA}}}(ds^2_{1,8} + v\tau_2 dx_B^2) \\ e^{\frac{4}{3}\phi^{\mathrm{IIA}}} &= \ell_M^{-2}\frac{v}{\tau_2}, \quad C_1^{\mathrm{IIA}} = \ell_s \tau_1 dx_B. \end{aligned} \tag{2.47}$$

Note that the prefactor of the metric is 1 if the dilaton ϕ^{IIA} is constant, because we can relate it directly to g_{IIA} in that case. However, we want to use our result for torus fibrations later on, so we will not make this identification yet. Instead we dualize it to Type IIB via T-duality along the x_B-circle. Such a duality relates the Type IIA and Type IIB content in the following manner

$$\begin{aligned} C_0^{\mathrm{IIB}} &= \ell_s^{-1}(C_1^{\mathrm{IIA}})_{x_B}, \quad e^{\phi^{\mathrm{IIB}}} = \frac{\ell_s}{L_A} e^{\phi^{\mathrm{IIA}}}, \\ L_B &= \frac{\ell_s^2}{L_A}. \end{aligned} \tag{2.48}$$

This yields[5]

$$\begin{aligned} ds^2_{\mathrm{IIB}} &= \frac{\ell_s^2}{\ell_M^3}\sqrt{\frac{v}{\tau_2}} ds^2_{1,8} + \frac{\ell_M^3 \ell_s^2}{v\sqrt{v\tau_2}} dx_B^2, \\ C_0^{\mathrm{IIB}} &= \tau_1, \quad e^{\phi^{\mathrm{IIB}}} = \frac{1}{\tau_2}. \end{aligned} \tag{2.49}$$

[5]Note that we have $L_A^2 = v\sqrt{v\tau_2}\,\ell_s^2 \ell_M^{-3}$.

To simplify this expression, the idea is to relate the asymptotic values of v and τ_2 to each other, similar to the identification with the string coupling. Let us denote these values by v^0 and τ_2^0. Then we obtain from Eq. (2.43)

$$\frac{\ell_s^4}{\ell_M^6}\frac{v^0}{\tau_2^0} = 1. \tag{2.50}$$

If we apply the F-theory limit $v^0 \to 0$, we can show that we recover Lorentz invariance, at least asymptotically far away. To show this, it is convenient to redefine the coordinate x_B in the following way

$$x = \frac{\ell_s^2}{\sqrt{v^0\tau_2^0}} x_B \tag{2.51}$$

Indeed, through this redefinition we obtain as asymptotic behavior

$$ds^2_{\text{IIB}} = ds^2_{1,8} + dx^2, \tag{2.52}$$

and in the case of constant v and τ_2, this equality holds everywhere.

One can continue this approach for the other fields of Type IIB, but we will not rederive this result here. If we want to identify the Type IIB content later on, we will make these identifications on the spot. Furthermore, one could also generalize the procedure above to include vectors in the toroidal fibration with legs on the base manifold.

2.4 Geometric approach of F-theory via M-theory

Motivated by the duality between M-theory and Type IIB string theory above, we want to study manifolds with toroidal parameters that vary over a base. This is exactly what can be achieved through the study of elliptically fibered manifolds. After discussing the basics of these fibrations, we will move on to the physical implications of these models, and how one can recover all features of Type IIB string theory. Most importantly, this approach directly gives us only consistent 7-brane systems, whereas determining the (p,q) 7-brane setup on the Type IIB side remained guesswork. At the end, we will consider warping of the F-theory geometry. This will be important later on in this thesis, since we will consider this warping in the context of M2-branes in Ch. 5, and in the context of 7-brane fluxes in Ch. 6.

2.4.1 Elliptic fibrations

The idea is that we have a complex n-dimensional Kähler base B_n, with an elliptic curve varying over it, which results in an elliptically fibered complex $(n+1)$-dimensional manifold X_{n+1}. Because we want to compactify Type IIB to $(10-2n)$-dimensional Minkowski spacetime with $\mathcal{N}=2$ supersymmetry, it can be argued that X_{n+1} must be a Calabi-Yau manifold.

As an example, let us therefore consider a Weierstrass model for such an elliptic curve. We will build this curve out of the weighted projective space $\mathbb{P}_{2,3,1}[6]$, which is defined by an equivalence relation on $\mathbb{C}^3$

$$(x, y, z) \sim (\lambda^2 x, \lambda^3 y, \lambda z), \quad \lambda \in \mathbb{C}\backslash\{0\}. \tag{2.53}$$

Then the curve can be defined by a polynomial equation, which can be written in so-called Weierstrass form as

$$P_W = y^2 - x^3 - fxz^4 - gz^6 = 0, \tag{2.54}$$

where f, g are complex parameters. First we study the degenerations of this elliptic curve. We found previously that τ_2 diverges at the position of ordinary D7-branes. This causes the metric component of the x_A-circle vanishes, which means that the A-cycle shrinks to zero size. For our elliptic curve, these degenerations correspond to the additional condition $dP_W = 0$. Its y-component implies $y = 0$. Furthermore, if we consider non-zero z, we can fix it to $z = 1$ by the equivalence relation. Then our polynomial equation $P_W = 0$ reduces to a third-degree polynomial in x, with three zeroes x_1, x_2, x_3. Therefore, the x-component of $dP_W = 0$ implies that two of these zeroes must coincide. Alternatively, these degenerations can be studied via the discriminant of the polynomial P_W, given by

$$\Delta = -(x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2 = 27g^2 + 4f^3. \tag{2.55}$$

Then the degenerations of the elliptic curve correspond to the vanishing of the discriminant, i.e. $\Delta = 0$. Now this object is useful, because it can be related to the complex structure parameter of the elliptic curve via

$$j(\tau) = \frac{4(24f)^3}{\Delta}. \tag{2.56}$$

Thus, if we promote f, g to polynomials in the coordinates on the base, the zeroes of Δ correspond to poles. Hence the multiplicity of the zeroes corresponds to the order of the pole, and therefore to the number of 7-branes.[6] Normally, we associated these poles with D7-branes. But in a setup with different kinds of 7-branes, it is not possible to choose such a global $\mathrm{SL}(2, \mathbb{Z})$ frame. Therefore, instead of identifying the presence of a brane with shrinking of the A-cycle, it turns out that we must identify the presence of a (p, q) 7-brane with the collapse of the 1-cycle $pA + qB$.

The corresponding divisors in the base B_n of these 7-branes are given by the locus of $\Delta = 0$, and let us denote its disconnected components by D_i. First we will consider the case $n = 1$, i.e. B_1. This means that we are dealing with K3 surfaces, because X_2 must be Calabi-Yau. Judging from the earlier discussion about the case of 24 D7-branes, we know that the degenerations of the fiber correspond to points in the base, which means that the divisors are of codimension one in the base. This aspect, that the divisors are of codimension one, can be generalized for higher-dimensional bases.

Let us also point out that we can construct a section for this elliptic fibration, i.e. a projection $\pi : X_{n+1} \to B_n$. This map can be constructed via the equivalence class of the

[6] Of course, if f vanishes as well, one needs to subtracts the multiplicity of the zero for f.

point $(1,1,0)$, which is contained in the curve for every value of f,g. Therefore, it is present in the elliptic curve for every point in the base, and can be used to project X_{n+1} onto its base B_n. Because every elliptic fibration with a section can be described by a Weierstrass model, these models can be very useful.

Next, let ω_i be the basis of Poincaré dual 2-forms for the locus $\Delta = 0$. An important result which links these 2-forms to the first Chern classes of B_n and X_{n+1} is Kodaira's equation, given by

$$c_1(X_{n+1}) = \pi^*\Big(c_1(B_n) - \frac{1}{12}\sum_i \omega_i\Big). \tag{2.57}$$

It has only been shown to hold for K3 surfaces, but by arguments given in [29], it can be generalized for the higher-dimensional cases. And if X_{n+1} is a Calabi-Yau manifold, it has a vanishing first Chern class, which means we arrive at

$$12c_1(B_n) = \sum_i \omega_i, \tag{2.58}$$

which reminds us of the tadpole condition for 7-branes and O-planes, stated for ordinary D-branes in Eq. (2.15). Actually, it is precisely the consistency condition for 7-branes that we did not have yet in section 2.4.2. Therefore, by approaching 7-brane configurations via geometry, the consistency conditions of the manifold automatically select the right configurations for us.

2.4.2 Gauge fields of 7-branes

We have already discussed how the position of 7-branes, together with their implications for the complex structure parameter τ, lift to the geometry in M-theory. However, one important feature of the 7-branes has not been recovered from M-theory, namely their gauge fields.

We can recover the fields corresponding to the Cartan generators of the gauge group via the 3-form potential A_3, by decomposing in the 2-forms ω_i associated with the 7-branes

$$A_3 = A^i \wedge \omega_i. \tag{2.59}$$

Furthermore, one finds a Cartan matrix as intersection matrix associated with the ω_i, which can be classified according to the ADE classification. Then the corresponding Lie algebra naturally leads us to a suggestion for the gauge enhancement. For instance, a Cartan matrix of A_{k-1} can be identified with the group $\mathrm{SU}(k)$, and a Cartan matrix of D_k can be identified with $\mathrm{SO}(2k)$. One can even recover the exceptional groups E_6, E_7 and E_8 following this reasoning. This is the reason that F-theory yields more possibilities in GUT model building then perturbative Type IIB, which does not give rise to exceptional gauge groups.

The other fields of the gauge group can be realized through M2-branes wrapping the spheres between the divisors. Namely, via reduction to Type IIA, they become the strings stretched between the branes. Typically this idea is explained with Taub-NUT space as example. Since we will consider this example explicitly in section 4.1.1, we will not include this discussion here.

For an explicit compactification of the action involving these ω_i, and its comparison to the compactification of the DBI- and CS-terms for the D7-branes, see for instance [30, 7].

2.4.3 Warping by M2-branes and fluxes

The main focus of this thesis is the warping of F-theory geometries. Therefore, we will discuss this concept in detail in this subsection, and why one needs a local description of the Calabi-Yau manifold Y_4. We will use the setup of [6], where it was considered for M-theory, and here we will apply it for F-theory.

We are interested in compactification of M-theory to three-dimensional Minkowski spacetime, with possibly spacetime-wrapping M2-branes. The warping of the metric can be induced by these M2-branes, or alternatively by a 4-form flux $\mathcal{G}_4$ on the internal manifold. Either way, we can use as ansatz for the warped system

$$\begin{aligned} &ds^2 = e^{-A} ds^2_{1,2} + e^{A/2} ds^2_{Y_4}, \\ &A_3 = e^{-3A/2}\mathrm{dvol}_{1,2}, \qquad\qquad G_4 = de^{-3A/2} \wedge \mathrm{dvol}_{1,2} + \mathcal{G}_4, \end{aligned} \tag{2.60}$$

where A only depends on the internal coordinates. We could have chosen to incorporate the warp factor in the metric of this internal space. However, if we write it in this form, we can use the property that the internal space without warp factor is a Calabi-Yau manifold Y_4. Furthermore as a sidenote, it can be shown that we must impose the following conditions on the flux

$$*_{Y_4}\mathcal{G}_4 = \mathcal{G}_4, \quad J \wedge \mathcal{G}_4 = 0, \tag{2.61}$$

where J is the Kähler form of the Y_4. This implies the flux must be self-dual and primitive. In this thesis, we will not delve deeply into these properties, but for a discussion we refer to [14].

Now we can analyze the warp factor via the equation of motion of A_3, similar to D-branes. This equation, with the contributions due to the M2-branes, is given by

$$d * G_4 = \frac{1}{2} G_4 \wedge G_4 - \ell_{\mathrm{M}}^6 X_8 + 2\kappa_{\mathrm{M}}^2 T_{\mathrm{M2}} \sum_i \delta_8^i, \tag{2.62}$$

where δ_8^i denotes the 8-form current due to the i-th M2-brane. Typically, when we consider a local geometry $\mathcal{Y}^4$ to describe the internal manifold Y_4, this object can be expressed in δ-functions and volume forms, but we will not attempt this yet.

If we plug in the ansatz for our warped system, we obtain

$$\Delta_{Y_4} e^{3A/2} = *_{Y_4}\left(\frac{1}{2}\mathcal{G}_4 \wedge \mathcal{G}_4 - \ell_{\mathrm{M}}^6 X_8 + 2\kappa_{\mathrm{M}}^2 T_{\mathrm{M2}} \sum_i \delta_8^i\right). \tag{2.63}$$

Throughout this thesis, we will assume the curvature to be small. Therefore X_8, the fourth order polynomial in the curvature, can be neglected. Then remains an equation with the flux $\mathcal{G}_4$ and the contributions of the M2-branes.

This equation can be placed into an F-theory setting by decomposing the flux $\mathcal{G}_4$ in the 7-brane fluxes, which we will denote by $\hat{\mathcal{F}}^I$ for the I-th 7-brane. First, however, we must clarify the setup and involved dualities.

Specifically, we consider the stack of 7-branes to wrap $\mathbb{R}^{1,2} \times S^1 \times S_b$, where S_b is the divisor that they wrap in the base of the Calabi-Yau 4-fold Y_4. Furthermore, S^1 is the circle that is used for the T-duality between the Type IIA side and the Type IIB side, from which we recover a spacetime dimension in the F-theory limit. We assume the gauge theory of the 7-branes to be pushed on the Coulomb branch via another 7-brane flux, such that we have $\mathrm{U}(1)^k$ as gauge group. We achieve this by a flux with one leg on the $\mathbb{R}^{1,2}$, and another on the S^1. Via T-duality, this circle S^1 is related to a T-dual circle, which is part of the Calabi-Yau 4-fold Y_4 on the M-theory side. And because of the additional flux, we have 6-branes that do not coincide on this circle, but instead have a position on this circle described by the flux.

Now let us denote the 2-forms associated with these 6-branes in Y_4 by Ω_I. Then we can expand $\mathcal{G}_4$ in the flux $\hat{\mathcal{F}}^I$ on S_b, similar to the decomposition of A_3 in the D-brane gauge fields. This yields

$$\mathcal{G}_4 = \hat{\mathcal{F}}^I \wedge \Omega_I. \tag{2.64}$$

Next, we can plug this expansion into our equation. However, solving such a differential equation without knowing the metric or possessing an explicit expression for the forms is difficult. Therefore, it would be convenient to construct a local description $\mathcal{Y}_4$ of the Calabi-Yau fourfold Y_4. We will derive these geometries in Chapter 4. Then we obtain as equation

$$\Delta_{\mathcal{Y}_4} e^{3A/2} = \frac{1}{2} *_{\mathcal{Y}_4} \left(\hat{\mathcal{F}}^I \wedge \hat{\mathcal{F}}^J \wedge \Omega_I \wedge \Omega_J \right) + 2\kappa_{\mathrm{M}}^2 T_{\mathrm{M2}} \sum_i \delta^{(8)}(x_\perp - x_i), \tag{2.65}$$

where the local geometry provides an explicit expression for the 2-forms Ω_I. Then remains the task of making an appropriate choice for $\hat{\mathcal{F}}^I$, but we will postpone this to Chapter 6.

