
Representations of locally compact groups

Bachelor thesis in Mathematics

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Introduction

In this bachelor thesis, we will give an exposition of the basic representation theory of locally compact groups. Locally compact groups are groups endowed with a topological structure that is compatible with the group structure, and the topology on these groups satisfy two regularity conditions: the Hausdorff property and the local compactness property. A concrete discussion of these groups can be found in Chapter 1. The first chapter is mainly devoted to introducing the main machinery for doing analysis on these type of groups. The theory developed during this chapter is of importance during the rest of the thesis.

Chapters 2 and 3 are devoted to representation theory. In Chapter 2 we will present the basic notions of representation theory and prove a handful of useful results. We shift our attention to compact groups in Chapter 3. The compact case will turn out to be easier to understand. The main result of Chapter 3 is the Peter-Weyl theorem. After proving this, we will continue to prove some interesting corollaries. For instance, we will prove the Gleason-Yamabe theorem for compact groups using the theorem of Peter-Weyl. Afterwards, we will introduce a generalization of the Fourier transform on compact groups. We conclude Chapter 3 by determining all unitary irreducible representations of $SU(2)$ and $U(2)$.

The reader of this text should be familiar with the basics of topology, measure theory and functional analysis. Some material which is usually not covered during the introductory course on functional analysis taught at Utrecht University, is included in the appendix.

I will take this opportunity to expand on the notation used in this thesis. Throughout this thesis, we consider 0 a natural number, and consequently write $\mathbb{N} := \{n \in \mathbb{Z} \mid n \geq 0\}$. Let X be a topological space. Then we will let $C(X)$ denote the vector space of continuous function $X \rightarrow \mathbb{C}$. The subspace of $C(X)$ consisting of all compactly supported continuous functions $X \rightarrow \mathbb{C}$ is denoted by $C_c(X)$. Suppose that V, W are normed vector spaces. An operator $V \rightarrow W$ will always mean a linear map $V \rightarrow W$. The set of bounded operators $V \rightarrow W$ is denoted by $B(V, W)$, and we denote $B(V) := B(V, V)$. Whenever a bijective bounded operator $T : V \rightarrow W$ has a bounded inverse, we say that T is an isomorphism. The set of isomorphisms $V \rightarrow W$ is denoted by $\text{Iso}(V, W)$. Again, we denote $\text{Iso}(V) := \text{Iso}(V, V)$.

1. Integration on groups

We start by introducing the concept of topological groups, and especially (locally) compact groups. We then proceed to introduce the main tool for doing analysis on locally compact groups, the Haar measure. This measure is compatible with both the topology and the group structure of a topological group. We will follow the approach taken in [Coh13] and [DE14].

1.1 Topological groups

Definition 1.1.1. A group G equipped with a topology is said to be a *topological group* if the multiplication and inversion maps,

$$\begin{aligned} G \times G &\longrightarrow G : (x, y) \mapsto xy, \\ G &\longrightarrow G : x \mapsto x^{-1}, \end{aligned}$$

are continuous. Such a group is said to be a *(locally) compact group* if the underlying topology is Hausdorff and (locally) compact.

For topological groups, the requirement to be Hausdorff is more subtle, as we will show shortly. We start by stating some important properties of topological groups. First, recall the tube lemma from topology.

Lemma 1.1.2 (Tube lemma). *Let X, Y be topological spaces and suppose that X is compact. Let $U \subset X \times Y$ be an open subset. Suppose that U contains the slice $X \times \{y\}$ for some $y \in Y$. Then there exists a neighbourhood V of y such that $X \times V \subset U$.*

We will use the lemma above multiple times. For example in the following proposition. For a group G , we will denote the identity element by 1 . We say that a subset $A \subset G$ is *symmetric* if $A^{-1} = A$. Here A^{-1} denotes the set $\{x^{-1} \mid x \in A\}$. Furthermore, we denote $gA := \{ga \mid a \in A\}$, $Ag := \{ag \mid a \in A\}$ and $AB := \{ab \mid a \in A, b \in B\}$ for $g \in G$ and subsets A, B of G .

Proposition 1.1.3. *Let G be a topological group.*

- (i) *For every neighbourhood U of the identity, there exists a neighbourhood V of the identity such that $V^2 = VV \subset U$.*
- (ii) *The identity has a basis of symmetric neighbourhoods.*
- (iii) *For every open subset $U \subset G$ containing a compact set K , there exists a neighbourhood V of the identity such that $KV \subset U$.*

Proof. We start by proving (i). Let U be a neighbourhood of 1. By continuity of multiplication, we have neighbourhoods V_1, V_2 of 1 such that $V_1 V_2 = U$. Then $V := V_1 \cap V_2$ is a neighbourhood of 1 satisfying $V^2 \subset U$, as desired.

We show (ii). Let U be an arbitrary neighbourhood of 1. By continuity of inversion, U^{-1} is a neighbourhood of 1. Now $V := U \cap U^{-1}$ is a symmetric neighbourhood of 1 contained in U .

We turn to assertion (iii). Let U be an open subset of G and $K \subset G$ compact such that $K \subset U$. Note that the slice $K \times \{1\}$ is contained in the preimage of U under the multiplication map. As this preimage is open, the tube lemma implies that there exists a neighbourhood V of 1 such that $K \times V$ is contained in this preimage. Thus $KV \subset U$. \square

Proposition 1.1.4. *Let H be a normal subgroup of a topological group G . Then G/H (equipped with the quotient topology) is again a topological group, and the projection map $G \rightarrow G/H$ is open.*

Proof. Let $\pi : G \rightarrow G/H$ be the projection map. Note that for $U \subset G$ open, we have $\pi^{-1}(\pi(U)) = \bigcup_{h \in H} hU$. Since $g \mapsto hg$ is a homeomorphism for every $h \in H$, we get that $\pi(U)$ is open. Hence π is an open map. To show that G/H is a topological group, it suffices to show that the map $f : G/H \times G/H \rightarrow G/H : (x, y) \mapsto xy^{-1}$ is continuous. Let $x, y \in G$ and U a neighbourhood of $xy^{-1}H$. Then $\pi^{-1}(U)$ is a neighbourhood of xy^{-1} . Hence we find open neighbourhoods V_1, V_2 of respectively x and y , such that $V_1 V_2^{-1} \subset \pi^{-1}(U)$. Since π is a homeomorphism, we get that $\pi(V_1) \cdot \pi(V_2)^{-1} \subset U$. Note that $\pi(V_1)$ and $\pi(V_2)$ are open neighbourhoods of respectively xH and yH . Hence f is continuous, as desired. \square

Proposition 1.1.5. *Let G be a topological group, and H a normal subgroup of G . Then G/H is Hausdorff if and only if H is closed.*

Proof. First, assume that H is closed. We show that G/H is Hausdorff. Consider two elements $xH, yH \in G/H$ such that $xH \neq yH$. The latter is equivalent to saying that $xy^{-1} \notin H$. Since H is closed, we find an open neighbourhood U of xy^{-1} such that $U \cap H = \emptyset$. Moreover, since G is a topological group, we find open neighbourhoods V_1, V_2 inside G of respectively x and y such that $V_1 V_2^{-1} \subset U \subset G \setminus H$. The images of these two neighbourhoods in G/H are open neighbourhoods of xH and yH by Proposition 1.1.4. The fact that $V_1 V_2^{-1} \subset G \setminus H$ now translates to the fact that these images in G/H are disjoint. Hence G/H is Hausdorff.

Coversely, assume that G/H is Hausdorff. For any element $x \in G \setminus H$, we have $xH \neq H$. Thus, by Hausdorffness, we find an open $U \subset G/H$ such that $H \notin U$. Now, the preimage of U under the natural projection is an open set which is disjoint from H . Hence H is closed. \square

Using the proposition above, we obtain the following result.

Corollary 1.1.6. *Let G be a topological group. Then the following are equivalent:*

1. G is Hausdorff,
2. G has the T_1 -property (i.e. every singleton is closed),
3. $\{1\}$ is closed inside G .

We see that the Hausdorff condition can be translated to a condition in the neighbourhood of the identity. This is an import principle in general, and we will see more examples of this. For example, it is readily verified that the topology on a topological group is locally compact if and only if the identity element has a compact neighbourhood.

As preparation, we will prove some statements about the continuous functions with compact support on a topological space.

Lemma 1.1.7. *Let G be a topological group and $f \in C_c(G)$. Then for every $\varepsilon > 0$, there exists a neighbourhood U of the identity such that $|f(x) - f(y)| < \varepsilon$ whenever $y \in xU$ or $y \in Ux$ for all $x, y \in G$.*

Proof. For every $x \in \text{supp } f$, we choose an open neighbourhood V_x of x such that for all $y \in V_x$ we have $|f(y) - f(x)| < \varepsilon/2$. By Proposition 1.1.3, we find an open neighbourhood U_x of 1 such that $U_x^2 \subset x^{-1}V_x$. This yields an open cover $\{xU_x\}_{x \in \text{supp } f}$ of the (compact) support of f . Hence there exists $x_1, \dots, x_n \in \text{supp } f$ such that $\text{supp } f \subset \bigcup_{i=1}^n x_i U_{x_i}$. Using Proposition 1.1.3 again, we find a symmetric neighbourhood U' of 1 such that $U' \subset \bigcap_{i=1}^n U_{x_i}$. We show that for $x, y \in G$, we have $|f(x) - f(y)| < \varepsilon$ whenever $y \in xU'$.

Indeed, the inequality holds if $x, y \notin \text{supp } f$. Hence assume that at least one of the elements x, y is contained in the support of f . Consider the case that $x \in \text{supp } f$, then there exists an index i such that $x \in x_i U_{x_i}$. This implies that $y \in V_{x_i}$ as $xU' \subset x_i U_{x_i}^2 \subset V_{x_i}$ and $x \in V_{x_i}$ as $x_i U_{x_i} \subset x_i U_{x_i}^2 \subset V_{x_i}$. It follows that $|f(x) - f(y)| \leq |f(x) - f(x_i)| + |f(x_i) - f(y)| < \varepsilon$. If $x \notin \text{supp } f$, then we have $y \in \text{supp } f$. Since $y = xu$ for some $u \in U'$, we get that $yu^{-1} = x$. Because U' is symmetric, we conclude that $x \in yU'$. Now, interchanging x and y in the reasoning of previous case, we obtain the inequality as well.

Applying the same reasoning to the map $x \mapsto f(x^{-1})$ (which is again compactly supported), we find a symmetric neighbourhood U'' of 1 such that for all $x, y \in U''$ we have $|f(x^{-1}) - f(y^{-1})| < \varepsilon$ whenever $y \in xU''$. As U'' is symmetric, we get $x^{-1} \in y^{-1}U''$ whenever $x \in U''y$, and hence $|f(x) - f(y)| < \varepsilon$. We find the desired neighbourhood U by intersecting U' and U'' . \square

For a topological group G , we endow $C_c(G)$ with the supremum-norm $\|\cdot\|_\infty$, making $C_c(G)$ a normed vector space. Note that G acts on $C_c(G)$ as follows. We have homomorphisms

$$L : G \longrightarrow B(C_c(G)) : g \mapsto L_g, \quad R : G \longrightarrow B(C_c(G)) : g \mapsto R_g,$$

such that for $g \in G$ we have,

$$L_g f(x) := f(g^{-1}x) \text{ and } R_g f(x) := f(xg). \quad (f \in C_c(G))$$

In fact, L_g and R_g are surjective isometries.

Proposition 1.1.8. *Let G be a topological group and $f \in C_c(G)$. Then $g \mapsto L_g f$ and $g \mapsto R_g f$ are continuous maps from G to $C_c(G)$.*

Proof. Fix an element $g \in G$. On the strength of Lemma 1.1.7, we find a neighbourhood U of the identity such that $|f(x) - f(y)| < \varepsilon$ whenever $y \in Ux$ for all $x, y \in G$. Consider the neighbourhood $V := gU^{-1}$. Let $h \in V$. Then for all $x \in G$, we have $h^{-1}x \in Ug^{-1}x$ thus $|f(g^{-1}x) - f(h^{-1}x)| < \varepsilon$. It follows that $\|L_h f - L_g f\|_\infty \leq \varepsilon$ whenever $h \in V$. Thus $g \mapsto L_g f$ is continuous. Continuity of $g \mapsto R_g f$ is shown in a similar way. \square

1.2 Radon measures

We now introduce a notion of a Borel measure on a general topological space, which is compatible with the topology in the following sense. We denote the σ -algebra of Borel sets of a topological space X by $\mathcal{B}(X)$.

Definition 1.2.1. A Radon measure on a topological space X is a measure $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$ such that

- (i) the measure μ is outer regular on all Borel subsets, i.e.

$$\mu(A) = \inf\{\mu(U) \mid A \subset U \text{ with } U \text{ open}\}$$

for all $A \in \mathcal{B}(X)$,

- (ii) the measure μ is inner regular on all open sets, i.e.

$$\mu(U) = \sup\{\mu(K) \mid K \subset U \text{ with } K \text{ compact}\}$$

for all open subsets U of X ,

- (iii) the measure μ is finite on all compact subsets of X .

Example 1.2.2. The following are examples of Radon measures.

- The trivial measure on an arbitrary topological space which assigns zero to each Borel subset.
- Let X be a Hausdorff space and $x \in X$. Then the Dirac measure δ_x , i.e. the measure that assigns 1 to Borel subsets containing x and assigns 0 to Borel subsets which do not contain x .
- The Lebesgue measure on \mathbb{R}^n restricted to the Borel subsets.

Using the following proposition, one can readily come up with a large arsenal of Radon measures on topological manifolds. We will not prove this here, instead we refer to the proof found in [Coh13, Proposition 7.2.3].

Proposition 1.2.3. Let X be a second countable, locally compact Hausdorff space. Then every measure $\mu : \mathcal{B}(X) \rightarrow [0, \infty]$ which is finite on compact subsets of X , is a Radon measure on X .

For a Radon measure μ on a topological space X , we write

$$\mu(f) := \int_X f \, d\mu$$

for an integrable function f . It is readily verified that $f \mapsto \mu(f)$ defines a complex-valued linear functional on $C_c(X)$. Note that on the set of positive compactly supported functions,

$$C_c^+(X) := \{f \in C_c(X) \mid f \geq 0\},$$

the function $f \mapsto \mu(f)$ is positive. I.e., $f \mapsto \mu(f)$ is a *positive linear functional*. In fact, when X is locally compact Hausdorff, every Radon measure arises as a positive linear functional. This is the content of the representation theorem of Riesz. We state this theorem below; a full proof may be found in [Coh13, Theorem 7.2.8] or [DE14, Theorem B.2.2].

Theorem 1.2.4 (Riesz representation theorem). *Let X be a locally compact Hausdorff space and I be a positive linear functional on $C_c(X)$. Then there exists a unique Radon measure μ such that $I(f) = \mu(f)$ for every $f \in C_c(X)$.*

We will prove the uniqueness assertion of the theorem here.

Lemma 1.2.5. *Let X be a locally compact Hausdorff space. Let $K \subset X$ be compact and suppose that U is an open neighbourhood of K . Then there exists a compactly supported continuous function $f : X \rightarrow [0, 1]$ such that $f = 1$ on K and $\text{supp } f \subset U$.*

Proof. We find an open subset V such that $K \subset V \subset U$ such that \bar{V} is compact (we can achieve this by covering K with finitely many relatively compact neighbourhoods of points in K). On the strength of Urysohn's lemma, there exists a continuous function $\tilde{f} : \bar{V} \rightarrow [0, 1]$ such that $\tilde{f} = 1$ on K and $\tilde{f} = 0$ on $\bar{V} \setminus V$. We extend \tilde{f} to a continuous function f defined on the whole space X by setting $f = \tilde{f}$ on \bar{V} and $f = 0$ on $X \setminus V$. Note that f is well-defined and continuous as it is the combination of two continuous maps defined on closed subsets of X , which agree on the intersection of these closed subsets. The support of the resulting map f is contained in the compact \bar{V} , hence f has compact support. \square

Lemma 1.2.6. *Let μ be a Radon measure on a topological space X . Suppose that $\mathcal{F} \subset C^+(X)$ is a family of functions such that for every $f, g \in \mathcal{F}$ there exists a $h \in \mathcal{F}$ such that $h \geq \max\{f, g\}$. Then*

$$\mu(\sup_{f \in \mathcal{F}} f) = \sup_{f \in \mathcal{F}} \mu(f).$$

Proof. Write g for the measurable function $\sup_{f \in \mathcal{F}} f$. It is clear that $\sup_{f \in \mathcal{F}} \mu(f) \leq \mu(g)$. We show the converse inequality. Consider a simple function $\phi \leq g$ (recall that a simple function is a linear combination of characteristic functions of finite measure with positive coefficients). Let $\varepsilon > 0$. Then ϕ can be written as $\phi = \sum_{i=1}^n a_i 1_{A_i}$ with $a_i > 0$ and $A_i \in \mathcal{B}(G)$ of finite measure for all i . Without loss of generality we may assume that the A_i 's are disjoint. By inner regularity we find compact sets K_1, \dots, K_n such that $K_i \subset A_i$ and $\mu(A_i) - \varepsilon/(na_i) \leq \mu(K_i)$ for every i . Setting $\psi := \sum_{i=1}^n a_i 1_{K_i}$, we obtain another simple function $\psi \leq g$ for which $\mu(\phi) \leq \mu(\psi) + \varepsilon$.

Let $1 \leq i \leq n$. For every $x \in K_i$, we have $\psi(x) = a_i \leq g(x)$. Hence we find a $f_x \in \mathcal{F}$ such that $a_i \leq f_x(x)$. Thus the subset $U_x := f_x^{-1}((1 - \varepsilon)a_i, \infty)$ is an open neighbourhood of x . As K_i is compact, there exists points $x_1, \dots, x_m \in K_i$ such that $\bigcup_{j=1}^m U_{x_j} \supset K_i$. Next, choose $f^{(i)} \in \mathcal{F}$ greater or equal to $\max\{f_{x_1}, \dots, f_{x_m}\}$ everywhere. Then $f^{(i)} \geq (1 - \varepsilon)a_i 1_{K_i}$. Finally, we choose a $f \in \mathcal{F}$ such that $f \geq \max\{f^{(1)}, \dots, f^{(n)}\}$. It follows that $f \geq (1 - \varepsilon)\psi$. Hence,

$$\int_X \phi d\mu \leq \int_X \psi d\mu + \varepsilon \leq \frac{1}{1 - \varepsilon} \int_X f d\mu + \varepsilon \leq \frac{1}{1 - \varepsilon} \sup_{f \in \mathcal{F}} \mu(f) + \varepsilon.$$

As this holds for any $\varepsilon > 0$ and any simple function $\phi \leq g$, the desired result follows. \square

Corollary 1.2.7 (Uniqueness assertion of Theorem 1.2.4). *Let X be a locally compact Hausdorff space. Then for a Radon measure μ on X we have for every open subset $U \subset X$*

$$\mu(U) = \sup\{\mu(f) \mid f \in C_c(X) \text{ such that } 0 \leq f \leq 1 \text{ and } \text{supp } f \subset U\},$$

and in particular every Radon measure on X is completely determined by how it acts on $C_c^+(X)$.

Proof. We prove the first assertion. Let $U \subset X$ be open. Taking $\mathcal{F} := \{f \in C_c(X) \mid 0 \leq f \leq 1 \text{ and } \text{supp } f \subset U\}$, we see that $\sup_{f \in \mathcal{F}} f = 1_U$ on account of Lemma 1.2.5. The desired equality now follows Lemma 1.2.6.

