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A Truthmaker Semantics Proposal to Abandon Logical Omniscience

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Contents

1 Introduction

Humans are non-ideal reasoners.[1] They fail to satisfy an important aspect of ideal reasoning: the property of logical omniscience. That is, humans are not logically omniscient, which means that they do not know all logical truths and they do not know all logical consequences of their knowledge.[2] The aim of this paper is to do justice to the property of not being logically omniscient by attempting to represent this human characteristic in a logical framework.

1.1 Relevance to Artificial Intelligence

To get a better understanding of the problem of logical omniscience it should be considered in the context of the field of artificial intelligence. Artificial intelligence is the area in computer science which is concerned with the creation of intelligent systems which "work and react" like humans do.[3] If logical frameworks are used to model knowledge in an intelligent system and if those frameworks entail logical omniscience then the resulting intelligent system does not match human intelligence. This is because the intelligent system is logically omniscient and humans are not. For this reason it would be desirable to have a logical framework as the base of an intelligent system in which logical omniscience is not implied.

Another application of artificial intelligence in which logical omniscience is not desired concerns human robot interaction.[4] Robots are expected to be used for social purposes in a not too distant future. Consider the case in which an intelligent system interacts with a human and suppose that the intelligent system needs to reason about the human's knowledge, in order to solve a problem. On the basis of the wrong assumption of humans being logically omniscient, the intelligent system could make an incorrect analysis about the human and therefore the system might make inappropriate decisions. Thus in order to correctly analyse human reasoning the property of being not logically omniscient should somehow be implemented in the intelligent system.[5]

[1] Fausto Giunchiglia et al., "Non-omniscient belief as context-based reasoning," *IJCAI* Vol. 93 (1993): 549.

[2] Robert Stalnaker, "The Problem of Logical Omniscience, I," *Synthese* Vol. 89 No. 3 (1991): 426.

[3] "Artificial Intelligence (AI)," accessed June 13, 2018, https://www.techopedia.com/definition/190/artificial-intelligence-ai.

[4] "Human-Robot Interaction," accessed June 13, 2018, http://humanrobotinteraction.org/1-introduction/.

[5] An example in which humans need to reason about other persons knowledge is known as the muddy children example. In a specific version of this example there exist three children and two of them have mud on their forehead. Nobody is able to see himself, but everybody can see the others. So the two muddy children both see one muddy child and the child without mud on his face sees two muddy children. Given their fathers' announcements and the actions of the others they will be able to deduct who has mud on their face and who has not, as they are assumed to be ideal reasoners and thus logically omniscient. - Ronald Fagin et al., *Reasoning about Knowledge* (Cambridge: The MIT Press, 1995), 3 - 7.

Imagine that this example was applied to two non logically omniscient humans and an intelligent system. The intelligent system is not able to rely on the logical omniscience of the humans but if he is

1.2 Structure of this Paper

Chapter 2 introduces the problem of logical omniscience. The first sections will sketch the context in which this problem manifests itself. In Section 2.1 classic epistemic logic is discussed. Epistemic logic is the field which is concerned with modeling the notion of knowledge in a logical framework.[6] The classic approach to epistemic logic is based on Kripke structures. Section 2.2 explains how those structures are used to formalise knowledge.[7] Subsequently, in Section 2.3 an epistemological approach will be used, introducing an alternative definition of knowledge. In this definition knowledge is identified as justified true belief.[8] This definition will be formally represented in Kripke structures in Section 2.4. It will be concluded in Section 2.5 that the resulting knowledge representation entails the presence of logical omniscience, which is problematic in the context of human reasoning. The purpose of this paper is to provide a way such that (most of) the six logical omniscience forms can be abandoned, therefore Chapter 3 is devoted to an alternative logical framework: truthmaker semantics. The notion of truthmaker semantics will be explained (3.1) and then a means will be presented to represent knowledge as justified true belief in this semantics (3.2). Section 3.3 provides an analysis of the logical omniscience forms applied to the truthmaker semantics. The paper will conclude with Chapter 4.

not aware of the humans not being logically omniscient or if he does not know what it means not to be logically omniscient he would not be able to reason about the humans' knowledge. The example would fail anyway because if non logically omniscient humans are involved nobody neither the humans nor the intelligent system can be sure of their mud status. Since nobody can rely on the knowledge and rationality of others, that is important for the example to succeed. Maybe one does not know his mud status, but if he had derived all consequences correctly he would know his mud status and since the others are relying on his judgment their judgments about their own mud status will fail as well. The purpose of showing this example is that not knowing about the lack of logical omniscience in humans can result in problems whenever intelligent systems and humans interact.

[6] Wesley Holliday, *"Epistemic logic and epistemology: Handbook of Formal Philosophy,"* Berkeley (2015).

[7] In this paper a single-agent approach is adopted. In the context of epistemic logic this means that if there exists knowledge of some proposition ϕ this knowledge is applied to a single agent and can be read as individual i knows ϕ.

[8] Richard Feldman, *Epistemology: Foundations of Philosophy Series* (New Jersey: Prentice Hall, 2003), 23.

2 Classic Epistemic Logic, Justified True Belief (JTB) and the Logical Omniscience Problem

Epistemology is the study of knowledge. Researchers in this field try to develop a theory that presents the conditions such that only the people who satisfy those conditions have knowledge.[9] Closely related to epistemology is the discipline of epistemic logic, which is concerned with "a formal logical analysis of reasoning about knowledge".[10] In the classic logical approach to knowledge the notion of possible worlds is used. In order to comprehend classic epistemic logic it is necessary to understand possible world semantics, therefore this chapter begins with a brief explanation of Kripke structures: the traditional possible world semantics. By using those structures it is demonstrated how knowledge is represented. Subsequently, another approach to knowledge is introduced. This approach originates from epistemology rather than from epistemic logic and is known as justified true belief (JTB).[11] After introducing JTB an implementation of this approach in a logical system of Kripke structures is presented. This chapter concludes with a fairly extensive explanation of a problem which comes along with the classic logical representation of knowledge as justified true belief: the logical omniscience problem.

2.1 Possible Worlds Semantics: Kripke Structures

Formally analysing modal notions, like possibility and necessity, can be done by taking into account the semantics of possible worlds. The intuition behind using possible worlds is that the real situation could have been different.[12] In possible worlds semantics a set of possible worlds is introduced.[13] The actual world belongs to this set. It differs from the others because unlike the rest it represents the way things really are. The other worlds represent situations different from the real one and therefore function as alternatives to the actual world.

The prevalent semantics which uses possible worlds is Kripke structures. Those structures can be divided into frames and models. A frame consists of a non-empty set of possible worlds and a set of binary relations on those worlds. A model is based on a frame. It therefore consists of the same set of worlds and the same set of relations as the frame it is based on. Besides those sets it also contains a valuation

[9] Richard Feldman, *Epistemology: Foundations of Philosophy Series*, 1.

[10] Fagin et al., *Reasoning about Knowledge*, 1.

[11] Jonathan Ichikawa and Matthias Steup, "The Analysis of Knowledge," in *The Stanford Encyclopedia of Philosophy* (Summer 2018 Edition), ed. Edward N. Zalta, https://plato.stanford.edu/archives/sum2018/entries/knowledge-analysis/.

[12] David Lewis, "Possible Worlds," in *The Possible and the Actual: Readings in the Metaphysics of Modality*, ed. Michael J. Loux (London: Cornells University Press, 1979), 182.

[13] James Garson, "Modal Logic," in *The Stanford Encyclopedia of Philosophy* (Spring 2016 Edition), ed. Edward N. Zalta, https://plato.stanford.edu/archives/spr2016/entries/logic-modal/.

function. This function maps every atomic proposition to a set of worlds in which the proposition is true.[14] To get a better understanding of the frame and model notions consider the formal definition below.[15]

Definition 2.1 Kripke frames and models

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1. Kripke frame $\mathcal{F} = \langle W, R \rangle$
   - W is a non-empty set of possible worlds[16]
   - $R \subseteq (W \times W)$ is a binary relation on $W : wRv$ means that w can access v.
2. Kripke model $\mathcal{M} = \langle W, R, V \rangle$
   - $\langle W, R \rangle$ is the frame $\mathcal{M}$ is based on.
   - $V : \phi \mapsto \mathscr{P}(W)$ for every $p \in \phi : V(p)$ is the set of all worlds in which p is the case.

The notions of possibility and necessity can now be modeled. Consider the actual world w as point of perspective and let ϕ be the proposition which will be evaluated.[17] ϕ is possible in the actual world w if and only if there is a world v accessible from w such that ϕ is the case in v ($\exists v$ s.t. $v \in W$ and wRv and $v \vDash \phi$). ϕ is necessary in the actual world w if and only if in every world v, w can access, ϕ is true ($\forall v$ s.t. $v \in W$ and wRv, $v \vDash \phi$).

2.2 Knowledge Representation in Kripke Structures

In classic epistemic logic possible worlds are used to formalise knowledge. The intuition behind this usage is that if an individual does not have complete knowledge about the actual world he considers other worlds possible.[18] The fewer worlds one considers possible, the less uncertain he is and thus the more he knows. To reason about knowledge in a formal way a knowledge operator K is introduced. 'An individual i knows a proposition ϕ' is then written as $K_i\phi$. It could be the case that i does not know whether ϕ is or is not true. To take this option into account the dual of K is introduced, $\neg K \neg$.[19] $\neg K_i \neg \phi$ means that i considers both ϕ and $\neg\phi$ possible, since i does not know if ϕ is not the case.

In classic epistemic logic Kripke structures are used to give meaning to statements like $K\phi$ and $\neg K \neg \phi$.[20] World w is taken as the world of perspective. $K\phi$ is true

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[14]Rosja Mastop, *"Modal Logic for Artificial Intelligence,"* (np), 18.

[15]Another way to get a better understanding of modal logic is by reading "Modal Logic for Artificial Intelligence" by Mastop - Mastop, *"Modal Logic for Artificial Intelligence,"* 10 - 55.

[16]The property of any world w in the set of possible worlds W is consistency, which means that no $\perp$ can be derived from any proposition in the set. No contradiction is allowed, because then the property of consistency is violated. For example, if p is in the world w it is impossible that $\neg p$ is in w or derivable from other proposition in w since that will lead to a contradiction. - Christopher Menzel, "Possible Worlds," in *The Stanford Encyclopedia of Philosophy* (Winter 2017 Edition), ed. Edward N. Zalta, https://plato.stanford.edu/archives/win2017/entries/possible-worlds/.

[17]Knowledge about propositional logic is assumed. If the reader lacks knowledge about this topic, the first chapter of "Modal Logic for Artificial Intelligence" by Mastop briefly describes propositional logic - Mastop, *"Modal Logic for Artificial Intelligence,"* 5 - 9.

[18]Fagin et al., *Reasoning about Knowledge*, 7.

[19]Fagin et al., *Reasoning about Knowledge*, 17.

[20]Fagin et al., *Reasoning about Knowledge*, 17.

in w if and only if ϕ is true in all worlds v which w considers possible.[21] Notice that knowledge here is analogous to necessity before.[22] Necessarily ϕ in w means that ϕ is true in all worlds w can access and knowing ϕ in w means that ϕ is true in all worlds w can access. Now consider the other statement: $\neg K \neg \phi$. $\neg K \neg \phi$ is true in w if and only if ϕ is true in at least one world which is accessible from w.

Usually when Kripke structures are used in an epistemology context the veridicality condition is assigned to relation R.[23] This condition is known as reflexivity, which corresponds to the truth of knowledge and means that every world can access itself ($\forall w$ s.t. $w \in W$ and wRw).[24] Assigning conditions to R will imply closure forms. In the case of reflexivity, for example, it follows directly from the knowledge of ϕ that ϕ is true ($K\phi \rightarrow \phi$).[25] As has been mentioned in the introduction the main interest of this paper is to abandon logical omniscience. As will be explained in Section 2.5 there exist several logical omniscience closure forms which cannot be abandoned from logical frameworks which are based on possible worlds. This is because the problem of logical omniscience arise from the usage of possible worlds. The closure forms that are present when conditions are assigned to R (R closure forms) do not depend on the usage of possible worlds, because even when a possible world framework is used those closure forms can be abandoned simply by withdrawing the conditions which cause those forms. Thus, the R closure forms are - unlike the logical omniscience forms - not essential to the logical omniscience problem and will therefore not taken into further account in this paper.

Observe the Kripke model $\mathcal{M} = \langle W, R, V \rangle$ in figure 2.1 to get a better understanding of how knowledge is represented in this framework.