Chapter 3

AdS/CFT with D3/D7-brane systems

Here we will discuss the basics of the D3/D7-brane system, focusing on the AdS/CFT correspondence for this model. This chapter is intended to discuss the most important features of this system, and point out where a description via F-theory might be useful, instead of giving full review. Therefore, we start with a short explanation of the main concepts of the AdS/CFT correspondence that we need, after which we proceed directly to our D3/D7-brane system. First we analyze how this gives rise to a QCD-like model, together with an analysis of both sides of the correspondence. We end this chapter by considering corrections on both sides, and identify where these descriptions break down.

Good reviews on AdS/CFT are for instance [31, 32]. This specific setup is discussed quite detailed in [33, 31], and both contain an overview of related solutions. The original articles on the parts that we will focus on are [8, 9, 10, 11, 12].

3.1 Main concepts of AdS/CFT

Originally, the AdS/CFT correspondence was proposed in the context of a stack of N D3-branes [34]. We consider an extension to this system, thus let us shortly discuss the idea. It comes down to two descriptions that can be used for the same theory, but at different regions of the parameter space. Namely, on one hand we can describe the stack of D3-branes by a gauge theory, namely $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory, with $\mathrm{SU}(N)$ as gauge group.[1] On the other hand, it can also be described by Type IIB supergravity on $\mathrm{AdS}_5 \times S^5$, which is the near-horizon geometry of the stack of D3-branes. Both descriptions require $g_s \ll 1$, but the former needs $g_s N \ll 1$, whereas the latter needs $g_s N \gg 1$. Furthermore, the Maldacena limit, which states that we should take $\alpha' \to 0$ but r/α' fixed for any length scale r, is essential in the precise formulation of the conjecture.

[1] It can be argued that the U(1) decouples as a free theory, as done in for instance [14].

Instead of reproducing this whole idea, we will claim the features we need from duality arguments.

First we consider the symmetries of the two theories. Because the theories are dual to each other, we should be able to match the symmetries of both sides. The isometry group of the AdS_5 is SO(4,2), which is precisely the group of conformal transformations in 4 dimensions, and a global symmetry group of the CFT.[2] The other symmetry group of the CFT is the R-symmetry, which rotates the 4 supersymmetry generators, and is given by $\mathrm{SU}(4)_R$. This is the covering group of SO(6), and that group is precisely the isometry group of S^5. Later on, we will consider D7-branes in this setup, which break this SO(6) on both sides of the correspondence.

Another feature that would be interesting to match, is the couplings in the two theories. In the constant case, we know that we can relate the 't Hooft coupling of the field theory to the string coupling via the DBI-term. Then we find for a Dp-brane

$$\lambda_{\mathrm{D}p} = g_{\mathrm{YM}}^2 N = (2\pi)^{p-2}(\alpha')^{(p-3)/2} g_s N. \tag{3.1}$$

But, as we found in Ch. 2 for instance, we can consider cases with a string coupling g_s that is not constant. Therefore, it would be interesting if we could relate this to similar behavior of the coupling in the field theory, involving for RG flows for instance. And indeed, it turns out that the radius of the AdS_5 can typically be related to the energy scale of the field theory, as we will see in section 3.3.

3.2 Basics of the D3/D7-brane systems

Here we consider the D3/D7-brane system in its simplest form. For instance, we only assume a small number of D7-branes, which allow us to make various simplifications, such as neglecting the backreaction of D7-branes or the so-called quenched approximation.

3.2.1 Setup

The idea is to add N_f D7-branes to N D3-branes for flavor in AdS/CFT, originally proposed in [35]. Like before, we have the 3-3 strings corresponding to the SU(N) gauge group. Additionally, we have 3-7 strings and 7-3 strings, which result in N_f flavor fields in the N or $\bar{N}$ fundamental representation of SU(N). Namely, because the 7-7 strings turn out to decouple in the Maldacena limit, we can neglect that the flavor fields would form N_f or $\bar{N}_f$ fundamental representations of SU(N_f) as well.

This latter aspect can be explained from the 't Hooft coupling of the respective gauge theories. For the D7-brane it is given by

$$\lambda_{\mathrm{D7}} = g_{\mathrm{D7}}^2 N_f = (2\pi)^5(\alpha')^2 \tag{3.2}$$

[2]More precisely, we should consider the covering group SU(2,2) as global symmetry.

Then, if we take the Maldacena limit $\alpha' \to 0$, and compare this coupling to the D3-brane coupling $\lambda_{\mathrm{D3}} = 2\pi g_s N$, it is clear that this coupling becomes small, and hence the D7-brane gauge theory decouples.

Furthermore, it is a common feature of D-brane intersections to be supersymmetric if the number of Neumann-Dirichlet (ND) directions (for strings stretched between the D-branes) is a multiple of 4, see for instance [36]. This can be observed from the fact that the sets of preserved supercharges must overlap. Furthermore, if the number of ND directions in non-zero, it follows that only half of the sets overlap, and hence the additional flavor branes break $\mathcal{N} = 4$ supersymmetry down to $\mathcal{N} = 2$.

Direction	0	1	2	3	4	5	6	7	8	9
D3-Brane	–	–	–	–	·	·	·	·	·	·
D7-Brane	–	–	–	–	–	–	–	–	·	·

Table 3.1 – D-brane setup: dashes denote parallel directions, dots point-like directions.

3.2.2 Isometries of the supergravity side

This system would have an $\mathrm{SO}(4) \times \mathrm{SO}(2)$ rotation symmetry around the D3-brane. The SO(4)-symmetry corresponds to rotating around the D3-brane in the directions parallel to the D7-branes. The remaining SO(2)-symmetry is associated with rotating around the D7-brane. An interesting feature is that we can give the flavor fields a mass by separating the D3- an D7-branes from each other by a distance l_f. From the worldsheet action of a string, we know that the mass per area unit is given by $1/2\pi\alpha'$. Then we know that the mass of the corresponding flavor field (i.e. the 3-7 and 7-3 strings) must be given by

$$m_f = \frac{l_f}{2\pi\alpha'} \tag{3.3}$$

Now we will observe that such a mass results in chiral symmetry breaking on the field theory side, and therefore it is interesting to note that this separating of the D3- and D7-branes breaks the SO(2)-symmetry of the supergravity as well.

3.2.3 Field theory content and global symmetries

The classic D3-brane model would correspond to a (3+1)-dimensional $\mathcal{N} = 4$ Super Yang-Mills theory. Our inclusion of D7-branes couples it to flavor fields, preserving only $\mathcal{N} = 2$ supersymmetry. We will describe this content through $\mathcal{N} = 1$ superspace formalism.

The $\mathcal{N} = 4$ vector multiplet decomposes into a vector multiplet W_α and 3 chiral superfields Φ_1, Φ_2, Φ_3 under $\mathcal{N} = 1$ supersymmetry. Then this can be rewritten in $\mathcal{N} = 2$ multiplets. The first can be formed by grouping W_α and a chiral superfield, w.l.o.g. Φ_3, into an $\mathcal{N} = 2$ vector multiplet. The remaining 2 chiral superfields can be grouped into an $\mathcal{N} = 2$ hypermultiplet. Then remains the flavour content, which consists of $\mathcal{N} = 1$ chiral multiplets

Q^f and $\tilde{Q}_f$, where f denotes the flavour, which can be grouped in $\mathcal{N}=2$ hypermultiplets $(Q^f,\tilde{Q}_f)$. This results in the following Lagrangian

$$\begin{aligned}\mathcal{L} = &\int d^4\theta \left(\mathrm{Tr}(\bar{\Phi}_I e^V \Phi_I e^{-V}) + Q_f^\dagger e^V Q^f + \tilde{Q}_f^\dagger e^{-V}\tilde{Q}^f\right)\\ &+ \mathrm{Im}\left(\tau \int d^2\theta\, \mathrm{Tr}(W^\alpha W_\alpha)\right) + \int d^2\theta\, W + \text{c.c.},\end{aligned} \tag{3.4}$$

where we integrate over Grassmann coordinates, τ denotes the complexified gauge coupling, and we have as superpotential W

$$W = \mathrm{Tr}(\varepsilon_{IJK}\Phi_I\Phi_J\Phi_K) + \tilde{Q}_f(m_f + \Phi_3)Q^f. \tag{3.5}$$

The important quantum numbers of the fields of this field theory have been summarized in the following table.

$\mathcal{N}=2$	Components	Spin	SU(2)$_\Phi\times$SU(2)$_R$	U(1)$_R$	Δ	U(N$_f$)	U(1)$_B$
(Φ_1,Φ_2)	X^4+iX^5,X^6+iX^7	0	$(\frac{1}{2},\frac{1}{2})$	0	1	1	0
	λ_1,λ_2	$\frac{1}{2}$	$(\frac{1}{2},0)$	-1	$\frac{3}{2}$	1	0
(Φ_3,W_α)	X^8+iX^9	0	(0,0)	2	1	1	0
	λ_3,λ_4	$\frac{1}{2}$	$(0,\frac{1}{2})$	1	$\frac{3}{2}$	1	0
	A_μ	1	(0,0)	0	1	1	0
$(Q,\tilde{Q})$	$(q,\bar{\tilde{q}})$	0	$(0,\frac{1}{2})$	0	1	N_f	1
	$\psi_i=(\psi,\tilde{\psi}^\dagger)$	$\frac{1}{2}$	(0,0)	∓ 1	$\frac{3}{2}$	N_f	1

Table 3.2 – Field content. Note that U(1)$_B\subset$ U(N$_f$). See also [37].

The SU(2)$_\Phi$ symmetry rotates the scalars X^4+iX^5 and X^6+iX^7 into each other, and acts similarly on their superpartners. The SU(2)$_R$ R-symmetry rotates the $\mathcal{N}=2$ supersymmetry spinors, and hence rotates the spinors λ_3,λ_4 into each other. Together, they can be realized from the SO(4) symmetry group of the supergravity, which can be decomposed into two SU(2) symmetry groups.

The U(1)$_R$ R-symmetry rotates the scalar X^8+iX^9, just like the SO(2) rotation would act on this coordinate. Note that there is a subtlety involving the quantum numbers for this symmetry, since fermions must be rotated around by 4π instead of 2π. An interesting observation is that the mass m_f breaks precisely this symmetry via the term in the superpotential, which we already predicted from the supergravity side.

The last symmetry group is U(N_f), which rotates the flavors into each other. Note that it would be broken to U(1) subgroups if we gave our flavors different masses m_f, but it is unbroken if we give them the same mass. Furthermore, we can realize a baryon number by rotating all quarks with the same phase (and anti-quarks with the opposite phase).

3.3 Beyond the probe approximation

Up until now, we did not take the backreaction of the D7-branes into account. We can motivate this from the fact that we consider a large number of D3-branes ($N \to \infty$), but take only a small finite amount of flavors N_f, which is commonly called the 't Hooft limit. Indeed, consistency conditions of D7-branes already bound the number of flavors from above, by $N_f \leq 24$. Hence the backreaction of the D3-branes is much larger than the backreaction of the D7-branes, and one can assume the D7-branes to probe the geometry, i.e. they do not backreact. Of course, such an approximation only holds in the vicinity of the D-branes, since the backreaction of D7-branes is present at large distances, but the backreaction of D3-branes is not. Since AdS/CFT needs the near-horizon geometry of the D3-branes, this justifies the claim at leading order.

On the field theory side, this is called the quenched approximation. This nomenclature originates from lattice QCD, where they used it to argue that the flavor determinant in the path integral could be set to 1. For our purposes, this means that we neglect corrections due to loops in the flavor fields.

Here, we will show that we can match the corrections on both sides, and point out where the two descriptions break down.

3.3.1 Backreaction on the supergravity side

The most convenient way to include the backreaction of the D7-branes, is by starting from the backreacted metric described in section 2.4.2. Then the idea is to let the large number of D3-branes backreact with this metric. This gives as ansatz

$$ds^2 = e^{-A/2} ds_{1,3}^2 + e^{A/2}(ds_4^2 + e^B d\bar{z}dz), \tag{3.6}$$

where e^B is the warp factor from Eq. (2.31)

$$e^B = \tau_2 \frac{\eta^2(\tau)\bar{\eta}^2(\bar{\tau})}{|z/d|^{N_f/6}}. \tag{3.7}$$

Similar to the discussion in section 2.2.1, we obtain for the D3-brane warp factor

$$(\nabla_4^2 + e^{-B}\bar{\partial}\partial)e^A = 0. \tag{3.8}$$

Due to the complicated expression for B, it is believed that this equation can only be solved numerically, or by approximations. For instance, one can try to simplify the equation by Fourier transforming the coordinates associated with the $\mathbb{R}^4$ via

$$e^A = 1 + \int \frac{d^4p}{(2\pi)^4} f_p(z,\bar{z}), \tag{3.9}$$

which results in the following equation

$$(-p^2 + e^B\bar{\partial}\partial) f_p(z,\bar{z}) = 0. \tag{3.10}$$

Because B is expressed in both holomorphic and anti-holomorphic functions, standard complex analysis techniques do not suffice. Furthermore, B it is not rotationally invariant in the complex z-plane, so one cannot reduce it to a single-variable equation either.

However, in the near-horizon region of the D7-brane, we know that $e^B \simeq \tau_2$, for which we know that $\tau_2 \simeq \frac{N_f}{2\pi}\log(r/\lambda)$, with $z = re^{i\phi}$. This means that, if we make a coordinate redefinition given by

$$\rho^2 = \frac{N_f}{2\pi}\log(r/\lambda)r^2, \tag{3.11}$$

the metric transverse to the D3-branes takes the following form

$$ds_\perp^2 \simeq ds_4^2 + d\rho^2 + \rho^2 d\phi^2. \tag{3.12}$$

Hence we can use the solution for D3-branes in flat spacetime, from which we obtain close to the D-branes

$$e^A = 1 + \frac{Q_{\mathrm{D3}}}{(y^2 + \tau_2 r^2)^2} = 1 + \frac{Q_{\mathrm{D3}}}{(y^2 + [g_s - \frac{k}{2\pi}\log(\rho/\lambda)]\rho^2)^2}, \tag{3.13}$$

where y denotes the distance from the D3-branes in the $\mathbb{R}^4$. We will recover precisely this result via M-theory in section 5.2.2.