We turn to the second assertion. Let ν be another Radon measure on X such that $\mu(f) = \nu(f)$ for every $f \in C_c^+(X)$. Then we get from the above that μ and ν equal on the open subsets of X . Hence, it follows from outer regularity that $\nu = \mu$. \square

For a measure space (X, \mathcal{A}, μ) we will denote $\mathcal{L}^p(\mu)$, for the p -th power integrable functions $X \rightarrow \mathbb{C}$, $1 \leq p < \infty$. If $p = \infty$ we will define $\mathcal{L}^\infty(\mu)$ to be the space of essentially bounded measurable functions $X \rightarrow \mathbb{C}$. We denote $\|\cdot\|_p$ for the L^p -norm when $1 \leq p < \infty$. The L^∞ -norm (i.e. the essential supremum norm) will be denoted with $\|\cdot\|_{\text{ess}}$. Endowed with these norms, $\mathcal{L}^p(\mu)$ becomes a semi-normed space. The null space N of the L^p -norms is exactly the space of measurable functions that vanish almost everywhere. The quotient $L^p(\mu) := \mathcal{L}^p(\mu)/N$ (a quotient of vector spaces) with the norm induced by the $\|\cdot\|_p$ norm (which we will again denote by $\|\cdot\|_p$) forms a Banach space. Recall from integration theory that in the case of $p = 2$, this is a Hilbert space. The equivalence class of a function $f \in \mathcal{L}^p(\mu)$ (w.r.t. the quotient map $\mathcal{L}^p(\mu) \rightarrow L^p(\mu)$) will be denoted by $f + N = [f]$.

Proposition 1.2.8. *Let X be a locally compact Hausdorff space and μ a Radon measure on X . Then $C_c(X)$ is dense $L^p(\mu)$ for $1 \leq p < \infty$ (here we identify $C_c(X)$ with its image under the natural inclusion $C_c(X) \hookrightarrow L^p(\mu)$).*

Proof. Recall that the complex simple functions (i.e. the \mathbb{C} -linear combinations of characteristic functions of finite measure) are dense in $L^p(X)$. Hence, it suffices to show that the characteristic functions of the Borel sets of X with finite measure sit inside the closure of $C_c(X)$.

Indeed, let A be a Borel set of X of finite measure. Let $\varepsilon > 0$. There exists an open subset U containing A such that $\mu(U) < \mu(A) + \varepsilon$. Furthermore, there exists a compact subset $K \subset A$ such that $\mu(A) - \varepsilon \leq \mu(K)$. Hence $\mu(U) - \mu(K) < 2\varepsilon$. Let $f \in C_c(X)$ such that $f = 1$ on K and $\text{supp } f \subset U$. Then

$$\int_X |1_A - f|^p d\mu = \int_{U \setminus K} |1_A - f|^p \leq 2^p \mu(U \setminus K) \leq 2^{p+1} \varepsilon.$$

Thus 1_A can be approximated by continuous compactly supported functions. \square

Proposition 1.2.9. *Let X be a topological space equipped with a Radon measure μ . Then, for a homeomorphism $f : X \rightarrow X$, we have $f(A) \in \mathcal{B}(X)$ for every $A \in \mathcal{B}(X)$, and*

$$\mu_f : \mathcal{B}(X) \longrightarrow [0, \infty] : A \mapsto (\mu \circ f)(A)$$

defines again a Radon measure.

Proof. For a subset \mathcal{C} of the power set of X , denote $f_*(\mathcal{C}) := \{f(A) \mid A \in \mathcal{C}\}$. It is readily verified that when \mathcal{C} is a σ -algebra, its image $f_*(\mathcal{C})$ is again a σ -algebra. Denoting the topology on X by \mathcal{T} , we get $f_*(\mathcal{T}) \subset f_*(\mathcal{B}(X)) = f_*(\sigma(\mathcal{T}))$. Since f is a homeomorphism, $f_*(\mathcal{T}) = \mathcal{T}$. As $f_*(\mathcal{B}(G))$ is a σ -algebra, we have $\mathcal{B}(X) \subset f_*(\mathcal{B}(X))$. We get the reverse inclusion by replacing f with its inverse f^{-1} in the reasoning above. Hence $f_*(\mathcal{B}(X)) = \mathcal{B}(X)$.

Checking that μ_f satisfies the regularity conditions of Definition 1.2.1 is a straight-forward procedure; it follows from the fact that $A \mapsto f(A)$ restricts to bijections between the open and compact sets of X . \square

1.3 Products of Radon measures

We finish this digression into Radon measures by proving a theorem of Fubini for Radon measures on locally compact spaces. The reader might be aware of this theorem in the context of σ -finite measure spaces. However, generally, not every Radon measure on a locally compact space is σ -finite.

Lemma 1.3.1. *Let X, Y be topological spaces, and Z a metric space. Suppose that $K \subset Y$ is compact. Let $f : X \times Y \rightarrow Z$ be a continuous map. Then, for each $\varepsilon > 0$ and $\xi \in X$, there exists a neighbourhood U of ξ such that for all $(x, y) \in U \times K$, we have $d(f(x, y), f(\xi, y)) < \varepsilon$.*

Proof. Consider the subset $V := \{(x, y) \in X \times Y \mid d(f(x, y), f(\xi, y)) < \varepsilon\} \subset X \times Y$. This set is open due to continuity of the map $(x, y) \mapsto d(f(x, y), f(\xi, y))$. Since V contains the slice $\{\xi\} \times K$, the tube lemma implies that there exists a neighbourhood U of ξ such that $U \times K \subset V$. \square

Lemma 1.3.2. *Let X and Y be locally compact Hausdorff spaces, with respectively Radon measures μ and ν . Then for every map $f \in C_c(X \times Y)$, we have that $x \mapsto \int_Y f(x, y) d\nu(y) \in C_c(X)$ and $y \mapsto \int_X f(x, y) d\mu(x) \in C_c(Y)$ and*

$$\int_X \int_Y f(x, y) d\nu(y) d\mu(x) = \int_Y \int_X f(x, y) d\mu(x) d\nu(y).$$

Proof. Let $K_1 \subset X$ and $K_2 \subset Y$ be, respectively, the projection of $\text{supp } f$ onto X and Y . Then K_1 and K_2 are both compact by continuity of the projections and $\text{supp } f \subset K_1 \times K_2$. It follows that for every $y \in Y$ the map $f^y : x \mapsto f(x, y)$ vanishes outside K_1 . Likewise, for every $x \in X$, the map $f_x : y \mapsto f(x, y)$ vanishes outside K_2 . Thus for all $(x, y) \in X \times Y$ we get that the maps f_x and f^y are integrable.

A fortiori, the mapping $x \mapsto \int_Y f_x d\nu$ is continuous with compact support. Indeed, consider $\xi \in X$ and let $\varepsilon > 0$. On account of Lemma 1.3.1, there exists a neighbourhood U of ξ such that $|f(x, y) - f(\xi, y)| < \varepsilon$ for every $(x, y) \in U \times K_2$. It follows that for $x \in U$, we have

$$\left| \int_Y f_x d\nu - \int_Y f_\xi d\nu \right| \leq \int_Y |f_x - f_\xi| d\nu = \int_{K_2} |f_x - f_\xi| d\nu \leq \varepsilon \nu(K_2).$$

As ε is an arbitrary constant greater than zero, and $\nu(K_2)$ is finite, this implies that the mapping $x \mapsto \int_Y f_x d\nu$ is continuous. It has compact support, since it vanishes outside K_1 . Applying the same reasoning to f^y , we get that $y \mapsto \int_X f^y d\mu$ is continuous with compact support. In particular, all integrals in the statement of the theorem exist.

The main content of the proof remains to show. Let $\varepsilon > 0$. For every $\xi \in K_1$, there exists an open neighbourhood U_ξ of ξ such that for all $x \in U$, $|f_x - f_\xi| < \varepsilon$ on K_2 , by Lemma 1.3.1. This defines an open cover of K_1 , thus there exists ξ_1, \dots, ξ_n such that $\bigcup_{i=1}^n U_{\xi_i} \supset K_1$. Set $A_1 := U_{\xi_1} \cap K_1$ and $A_i := U_{\xi_i} \setminus \bigcup_{j=1}^{i-1} U_{\xi_j} \cap K_1$ for $i \geq 1$. Note that the collection $\{A_i\}_{i=1}^n$ consists of Borel subsets partitioning K_1 . Consider the function

$$g : X \times Y \longrightarrow \mathbb{C} : (x, y) \mapsto \sum_{i=1}^n f(\xi_i, y) 1_{A_i}(x).$$

For every $x \in X$, the mapping $g_x : y \mapsto g(x, y)$ is Borel measurable and integrable with $\int_Y g_x d\nu = \sum_{i=1}^n \int_Y f(\xi_i, y) d\nu(y) 1_{A_i}(x)$. As the A_i 's are Borel subsets, the function $x \mapsto \int_Y g_x d\nu$ is a

simple function. It is integrable since $\mu(A_i) < \infty$ and

$$\int_X \int_Y g(x, y) dv(y) d\mu(x) = \int_X \int_Y g_x dv d\mu(x) = \sum_{i=1}^n \int_Y f(\xi_i, y) dv(y) \mu(A_i).$$

Similarly, for $y \in Y$, the mapping $g^y : x \mapsto g(x, y)$ is integrable and the function $y \mapsto \int_X g^y d\mu$ is integrable, yielding

$$\int_Y \int_X g(x, y) d\mu(x) dv(y) = \int_Y \int_X g^y d\mu dv(y) = \sum_{i=1}^n \int_Y f(\xi_i, y) dv(y) \mu(A_i).$$

Thus the iterated integrals over g agree.

Now, note that

$$\begin{aligned} \left| \int_X \int_Y f(x, y) dv(y) d\mu(x) - \int_X \int_Y g(x, y) dv(y) d\mu(x) \right| \leq \\ \sum_{i=1}^n \int_{A_i} \int_{K_2} |f(x, y) - f(\xi_i, y)| dv(y) d\mu(x) \leq \varepsilon v(K_2) \sum_{i=1}^n \mu(A_i) = \varepsilon \mu(K_1) v(K_2). \end{aligned}$$

Similarly, we deduce that

$$\left| \int_Y \int_X f(x, y) d\mu(x) dv(y) - \int_Y \int_X g(x, y) d\mu(x) dv(y) \right| \leq \varepsilon \mu(K_1) v(K_2).$$

From this, and the fact that the iterated integrals over g agree, we get

$$\left| \int_X \int_Y f(x, y) dv(y) d\mu(x) - \int_Y \int_X f(x, y) d\mu(x) dv(y) \right| \leq 2\varepsilon \mu(K_1) v(K_2).$$

As this holds for arbitrary $\varepsilon > 0$, the iterated integrals agree. \square

Using the representation theorem of Riesz (see Theorem 1.2.4), we come to the following definition.

Definition 1.3.3. Let X and Y be two locally compact Hausdorff spaces with, respectively, Radon measures μ and ν . Then the *product measure* $\mu \times \nu : \mathcal{B}(X \times Y) \rightarrow [0, \infty]$ of μ and ν is the unique Radon measure such that

$$(\mu \times \nu)(f) = \int_X \int_Y f(x, y) dv(y) d\mu(x) = \int_Y \int_X f(x, y) d\mu(x) dv(y)$$

for all $f \in C_c(X \times Y)$.

Theorem 1.3.4 (Fubini's theorem). *Let X and Y be two locally compact Hausdorff spaces with, respectively, Radon measures μ and ν . Suppose that $f : X \times Y \rightarrow \mathbb{C}$ is a $\mathcal{B}(X \times Y)$ -measurable function. The following statements are true.*

(i) If f is integrable (w.r.t. $\mu \times \nu$) then $x \mapsto \int_Y f(x, y) d\nu(y)$, $y \mapsto \int_X f(x, y) d\mu(x)$ are integrable a.e. and

$$\int_{X \times Y} f d(\mu \times \nu) = \int_{A_X} \int_Y f(x, y) d\nu(y) d\mu(x) = \int_{A_Y} \int_X f(x, y) d\mu(x) d\nu(y),$$

where $A_X \in \mathcal{B}(X)$ and $A_Y \in \mathcal{B}(Y)$ are the sets where the integrand is defined.

(ii) If f vanishes outside a σ -finite set and

$$\int_Y \int_X |f(x, y)| d\mu(x) d\nu(y) < \infty \quad \text{or} \quad \int_X \int_Y |f(x, y)| d\nu(y) d\mu(x) < \infty$$

then f is integrable.

Proof. We start by proving assertion (i). We start by showing the equality for the characteristic functions 1_A with $A \in \mathcal{B}(X)$ of finite measure.

Consider the case that A is open. Consider the family $\mathcal{F} := \{f \in C_c(X \times Y) \mid 0 \leq f \leq 1 \text{ and } \text{supp } f \subset A\}$. Using Corollary 1.2.7 and Lemma 1.3.2 we obtain

$$(\mu \times \nu)(A) = \sup_{f \in \mathcal{F}} (\mu \times \nu)(f) = \sup_{f \in \mathcal{F}} \int_X \int_Y f(x, y) d\nu(y) d\mu(x).$$

We now define $\mathcal{G} := \{x \mapsto \int_Y f(x, y) d\nu(y) \mid f \in \mathcal{F}\}$. Then $\mathcal{G} \subset C_c^+(X)$ on account of Lemma 1.3.2 and meets the condition of Lemma 1.2.6 as \mathcal{F} has this property. It follows that

$$(\mu \times \nu)(A) = \sup_{g \in \mathcal{G}} \int_X g d\mu = \int_X \sup_{g \in \mathcal{G}} g d\mu = \int_X \sup_{f \in \mathcal{F}} \int_Y f(x, y) d\nu(y) d\mu(x).$$

Finally, applying Lemma 1.2.6 to the family $\{y \mapsto f(x, y) \mid f \in \mathcal{F}\}$ we get

$$(\mu \times \nu)(A) = \int_X \int_Y \sup_{f \in \mathcal{F}} f(x, y) d\nu(y) d\mu(x) = \int_X \int_Y 1_A(x, y) d\nu(y) d\mu(x).$$

If A is compact, then we can choose a relatively compact open neighbourhood U of A . Thus $1_A = 1_U - 1_{U \setminus A}$, a linear combination of characteristic functions of opens with finite measure (hence each characteristic function is integrable). It now directly follows from the above and linearity of the integral that $(\mu \times \nu)(A) = \int_X \int_Y 1_A(x, y) d\nu(y) d\mu(x)$.

Next, let A be an arbitrary Borel subset of A with finite measure. Using inner regularity of the product measure, we find a collection $\{K_n\}_{n \in \mathbb{N}}$ of compact subsets of A such that $(\mu \times \nu)(A) - (\mu \times \nu)(K_n) \leq 1/(n+1)$ for all $n \in \mathbb{N}$. The sets $K := \bigcup_{n \in \mathbb{N}} K_n$ and $N := A \setminus K$ partition A . Note that $1_{K_n} \rightarrow 1_K$ pointwise as $n \rightarrow \infty$. Using the monotone convergence theorem and the above, we obtain $(\mu \times \nu)(K) = \int_X \int_Y 1_K(x, y) d\nu(y) d\mu(x)$. As N is negligible, we can find an open neighbourhood U of N such that $(\mu \times \nu)(U) < \varepsilon$ for every $\varepsilon > 0$. It follows that $\int_X \int_Y 1_N(x, y) d\nu(y) d\mu(x) \leq \int_X \int_Y 1_U(x, y) d\nu(y) d\mu(x) = (\mu \times \nu)(U) < \varepsilon$. As this holds for any $\varepsilon > 0$, we deduce that $\mu(N) = \int_X \int_Y 1_N(x, y) d\nu(y) d\mu(x) = 0$. Hence, also $\mu(A) = \mu(K) + \mu(N) = \int_X \int_Y 1_A(x, y) d\nu(y) d\mu(x)$.

Now, consider a positive measurable function $f \geq 0$. Then there exists a sequence $(\phi_n)_{n \in \mathbb{N}}$ of simple functions such that $\phi_n \rightarrow f$ pointwise monotone as $n \rightarrow \infty$. It follows from the monotone convergence theorem and the above that

$$\int_{X \times Y} f d(\mu \times \nu) = \lim_{n \rightarrow \infty} \int_X \int_Y \phi_n(x, y) d\nu(y) d\mu(x).$$

Applying the monotone convergence theorem applied to the measurable sections $y \mapsto \int_Y \phi_n(x, y) d\nu(y)$, $x \in X$, we get

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X \lim_{n \rightarrow \infty} \int_Y \phi_n(x, y) d\nu(y) d\mu(x).$$

Using the monotone convergence theorem again, we conclude that the sections $x \mapsto \int_Y f(x, y) d\nu(y)$ are measurable and

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X \int_Y f(x, y) d\nu(y) d\mu(x).$$

Reversing the roles of X and Y in the reasoning above, we also get that $\int_{X \times Y} f d(\mu \times \nu) = \int_Y \int_X f(x, y) d\mu(x) d\nu(y)$.

Now let f be a real valued integrable function. Then we decompose f into its positive part f^+ and its negative part f^- such that $f = f^+ - f^-$. From the above we have $\int_{X \times Y} f^\pm d(\mu \times \nu) = \int_X \int_Y f^\pm(x, y) d\nu(y) d\mu(x) < \infty$. Hence $x \mapsto \int_Y f^\pm(x, y) d\nu(y)$ is finite on a Borel subset A_X^\pm such that $\mu(X \setminus A_X^\pm) = 0$. Define $A_X := A_X^+ \cap A_X^-$. Note that the complement of A_X is again negligible. It follows that for $x \in A_X$ the section $y \mapsto f(x, y)$ is ν -integrable and $x \mapsto \int_Y f(x, y) d\nu(y)$ is integrable on A_X such that

$$\begin{aligned} \int_{A_X} \int_Y f(x, y) d\nu(y) d\mu(x) &= \int_{A_X} \left(\int_Y f^+(x, y) d\nu(y) - \int_Y f^-(x, y) d\nu(y) \right) d\mu(x) \\ &= \int_{X \times Y} f^+ d(\mu \times \nu) - \int_{X \times Y} f^- d(\mu \times \nu) = \int_{X \times Y} f d(\mu \times \nu). \end{aligned}$$

The iterated integral with X and Y reversed can be treated in the same way. If f is a complex valued integrable function, then the desired result follows after considering the real and imaginary parts of f and using the above. This finishes the proof of assertion (i).

We turn to assertion (ii). Assume that f is measurable and $f^{-1}(\mathbb{C} \setminus \{0\})$ is contained in a σ -finite subset of $X \times Y$. Without loss of generality, we may assume that

$$C := \int_X \int_Y |f(x, y)| d\nu(y) d\mu(x) < \infty.$$

By assumption there exists an increasing sequence of Borel subsets of finite measure $\{A_n\}_{n \in \mathbb{N}}$ such that $f^{-1}(\mathbb{C} \setminus \{0\}) \subset \bigcup_{n=0}^{\infty} A_n$. This induces a sequence of measurable functions $(|f|_n)_{n \in \mathbb{N}}$ given by $|f|_n := |f| 1_{A_n}$. Note that $|f|_n \rightarrow |f|$ pointwise monotone as $n \rightarrow \infty$. It follows from assertion (i) that

$$\int_{X \times Y} |f|_n d(\mu \times \nu) = \int_X \int_Y 1_{A_n}(x, y) |f(x, y)| d\nu(y) d\mu(x) \leq C.$$

Using the monotone convergence theorem, we conclude that

$$\int_{X \times Y} |f| d(\mu \times \nu) \leq C < \infty.$$

□

1.4 Haar measures

We now introduce the notion of a Haar measure, as promised. First, note that for any Borel subset A of a topological group G and group element $g \in G$, the translations gA and Ag are again Borel subsets of G on account of Proposition 1.2.9 and continuity of multiplication.