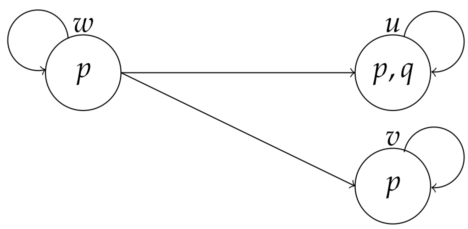

FIGURE 2.1: Kripke model $\mathcal{M}$.

Suppose that w is the world of perspective. $w \vDash Kp$ is a true statement, since in every

[21] Obviously, a world is not able to consider something possible. If the statement $K\phi$ is true in world w this statement can be interpreted as ϕ is known in w rather than w knows ϕ. Another interpretation of $w \vDash K\phi$ could that w should be read as the individual in w, then the individual in w knows ϕ because ϕ is true in all worlds he considers possible.

[22] Stalnaker, "The Problem of Logical Omniscience, I," 425.

[23] Vincent Hendricks and John Symons, "Epistemic Logic," in *The Stanford Encyclopedia of Philosophy* (Fall 2015 Edition), ed. Edward N. Zalta, https://plato.stanford.edu/archives/fall2015/entries/logic-epistemic/.

[24] "whatever is known is true" - Timothy Williamson, "Gettier Cases in Epistemic Logic," *Inquiry* Vol. 56 No. 1 (2013): 4.

[25] When knowledge is considered other conditions which are usually assigned to R are positive- and negative introspection. Positive introspection is referred to as transitivity and means that whenever one knows ϕ, one also knows that he knows that ϕ ($K\phi \rightarrow KK\phi$). Negative introspection refers to the euclidian property which means that whenever one does not know ϕ, one knows that he does not know ϕ ($\neg K\phi \rightarrow K\neg K\phi$). The euclidian property together with transitivity and reflexivity form the property of equivalence. The Kripke system which has this property is referred to as $S5$ and is often used in epistemic logic. - Kwang Mong Sim, "Epistemic Logic and Logical Omniscience: A Survey," *International Journal of Intelligent Systems* Vol. 12 (1997): 59-60.

world accessible from w, p is the case ($w \vDash p$, $v \vDash p$ and $u \vDash p$).[26] $w \vDash \neg K \neg q$ is also a true statement, since in at least one world accessible from w, q is the case ($u \vDash q$).[27]

2.3 JTB as Knowledge

In the previous section it is showed how knowledge is represented in the classic epistemic logical system. Recall that the condition under which an individual has knowledge of ϕ in that system is that ϕ should be true in all worlds the individual considers possible. It is hard to see how this condition applies to knowledge of an individual outside of the logical system.[28] Therefore, in this section another approach to knowledge is introduced. This approach originates from epistemology and states real-life applicable knowledge conditions.

In epistemology the standard view of knowledge is based on the idea that people usually think they know quite a lot.[29] This view fits well with the traditional analysis of knowledge, according to which knowledge is justified true belief.[30] An individual knows a proposition ϕ if and only if the individual believes ϕ, ϕ is true and the individual is justified in believing ϕ.[31] Believing a proposition is the first condition of knowing the proposition. One cannot know ϕ if he does not believe that ϕ. By believing the proposition one accepts that the proposition is true.[32] Knowledge requires truth as well, false propositions cannot be known.[33] Besides the conditions of belief and truth, justification is needed. The justification for believing a proposition needs to be strong and adequate.[34] About the precise meanings of those two notions can be argued. This is not the right place to do so, since for the purpose of this paper it is merely relevant that justification is required. Therefore the identification of the specific content of the justification component of knowledge is ignored here.

The most important objection to the JTB approach is presented by Gettier.[35,36] Gettier gives two cases in which an individual has met the requirements of justified true belief, because the individual is justified in believing a proposition and it happens to be true (coincidently).[37] Nevertheless in those cases knowledge cannot be

[26] $w \vDash Kp$ means that p is known in w, because p is true in every world which is considered possible in w.

[27] Again, $w \vDash \neg K \neg q$ means that in w it is not known whether q, since in w at least one world is considered possible in which q is not the case and also at least one world in which q is the case.

[28] The condition, applied to a real-life individual, brings up some questions. For example how do the worlds look like, an individual considers possible? Is this a real existing world or some imaginary world? And how to find out what is true in those worlds?

[29] Feldman, *Epistemology: Foundations of Philosophy Series*, 2.

[30] "The examples of knowledge endorsed by *The Standard View* seem to be cases of justified true belief. And cases in which we lack knowledge seem to be cases in which we lack at least one of these three factors." - Feldman, *Epistemology: Foundations of Philosophy Series*, 23.

[31] Feldman, *Epistemology: Foundations of Philosophy Series*, 15.

[32] According to Feldman there are three attitudes towards believing a proposition: one can believe, disbelieve or suspend his judgement about the proposition. Believing includes not only "fully confident acceptance" but also "hesitant acceptance", as long as one accepts the proposition to be true he believes the proposition no matter what the extent of confidence is. - Feldman, *Epistemology: Foundations of Philosophy Series*, 13 - 16.

[33] Sometimes people say they know a proposition ϕ, but it then turns out that ϕ is not true. In that case it means they did not know ϕ, since to know ϕ it is necessary that ϕ is true. Feldman explains this by using "The Mystery Story" example. - Feldman, *Epistemology: Foundations of Philosophy Series*, 13.

[34] Feldman, *Epistemology: Foundations of Philosophy Series*, 21.

[35] Edmund Gettier, "Is Justified True Belief Knowledge?" *Analysis* Vol. 23 (1963): 121 - 123.

[36] For other objections and also alternatives to the standard view consider chapter 3 to 9 in "Epistemology" by Feldman. (Feldman, *Epistemology: Foundations of Philosophy Series*, 25 - 190.)

[37] Feldman, *Epistemology: Foundations of Philosophy Series*, 25.

ascribed to the individual.[38] Since there are examples of one having justified true belief without having knowledge, JTB as knowledge does not appear to be the correct definition of knowledge. Although Gettier's cases are an important problem for the JTB approach, this analysis of knowledge will still be used in this paper. The main concern of this paper is the problem of logical omniscience and the problematic aspect of the JTB approach has nothing to do with it. This is because logical omniscience arises due to the use of a possible worlds framework, which does not depend on the chosen knowledge definition. Therefore the difficulties which come with the chosen knowledge definition do not affect the logical omniscience problem. The JTB approach is chosen for the sake of concreteness and because it is in line with the commonsense view on knowledge.[39]

2.4 JTB in Kripke Structures

In this section the Kripke structures which were presented in sections 2.1 and 2.2 will be adjusted in such a way that knowledge will be represented as justified true belief. Recall that one relation operator R was used in the classic way of representing knowledge in a logical system.[40] R was seen as the epistemic accessibility of an individual, all what knowledge is was represented in that relation.[41] A way to provide a JTB framework which uses Kripke structures, is by creating two different relation operators: one for belief and one for justification. An individual believes ϕ if and only if in all worlds the individual considers doxastically possible ϕ is true. An individual has a strong and adequate justification for ϕ if and only if ϕ is true in all worlds the individual can access via the justification relation. Knowing ϕ would then mean that an individual has to be justified in believing ϕ and ϕ should be the case. Consider the formal definition of JTB Kripke structures and the notions of justification, belief and knowledge in definition 2.2 below.

Definition 2.2 Justification, belief and knowledge in JTB Kripke structures

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1. JTB Kripke frame $\mathcal{F} = \langle W, R_J, R_B \rangle$
2. JTB Kripke model $\mathcal{M} = \langle W, R_J, R_B, V \rangle$

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[38] One of the cases: "Suppose that Smith and Jones have applied for a certain job. And suppose that Smith has strong evidence for the following conjunctive proposition: (d) Jones is the man who will get the job, and Jones has ten coins in his pocket. Smith's evidence for (d) might be that the president of the company assured him that Jones would, in the end, be selected and that he, Smith, had counted the coins in Jones's pocket ten minutes ago. Proposition (d) entails: (e) The man who will get the job has ten coins in his pocket. Let us suppose that Smith sees the entailment from (d) to (e), and accepts (e) on the grounds of (d), for which he has strong evidence. In this case, Smith is clearly justified in believing that (e) is true. But imagine, further, that unknown to Smith, he himself, not Jones, will get the job. And, also, unknown to Smith, he himself has ten coins in his pocket. Proposition (e) is true, though proposition (d), from which Smith inferred (e), is false. In our example, then, all of the following are true: (i) (e) is true, (ii) Smith believes that (e) is true, and (iii) Smith is justified in believing that (e) is true. But it is equally clear that Smith does not know that (e) is true; for (e) is true in virtue of the number of coins in Smith's pocket, while Smith does not know how many coins are in Smith's pocket, and bases his belief in (e) on a count of the coins in Jones's pocket, whom he falsely believes to be the man who will get the job." - Gettier, "Is Justified True Belief Knowledge?" 121.

[39] Feldman, *Epistemology: Foundations of Philosophy Series*, 1 - 5.

[40] In classic epistemic logic Kripke structures are used. Those structures consist of frames and models. A frame contains a set of relations and a set of worlds and a model consist besides those sets of a valuation function: respectively $\mathcal{F} = \langle W, R \rangle$ and $\mathcal{M} = \langle W, R, V \rangle$.

[41] Timothy Williamson, "Gettier Cases in Epistemic Logic," *Inquiry* Vol. 56 No. 1 (2013): 3.

3. For any $w \in W$, $w \models J\phi$ if and only if $\forall v$ such that $v \in W$ and $wR_J v$, $v \models \phi$: justification for ϕ.

4. For any $w \in W$, $w \models B\phi$ if and only if $\forall v$ such that $v \in W$ and $wR_B v$, $v \models \phi$: belief in ϕ.

5. For any $w \in W$, $w \models K\phi$ if and only if $w \models B\phi \wedge J\phi \wedge \phi$: knowledge about ϕ.

A component of the JTB analysis of knowledge is belief. The logic which is concerned with belief is called doxastic logic.[42] Usually when a doxastic framework is used it assigns the condition of not allowing inconsistent beliefs to the relation R.[43] This condition refers to the serial property and means that every world can access at least one world ($\forall w\ w \in W$ s.t. $\exists v\ v \in W$ and wRv).[44] Assigning conditions to R give rise to closure forms, in the case of seriality it means that from believing ϕ directly follows that $\neg\phi$ is not believed ($B\phi \rightarrow \neg B \neg \phi$).[45] As has been stated in section 2.2 the closure forms which depend on R are not considered to be an important difficulty in the context of the logical omniscience problem, therefore they are ignored in the remainder of this paper.

Observe the JTB Kripke model $\mathcal{M} = \langle W, R_J, R_B, V \rangle$ in figure 2.2 to get a better understanding of how knowledge as justified true belief is represented in this framework.

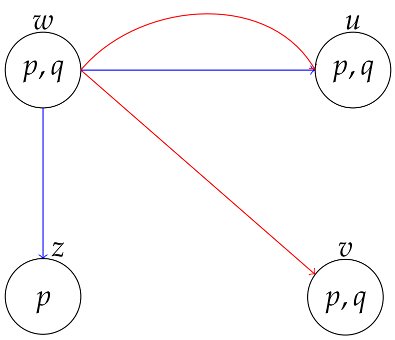

FIGURE 2.2: JTB Kripke model $\mathcal{M}$.

The red lines represent the belief relation (R_B) and the blue lines represent the relation of justification (R_J). Consider world w as the world of perspective. In w both p and q are believed: $w \models Bp$ and $w \models Bq$. This is true because in all worlds with a *doxastic arrow* from w p and q are the case: $v \models p$, $v \models q$, $u \models p$ and $u \models q$. In world w there is justification for p: $w \models Jp$. In all worlds *accessible via the justification relation* from w p is true: $z \models p$ and $u \models p$. In w p is known, since p is justified believed and

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[42] Sim, "Epistemic Logic and Logical Omniscience: A Survey," 57 - 60.

[43] In the case of JTB Kripke structures the conditions would be assigned to the belief relation R_B.

[44] This property does not seem to align with human belief since humans can believe contradictions, see: The editors of Encyclopaedia Britannica, "Cognitive Dissonance," in *Encyclopaedia Brittanica* (March 27, 2018), https://www.britannica.com/science/cognitive-dissonance.