3.3.2 RG flow vs dilaton profile

Going beyond the quenched approximation, gives us for the field theory side a one-loop beta function proportional to $\beta \sim \lambda_{D3}^2 N_f/N$ [12, 37]. Then it follows from the RG equation that the associated coupling $\alpha = g_{\mathrm{YM}}^2/(2\pi)$ is given by

$$\alpha(Q^2) = \frac{2\pi}{N_f\log(\Lambda_L^2/Q^2)}, \tag{3.14}$$

where Q denotes the energy scale, and Λ_L is given by

$$\Lambda_L^2 = \mu^2 e^{4\pi/(N_f\alpha(\mu^2))}, \tag{3.15}$$

with μ^2 as reference scale. For $Q = \Lambda_L$ the coupling diverges, which means we have a Landau pole.

It is interesting to note that this result can be related to the logarithmic approximation of the D3/D7-brane backreaction, if we make the identifications $Q = r/2\pi\alpha'$ and $\lambda = 2\pi\alpha'\Lambda_L$. Indeed we would find

$$\frac{1}{\alpha(Q^2)} = \frac{N_f}{2\pi}\log(\Lambda_L^2/Q^2) = -\frac{N_f}{2\pi}\log(r^2/\lambda^2) = e^{-\phi(r)} \tag{3.16}$$

Now it is expected that, instead of this Landau pole, which can be related to the breakdown of the logarithmic approximation of the dilaton, chiral symmetry breaking occurs instead.

Close to the D7-branes, we observed that the transverse metric has rotational symmetry, thus that the field theory has chiral symmetry. However, judging from the relation between

$j(\tau)$ and the position of the D7-branes (Eq. (2.25)), we already argued that we have at most a $\mathbb{Z}_{2N_f}$ symmetry.[3] From numerical plots of the relation [31], it has been verified that this symmetry breaking pattern occurs at larger length scales indeed. This implies that the chiral symmetry breaks to $\mathbb{Z}_{2N_f}$ as well. It is suspected that this symmetry breaking occurs on the field theory side due to a quantum anomaly. Then it can be argued that breaking the $\mathrm{U}(1)_R$ symmetry cures the theory of its Landau pole, but we will not include this discussion here.

[3] Again, the factor two is related to the fact that rotations of fermions need 4π instead of 2π.

Chapter 4

Local geometries in M- and F-theory

The main focus of this thesis is to study the warping of F-theory geometries. However, solving the corresponding equation for the warp factor requires knowledge of the metric, and this poses a problem with the Calabi-Yau manifolds that are typically studied. Namely, there is no explicit expression known for the metric of any compact Calabi-Yau manifold. To circumvent this problem, we will utilize an appropriate local description, that should be valid close to the 7-brane singularities. Of course, without the context of Calabi-Yau manifolds, the resulting geometries are still interesting to study on their own.

We will motivate these local geometries from the lift of certain objects in Type II string theory to geometries in M-theory. For instance, it is known that D6-branes lift to Kaluza-Klein monopoles in M-theory, which can be described by Taub-NUT space. Then we can use that the singularities in the Calabi-Yau manifold are associated with these objects, and that therefore the corresponding geometry should provide an appropriate description in the proximity of the singularities. Furthermore, because Type IIA and Type IIB can be related through T-duality on a circle, we will periodify the local geometry from lifted Type IIA objects to be able to describe Type IIB objects.

As a sidenote, we should mention that the local geometries only describe the space transverse to the objects, but not the divisors that they wrap in the Calabi-Yau manifold. In the context of M2-branes as in Ch. 5, we can neglect this subtlety, since we are mainly interested in D3/D7-brane systems, instead of implementing the local geometry into a Calabi-Yau manifold. Therefore the divisor will simply be $\mathbb{R}^4$. In the context of 7-brane fluxes as in Ch. 6, we would eventually be interested in coincident 7-branes for gauge enhancement, so they would wrap the same divisor S_b in Y_4.

4.1 Local geometries in M-theory

In this section we will review the lift of Type IIA objects, specifically the D6-brane and the O6-plane. The former can be interpreted as a Kaluza-Klein monopole, and therefore lifts to Taub-NUT space. The latter lifts to the Atiyah-Hitchin manifold. It turns out that we can combine Atiyah-Hitchin space with Kaluza-Klein monopoles as well, and we are therefore able to lift a system of D6-branes and an O6-plane to M-theory. The relevant features of these lifts are covered in [38], upon which we draw heavily. A good review on Taub-NUT space in M-theory is also given in [39].

4.1.1 Taub-NUT space

First, we will consider the lift of D6-branes to Taub-NUT space. Originally, this space was proposed as the lift of a magnetic monopole to 5-dimensional spacetime. In our context, the space transverse to the D6-brane is 3-dimensional, just like the space transverse to the monopole. Furthermore, D6-branes are magnetically charged under the RR 1-form potential C_1, just like the monopole, which is magnetically charged under the electromagnetic field. Thus, the idea is that D6-branes can be lifted similarly to M-theory via this Taub-NUT space. Then we obtain as metric

$$ds^2_{\mathrm{M}} = ds^2_{1,6} + ds^2_{\mathrm{TN}_k}. \tag{4.1}$$

Now that we have motivated our use of Taub-NUT space, let us discuss it in detail. This space admits a circle fibration over a 3-dimensional base. Specifically, each D6-branes lifts to a Taub-NUT center, and the circle degenerates at each center. Moreover, the circle shrinks such that it looks locally like $\mathbb{R}^4$, and hence we avoid conical singularities. We can make this description explicit by considering the metric. For k-centered Taub-NUT space TN_k, it is given by

$$\begin{aligned} ds^2_{\mathrm{TN}_k} &= V ds_3^2 + \frac{1}{V}(dt+U)^2, \\ V &= 1 + \sum_I V_I, \qquad\qquad U = \sum_I U_I, \end{aligned} \tag{4.2}$$

where we defined[1]

$$V_I = \frac{r_A}{4\pi|r-r_I|}, \quad dU_I = -*_3\, dV_I. \tag{4.3}$$

Here r denote the coordinate on the base, r_I the position of each monopole in the base, and t the coordinate on the circle, with periodicity r_A. We should also mention that this circle length can be related to the mass of the monopole m via $r_A = 4\pi m$.[2] Then consistency of our geometry requires that all our monopoles should have the same mass. This condition

[1] We use the conventions of [7]. Note that we do not state an explicit expression for the U_I here, since it can be derived from the V_I up to a closed 1-form. We do discuss these U_I in detail in Appendix A.2.

[2] Note that other literature, such as[38, 40], uses $2m$ where we use m.

follows from the fact that we want our metric to be smooth at each center. Namely, if we take the limit $|r - r_I| = w^2/4m \ll m$, we find as leading order behavior of the metric

$$ds^2_{\mathrm{TN}_k} \simeq dw^2 + w^2 d\Omega_3^2. \tag{4.4}$$

If there was a center with a different mass, one would not find the metric of the 3-sphere S^3 for the second term, which indicates a conical singularity. This is precisely what occurs if we stack k monopoles on top of each other, which gives rise to the Lens space $S^3/\mathbb{Z}_k$ instead. It turns out that we can obtain useful information about these specific singular cases via the homology group of 2-cycles, which is non-trivial for Taub-NUT space.

Let us start with k separate monopoles, and we can move them together later on. Then, as mentioned before, the circle fibration pinches at each center. This means that, if we restrict the fibration to a line between two centers, we create a 2-cycle that is topologically equivalent to the 2-sphere S^2. Neglecting subtleties involving other centers located at these lines, we can give the 2-cycles by

$$S_{IJ} = \{(r,t)| r = (1-x)r_I + x r_J \text{ for } x \in [0,1], t \in [0,1]\}. \tag{4.5}$$

To build our homology group, we must define a basis for these 2-cycles. Our choice of basis is given by $S_i = S_{i,i+1}$ for $1 \leq i < k$. Indeed, the other cycles can be expressed in this basis via $S_{IJ} = S_I \cup \ldots \cup S_{J-1}$ for $I < J$, and similarly for $I > J$ by $S_{IJ} = -S_{JI}$. Consequently, our homology group is isomorphic to $\mathbb{Z}^{k-1}$.

We can calculate the intersection numbers for this basis simply by counting the points at which the 2-cycles intersect, and include minus signs if they intersect with opposite orientation. Then we find that S_i intersects with itself at r_i and r_{i+1}, whereas S_i and S_{i-1} intersect only at r_i, and with opposite orientation. All other pairs of basis 2-cycles do not intersect with each other. Hence we find as intersection matrix

$$C_{ij} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}. \tag{4.6}$$

This matrix C_{ij} is the $(k-1)\times(k-1)$ Cartan matrix of A_{k-1}. Due to its relation with su(k), it suggests an SU(k) gauge enhancement if we stack the monopoles on top of each other. Namely, following the outline given in subsection 2.4.2, we can decompose the M-theory 3-form A_3 in the Poincaré dual 2-forms ω_i of the 2-cycles S_i. This results in the gauge fields associated with the Cartan generators of su(k). The remaining $k(k-1)$ gauge fields, related to the roots of SU(k), can be recovered from M2-branes wrapping the 2-cycles S_{IJ} between the monopoles.

Alternatively, the intersection matrix can also be calculated by use of the 2-forms on this space. To achieve this, we must realize an explicit expression for them first. It turns out that each of the monopoles has an associated 2-form Ω_I [41]. From these, we can construct the 2-forms ω_{IJ} that are Poincaré dual to the 2-cycles S_{IJ}, with $\omega_i = \omega_{i,i+1}$ as basis. We will study these forms extensively in Appendix A.2, but here we just state them directly

$$\begin{aligned} \Omega_I &= d\eta_I = \frac{1}{r_A} d\left(\frac{V_I}{V}(dt+U) - U_I\right), \\ \omega_{IJ} &= \Omega_I - \Omega_J. \end{aligned} \tag{4.7}$$

Through this approach, one can calculate the intersection numbers via

$$\int_{S_i} \omega_j = \int_{\mathrm{TN}_k} \omega_i \wedge \omega_j = -C_{ij}. \tag{4.8}$$

Note that we get an additional overall minus sign due to the anti-self-duality of the 2-forms Ω_I, which is related to our convention of orientation via the relation $dU_I = -\ast_3 dV_I$. Also, the remaining U(1) of U(k) = SU(k) × U(1) can be identified from these 2-forms, since we have a remaining independent 2-form $\Omega_1 + \cdots + \Omega_k$. Often, it is argued to become massive due to a Stückelberg mechanism (see for instance [7]), but this discussion will not be included in this thesis.

4.1.2 Atiyah-Hitchin space with KK monopoles

In addition to the lift of a D6-brane, we want to consider the lift of an O6-plane to M-theory. This lift was first proposed in [42, 43], and it gives rise to the Atiyah-Hitchin manifold [44]. This geometry admits a circle fibration over a 3-dimensional base, similar to Taub-NUT space, but it has another feature. Similar to the involution of the orientifold projection, we have to identify points on this space which are related by a map σ. The action of σ is

$$\sigma: \quad r \to -r, \quad t \to r_A - t \tag{4.9}$$

where r denotes the position in the 3-dimensional base, and t the coordinate of the circle, again with periodicity $r_A = 4\pi m$. The fixed point of this identification is called the bolt, and if we reduce along the circle, which corresponds to moving far away from the bolt, we recover our O6-plane. Closer to this bolt, various exponentially small corrections come into play, but this aspect is beyond the scope of this thesis, although it might be studied in future research. Furthermore, since we know that D6-branes lift to KK monopoles, it turns out that we can simply include these monopoles in the Atiyah-Hitchin geometry. Of course, due to the involution σ, we have to consider their images as well. For more details about this aspect, see for instance [38]. The asymptotic behavior of the metric for this Atiyah-Hitchin space with k KK monopoles is given by[3]

$$\begin{aligned} ds^2_{\mathrm{AH}} &= V ds_3^2 + \frac{1}{V}(dt+U), \quad dU = -\ast_3 dV \\ V &= 1 - 4V_0 + \sum_{I=1}^{k} (V_I + V_{-I}), \\ V_0 &= \frac{r_A}{4\pi|r|}, \quad V_I = \frac{r_A}{4\pi|r-r_I|}, \quad V_{-I} = \frac{r_A}{4\pi|r+r_I|}, \end{aligned} \tag{4.10}$$

[3]Again, we do not state the explicit expressions for the connections $U_0, U_{\pm I}$ here, but in Appendix A.3.

where V_0 corresponds to the contribution of the bolt, and V_I and V_{-I} denote the contributions of the monopoles and image monopoles respectively. Note that the contribution due to the bolt differs by a factor -4 from the contributions of the monopoles, just like the difference between the charge of an O6-plane compared to the charge of a D6-brane.

Next, we consider the homology group of 2-cycles of this manifold, to verify that it indeed gives rise to an $\mathrm{SO}(2k)$ gauge enhancement. Compared to the Taub-NUT space, the homology basis of 2-cycles for this manifold is slightly more complicated, due to the involution σ. Namely, in addition to the cycles S_i between the monopoles at r_i and r_{i+1} (for $1 \leq i < k$), we can consider cycles between monopoles and image monopoles. It turns out that we only need to add $S_k = S_{k-1,-k}$ to the homology basis, which is stretched between r_{k-1} and $-r_k$. Cycles between image monopoles are equivalent to cycles between monopoles due to the projection, and cycles between a monopole and an image monopole can be related to S_k via the cycles S_i and their projections.

Then we can consider the intersection numbers for the basis of 2-cycles. The intersection numbers for the cycles S_i are identical to the results of Taub-NUT space, and therefore the remaining intersection numbers must involve S_k. This cycle intersects with itself at r_{k-1} and $-r_k$, and hence the associated intersection number is 2. It intersects with S_{k-1} at r_{k-1} and r_k with opposite signs, because the involution σ, which relates r_k and $-r_k$, changes the orientation of the circle. Hence this intersection number is zero, and the remaining non-zero intersection is with S_{k-2} at r_{k-1}, with intersection number -1. Then we obtain as $k \times k$ intersection matrix

$$C_{IJ} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 2 \end{pmatrix}, \tag{4.11}$$

which is precisely the D_k Cartan matrix. Due to its relation to the special orthogonal group, it suggests a gauge enhancement to $\mathrm{SO}(2k)$ if we stack the monopoles on top of each order. And this is exactly the result we expect if we move k D-branes and their images on top of an O-plane, as discussed in subsection 2.2.2.

Now let us consider the Poincaré dual 2-forms of these 2-cycles. Since we choose to integrate over the covering space instead of the quotient space, it is convenient to construct 2-forms that are even or odd under the involution σ. Similar to the Taub-NUT space, we can build our forms out of the 2-forms associated with the centers[4]

$$\Omega_{\pm I} = d\eta_{\pm I} = \frac{1}{r_A} d\left(\frac{V_{\pm I}}{V}(dt + U) - U_{\pm I}\right) \tag{4.12}$$

[4]Note that V and U contain the contribution due to the bolt, which was not present for the Taub-NUT space.