Definition 1.4.1. A *left-invariant Haar measure* (resp. *right-invariant Haar measure*) on a topological group G is a non-trivial Radon measure $\mu : \mathcal{B}(G) \rightarrow [0, \infty]$ such that $\mu(gA) = \mu(A)$ (resp. $\mu(Ag) = \mu(A)$) holds for every Borel subset $A \subset G$. Right-invariant and left-invariant Haar measures are both called *Haar measures*.

Note that the Lebesgue measure, restricted to the Borel subsets, is a Haar measure on \mathbb{R}^n . The following proposition follows from Proposition 1.2.9, and should be clear.

Proposition 1.4.2. Every left-invariant Haar measure μ on a topological group G induces a right-invariant Haar measure $\tilde{\mu}$ on G such that $\tilde{\mu}(A) = \mu(A^{-1})$ for every $A \in \mathcal{B}(G)$. The map $\mu \mapsto \tilde{\mu}$ defines a bijection between the left- and right-invariant Haar measures on G .

Example 1.4.3. If G is a *discrete group*, i.e. a topological group with the discrete topology, we can readily classify all Haar measures on G . Note that $\mathcal{B}(G) = \mathcal{P}(G)$ in this case. Every multiple of the counting measure defines a left- and right-invariant Haar measure. Indeed, consider the map

$$\mu : \mathcal{B}(G) \longrightarrow [0, \infty] : A \mapsto \begin{cases} c|A| & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite,} \end{cases}$$

where $c \in]0, \infty[$. Since the compact sets in the discrete topology are precisely the finite sets, we get inner regularity on all finite subsets of G . Since $c > 0$, we also have inner regularity on the infinite subsets. Outer regularity on the subsets follows immediately. Clearly, μ is finite on compact sets since $c < \infty$. It is readily verified that μ is left- and right-invariant.

We show that every Haar measure μ on G is a (finite and non-zero) multiple of the counting measure. Define $c := \mu(\{1\})$. Note that c is finite, since μ is finite on compact sets. Using translation invariance, we get that $\mu(\{g\}) = c$ for every $g \in G$. Thus, if $A \subset G$ is finite, we get $\mu(A) = \sum_{g \in A} \mu(\{g\}) = c|A|$. As the compact sets of G are exactly the finite sets, this implies that $c \neq 0$. Indeed, if c were zero, all subsets of A would be negligible by inner regularity, and μ would be the trivial measure. Now, for an infinite subset of A , we get $\mu(A) = \infty$. Indeed, choose any injection $f : \mathbb{N} \rightarrow A$, then $\mu(f(\mathbb{N})) = \sum_{g \in f(\mathbb{N})} \mu(\{g\}) = \infty \leq \mu(A)$. We conclude that $\mu = c \cdot \text{counting measure on } G$.

Remark 1.4.4. Not every group has a Haar measure. Consider the rationals (with addition) equipped with the topology inherited from \mathbb{R} . Assume that μ is a left-invariant Haar measure on \mathbb{Q} , and set $c := \mu(\{0\})$. As \mathbb{Q} is countable, c uniquely determines μ . Note that $c > 0$ as μ is not trivial. For every open neighbourhood U of 0, we have that there exists some $\epsilon > 0$ such that $]-\epsilon, \epsilon[\cap \mathbb{Q} \subset U$. As $]-\epsilon, \epsilon[$ contains infinitely many rational numbers, we have that $\mu(]-\epsilon, \epsilon[\cap \mathbb{Q}) = \infty$. Hence, $\mu(U) = \infty$. Using outer regularity, we get that $c = \infty$, which violates Definition 1.2.1.

Note that the invariance of the Haar measures imply the following for the integral.

Proposition 1.4.5. *Let G be topological group. Suppose that μ is a left-invariant Haar measure (resp. right-invariant Haar measure) on G . Then a Borel measurable function $f : G \rightarrow \mathbb{C}$ is integrable if and only if $L_g f$ (resp. $R_g f$) is integrable, and in this case we have $\int_G L_g f d\mu = \int_G f d\mu$ (resp. $\int_G R_g f d\mu = \int_G f d\mu$).*

Proof. Suppose that μ is left-invariant; the case that μ is right-invariant is shown similarly. If f is a characteristic function of a Borel subset of G , then we have $\int_G L_g f d\mu = \int_G f d\mu$ by left-invariance. Thus this equality continues to hold whenever f is a simple function. If f is a positive and Borel measurable, then we can find a sequence of pointwise monotone simple functions converging to f pointwise (this is a result used in basic integration theory). The monotone convergence theorem implies that the equality holds for f . Considering the positive and negative part of a real Borel measurable function f , we see that f is integrable precisely when $L_g f$ is integrable and $\int_G L_g f d\mu = \int_G f d\mu$. The result now readily follows for complex-valued Borel measurable functions. \square

Furthermore, the Haar integral has the following properties.

Proposition 1.4.6. *Let μ be a Haar measure on a topological group G . The following statements are true.*

- (i) *If A is a Borel set of G with non-empty interior, then A has positive measure.*
- (ii) *If $f \in C^+(G)$ then $\mu(f) = 0$ if and only if $f = 0$.*

Proof. We prove (i). Assume to the contrary that there exists a Borel set $A \in \mathcal{B}(G)$ with non-empty interior such that $\mu(A) = 0$. As $\overset{\circ}{A} \neq \emptyset$, we can cover every compact subset of G with finitely many translates of A (which are again negligible). But this implies that every compact subset of G is negligible. Regularity of the measure then implies that $\mu = 0$, a contradiction since μ is non-trivial.

We turn to assertion (ii). Clearly, $\mu(f) = 0$ whenever $f = 0$. Assume that $f \in C^+(G)$ and $f \neq 0$. Then $U := f^{-1}(]0, \infty[)$ is a non-empty open subset of G . It follows from assertion (i) that $\mu(U) > 0$. Since $\mu(U) > 0$, we can find a compact set $K \subset U$ such that $\mu(K) > 0$ as μ is inner regular on opens. By compactness of K we have $m := \inf_{x \in K} f(x) > 0$. Now note that $\mu(f) \geq \int_K m d\mu = m\mu(K) > 0$. \square

In case of a left-invariant Haar measure on a topological group G , the integral over $R_g f$ where f is an integrable function, might fail to be constant as $g \in G$ varies. We still have the following result.

Proposition 1.4.7. *Let μ be a Radon measure on a locally compact group G and $f \in C_c(G)$. Then the maps*

$$G \longrightarrow \mathbb{C} : g \mapsto \mu(L_g f) = \int_G L_g f d\mu, \quad G \longrightarrow \mathbb{C} : g \mapsto \mu(R_g f) = \int_G R_g f d\mu$$

are continuous.

Proof. We show continuity of the first map. Continuity of the second map is treated similarly. Let $g \in G$ and $\varepsilon > 0$. Fix a compact neighbourhood N of g . Note that $N \cdot \text{supp } f$ is again a compact set (since it is the image of the compact set $N \times \text{supp } f$ under multiplication) and hence has finite measure. We find a neighbourhood $U \subset N$ of g such that $\|L_h f - L_g f\|_\infty < \varepsilon/(\mu(N \cdot \text{supp } f) + 1)$

whenever $h \in U$ by Proposition 1.1.8. Consider an element $h \in U$. Then if $x \notin N \cdot \text{supp } f$, we have $g^{-1}x, h^{-1}x \notin \text{supp } f$ and thus $f(g^{-1}x) = f(h^{-1}x) = 0$. It follows that $\text{supp}(L_h f - L_g f)$ is contained in $N \cdot \text{supp } f$. It follows that

$$|\mu(L_g f) - \mu(L_h f)| \leq \int_G |L_g f - L_h f| d\mu \leq \|L_h f - L_g f\|_\infty \mu(N \cdot \text{supp } f) < \varepsilon.$$

This implies continuity of the map $g \mapsto \mu(L_g f)$. \square

Lemma 1.4.8. *Let μ be a non-trivial Radon measure on a locally compact group G . Suppose that $\mu(L_g f) = \mu(f)$ for all $f \in C_c^+(G)$ and $g \in G$, then μ is a left-invariant Haar measure.*

Proof. Let $g \in G$. For $f \in C_c^+(G)$ we have $\text{supp } L_g f = g \cdot \text{supp } f$. Thus in light of Corollary 1.2.7, we obtain $\mu(U) \leq \mu(gU)$, and hence, $\mu(U) = \mu(gU)$, for all open subsets U of G . Using the outer regularity of μ , we deduce that μ is left-invariant. As μ is not trivial, this implies that μ is a left-invariant Haar measure. \square

Corollary 1.4.9. *Let μ, ν be left-invariant Haar measures on, respectively, locally compact groups G and H . Then the product measure $\mu \times \nu$ is a left-invariant Haar measure on $G \times H$.*

We deduce from Lemma 1.4.8 and Riesz representation theorem (see Theorem 1.2.4) that we can characterize Haar measures using positive linear functionals.

Corollary 1.4.10. *Let G be a locally compact group. Suppose that $I : C_c(G) \rightarrow \mathbb{C}$ is a positive linear functional such that $I(L_g f) = I(f)$ for all $f \in C_c^+(G)$ and $I \neq 0$. Then there exists a unique left-invariant Haar measure μ on G such that $\mu(f) = I(f)$ for all $f \in C_c(G)$.*

Example 1.4.11. Consider the circle group $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. It is readily verified that the functional

$$I : C(S^1) \longrightarrow \mathbb{C} : f \mapsto \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) dt$$

is positive. This functional is also left-invariant, as we will show. Let $z \in S^1$ and write $z = e^{i\phi}$ for some angle $\phi \in \mathbb{R}$. Then $I(L_z f) = 1/(2\pi) \int_0^{2\pi} f(e^{i(t-\phi)}) dt = 1/(2\pi) \int_\phi^{2\pi+\phi} f(e^{it}) dt$. Consider the C^1 -map $F : \mathbb{R} \rightarrow \mathbb{C} : s \mapsto \int_0^s f(e^{it}) dt$. One readily verifies that the map $s \mapsto F(2\pi + s) - F(s)$ is constant by calculating its derivative. Hence $I(L_z f) = F(2\pi + \phi) - F(\phi) = F(2\pi) - F(0) = \int_0^{2\pi} f(e^{it}) dt = I(f)$. Thus Corollary 1.4.10 implies that there exists a Haar measure μ on S^1 for which $\mu(f) = I(f)$ for all $f \in C(S^1)$. Note that this measure is left- and right-invariant as S^1 is abelian.

We now present another way of constructing this measure. Consider the continuous map $g : [0, 2\pi] \rightarrow S^1 : t \mapsto e^{it}$. Let λ_0 denote the restriction of the Lebesgue measure to the Borel subsets of the interval $[0, 2\pi]$. As g is in particular measurable, we can consider the pushforward measure $g_*\lambda_0 : \mathcal{B}(S^1) \rightarrow [0, \infty] : A \mapsto \lambda_0(g^{-1}(A))$. Note that $g_*\lambda_0(S^1) = \lambda_0([0, 2\pi]) = 2\pi < \infty$. Hence Proposition 1.2.3 implies that $g_*\lambda_0$ is a Radon measure. Thus $\mu' := 1/(2\pi)g_*\lambda_0$ is again a Radon measure. Note that for $A \in \mathcal{B}(S^1)$ we have

$$\mu'(A) = \frac{1}{2\pi} \int_0^{2\pi} 1_{g^{-1}(A)}(t) dt = \frac{1}{2\pi} \int_0^{2\pi} 1_A(e^{it}) dt.$$

Using a standard argument (similar to the proof of Proposition 1.4.5; we first consider simple functions, then positive functions and use the monotone convergence theorem), we obtain the integral formula

$$\int_{S^1} f d\mu' = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) dt. \quad (f \in \mathcal{L}^1(\mu'))$$

On the strength of Corollary 1.2.7, we now deduce that $\mu = \mu'$.

1.5 Haar's theorem

If we consider groups that are locally compact, we obtain the following result. This section is devoted to proving this result. There are multiple known ways to prove this theorem; we will follow the proof in [Coh13].

Theorem 1.5.1 (Haar's theorem). *Let G be a locally compact group, then there exists left-invariant Haar measure on G . Furthermore, every other left-invariant Haar measure is a positive multiple of this measure.*

Consider a topological group G . Let $K \subset G$ be compact, and $U \subset G$ be a non-empty subset. Then we define

$$\#(K : U) := \min\{n \in \mathbb{N} \mid \text{there exists } g_1, \dots, g_n \text{ such that } K \subset \bigcup_{i=1}^n g_i U\}.$$

This is called the *covering number of U over K* . Note that this quantity is well-defined the translates of U forms an open cover of the subset K , and hence there exists finitely many translates of U covering K .

Lemma 1.5.2. *Let G be a topological group. Let $K_1, K_2 \subset G$ be compact subsets of G , let and $U \subset G$ be a non-empty open subset. Then the following statements are true.*

- (i) *If $\overset{\circ}{K}_2 \neq \emptyset$ then $\#(K_1 : U) \leq \#(K_1 : \overset{\circ}{K}_2) \#(K_2 : U)$.*
- (ii) *For every $g \in G$ we have $\#(gK_1 : U) = \#(K_1 : U)$.*
- (iii) *If $K_1 \subset K_2$ then $\#(K_1 : U) \leq \#(K_2 : U)$.*
- (iv) *We have $\#(K_1 \cup K_2 : U) \leq \#(K_1 : U) + \#(K_2 : U)$, with equality if $K_1 U^{-1} \cap K_2 U^{-1} = \emptyset$.*

Proof. The assertions (i)-(iii) are readily verified. We turn to assertion (iv). The inequality is clear, hence we prove the last part of the assertion. Assume that $K_1 U^{-1} \cap K_2 U^{-1} = \emptyset$. Set $n := \#(K_1 \cup K_2 : U)$ and let $g_1, \dots, g_n \in G$ be the elements such that $K_1 \cup K_2 \subset \bigcup_{i=1}^n g_i U$. For $j = 1, 2$ we define $I_j := \{1 \leq i \leq n \mid g_i U \cap K_j \neq \emptyset\}$. By minimality of n , we have $1, \dots, n \in I_1 \cup I_2$. We show that I_1 and I_2 are disjoint. Indeed, assume to the contrary that there exists an index $i \in I_1 \cap I_2$. Then $g_i U \cap K_1 \neq \emptyset$ and $g_i U \cap K_2 \neq \emptyset$, but this implies that $g_i \in K_1 U^{-1} \cap K_2 U^{-1}$, which is a contradiction. Thus I_1 and I_2 partition the indices $1, \dots, n$. It follows that $\#(K_1 \cup K_2 : U) = n = |I_1| + |I_2| \geq \#(K_1 : U) + \#(K_2 : U)$. \square

Lemma 1.5.3. *Let G be a locally compact group G . Denote the collection of compact subsets of G by \mathcal{C} . Then there exists a non-zero map $m : \mathcal{C} \rightarrow [0, \infty[$ which satisfies*

- (i) $m(\emptyset) = 0$,

- (ii) $m(gK) = m(K)$ for all $g \in G$,
- (iii) $m(K_1) \leq m(K_2)$ if $K_1 \subset K_2$,
- (iv) $m(K_1 \cup K_2) \leq m(K_1) + m(K_2)$, with equality if $K_1 \cap K_2 = \emptyset$,

for all $K, K_1, K_2 \in \mathcal{C}$.

Proof. Fix a compact neighbourhood C of 1. Let B be a basis of open neighbourhoods of 1. For $U \in B$, we define a map $m_U : \mathcal{C} \rightarrow [0, \infty[$ given by $m_U(K) := \#(K : U) / \#(C : U)$ for every $K \in \mathcal{C}$. As C has non-empty interior, we have $m_U(K) \in [0, \#(K : \overset{\circ}{C})]$ on account of Lemma 1.5.2.

The space

$$P := \prod_{K \in \mathcal{C}} [0, \#(K : \overset{\circ}{C})]$$

(equipped with the product topology) is compact on account of Tikhonov's theorem. Note that $m_U \in P$ for every $U \in B$. For a neighbourhood V of the identity, we define

$$M(V) := \overline{\{m_U \mid U \in B, U \subset V\}} \subset P.$$

We now set

$$M := \bigcap_{V \in B} M(V).$$

The space M is not empty. Indeed, assume to the contrary that M is empty. Then $\{P \setminus M(V)\}_{V \in B}$ covers P . As P is compact, there exists $V_1, \dots, V_n \in B$ such that $P \subset \bigcup_{i=1}^n P \setminus M(V_i)$. This implies that $\bigcap_{i=1}^n M(V_i) = \emptyset$. However, there exists a $U \in B$ with $U \subset \bigcap_{i=1}^n V_i$. This leads to a contradiction as $m_U \in M(V_i)$ for all indices i . We conclude that M is non-empty. Therefore, fix a map $m \in M$. We show that this m satisfies (i)-(iv).

Claim. Let $K_1, \dots, K_n \in \mathcal{C}$, $r_1, \dots, r_n \in \mathbb{R}$ and let $S \subset \mathbb{R}$ be closed. If there exists a neighbourhood V of the identity, such that $\sum_{i=1}^n r_i m_U(K_i) \in S$ for every $U \in B$ with $U \subset V$, then $\sum_{i=1}^n r_i m(K_i) \in S$.

Proof of the claim. Note that for every i , the evaluation map $\epsilon_i : P \rightarrow \mathbb{R} : f \mapsto f(K_i)$ is continuous, by definition of the product topology. Hence the map $s := \sum_{i=1}^n r_i \epsilon_i$ is continuous. It now follows that

$$\{m_U \mid U \in B, U \subset V\} \subset M(V) \subset s^{-1}(S),$$

as S is closed. Thus in particular, we have that $s(m) = \sum_{i=1}^n r_i m(K_i) \in S$. ◇

Since $m_U(\emptyset) = 0$ and $m_U(C) = 1$ for every $U \in B$, we get $m(\emptyset) = 0$ and $m(C) = 1$ by the previous claim. In particular, m is non-zero. Similarly, (ii) follows from Lemma 1.5.2(ii) and previous claim. Now, if $K_1, K_2 \in \mathcal{C}$ and $K_1 \subset K_2$, then $m_U(K_1) - m_U(K_2) \in]-\infty, 0]$ for all $U \in B$. Thus previous claim implies that $m(K_1) \leq m(K_2)$. Finally, we show (iv). Clearly, the inequality ' \leq ' follows again from the previous claim and lemma. Assume that $K_1 \cap K_2 = \emptyset$. As G is Hausdorff, we find disjoint open neighbourhoods U_1, U_2 of respectively K_1, K_2 . It follows from Proposition 1.1.3 that there exists neighbourhoods V_1, V_2 such that $K_1 V_1 \subset U_1$ and $K_2 V_2 \subset U_2$. Thus $V := V_1^{-1} \cap V_2^{-1}$ has the property that for all open subsets $U \in B$ contained in V , we have $m_U(K_1 \cup K_2) = m_U(K_1) + m_U(K_2)$ by Lemma 1.5.2(iv). From the previous claim it follows that we have the same equality for the map m . □

Lemma 1.5.4. *Let G be a locally compact group. Denote the collection of subsets of G again by \mathcal{C} . Suppose there exists a map $m : \mathcal{C} \rightarrow [0, \infty[$ satisfying conditions (i)-(iv) of Lemma 1.5.3. Then there exists a left-invariant Haar measure on G .*

Proof. We construct an outer measure $\mu^* : \mathcal{P}(G) \rightarrow [0, \infty]$ as follows, for $U \subset G$ open, we set

$$\mu^*(U) := \sup\{m(K) \mid K \text{ compact}, K \subset U\}.$$

For an arbitrary subset A of G , we define

$$\mu^*(A) := \inf\{\mu^*(U) \mid U \text{ open}, A \subset U\}.$$

It is readily verified that this is well-defined (i.e. the latter definition agrees with the first on the open subsets) and that μ^* is monotone. Condition (i) implies that $\mu^*(\emptyset) = 0$. To conclude that μ^* is indeed an outer measure, it remains to show subadditivity of μ^* .