[45] Besides the condition of no inconsistent beliefs the conditions of positive and negative transparancy are often assigned to belief. Positive transparency refers to transitivity: whenever one believes that ϕ he will also believe that he believes ϕ. Negative transparency refers to the euclidean property: whenever one fails to believe ϕ then he will also believe that he does not believe ϕ. The system which satisfy those conditions is referred to as $KD45$. - Michael Caie, "Doxastic Logic," in *The Open Handbook of Formal Epistemology*, ed. Richard Pettigrew and Jonathan Weisberg, 2017.

p is true: respectively $w \vDash Jp$, $w \vDash Bp$ and $w \vDash p$. Notice that although q is believed and true in w there is no justification for q in w. Both q and $\neg q$ are considered possible via the justification arrow. To know it is necessary to satisfy all three conditions, therefore there is no knowledge of q in w ($w \vDash \neg Kq$ and also $w \vDash \neg K\neg q$).

2.5 Logical Omniscience Problem in JTB Kripke Structures

All formal representations of knowledge which are founded in the notion of possible worlds come with a problem, known as the logical omniscience problem.[46] As has been shown, JTB Kripke structures are based on possible worlds, therefore this logical system is also faced with the problem. Logical omniscience applied to an individual means that the individual is logically omniscient if and only if he knows all logical truths and all the deductive consequences of anything that he knows.[47]

The main reason why logical omniscience of an individual is a problem is because it is not completely compatible with human reasoning.[48] A logically omniscient individual is a more powerful reasoner than real individuals are. Humans are non-ideal reasoners, for several reasons.[49] Kahneman highlights the contrast between rationality and intuition. Rationality requires effort and computation and is contrasted to intuition, which comes spontaneously. According to Kahneman humans are bounded rationally since their thoughts are mostly intuitive. These intuitive thoughts often include errors which leads to faulty reasoning.[50] Besides faulty reasoning another argument why humans are non-ideal reasoners is that they cannot think of everything simultaneously, because they lack computational power and do not have unlimited memory capacity.[51] In line with this argument: one is not always aware of all the premisses at the same time and therefore is not always able to draw a conclusion from those premisses.[52] A last reason why humans are not logically omniscient can be that humans can be aware of some logical consequences, but then could still refuse to accept those consequences.[53]

To evaluate that logical omniscience applies to the JTB Kripke system a closer look to the notion of logical omniscience is required. Fagin describes logical omniscience as a "certain closure property of an agent's knowledge," which means that if an individual knows certain facts and some conditions hold then he has to know some other facts too.[54] The 'certain closure property' can be divided into several

[46] Rohit Parikh, *Knowledge and the Problem of Logical Omniscience*, (Brooklyn College and CUNY Graduate Center, 2014), 1.

[47] Stalnaker, "The Problem of Logical Omniscience, I," 425.

[48] Ronald Fagin and Joseph Halpern, "Belief, Awareness and Limited Reasoning," *Artificial Intelligence* Vol. 34 (1988): 40.

[49] Jens Christian Bjerring, "Impossible Worlds and Logical Omniscience: An Impossibility Result," *Synthese* Vol. 190 No. 13, (2013): 2507.

[50] Daniel Kahneman, "Maps of Bounded Rationality: Psychology for Behavioral Economics," *The American Economic Review* Vol. 93 No. 5, (2003): 1450-1452.

[51] Stalnaker, "The Problem of Logical Omniscience," 426.

[52] "An obvious objection to the simple version of the claim is that an agent with bounded rationality may know ϕ and know that ϕ implies ψ, yet not 'put two and two together' and draw a conclusion about ψ. Such an agent may not even believe ψ, let alone know it." - Wesley Holliday, "Epistemic Closure and Epistemic Logic I: Relevant alternatives and Subjunctivism," *Journal of Philosophical Logic* Vol. 44 No. 1 (2015): 2.

[53] Fagin et al., *Reasoning about Knowledge*, 310.

[54] Fagin et al., *Reasoning about Knowledge*, 311.

forms:[55]

1. Closure under logical implication : $\models K\phi \wedge K(\phi \rightarrow \psi) \rightarrow K\psi$
   If a individual i knows ϕ and also knows if ϕ then ψ, then i must also know ψ.

2. Closure under valid formulas : $\models \phi \Rightarrow \models K\phi$
   If a proposition ϕ is logically valid, then i knows the proposition ϕ.[56]

3. Closure under valid implication : $\models \phi \rightarrow \psi \Rightarrow \models K\phi \rightarrow K\psi$
   If a proposition $\phi \rightarrow \psi$ is logically valid and if i knows ϕ, then i must also know ψ.

4. Closure under valid equivalence : $\models \phi \leftrightarrow \psi \Rightarrow \models K\phi \leftrightarrow K\psi$
   If a proposition $\phi \leftrightarrow \psi$ is logically valid and if i knows ϕ, then i must also know ψ and if i knows ψ, then i must also know ϕ.

5. Closure under conjunction : $\models (K\phi \wedge K\psi) \rightarrow K(\phi \wedge \psi)$
   If i knows ϕ and also knows ψ, then i also knows the conjunction of ϕ and ψ.

6. Closure under disjunction : $\models K\phi \rightarrow K(\phi \vee \psi)$
   If i knows ϕ then i also knows the disjunction of ϕ with any proposition ψ.

Consider proof 2.1 to see that the JTB Kripke system implies the first logical omniscience form: closure under logical implication.[57]

Proof 2.1 Closure under logical implication applied to the JTB Kripke system

---

To show: validity of $\models K\phi \wedge K(\phi \rightarrow \psi) \rightarrow K\psi$.

Let $\mathcal{F} = \langle W, R_J, R_B \rangle$ be any JTB Kripke frame and suppose $\mathcal{F} \models K\phi \wedge K(\phi \rightarrow \psi)$.

To show: $\mathcal{F} \models K\psi$. Consider an arbitrary JTB Kripke model $\mathcal{M} = \langle W, R_J, R_B, V \rangle$ based on $\mathcal{F}$. Take an arbitrary world $w \in W$. From frame validity and $\mathcal{F} \models K\phi \wedge K(\phi \rightarrow \psi)$ follows that $w \models K\phi \wedge K(\phi \rightarrow \psi)$.

To show: $w \models K\psi$. By definition of $\wedge$ and $w \models K\phi \wedge K(\phi \rightarrow \psi)$ follows that $w \models K\phi$ and $w \models K(\phi \rightarrow \psi)$. $w \models K\phi$ means that $w \models B\phi$, $w \models J\phi$ and $w \models \phi$ (following K's definition in JTB Kripke systems and the definition of $\wedge$). Using the same definitions $w \models K(\phi \rightarrow \psi)$ means that $w \models B(\phi \rightarrow \psi)$, $w \models J(\phi \rightarrow \psi)$ and $w \models \phi \rightarrow \psi$.

From $w \models B\phi$ and $w \models B(\phi \rightarrow \psi)$ follows $w \models B\psi$. $w \models B\phi$ means that ϕ is true in every world v w can access via belief arrows and $w \models B(\phi \rightarrow \psi)$ means that $\phi \rightarrow \psi$ is true in every world v w can access via beliefs arrows. So in every world v w can access via belief arrows both ϕ and $\phi \rightarrow \psi$ are true: $v \models \phi$ and $v \models \phi \rightarrow \psi$. By definition of $\rightarrow$ it follows that $v \models \psi$. Since in every world v which is accessible from w via the belief relation $v \models \psi$ is the case it follows - using the definition of JTB B - that $w \models B\psi$.

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[55]Wiebe van der Hoek et al., "An integrated Modal Approach to Rational Agents," in *Foundations of Rational Agency*, ed. Michael Wooldridge and Anand Rao (Dordrecht: Kluwer Academic Publishers, 1999), 140.

[56]In this context logical validity of a propostion ϕ means that ϕ is true in every possible world (regardless of what the frame and the model looks like).

[57]Proofs that closure forms 2 to 6 apply to JTB Kripke system can be found in Appendix A.

From $w \models J\phi$ and $w \models J(\phi \to \psi)$ follows $w \models J\psi$. $w \models J\phi$ means that ϕ is true in every world v w can access via justification arrows and $w \models J(\phi \to \psi)$ means that $\phi \to \psi$ is true in every world v w can access via justification arrows. So in every world v w can access via justification arrows both ϕ and $\phi \to \psi$ are true: $v \models \phi$ and $v \models \phi \to \psi$. By definition of $\to$ it follows that $v \models \psi$. Since in every world v which is accessible from w via the justification relation $v \models \psi$ is the case it follows - using the definition of JTB J - that $w \models J\psi$.

From $w \models \phi$ and $w \models \phi \to \psi$ follows by definition of $\to$ that $w \models \psi$. By definition of JTB K and since $w \models B\psi$, $w \models J\psi$ and $w \models \psi$ it follows that $w \models K\psi$.

Since an arbitrary world and JTB Kripke model were taken it can be concluded that $\mathcal{F} \models \psi$. So $K\phi \wedge K(\phi \to \psi) \to K\psi$ holds on any JTB Kripke frame and therefore: $\models K\phi \wedge K(\phi \to \psi) \to K\psi$.

As has been explained above humans are not logically omniscient reasoners. Therefore whenever knowledge is represented in a logical framework it would be desirable to abandon the property of logical omniscience. However from proof 2.1 and the other proofs in appendix A follows that the JTB Kripke system implies all six logical omniscience forms. It should thus be concluded that the Kripke system of JTB does not formalise human reasoning in a suitable way.

3 JTB in Truthmaker Semantics and Logical Omniscience

Logical omniscience is a property which does not apply to human knowledge. When attempts are made to represent human knowledge in a logical system it is desirable that the property of logical omniscience is not present. In the last chapter it became clear that the problem of logical omniscience exists whenever Kripke structures are used to represent knowledge (as justified true belief (JTB)). Therefore in this chapter another logical framework will be investigated, namely the semantics of truthmakers. After introducing truthmaker semantics a way to represent JTB in this system will be presented. This chapter will conclude with an analysis of logical omniscience in the JTB truthmaker semantics.

3.1 Truthmaker Semantics

As seen in the previous chapters Kripke structures are based on possible worlds and the content of a proposition is defined by the set of possible worlds in which the proposition is true. In this chapter truthmaker semantics is considered, which uses another notion: the notion of states. In this case the content of a proposition is identified through the set of verifying states in which the proposition is true.[58] A difference between states and possible worlds concerns settling the truth value of a proposition. For any proposition ϕ the truth value will be determined by the possible world, i.e. ϕ is either true or false in the possible world.[59] A state can besides verifying or falsifying the proposition ϕ also omit to settle the truth value of ϕ. In that case the state is a non-verifier and a non-falsifier of ϕ at the same time.[60] For making a proposition ϕ true a state is required to be an exact verifier of ϕ, which means that the state is wholly relevant to the proposition. To make the notion of exact verifier more clear consider the following example: the state of the rose smelling sweet is an exact verifier for the proposition that the rose has a sweet smell. The state of the rose smelling sweet and having a pink colour is only partially relevant for the truth of the same proposition. That is, the part of having a pink colour is irrelevant and therefore this state is not an exact verifier for 'the rose has a sweet smell.'[61] Another way in which states are different from possible worlds is that states "make up a world rather than being worlds themselves."[62] A state can be a part of a possible world,

[58]Consider the proposition ϕ and suppose that state a, b and c are verifiers of ϕ, the content of the statement ϕ will then be $\{a, b, c\}$ - Kit Fine, "Truthmaker Semantics," *A Companion to the Philosophy of Language* (2017): 557.

[59]This is an exclusive or, so ϕ cannot be true and false in a world.

[60]"The state of the weather in New York, for example, will not settle whether it is raining in London." Suppose s is the state of the weather in New York. s cannot settle whether it is raining in London, therefore s is a non-verifier and a non-falsifier of the proposition 'It rains in London' and the proposition 'It does not rain in London.' - Fine, "Truthmaker Semantics," 558.

[61]Fine, "Truthmaker Semantics," 558.

[62]Fine, "Truthmaker Semantics," 557.

such a world can be reached if enough states are combined. States have an internal mereological structure, which means that states can be part of each other.[63] Possible worlds are besides the external accessibility relations completely distinguished from each other.

Frames and models were introduced when Kripke structures were considered. The truthmaker semantics variants are state spaces and state models. A state space consists of a set of states and a binary relation on the states. That relation is a partial order: it has the property of reflexivity, anti-symmetry and transitivity. In this context it respectively means that any state s is part of itself, if any state s is part of state t and t is part of s then it must be the case that $s = t$ and if any state s is part of state t and t is part of state u then s is part of u.[64] The set of states consists of both possible and impossible states.[65] If it is preferable to work with possible states only another set can be added to the definition of state spaces: the set of possible states, which exclusively contains possible states. A state model is based on a state space and it therefore contains the same set of (possible) states and the set of relations. Besides those sets a state model contains an interpretation function, which maps every atomic proposition into a pair (V, F) of subsets of S. V are the verifiers which means the set of states in which the proposition is true. F are the falsifiers which stands for the set of states in which the proposition is false.[66] Consider definition 3.1 to get a better understanding of the state space and model notions.