Then, using the expressions in Appendix A.3, it follows that the connections $U_0, U_{\pm I}$ are odd under the involution σ, but the potentials $V_0, V_{\pm I}$ are even. Therefore it maps $\eta_{\pm I} \to -\eta_{\mp I}$. Furthermore, the exterior derivative preserves the sign of a map that reflects all coordinates. Consequently, we obtain $\Omega_{\pm I} \to -\Omega_{\mp I}$. Then we can define the 2-forms $\Omega_I^{\pm}$ and $\omega_i^{\pm}, \omega_k^{\pm}$, associated with the monopoles and 2-cycles S_i, S_k respectively. The ±-sign denotes their parity under the involution σ. They are given by

$$\begin{aligned} \Omega_I^{\pm} &= \frac{1}{\sqrt{2}}(\Omega_I \mp \Omega_{-I}), \\ \omega_i^{\pm} &= \Omega_i^{\pm} - \Omega_{i+1}^{\pm}, \quad \omega_k^{\pm} = \Omega_{k-1}^{\pm} + \Omega_k^{\pm} \end{aligned} \tag{4.13}$$

Note that relating $\Omega_k^{\pm}$ to $-\Omega_{-k}^{\pm}$ for $\omega_k^{\pm}$, gives the interpretation of the 2-cycle S_k that connects the $(k-1)$-th monopole to the k-th image monopole.

We can also verify the intersection matrix argued above by computation using these 2-forms. This yields indeed

$$\int \omega_I^{\pm} \wedge \omega_J^{\pm} = -C_{IJ}, \tag{4.14}$$

and again a minus sign due to our conventions. The gauge fields corresponding to the Cartan generators follow from decomposing the M-theory 3-form A_3 in the odd 2-forms ω_i^-, ω_k^-. We need to expand in odd 2-forms to counter that the D-brane gauge field is odd under the orientifold projection. The remaining $2k(k-1)$ gauge fields follow from M2-branes wrapping odd 2-cycles,[5] where we need the negative parity for the same reason.

Note also that, in this case, there is no remaining independent 2-form. So there is no additional gauge symmetry in the gauge enhancement to SO$(2k)$, like there was an additional U(1) in the gauge enhancement to SU(k).

4.2 Local geometries for F-theory

In the section above, we analyzed the lift of Type IIA objects to M-theory. To relate these geometries to F-theory, we already mentioned that we must consider a periodic array of these objects instead. Therefore, we will start this section by highlighting the important parts in constructing these periodic arrays in the M-theory geometry, following [7]. Then we proceed with a discussion of the resulting periodic Taub-NUT and Atiyah-Hitchin spaces, and the subtleties that might arise in this procedure.

4.2.1 Construction of periodic arrays

We know from the previous section that each object is associated with a potential, which we denote by $V_n = m/|r - r_n|$.[6] For simplicity, we will first assume that our object is located

[5] These are the 2-cycles between r_I and $r_{\pm J}$ (for $I \neq J$), combined with minus the 2-cycle between r_{-I} and $r_{\mp J}$.

[6] We give suitable arguments to neglect the corrections for the Atiyah-Hitchin bolt in subsection 4.2.3.

at the origin of the 3-dimensional base, and hence $r_n = 0$. Then we place the array of these objects along the z-axis of the 3-dimensional base, with spacing r_B between them. Thus, we let z describe this periodic direction, and we choose r as coordinate for the remaining 2-dimensional base. Then we obtain as potential for the periodic array

$$V_n^\infty = \frac{r_A}{4\pi}\left(\sum_l \frac{1}{\sqrt{r^2+(z+lr_B)^2}} - \sum_{l\neq 0}\frac{1}{|l|r_B}\right). \tag{4.15}$$

Due to the fact that the summation on itself is not regular, we subtracted an in infinite constant to regulate the series. Therefore we are free to choose any additional finite constant in this regularization scheme. This will result in a new parameter Λ that we encounter later, which can be interpreted as a length scale associated with the local geometry.

First, let us rewrite the expression for the potential. Instead of summing over all objects in the array, we prefer to sum over the Fourier modes of the circle instead. Namely, this allows for an easier expansion in orders of r_B, which is useful for the F-theory limit, in which the circle lengths are very small. The trick we can use to achieve this, is a so-called Poisson resummation. And since it is most conveniently applied for functions with period 1, we redefine $z \to z/r_B$, such that z has period 1. Then we have the following identity

$$\sum_{l\in\mathbb{Z}} f(z+l) = \sum_{l\in\mathbb{Z}} \hat{f}(l)e^{2\pi i l z}, \tag{4.16}$$

where $\hat{f}(l)$ is the Fourier transform of $f(z)$. Calculating these Fourier transforms is rather tedious, and there are also some subtleties involved with the $l = 0$ mode. Deriving the connection U_n^∞ is cumbersome as well, due to similar complications. For this reason, we refer to Appendix B.1 for these derivations, and we state the final result directly

$$\begin{aligned} V_n^\infty &= -\frac{r_A}{2\pi r_B}\log\Big(\frac{|r|}{r_B\Lambda}\Big) + \frac{r_A}{\pi r_B}\sum_{l>0} K_0\Big(\frac{2\pi l|r|}{r_B}\Big)\cos(2\pi l z), \\ U_n^\infty &= \frac{r_A}{2\pi}(\phi-\phi_0)dz - \frac{r_A}{\pi r_B}|r|\sum_{l>0} K_1\Big(\frac{2\pi l|r|}{r_B}\Big)\sin(2\pi l z)d\phi. \end{aligned} \tag{4.17}$$

Here Λ corresponds to the arbitrary choice of constant in the regularization scheme, ϕ is the angular coordinate of the 2-dimensional base, ϕ_0 denotes our choice of U^∞ up to a closed 1-form, and K_0, K_1 are the modified Bessel functions of the second kind.

We can generalize the result for V_n^∞ by redefining the coordinates $r \to r - r_n$ and $z \to z - z_n$, with (r, z) the coordinate in the periodic 3-dimensional base, and (r_n, z_n) the position of our object. Then we obtain

$$V_n^\infty = -\frac{r_A}{2\pi r_B}\log\Big(\frac{|r-r_n|}{\Lambda r_B}\Big) + \frac{r_A}{\pi r_B}\sum_{l>0} K_0\Big(\frac{2\pi l|r-r_n|}{r_B}\Big)\cos(2\pi l(z-z_n)). \tag{4.18}$$

Throughout this section, we will refer to the equation above for each object, and specify only its position (r_n, z_n) in the periodic base. Then one can simply read of the associated potential V_n^∞ from this equation.

Generalizing the expression for the connection U_n^∞ is a bit more difficult, since we would need to redefine ϕ, which is quite involved. Often, it suffices to know that the connection could, in principle, be calculated from $dU_n^\infty = -*_3 dV_n^\infty$. Pointing out an explicit expression will only be useful for objects at $(0, z_I)$, which is given by

$$U_n^\infty = \frac{r_A}{2\pi}(\phi - \phi_0)dz - \frac{r_A}{\pi r_B}|r| \sum_{l>0} K_1(\frac{2\pi l|r|}{r_B}) \sin(2\pi l(z - z_I))d\phi. \tag{4.19}$$

Next, let us consider the F-theory limit ($r_A, r_B \to 0$) in this case $r_n = 0$, which implies that $|r|/r_B \gg 1$. Then the Bessel functions become exponentially small,[7] and hence the summations vanish as well.[8] This means that the limit yields

$$V_n^\infty = -\frac{r_A}{2\pi r_B}\log(\frac{r}{r_B\Lambda}), \quad U_n^\infty = \frac{r_A}{2\pi}(\phi - \phi_0)dz \tag{4.20}$$

Let us already note that, to make sense of our expression, we must take Λ to be of a similar scale as $1/r_B$ in such a limit. But even in that case, there are regimes of large r that seem inappropriate. Namely, V_n^∞ can become negative, and cause the metric to become degenerate. Therefore, $r_B\Lambda$ sets the length scale up to which we can use our local geometry.

4.2.2 Periodic Taub-NUT space

The first geometry for which we will construct periodic arrays, is the k-centered Taub-NUT space. It results in a local geometry TN_k^∞, that should describe a stack of k D7-branes via M-theory. To recall, this idea is motivated from the T-duality between D7-branes and a periodic array of D6-branes, together with the lift of D6-branes to Taub-NUT space. The metric of this geometry can be obtained via the procedure of the previous section. This yields

$$\begin{aligned} ds^2_{\mathrm{TN}_k^\infty} &= V^\infty(ds_2^2 + r_B dz^2) + \frac{1}{V^\infty}(dt + U^\infty)^2, \\ V^\infty &= 1 + \sum_{I=1}^{k} V_I^\infty, \quad U^\infty = \sum_{I=1}^{k} U_I^\infty, \end{aligned} \tag{4.21}$$

where V_I^∞ and U_I^∞ are the potential and the connection of the I-th monopole respectively, which is located at (r_I, z_I) in the periodic base.

The 2-cycles in this geometry can be constructed in a similar way as those in the ordinary Taub-NUT space. Namely, we can restrict the circle fibration to lines between monopoles, but now in the periodified base $\mathbb{R}^2 \times S^1_{r_B}$ instead of $\mathbb{R}^3$. Hence we can consider a similar basis of 2-cycles, and we will find the same intersection numbers as for TN_k.

Alternatively, we can define the 2-forms on TN_k^∞ to calculate these intersection numbers. Again, it is convenient to consider the 2-forms Ω_I^∞ associated with the monopoles, as well as the 2-forms ω_i^∞ that are dual to the basis of 2-cycles. The idea is to define the 2-forms in

[7] Not to be confused with the exponentially small corrections of the Atiyah-Hitchin metric.

[8] For $x = e^{-|r-r_n|/r_B} \ll 1$, one can use as bound $\sum_{l>0} x^l = x/(1-x)$, which is of the same size as x, and hence can be neglected in this limit as well.

a similar fashion as those of ordinary Taub-NUT space, but use the periodified potentials and connections instead. Therefore we suggest

$$\begin{aligned}\Omega_I^\infty &= d\eta_I^\infty = \frac{1}{r_A} d\Big(\frac{V_I^\infty}{V^\infty}(dt + U^\infty) - U_I^\infty\Big),\\ \omega_i^\infty &= \Omega_i^\infty - \Omega_{i+1}^\infty.\end{aligned} \tag{4.22}$$

Indeed, one can proceed to calculate integrals with these 2-forms over TN_k^∞, and this yields results identical to what we found for ordinary Taub-NUT space. These calculations are included in Appendix B.2. For instance, computing the intersection numbers results in minus the Cartan matrix of A_{k-1}

$$\int_{\mathrm{TN}_k^\infty} \omega_i^\infty \wedge \omega_j^\infty = -C_{ij}. \tag{4.23}$$

Analogous to TN_k, this suggests an $\mathrm{SU}(k)$ gauge enhancement, when we move the monopoles on top of each other in the periodic base. Its gauge fields follow from decomposing A_3 in ω_i^∞, and from M2-branes wrapping the 2-cycles between the monopoles, as usual. We also have an additional U(1), now corresponding to the remaining independent 2-form $\Omega_1^\infty + \ldots + \Omega_k^\infty$.

However, instead of moving the monopoles on top of each other directly, we can play around with this aspect via the F-theory limit. Namely, we can consider the case in which the monopoles coincide in the 2-dimensional base, i.e. $r_I = 0$ for each of them, but are separate on the z-circle. Then the F-theory limit ($r_A, r_B \to 0$, r_A/r_B fixed) effectively moves them on top of each other. Using Eq. (4.20), we obtain as metric for this setup

$$\begin{aligned} ds^2_{\mathrm{TN}_n^\infty} &= \frac{r_A}{r_B}\tau_2 ds_2^2 + \frac{v}{\tau_2}\big[(dx + \tau_1 dy)^2 + \tau_2^2 dy^2\big]\\ v = r_A r_B, \quad \tau_1 &= \frac{\phi - \phi_0}{2\pi}, \quad \tau_2 = \frac{r_B}{r_A} - \frac{k}{2\pi}\log\frac{|r|}{\Lambda r_B},\end{aligned} \tag{4.24}$$

where we defined $x = t/r_A$ and $y = z$, both with period 1, to make the T^2-fibration explicit.

Now we can recover various aspects of the D7-branes in Type IIB from this expression. If one is interested in a complete dualization, we refer to the procedure outlined in subsection 2.3.1. Here we will only make a comparison.

To recall, the idea is that the 2-dimensional base describes the space transverse to the D7-branes, and that their presence induces a profile for τ. Then the first similarity that we recognize, is that we have the same τ-profile as we found close to the D7-branes in subsection 2.4.2. This suggests that we have as string coupling $g_{\mathrm{IIB}} = r_A/r_B$, and that the length scale λ associated with the D7-branes is related to the choice of Λ in the regularization scheme.

It explains the breakdown of our geometry as well. Namely, the logarithmic approximation for the axio-dilaton was only valid close to the D7-branes, and therefore we should only use our local geometry close to the singularities. In hindsight, this is precisely the intended use of our geometry, since we want to glue it into a compact Calabi-Yau manifold. It would be interesting if the local geometry could be used beyond the logarithmic approximation, but the fact that the rotational symmetry in the 2-dimensional base does not break, indicates otherwise. Namely, from the backreaction of D7-branes we know that there is no rotational symmetry around the D7-branes at larger distances.

At last, we should also point out the limit $r \ll \Lambda r_B$. In this limit, we can define coordinates such that the metric on the base looks like $\mathbb{R}^2$, similar to transverse space of the D7-branes. This radial coordinate is given by $\rho^2 = V^\infty r^2$, and we find for the metric in this limit

$$ds_2^2 \simeq d\rho^2 + \rho^2 d\phi^2. \tag{4.25}$$

This feature will be especially useful in subsection 5.2.2, where we consider M2-branes with $\mathbb{R}^4 \times \mathrm{TN}_k^\infty$ as transverse geometry, and hence need to find harmonic functions on this space.

4.2.3 Periodic Atiyah-Hitchin space with KK monopoles

Motivated by the fact that periodic arrays in Taub-NUT space reproduced Type IIB close to D7-branes, we want to apply this concept in Atiyah-Hitchin space as well. Together with KK monopoles, this should yield a local geometry AH_k^∞ that describes a system of D7-branes and an O7-plane via M-theory.

However, there are already some subtleties involved with the T-duality that relates the two descriptions. Specifically, recall from subsection 2.2.2, that an O7-plane wrapping a circle is dual to two O6-planes on the dual circle. This follows from the reflection

$$\sigma: \quad r \to -r, \quad z \to 1-z, \quad t \to r_A - t, \tag{4.26}$$

which has two fixed points on the z-circle. Therefore, we should lift to a geometry with two Atiyah-Hitchin bolts instead of one, the first located at $(r_0, z_0) = (0,0)$, the other at $(r_{0'}, z_{0'}) = (0, \frac{1}{2})$. It would be interesting to investigate this aspect starting from the Atiyah-Hitchin space itself, but for now this approach suffices. Namely, we are only interested in the metric far away from the bolts, just as in subsection 4.1.2. And in this limit, we can treat the Atiyah-Hitchin bolts as KK monopoles with -4 units of monopole mass.