Let U_1, U_2 be open subsets of G . Let $K \subset U_1 \cup U_2$ be compact. Then $K \setminus U_1$ and $K \setminus U_2$ are disjoint compact sets, hence there exists disjoint open neighbourhoods V_1, V_2 of respectively $K \setminus U_1$ and $K \setminus U_2$. Note that $K \setminus V_1 \subset U_1$ and $K \setminus V_2 \subset U_2$ are compact sets, and $K \setminus V_1 \cup K \setminus V_2 = K$. Hence, it follows from condition (iv) that $m(K) \leq m(K \setminus V_1) + m(K \setminus V_2) \leq \mu^*(U_1) + \mu^*(U_2)$. As $K \subset U_1 \cup U_2$ was arbitrary, we conclude that

$$\mu^*(U_1 \cup U_2) \leq \mu^*(U_1) + \mu^*(U_2).$$

Next, consider a collection $\{U_i\}_{i \in \mathbb{N}}$ of open subsets of G . Let $K \subset \bigcup_{i=0}^{\infty} U_i$ be compact. By compactness, there exists a $n \in \mathbb{N}$ such that $K \subset \bigcup_{i=0}^n U_i$. By successively applying the inequality we proved above, we obtain

$$m(K) \leq \mu^*\left(\bigcup_{i=0}^n U_i\right) \leq \sum_{i=0}^n \mu^*(U_i) \leq \sum_{i=0}^{\infty} \mu^*(U_i).$$

Thus it follows that $\mu^*(\bigcup_{i=0}^{\infty} U_i) \leq \sum_{i=0}^{\infty} \mu^*(U_i)$. Finally, consider a collection $\{A_i\}_{i \in \mathbb{N}}$ of arbitrary subsets of G . Let $\varepsilon > 0$. For every $i \in \mathbb{N}$ we find an open subset $U_i \supset A_i$ such that $\mu^*(U_i) \leq \mu^*(A_i) + \varepsilon/2^{i+1}$. It now follows that

$$\mu^*\left(\bigcup_{i=0}^{\infty} A_i\right) \leq \mu^*\left(\bigcup_{i=0}^{\infty} U_i\right) \leq \sum_{i=0}^{\infty} \mu^*(U_i) \leq \sum_{i=0}^{\infty} \mu^*(A_i) + \sum_{i=0}^{\infty} \frac{\varepsilon}{2^{i+1}} = \sum_{i=0}^{\infty} \mu^*(A_i) + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, the desired inequality follows.

On strength of the Carathéodory's theorem (see [Coh13, Theorem 1.3.6]), μ^* restricted to the σ -algebra \mathcal{A} of sets that meet the Carathéodory criterion, is a measure. We show that $\mathcal{B}(G) \subset \mathcal{A}$. It suffices to show that every open subset U of G obeys the Carathéodory condition. First, consider an open set V . Let $K_1 \subset U \cap V$ compact. For every compact set $K_2 \subset V \setminus K_1$ we have $m(K_1) + m(K_2) = m(K_1 \cup K_2) \leq \mu^*(V)$ by condition (iv). Because this holds for every compactum $K_2 \subset V \setminus K_1$, we get $m(K_1) + \mu^*(V \setminus K_1) \leq \mu^*(V)$. Note that $V \setminus U \subset V \setminus K_1$. As $K_1 \subset U \cap V$ was an arbitrary compact set, we obtain $\mu^*(V \cap U) + \mu^*(V \setminus U) \leq \mu^*(V)$. Now, consider a subset A of G . Let $\varepsilon > 0$. Then there exists an open subset $V \supset A$ such that $\mu^*(V) \leq \mu^*(A) + \varepsilon$. It follows that

$$\mu^*(A \cap U) + \mu^*(A \setminus U) \leq \mu^*(V \cap U) + \mu^*(V \setminus U) \leq \mu^*(V) \leq \mu^*(A) + \varepsilon.$$

Hence $U \in \mathcal{A}$. Thus we conclude that $\mathcal{B}(G) \subset \mathcal{A}$.

Denote μ for the restriction of μ^* to $\mathcal{B}(G)$. This is a measure on account of what we showed above. From the definition of μ^* and condition (ii) on m , it readily follows that μ^* is invariant under left-translations. Hence μ is left-invariant. Furthermore, it directly follows from the definition of μ^* that it is outer regular on the Borel sets. Again using the definition, and observing that $m(K) \leq \mu^*(K)$ for all compact subsets K , we deduce that μ is inner regular on the open sets. Note that μ is not the trivial measure, as m is non-zero. It remains to show that μ is finite on compact sets. Indeed, let K be compact. Then we can cover K with finitely many relatively compact open subsets U_1, \dots, U_n . It follows that $\mu(K) \leq \sum_{i=1}^n \mu(U_i) \leq \sum_{i=1}^n m(\overline{U_i}) < \infty$ (here we used condition (iii)). We conclude that μ is a left-invariant Haar measure. \square

Proof of Theorem 1.5.1. Combining Lemma 1.5.3 and Lemma 1.5.4, we conclude that there exists a left-invariant Haar measure μ on G . We turn to the second assertion of the theorem. Let ν be another left-invariant Haar measure on G . For $f \in C_c(G)$ with $\mu(f) \neq 0$ we define

$$D_f : G \longrightarrow \mathbb{C} : x \mapsto \frac{\nu(R_x f)}{\mu(f)}.$$

On the strength of Proposition 1.4.7 this map is continuous. Using left-translation invariance of ν (see Proposition 1.4.5), we get for $g \in C_c(G)$,

$$\mu(f)\nu(g) = \int_G \int_G f(x)g(x^{-1}y) d\nu(y) d\mu(x).$$

Invoking the theorem of Fubini (see Theorem 1.3.4), using left-translation of μ and applying Fubini's theorem again, we obtain

$$\begin{aligned} \nu(g) &= \frac{1}{\mu(f)} \int_G \int_G f(yx)g(x^{-1}) d\mu(x) d\nu(y) \\ &= \frac{1}{\mu(f)} \int_G \int_G f(yx)g(x^{-1}) d\nu(y) d\mu(x) = \int_G D_f(x)g(x^{-1}) d\mu(x). \end{aligned}$$

Thus we have

$$\int_G (D_f(x) - D_{f'}(x))g(x^{-1}) d\mu(x) = 0$$

for every $f, f' \in C_c(G)$ with $\mu(f), \mu(f') \neq 0$. Substituting $x \mapsto \phi(x^{-1})\overline{(D_f(x^{-1}) - D_{f'}(x^{-1}))}$ for g , where $\phi \in C_c^+(G)$ is arbitrary, we deduce that $\phi|D_f - D_{f'}|^2 = 0$ using Proposition 1.4.6. It follows from Lemma 1.2.5 that $D_f = D_{f'}$. Setting $c := \nu(f)/\mu(f) = D_f(1)$, we obtain that $\nu(f')/\mu(f') = D_{f'}(1) = c$ for every other $f' \in C_c(G)$ with $\mu(f') \neq 0$. Thus in particular, we have $\nu(f') = c\mu(f')$ for every $f' \in C_c^+(G)$. On account of Corollary 1.2.7 we get $\nu = c\mu$. \square

1.6 Modular functions

Note that for a locally compact group G with a left-invariant Haar measure μ , the map $\mu_x : A \mapsto \mu(Ax)$, $x \in G$, is again a Radon measure on account of Proposition 1.2.9. It is readily verified that μ_x is left-invariant again. Thus μ_x must be a multiple of μ on account of Haar's theorem. Thus there exists a $c > 0$ such that $\mu_x = c\mu$. This number c does not depend on μ . Indeed, for another left-invariant Haar measure ν , we have $\nu = c'\mu$ for some $c' > 0$. Hence $\nu_x = c'\mu_x = cc'\mu = c\nu$. This motivates the following definition.

Definition 1.6.1. Let G be a locally compact group with a left-invariant Haar measure μ . Then the unique function $\Delta : G \rightarrow]0, \infty[$ such that $\mu_x = \Delta(x)\mu$ for all $x \in G$, is called the *modular function* of G . The group G is said to be *unimodular* if $\Delta = 1$ everywhere.

It is readily verified that if a group is unimodular, its left-invariant Haar measures coincide with its right-invariant Haar measures. The following proposition is proven in the same way as Proposition 1.4.5.

Proposition 1.6.2. Let μ be a left-invariant Haar measure on a locally compact group G . Let $g \in G$. Then a Borel measurable function $f : G \rightarrow \mathbb{C}$ is integrable if and only if $R_g f$ is integrable, and in this case we have

$$\mu(R_g f) = \Delta(g^{-1})\mu(f).$$

Proposition 1.6.3. The modular function Δ on a locally compact group G is a continuous homomorphism of groups (with multiplication on $]0, \infty[$).

Proof. It is readily verified that Δ is a group homomorphism. We show that it is continuous. Let μ be a left-invariant Haar measure on G . Fix a map $f \in C_c(G)$ with $\mu(f) \neq 0$. Then we obtain $\Delta(x) = \mu(R_{x^{-1}} f) / \mu(f)$ in light of Proposition 1.6.2. Continuity of Δ now directly follows from Proposition 1.4.7. \square

Corollary 1.6.4. Every compact group is unimodular.

Proof. The image of a compact group under its modular function is a compact subgroup of $]0, \infty[$ in light of Proposition 1.6.3. There is only one such subgroup: $\{1\}$. \square

We are now ready to formulate a generalization of Proposition 1.4.7 in case of Haar measures.

Proposition 1.6.5. Let μ be a left-invariant Haar measure on a locally compact group G . Suppose that $f \in L^p(\mu)$ for some $1 \leq p < \infty$. Then the maps

$$G \longrightarrow L^p(\mu) : g \mapsto [L_g f], \quad G \longrightarrow L^p(\mu) : g \mapsto [R_g f]$$

are continuous.

Proof. It is clear that the maps above are well defined, since the maps $G \rightarrow \mathcal{L}^p(\mu) : g \mapsto L_g f$ and $G \rightarrow \mathcal{L}^p(\mu) : g \mapsto R_g f$ factor through the quotient map $\mathcal{L}^p(\mu) \rightarrow L^p(\mu)$ as consequence of Proposition 1.4.5 and Proposition 1.6.2.

We show continuity of the map $g \mapsto [R_g f]$. Let $g \in G$ and $\varepsilon > 0$. We fix an compact neighbourhood N of g . On account of Proposition 1.2.8 there exists a $\phi \in C_c(G)$ such that $\|f - \phi\|_p < \varepsilon^{1/p}$. Similarly as in the proof of Proposition 1.4.7, we find an open neighbourhood $U \subset N$ of g such that $\|R_h \phi - R_g \phi\|_\infty < \min\{1, \varepsilon / \mu(\text{supp}(\phi) \cdot N^{-1})\}$ and $\text{supp}(R_h \phi - R_g \phi) \subset \text{supp}(\phi) \cdot N^{-1}$ whenever $h \in U$. It follows that for $h \in U$

$$\begin{aligned} \|R_h f - R_g f\|_p^p &\leq \int_G |R_h f - R_h \phi|^p d\mu + \int_G |R_h \phi - R_g \phi|^p d\mu + \int_G |R_g \phi - R_g f|^p d\mu \\ &\leq (\Delta(g^{-1}) + \Delta(h^{-1})) \|f - \phi\|_p^p + \varepsilon \leq (\Delta(g^{-1}) + \Delta(h^{-1}) + 1)\varepsilon. \end{aligned}$$

Here we used Proposition 1.6.2. Note that Δ is bounded on N^{-1} since Δ is continuous. Hence it follows from inequality above that $g \mapsto [R_g f]$ is continuous.

The proof of the continuity of $g \mapsto [L_g f]$ is similar, but one exploits the left-invariance of the Haar integral in this case instead of using Proposition 1.6.2. \square

1.7 Convolutions

Definition 1.7.1. Let μ be a left-invariant Haar measure on a locally compact group G . Let f, g be integrable functions on G . Then the convolution of f and g is the map

$$f * g : G \longrightarrow \mathbb{C} : x \mapsto \begin{cases} \int_G f(y)g(y^{-1}x) d\mu(y) & \text{if } y \mapsto f(y)g(y^{-1}x) \text{ is integrable} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.7.2. Let μ be a left-invariant Haar measure on a locally compact group G . Let f, g be integrable functions on G . Then the convolution $f * g$ is integrable and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

We will need the following lemma to prove this proposition.

Lemma 1.7.3. Let μ be a left-invariant Haar measure on a locally compact group G . Then every integrable function f on G vanishes outside a σ -compact set.

Proof. First we note that G has a σ -compact open subgroup H . Indeed, one takes a symmetric compact neighbourhood K of the identity and defines $H := \bigcup_{n \in \mathbb{N}} K^n$. It is clear that H is a subgroup of G and open as any $x \in H$ has the neighbourhood xK contained in H . Since K^n is compact (it is the image of $K \times \cdots \times K$ under multiplication), H is σ -compact.

Recall from integration theory that every integrable function f vanishes outside a σ -finite subset of G . Hence there exists a collection of nonempty Borel subsets $\{A_n\}_{n \in \mathbb{N}}$ all of finite measure such that $f^{-1}(\mathbb{C} \setminus \{0\}) \subset \bigcup_{n \in \mathbb{N}} A_n$. By outer regularity of μ , we may assume that every A_n is open.

For $n \in \mathbb{N}$, let $P_n \subset G$ be the subset consisting of the points $x \in G$ such that $xH \cap A_n \neq \emptyset$. As $xH \cap A_n$ is open, it has positive measure (see Proposition 1.4.6). By additivity of the measure it follows that $\sum_{x \in P_n} \mu(xH \cap A_n) \leq \mu(A_n)$. This implies that P_n consists of countably many elements, since otherwise we would have $\mu(A_n) = \infty$. We now set $B_n := \bigcup_{x \in P_n} xH$. Note that B_n contains A_n and is σ -compact since it is a countable union of σ -compact sets. As f vanishes outside $\bigcup_{n \in \mathbb{N}} B_n$, this concludes the proof. \square

Proof of Proposition 1.7.2. Note that the map $f \times g$ is $\mathcal{B}(G) \otimes \mathcal{B}(G)$ -measurable, thus in particular $\mathcal{B}(G \times G)$ -measurable. Furthermore, the maps $s : (x, y) \mapsto (y, y^{-1}x)$ and the multiplication map $G \times G \rightarrow G$ are $\mathcal{B}(G \times G)$ -measurable as they are continuous. Hence the map $h : (x, y) \mapsto f(y)g(y^{-1}x)$ is $\mathcal{B}(G \times G)$ -measurable since it is the composition of multiplication, $f \times g$ and s .

Using Lemma 1.7.3 we find a σ -compact subset A of G such that f and g vanish outside A . It is readily verified that h vanishes outside $A^2 \times A$. By continuity of multiplication A^2 is σ -compact, and hence the subset $A^2 \times A$ is again σ -compact. Thus h vanishes outside a σ -compact subset of G , and thus in particular outside a σ -finite subset.

Using left-invariance of μ we get

$$\int_G \int_G |h(x, y)| d\mu(x) d\mu(y) = \int_G |f(y)| \int_G |g(x)| d\mu(x) d\mu(y) = \|f\|_1 \|g\|_1 < \infty.$$

It follows from Theorem 1.3.4(ii) that h is integrable w.r.t. $\mu \times \mu$. The assertions now readily follow from Theorem 1.3.4(i). \square

Corollary 1.7.4. Let μ be a left-invariant Haar measure on a locally compact group G . When $L^1(\mu)$ is endowed with convolution

$$* : L^1(\mu) \times L^1(\mu) \longrightarrow L^1(\mu) : ([f], [g]) \mapsto [f * g],$$

it forms a Banach algebra.

Proof. One readily proves that $f * g = f' * g'$ for integrable functions f, g, f', g' on G such that $f = f'$ a.e. and $g = g'$ a.e. Hence the convolution on $L^1(\mu)$ is well-defined. One readily shows that $(L^1(\mu), *)$ forms an associative algebra (it suffices to show the desired identities for compactly supported functions on the strength of Proposition 1.7.2 and Proposition 1.2.8). As $L^1(\mu)$ is complete and we have $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$ for all $f, g \in L^1(\mu)$, it follows that $(L^1(\mu), *)$ is a Banach algebra. \square

Definition 1.7.5. Let μ be a left-invariant Haar measure on a locally compact group G . A *Dirac function* on G is a function $\phi \in C_c^+(G)$ such that

- (i) $\phi(x) = \phi(x^{-1})$ for all $x \in G$,
- (ii) $\mu(\phi) = 1$.

Proposition 1.7.6. Let μ be a left-invariant Haar measure on a locally compact group G . Then for every open U of the identity there exists a Dirac function ϕ supported in U .

Proof. Let V be a symmetric neighbourhood of the identity such that V is contained in U (see Proposition 1.1.3). Now choose a $g \in C_c^+(G)$ such that $g(1) > 0$ and $\text{supp } g \subset V$. Consider the function $f \in C_c^+(G)$ given by $f(x) := g(x) + g(x^{-1})$. It is readily verified that $\phi := (1/\mu(f))f$ is a Dirac function supported in U . \square

Lemma 1.7.7. Let μ be a left-invariant Haar measure on a locally compact group G . Let $f \in L^1(\mu)$. Then for every $\varepsilon > 0$, there exists a neighbourhood U of the identity such that for all Dirac functions ϕ supported in U we have $\|f * \phi - f\|_1 < \varepsilon$.

Proof. In light of Proposition 1.7.2 and Proposition 1.2.8, it suffices to prove the theorem for $f \in C_c(G)$. Let U be a neighbourhood of the identity such that $\|R_x f - f\|_1 < \varepsilon$ for every $x \in U$. For any Dirac function ϕ supported in U , we have

$$\|f * \phi - f\|_1 \leq \int_G \left| \int_G f(y) \phi(y^{-1}x) d\mu(y) - f(x) \right| d\mu(x).$$

From symmetry of ϕ and translation invariance of μ it follows that $\int_G \phi(y^{-1}x) d\mu(y) = \mu(\phi) = 1$ for all $x \in G$. Hence

$$\|f * \phi - f\|_1 \leq \int_G \int_G \phi(y^{-1}x) |f(y) - f(x)| d\mu(y) d\mu(x).$$

Using the theorem of Fubini twice and translation invariance, we obtain

$$\begin{aligned} \|f * \phi - f\|_1 &\leq \int_G \int_G \phi(x) |f(y) - f(yx)| d\mu(y) d\mu(x) \\ &\leq \int_G \phi(x) \|R_x f - f\|_1 d\mu(x) < \varepsilon. \end{aligned} \quad \square$$

2. Representations

We follow the approach taken in [Ban10].