Definition 3.1 State spaces and models

---

1. State space $\mathcal{S} = \langle S, \sqsubseteq \rangle$
   - S is a non-empty set of states.
   - $\sqsubseteq$ is a binary relation on S and is a partial order:
     * Reflexivity: $s \sqsubseteq s$
     * Anti-symmetry: $s \sqsubseteq t \land t \sqsubseteq s \rightarrow s = t$
     * Transitivity: $s \sqsubseteq t \land t \sqsubseteq u \rightarrow s \sqsubseteq u$
   - For arbitrary states $s \in S$ and $t \in S$ the fusion $s \sqcup t$ exists: the fusion (or least upperbound) of states.
2. Modalised state space $\mathcal{S} = \langle S, S^{\Diamond}, \sqsubseteq \rangle$
   - $S^{\Diamond}$ is a non-empty subset of S, containing only possible states such that if $s \in S^{\Diamond}$ and $t \sqsubseteq s$ then $t \in S^{\Diamond}$: parts of possible states are also possible states.
3. State model $\mathcal{M} = \langle S, \sqsubseteq, I \rangle$
   - $\langle S, \sqsubseteq \rangle$ is the state space $\mathcal{M}$ is based on.
   - $I : \phi \mapsto (V, F)$ for every $p \in \phi$:
     * $|p|^+ = V$ is the set of all states which verify p.
     * $|p|^- = F$ is the set of all states which falsify p.

[63] The state of the rose smelling sweet is a part of the state of the rose smelling sweet and having a pink colour - Fine, "Truthmaker Semantics," 559.

[64] Fine, "Truthmaker Semantics," 560.

[65] Possible states are consistent and an impossible state is a state in which a contradiction occurs.

[66] Fine, "Truthmaker Semantics," 562.

4. Modalised state model $\mathcal{M} = \langle S, S^{\Diamond}, \sqsubseteq, I \rangle$

In definition 3.2 it is stated under which conditions an arbitrary proposition ϕ will be verified or falsified by a state.[67]

Definition 3.2 Verifiers and falsifiers

(i)$^+$ $s \Vdash p$ if $s \in |p|^+$: state s is a verifier of p if s belongs to the set of verifiers of p.

(i)$^-$ $s \dashv\mid p$ if $s \in |p|^-$: state s is a falsifier of p if s belongs to the set of falsifiers of p.

(ii)$^+$ $s \Vdash \neg\phi$ if $s \dashv\mid \phi$: state s verifies $\neg\phi$ if s is a falsifier of ϕ.

(ii)$^-$ $s \dashv\mid \neg\phi$ if $s \Vdash \phi$: state s falsifies $\neg\phi$ if s is a verifier of ϕ.

(iii)$^+$ $s \Vdash \phi \wedge \psi$ if for some states t and u, $t \Vdash \phi$, $u \Vdash \psi$ and $s \Vdash t \sqcup u$: state s verifies $\phi \wedge \psi$ if s is the fusion of two states, one of which is a verifier of ϕ and the other one is a verifier of ψ.[68]

(iii)$^-$ $s \dashv\mid \phi \wedge \psi$ if $s \dashv\mid \phi$ or $s \dashv\mid \psi$: state s falsifies $\phi \wedge \psi$ if it is a falsifier of ϕ or a falsifier of ψ.

(iv)$^+$ $s \Vdash \phi \vee \psi$ if $s \Vdash \phi$ or $s \Vdash \psi$: state s verifies $\phi \vee \psi$ if s is a verifier of ϕ or a verifier of ψ.

(iv)$^-$ $s \dashv\mid \phi \vee \psi$ if for some t and u, $t \dashv\mid \phi$, $u \dashv\mid \psi$ and $s = t \sqcup u$: state s falsifies $\phi \vee \psi$ if s is the fusion of two states one of which is a falsifier of ϕ and the other is a falsifier of ψ.

This section concludes with a state model. Observe figure 3.1 and consider the following examples of states making true or false specific propositions, such that the notions of verifiers and falsifiers become more clear. s_1 is a verifier of proposition p, since p is made true in s_1. s_2 is a falsifier of the proposition q, since s_2 verifies the negated proposition: $\neg q$.[69] s is a verifying state for the proposition $p \wedge q$, since s is the fusion of one state which verifies p (s_1) and another state which verifies q (s_2). s_2 verifies the proposition $\neg p \vee \neg q$ since it makes true at least on of the components of the disjunction ($s \Vdash \neg q$).

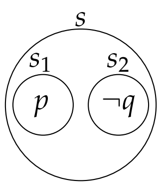


FIGURE 3.1: State model $\mathcal{M}$.

[67] Notice that only $\neg$, $\vee$ and $\wedge$ clauses are given, those are the minimal clauses to define other connectives like $\rightarrow$ and $\leftrightarrow$. - Fine, "Truthmaker Semantics," 563.

[68] The fusion of state t and u means that both t and u are part of the resulting state.

[69] "A state verifies a negative statement $\neg A$ just in case it falsifies the negated statement A; a state falsifies the negative statement $\neg A$ just in case it verifies the negated statement A." - Fine, "Truthmaker Semantics," 562.

3.2 JTB in Truthmaker Semantics

Now that the system of truthmaker semantics is introduced a definition of knowledge in this system will be presented. Recall that the analysis of knowledge as justified true belief is used in this paper. The definition of (modalised) state spaces and state models needs to be adjusted, such that knowledge as JTB can be represented. A current state space consists of a set of states, a set of possible states and a partial order relation on those states. Besides the sets of states a set of possible worlds is required.

Possible worlds are the components of the possible worlds set. The property of maximal consistency is ascribed to any possible world. In this context maximal consistency means that all true propositions in the world are consistent with one another. It also means that if a proposition is added to a world and if it was not yet in the world a contradiction can be derived.[70] Possible worlds consists of states. States represent a certain state of affairs, the summation of enough states make a world. The states that are part of a world need to be possible states, that is inside such a possible state no contradiction can be derived: the state is consistent. Another requirement on the states which are part of a world is that the states need to be mutually consistent. This means that what is verified by a state cannot lead to a contradiction with what any other state verifies.[71] Worlds also have the property of settling the truth value for any proposition, this means that every proposition is either true or false in the world.

Later in this chapter models will be used to explain different notions. Those models consist among other things of possible worlds and states.[72] Consider figure 3.2 to see how worlds and states are represented.

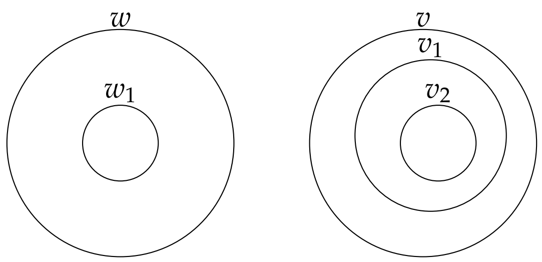


FIGURE 3.2: Worlds and states.

Circles are used to represent worlds. Worlds cannot be part of one another. Besides accessibility relations worlds are distinguished. To model this property the circles which represent worlds are distinguished from each other and will not be part of any bigger circle. Thus in figure 3.2 the worlds are w and v. In contrast to worlds states can be part of each other. Possible states make up a world. Only the states which are part of a world will be used in this paper, thus states existing apart from a world would not be displayed.[73] Like worlds states will be represented by circles, but other than worlds the states will be part of other bigger circles. In this case the

[70] Mastop, "Modal Logic for Artificial Intelligence," 47.

[71] Consider the following example: state s and u are part of world w, $s \Vdash p$ and $u \Vdash \neg p$. This leads to a mutually contradiction, since from p (what s verifies) and $\neg p$ (what u verifies) a contradiction can be derived.

[72] 'among other things' refers to accessibility relations, this notion will be introduced later.

[73] Impossible states will not be represented, since they are not relevant to the problems which will be considered.

bigger circles can either be a world or another state which is also part of a world. Therefore, in figure 3.2 the states are w_1, v_1 and v_2. The letters w, v and z are used to refer to worlds and to refer to states those letters plus an underscore and a number are used. Another property of worlds is that they are maximally consistent but not all propositions which are true in the world will be represented, since there would be an infinite number. Only propositions relevant to the example will be displayed.

Next to adding a set of possible worlds to the state space, accessibility relations for justification and belief are also needed. Both the justification and belief relation are from worlds to worlds and/or states. States do not have an outgoing accessibility relation. In definition 3.3 the adjusted definitions of state spaces and state models are stated, the meaning of the relations will be defined in definition 3.4.

Definition 3.3 JTB state space and model

---

1. JTB state space $\mathcal{S} = \langle S, S^{\Diamond}, W, \sqsubseteq, R_J, R_B \rangle$
   - $\langle S, S^{\Diamond}, \sqsubseteq \rangle$ is a modalised state space.
   - $W \subseteq S^{\Diamond}$ such that for $\forall w \in W$ and $\forall s \in S$ if $w \sqcup s \in S^{\Diamond}$ then $s \sqsubseteq w$: worlds are maximally consistent.
   - $R_J \subseteq W \times W$ and $R_B \subseteq W \times W$: both R_J and R_B are a binary relation on W.
2. JTB state model $\mathcal{M} = \langle S, S^{\Diamond}, W, \sqsubseteq, R_J, R_B, I \rangle$
   - $\langle S, S^{\Diamond}, W, \sqsubseteq, R_J, R_B \rangle$ is the JTB state space $\mathcal{M}$ is based on.
   - $I : \phi \mapsto (V, F)$ for every $p \in \phi$ is the truthmaker semantics interpretation function such that:
     * $\neg\exists s \in V(p)$ and $\exists t \in F(p)$ such that $s \sqcup t \in S^{\Diamond}$: exclusivity, a statement cannot be both true and false.
     * $\forall s \in S^{\Diamond}\ \exists t \in V(p)$ such that $s \sqcup t \in S^{\Diamond}$ or $\forall s \in S^{\Diamond}\ \exists t \in F(p)$ such that $s \sqcup t \in S^{\Diamond}$: exhaustivity, any statement needs to be either true or false.[74]

As seen in Kripke structures there exist three different operators: one for justification, one for belief and one for knowledge (respectively J, B and K). In truthmaker semantics the same operators are made use of. Consider definition 3.4 to take into account how the operators are defined.

Definition 3.4 Justification, belief and knowledge in truthmaker semantics

---

1. Having justification for ϕ in any world $w \in W$:

   $w \models J\phi$ if and only if

[74]"Exclusivity corresponds to the assumption that no statement is both true and false (which is how things would be if a verifier were compatible with a falsifier); and Exhaustivity corresponds to the assumption that every statement is either true or false (since no possible state could exclude the statement being either true or false)." - Fine, "Truthmaker Semantics," 562. These two conditions guarantee that if truth is defined in a possible world it behaves just like in Kripke structures.

  - $\forall v$ such that $v \in W$ and $wR_J v$ and $\exists s$ with $s \in S^\Diamond$ and $s \sqsubseteq v$ and $s \Vdash \phi$ or
  - $\forall s$ such that $s \in S^\Diamond$ and $wR_J s$ and $\exists t$ with $t \in S^\Diamond$, $t \sqsubseteq s$ and $t \Vdash \phi$

2. Believing ϕ in any world $w \in W$:

   $w \vDash B\phi$ if and only if

   - $\forall v$ such that $v \in W$ and $wR_B v$ and $\exists s$ with $s \in S^\Diamond$ and $s \sqsubseteq v$ and $s \Vdash \phi$ or
   - $\forall s$ such that $s \in S^\Diamond$ and $wB_J s$ and $\exists t$ with $t \in S^\Diamond$, $t \sqsubseteq s$ and $t \Vdash \phi$

3. Truth of ϕ in any world $w \in W$:

   $w \vDash \phi$ if and only if $\exists s$ with $s \in S^\Diamond$ such that $s \sqsubseteq w$ and $s \Vdash \phi$

4. Knowing ϕ in any world $w \in W$:

   $w \vDash K\phi$ if and only if $w \vDash J\phi \wedge B\phi \wedge \phi$

To get a better understanding of how the operators work, consider the state model in figure 3.3.

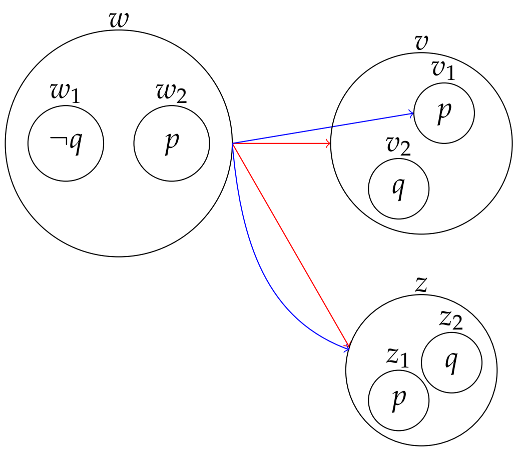


FIGURE 3.3: JTB state model $\mathcal{M}$.