Of course, there are corrections to this interpretation of the Atiyah-Hitchin bolts. And we should address these corrections properly, since we consider a periodic array of these bolts instead of a single one. The only aspect that we need, is that the corrections are exponentially small. As long as we consider a point far away from the two bolts (with respect to length scale r_A), this suggests that the corrections vanish by virtue of the geometric series. Note that this is exactly the same argument as used for the sum of Bessel functions in the footnote in subsection 4.2.1.

The reasoning above implies that our description of the geometry is only valid for points (r,z) that satisfy $r^2 + r_B^2(z-z')^2 \gg r_A^2$ for $z' = 0, \frac{1}{2}, 1$. Since our main interest is describing the geometry close to the monopoles and image monopoles, their positions (r_I, z_I) and (r_{-I}, z_{-I}) should satisfy this condition. In the context of the weak-coupling limit $g_{\text{IIB}} = r_A/r_B \ll 1$, this means that we can stack the monopoles on top of bolts in the 2-dimensional base. Namely, we can set $r_I = r_{-I} = 0$, as long as we choose appropriate values for z_I and $z_{-I} = 1 - z_I$. Without this limit, we must separate them from the bolts in the 2-dimensional base, i.e. non-zero r_I and $r_{-I} = -r_I$.[9]

Under the conditions stated in the above paragraphs, we can write the metric as

$$\begin{aligned} ds^2_{\text{AH}^\infty_k} &= V^\infty(ds_2^2 + r_B^2 dz^2) + \frac{1}{V^\infty}(R_A dt + U)^2, \\ V^\infty &= 1 - 4V_0^\infty - 4V_{0'}^\infty + \sum_{I=1}^k (V_I^\infty + V_{-I}^\infty), \end{aligned} \tag{4.27}$$

where we can use the expressions from subsection 4.2.1 for the potentials, by simply plugging in the specified positions of the objects.

This space has the same non-trivial homology group of 2-cycles as the ordinary Atiyah-Hitchin manifold with KK monopoles, and they have the same intersection numbers. It can be argued similarly to the comparison between TN_k and TN_k^∞, since we can just restrict the circle fibration to lines in the periodic base $\mathbb{R}^2 \times S^1_{r_B}$ instead of $\mathbb{R}^3$. We can also define the Poincaré dual 2-forms of these 2-cycles. Motivated by the previous subsection, we just replace the potentials and connections in Eq. 4.13 by their periodified versions. Therefore, we suggest[10]

$$\begin{aligned} \Omega^\infty_{\pm I} &= d\eta^\infty_{\pm I} = \frac{1}{r_A} d\Big(\frac{V^\infty_{\pm I}}{V^\infty}(dt + U^\infty) - U^\infty_{\pm I}\Big), \\ \Omega_I^{\infty,\pm} &= \frac{1}{\sqrt{2}}(\Omega_I^\infty \mp \Omega_{-I}^\infty), \\ \omega_i^{\infty,\pm} &= \Omega_i^{\infty,\pm} - \Omega_{i+1}^{\infty,\pm}, \quad \omega_k^{\infty,\pm} = \Omega_{k-1}^{\infty,\pm} + \Omega_k^{\infty,\pm} \end{aligned} \tag{4.28}$$

Indeed, it turns out that they obey the required properties, which we verify in Appendix B.3. For instance, we can calculate the intersection matrix via

$$\int \omega_i^{\infty,\pm} \wedge \omega_j^{\infty,\pm} = -C_{ij}, \tag{4.29}$$

where we find the Cartan matrix C_{ij} of D_k, and hence again an SO$(2k)$ gauge enhancement. As before, the gauge fields associated with the Cartan generators follow from decomposing the M-theory 3-form A_3 in the odd 2-forms $\omega_i^{\infty,-}, \omega_k^{\infty,-}$, and the remaining fields correspond to M2-branes wrapping the odd 2-cycles between the monopoles.

At last, we consider the F-theory limit, which implies that $r \gg r_B$. In this limit, it looks like the two bolts coincide, and indeed we find that $V_0^\infty = V_{0'}^\infty$. Hence we find a factor of -8 difference between the combined contribution of the bolts compared to the contribution of a monopoles. This is in agreement with the fact that O7-planes have -8 units of D7-brane charge, whereas O6-planes have only 4 units of D6-brane charge.

[9] Future research into these corrections could therefore be useful.

[10] Again, in comparison to Taub-NUT space, U and V now contain the contributions due to the bolts as well.

Summary

We started this chapter by recalling basic features of Taub-NUT space and Atiyah-Hitchin space. Then we moved on to our construction of periodic arrays in both of these spaces. Periodification of Taub-NUT space resulted in TN_k^∞, which captured the description of Type IIB close to the D7-branes. Application to Atiyah-Hitchin space was novel and captured aspects of the D7/O7-system, where the weak-coupling limit seems to play an interesting role. However, there are still some open ends to tie up for the latter, since we neglected the corrections close to the O7-plane (cq. Atiyah-Hitchin bolt).

Chapter 5

M2-branes in local geometries

In this chapter we will study the application of the local geometries in the context of M2-branes. Namely, we will assume that these geometries describe part of the space transverse to a stack of M2-branes, such that the M2-branes coincide with the monopoles. Since the geometries are related to D6- or D7-branes, this means that we study D2/D6- or D3/D7-brane configurations via M-theory.

Our main point of focus will be the warping due to the M2-branes, as considered in subsection 2.4.3. This resulted in the following metric

$$ds_{\mathrm{M}}^2 = e^{-A} ds_{\parallel}^2 + e^{A/2} ds_{\perp}^2. \tag{5.1}$$

We will set the flux $\mathcal{G}_4 = 0$ throughout this chapter, and we neglect the curvature polynomial X_8 as well. Then the equation for the warp factor reduces to

$$\Delta_{\perp} e^{3A/2} = 2\kappa_{\mathrm{M}}^2 T_{\mathrm{M2}} \sum_i \delta^{(8)}(x - x_{\mathrm{M2}}^i). \tag{5.2}$$

This means that we are interested in harmonic functions on the transverse space, which are singular at the positions of the M2-branes.

5.1 M2-branes in local M-theory geometries

First, we study the most basic setup, which is an M2-brane in $\mathbb{R}^{1,10}$. We will relate this system to a D2-brane by compactification on a circle. Then we consider this stack in $\mathbb{R}^{1,6} \times \mathrm{TN}$, for which, rather impressively, a full description of the warp factor is known in the single-centered case [40]. To clarify this statement, we mean that the solution interpolates from M2-branes in $\mathbb{R}^{1,10}$ close to the center, to D2-branes in $\mathbb{R}^{1,9}$ far away from the center. In between, it describes the interplay with the Taub-NUT geometry, and therefore the interplay with D6-branes.

5.1.1 Basics of M2-branes

Thus, we start with a stack of N M2-branes in $\mathbb{R}^{1,10}$, with $\mathbb{R}^8$ as transverse space. Assuming the stack is located at the origin of the transverse space, we obtain as warp factor

$$e^{3A/2} = 1 + \frac{Q_{\mathrm{M2}}}{r_\perp^6}, \tag{5.3}$$

where $r_\perp$ denotes radius of the transverse space. The constant in this solution is present such that we recover the background geometry $\mathbb{R}^{1,10}$ asymptotically far away. The charge Q_{M2} follows from the prefactor of the delta-function, and can be fixed by integrating the equation over a sphere around the M2-branes.[1]

The worldvolume theory of this object is not fully understood as of yet, although there have been adequate attempts, such as the ABJM model [45]. For our purposes, we don't need such a description, since we are mainly interested in the warp factor, and duality with Type IIA objects. As mentioned before, this duality can be achieved by replacing one dimension with a circle, which results in $\mathbb{R}^{1,9} \times S^1$. Compactification on this circle should yield Type IIA string theory. For instance, the F1-string can be recovered from an M2-brane wrapping this circle. For this chapter, an M2-brane positioned on the circle is very useful, since it yields the D2-brane.

We can calculate the warp factor in this setting by an image-charge trick, where we treat this latter dimension as $\mathbb{R}$, along which we place a periodic array of M2-branes. This results in a sum of terms, instead of the single term $Q_{\mathrm{M2}} r_\perp^{-6}$ that is present now. We will not discuss this procedure in detail, since we will later consider the case we have a torus T^2, instead of a single circle S^1, in section 5.2.1. If we assume the circle length r_A to be very small, we do not expect our warp factor to depend on the circle coordinate, and this is precisely what occurs if one uses this procedure. Namely, the final result is

$$e^{3A/2} = 1 + \frac{3\pi Q_{\mathrm{M2}}}{8 r_A r_\perp^5} \tag{5.4}$$

with now $r_\perp$ the radius of the transverse space $\mathbb{R}^7$. And since we have $Q_{\mathrm{D2}} = 3\pi Q_{\mathrm{M2}}/8r_A$, we recovered indeed the warp factor for the D2-brane.

Both these solutions will be useful in the next section, since Taub-NUT space interpolates between $\mathbb{R}^4$ close to the center, and $\mathbb{R}^3 \times S^1$, with a small circle length, far away from the center. Namely, this is related to an M2-brane with $\mathbb{R}^8$ as transverse space or a D2-brane with $\mathbb{R}^7$ as transverse space, and therefore th M2-brane with $\mathbb{R}^4 \times \mathrm{TN}_1$ as transverse space should reproduce these results.

[1] Charge $Q_{\mathrm{M2}} = 2^{-1}\pi^{-4}\ell_{\mathrm{M}}^6 N$ [13], where we use $\ell_{\mathrm{M}} = 2\pi\ell_{11}$, such that $T_{\mathrm{M2}} = 2\pi/\ell_{\mathrm{M}}^3$.

5.1.2 M2-branes in Taub-NUT space

Here we consider the setup of a stack of N M2-branes, with $\mathbb{R}^4 \times \mathrm{TN}_1$ as transverse space, such that the M2-branes coincides with the monopole in the base of the Taub-NUT space. We choose to only study this simple case, because, as mentioned before, we can write down a full solution. Achieving a full solution for the two-centered case is already very difficult (as accomplished in [40]), and cases with even more centers can typically only be solved in limiting cases, such as close to a single center, or far away such that they look stacked on top of each other. And in these latter cases, one can use the single-centered solution, with some alterations for the stack of centers. Separating the M2-branes from the Taub-NUT center turns out to be complicated for similar reasons.

Thus, we start by solving the harmonic equation for $\mathbb{R}^4 \times \mathrm{TN}_1$. In doing so, we will make certain assumptions. The assumptions will be based on spherical symmetry around the stack of M2-branes. This means that we only allow our solution to depend on the length of the coordinate y of $\mathbb{R}^4$, and only on the distance from the Taub-NUT center r in the base of TN_1, since we assume the M2-branes to be located at $r = 0$. Close to the Taub-NUT center, the TN_1 looks like $\mathbb{R}^4$, where the Taub-NUT circle becomes part of the sphere around the center. Therefore, we impose that our solution does not allow on this circle coordinate as well. This is also consistent with the fact that the circle becomes small far away, and hence we should smear the solution along it. Then the equation for the warp factor reduces to[2]

$$\nabla_y e^{3A/2} + \frac{1}{V(r)}(\partial_r^2 + \frac{1}{r}\partial_r)e^{3A/2} = 0. \tag{5.5}$$

However, this equation still involves two variables, and is therefore hard to solve. To circumvent this problem, we can Fourier transform the dependence on the $\mathbb{R}^4$. Then we must simply fix the Fourier coefficient later on, in a limiting case where we know the behavior of the solution. Hence we obtain

$$e^{3A/2} = 1 + Q_{\mathrm{M2}} \int \frac{d^4p}{(2\pi)^4} f_p(r) e^{ipy}, \quad (\partial_r^2 + \frac{2}{r}\partial_r) f_p(r) = (1 + \frac{m}{r}) p^2 f_p(r). \tag{5.6}$$

And this equation can be solved by

$$f_p(r) = c_p e^{-|p|r} \mathcal{U}(1 + \frac{|p|m}{2}, 2, 2|p|r) \tag{5.7}$$

with $\mathcal{U}$ the confluent hypergeometric function, and c_p a yet undetermined constant. Now the limiting case we want to consider is $m \to \infty$, in which we should recover the solution of a stack of M2-branes with $\mathbb{R}^8$ as transverse space, combined with the redefinition $w^2 = 4mr$. Therefore, in this limit, it follows that $f_p(r)$ should solve

$$e^{3A/2} = 1 + Q_{\mathrm{M2}} \int \frac{d^4p}{(2\pi)^4} f_p(r) e^{ipy} = 1 + \frac{Q_{\mathrm{M2}}}{(y^2 + w(r)^2)^3}. \tag{5.8}$$

[2] We have chosen not to simplify the derivatives to y, since we will Fourier transform this coordinate in the next part.

And we know this Fourier transform, hence the limit $m \to \infty$ of f_p must be given by

$$f_p(r(w)) = \frac{\pi^2}{2w^2}|p|wK_1(|p|w). \tag{5.9}$$

By the limit properties of $\mathcal{U}$,[3] this means that must set the coefficient c_p equal to

$$c_p = \frac{\pi^2}{8}\frac{1}{m^2}(|p|m)^2\Gamma(|p|m/2), \tag{5.10}$$

Note that this imposes the spherical symmetry in $\mathbb{R}^4$ as well, since f_p now only depends on $|p|$.

This means that we have a solution for the warp factor, in the form of a Fourier integral. By using the spherical symmetry of this solution, we can reduce the integral to

$$e^{3A/2} = 1 + Q_{\mathrm{M2}}\int_0^\infty dp\frac{(py)^2 J_1(py)}{4\pi^2 y^3} f_p(r), \tag{5.11}$$

with J_1 the a Bessel function of the first kind. This is the full description of the warp factor, if we plug in our expression for $f_p(r)$.

Now, as we mentioned before, we recover the D2-brane solution for $r \gg m$. If we use the identities $\mathcal{U}(1,2,2pr) = 1/(2pr)$ and $(pm/2)\Gamma(pm/2) \to 1$, we find in this limit

$$\begin{aligned} e^{3A/2} &= 1 + Q_{\mathrm{M2}}\int dp\frac{(py)^2 J_1(py)}{4\pi^2 y^3}\frac{\pi^2}{8mr}e^{-|p|r} \\ &= 1 + \frac{3Q_{\mathrm{M2}}}{32m(r^2+y^2)^{5/2}} \end{aligned} \tag{5.12}$$

This is our D2-brane solution, with $Q_{\mathrm{D2}} = 3Q_{\mathrm{M2}}/32m$ and $r_A = 4\pi m$, as we found in the previous subsection. Thus the solution interpolates indeed between the M2-brane solution and the D2-brane solution, and it describes or M2-branes in Taub-NUT space, i.e. the M-theory lift of D2-branes in the presence of D6-branes, at intermediate distances.