Definition 2.0.1. Let G be a topological group. A *representation* of G is a pair (π, V) consisting of a non-trivial complex Banach space V and a group homomorphism $\pi : G \rightarrow \text{Iso}(V)$, such that the induced map

$$G \times V \longrightarrow V : (g, v) \mapsto \pi(g)v$$

is continuous. If V is finite-dimensional, (π, V) is said to be a *finite-dimensional representation*. If V is a Hilbert space, and $\pi(G) \subset U(V) = \{T \in B(V) \mid T \text{ is unitary}\}$, the representation (π, V) is called *unitary*.

Lemma 2.0.2. Let G be a locally compact group and V a Banach space. Let $\pi : G \rightarrow \text{Iso}(V)$ be a homomorphism. Then π is a representation if and only if the map $g \mapsto \pi(g)v$ is continuous at 1 for every $v \in V$.

Proof. The implication ‘ \Rightarrow ’ is clear. Assume that the map $g \mapsto \pi(g)v$ is continuous at 1 for every $v \in V$. Since π is a homomorphism, this implies that the map $g \mapsto \pi(g)v$ is continuous on the whole group G for every $v \in V$. Fix a compact neighbourhood K of 1. Compactness of K implies that $\sup_{g \in K} \|\pi(g)v\| < \infty$ for every $v \in V$. Consequently, on the strength of the uniform boundedness principle, we obtain $C := \sup_{g \in K} \|\pi(g)\| < \infty$. Let $g \in G$ and $h \in gK$ and $v, w \in V$, then we have

$$\begin{aligned} \|\pi(h)w - \pi(g)v\| &\leq \|\pi(g)\| \|\pi(g^{-1}h)w - \pi(g^{-1}h)v + \pi(g^{-1}h)v - v\| \\ &\leq \|\pi(g)\| C \|w - v\| + \|\pi(g)\| \|\pi(g^{-1}h)v - v\|. \end{aligned}$$

From this inequality it now readily follows that $(g, v) \mapsto \pi(g)v$ is continuous. \square

Consider a locally compact group G and a left-invariant Haar measure μ on G . Recall that for every $g \in G$, the map L_g on $L^2(\mu)$ is a unitary operator (see Proposition 1.4.5). It follows from Proposition 1.6.5 that the map

$$L : G \longrightarrow U(L^2(\mu)) : g \mapsto L_g$$

is a representation of G . This is called the *left regular representation* of G . Similarly, we have the *right regular representation*

$$R : G \longrightarrow \text{Iso}(L^2(\mu)) : g \mapsto R_g.$$

Definition 2.0.3. Let (π, V) be a representation of a topological group G . A linear subspace $W \subset V$ is an *invariant subspace* (w.r.t. π) of V whenever $\pi(G)W \subset W$. As W is again a Banach space, the restriction

$$\pi|_W : G \longrightarrow \text{Iso}(W) : g \mapsto \pi(g)|_W$$

of π to a non-trivial invariant subspace W gives rise to a representation $(\pi|_W, W)$ of G . This is called a *subrepresentation* of π . If the only invariant subspaces of π are either V or 0 , then we say that π is *irreducible*. A non-trivial invariant subspace W is said to be a *irreducible subspace* of V whenever $\pi|_W$ is irreducible.

Example 2.0.4. Consider the group $U(n) = \{A \in M(n, \mathbb{C}) \mid AA^* = 1\}$, the group of *unitary matrices*. This group has a natural action on \mathbb{C}^n (the identification of matrices with linear operators w.r.t. the standard basis), inducing a unitary representation (π, \mathbb{C}^n) of $U(n)$. This representation is irreducible. Indeed, let $W \subset \mathbb{C}^n$ be a non-trivial invariant subspace. Then there exists a $v_1 \in W$ such that $\|v_1\| = 1$. We can extend to an orthonormal basis $\{v_1, v_2, \dots, v_n\}$ of \mathbb{C}^n . For every permutation σ of $1, \dots, n$, we have a unitary matrix A^σ satisfying $A^\sigma v_i = e_{\sigma(i)}$ for all i . It follows that $Av_1 = e_{\sigma(1)} \in W$ for every permutation σ . Thus $W = \mathbb{C}^n$.

Proposition 2.0.5. Let (π, V) be a unitary representation of a topological group G . If $W \subset V$ is an invariant subspace, then W^\perp is also an invariant subspace. In particular, V decomposes as the direct sum of two invariant subspaces: $V = W \oplus W^\perp$.

Proof. Note that W^\perp is again closed, and as W is closed we have $V = W \oplus W^\perp$. Let $w' \in W^\perp$. Then for every $w \in W$, we have $\langle \pi(g)w', w \rangle = \langle w', \pi(g^{-1})w \rangle = 0$. Here we used the unitary property of π and the fact that W is invariant. Hence $\pi(g)w' \in W^\perp$. Thus W^\perp is invariant. \square

Definition 2.0.6. Let $(\pi, V_\pi), (\rho, V_\rho)$ be two representations of a topological group G . A bounded operator $T : V_\pi \rightarrow V_\rho$ is said to be *intertwining* (w.r.t. π and ρ) if the following diagram commutes for every $g \in G$.

$$\begin{array}{ccc} V_\pi & \xrightarrow{\pi(g)} & V_\pi \\ \downarrow T & & \downarrow T \\ V_\rho & \xrightarrow{\rho(g)} & V_\rho \end{array}$$

The linear space of intertwining maps $V_\pi \rightarrow V_\rho$ is denoted by $B_G(V_\pi, V_\rho)$. If there exists an bijective intertwining operator, the representations of π and ρ are said to be equivalent and we write $\pi \cong \rho$.

Note that on the strength of the open mapping theorem, every bijective intertwining operator is an isomorphism of normed spaces.

Lemma 2.0.7. Let $(\pi, V_\pi), (\rho, V_\rho)$ be two representations of a topological group G . Suppose that $T : V_\pi \rightarrow V_\rho$ is a intertwining operator. Then $\ker(T)$ is a invariant subspace of V_π and the image $\text{im}(T)$ is an invariant subspace of V_ρ whenever it is closed.

Proof. As $\ker(T) = T^{-1}(0)$, the kernel of T is closed. Let $v \in \ker(T)$, then $T(\pi(g)v) = \rho(g)(Tv) = 0$ for every $g \in G$. Thus $\ker(T)$ is invariant. As $T(\pi(g)v) = \rho(g)(Tv)$ for all $v \in V_1$, we see that $\text{im}(T)$ is an invariant subspace if $\text{im}(T)$ is closed. \square

Proposition 2.0.8. *Let (π, V_π) and (ρ, V_ρ) be two irreducible finite-dimensional representations of a topological group G . If $\pi \not\cong \rho$ then $B_G(V_\pi, V_\rho) = 0$.*

Proof. Indeed, assume that $\pi \not\cong \rho$. Let $T : V_\pi \rightarrow V_\rho$ be an intertwining map. Then $\ker(T)$ and $\text{im}(T)$ are invariant subspaces by previous lemma (recall that every linear subspace of a finite-dimensional space is closed). If $\ker(T) = 0$ then $\text{im}(T) \neq 0$ (as V_ρ has dimension ≥ 1). This implies that $\text{im}(T) = V_\rho$. Hence T is bijective, which contradicts the assumption that π and ρ are not equivalent. Thus we must have that $\ker(T) \neq 0$. Hence $\ker(T) = V_\pi$, i.e. T is the trivial map. \square

Recall that a finite-dimensional space is a Hilbert space, and every norm is equivalent on a finite-dimensional space. Hence, it might occur that we can equip a finite-dimensional space with a (Hermitian) inner product for which a representation is a unitary. This motivates the following definition.

Definition 2.0.9. Let (π, V) be a finite-dimensional representation of a topological group G . Then π is said to be *unitarizable* if there exists an inner product $\langle \cdot, \cdot \rangle_G$ on V for which the representation π is unitary.

Lemma 2.0.10 (Schur's lemma). *Let (π, V) be a finite-dimensional representation of a topological group G . Then $B_G(V) = \mathbb{C}\text{id}_V$ if π is irreducible. The converse statement holds if π is unitarizable.*

Proof. Let $T \in B_G(V)$. Then there exists an eigenvalue $\lambda \in \mathbb{C}$ of T . Since $\ker(T - \lambda\text{id}_V)$ is non-trivial and invariant in light of Lemma 2.0.7, we must have that $\ker(T - \lambda\text{id}_V) = V$. Hence $T = \lambda\text{id}_V$.

Next, suppose that π is unitarizable and let $\langle \cdot, \cdot \rangle_G$ be an inner product on V for which π is unitary. We prove the last assertion. Assume that $B_G(V) = \mathbb{C}\text{id}_V$. Let W be an invariant subspace of V . We can decompose V as $V = W \oplus W^\perp$ (here the orthogonal complement is taken w.r.t. $\langle \cdot, \cdot \rangle_G$). This decomposition comes with a projection $P : V \rightarrow W$. As both W and W^\perp are invariant on account of Proposition 2.0.5, P is intertwining. We interpret P as a intertwining operator $V \rightarrow V$. It then follows that $P = \lambda\text{id}_V$ for some $\lambda \in \mathbb{C}$. Hence $\lambda V = W$ and thus $W = 0$ or $W = V$. \square

Let V_1, V_2 be two Banach spaces with respectively norms $\|\cdot\|_{V_1}, \|\cdot\|_{V_2}$. Recall that we can endow the (external) direct sum $V_1 \oplus V_2$ with the norm $\|\cdot\| = \|\cdot\|_{V_1} + \|\cdot\|_{V_2}$. With this norm, the direct sum is again a Banach space, and the topology induced by this norm coincides with the product topology.

Definition 2.0.11. Let $(V_1, \pi_1), (V_2, \pi_2)$ be two representations of a topological group G . Then the induced representation

$$\pi_1 \oplus \pi_2 : G \longrightarrow \text{Iso}(V_1 \oplus V_2) : g \mapsto \pi_1(g) \oplus \pi_2(g)$$

is called the *direct sum* of π_1 and π_2 .

Proposition 2.0.12. *Let (π, V) be a representation of a topological group G . Suppose that V decomposes into non-trivial invariant subspaces V_1, \dots, V_n , i.e. $V = \bigoplus_{i=1}^n V_i$. Then*

$$\pi \cong \bigoplus_{i=1}^n \pi_i,$$

where π_i is the restriction of π to V_i .

Proof. One takes the identification of the internal and external direct sum of the V_i 's as the desired intertwining isomorphism. \square

Proposition 2.0.13. *Every unitarizable finite-dimensional representation (π, V) of a topological group G decomposes into irreducible representations. I.e., there exists a $n \in \mathbb{N}$ such that*

$$\pi \cong \bigoplus_{i=1}^n \pi_i,$$

where each π_i is an irreducible representation.

Proof. We proceed by induction on the dimension of V . If $\dim(V) = 1$, we are done. Let $k \in \mathbb{N}$ be a positive number, and assume the statement holds for representations of dimension $\leq k$. Consider a unitarizable representation (π, V) with $\dim(V) = k + 1$. If π is irreducible, we are done. Hence assume that π is not irreducible. Then there exists a non-trivial invariant subspace $W \subset V$ of minimal dimension. Recall that W^\perp (the orthogonal complement of W w.r.t. $\langle \cdot, \cdot \rangle_G$) is again invariant. Write $\pi_1 := \pi|_W$ and $\pi' := \pi|_{W^\perp}$. Note that π_1 is irreducible. By the induction hypothesis, π' decomposes into irreducible representations π_2, \dots, π_n . Hence $\pi \cong \pi_1 \oplus \pi' \cong \bigoplus_{i=1}^n \pi_i$. \square

Using the terminology of Section A.2, we define the following.

Definition 2.0.14. Let $\{(\pi_i, V_i)\}_{i \in I}$ be a family of unitary representations of a topological group G . The unitary representation

$$\widehat{\bigoplus_{i \in I} \pi_i} : G \longrightarrow U \left(\widehat{\bigoplus_{i \in I} V_i} \right) : g \mapsto \bigoplus_{i \in I} \pi_i(g)$$

is called the *Hilbert direct sum* of $\{\pi_i\}_{i \in I}$.

Proposition 2.0.15. *Let (π, V) be a unitary representation of a topological group G . Suppose that V has a orthogonal decomposition into non-trivial invariant subspaces $\{V_i\}_{i \in I}$, i.e. $V = \overline{\bigoplus_{i \in I} V_i}$ and $V_i \perp V_j$ whenever $i \neq j \in I$. Then*

$$\pi \cong \widehat{\bigoplus_{i \in I} \pi_i},$$

where π_i is the restriction of π to V_i .

Proof. Using the notation of Section A.2, we can take $\prod_{i \in I} P_i : V \rightarrow \widehat{\bigoplus_{i \in I} V_i}$ as desired unitary intertwining operator. Here $P_i : V \rightarrow V_i$ denotes the projection onto V_i . \square

3. Representations of compact groups

We will now shift our attention to compact groups.

Troughout this chapter G will be a compact group.

As G is compact, there exists a left- and right-invariant Haar measure μ on G which satisfies $\mu(G) = 1$. This is called the *normalized Haar measure on G* . Indeed, there exists a left-invariant Haar measure μ' on G in account of Haar's theorem. Since G is compact, it is unimodular and hence μ' is also right-invariant. Note that $\mu'(G) < \infty$, hence we can scale μ' by a factor $1/\mu'(G)$ to obtain the desired normalized Haar measure μ . Troughout this chapter, we will endow G with this Haar measure μ . We denote

$$L^p(G) := L^p(\mu).$$

Proposition 3.0.1. *Let (π, V) be a finite-dimensional representation of G . Every inner product $\langle \cdot, \cdot \rangle$ on V induces an inner product $\langle \cdot, \cdot \rangle_G$ on V given by*

$$\langle v, w \rangle_G := \int_G \langle \pi(x)v, \pi(x)w \rangle d\mu(x) \quad (v, w \in V).$$

When V is endowed with this inner product, (π, V) is a unitary representation. In particular, every finite-dimensional representation of G is unitarizable.

Proof. Let $v, w \in V$. Note that the map $\langle \cdot, \cdot \rangle_G$ is well-defined as the map $x \mapsto \langle \pi(x)v, \pi(x)w \rangle$ is continuous (since π is a representation) and hence integrable over the compact group G . We claim that this is an inner product on V for which π is invariant. Note that $x \mapsto \langle \pi(x)v, \pi(x)v \rangle$ is a positive continuous function. Hence $\langle v, v \rangle_G \geq 0$. Furthermore, on account of Proposition 1.4.6 we get that $\langle v, v \rangle_G = 0$ if and only if $v = 0$. Linearity in the first argument and conjugate symmetry of $\langle \cdot, \cdot \rangle_G$ readily follow from the definition. Finally, one observes that π is unitary w.r.t. $\langle \cdot, \cdot \rangle_G$ by right-invariance of the Haar integral. \square

Corollary 3.0.2. *Every finite-dimensional representation of G decomposes into irreducible representations.*

From now on, we will equip every finite-dimensional representation with an associated inner product $\langle \cdot, \cdot \rangle_G$. This makes every finite-dimensional representation a unitary representation.

3.1 Matrix coefficients and characters

Definition 3.1.1. For a unitary representation (π, V) of G , the (continuous) maps

$$m_{v,w}^\pi : G \longrightarrow \mathbb{C} : x \mapsto \langle \pi(x)v, w \rangle \quad (v, w \in V)$$

are called the *matrix coefficients* of π . The linear subspace of $C(G) \subset L^2(G)$ spanned by these matrix coefficients is denoted by $C(G)_\pi$.

If it is clear from the context to which representation π a matrix coefficient $m_{v,w}^\pi$ corresponds, we leave out π in our notation, and consequently write $m_{v,w} := m_{v,w}^\pi$.

Lemma 3.1.2. Let V be a finite-dimensional vector space over a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Suppose that $\langle \cdot, \cdot \rangle$ is an inner product on V with orthonormal basis v_1, \dots, v_n . Then for any $A \in \text{End}(V)$, we have

$$\text{tr}(A) = \sum_{i=1}^n \langle Av_i, v_i \rangle.$$

Proof. Let $\beta : V \rightarrow \mathbb{K}^n$ be the coordinate map such that $\beta(v_i) = e_i$ for all i . Note that A has a matrix A_β with respect to these coordinates satisfying $\beta^{-1}A_\beta\beta = A$. By definition of the trace of a linear map, we have

$$\text{tr}(A) = \sum_{i=1}^n (A_\beta)_{ii} = \sum_{i=1}^n (\beta^{-1}(Av_i))_i.$$

Note that for every $v \in V$ we can write $v = \sum_{i=1}^n \lambda_i v_i$ with scalars $\lambda_i \in \mathbb{K}$. Hence $\beta^{-1}(v)_i = \lambda_i = \langle v, v_i \rangle$. Combining this with the equation above, the assertion follows. \square

For a finite-dimensional representation of G , the linear map $T_\pi : \text{End}(V) \rightarrow C(G)_\pi$ given by

$$(T_\pi A)(x) := \text{tr}(\pi(x)A) \quad (A \in \text{End}(V), x \in G)$$

will be of importance. In fact, when π is irreducible and $\text{End}(V)$ is endowed with the Hilbert-Schmidt norm (see Section A.2), this map is unitary, as we will see shortly. As V is finite-dimensional, the Hilbert-Schmidt norm is induced by the inner product

$$\langle A, B \rangle_{\text{HS}} = \text{tr}(AB^*).$$

Lemma 3.1.3. Let (π, V) be a finite-dimensional representation of G . Then the map T_π is surjective.

Proof. First we show that T_π maps into $C(G)_\pi$. Let v_1, \dots, v_n be an orthonormal basis of V (w.r.t. $\langle \cdot, \cdot \rangle_G$). Let $A \in \text{End}(V)$. Then for every $x \in G$ we have $\text{tr}(\pi(x)A) = \sum_{i=1}^n \langle \pi(x)Av_i, v_i \rangle_G = \sum_{i=1}^n m_{Av_i, v_i}(x)$. Thus $x \mapsto \text{tr}(\pi(x)A)$ indeed defines a function of $C(G)_\pi$. Furthermore, it is clear that T_π is linear, due to linearity of the trace.

We turn to the main content of the lemma. Let $v, w \in V$ and consider the matrix coefficient $m_{v,w}$. Write $v = \sum_{i=1}^n \lambda_i v_i$ and $w = \sum_{i=1}^n \mu_i v_i$, with $\lambda_i, \mu_i \in \mathbb{C}$. Let A be the endomorphism such that $Av_i = \sum_{j=1}^n \lambda_j \overline{\mu_i} v_j$. It is readily verified that $T_\pi A = m_{v,w}$. \square

Proposition 3.1.4. Let (π, V_π) and (ρ, V_ρ) be two equivalent finite-dimensional representations of G . Then $C(G)_\pi = C(G)_\rho$.

Proof. As $\pi \cong \rho$, there exists a linear isomorphism $T : V_\pi \rightarrow V_\rho$ such that $T\pi(x)T^{-1} = \rho(x)$ for every $x \in G$. Consider the surjective linear maps T_π and T_ρ as in Lemma 3.1.3. As T is an isomorphism, it induces a linear isomorphism $T_* : \text{End}(V_\rho) \rightarrow \text{End}(V_\pi) : A \mapsto T^{-1}AT$. It is readily verified that $T_\pi \circ T_* = T_\rho$ (use the fact that $\text{tr}(CD) = \text{tr}(DC)$ for linear maps C, D). By previous lemma all occurring maps in this composition are surjective, hence the desired equality follows. \square

Lemma 3.1.5. *Let V, W be two complex finite-dimensional vector spaces. Suppose that $L : V \times W \rightarrow \mathbb{C}$ is a map which satisfies $L(v + v', w + w') = L(v, w) + L(v, w') + L(v', w) + L(v', w')$ and $L(\lambda v, \mu w) = \lambda \bar{\mu} L(v, w)$ for all $v, v' \in V, w, w' \in W$ and $\lambda, \mu \in \mathbb{C}$. Then for any inner product on W , there exists a unique linear map $A : V \rightarrow W$ such that*

$$\langle Av, w \rangle = L(v, w)$$

for all $v \in V$ and $w \in W$.