The relation of justification is represented by the *blue arrow* and the belief relation by the *red one*. Let w be the world of perspective. In world w there is justification for p, because in all states or worlds w can access via *justification arrows* there exists a state which is a verifier of p. State v_1 and world z are the only states and worlds accessible from w via R_J. $w \vDash Jp$ since in state v_1 and in world z there exists a state which is a verifier of p and validates the requirements according to the justification definition stated in definition 3.4 (respectively: $v_1 \in S^\Diamond, v_1 \sqsubseteq v_1$ and $v_1 \Vdash p$; $z_1 \in S^\Diamond, z_1 \sqsubseteq z$ and $z_1 \Vdash p$). In w p is believed, since in all worlds w can access with the *belief arrows* there exists a state which is a verifier of p. The only worlds accessible from w with R_B are v and z. $w \vDash Bp$ because in both worlds v and z there exists a possible state which is a verifier of p and validates the requirements according to the definition of belief presented in definition 3.4 (respectively: $v_1 \in S^\Diamond, v_1 \sqsubseteq v$ and $v_1 \Vdash p$; $z_1 \in S^\Diamond, z_1 \sqsubseteq z$

and $z_1 \Vdash p$). p is true in w because there is a possible state in w which is a verifier of p ($w_1 \in S^{\Diamond}, w_1 \sqsubseteq w$ and $w_1 \Vdash p$). There is knowledge of p in w since in world w p is true and there is justified belief of p ($w \vDash Jp \wedge Bp \wedge p$ and therefore $w \vDash Kp$). In w it is not known whether q, since at least one of the components of justified true belief is not satisfied. There is no justification for q in w since both q and $\neg q$ are considered possible when the justification arrows are followed ($wR_J z$, $z_2 \sqsubseteq z$ and $z_2 \Vdash q$; $wR_J v_1$ and $v_1 \sqsubseteq v_1$ and $v_1 \nVdash q$).[75] This representation of knowledge as justified true belief in truthmaker semantics will be used in the next section when logical omniscience is considered.

3.3 Logical Omniscience in JTB Truthmaker Semantics

In this section the logical omniscience forms will be taken into account. On the basis of those forms logical omniscience in the JTB truthmaker semantics will be analysed. All logical omniscience forms present a statement. If there is logical omniscience in the system the statements need to be valid. Validity of those statements in the context of truthmaker semantics means that they have to be valid on all possible JTB state spaces.[76] An attempt will be made to abandon the forms of logical omniscience, therefore the invalidity of the statements needs to be proven. For showing that a statement is not valid on all possible JTB state spaces it is necessary to give a counterexample. To serve as a proper counterexample it is enough to give a specific JTB state model based on some JTB state space. In that model there needs to exist a world in which the statement is false. Particular propositions are required to be taken for the arbitrary formula's ϕ and ψ in the statements. Consider the proofs of the (in)validity of the logical omniscience forms in proof 3.1 to proof 3.6.[77]

Proof 3.1 Invalidity of closure under logical implication in JTB truthmaker semantics

---

To show: $\nvDash K\phi \wedge K(\phi \rightarrow \psi) \rightarrow K\psi$

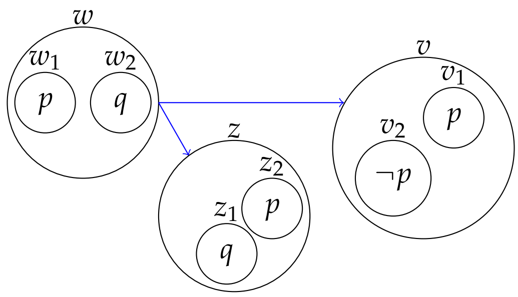
With specific propositions for ϕ and ψ a JTB state model based on some JTB state space needs to be created such that there is a world in that model in which the statement is not true. To make the statement false in that world, the antecedent of the statement needs to be true and the consequent needs to be false in the world. For ϕ p is chosen and for ψ q is picked.

To show: there is a world w in a model such that $w \nvDash Kp \wedge K(p \rightarrow q) \rightarrow Kq$, which means that: $w \vDash Kp \wedge K(p \rightarrow q)$ and $w \nvDash Kq$.

[75] Another reason for not knowing q is that the truth condition of JTB is not satisfied, since there is no state part of w which is a verifier of q.

[76] For a proposition to be valid on all JTB state spaces it means that the proposition is valid on any JTB state model which is based on the JTB state space. And validity on any JTB state model means that the proposition is true in every world in the model. A proposition ϕ is true in a world w if and only if $\exists s\, s \sqsubseteq w$ and $s \Vdash \phi$.

[77] In those proofs definition 3.2, 3.3 and 3.4 will be applied without always specifying which rule has been used.



JTB counter model 3.1

Consider JTB counter model 3.1. The *blue arrows* represents both the relation for justification and belief ($R_J = \{< w, v >, < w, z >\}$ and $R_B = \{< w, v >, < w, z >\}$).

To show: $w \vDash Kp$
w can access world v and world z via the justification relation. The worlds which w can access with the justification arrows should have a state as their part which is a verifier of p. wR_Jz, state z_2 is a part of z and $z_2 \Vdash p$; wR_Jv, $v_1 \sqsubseteq v$ and $v_1 \Vdash p$. Therefore: $w \vDash Jp$. w can access world v and world z via the belief relation. The worlds which w can access with the belief arrows should have a state as their part which is a verifier of p. wR_Bz, $z_2 \sqsubseteq z$ and $z_2 \Vdash p$; wR_Bv, $v_1 \sqsubseteq v$ and $v_1 \Vdash p$. Therefore: $w \vDash Bp$. $w \vDash p$, since there exists a state which is a part of w and a verifier of p ($w_1 \Vdash p$). There is knowledge of p in w, because there is justified true belief of p in w ($w \vDash Kp$ because $w \vDash Jp \wedge Bp \wedge p$).

To show: $w \vDash K(p \to q)$
For this statement to be true there needs to be justified true belief of $p \to q$ in w. $w \vDash J(p \to q)$ because wR_Jv, $v_2 \sqsubseteq v$ and $v_2 \Vdash \neg p$, therefore $v_2 \Vdash p \to q$ and since wR_Jz, $z_1 \sqsubseteq z$ and $z_1 \Vdash q$, therefore $z_1 \Vdash p \to q$. Thus $p \to q$ is made true in all worlds w can access via the justification relation. $w \vDash B(p \to q)$, because wR_Bv, $v_2 \sqsubseteq v$ and $v_2 \Vdash \neg p$, therefore $v_2 \Vdash p \to q$ and since wR_Bz, $z_1 \sqsubseteq z$ and $z_1 \Vdash q$, therefore $z_1 \Vdash p \to q$. Thus $p \to q$ is made true in all worlds w can access with belief arrows. $w \vDash p \to q$, because $w_2 \sqsubseteq w$, $w_2 \Vdash q$ and therefore $w_2 \Vdash p \to q$. Since $w \vDash J(p \to q) \wedge B(p \to q) \wedge (p \to q)$ it follows that $w \vDash K(p \to q)$.

To show: $w \nvDash Kq$
There would not be knowledge of q in w if at least one of the three conditions of knowledge is violated. Consider justification. It is required to show that all states which are part of at least one of the worlds w can access via justification arrows do not make true q. There is such a world accessible from w, namely v. The two states of which v consist are no verifiers of q ($v_1 \nVdash q$ and $v_2 \nVdash q$). Therefore $w \nvDash Jq$ and therefore $w \nvDash Kq$.

Since the antecedent is true and the consequent is false in w, it means that $w \nvDash K\phi \wedge K(\phi \to \psi) \to K\psi$. Thus a counterexample is presented to prove that $\vDash K\phi \wedge K(\phi \to \psi) \to K\psi$ does not hold.

Proof 3.2 Invalidity of closure under valid formulas in JTB truthmaker semantics

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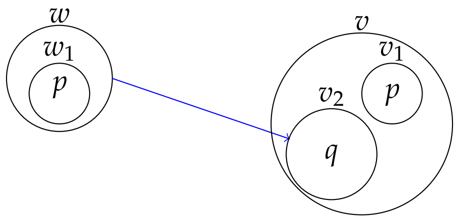
To show: $\vDash \phi \nRightarrow \vDash K\phi$

With a specific proposition for ϕ a JTB state model based on some JTB state space needs to be created such that there is a world in that model in which the statement is not true. The antecedent of the statement ($\vDash \phi$) needs to be true and the consequent ($\vDash K\phi$) needs to be false. For the antecedent to be true the proposition ϕ should be valid on all JTB state spaces. To be valid on all JTB state spaces ϕ should be valid on any possible JTB state model which can be created from the JTB state spaces. Validity of ϕ in a JTB state model means that all worlds in the model make true ϕ. A tautology is a proposition which is always true in any world. Therefore for ϕ a tautology is chosen: $p \vee \neg p$.[78]

To show: $p \vee \neg p$ is true in all worlds in the counter model but $K(p \vee \neg p)$ is false in at least one world.



JTB counter model 3.2

Consider JTB counter model 3.2. The *blue arrow* represents the relation for justification and for belief ($R_J = \{< w, v_2 >\}$ and $R_B = \{< w, v_2 >\}$).

$p \vee \neg p$ is true in all worlds in this model. In world w $p \vee \neg p$ is true because there exists a state ($w_1 \sqsubseteq w$) which is a verifier of p and therefore is a verifier of $p \vee \neg p$. $p \vee \neg p$ is also true in the other world v, since there is a state (v_1) which is part of v and a verifier of p and thus $v_1 \Vdash p \vee \neg p$.

To show: there is a world in the model in which $K(p \vee \neg p)$ is false.
Consider world w. To lack knowledge about $p \vee \neg p$ it is necessary that at least one of the three components of knowledge is not satisfied. Consider the belief component. It is required to show that all states which are part of the state w can access via the belief arrow, are not a verifier of $p \vee \neg p$. The state w can access is v_2 and v_2 is the only part of v_2. v_2 is not a verifier of $p \vee \neg p$ since it is not a verifier of at least one of the components of the disjunction ($v_2 \nVdash p$ and $v_2 \nVdash \neg p$). Now it is showed that $w \nvDash B(p \vee \neg p)$ and therefore is showed that $w \nvDash K(p \vee \neg p)$.

Since $p \vee \neg p$ is valid on all JTB state spaces and world w is in a JTB state model which is based on some JTB state space $p \vee \neg p$ is true in w. $K(p \vee \neg p)$ is false in w. This means a counter model is presented to prove that $\vDash \phi \Rightarrow \vDash K\phi$ does not hold.

Proof 3.3 Invalidity of closure under valid implication in JTB truthmaker semantics

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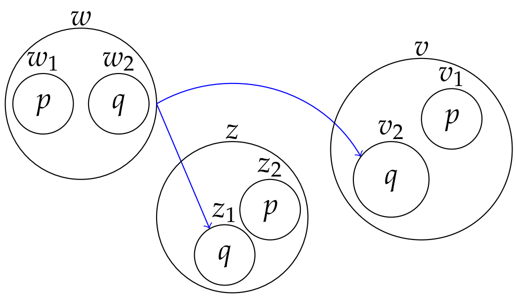
To show: $\vDash \phi \rightarrow \psi \nRightarrow \vDash K\phi \rightarrow K\psi$

With specific propositions for ϕ and ψ a JTB state model based on some JTB state space needs to be created such that there is a world in that model in which the statement is not true. The antecedent of the statement ($\vDash \phi \rightarrow \psi$) needs to be true and

[78] Worlds settle the truth value for each proposition. So if in a world p is false then $\neg p$ should be true and therefore $p \vee \neg p$ is true. And if $\neg p$ is false in a world then p should be true and then $p \vee \neg p$ is also true in that world. This holds for every world.

the consequent ($\vDash K\phi \to K\psi$) needs to be false. For the antecedent to be true the proposition $\phi \to \psi$ should be valid on all JTB state spaces. To be valid on all JTB state spaces $\phi \to \psi$ should be valid on any possible JTB state model which can be created from the JTB state spaces. Validity of $\phi \to \psi$ in a JTB state model means that all worlds in the model make true $\phi \to \psi$. A tautology is a proposition which is always true in any world. Therefore for ψ a tautology is chosen: $p \vee \neg p$, because if the consequent of the implication is true the implication is true as well. For ϕ q is picked.