Now it would be interesting to use this solution for multi-centered Taub-NUT spaces, with an M2-brane located at each center. However, this would alter the function V as well, that is present in our equation, and hence a simple superposition of solutions doesn't suffice, due to cross terms that do not cancel. And our assumption of spherical symmetry is no longer suitable either. In [40] they actually analyze the case with 2 centers, and they needed quite difficult coordinate redefinitions. Separating the stack of M2-branes from the Taub-NUT space is also difficult, because then we lose rotational symmetry in the Taub-NUT base as well.

[3] One needs $\lim_{a\to\infty}\Gamma(1+a-b)\mathcal{U}(a,b,z/a) = 2z^{(1-b)/2}K_{b-1}(2\sqrt{z})$, where we have $z = (pw)^2/4$, $a = 1 + |p|m/2$ and $b = 2$.

5.2 M2-branes and local F-theory geometries

Here we consider the Type IIB systems via M-theory. First we dualize the M2-brane solution to a D3-brane solution, to sketch the main idea. Then we consider the M2-brane solution in $\mathbb{R}^4 \times \mathrm{TN}^\infty$, and dualize it to the D3/D7-brane solution, albeit only in the near-horizon region of the D7-brane. To recall, this dualization needs two circles in the M-theory geometry, one to relate it to Type IIA, and one for T-duality, as we discussed in subsection 2.3.1.

5.2.1 M2-branes in a toroidal background

Thus, to dualize our M2-brane warp factor in $\mathbb{R}^{1,10}$ to a D3-brane, we need to consider $\mathbb{R}^{1,8} \times T^2$ as background geometry instead. Then the metric transverse to the M2-branes can be written as

$$ds_\perp^2 = ds_6^2 + \frac{v}{\tau_2}((dx_A + \tau_1 dx_B)^2 + \tau_2^2 dx_B^2) \tag{5.13}$$

where v is the volume of the torus, our coordinate on the torus is $x = (x_A, x_B)$, both components with periodicity 1. Later on, we will take the F-theory limit $v \to 0$, but first we study it for general v.

We want to find the solution in this new geometry. And, as mentioned in subsection 5.2.1, we can use an image-charge trick to achieve this. Namely, we will assume a toroidal lattice in $\mathbb{R}^2$ instead, and place our stack of M2-branes at each lattice point. Then we can simply use the superposition of solutions to construct the solution in $\mathbb{R}^{1,8} \times T^2$. First, let us recall the solution for a stack of M2-branes at $x = 0$, given by

$$f(y,x) = \frac{1}{(y^2 + (Mx)^2)^3}, \quad M = \sqrt{v}\begin{pmatrix} \frac{\sqrt{\tau_2}}{|\tau|} & 0 \\ \frac{\tau_1}{\sqrt{\tau_2}|\tau|} & \frac{|\tau|}{\sqrt{\tau_2}}, \end{pmatrix} \tag{5.14}$$

where y denotes the coordinate of the transverse $\mathbb{R}^6$. Note that we introduced a matrix M, which will make coordinate redefinitions later on easier. For now, it is only relevant that $(Mx)^2$ describes the distance on the torus. Using this solution, we can construct the superposition

$$e^{3A/2} = 1 + Q_{\mathrm{M2}} \sum_{n \in \mathbb{Z}^2} f(y, x+n). \tag{5.15}$$

Because we are interested in the behaviour of this solution for small volume v, it is convenient to sum over the Fourier modes instead, since this allows for an expansion in orders of v. A summation over Fourier modes can be achieved through a Poisson resummation, which states that

$$\sum_{n \in \mathbb{Z}^2} f(y, x+n) = \sum_{k \in \mathbb{Z}^2} \hat{f}(y,k) e^{2\pi i k \cdot x}, \tag{5.16}$$

where $\hat{f}(y,k)$ denotes the Fourier transform of $f(y,x)$ with respect to x. It is given by

$$\begin{aligned} \hat{f}(y,k) &= \int d^2x f(y,x) e^{-2\pi i x\cdot k} \\ &= \int \frac{d^2x}{v} \frac{1}{(y^2+x^2)^3} e^{-2\pi i x\cdot (M^{-1}k)}, \end{aligned} \tag{5.17}$$

where we shifted $x \to M^{-1}x$, and used that $\det(M) = v$. Then we find for $k = 0$ that

$$\hat{f}(y,0) = \frac{\pi}{2vy^4}. \tag{5.18}$$

And for non-zero k we obtain

$$\hat{f}(y,k) = \frac{\pi^3 Q_{\mathrm{M2}}}{vy^4}\sigma(k)^2 K_2(2\pi\sigma(k)), \tag{5.19}$$

where K_2 denotes the modified Bessel function of the second kind. For convenience, we defined

$$\sigma(k) = |y||M^{-1}k| = \frac{|y|}{\sqrt{v}}\left(\frac{1}{\tau_2}(|\tau|^2 + \frac{\tau_1^2}{|\tau|^2})k_A^2 + \frac{\tau_1}{|\tau|^2}k_A k_B + \frac{\tau_2}{|\tau|^2}k_B^2\right)^{1/2}. \tag{5.20}$$

Then, through Poisson resummation, we obtain as alternative expression for the warp factor

$$e^{3A/2} = 1 + \frac{\pi Q}{2vy^4} + \frac{\pi^3 Q}{vy^4}\sum_{k\neq 0}\sigma(k)^2 K_2(2\pi\sigma(k))e^{2\pi i k\cdot x}. \tag{5.21}$$

We want to apply the F-theory limit for this solution, which states that $y^2 \gg v$. This means that $\sigma(k)$ diverges, which implies that the Bessel functions become exponentially small. Then the entire summation vanishes, and the only part that remains is given by

$$e^{3A/2} = 1 + \frac{\pi Q_{\mathrm{M2}}}{2vy^4}. \tag{5.22}$$

This is indeed the D3-brane warp factor with $\pi Q_{\mathrm{M2}}/2v = Q_{\mathrm{D3}}$, and y as coordinate for the transverse space $\mathbb{R}^6$.

5.2.2 M2-branes and periodic Taub-NUT space

Next, we consider M2-branes with $\mathbb{R}^{1,6}\times\mathrm{TN}^\infty$ as background geometry. Similar to the duality of TN^∞ to the near-horizon geometry of D7-brane, this system will dualize to the D3-brane solution with the near-horizon D7-brane geometry as background. We will try to analyze the equation for the warp factor as thoroughly as possible. In the end, it turns out that we can solve it only in a limiting case, where a simple trick suffices.

We will use the cylindrical coordinates (r,ϕ,z) to describe the periodic 3-dimensional Taub-NUT base, and x as coordinate for the 4-dimensional Euclidean space. Similar to our solution for ordinary Taub-NUT space TN, we will argue for spherical symmetry whenever appropriate. Note that this aspect is a bit subtle. Namely, from the backreaction of the D7-branes, we do not expect rotational symmetry at intermediate length scales. And hence, we should not expect the warp factor due to the D3-branes to possess this symmetry either, at least not at this length scale. However, our local geometry, the periodic Taub-NUT space TN^∞, only captures the near-horizon geometry. Hence we can proceed and assume this rotational symmetry, since our description of the D3/D7-brane system via M-theory breaks down at larger length scales anyway.

Thus, from the arguments above, we can assume our M2-brane warp factor to only depend on $|y|$, r and z.[4]. Then the equation is given by

$$(V^{\infty}\nabla_y^2 + (\partial_r^2 + \frac{1}{r}\partial_r) + \partial_z^2)e^{3A/2} = 0. \tag{5.23}$$

As mentioned before in subsection 5.1.2, superpositions of solutions for the single-centered case do not give the solution for the multi-centered case, due to cross terms with V. Hence the image-charge trick used for $\mathbb{R}^{1,8} \times T^2$ does not work for $\mathbb{R}^{1,6} \times \mathrm{TN}^{\infty}$.

Therefore, let us employ a different strategy for solving this equation. First, to simplify our equation, we can Fourier transform y of the $\mathbb{R}^4$ again, such that we need to solve for a function $f_p(r,z)$, which should depend on the length $|p|$ only, by rotational symmetry. Since the z-coordinate is periodic, we can Fourier transform this coordinate as well

$$f_p(r,z) = \sum_l f_{p,l}(r)e^{2\pi i l z/R_B}. \tag{5.24}$$

Then we obtain as equation

$$\begin{aligned}&(\partial_r^2 + \frac{1}{r}\partial_r)f_{p,l} - l^2 f_{p,l} - p^2(1 - \frac{r_A}{2\pi r_B}\log\frac{r}{r_B\Lambda})f_{p,l}\\ &- p^2\frac{1}{2\pi}\frac{r_A}{\pi r_B}\sum_{k\neq 0}K_0(2\pi\rho|k|/r_B)f_{p,l+k} = 0\end{aligned} \tag{5.25}$$

Solving the equation in this form is rather tough. However, we can solve it in the F-theory limit, which implies that $\rho \gg R_B$. In this limit, the solution does not depend on z, and only the $f_p = f_{p,l=0}$ mode remains. Then the equation reduces to

$$(\partial_r^2 + \frac{1}{\rho}\partial_\rho)f_p - l^2 f_p - p^2(1 - \frac{R_A}{2\pi R_B}\log\frac{\rho}{R_B\Lambda})f_p = 0. \tag{5.26}$$

Then, in the limit $r \ll r_B\Lambda$, it turns out that we are able to solve the equation. However, instead of all the work we went through to get here, we could have applied these limits on the metric directly. Via this way, one finds the same solution. Therefore, let us only discuss this approach below, because it turns out that the metric in this limit results in the equation above.

Direct approach at the solution

Motivated by the analysis above, one can take the limits $r \gg r_B$ and $r \ll r_B\Lambda$ immediately. In this case, we know that we can dualize our background geometry to the near-horizon D7-brane geometry. Therefore, we can make the same coordinate redefinition

$$\rho^2 = [1 - \frac{r_A}{2\pi r_B}\log(r/r_B\Lambda)]r^2. \tag{5.27}$$

Then in these limits, we know the metric takes the form (see Equation (4.25))

$$ds_\perp^2 \simeq ds_4^2 + d\rho^2 + \rho^2 d\phi^2. \tag{5.28}$$

[4]Since we will Fourier transform y later on, we should impose the rotational symmetry for the associated momentum instead later on.

Hence like for the D3/D7-brane model (Eq. (3.13)), we find for the warp factor in this limit

$$e^{3A/2} = 1 + \frac{Q}{(x^2+\rho^2)^2} = 1 + \frac{Q}{(x^2 + [1 - \frac{r_A}{2\pi r_B}\log(r/r_B\Lambda)]r^2)^2} \tag{5.29}$$

Note that there is only a slight difference in conventions between Einstein frame and string frame, which involves factors of the string coupling, and therefor r_A/r_B. It is remarkable that we recovered this result, already known from the D-brane setup in Type IIB [9], via a partly geometrical setup in M-theory. This leads one to the idea that a geometrical setup that captures corrections on the M-theory side could be used to study corrections to the backreaction of the D-brane setup. Namely, as we reviewed in Chapter 3, this approximation is currently the best known result in the AdS/CFT literature. Since our geometry doesn't break rotational symmetry in the 2-dimensional base, it might not be directly applicable to the breaking of chiral symmetry for the field theory.

Summary

First we reviewed the backreaction of M2-branes with background geometries $\mathbb{R}^{1,10}$ and $\mathbb{R}^{1,6} \times \mathrm{TN}_1$. Then we considered a stack of M2-branes in $\mathbb{R}^{1,8} \times T^2$, and related this setup to a stack of D3-branes in $\mathbb{R}^{1,9}$ via the F-theory limit. At last, we analyzed a stack of M2-branes in $\mathbb{R}^{1,6} \times \mathrm{TN}_k^\infty$, from which we recovered the backreaction of D3-branes on the near-horizon geometry of k D7-branes. Reproducing this result via a geometric setup in M-theory had not been considered before in the literature.

Chapter 6

Fluxes and local F-theory geometries

In this chapter we want to study the backreaction of 7-brane fluxes on the geometry. To recall, we assume that the k 7-branes wrap the same divisor S_b in the base of the Calabi-Yau fourfold Y_4, and that the gauge theory of this stack is moved on the Coulomb branch with $\mathrm{U}(1)^k$ gauge symmetry. Via T-duality this is related to 6-branes wrapping S_b in the base of Y_4, which are separated on the circle in the periodic base of the local geometry. In our notation from Chapter 4, this means that they have different positions z_I on the circle in the periodic base of TN_k^∞ or AH_k^∞.

Then the idea is that the field strength of each 7-brane has a flux $\hat{\mathcal{F}}^I$ on S_b, valued in the U(1) of its gauge group. Through identification of the T-dual 6-branes with 2-forms Ω_I on the local geometry, we can expand the 4-form field strength in these fluxes via

$$G_4 = de^{3A/2} \wedge \mathrm{dvol}_{1,2} + \mathcal{G}_4 = de^{3A/2} \wedge \mathrm{dvol}_{1,2} + \hat{\mathcal{F}}^I \wedge \Omega_I. \tag{6.1}$$

Since we do not consider M2-branes, and neglect the curvature polynomial, we obtain as equation for the warp factor

$$\Delta_{\mathcal{Y}_4} e^{3A/2} = \frac{1}{2} *_{\mathcal{Y}_4} (\hat{\mathcal{F}}^I \wedge \hat{\mathcal{F}}^J \wedge \Omega_I \wedge \Omega_J), \tag{6.2}$$

on the local model $\mathcal{Y}_4$ for Y_4. It is difficult to determine the profile of the warp factor on the generic divisor S_b. Therefore, we argue that the volume $\mathcal{V}_{S_b}$ of the divisor is small with respect to the length scale of TN_k^∞ and AH_k^∞, which allows us to integrate S_b out. Then we need a proper ansatz for

$$\langle \hat{\mathcal{F}}^I \wedge \hat{\mathcal{F}}^J \rangle_{S_b} = \frac{1}{\mathcal{V}_{S_b}} \int_{S_b} \hat{\mathcal{F}}^I \wedge \hat{\mathcal{F}}^J = \frac{c^{IJ}}{\mathcal{V}_{S_b}}. \tag{6.3}$$

Typically we can set $c^{IJ} = \delta^{IJ} n^I$ with $n^I = -8\pi^2 k^I$, where k^I denotes the instanton number of the U(1) on the I-th brane.