Proof. We fix inner products on V, W . Let $v_1, \dots, v_n \in V$ and $w_1, \dots, w_m \in W$ be orthonormal bases of V and W . Let $A : V \rightarrow W$ be the unique linear map such that $Av_i = \sum_{j=1}^m L(v_i, w_j)w_j$ for every $1 \leq i \leq n$ (note that any linear map satisfying the desired relation meets this condition). One readily verifies that this is the desired map. \square

Theorem 3.1.6 (Schur orthogonality relations). *Let (π, V_π) and (ρ, V_ρ) be irreducible finite-dimensional representations of G . The following statements are true.*

- (i) *If $\pi \not\cong \rho$ then $C(G)_\pi \perp C(G)_\rho$ in $L^2(G)$.*
- (ii) *For $v, w, v', w' \in V_\pi$ we have*

$$\langle m_{v,w}, m_{v',w'} \rangle = \frac{1}{\dim(V_\pi)} \langle v, v' \rangle_G \overline{\langle w, w' \rangle_G}.$$

Proof. We prove assertion (i). Let $w \in V_\pi$ and $w' \in V_\rho$. Using Lemma 3.1.5 one readily verifies that there exists a linear map $I_{w,w'} : V_\pi \rightarrow V_\rho$ such that

$$\langle I_{w,w'} v, v' \rangle_G = \langle m_{v,w}^\pi, m_{v',w'}^\rho \rangle = \int_G \langle \pi(x)v, w \rangle_G \overline{\langle \rho(x)v', w' \rangle_G} d\mu(x) \quad (3.1)$$

for every $v, v' \in V$. Note that due to right-invariance

$$\begin{aligned} \langle I_{w,w'}(\pi(g)v), v' \rangle_G &= \int_G \langle \pi(x)v, w \rangle_G \overline{\langle \rho(xg^{-1})v', w' \rangle_G} d\mu(x) \\ &= \langle I_{w,w'}v, \rho(g^{-1})v' \rangle_G = \langle \rho(g)(I_{w,w'}v), v' \rangle_G \end{aligned}$$

for every $g \in G$. From this it follows that $I_{w,w'}$ is intertwining. Thus if $\pi \not\cong \rho$ then, by Proposition 2.0.8, we must have $I_{w,w'} = 0$. From (3.1) it follows that $C(G)_\pi \perp C(G)_\rho$.

We turn to assertion (ii). On account of the above, we have intertwining maps $J_{w,w'} : V_\pi \rightarrow V_\pi$ such that $\langle J_{w,w'} v, v' \rangle_G = \langle m_{v,w}, m_{v',w'} \rangle$ for all $v, w, v', w' \in V_\pi$. On the strength of Schur's lemma (see Lemma 2.0.10), we have $J_{w,w'} = \lambda \text{id}_{V_\pi}$ for some $\lambda \in \mathbb{C}$. Set $n := \dim(V_\pi)$ and fix an orthonormal basis v_1, \dots, v_n of V_π . One now observes that

$$\text{tr}(J_{w,w'}) = \sum_{i=1}^n \langle J_{w,w'} v_i, v_i \rangle_G = n\lambda = \int_G \sum_{i=1}^n \langle \pi(x)v_i, w \rangle_G \overline{\langle \pi(x)v_i, w' \rangle_G} d\mu(x)$$

and that $\sum_{i=1}^n \langle \pi(x)v_i, w \rangle_G \overline{\langle \pi(x)v_i, w' \rangle_G} = \langle \pi(x^{-1})w', \pi(x^{-1})w \rangle_G = \overline{\langle w, w' \rangle_G}$ for all $x \in G$. It follows that $\lambda = \overline{\langle w, w' \rangle_G} / n$. As $\langle m_{v,w}, m_{v',w'} \rangle = \langle J_{w,w'}v, v' \rangle_G = \lambda \langle v, v' \rangle_G$, the result follows. \square

Corollary 3.1.7. *Let (π, V) an irreducible finite-dimensional representation of G . Then*

$$\sqrt{\dim(V)} T_\pi : \text{End}(V) \longrightarrow C(G)_\pi$$

is a unitary operator (here $\text{End}(V)$ is endowed with the Hilbert-Schmidt norm).

Proof. Indeed, let $A \in \text{End}(V)$. Fix a orthonormal basis v_1, \dots, v_n of V . As $T_\pi A = \sum_{i=1}^n m_{Av_i, v_i}$, we obtain from the Schur orthogonality relations that

$$\|T_\pi A\|_2^2 = \frac{1}{\dim(V)} \sum_{i,j=1}^n \langle m_{Av_i, v_i}, m_{Av_j, v_j} \rangle = \frac{1}{\dim(V)} \sum_{i=1}^n \langle Av_i, Av_i \rangle_G = \frac{1}{\dim(V)} \|A\|_{\text{HS}}^2.$$

We already showed that T_π is surjective, thus the desired result follows. \square

Definition 3.1.8. Let (π, V) be a finite-dimensional representation of G . The map

$$\chi_\pi : G \longrightarrow \mathbb{C} : x \mapsto \text{tr}(\pi(x))$$

is called the *character* of π .

Let χ, ρ be two finite-dimensional representations. It readily follows from the definition that $\chi_\pi = \chi_\rho$ whenever $\pi \cong \rho$. We will prove a stronger result below. In the rest of the text we say that a function $f : G \rightarrow \mathbb{C}$ is *conjugation invariant* or a *class function* if for all $x, y \in G$ we have $f(yxy^{-1}) = f(x)$.

Proposition 3.1.9. *Let π be an irreducible finite-dimensional representation of G . Suppose that $f \in C(G)_\pi$. Then f is conjugation invariant and we have $\|f\|_2 = 1$ if and only if $f \in S^1 \chi_\pi$.*

Proof. It is readily verified that χ_π is conjugation invariant. Furthermore, since $\chi_\pi = T_\pi \text{id}_V$, we obtain from Corollary 3.1.7 that $\|\chi_\pi\|_2 = 1$. Thus it follows that f is conjugation invariant and $\|f\|_2 = 1$ whenever $f \in S^1 \chi_\pi$.

Conversely, assume that f is conjugation invariant and $\|f\|_2 = 1$. On account of Corollary 3.1.7 we get $f = T_\pi A$ for some $A \in \text{End}(V)$ with $\|A\|_{\text{HS}}^2 = \dim(V)$. Using the conjugation invariance of f , we deduce that $T_\pi(\pi(g^{-1})A\pi(g)) = T_\pi A$ for all $g \in G$. Hence $A \in B_G(V)$. Schur's lemma now implies that $A = \lambda \text{id}_V$ for some $\lambda \in \mathbb{C}$. Thus we obtain $\|A\|_{\text{HS}}^2 = |\lambda|^2 \dim(V) = \dim(V)$, hence $\lambda \in S^1$. As $f = T_\pi A = \lambda T_\pi \text{id}_V = \lambda \chi_\pi$, we deduce that $f \in S^1 \chi_\pi$ as desired. \square

The following readily follows from the Schur orthogonality relations and the proposition above.

Proposition 3.1.10. *Let π, ρ be two irreducible finite-dimensional representations of G . Then*

$$\langle \chi_\pi, \chi_\rho \rangle = \begin{cases} 1 & \text{if } \pi \cong \rho, \\ 0 & \text{if } \pi \not\cong \rho. \end{cases}$$

Proposition 3.1.11. *Let π be a finite-dimensional representation of G . Then there exists pairwise non-equivalent, irreducible finite-dimensional representations π_1, \dots, π_n of G , such that we have a decomposition*

$$\pi \cong \bigoplus_{i=1}^n \pi_i^{\oplus m_i},$$

where $m_i = \langle \chi_\pi, \chi_{\pi_i} \rangle$. Here $\pi_i^{\oplus m_i}$ denotes the map $\pi_i \oplus \dots \oplus \pi_i$ (m_i -fold).

Proof. As π is a finite-dimensional representation, π completely decomposes into irreducible representations. Thus there exists irreducible representations ρ_1, \dots, ρ_m of G such that $\pi \cong \bigoplus_{j=1}^m \rho_j$. We select a subset of representations π_1, \dots, π_n of ρ_1, \dots, ρ_m such that $\pi_i \not\cong \pi_j$ if $i \neq j$ and such that every ρ_j is equivalent to one of the π_1, \dots, π_n . Consider the number

$$m_i := |\{1 \leq j \leq m \mid \rho_j \cong \pi_i\}|.$$

Then we have $\pi \cong \bigoplus_{j=1}^m \rho_j \cong \bigoplus_{i=1}^n \pi_i^{\oplus m_i}$. In light of Proposition 3.1.10, we obtain

$$\langle \chi_\pi, \chi_{\pi_i} \rangle = \langle \chi_{\bigoplus_{j=1}^m \rho_j}, \chi_{\pi_i} \rangle = \sum_{j=1}^m \langle \chi_{\rho_j}, \chi_{\pi_i} \rangle = |\{1 \leq j \leq m \mid \rho_j \cong \pi_i\}| = m_i$$

for all i . □

This proposition has the following two corollaries; we omit the proofs as the results readily follow from the above.

Corollary 3.1.12. *Let π, ρ be two finite-dimensional representations of G . Then $\chi_\pi = \chi_\rho$ if and only if $\pi \cong \rho$.*

Corollary 3.1.13. *Let π be a finite-dimensional representation of G . Then π is irreducible if and only if $\|\chi_\pi\|_2 = 1$.*

3.2 The Peter-Weyl theorem

In the following, an *enumeration of representation of G* is a set \hat{G} containing all distinct irreducible finite-dimensional representations of G up to equivalence. I.e., for every two distinct $\pi, \rho \in \hat{G}$ we have $\pi \not\cong \rho$, and \hat{G} is such that for every irreducible finite-dimensional representations ρ of G there exists a $\pi \in \hat{G}$ such that $\pi \cong \rho$. It is readily shown that there exists such a set. Indeed, one takes $R_n := \{\pi : G \rightarrow \mathbb{C}^n \mid \pi \text{ is an irreducible representation of } G\} / \cong$, and uses the axiom of choice to choose a system of representatives S_n of the equivalence classes in R_n . We then set $\hat{G} := \bigcup_{n \geq 1} S_n$. In the following discussion, we fix such an enumeration \hat{G} . With abuse of notation, we will sometimes write ‘let $(\pi, V) \in \hat{G}$ ’, meaning we consider an element $\pi \in \hat{G}$ with V being the associated finite-dimensional vector space on which the representation acts.

In the following the discussion, the representation

$$R \times L : G \times G \longrightarrow U(L^2(G)) : (g, h) \mapsto R_g L_h = L_h R_g$$

will be of interest. It is readily verified that this is a map is indeed a homomorphism. It is shown, similarly as Proposition 1.6.5, that this map is indeed continuous in the sense of Lemma 2.0.2 by using the following lemma.

Lemma 3.2.1. *For $f \in C(G)$, the map $G \times G \rightarrow C(G) : (g, h) \mapsto R_g L_h f$ is continuous.*

Proof. Consider $(g_0, h_0) \in G \times G$, and let $\varepsilon > 0$. In light of Lemma 1.1.7, there exists a neighbourhood U of 1 such that $|f(x) - f(y)| < \varepsilon$ whenever $y \in Ux$. Note that $x \mapsto h_0^{-1}x^2h_0$ is continuous, hence we have a neighbourhood V of the identity such that $h_0^{-1}V^2h_0 \subset U$. Next, consider the continuous map $c : (x, y) \mapsto xyx^{-1}$. Since $G \times \{1\} \subset c^{-1}(V)$, there exists a neighbourhood V' of the identity such that $G \times V' = c^{-1}(V)$ on account of the tube lemma. It follows that $xV'x^{-1} \subset V$ for all $x \in G$. Now, consider $(g, h) \in V'g_0 \times V^{-1}h_0$, then for $x \in G$ we have

$$h^{-1}xg \in h_0^{-1}VxV'g_0 \subset h_0^{-1}V^2xg_0 \subset Uh_0^{-1}xg_0,$$

hence $|f(h^{-1}xg) - f(h_0^{-1}xg_0)| < \varepsilon$. Thus $\|R_g L_h f - R_{g_0} L_{h_0} f\|_\infty \leq \varepsilon$. \square

Definition 3.2.2. Let $W \subset L^2(G)$. If W is an invariant subspace for the R or L representation, we will call W , respectively, a *right-* or *left-invariant* subspace of $L^2(G)$. If W is left- and right-invariant we call W *bi-invariant*. Note that W is an invariant subspace for the representation $R \times L$ representation in this case.

Proposition 3.2.3. *Let $(\pi, V) \in \hat{G}$. Then the space $C(G)_\pi$ is bi-invariant. A fortiori, $C(G)_\pi$ is an irreducible subspace w.r.t. the $R \times L$ representation of $G \times G$.*

Proof. Bi-invariance of $C(G)_\pi$ is readily verified. Let δ denote the restriction of $R \times L$ to $C(G)_\pi$. In light of Corollary 3.1.13, it suffices to show $\|\chi_\delta\|_2 = 1$. Fix an orthonormal basis v_1, \dots, v_n of V . On account of Corollary 3.1.7, the functions $\{n^{1/2}m_{v_i, v_j}\}_{i,j=1, \dots, n}$ form an orthonormal basis of $C(G)_\pi$. Thus, for $x, y \in G$ we obtain

$$\chi_\delta(x, y) = n \sum_{i,j=1}^n \langle \delta(x, y) m_{v_i, v_j}, m_{v_i, v_j} \rangle = n \sum_{i,j=1}^n \langle m_{\pi(x)v_i, \pi(y)v_j}, m_{v_i, v_j} \rangle.$$

Here we used the fact that π is unitary. Using the Schur orthogonality relations, we get

$$\chi_\delta(x, y) = \left(\sum_{i=1}^n \langle \pi(x)v_i, v_i \rangle_G \right) \cdot \overline{\left(\sum_{j=1}^n \langle \pi(y)v_j, v_j \rangle_G \right)} = \chi_\pi(x) \overline{\chi_\pi(y)}.$$

Recall that the product measure $\mu \times \mu$ is again a Haar measure on $G \times G$ (see Corollary 1.4.9). In fact, this measure is the normalized Haar measure as $(\mu \times \mu)(G \times G) = \int_G \int_G 1 d\mu d\mu = 1$ on account of Fubini's theorem. Thus, again using Fubini, we obtain

$$\|\chi_\delta\|_2^2 = \int_{G \times G} |\chi_\pi(x)|^2 |\chi_\pi(y)|^2 d(\mu \times \mu)(x, y) = \|\chi_\pi\|_2^4 = 1. \quad \square$$

Theorem 3.2.4 (Peter-Weyl theorem). *The space $L^2(G)$ decomposes canonically as*

$$L^2(G) \cong \widehat{\bigoplus_{\pi \in \hat{G}} C(G)_\pi},$$

and each subspace $C(G)_\pi \subset L^2(G)$ is irreducible (w.r.t the $R \times L$ representation).

We will follow the approach of [Tao14] to prove this theorem. We first present three lemma's.

Lemma 3.2.5. *Suppose that $K \in \mathcal{L}^2(G)$ is conjugation invariant and $K(x) = \overline{K(x^{-1})}$ for all $x \in G$. Then the map*

$$T_K : L^2(G) \longrightarrow L^2(G) : [f] \mapsto [f * K]$$

is a self-adjoint Hilbert-Schmidt operator (see Section A.2) which intertwines with the left and right regular representation.

Proof. We first show that the map T_K is well-defined. Indeed, as $L^2(G) \subset L^1(G)$ we have $f * K \in \mathcal{L}^1(G)$. It remains to show that $f * K \in \mathcal{L}^2(G)$. This is readily verified, as

$$\|f * K\|_2^2 = \int_A \left| \int_G f(y) K(y^{-1}x) d\mu(y) \right|^2 d\mu(x) \leq \int_A \|f\|_2^2 \|\overline{L_x K}\|_2^2 d\mu(x) = \|f\|_2^2 \|K\|_2^2,$$

where A is some Borel set with $G \setminus A$ negligible. Here we used the Cauchy-Schwarz inequality.

Using the theorem of Fubini it is readily verified that T_K is indeed self-adjoint. It remains to show that T_K is a Hilbert-Schmidt operator. Indeed, let A be an orthonormal basis of $L^2(G)$ then

$$\begin{aligned} \|T_K\|_{\text{HS}}^2 &= \sum_{f \in A} \langle f * K, f * K \rangle = \sum_{f \in A} \int_G |\langle f, \overline{L_x K} \rangle|^2 d\mu(x) = \int_G \sum_{f \in A} |\langle f, \overline{L_x K} \rangle|^2 d\mu(x) \\ &= \int_G \|\overline{L_x K}\|^2 d\mu(x) = \|K\|_2^2 < \infty. \end{aligned}$$

Here we used the theorem of Fubini again.

We have in general that $L_g(f_1 * f_2) = L_g f_1 * f_2$ for all $f_1, f_2 \in \mathcal{L}^1(G)$ and $g \in G$. Thus follows that $L_g \circ T_K = T_K \circ L_g$. One readily verifies that T_K intertwines the right regular representation; this follows from the conjugation invariance of K . \square

Lemma 3.2.6. *Let U be an open neighbourhood of the identity of G . Then there exists a conjugation invariant Dirac function $\phi \in C^+(G)$ supported in U .*

Proof. Let V be a symmetric compact neighbourhood of the identity contained in U . The map $c : (x, y) \mapsto yxy^{-1}$ is continuous and $\{1\} \times G \subset c^{-1}(V)$, thus we find a neighbourhood V' of 1 such that $V' \times G \subset c^{-1}(V)$. It follows that $yV'y^{-1} \subset V$ for all $y \in G$. Let $f \in C^+(G)$ be such that $f(1) > 0$ and $\text{supp } f \subset V'$. Consider the map $\psi : G \rightarrow \mathbb{C}$ given by

$$\psi(x) := \int_G (f \circ c)(x, y) d\mu(y) = \int_G f(yxy^{-1}) d\mu(y).$$

One readily verifies that this map is continuous by using Lemma 1.3.1 and the fact that $f \circ c$ is continuous. Hence $\psi \in C^+(G)$ and as $\psi(1) = f(1)$, ψ does not vanish everywhere. Furthermore, it is readily verified that this ψ is conjugation invariant. Let $x \in G \setminus V$, then it follows that $x \notin yV'y^{-1}$ for all $y \in G$. Hence $yxy^{-1} \notin V'$ for all $y \in G$. As $\text{supp } f \subset V'$, this implies that $\psi(x) = 0$. Hence $\text{supp } \psi \subset V$ since V is closed.