To show: $q \to (p \vee \neg p)$ is true in all worlds in the counter model but $Kq \to K(p \vee \neg p)$ is false in at least one world.



JTB counter model 3.3

Consider JTB counter model 3.3. The *blue arrows* represent the relation for justification and for belief ($R_J = \{< w, v_2 >, < w, z_1 >\}$ and $R_B = \{< w, v_2 >, < w, z_1 >\}$).

$q \to (p \vee \neg p)$ is true in all worlds in this model. In world w it is true that $q \to (p \vee \neg p)$ because in w there exists a state (w_1) which is a verifier of p from this follows that $w_1 \Vdash p \vee \neg p$ and therefore $w_1 \Vdash q \to (p \vee \neg p)$. In world v $q \to (p \vee \neg p)$ is also true, since there is a state which is part of v and a verifier of p and thus a verifier of $p \vee \neg p$ and therefore a verifier of $q \to (p \vee \neg p)$: $v_1 \Vdash q \to (p \vee \neg p)$. In world z $q \to (p \vee \neg p)$ is true as well, since $z_2 \sqsubseteq z$ and $z_2 \Vdash p$ from this follows that $z_2 \Vdash p \vee \neg p$ and therefore $z_2 \Vdash q \to (p \vee \neg p)$.

To show: there is a world in the model in which $Kq \to K(p \vee \neg p)$ is false.
For the statement $Kq \to K(p \vee \neg p)$ to be false in a world, Kq should be the case but $K(p \vee \neg p)$ should be false in that world. Consider world w. $w \vDash Bq$ since all states w can access via belief arrows are verifiers of q: wR_Bv_2, $v_2 \Vdash q$ and $wR_Bz_1, z_1 \Vdash q$. $w \vDash Jq$ since all states w can access via justification arrows are verifiers of q: wR_Jv_2, $v_2 \Vdash q$ and $wR_Jz_1 \Vdash q$. $w \vDash q$ because $w_2 \sqsubseteq w$ and $w_2 \Vdash q$. Since $w \vDash Jq \wedge Bq \wedge q$ also $w \vDash Kq$ is the case.

To show: $w \nvDash K(p \vee \neg p)$.
To lack knowledge about $p \vee \neg p$ it is necessary that at least one of the three components of knowledge is not satisfied. Consider the belief component. It is required to show that there exists at least one state which w can access via the beliefs arrows and in that state there exists no part which is a verifier of $p \vee \neg p$. v_2 is one of the states w can access and v_2 is the only part of v_2. v_2 is not a verifier of $p \vee \neg p$ since it is not a verifier of at least one of the components of the disjunction ($v_2 \nVdash p$ and $v_2 \nVdash \neg p$). From w there are two states accessible via the belief relation, at least one of them is not a verifier of $p \vee \neg p$ therefore $w \nvDash B(p \vee \neg p)$. Then follows that $w \nvDash K(p \vee \neg p)$. From $w \vDash Kq$ and $w \nvDash K(p \vee \neg p)$ follows that $w \nvDash Kq \to K(p \vee \neg p)$.

Since $q \to (p \vee \neg p)$ is valid on all JTB state spaces and world w is in a JTB state model which is based on some JTB state space $q \to (p \vee \neg p)$ is true in w. $Kq \to K(p \vee \neg p)$ is false in w. This means, a counter model is presented to prove that $\vDash \phi \to \psi \Rightarrow \vDash K\phi \to K\psi$ does not hold.

Proof 3.4 Invalidity of closure under valid equivalence in JTB truthmaker semantics

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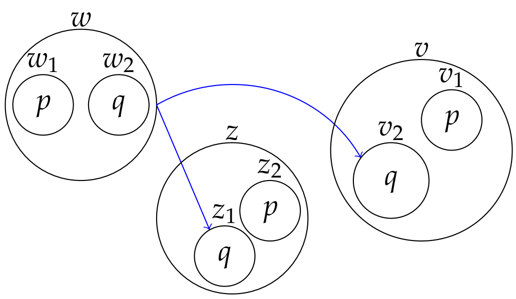
To show: $\vDash \phi \leftrightarrow \psi \not\Rightarrow \vDash K\phi \leftrightarrow K\psi$

With specific propositions for ϕ and ψ a JTB state model based on some JTB state space needs to be created such that there is a world in that model in which the statement is not true. The antecedent of the statement ($\vDash \phi \leftrightarrow \psi$) needs to be true and the consequent ($\vDash K\phi \leftrightarrow K\psi$) needs to be false. For the antecedent to be true the proposition $\phi \leftrightarrow \psi$ should be valid on all JTB state spaces. To be valid on all JTB state spaces, $\phi \leftrightarrow \psi$ should be valid on any possible JTB state model which can be created from the JTB state spaces. Validity of $\phi \leftrightarrow \psi$ in a JTB state model means that all worlds in the model make true $\phi \leftrightarrow \psi$. A tautology is a proposition which is always true in any world. Therefore for both ϕ and ψ a tautology is chosen, because then it is always the case that if ϕ is true ψ is also true and if ψ is true ϕ is true as well. $\phi = p \vee \neg p$ and $\psi = q \vee \neg q$.

To show: $(p \vee \neg p) \leftrightarrow (q \vee \neg q)$ is true in all worlds in the counter model but $K(p \vee \neg p) \leftrightarrow K(q \vee \neg q)$ is false in at least one world.



JTB counter model 3.4

Consider JTB counter model 3.4. The *blue arrows* represent the relation for justification and for belief ($R_J = \{< w, v_2 >, < w, z_1 >\}$ and $R_B = \{< w, v_2 >, < w, z_1 >\}$).

$(p \vee \neg p) \leftrightarrow (q \vee \neg q)$ is true in all worlds in this model. In world w there exists a state as a part (w_1) which is a verifier of p and therefore $w_1 \Vdash p \vee \neg p$, from this follows that $w_1 \Vdash (q \vee \neg q) \to (p \vee \neg p)$ (thus $w \vDash (q \vee \neg q) \to (p \vee \neg p)$). In world w there exists a state as a part (w_2) which is a verifier of q and therefore $w_2 \Vdash (q \vee \neg q)$, from this follows that $w_2 \Vdash (p \vee \neg p) \to (q \vee \neg q)$ (thus $w \vDash (p \vee \neg p) \to (q \vee \neg q)$). Since $w \vDash (p \vee \neg p) \to (q \vee \neg q)$ and $w \vDash (q \vee \neg q) \to (p \vee \neg p)$ also $w \vDash (p \vee \neg p) \leftrightarrow (q \vee \neg q)$ is the case.

There is a state part of v which is a verifier of p: $v_1 \Vdash p$. Since v_1 is a verifier of p it is also a verifier of $p \vee \neg p$. From $v_1 \Vdash p \vee \neg p$ follows that $v_1 \Vdash (q \vee \neg q) \to (p \vee \neg p)$. Thus $v \vDash (q \vee \neg q) \to (p \vee \neg p)$ since there is a state part of v which is a verifier of $(q \vee \neg q) \to (p \vee \neg p)$. There is a state part of v which is a verifier of q: $v_2 \Vdash q$. Since v_2 is a verifier of q it is also a verifier of $q \vee \neg q$. From $v_2 \Vdash q \vee \neg q$ follows that $v_2 \Vdash (p \vee \neg p) \to (q \vee \neg q)$. Thus $v \vDash (p \vee \neg p) \to (q \vee \neg q)$ because there is a state part of v which is a verifier of $(p \vee \neg p) \to (q \vee \neg q)$. Since $v \vDash (p \vee \neg p) \to (q \vee \neg q)$ and $v \vDash (q \vee \neg q) \to (p \vee \neg p)$ also $v \vDash (p \vee \neg p) \leftrightarrow (q \vee \neg q)$ is the case.

There is a state part of z which is a verifier of p: $z_2 \Vdash p$, therefore z_2 is also a verifier of $p \vee \neg p$ and from this follows that $z_2 \Vdash (q \vee \neg q) \rightarrow (p \vee \neg p)$ (thus: $z \vDash (q \vee \neg q) \rightarrow (p \vee \neg p)$). There is a state part of z which is a verifier of q: $z_1 \Vdash q$. Since z_1 is a verifier of q it is also a verifier of $q \vee \neg q$. From $z_1 \Vdash q \vee \neg q$ follows that $z_1 \Vdash (p \vee \neg p) \rightarrow (q \vee \neg q)$. Thus $z \vDash (p \vee \neg p) \rightarrow (q \vee \neg q)$ because there is a state part of z which is a verifier of $(p \vee \neg p) \rightarrow (q \vee \neg q)$. Since $z \vDash (p \vee \neg p) \rightarrow (q \vee \neg q)$ and $z \vDash (q \vee \neg q) \rightarrow (p \vee \neg p)$ also $z \vDash (p \vee \neg p) \leftrightarrow (q \vee \neg q)$ is the case.

To show: in the model there is a world in which $K(p \vee \neg p) \leftrightarrow K(q \vee \neg q)$ is false.
For the statement $K(p \vee \neg p) \leftrightarrow K(q \vee \neg q)$ to be false in a world it should not be the case that $K(p \vee \neg p) \rightarrow K(q \vee \neg q)$ or $K(q \vee \neg q) \rightarrow K(p \vee \neg p)$. Consider world w. $w \vDash B(q \vee \neg q)$ since all states w can access via belief arrows are verifiers of $q \vee \neg q$: $wR_B v_2$, $v_2 \Vdash q$ thus $v_2 \Vdash q \vee \neg q$ and $wR_B z_1, z_1 \Vdash q$ thus $z_1 \Vdash q \vee \neg q$. $w \vDash J(q \vee \neg q)$ since all states w can access via justification arrows are verifiers of $q \vee \neg q$: $wR_J v_2$, $v_2 \Vdash q$ thus $v_2 \Vdash q \vee \neg q$ and $wR_J z_1$, $z_1 \Vdash q$ thus $z_1 \Vdash q \vee \neg q$. $w \vDash q \vee \neg q$ because $w_2 \sqsubseteq w, w_2 \Vdash q$ and therefore $w_2 \Vdash q \vee \neg q$. Since $w \vDash J(q \vee \neg q) \wedge B(q \vee \neg q) \wedge (q \vee \neg q)$ also $w \vDash K(q \vee \neg q)$ is the case.

To show: $w \nvDash K(p \vee \neg p)$.
To lack knowledge about $p \vee \neg p$ it is necessary that at least one of the three components of knowledge is not satisfied. Consider the belief component. It is required to show that all states which are part of at least one of the states w can access via the belief arrows, are not a verifier of $p \vee \neg p$. v_2 is one of the states w can access and v_2 is the only part of v_2. v_2 is not a verifier of $p \vee \neg p$ since it is not a verifier of at least one of the components of the disjunction ($v_2 \nVdash p$ and $v_2 \nVdash \neg p$). The condition of believing $p \vee \neg p$ in w is violated, since one of the states w can access via belief arrows is not a verifier of $p \vee \neg p$ (or has no verifier of $p \vee \neg p$ as its part). Now is showed that $w \nvDash B(p \vee \neg p)$ and therefore $w \nvDash K(p \vee \neg p)$. From $w \vDash K(q \vee \neg q)$ and $w \nvDash K(p \vee \neg p)$ follows that $w \nvDash K(q \vee \neg q) \rightarrow K(p \vee \neg p)$ and therefore $w \nvDash K(p \vee \neg p) \leftrightarrow K(q \vee \neg q)$.

Since $(p \vee \neg p) \leftrightarrow (q \vee \neg q)$ is valid on all JTB state spaces and world w is in a JTB state model which is based on some JTB state space, $(p \vee \neg p) \leftrightarrow (q \vee \neg q)$ is true in w. $K(p \vee \neg p) \leftrightarrow K(q \vee \neg q)$ is false in w. This means a counter model is presented to prove that $\vDash \phi \leftrightarrow \psi \Rightarrow \vDash K\phi \leftrightarrow K\psi$ does not hold.

Proof 3.5 Invalidity of closure under conjunction in JTB truthmaker semantics

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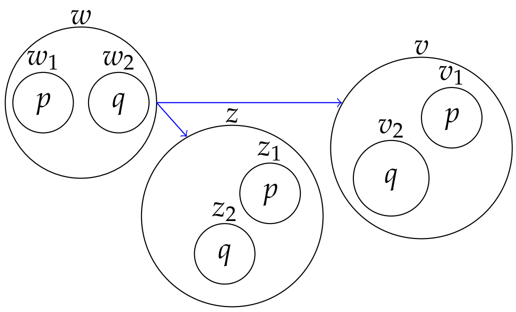
To show: $\nvDash (K\phi \wedge K\psi) \rightarrow K(\phi \wedge \psi)$

With specific propositions for ϕ and ψ a JTB state model based on some JTB state space needs to be created such that there is a world in that model in which the statement is not true. To make the statement false in that world, the antecedent of the statement needs to be true and the consequent needs to be false in the world. For ϕ p is chosen and for ψ q is picked.