6.1 Fluxes and periodic Taub-NUT space

We begin with a reproduction of the results from [7], where they considered the periodic Taub-NUT geometry TN_k^∞ in the presence of these fluxes. Our main focus is the derivation of the warp factor, to be able to make a comparison with the warp factor for AH_k^∞, which we derive thereafter. We will also clarify the reasoning used in this derivation, such that we can repeat the procedure in the next section.

For completeness, we start with embedding the flux $\mathcal{G}_4$ in the Cartan subalgebra of $\mathrm{U}(1) \times \mathrm{SU}(k)$ of the gauge group. Therefore, we want to express the flux in the basis $\omega_i = \Omega_i - \Omega_{i+1}$ and $\Omega_1 + \ldots + \Omega_k$, instead of the expansion given in the beginning of this chapter. Consequently, we decompose the flux instead as

$$\mathcal{G}_4 = \mathcal{F}^i \omega_i + \mathcal{F}^0 \sum_J \Omega_J, \tag{6.4}$$

where $\mathcal{F}^0$ is the flux in the additional U(1), and $\mathcal{F}^i$ are the fluxes in the Cartan subalgebra of $\mathrm{SU}(k)$. Using the expression for ω_i, we can identify

$$\hat{\mathcal{F}}^m = \mathcal{F}^0 + \mathcal{F}^m - \mathcal{F}^{m-1}, \quad \hat{\mathcal{F}}^1 = \mathcal{F}^0 + \mathcal{F}^1, \quad \hat{\mathcal{F}}^k = \mathcal{F}^k - \mathcal{F}^0. \tag{6.5}$$

Next we consider the main goal of this section, obtaining the warp factor due to the fluxes. Following the suggestions made in the beginning of this chapter, such as integrating out the dependence on S_b, we need to solve

$$\begin{aligned} d *_4 de^{3A/2} &= \frac{c^{IJ}}{2\mathcal{V}_{S_b}} \Omega_I \wedge \Omega_J \\ &= -\frac{c^{IJ}}{r_A^2 \mathcal{V}_{S_b}} d(\frac{V_I}{V}) \wedge *_4 d(\frac{V_J}{V}), \end{aligned} \tag{6.6}$$

where $*_4$ denotes the Hodge star of TN_k^∞, and we used the identity (B.12)

$$\Omega_I \wedge \Omega_J = -\frac{2}{r_A^2} V d(\frac{V_I}{V}) \wedge *_4 d(\frac{V_J}{V}). \tag{6.7}$$

Judging from this equation for the warp factor, it is useful to point out that

$$d *_4 d(\frac{V_I V_J}{V}) = 2V d(\frac{V_I}{V}) \wedge *_4 d(\frac{V_J}{V}). \tag{6.8}$$

It is tempting to write down the expression for the warp factor directly with this relation by using this relation. However, there are some subtleties involved, due to the fact that an inhomogeneous equation can be solved up to a homogeneous solution. The singular behavior of functions plays an important role in this story as well. Therefore, we will classify the solutions to the harmonic equation, and determine whether they are suitable or not.

The first type of solutions we consider are polynomials in the coordinates. Using the coordinates of the two circles would not result in a periodic warp factor, and hence can be disregarded. Using the remaining coordinates gives rise to large warping far away from the branes, which is inconsistent as well. Then we are left with the constant polynomial, which choose to be 1 to recover the background geometry far away.[1]

The other type of solutions are functions that are singular at a specific position, like the potentials V_I at (r_I, z_I). In the non-periodified case, we could have used any function $1/|(r,z)-(r',z')|$, which is singular at (r',z'). Then the periodification procedure results in functions exactly like V_I, but singular at (r',z') instead of (r_I,z_I) in the periodic base. For our V_I we have

$$\Delta_3 V_I = -r_A \delta^{(2)}(r-r_I)\delta(z-z_I). \tag{6.9}$$

Therefore we can immediately disregard the V_I-like functions for other positions, since we do not have such delta-functions in our equation. One might think that we can disregard the V_I as well, but from a close inspection of the suggested term we find

$$\lim_{(r,z)\to(r_I,z_I)} \frac{V_I V_J}{V} = V_J. \tag{6.10}$$

Hence if $I=J$, we actually have the singular behaviour at (r_I,z_I). Consequently we should subtract a term proportional to V_I to cancel this behavior, since the inhomogeneous term of our equation is regular at these positions. This determines our solution for the warp factor uniquely

$$e^{3A/2} = 1 - \frac{n^I}{2r_A^2\mathcal{V}_{S_b}}\left(\frac{V_I^2}{V} - V_I\right) - \frac{\tilde{c}^{IJ}}{2r_A^2\mathcal{V}_{S_b}}\frac{V_I V_J}{V}, \tag{6.11}$$

where $\tilde{c}^{IJ}$ denotes the off-diagonal part of c^{IJ}, and $n^I = c^{II}$. A suitable interpretation for this off-diagonal part is yet to be determined. In the case that the 6-brane fluxes are due to instanton numbers in their respective U(1)'s, we have

$$e^{3A/2} = 1 - \frac{n^I}{2r_A^2\mathcal{V}_{S_b}}\left(\frac{V_I^2}{V} - V_I\right). \tag{6.12}$$

We can analyze this solution by evaluating the warp factor at the J-th monopole

$$\begin{aligned}
(e^{3A/2}-1)\big|_{(r_J,z_J)} &= \frac{n^I}{2r_A^2\mathcal{V}_{S_b}}\frac{V_I}{V}\left(1+\sum_{K\neq I}V_K\right)\Bigg|_{(r_J,z_J)} \\
&= \frac{n^I}{2r_A^2\mathcal{V}_{S_b}}\left(\delta_{IJ}(1+\sum_{K\neq I}V_K) + V_I\sum_{K\neq I}\delta_{KJ}\right)\Bigg|_{(r_J,z_J)} \\
&= \frac{1}{2r_A^2\mathcal{V}_{S_b}}\left(n^J + \sum_{K\neq J}(n^J+n^K)V_K\big|_{(r_J,z_J)}\right).
\end{aligned} \tag{6.13}$$

The first term suggests that the warp factor sees the flux n^J at the monopole, as expected. Its backreaction falls off away from the monopole, which leads to suppressed warping at the other monopoles. Vice versa, the fluxes of these other monopoles source warping at the J-th monopole, which explains the terms $n^K V_K|_{(r_J,z_J)}$, with $V_K|_{(r_J,z_J)}$ as suppression factor. This means that the warp factor sees the fluxes at the other monopoles. The remaining terms $n^J V_K|_{(r_J,z_J)}$ can be understood as the manifestation of the flux n^J backreacting with the background geometry.

[1] This is a bit subtle in the context of gluing the local geometry into the compact Calabi-Yau fourfold.

6.2 Fluxes and periodic Atiyah-Hitchin space with KK monopoles

Here we want to study the backreaction of the fluxes in the context of AH_k^∞ instead of TN_k^∞. In the Type IIB perspective, this means that we have added an O7-plane to our system of 7-branes. Due to the orientifold projection, the expansion in the fluxes is slightly altered. Namely, we need to expand $\mathcal{G}_4$ in the odd 2-forms $\Omega_I^- = (\Omega_I + \Omega_{-I})/\sqrt{2}$, because the flux $\hat{\mathcal{F}}^I$ of the 7-brane is odd under the orientifold projection. Then we obtain

$$\mathcal{G}_4 = \hat{\mathcal{F}}^I \wedge \Omega_I^-. \tag{6.14}$$

We want to embed this flux in the Cartan subalgebra of SO$(2k)$, and therefore rewrite $\mathcal{G}_4$ in terms of $\omega_i^- = \Omega_i^- - \Omega_{i+1}^-$ and $\omega_k^- = \Omega_{k-1}^- - \Omega_{-k}^- = \Omega_{k-1}^- + \Omega_k^-$. This allows us to decompose as

$$\mathcal{G}_4 = \mathcal{F}^I \wedge \omega_I^-, \tag{6.15}$$

if we identify

$$\begin{aligned} \hat{\mathcal{F}}^1 = \mathcal{F}^1, \quad \hat{\mathcal{F}}^{k-1} = \mathcal{F}^k + \mathcal{F}^{k-1} - \mathcal{F}^{k-2}, \\ \hat{\mathcal{F}}^m = \mathcal{F}^m - \mathcal{F}^{m-1}, \quad m = 2, \ldots, k-2, k. \end{aligned} \tag{6.16}$$

Now we can investigate the main objective of this chapter, deriving the warp factor due to 7-brane fluxes via the local geometry $\mathcal{Y}_4 = S_b \times \mathrm{AH}_k^\infty$. Similar to before, we can integrate out the divisor S_b, which yields

$$\begin{aligned} d *_4 de^{3A/2} &= \frac{c^{IJ}}{2\mathcal{V}_{S_b}}(\Omega_I^- \wedge \Omega_J^-) \\ &= -\frac{c^{IJ}}{2r_A^2\mathcal{V}_{S_b}} V d(\frac{V_I + V_{-I}}{V}) \wedge *_4 d(\frac{V_J + V_{-J}}{V}), \end{aligned} \tag{6.17}$$

where $*_4$ denotes the Hodge star of AH_k^∞. We can employ the same strategy as in the previous section for solving this equation, since a similar identity holds

$$d *_4 d\Big(\frac{(V_I + V_{-I})(V_J + V_J)}{V}\Big) = 2V d(\frac{V_I + V_{-I}}{V}) \wedge *_4 d(\frac{V_J + V_{-J}}{V}), \tag{6.18}$$

and we have comparable singular behavior. It results in

$$e^{3A/2} = 1 - \frac{1}{4r_A^2\mathcal{V}_{S_b}}\Big(n^I[\frac{(V_I + V_{-I})^2}{V} - V_I - V_{-I}] - \tilde{c}^{IJ}\frac{(V_I + V_{-I})(V_J + V_{-J})}{V}\Big). \tag{6.19}$$

In the case that the fluxes satisfy $\tilde{c}^{IJ} = 0$, we have

$$e^{3A/2} = 1 - \frac{n^I}{4r_A^2\mathcal{V}_{S_b}}(\frac{(V_I + V_{-I})^2}{V} - V_I - V_{-I}), \tag{6.20}$$

and the behavior at the J-th monopole is given by

$$\begin{aligned} (e^{3A/2} - 1)\big|_{(r_J,z_J)} &= \frac{n^I}{4r_A^2\mathcal{V}_{S_b}}\left(\frac{V_I}{V}(V - 2V_{-I} - V_I) + V_{-I} - \frac{V_{-I}^2}{V}\right)\bigg|_{(r_J,z_J)} \\ &= \frac{n^I}{4r_A^2\mathcal{V}_{S_b}}\left(\delta_{IJ}(V - 2V_{-I} - V_I) + V_I\sum_{K\neq I}\delta_{KJ} + V_{-I}\right)\bigg|_{(r_J,z_J)} \\ &= \frac{n^J}{4r_A^2\mathcal{V}_{S_b}}(1 - 4V_0 - 4V_{0'})|_{(r_J,z_J)} \\ &\quad + \frac{1}{4r_A^2\mathcal{V}_{S_b}}\sum_{K\neq J}(n^J + n^K)(V_K + V_{-K})|_{(r_J,z_J)}. \end{aligned} \tag{6.21}$$

Again the warp factor sees the flux n^J at the monopole. The warping due to the other monopoles has $(V_K + V_{-K})|_{(r_J,z_J)}$ as suppression factor, since n^K is the flux of the monopole image-monopole pair. The remaining terms correspond to the backreaction of the background geometry with the flux n^J at the monopole. Most interestingly, this includes contributions from the Atiyah-Hitchin bolts.

Interpreting the behavior of the warp factor at the bolts could be interesting as well. However, we neglected the exponential corrections that become relevant close to these bolts, so we will not attempt this yet.

6.3 Attempt at non-trivial warp profile over divisor

At last, we will shortly point out an idea to allow the warp factor to vary of the divisor S_b, by making an explicit choice for this divisor. The idea is that this divisor has non-trivial 2-forms, such that we have an expression for $\hat{\mathcal{F}}^I$. Interpretation of such a setup will not be included.

Our choice is S_b =TN, and we expand in the 2-forms of this space as

$$\mathcal{G}_4 = b^{IJ}\Omega_I \wedge \Omega_J^\infty . \tag{6.22}$$

For now we will not make a choice for b^{IJ}. If we plug this flux into our equation for the warp factor, we find

$$\Delta_\perp e^{3A/2} = *_\perp \left(\frac{2b^{IJ}b^{KL}}{r_a^2 r_b^2} V V^\infty d(\frac{V_I}{V}) \wedge *d(\frac{V_K}{V}) \wedge d(\frac{V_J^\infty}{V^\infty}) \wedge *_\infty d(\frac{V_L^\infty}{V^\infty}) \right). \tag{6.23}$$

Now one can try to solve this solution in the case that both spaces have only one center first. However, the dependence on two variables makes it difficult to solve, and therefore we do not have any interesting solutions yet. Clearly the useful identity of the previous sections does not work, because we do not integrate out the divisor.

Summary

First we reproduced the results of [7], where they derived the warp factor due to 6-brane fluxes via M-theory with periodic Taub-NUT space. Here we pointed out a possible generalization for coupling between fluxes of different 6-branes, but we are uncertain if this idea is of any use. Then we considered the flux-induced warping for a configuration of 7-branes with an O7-plane, which we achieved via M-theory with the new geometry AH_k^∞. At last, we considered the possibility of a non-trivial profile for the warp factor over the divisor S_b of the 6-branes, but we did not achieve anything yet.

Chapter 7

Summary and outlook

7.1 Summary

We started with a short review on F-theory, in which we covered all related concepts we needed. Then we pointed out the main features of the D3/D7-brane system, focusing on the AdS/CFT correspondence for this configuration. We included this part to be able to investigate whether F-theory could provide meaningful insights in this context.[1] Below we will summarize the most important parts of this thesis.

First we considered the construction of local geometries, and reproduced the periodification of Taub-NUT space from [7], which describes D7-branes via M-theory. Then we applied this procedure to Atiyah-Hitchin space with k KK monopoles, to describe k D7-branes in the presence of an O7-plane via M-theory, which resulted in a new geometry AH_k^∞.

Thereafter, we dualized a stack of M2-branes in $\mathbb{R}^{1,8} \times T^2$ to a stack of D3-branes in $\mathbb{R}^{1,9}$ by use of the F-theory limit for the torus area, where we initially kept the dependence of the warp factor on the T^2. Next we considered this stack in $\mathbb{R}^{1,6} \times \mathrm{TN}_k^\infty$, from which we recovered the backreaction of D3-branes on the near-horizon geometry of D7-branes, as known in for instance [9]. Reproducing this result from an M-theoretic approach had not been considered previously in the literature.