Now consider the continuous map $h \in C^+(G)$ given by $h(x) := \psi(x) + \psi(x^{-1})$. This map does not vanish everywhere, and by symmetry of V we have $\text{supp } h \subset V \subset U$. Thus $\phi := 1/\mu(h)h$ is the desired Dirac function. \square

In the following discussion we denote

$$\mathcal{M}(G) := \bigoplus_{\pi \in \hat{G}} C(G)_\pi.$$

Lemma 3.2.7. *Every finite-dimensional right-invariant subspace V of $L^2(G)$ is contained in $\mathcal{M}(G)$.*

Proof. As $(L|_V, V)$ is a unitary finite-dimensional representation, $L|_V$ completely decomposes into a finite number of irreducible subrepresentations. Hence, we might as well assume that $(L|_V, V)$ is irreducible. Then there exists some $(\pi, V_\pi) \in \hat{G}$ such that $L|_V \cong \pi$. But this implies that there exists an injective bounded operator

$$\iota : V_\pi \hookrightarrow L^2(G),$$

whose image equals V and $L_g \circ \iota = \iota \circ \pi(g)$ for all $g \in G$.

Let U be an arbitrary open neighbourhood of the identity and suppose that $\phi \in C^+(G)$ is a conjugation invariant Dirac function supported in U . Let $v \in V_\pi$ and write $\iota(v) = [f]$ for some $f \in \mathcal{L}^2(G)$. It follows that

$$(f * \phi)(x) = \int_G f(y)\phi(y^{-1}x) d\mu(y) = \int_G f(yx)\phi(y) d\mu(y) = \langle R_x f, \phi \rangle.$$

for almost every $x \in G$. Hence

$$(f * \phi)(x) = \langle R_x(\iota(v)), \phi \rangle = \langle \iota(\pi(x)v), \phi \rangle = \langle \pi(x)v, \iota^* \phi \rangle_G = m_{v, \iota^* \phi}^\pi(x)$$

for almost every $x \in G$. Hence $[f * \phi] \in C(G)_\pi$. From Lemma 1.7.7 and the fact that $C(G)_\pi$ is finite-dimensional, we get $\iota(v) = [f] \in C(G)_\pi$ as desired. Since $\iota(V_\pi) = V$ the result follows. \square

Proof of Theorem 3.2.4. The latter part of the theorem has already been shown. We turn to the main content of the theorem. It suffices to show that $L^2(G) = \overline{\mathcal{M}(G)}$, as $\overline{\mathcal{M}(G)}$ is canonically unitarily isomorphic to $\bigoplus_{\pi \in \hat{G}} C(G)_\pi$ (this follows from Theorem 3.1.6). As $L^2(G) = \overline{\mathcal{M}(G)} \oplus \mathcal{M}(G)^\perp$, this amounts to showing that $\mathcal{M}(G)^\perp = 0$. Assume to the contrary that there exists a non-trivial $[f] \in L^2(G)$ orthogonal to $\mathcal{M}(G)$.

Let U be an arbitrary open neighbourhood of the identity and $\phi \in C^+(G)$ a conjugation invariant Dirac function supported in U . Consider $T_\phi : L^2(G) \rightarrow L^2(G)$ as in Lemma 3.2.5. Then T_ϕ is a self-adjoint compact operator (see Proposition A.2.1) which intertwines the right regular representation. On the strength of the spectral theorem (see Theorem A.1.2), we have an orthogonal decomposition

$$L^2(G) = \ker(T_\phi) \oplus \overline{\bigoplus_n E_n}$$

where each E_n is a finite-dimensional eigenspace of T_ϕ corresponding to an eigenvalue $\lambda_n \in \mathbb{C}^*$ (in fact λ_n is real as T_ϕ is self-adjoint). Note that each eigenspace E_n is right-invariant since T_ϕ intertwines the right regular representation. On account of Lemma 3.2.7, this implies that $E_n \subset \mathcal{M}(G)$, and hence $[f]$ is orthogonal to all eigenspaces E_n of T_ϕ . But this implies that $[f] \in \ker(T_\phi)$, hence $[f * \phi] = 0$. As this holds for every conjugation invariant Dirac function ϕ supported in an arbitrary open neighbourhood of 1, we get $[f] = 0$ on account of Lemma 1.7.7, a contradiction. \square

Using the theorem of Peter-Weyl, we can prove a generalization of Proposition 2.0.13 for compact groups. In the proof, the role of induction in the proof of Proposition 2.0.13 will be replaced by the Zorn's lemma.

Proposition 3.2.8. *Let (π, V) be a unitary representation of G . The following statements are true.*

- (i) *The space V contains an irreducible finite-dimensional subspace.*
- (ii) *A fortiori, π decomposes into irreducible finite-dimensional representations. I.e., there exists irreducible finite-dimensional representations $\{\pi_i\}_{i \in I}$ of G such that*

$$\pi \cong \widehat{\bigoplus_{i \in I} \pi_i}.$$

Proof. We first prove assertion (i). Fix a non-zero vector $v \in V$. Consider the operator $T : V \rightarrow L^2(G) : w \mapsto [m_{v,w}]$. Note that $Tv \neq 0$ and hence $T \neq 0$. Using Cauchy-Schwarz, one readily verifies that T is bounded and in particular, $\|T\| \leq \|v\|$. Furthermore, T is intertwining w.r.t. the representations (π, V) and $(L, L^2(G))$. On account of Peter-Weyl's theorem, there exists a $\rho \in \hat{G}$ such that $PT \neq 0$ where $P : L^2(G) \rightarrow C(G)_\rho$ denotes the projection. Note that again $PT \in B_G(V, C(G)_\rho)$ (with the L representation acting on $C(G)_\rho$). As $(L, C(G)_\rho)$ is a finite-dimensional representation, we have a decomposition $L|_{C(G)_\rho} \cong \bigoplus_{i=1}^n \xi_i$ into irreducible finite-dimensional representations (ξ_i, X_i) . Thus there exists some $1 \leq j \leq n$ such that for the projection $S : C(G)_\rho \rightarrow X_j$ we have $A := SPT \neq 0$. As $S \in B_G(C(G)_\rho, X_j)$, we conclude that $A \in B_G(V, X_j)$. Consider the subspace $W := \ker(A)^\perp \subset V$. Note that W is invariant on account of Lemma 2.0.7 and Proposition 2.0.5. Furthermore, $W \neq 0$ as $A \neq 0$. The restriction $A : W \rightarrow X_j$ is intertwining and injective. Thus $\text{im}(A) \neq 0$ is invariant, and hence $\text{im}(A) = X_j$. Thus A is an intertwining isomorphism and hence $\pi|_W$ is equivalent to the irreducible representation ξ_j . Hence W is an irreducible finite-dimensional subspace of V .

We turn to assertion (ii). On account of Proposition 2.0.15, it suffices to show that $V = \overline{\bigoplus_{W \in \mathcal{W}} W} = \bigoplus \mathcal{W}$, where \mathcal{W} is a collection of orthogonal irreducible finite-dimensional subspaces of V . Consider the set

$$P := \{\mathcal{W} \in \mathcal{P}(V) \mid \mathcal{W} \text{ consists of orthogonal irreducible finite-dimensional subspaces of } V\}.$$

Here $\mathcal{P}(V)$ denotes the power set of V . Note that (P, \subset) forms a poset. Furthermore, P is non-empty in light of assertion (i). Every non-empty chain $C \subset P$ has the upper bound $\bigcup C$ in P , hence Zorn's lemma asserts that P has a maximal element $\mathcal{W} \in P$. We claim that $\bigoplus \mathcal{W} = V$. Indeed, assume to the contrary that $W := (\bigoplus \mathcal{W})^\perp \neq 0$. Note that $\bigoplus \mathcal{W}$ is invariant, hence W is invariant (see Proposition 2.0.5). Applying assertion (i) to the unitary representation $(\pi|_W, W)$, we conclude that W contains an irreducible finite-dimensional representation; this contradicts the maximality of \mathcal{W} . \square

Corollary 3.2.9. *Every irreducible unitary representation of G is finite-dimensional.*

3.3 Corollary: The Gleason-Yamabe theorem

Definition 3.3.1. In the following, a *linear group* will be a topological group which is isomorphic (isomorphic as topological groups, i.e. homeomorphic and group isomorphic) to a closed subgroup of $\text{GL}(n, \mathbb{R})$ for some $n \geq 0$.

It follows that linear groups can be given the structure of a Lie group (see [Ban10, Theorem 9.1]) which is compatible with the topology of the group. The next theorem asserts that the compact group G can be approximated by a linear groups in the following sense.

Theorem 3.3.2 (Gleason-Yamabe theorem for compact groups). *For every neighbourhood U of the identity there exists a closed normal subgroup H of G contained in U such that G/H is a linear group.*

Lemma 3.3.3. *Let V be a finite-dimensional vector space and $\pi : G \rightarrow \text{Iso}(V)$ be a homomorphism. Then π is a representation if and only if π is continuous.*

Proof. It is clear that π is a representation whenever π is continuous. This readily follows from Lemma 2.0.2. Thus we show the converse implication. Suppose that π is a representation. Fix a inner product $\langle \cdot, \cdot \rangle$ on V and let v_1, \dots, v_n be an orthonormal basis for V . Then for $g, h \in G$ we have $\|\pi(g) - \pi(h)\| \leq \sum_{i=1}^n \|(\pi(g) - \pi(h))v_i\|$ as one readily verifies (use Cauchy-Schwarz). It follows directly from this inequality that π is continuous. \square

Proof of Theorem 3.3.2. For every $g \in G \setminus U$ we find a finite-dimensional bi-invariant subspace $V_g \subset L^2(G)$ such that L_g is not the identity on V_g . Indeed, if this were not the case it would follow from the theorem of Peter-Weyl that $L_g = \text{id}_{L^2(G)}$. But this leads to a contradiction, as one can find a $f \in C(G)$ with $f(1) > 0$ and $\text{supp } f \subset G \setminus \{g^{-1}\}$ and for this f , $L_g f(1) = f(g^{-1}) \neq f(1)$ hence $\|L_g f - f\|_2 \neq 0$ (see Proposition 1.4.6). Let π_g denote the restriction $L : G \rightarrow \text{Iso}(V_g)$. On account of Lemma 3.3.3, π_g is continuous. Hence $U_g := \pi_g^{-1}(\text{Iso}(V_g) \setminus \{\text{id}_{V_g}\})$ is an open neighbourhood of g on which π_g is not the identity operator.

Using compactness of $G \setminus U$, we find $g_1, \dots, g_n \in G \setminus U$ such that $G \setminus U \subset \bigcup_{i=1}^n U_{g_i}$. We have a composition

$$G \xrightarrow{\bigoplus_{i=1}^n \pi_{g_i}} \text{Iso} \left(\bigoplus_{i=1}^n V_{g_i} \right) \xrightarrow{\cong} \text{GL}(m, \mathbb{C}) \xrightarrow{\cong} \text{GL}(2m, \mathbb{R})$$

here the latter two maps are the natural linear isomorphisms (the latter being only \mathbb{R} -linear) and the integer m equals the sum of the dimensions of V_{g_1}, \dots, V_{g_n} . Let $\Phi : G \rightarrow \text{GL}(2m, \mathbb{R})$ denote this composition. Clearly, Φ is again continuous. Consider the normal subgroup $H := \ker(\Phi) = \ker(\bigoplus_{i=1}^n \pi_{g_i})$. As every π_{g_i} is not the identity on U_{g_i} , we have $H \subset U$. The universal properties of the quotient group and -topology (recall that G/H is again a topological group; see Proposition 1.1.4), now induces a continuous injective homomorphism $G/H \rightarrow \text{GL}(2m, \mathbb{R})$. This induces a continuous isomorphism of groups $G/H \rightarrow \Phi(G)$. As G/H is compact and $\Phi(G)$ Hausdorff, we deduce that $G/H \cong \Phi(G)$ as topological groups. The subgroup $\Phi(G) \subset \text{GL}(2m, \mathbb{R})$ is compact, thus G/H is a linear group. \square

Example 3.3.4. Consider the infinite torus

$$T^\omega := \prod_{n \in \mathbb{N}} S^1.$$

Endowed with the product topology, this is a compact group on account of Tikhonov's theorem. Let U be a neighbourhood of the identity. By definition of the product topology, we find a family $\{U_n\}_{n \in \mathbb{N}}$ of open neighbourhoods of the identity in S^1 such that $\prod U_n \subset U$ and $U_n = S^1$ for all

but finitely many indices $n \in \mathbb{N}$. Next, consider the normal subgroup $H := \prod H_n$ where $H_n = S^1$ whenever $U_n = S^1$ and H_n is trivial whenever $U_n \neq S^1$. Then $H \subset \prod U_n \subset U$ and

$$T^\omega/H \cong (S^1)^m$$

where m is the number of indices $n \in \mathbb{N}$ such that $H_n = \{1\}$.

The Gleason-Yamabe theorem has the following corollary.

Corollary 3.3.5. *If G has no small subgroups, i.e. there exists some neighbourhood of the identity which contains no subgroups but the trivial subgroup, then G is a linear group.*

Remark 3.3.6. We presented here a special case of the more general Gleason-Yamabe theorem for locally compact groups. This theorem is used to give an answer to Hilbert's fifth problem: is every locally euclidean group a Lie group? For a complete discussion of this problem, we refer to [Tao14].

Using the terminology of projective limits (for a definition of projective limits, see for instance [DE14, p. 42]), we can formulate the Gleason-Yamabe theorem as follows.

Corollary 3.3.7. *Let B be a basis of neighbourhoods of the identity. Then we have an isomorphism*

$$G \cong \varprojlim_{U \in B} G_U$$

where each G_U , $U \in B$, is a linear group (here B is endowed with \subset to form a poset).

3.4 Non-abelian Fourier analysis

The Peter-Weyl decomposition will enable us to define a (generalized) Fourier transform on the compact group G . We first define the integral over a family of linear maps. Consider the following scenario. Let (X, \mathcal{A}, ν) be a measure space and let V be some finite-dimensional complex vector space with inner product $\langle \cdot, \cdot \rangle$. Suppose that $L : X \rightarrow \text{End}(V)$ is such that $x \mapsto \langle L(x)v, w \rangle \in \mathcal{L}^1(\nu)$ for all $v, w \in V$. On account of Lemma 3.1.5, there exists a unique linear map $A \in \text{End}(V)$ such that $\langle Av, w \rangle = \int_X \langle L(x)v, w \rangle d\nu(x)$ for all $v, w \in V$. We will denote

$$\int_X L(x) d\nu(x) := A.$$

For $(\pi, V) \in \hat{G}$, we consider the map $S_\pi : C(G)_\pi \rightarrow \text{End}(V)$ given by

$$S_\pi f := \int_G f(x) \pi(x)^* d\mu(x) = \int_G f(x) \pi(x^{-1}) d\mu(x).$$

To simplify formulas in the following discussion, we will endow $\text{End}(V)$ with the *dilated Hilbert-Schmidt norm*

$$\|\cdot\|_{\text{DHS}} := \sqrt{\dim(V)} \|\cdot\|_{\text{HS}}.$$

This norm is induced by the inner product $\langle \cdot, \cdot \rangle_{\text{DHS}} := \dim(V) \langle \cdot, \cdot \rangle_{\text{HS}}$. It follows from Corollary 3.1.7 that the operator $\dim(V) T_\pi : \text{End}(V) \rightarrow C(G)_\pi$ is unitary with respect to the dilated Hilbert-Schmidt norm.

Lemma 3.4.1. *Let $(\pi, V) \in \hat{G}$. The map S_π is the inverse of the map $\dim(V)T_\pi$ (see Corollary 3.1.7). In particular, S_π is unitary (w.r.t. the dilated Hilbert-Schmidt norm).*

Proof. As $\dim(V)T_\pi$ is unitary, it suffices to show that $\dim(V)T_\pi^* = S_\pi$. Fix an orthonormal basis v_1, \dots, v_n of V . Let $A \in \text{End}(V)$ and $f \in C(G)_\pi$. We have

$$\begin{aligned} \langle S_\pi f, A \rangle_{\text{DHS}} &= \dim(V) \text{tr}((S_\pi f)A^*) = \dim(V) \sum_{i=1}^n \langle (S_\pi f)v_i, Av_i \rangle_G \\ &= \dim(V) \sum_{i=1}^n \int_G f(x) \langle \pi(x^{-1})v_i, Av_i \rangle_G d\mu(x) \\ &= \dim(V) \int_G f(x) \sum_{i=1}^n \overline{\langle \pi(x)Av_i, v_i \rangle_G} d\mu(x) = \langle f, \dim(V)T_\pi A \rangle. \end{aligned}$$

As this holds for every $A \in \text{End}(V)$ we deduce that $\dim(V)T_\pi^* = S_\pi$, as desired. \square

Let P_π denote the projection $L^2(G) \rightarrow C(G)_\pi$. The Peter-Weyl theorem asserts that the map

$$T := \prod_{\pi \in \hat{G}} P_\pi : L^2(G) \longrightarrow \bigoplus_{\pi \in \hat{G}} C(G)_\pi$$

is a unitary isomorphism (here we use the notation introduced in Section A.3). On account of Lemma 3.4.1, we have a unitary isomorphism $S := \bigoplus_{\pi \in \hat{G}} S_\pi : \bigoplus_{\pi \in \hat{G}} C(G)_\pi \rightarrow \bigoplus_{\pi \in \hat{G}} \text{End}(V_\pi)$. Composing these two maps, we obtain a unitary isomorphism

$$\mathcal{F} := ST : L^2(G) \longrightarrow \bigoplus_{\pi \in \hat{G}} \text{End}(V_\pi) : [f] \mapsto \hat{f}.$$

This is called the *Fourier transform*.

Theorem 3.4.2 (Plancherel theorem). *The Fourier transform $\mathcal{F} : L^2(G) \rightarrow \bigoplus_{\pi \in \hat{G}} \text{End}(V_\pi)$ is a unitary operator given by*

$$(\mathcal{F}[f])_\pi = \hat{f}_\pi = \int_G f(x) \pi(x^{-1}) d\mu(x). \quad ([f] \in L^2(G)) \quad (3.2)$$

The Fourier inversion formula is given by

$$(\mathcal{F}^{-1} \hat{f})(x) = \sum_{\pi \in \hat{G}} \dim(V_\pi) \text{tr}(\pi(x) \hat{f}_\pi) \quad (\hat{f} \in \bigoplus_{\pi \in \hat{G}} \text{End}(V_\pi)) \quad (3.3)$$

where the series converges in the L^2 -sense.

Proof. The first assertion has already been covered in the discussion above. Let $[f] \in L^2(G)$. On account of the Peter-Weyl theorem, we have $f = \sum_{\rho \in \hat{G}} P_\rho f$. Let $\pi \in \hat{G}$, and let $v, w \in V_\pi$ then we have

$$\int_G f(x) \langle \pi(x^{-1})v, w \rangle_G d\mu(x) = \langle f, m_{w,v}^\pi \rangle = \sum_{\rho \in \hat{G}} \langle P_\rho f, m_{w,v}^\pi \rangle.$$

In light of the Schur orthogonality relations, we deduce that

$$\int_G f(x) \langle \pi(x^{-1})v, w \rangle_G d\mu(x) = \langle P_\pi f, m_{w,v}^\pi \rangle = \langle (S_\pi P_\pi f)v, w \rangle_G = \langle (\mathcal{F}f)_\pi v, w \rangle_G.$$

As this holds for every $v, w \in V$, we conclude that (3.2) holds. Since the inverse of S is given by $\bigoplus_{\pi \in \hat{G}} \dim(V_\pi) T_\pi$, and the inverse of T is given by $g \mapsto \sum_{\pi \in \hat{G}} g_\pi$, we obtain

$$\hat{F}^{-1} \hat{f} = \sum_{\pi \in \hat{G}} \dim(V_\pi) T_\pi \hat{f}_\pi,$$

for every $\hat{f} \in \bigoplus_{\pi \in \hat{G}} \text{End}(V_\pi)$; this is the Fourier inversion formula (3.3). \square

Proposition 3.4.3. *Let $f, g \in \mathcal{L}^2(G)$ then*

$$(\widehat{f * g})_\pi = \hat{g}_\pi \hat{f}_\pi$$

for all $\pi \in \hat{G}$.