To show: there is a world w in a model such that $w \nvDash (Kp \wedge Kq) \rightarrow K(p \wedge q)$, which means that: $w \vDash Kp \wedge Kq$ and $w \nvDash K(p \wedge q)$.



JTB counter model 3.5

Consider JTB counter model 3.5. The *blue arrows* represents both the relation for justification and belief ($R_J = \{< w, v >, < w, z >\}$ and $R_B = \{< w, v >, < w, z >\}$).

To show: $w \vDash Kp \wedge Kq$
$w \vDash Kp$ because $w \vDash Jp \wedge Bp \wedge p$. $w \vDash p$ since $w_1 \sqsubseteq w$ and w_1 is a verifier of p. $w \vDash Bp$, the worlds w can access via the belief relation are v and z and in those worlds there is a state which is a verifier of p (wR_Bz, $z_1 \sqsubseteq z$ and $z_1 \Vdash p$; wR_Bv, $v_1 \sqsubseteq v$ and $v_1 \Vdash p$). $w \vDash Jp$, the worlds w can access via the justification relation are v and z and in those worlds there is a state which is a verifier of p (wR_Jz, $z_1 \sqsubseteq z$ and $z_1 \Vdash p$; wR_Jv, $v_1 \sqsubseteq v$ and $v_1 \Vdash p$). There is knowledge of p in w, because p is true and there is justified belief of p in w ($w \vDash Kp$ because $w \vDash Jp \wedge Bp \wedge p$).

$w \vDash Kq$ because $w \vDash Jq \wedge Bq \wedge q$. $w \vDash q$ since $w_2 \sqsubseteq w$ and w_2 is a verifier of q. $w \vDash Bq$, the worlds w can access via the belief relation are v and z and in those worlds there is a state which is a verifier of q (wR_Bz, $z_2 \sqsubseteq z$ and $z_2 \Vdash q$; wR_Bv, $v_2 \sqsubseteq v$ and $v_2 \Vdash q$). $w \vDash Jq$, the worlds w can access via the justification relation are v and world z and in those worlds there is a state which is a verifier of q (wR_Jz, $z_2 \sqsubseteq z$ and $z_2 \Vdash q$; wR_Jv, $v_2 \sqsubseteq v$ and $v_2 \Vdash q$). There is knowledge of q in w, because q is true and there is justified belief of q in w ($w \vDash Kq$ because $w \vDash Jq \wedge Bq \wedge q$). Since $w \vDash Kp$ and $w \vDash Kq$ also $w \vDash Kp \wedge Kq$.

To show: $w \nvDash K(p \wedge q)$
There would not be knowledge of q in w if at least one of the three conditions of knowledge is violated. Consider justification. It is required to show that all states which are part of at least one of the worlds w can access via justification arrows do not make true $p \wedge q$. There is such a world accessible from w, namely v. The two states of which v consist are no verifiers of $p \wedge q$. To be a verifier of $p \wedge q$ a state should be the fusion of a verifier of p and a verifier of q. Both states v_1 and v_2 are no such fusion, state $v_1 \nVdash q$ and $v_2 \nVdash p$ and therefore v_1 and v_2 are not a verifier of $p \wedge q$. Since at least one of the worlds accessible from w with the justification relation does not make true $p \wedge q$ it can be concluded that $w \nvDash J(p \wedge q)$ and therefore $w \nvDash K(p \wedge q)$.

Since the antecedent is true and the consequent is false in w, it means that $w \nvDash (Kp \wedge Kq) \rightarrow K(p \wedge q)$. Thus a counter model is presented to prove that $\vDash (K\phi \wedge K\psi) \rightarrow K(\phi \wedge \psi)$ does not hold.

Proof 3.6 Validity of closure under disjunction in JTB truthmaker semantics

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To show: $\vDash K\phi \rightarrow K(\phi \vee \psi)$

Let $\mathcal{S} = \langle S, S^{\Diamond}, W, \sqsubseteq, R_J, R_B \rangle$ be a JTB state space and suppose $\mathcal{S} \vDash K\phi$.

To show: $\mathcal{S} \vDash K(\phi \vee \psi)$. Consider an arbitrary JTB state model $\mathcal{M} = \langle S, S^{\Diamond}, W, \sqsubseteq, R_J, R_B, I \rangle$ which is based on S. Take an arbitrary world $w \in W$. From JTB state space validity and $\mathcal{S} \vDash K\phi$ follows that $w \vDash K\phi$.

To show: $w \vDash K(\phi \vee \psi)$. $w \vDash K\phi$ means that $w \vDash J\phi \wedge B\phi \wedge \phi$.

- $w \vDash \phi$ means that there exists at least one state s which is a part of w and a verifier of ϕ. By using the verifier definition of $\vee$ it follows that $s \Vdash \phi \vee \psi$ and since s is a part of w it is the case that $w \vDash \phi \vee \psi$. So if $w \vDash \phi$ then also $w \vDash \phi \vee \psi$.
- $w \vDash B\phi$ means that
  1. ϕ is true in all worlds v w can access by belief arrows. ϕ being true in v means that there is a state u which is part of v and a verifier of ϕ.
     or
  2. In all states x that w can access via belief arrows there exists at least one state t as its part which is a verifier of ϕ.

  If 1 is the case then by the verifier definition of $\vee$ follows that $u \Vdash \phi \vee \psi$ and since u is a part of v, $v \vDash \phi \vee \psi$. If 2 is the case then by the verifier definition of $\vee$ follows that $t \Vdash \phi \vee \psi$ and since t is part of x, $x \Vdash \phi \vee \psi$ is also the case.

  If $w \vDash B\phi$ then in all worlds and states w can access via belief arrows there exists a state as its part which is a verifier for $\phi \vee \psi$. Thus whenever $w \vDash B\phi$ it follows by the JTB belief definition that $w \vDash B(\phi \vee \psi)$.
- $w \vDash J\phi$ means that
  1. ϕ is true in all worlds v w can access by justification arrows. ϕ being true in v means that there is a state u which is part of v and a verifier of ϕ.
     or
  2. In all states x that w can access via the justification relation there exists at least one state t as its part which is a verifier of ϕ.

  If 1 is the case then by the verifier definition of $\vee$ follows that $u \Vdash \phi \vee \psi$ and since u is a part of v, $v \vDash \phi \vee \psi$. If 2 is the case then by the verifier definition of $\vee$ follows that $t \Vdash \phi \vee \psi$ and since t is part of x, $x \Vdash \phi \vee \psi$ is also the case.

  If $w \vDash J\phi$ then in all worlds and states w can access via justification arrows there exists a state as its part which is a verifier for $\phi \vee \psi$. Thus whenever $w \vDash J\phi$ it follows by the JTB justification definition that $w \vDash J(\phi \vee \psi)$.

By definition of JTB knowledge follows from $w \vDash J(\phi \vee \psi)$, $w \vDash B(\phi \vee \psi)$ and $w \vDash \phi \vee \psi$ that $w \vDash K(\phi \vee \psi)$. Since an arbitrary JTB state model and world are used it can be concluded that $\mathcal{S} \vDash K(\phi \vee \psi)$. Thus $K\phi \rightarrow K(\phi \vee \psi)$ holds on any JTB state space and therefore: $\vDash K\phi \rightarrow K(\phi \vee \psi)$

In summary, whenever truthmaker semantics is taken into account and the knowledge definition presented in definition 3.4 is used it follows from proof 3.1 to 3.5 that the logical omniscience forms of closure under logical implication, valid formulas, valid implication, valid equivalence and conjunction can be avoided. The only logical omniscience form which still applies to the presented truthmaker semantics is the closure under disjunction form. Hence if an individual knows ϕ it follows directly that the individual also knows $\phi \vee \psi$. On its own this logical omniscience form is not far removed from human reasoning. To get to this form of omniscience the only capabilities that should be included in human knowledge are awareness of the knowledge of ϕ and of the ability to connect this knowledge to the inference rule of $\phi \rightarrow (\phi \vee \psi)$. Once one is aware of knowing ϕ and also of the rule just mentioned he should conclude that he knows $\phi \vee \psi$. The conclusion would thus be that the proofs show that logical omniscience cannot completely be avoided when this truthmaker semantics system is used, nevertheless only a mild form of logical omniscience is present.

4 Conclusion

This paper has described the problem of logical omniscience and has sought to provide a solution. As has been stated logical omniscience (in all six forms) is present in all logical frameworks which are founded exclusively in the notion of possible worlds.[79] The traditional epistemic logical framework of Kripke structures is based on possible worlds alone and therefore entails all six logical omniscience forms. In this paper the definition of knowledge as justified true belief has been used and was initially represented in Kripke structures. Therefore logical omniscience applies to the resulting JTB representation. It has been explained that the property of being logically omniscient does not apply to human reasoning, therefore when an attempt is made to represent human reasoning in a logical framework it is desirable to abandon the property of logical omniscience. With this aim another framework is introduced: truthmaker semantics. To represent justified true belief in this framework both possible worlds as states were used. By giving several counterexamples it is demonstrated that five of the six logical omniscience forms can be eliminated when the presented truthmaker semantics is used.[80] Ideally, all forms of logical omniscience should not be present in a logical system that is used to represent human knowledge and reasoning. Since only one form - closure under disjunction - is applicable to the presented truthmaker semantics it can be concluded that this framework only provides a mild form of logical omniscience and therefore is a step closer to the ideal representation of human knowledge.

This truthmaker semantics can be of importance in the field of artificial intelligence. In this field attempts are made to create intelligent systems which work and react like humans do. Logical frameworks can be used to model human knowledge and reasoning in the intelligent system. Since humans are not logically omniscient and less logical omniscience forms are present in the truthmaker semantics than in the classic epistemic logical system, human knowledge will be represented more realistically if the truthmaker semantics is used. This may positively influence human robot interaction.

4.1 Further Research

These results may invite to further research. To start, consider the knowledge definition. As has been stated in section 2.3, the analysis of knowledge as justified true belief has at least one difficulty: the cases in which there is justified true belief but no knowledge. Since this theory of knowledge has some challenges of its own further investigations could focus on more appropriate knowledge definitions.[81]

[79] As opposed to the truthmaker semantics presented in this paper which uses both possible worlds and states. Possible worlds are then defined as consisting of possible states.

[80] The forms are closure under: logical implication, valid formulas, valid implication, valid equivalence and conjunction.

[81] Suggestion: relevant alternatives theory of Lewis described by Schaffer - Jonathan Schaffer, "Lewis on Knowledge Ascriptions," in *A Companion to David Lewis*, ed. by Barry Loewer and Jonathan Schaffer (Hoboken: John Wiley and Sons, 2015), 473 - 490.

Secondly, knowledge in a logical system is in this paper interpreted as a single agent's knowledge. In reality not only personal knowledge but also the possibility to reason about knowledge of others is important.[82] In future research the point of focus could be multi-agent epistemic systems in order to do justice to the importance of other people's reasoning.

Furthermore, only static knowledge was taken into account in the course of this paper, which assumes that the represented knowledge is constant. However, in reality one persons knowledge is continuously changing. New information can either acknowledge, shrink or increase knowledge. To satisfy this property of realistic knowledge in an epistemic logical system further research could consider dynamic systems.

Finally, only one type of knowledge - propositional knowledge - is examined in this paper. This type can be recognised by the form: an individual i knows proposition ϕ. According to Feldman humans contain at least two other types of knowledge, namely acquaintance knowledge and ability knowledge.[83] Future research could analyse those types of knowledge and make an attempt to capture them in a logical system. If those systems contain logical omniscience forms as well, appropriate adjustments should be made to abandon it. Such research is relevant to the field of artificial intelligence because in order to create intelligent systems which are as close to humans as possible, it is important to implement all three types of human knowledge.

[82] Interpreting the thoughts of other is important when one would like to communicate with others. In the muddy children example which is stated in footnote 5 (in the introduction) there is a situation in which humans have to reason about other humans reasoning in order to know something about theirselves.

[83] Feldman describes both types in Chapter 2 of Epistemology - Feldman, *Epistemology: Foundations of Philosophy Series*, 9 - 12.

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A Proof: Logical Omniscience Forms in JTB Kripke Structures

Proof A.1 Closure under valid formulas applied to the JTB Kripke system

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To show: validity of $\vDash \phi \Rightarrow \vDash K\phi$.