At last, we discussed the backreaction of 7-brane fluxes via the constructed geometries TN_k^∞ and AH_k^∞ in M-theory. We slightly generalized the results for TN_k^∞ from [7], although we are not yet sure if this generalization is useful. The warp factor we found in the context of AH_k^∞ was new. Note that these results could also have been derived easily for the non-periodified geometries.

[1] Since the worldvolume theory of the 7-branes decouples from the field theory on the D3-branes in the Maldacena limit, F-theory can only be useful in the study of corrections.

7.2 Outlook

Probably the easiest direction to extend this research, is to study the corrections in Atiyah-Hitchin space close to the bolt, which we neglected here. Since the expression for the metric with no KK monopoles is known explicitly,[2] it would be interesting to investigate whether the periodification procedure recovers the split of an O7-plane into two objects, originally found in [46]. This might be related to the fact that we have two Atiyah-Hitchin bolts in the geometry that we consider.

The flux-induced warp factors we found only depended on the space transverse to the 6-branes, because the divisor S_b was integrated out. Attempting to allow for a non-trivial profile over the divisor S_b could be interesting, but seemed to be a bit of a dead end due to multivariable differential equations. For similar reasons attempted studies of M2-branes with transverse spaces such as $\mathrm{TN} \times \mathrm{TN}$ have failed, which have not been included in this thesis.

It is important to point out that we only needed very basic properties of the functions associated with the monopoles in deriving the warp factor due to the fluxes. As already suggested in [7], the study of periods in the Calabi-Yau fourfold could therefore prove to be useful in generalizing the results here, such that a local description of the geometry is no longer needed. We could also try to study corrections to the gauge coupling as they did for TN_k^∞, but now for AH_k^∞.

[2] The metric for Atiyah-Hitchin space with one monopole/image-monopole pair is also known, and this geometry is called Dancer's manifold.

Appendix A

Details of local M-theory geometries

This appendix is intended to include details of Taub-NUT space and Atiyah-Hitchin space, focusing on the differential forms on these manifolds. Namely, many of these forms can only be defined patchwise. Here, we give these definitions, and calculate various of their properties.

A.1 Single-centered Taub-NUT space

Let us start with Taub-NUT space with only a single center, located at the origin.

We can write the potential and connection in cylindrical coordinates (r, ϕ, z) as

$$V = 1 + V_1 = 1 + \frac{r_A}{4\pi\sqrt{r^2+z^2}}, \quad U = -\frac{r_A}{4\pi}(-1 + \frac{z}{\sqrt{r^2+z^2}})d\phi, \tag{A.1}$$

such that U solves $*_3 dU = -dV$. Note that the first term for U is added such that it is smooth, i.e. such that the prefactor vanishes for $\rho = 0$, since $d\phi$ is ill-defined in this case. However, this only works for $z \geq 0$, and therefore we need to define

$$\begin{aligned} \mathcal{U}_+ = \{(r,\phi,z)|0 \leq z\}: \quad U^+ &= -\frac{r_A}{4\pi}(-1 + \frac{z}{\sqrt{r^2+z^2}})d\phi, \\ \mathcal{U}_- = \{(r,\phi,z)|0 \geq z\}: \quad U^- &= -\frac{r_A}{4\pi}(1 + \frac{z}{\sqrt{r^2+z^2}})d\phi. \end{aligned} \tag{A.2}$$

Furthermore, for later use, let us define the intersection of these patches by

$$H = \{(r,\phi,z)|z = 0\}. \tag{A.3}$$

In order for the metric to be well-defined, we need $dt + U$ to be well-defined. Hence we must define the coordinate t patchwise as well, to counter the patchwise definition of U. This yields

$$\begin{aligned} \mathcal{U}_+: \quad t^+ &= t - \frac{r_A}{2\pi}\phi, \\ \mathcal{U}_-: \quad t^- &= t, \end{aligned} \tag{A.4}$$

where $0 \leq t < r_A$. Note that the periodicity of ϕ of 2π hereby implies the periodicity of t by r_A as well.

Then we can define a 2-form by

$$\Omega = d\eta = \frac{1}{r_A} d\left(\frac{V_1}{V}(dt + U) - U\right), \tag{A.5}$$

which satisfies $*\Omega = -\Omega$ following from $*_3 dV = -dU$. From the patchwise definitions for U, it follows that we have patchwise definitions for η as well, given by

$$\eta^{\pm} = \frac{1}{r_A}\left(\frac{V_1}{V}(dt + U) - U^{\pm}\right). \tag{A.6}$$

Note that Ω is globally defined, independent of the patch used for η, since the patchwise definitions of η differ by a closed form.

Now we want to verify an integral identity for Ω. By Stokes' theorem we know that

$$\int \Omega \wedge \Omega = \int_{S^1 \times \mathcal{U}_+} d(\eta^+ \wedge \Omega) + \int_{S^1 \times \mathcal{U}_-} d(\eta^- \wedge \Omega) = \int_{S^1 \times H} (\eta^+ - \eta^-) \wedge \Omega. \tag{A.7}$$

Then we can simply calculate this integral, using that $\eta^+ - \eta^- = -\frac{d\phi}{2\pi}$, and hence

$$\int \Omega \wedge \Omega = -\int_{S^1 \times H} \frac{1}{2\pi r_A} d\phi \wedge \frac{V_1}{V} \wedge dt = \int_0^{\infty} d\frac{V_1}{V} = \frac{V_1}{V}\Big|_0^{\infty} = -1, \tag{A.8}$$

where we reduced it to an integral over r in the end, and used that $V_1/V \to 1$ for $z = 0$ and $r \to 0$.

A.2 Multi-centered Taub-NUT space

Next, we consider the multi-centered Taub-NUT space. Then we have potentials and connections satisfying $*_3 dU_I = \pm dV_I$, and $V = 1 + \sum_I V_I$, $U = \sum_I U_I$. As mentioned various times throughout this thesis, it is not necessary to generalize the expression for U_I. Namely, we can shift coordinates such that the I-th monopole is located at the origin. Then we can just use the expression of the previous section. Only this definition is needed to calculate the integral identities. Of course, t needs to be defined patchwise appropriately for each monopole, such that $dt + U$ is well-defined.

We can define 2-forms associated with the monopoles by

$$\Omega_I = d\eta_I = \frac{1}{R_A} d\left(\frac{V_I}{V}(dt + U) - U_I\right). \tag{A.9}$$

Note: it is important for the 1-form that we combine with V_I/V to be independent of the patch we use, because otherwise the exterior derivative that we apply results in an ill-defined Ω_I. Since we use $dt + U$, which includes the contributions of all monopoles, we are fine. And because we want our 2-form to be anti-self-dual, we must include all the contributions for V as well. This aspect will be very relevant for the Atiyah-Hitchin space, where it forces us to include the contribution due to the bolt in V and U in these expressions.

Now we are interested in calculating the following integral

$$\int \Omega_I \wedge \Omega_J. \tag{A.10}$$

We can consider the I-th monopole to be located at the origin. Then calculations similar to the single-centered case reduce it to an integral over r (at $z = z_I = 0$)

$$\int \Omega_I \wedge \Omega_J = \int_0^\infty d\frac{V_J}{V} = \frac{V_J}{V}\Big|_0^\infty = -\delta_{IJ}. \tag{A.11}$$

The last equation follows from the limit $V_I/V \to 1$ for $z = 0$ and $r \to 0$, similarly to the single-centered case. Of course, we assume the monopoles to be separate such that we get the Kronecker delta.

The should also mention the following identity

$$\Omega_I \wedge \Omega_J = \frac{2}{r_A^2} V d(\frac{V_I}{V}) \wedge (dt + U) \wedge *_3 d(\frac{V_J}{V}) = -\frac{2}{r_A^2} V d(\frac{V_I}{V}) \wedge *_4 d(\frac{V_J}{V}), \tag{A.12}$$

where we used $*_3 dU_I = -dV_I$ and $*_4 dV_I = -(dt + U) \wedge *_3 dV_I$.

A.3 Atiyah-Hitchin with KK monopoles

Here we will analyze the differential forms on Atiyah-Hitchin space. Actually, since we consider its asymptotic behavior, we can treat the bolt as a monopole with −4 units of mass, and therefore many results are similar to the multi-centered Taub-NUT space. Hence, this section will focus on the subtleties due to this limit.

Note that, instead of integrating over the quotient space, we will integrate over the whole space, for convenience, and do not correct with factors of 1/2.

We will denote the potential and connection of the bolt by V_0, U_0, with the factor of −4 extracted out. The potentials and connections of the monopoles are denoted by V_I, U_I. Hence we consider $V = 1 - 4V_0 + \sum_I (V_I + V_{-I})$ and $U = -4U_0 + \sum_I (U_I + U_{-I})$, and $dt + U$, with t defined patchwise such that $dt + U$ is defined globally.

The 2-forms of the monopoles are given by

$$\Omega_{\pm I} = d\eta_{\pm I} = \frac{1}{R_A} d(\frac{V_{\pm I}}{V}(dt + U) - U_{\pm I}) \tag{A.13}$$

As mentioned in the previous section, it is important that U contains the contribution of the bolt such that $\Omega_{\pm I}$ is well-defined. And by (anti-)self-duality of $\Omega_{\pm I}$, this implies that V must contain the contribution of the bolt as well. To trust our calculations, we should assume the monopoles to be positioned sufficiently far away from the bolt. Also, we should consider a patchwise definition for $U_{\pm I}$ such that the boundary $H_{\pm I}$ has the position of the monopole as closest point to the bolt, just to be sure.

Then, up to corrections, we can calculate

$$\int \Omega_{\pm I} \wedge \Omega_{\pm J} = -\int_{S^1 \times H_{\pm I}} \frac{1}{2\pi R_A} d\phi \wedge \frac{V_{\pm J}}{V} \wedge dt = -\delta_{IJ}, \tag{A.14}$$

and zero if we consider a monopole and an image monopole.

The identity for $\Omega_I \wedge \Omega_J$ is identical to the previous section as well.

We believe that many of these results carry easily over when we include the corrections of the Atiyah-Hitchin space.

Appendix B

Details of local F-theory geometries

This appendix is intended to derive the construction of the periodic arrays, and point out the details for the periodic Taub-NUT space and periodic Atiyah-Hitchin space, similar to what we did for their non-periodified versions. We will keep it short, since a thorough derivation is already included in [7].

B.1 Construction of periodic arrays

First we periodify a potential. Starting point is

$$V = 1 + \frac{r_A}{4\pi r_B}\sum_l \left(\frac{1}{\sqrt{(r/r_B)^2 + (z+l)^2}} - \frac{1}{|l|}\right). \tag{B.1}$$

Then we want to apply a Poisson resummation according to

$$\sum_l f(r, z+l) = \sum_l \hat{f}(r,l) e^{2\pi i l z}, \tag{B.2}$$

where $\hat{f}(r,l)$ denotes the Fourier transform of $f(r,z) = 1/\sqrt{(r/r_B)^2 + z^2}$ with respect to z. It is given by

$$\hat{f}(r,l) = \int dz \frac{1}{\sqrt{(r/r_B)^2 + z^2}} e^{-2\pi i l z} = 2K_0(2\pi r l / r_B). \tag{B.3}$$

Now there is a subtlety for $k = 0$. For this value K_0 diverges as

$$K_0(x) \simeq -\log\frac{x}{2} - \gamma. \tag{B.4}$$

Precisely for this divergency, we introduced our cutoff term, which is divergent as well. We can take this limit in the following way (identify k with $1/N$)

$$\lim_{N\to\infty}\left(K_0\left(\frac{2\pi r}{r_B N}\right) - \sum_{l=1}^{\infty}\frac{1}{l}\right) = \lim_{N\to\infty}\left(-\log\left(\frac{\pi r}{r_B N}\right) - \gamma - \sum_{l=1}^{\infty}\frac{1}{l}\right) = -\log\frac{r}{r_B\Lambda}, \tag{B.5}$$

where we defined $\Lambda = 1/(\pi e^{2\gamma})$, and used the asymptotic behaviour of K_0 in combination with

$$\gamma = \lim_{N\to\infty}\left(\sum_{l=1}^{N}\frac{1}{l} - \log N\right). \tag{B.6}$$

This results in the following expression

$$\begin{aligned} V &= 1 - \frac{R_A}{2\pi R_B}\left(\log(r/r_B\Lambda) - \sum_{l\neq 0} K_0(2\pi r l/r_B)e^{2\pi i l z}\right) \\ &= 1 - \frac{R_A}{2\pi R_B}\left(\log(\hat{\rho}/\Lambda) - 2\sum_{l>0} K_0(2\pi r|l|/r_B)\cos(2\pi l z)\right). \end{aligned} \tag{B.7}$$

Similarly we can rewrite the connection U, but we will claim the final result here directly.

$$\mathcal{U}_n = \{(r,\phi,z)|n \leq z \leq n+1\}: \quad U^n = -\frac{r_A}{4\pi}\left(1 + 2n + 2z + 4r/r_B\sum_l K_1(2\pi r l/r_B)\sin(2\pi l z)\right)d\phi. \tag{B.8}$$

which satisfies $*_3 dU = -dV$. We can choose to add a closed forms proportional to $d(z\phi)$ and $d\phi$, which results for the 0-th patch in

$$U = \frac{r_A}{2\pi}\phi dz - \frac{r_A}{4\pi}\left(4r/r_B\sum_l K_1(2\pi r l/r_B)\sin(2\pi l z)\right)d\phi. \tag{B.9}$$

B.2 Periodic Taub-NUT space

One can generalize the procedure above to the multi-centered case with $V = 1 + \sum_I V_I$ and $U = \sum_I U_I$. Then we can define a 2-form

$$\Omega_I = d\eta_I = \frac{1}{r_A}d\left(\frac{V_I}{V}(dt + U) - U_I\right). \tag{B.10}$$

And by restricting to a single period for $z \in [0,1]$, we can integrate

$$\int \Omega_I \wedge \Omega_J = \int_{S^1}\frac{dt}{r_A}\int_0^\infty d\frac{V_J}{V} = \frac{V_J}{V}\Big|_0^\infty = -\delta_{IJ} \tag{B.11}$$

We also have the following identity again

$$\Omega_I \wedge \Omega_J = \frac{2}{r_A^2}Vd\left(\frac{V_I}{V}\right)\wedge(dt+U)\wedge *_3 d\left(\frac{V_J}{V}\right) = -\frac{2}{r_A^2}Vd\left(\frac{V_I}{V}\right)\wedge *_4 d\left(\frac{V_J}{V}\right). \tag{B.12}$$

B.3 Periodic Atiyah-Hitchin space with KK monopoles

Since we have not found a way yet to include the exponential corrections, this would be a reproduction of the above section, in the same way as Appendix A.3. Therefore we will not include a further analysis.

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