Let $\mathcal{F} = \langle W, R_J, R_B \rangle$ be any JTB Kripke frame and suppose $\mathcal{F} \vDash \phi$.

To show: $\mathcal{F} \vDash K\phi$. Consider an arbitrary JTB Kripke model $\mathcal{M} = \langle W, R_J, R_B, V \rangle$ based on $\mathcal{F}$. Take an arbitrary world $w \in W$. From frame validity and $\mathcal{F} \vDash \phi$ follows that $w \vDash \phi$.

To show: $w \vDash K\phi$. From frame validity and $\mathcal{F} \vDash \phi$ follows that in every world $v \in W$ ϕ is true. Since ϕ is true in every world in W ϕ is also true in all worlds x w can access via justification arrows ($\forall x \in W$, wR_Jx and $x \vDash \phi$). By definition of JTB J then follows $w \vDash J\phi$. The same holds for the belief relation, in every world z w can access via belief arrows ϕ is true ($\forall z \in W$, wR_Jz and $z \vDash \phi$). From the JTB B definition can be concluded that $w \vDash B\phi$. Since $w \vDash J\phi$, $w \vDash B\phi$ and $w \vDash \phi$ are the case there is knowledge of ϕ in w: $w \vDash K\phi$.

Since an arbitrary world and JTB Kripke model were taken it can be concluded that $\mathcal{F} \vDash K\phi$. Thus if ϕ is valid on any JTB Kripke frame $K\phi$ also is, therefore: $\vDash \phi \Rightarrow \vDash K\phi$.

Proof A.2 Closure under valid implication applied to the JTB Kripke system

---

To show: validity of $\vDash \phi \rightarrow \psi \Rightarrow \vDash K\phi \rightarrow K\psi$.

Let $\mathcal{F} = \langle W, R_J, R_B \rangle$ be any JTB Kripke frame and suppose $\mathcal{F} \vDash \phi \rightarrow \psi$.

To show: $\mathcal{F} \vDash K\phi \rightarrow K\psi$. Consider an arbitrary JTB Kripke model $\mathcal{M} = \langle W, R_J, R_B, V \rangle$ based on $\mathcal{F}$. Take an arbitrary world $w \in W$. From frame validity and $\mathcal{F} \vDash \phi \rightarrow \psi$ follows that $w \vDash \phi \rightarrow \psi$.

To show: $w \vDash K\phi \rightarrow K\psi$. Suppose that $w \vDash K\phi$, to show: $w \vDash K\psi$. From frame validity and $\mathcal{F} \vDash \phi \rightarrow \psi$ follows that $\phi \rightarrow \psi$ is true in every world $v \in W$. $w \vDash K\phi$ means that $w \vDash J\phi$, $w \vDash B\phi$ and $w \vDash \phi$. From $w \vDash \phi \rightarrow \psi$ and $w \vDash \phi$ can be concluded that $w \vDash \psi$. $w \vDash J\phi$ means that in every world $z \in W$ such that wR_Jz $z \vDash \phi$ is the case. Since z is an element of W it follows that $z \vDash \phi \rightarrow \psi$, from this and $z \vDash \phi$ can be derived that $z \vDash \psi$. Since all worlds z can be accessed via the justification relation from w, by the JTB J definition can then be concluded that $w \vDash J\psi$. $w \vDash B\phi$ means that in every world $x \in W$ such that wR_Bx $x \vDash \phi$ is the case. Since x is an element of W it follows that $x \vDash \phi \rightarrow \psi$, from this and $x \vDash \phi$ can be derived that $x \vDash \psi$. By

JTB B definition can then be concluded that $w \models B\psi$, since in every world x which is accessible from w via belief arrows ψ is the case. Since $w \models \psi$, $w \models J\psi$ and $w \models B\psi$ are all the case: $w \models K\psi$. From $w \models \phi \rightarrow \psi$ and $w \models K\phi$ follows that $w \models K\psi$.

Since an arbitrary world and JTB Kripke model were taken it can be concluded that $\mathcal{F} \models K\phi \rightarrow K\psi$. Thus if $\phi \rightarrow \psi$ is valid on any JTB Kripke frame $K\phi \rightarrow K\psi$ also is, therefore: $\models \phi \rightarrow \psi \Rightarrow \models K\phi \rightarrow K\psi$.

Proof A.3 Closure under valid equivalence applied to the JTB Kripke system

---

To show: validity of $\models \phi \leftrightarrow \psi \Rightarrow \models K\phi \leftrightarrow K\psi$.

Let $\mathcal{F} = \langle W, R_J, R_B \rangle$ be any JTB Kripke frame and suppose $\mathcal{F} \models \phi \leftrightarrow \psi$.

To show: $\mathcal{F} \models K\phi \leftrightarrow K\psi$. Consider an arbitrary JTB Kripke model $\mathcal{M} = \langle W, R_J, R_B, V \rangle$ based on $\mathcal{F}$. Take an arbitrary world $w \in W$. From frame validity and $\mathcal{F} \models \phi \leftrightarrow \psi$ follows that $w \models \phi \leftrightarrow \psi$.

To show: $w \models K\phi \leftrightarrow K\psi$.
1. To show: $w \models K\phi \rightarrow K\psi$.[84] Suppose $w \models K\phi$, to show: $w \models K\psi$. From frame validity and $\mathcal{F} \models \phi \leftrightarrow \psi$ follows that in every world $v \in W$ $\phi \leftrightarrow \psi$ is true. $w \models K\phi$ means that $w \models J\phi$, $w \models B\phi$ and $w \models \phi$. From $w \models \phi \leftrightarrow \psi$ and $w \models \phi$ can be concluded that $w \models \psi$. $w \models J\phi$ means that in every world $z \in W$ such that wR_Jz $z \models \phi$ is the case. Since z is an element of W it follows that $z \models \phi \leftrightarrow \psi$, from this and $z \models \phi$ can be derived that $z \models \psi$. By definition of JTB J can then be concluded that $w \models J\psi$. $w \models B\phi$ means that in every world $x \in W$ such that wR_Bx $x \models \phi$ is the case. Since x is an element of W it follows that $x \models \phi \leftrightarrow \psi$, from this and $x \models \phi$ can be derived that $x \models \psi$. By JTB B definition can then be concluded that $w \models B\psi$. Since $w \models \psi$, $w \models J\psi$ and $w \models B\psi$ are all the case it can be concluded that $w \models K\psi$.
2. To show: $w \models K\psi \rightarrow K\phi$. Suppose $w \models K\psi$, to show: $w \models K\phi$. From frame validity and $\mathcal{F} \models \phi \leftrightarrow \psi$ follows that in every world $v \in W$ $\phi \leftrightarrow \psi$ is true. $w \models K\psi$ means that $w \models J\psi$, $w \models B\psi$ and $w \models \psi$. From $w \models \phi \leftrightarrow \psi$ and $w \models \psi$ follows that $w \models \phi$. $w \models J\psi$ means that in every world z such that wR_Jz $z \models \psi$ is the case. Since z is in W $z \models \phi \leftrightarrow \psi$ is the case, with $z \models \psi$ it is also the case that $z \models \phi$. By the definition of JTB J it then follows that $w \models J\phi$. $w \models B\psi$ means that in every world $x \in W$ such that wR_Bx $x \models \psi$ is the case. Since x is an element of W it follows that $x \models \phi \leftrightarrow \psi$, from this and $x \models \psi$ can be derived that $x \models \phi$. By JTB B definition can then be concluded that $w \models B\phi$. Since $w \models \phi$, $w \models J\phi$ and $w \models B\psi$ are all the case it can be concluded that $w \models K\phi$.

From 1. follows that $w \models K\phi \rightarrow K\psi$, from 2 folows that $w \models K\psi \rightarrow K\phi$ from both can be derived that $w \models K\phi \leftrightarrow K\psi$. Since an arbitrary world and JTB Kripke model were taken it can be concluded that $\mathcal{F} \models K\phi \leftrightarrow K\psi$. Thus whenever $\phi \leftrightarrow \psi$ is valid on any JTB Kripke Frame $K\phi \leftrightarrow K\psi$ is also valid, therefore: $\models \phi \leftrightarrow \psi \Rightarrow \models K\phi \leftrightarrow K\psi$.

Proof A.4 Closure under conjunction applied to the JTB Kripke system

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To show: validity of $\models (K\phi \wedge K\psi) \rightarrow K(\phi \wedge \psi)$.

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[84] Find a similar proof in Proof A.2.

Let $\mathcal{F} = \langle W, R_J, R_B \rangle$ be any JTB Kripke frame and suppose $\mathcal{F} \models K\phi \wedge K\psi$.

To show: $\mathcal{F} \models K(\phi \wedge \psi)$. Consider an arbitrary JTB Kripke model $\mathcal{M} = \langle W, R_J, R_B, V \rangle$ based on $\mathcal{F}$. Take an arbitrary world $w \in W$. From frame validity and $\mathcal{F} \models K\phi \wedge K\psi$ follows that $w \models K\phi \wedge K\psi$.

To show: $w \models K(\phi \wedge \psi)$. $w \models K\phi \wedge K\psi$ means that $w \models K\phi$ and $w \models K\psi$. $w \models K\phi$ means that $w \models J\phi \wedge B\phi \wedge \phi$. $w \models K\psi$ means that $w \models J\psi \wedge B\psi \wedge \psi$. From $w \models \phi$ and $w \models \psi$ follows that $w \models \phi \wedge \psi$. $w \models J\phi$ means that in every world $v \in W$ which w can access via the justification relation ϕ is true. In those worlds ψ is also the case, since $w \models J\psi$ and all those worlds v are accessible from w with justification arrows. Thus for every v with wR_Jv: $v \models \phi \wedge \psi$ and by definition of JTB J then follows that $w \models J(\phi \wedge \psi)$. $w \models B\phi$ means that in every world $z \in W$ which w can access via belief arrows ϕ is true. In those worlds ψ is also true, since $w \models B\psi$ and all those worlds z are accessible from w with the belief relation. Thus for every z with wR_Bz: $z \models \phi \wedge \psi$ and by definition of JTB B follows then $w \models B(\phi \wedge \psi)$. From $w \models J(\phi \wedge \psi)$, $w \models B(\phi \wedge \psi)$ and $w \models \phi \wedge \psi$ follows that $w \models K(\phi \wedge \psi)$.

Since an arbitrary world and JTB Kripke model were taken it can be concluded that $\mathcal{F} \models K(\phi \wedge \psi)$. Thus whenever $K\phi \wedge K\psi$ is valid on any JTB Kripke frame then also is $K(\phi \wedge \psi)$, therefore: $\models (K\phi \wedge K\psi) \rightarrow K(\phi \wedge \psi)$.

Proof A.5 Closure under disjunction applied to the JTB Kripke system

---

To show: validity of $\models K\phi \rightarrow K(\phi \vee \psi)$.

Let $\mathcal{F} = \langle W, R_J, R_B \rangle$ be any JTB Kripke frame and suppose $\mathcal{F} \models K\phi$.

To show: $\mathcal{F} \models K(\phi \vee \psi)$. Consider an arbitrary JTB Kripke model $\mathcal{M} = \langle W, R_J, R_B, V \rangle$ based on $\mathcal{F}$. Take an arbitrary world $w \in W$. From frame validity and $\mathcal{F} \models K\phi$ follows that $w \models K\phi$.

To show: $w \models K(\phi \vee \psi)$. $w \models K\phi$ means that $w \models J\phi \wedge B\phi \wedge \phi$. From $w \models \phi$ can be concluded that $w \models \phi \vee \psi$. $w \models J\phi$ means that in every world $v \in W$ which w can access via the justification relation ϕ is true. From $v \models \phi$ follows that $v \models \phi \vee \psi$. Since every world v is accessible from w via the justification relation it follows by the definition of JTB J that $w \models J(\phi \vee \psi)$. $w \models B\phi$ means that in every world $z \in W$ which w can access via the belief relation ϕ is true. From $z \models \phi$ can be derived that $z \models \phi \vee \psi$. Since every world z is accessible from w via the belief relation it follows from the definition of JTB B that $w \models B(\phi \vee \psi)$. From $w \models J(\phi \vee \psi)$, $w \models B(\phi \vee \psi)$ and $w \models \phi \vee \psi$ follows that $w \models K(\phi \vee \psi)$.

Since an arbitrary world and JTB Kripke model were taken it can be concluded that $\mathcal{F} \models K(\phi \vee \psi)$. Thus whenever $K\phi$ is valid on any JTB Kripke frame then also is $K(\phi \vee \psi)$, therefore: $\models K\phi \rightarrow K(\phi \vee \psi)$.