Proof. This is a straight forward calculation. Let $v, w \in V_\pi$. Then using Fubini and translation invariance, we obtain

$$\begin{aligned} \langle (\widehat{f * g})_\pi v, w \rangle_G &= \int_G (f * g)(x) \langle \pi(x^{-1})v, w \rangle_G d\mu(x) \\ &= \int_G \int_G f(y) g(y^{-1}x) \langle \pi(x^{-1})v, w \rangle_G d\mu(y) d\mu(x) \\ &= \int_G \int_G f(y) g(x) \langle \pi(x^{-1})\pi(y^{-1})v, w \rangle_G d\mu(x) d\mu(y) \\ &= \int_G f(y) \langle \pi(y^{-1})v, \hat{g}_\pi^* w \rangle_G d\mu(y) = \langle \hat{f}_\pi v, \hat{g}_\pi^* w \rangle_G = \langle \hat{g}_\pi \hat{f}_\pi v, w \rangle_G. \end{aligned}$$

As this holds for any $v, w \in V_\pi$, the result follows. \square

We now investigate the case when G is abelian. We will see shortly that the Plancherel theorem assumes a simpler form when the group is commutative. We denote the center of G by $Z(G)$. Recall that $Z(G)$ consists of the elements of G that commute with every group element of G . Endowed with the subspace topology, the normal subgroup $Z(G)$ of G is again a topological group.

Lemma 3.4.4. *The following statements are true.*

- (i) *Let (π, V) be an irreducible finite-dimensional representation of G , then for every $x \in Z(G)$ we have $\pi(x) = \dim(V)^{-1} \chi_\pi(x) \text{id}_V$. Furthermore, the map*

$$\delta : Z(G) \rightarrow S^1 : x \mapsto \dim(V)^{-1} \chi_\pi(x)$$

is a continuous homomorphism.

- (ii) *Suppose that G is abelian. Then every irreducible finite-dimensional representation of G is 1-dimensional, and we have a bijection*

$$\hat{G} \longrightarrow \{\delta : G \rightarrow S^1 \mid \delta \text{ is a continuous homomorphism}\} : \pi \mapsto \chi_\pi. \quad (3.4)$$

Proof. We start by proving assertion (i). As $x \in Z(G)$ we have $\pi(x)\pi(g) = \pi(xg) = \pi(gx) = \pi(g)\pi(x)$ for all $g \in G$. I.e., $\pi(x) \in B_G(V)$. Schur's lemma now implies that $\pi(x) = \lambda \text{id}_V$ for some $\lambda \in \mathbb{C}$. Taking the trace of both sides, we obtain $\chi_\pi(x) = \lambda \dim(V)$. Hence $\pi(x) = \dim(V)^{-1} \chi_\pi(x) \text{id}_V = \delta(x) \text{id}_V$. From this it follows that $\delta(1) = 1$ and $\delta(xy) = \delta(x)\delta(y)$ for all $x, y \in G$. We show that δ maps indeed into S^1 . Indeed, let $v \in V$ be of norm 1 then $\delta(x) = \langle \pi(x)v, v \rangle_G = \langle v, \pi(x^{-1})v \rangle_G = \overline{\delta(x^{-1})}$. Continuity of δ follows from continuity of $\chi_\pi \in C(G)_\pi$.

We turn to assertion (ii). We suppose that G is abelian. Let $(\pi, V) \in \hat{G}$. Assertion (i) then implies that $\pi(x) = \dim(V)^{-1} \chi_\pi(x) \text{id}_V$ for all $x \in Z(G) = G$. Thus every one dimensional subspace of V is invariant and as π is irreducible, this immediately implies that $\dim(V) = 1$. It follows from assertion (i) that $\chi_\pi : G \rightarrow S^1$ is a continuous homomorphism. Thus the map (3.4) is well-defined. Note that every continuous homomorphism $\delta : G \rightarrow S^1$ gives rise to a representation $\Delta : G \rightarrow \text{Iso}(\mathbb{C}) : x \mapsto \delta(x) \text{id}_\mathbb{C}$ for which $\chi_\Delta = \delta$ (we might as well assume that $\Delta \in \hat{G}$). Thus the mapping (3.4) is surjective. Injectivity of this map is a consequence of Corollary 3.1.12. \square

Remark 3.4.5. The set

$$\{\delta : G \rightarrow S^1 \mid \delta \text{ is a continuous homomorphism}\}$$

is also called the *Pontryagin dual* (or *dual group*) of the group G . This set can be given a group structure using pointwise multiplication as group operation. In fact, it can be made into a topological group using the compact-open topology. The dual group plays an important role in the representation theory of locally compact abelian group. For further reading, we refer to [DE14, Chapter 3].

Example 3.4.6. We can now readily verify all irreducible finite-dimensional representations of the n -dimensional torus

$$T^n := (S^1)^n.$$

By previous lemma, this is equivalent to determining the dual group of T^n . First, we determine the dual group of $(\mathbb{R}, +)$. Consider δ in the dual of \mathbb{R} . As $\delta(0) = 1$, we find some (small) constant $c > 0$ such that $\lambda := \int_0^c \delta(\xi) d\xi \neq 0$. Then $\lambda \delta(t) = \int_t^{c+t} \delta(\xi) d\xi = \int_0^{c+t} \delta(\xi) d\xi - \int_0^t \delta(\xi) d\xi$ for all $t \in \mathbb{R}$. Thus δ is differentiable and $\delta'(t) = (\delta(c+t) - \delta(t))/\lambda = (\delta(c) - 1)/\lambda \cdot \delta(t)$. It follows that $\delta(t) = e^{Ct}$ for all $t \in \mathbb{R}$, where $C = (\delta(c) - 1)/\lambda \in \mathbb{C}$. Note that C is purely imaginary. Conversely, every function $t \mapsto e^{iat}$ with $a \in \mathbb{R}$ is inside the dual group of \mathbb{R} . Hence we have a bijection

$$\mathbb{R} \longrightarrow \{\delta : \mathbb{R} \rightarrow S^1 \mid \delta \text{ is a continuous homomorphism}\} : a \mapsto (t \mapsto e^{iat}).$$

Consider the covering map $\exp : \mathbb{R} \rightarrow S^1 : t \mapsto e^{2\pi it}$. For δ in the dual of S^1 we then obtain a map $\tilde{\delta} := \delta \circ \exp$ in the dual of $(\mathbb{R}, +)$. Hence $\tilde{\delta}(t) = e^{iat}$ for all $t \in \mathbb{R}$ where $a \in \mathbb{R}$. As $\tilde{\delta}(1) = 1$, we deduce that $a = 2\pi k$ for some $k \in \mathbb{Z}$. It follows that $\delta(z) = z^k$ for all $z \in S^1$. Thus the dual of S^1 consists of all maps $z \mapsto z^k, k \in \mathbb{Z}$.

We now treat the general case. Consider the torus T^n . Let δ be in the dual of T^n . For $(z_1, \dots, z_n) \in T^n$ we then obtain

$$\delta(z_1, \dots, z_n) = \delta(z_1, 1, \dots, 1) \cdots \delta(1, \dots, 1, z_n) = \delta_1(z_1) \cdots \delta_n(z_n),$$

where δ_i denotes the map $S^1 \rightarrow S^1 : z \mapsto \delta(1, \dots, 1, z, 1, \dots, 1)$ (the z on the i -th place). Note that $\delta_i \in \widehat{S^1}$, hence $\delta_i(z) = z^{k_i}$ for some integer $k_i \in \mathbb{Z}$. We conclude that

$$\delta(z_1, \dots, z_n) = z_1^{k_1} \cdots z_n^{k_n}.$$

Conversely, every such map sits inside the dual of T^n .

Corollary 3.4.7 (Plancherel theorem for abelian compact groups). *Suppose that G is abelian. Then there exists a unitary isomorphism $\widehat{\bigoplus_{\pi \in \hat{G}} \text{End}(V_\pi)} \cong \ell^2(\hat{G})$. Under this isomorphism, the Fourier transform becomes a unitary isomorphism,*

$$\mathcal{F} : L^2(G) \longrightarrow \ell^2(\hat{G})$$

with the following Fourier formulas

$$\begin{aligned} (\mathcal{F}[f])_\pi &= \hat{f}_\pi = \langle f, \chi_\pi \rangle, \quad ([f] \in L^2(G)) \\ \mathcal{F}^{-1} \hat{f} &= \sum_{\pi \in \hat{G}} \hat{f}_\pi \chi_\pi. \quad (\hat{f} \in \ell^2(\hat{G})) \end{aligned}$$

Example 3.4.8. In light of Example 3.4.6 and (3.4), we have a bijection $\mathbb{Z}^n \rightarrow \widehat{T^n}$ mapping a tuple $(k_1, \dots, k_n) \in \mathbb{Z}^n$ to a representation with character $(z_1, \dots, z_n) \mapsto z_1^{k_1} \cdots z_n^{k_n}$. This bijection induces a unitary isomorphism $\ell^2(\widehat{T^n}) \cong \ell^2(\mathbb{Z}^n)$. Under this isomorphism, the Fourier transform is a unitary operator $\mathcal{F} : L^2(T^n) \rightarrow \ell^2(\mathbb{Z}^n)$ given by

$$\hat{f}_{(k_1, \dots, k_n)} = \int_{T^n} f(z_1, \dots, z_n) z_1^{-k_1} \cdots z_n^{-k_n} d\mu_{T^n}(z_1, \dots, z_n).$$

for $[f] \in L^2(T^n)$. Here μ_{T^n} denotes the normalized Haar measure on T^n . Using Corollary 1.4.9 and Example 1.4.11 we find an explicit Fourier formula,

$$\hat{f}_{(k_1, \dots, k_n)} = (2\pi)^{-n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(e^{it_1}, \dots, e^{it_n}) e^{-i \sum_{i=1}^n k_i t_i} dt_1 \cdots dt_n.$$

The inverse formula is given by,

$$(\mathcal{F}^{-1} \hat{f})(z_1, \dots, z_n) = \sum_{(k_1, \dots, k_n) \in \mathbb{Z}^n} \hat{f}_{(k_1, \dots, k_n)} z_1^{k_1} \cdots z_n^{k_n},$$

for $\hat{f} \in \ell^2(\mathbb{Z}^n)$. The series converges in the L^2 -sense.

3.5 Representations of SU(2)

We will now calculate all irreducible representations of the matrix group $\text{SU}(2) = \{A \in \text{U}(2) \mid \det(A) = 1\}$. It is readily verified that every matrix A in $\text{SU}(2)$ can be written as

$$A = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{C}$ are such that $|\alpha|^2 + |\beta|^2 = 1$. In particular, it now readily follows that $\text{SU}(2)$ is compact.

Consider the vector space $P_n \subset \mathbb{C}[X, Y]$ of homogeneous polynomials of degree n . We let A act on a polynomial $f \in P_n$, by setting $A \cdot f := f(A(X, Y)) = f(\alpha X - \bar{\beta}Y, \beta X + \bar{\alpha}Y)$. One readily verifies that $A \cdot f \in P_n$, $1 \cdot f = f$ and that $(AB) \cdot f = A \cdot (B \cdot f)$ for all $A, B \in \text{SU}(2)$. Thus we have defined an action of $\text{SU}(2)$ on P_n . This yields a representation π_n of $\text{SU}(2)$ on P_n , given by

$$\pi_n(A)f := A \cdot f.$$

It remains to show that this is continuous in the sense of Lemma 2.0.2. It can easily be calculated that the absolute values of the coefficients of the polynomials $A \cdot X^m Y^{n-m}$, $0 \leq m \leq n$ are bounded by $\|A\|^n$. Taking the ℓ^1 -norm on the coefficients of the polynomials of P_n as the norm for P_n , we then conclude that $A \mapsto A \cdot X^m Y^{n-m}$ is continuous. We will show that $\{\pi_n\}_{n \in \mathbb{N}}$ is an enumeration of representations of $\text{SU}(2)$. In the following, we will denote χ_n for the character of π_n .

Observe that there is a natural embedding of topological groups (i.e. a homomorphism of groups which is also a topological embedding)

$$\iota : S^1 \hookrightarrow \text{SU}(2) : z \mapsto \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}.$$

This embedding induces a restriction

$$r : C(\text{SU}(2))_{\text{class}} \longrightarrow C(S^1) : f \mapsto f \circ \iota.$$

Here $C(\text{SU}(2))_{\text{class}}$ denotes the set of continuous class functions (i.e. continuous conjugation invariant functions) on $\text{SU}(2)$.

Lemma 3.5.1. *The restriction r is an isometry (w.r.t. the sup-norms).*

Proof. Note that every element $A \in \text{SU}(2)$ is a conjugate of $\iota(z)$ for some $z \in S^1$. Indeed, as $A \in \text{SU}(2)$ we find eigenvectors $z, w \in S^1$ (by the spectral theorem, and the fact that A is unitary). It follows that $\det(A) = zw = 1$, thus $w = \bar{z}$. Taking some non-zero eigenvector (α, β) corresponding to the eigenvalue z with $|\alpha|^2 + |\beta|^2 = 1$, we verify that $A(-\bar{\beta}, \bar{\alpha}) = \bar{z}(-\bar{\beta}, \bar{\alpha})$. Taking these two eigenvectors as columns of U , we get a matrix $U \in \text{SU}(2)$ for which $A = U^{-1}\iota(z)U$.

It follows from the above that $\|f\|_\infty = \|f\|_{\iota(S^1)} = \|r(f)\|_\infty$ for every class function $f \in C(\text{SU}(2))_{\text{class}}$. Thus r is indeed an isometry. \square

We will use this map to prove the following.

Lemma 3.5.2. *The characters $\{\chi_n\}_{n \in \mathbb{N}}$ are dense in $C(\text{SU}(2))_{\text{class}}$.*

Recall the theorem of Stone-Weierstrass.

Theorem 3.5.3 (Stone-Weierstrass theorem). *Let X be a compact Hausdorff space. Suppose that \mathcal{A} is a point-separating, unital $*$ -subalgebra of $C(X)$. Then \mathcal{A} is dense in $C(X)$ (w.r.t. the sup-norm).*

Proof of Lemma 3.5.2. Let $\tilde{\chi}_n$ denote $r(\chi_n)$. On account of the theorem of Stone-Weierstrass and the fact that r is an isometry, it suffices to show that

$$\mathcal{A} := \text{span}\{\tilde{\chi}_n \mid n \in \mathbb{N}\}$$

is a point-seperating, unital $*$ -subalgebra of $C(S^1)$.

For $z \in S^1$ we have $\pi_n(\iota(z))(X^m Y^{n-m}) = (zX)^m (\bar{z}Y)^{n-m} = z^{2m-n} X^m Y^{n-m}$. As the polynomials $X^m Y^{n-m}$, $0 \leq m \leq n$, form a basis for P_n , we conclude that $\tilde{\chi}_n(z) = \text{tr}(\pi(\iota(z))) = \sum_{m=0}^n z^{2m-n}$ for all $n \in \mathbb{N}$. One now readily verifies that

$$\tilde{\chi}_{2n}(z) = 1 + \sum_{m=1}^n (z^{2m} + z^{-2m}), \quad \tilde{\chi}_{2n+1}(z) = \sum_{m=1}^{n+1} z^{2m-1} + z^{-(2m-1)}$$

for all n . It follows that

$$\mathcal{A} = \text{span}\{z \mapsto z^m + z^{-m} \mid m \in \mathbb{N}\}.$$

Using this description of \mathcal{A} , it is readily verified that \mathcal{A} is a point-seperating, unital $*$ -subalgebra of $C(S^1)$. \square

Proposition 3.5.4. *The set $\{\pi_n\}_{n \in \mathbb{N}}$ is an enumeration of representations of G .*

Proof. Suppose to the contrary that there exists an irreducible finite-dimensional representation ρ of G such that $\rho \not\cong \pi_n$ for all $n \in \mathbb{N}$. On account of Lemma 3.5.2, there exists a sequence of numbers $(n_k)_{k \in \mathbb{N}}$ such that $\chi_\rho = \lim_{k \rightarrow \infty} \chi_{n_k}$ (w.r.t. the sup-norm). As the inclusion $(C(G), \|\cdot\|_\infty) \hookrightarrow (L^2(G), \|\cdot\|_2)$ is bounded, it follows that $\chi_\rho = \lim_{k \rightarrow \infty} \chi_{n_k}$ w.r.t. the L^2 -norm. Hence $\|\chi_\rho\|_2^2 = \lim_{k \rightarrow \infty} \langle \chi_\rho, \chi_{n_k} \rangle = 0$ (here we used Proposition 3.1.10). This contradicts Proposition 3.1.9. \square

We are can now readily determine an enumerations of representations for $U(2)$.

Lemma 3.5.5. *The map*

$$\phi : S^1 \times \text{SU}(2) \longrightarrow U(2) : (z, A) \mapsto zA$$

is a surjective homomorphism (i.e. a continuous group homomorphism) with $\ker(\phi) = \{\pm 1\}$. Hence ϕ descends to an isomorphism $(S^1 \times \text{SU}(2))/\{\pm 1\} \rightarrow U(2)$. Thus ϕ is a quotient map.

Proof. It is readily verified that ϕ is well-defined (i.e. its maps into $U(2)$) and that it is continuous. We show surjectivity. Let $B \in U(2)$. Then $w := \det(B) \in S^1$ since B is unitary. Choosing a $z \in S^1$ such that $z^2 = w$, we obtain $\det(B/z) = \det(B)/w = 1$ hence $B/z \in \text{SU}(2)$. As $\phi(z, B/z) = B$, we conclude that ϕ is surjective. It remains to show that $\ker(\phi) = \{\pm 1\}$. Consider $(z, A) \in \ker(\phi)$. Then it follows that $A = \bar{z}1$. As $\det(A) = \bar{z}^2 = 1$, we deduce that $z = \pm 1$ hence $(z, A) = \pm(1, 1)$. The latter follows from the fact that ϕ descends to a continuous bijection $(S^1 \times \text{SU}(2))/\{\pm 1\} \rightarrow U(2)$, as $(S^1 \times \text{SU}(2))/\{\pm 1\}$ is compact and $U(2)$ is Hausdorff, the result follows. \square

For $(k, n) \in \mathbb{Z} \times \mathbb{N}$, we consider the following representation

$$\rho_{k,n} : S^1 \times \text{SU}(2) \longrightarrow \text{Iso}(P_n) : (z, A) \mapsto z^k \pi_n(A).$$

of $S^1 \times \text{SU}(2)$.

Proposition 3.5.6. *Let ϕ the quotient map defined in Lemma 3.5.5. Consider the set of numbers $S := \{(k, n) \in \mathbb{Z} \times \mathbb{N} \mid k + n \in 2\mathbb{Z}\}$. The representation $\rho_{k,n}$ factors through ϕ if and only if $(k, n) \in S$. A fortiori, the representations $\{\rho_{k,n}\}_{(k,n) \in S}$ descend to an enumeration $\{\tilde{\rho}_{k,n}\}_{(k,n) \in S}$ of representations of $U(2)$.*

Proof. The first assertion of the proposition is readily verified. We turn to the main content of the proposition. Let (π, V) be an irreducible finite-dimensional representation of $U(2)$. Denote the irreducible representation $\pi \circ \phi$ of $S^1 \times SU(2)$ by $\tilde{\pi}$. Since $S^1 \times \{1\}$ lies in the center of $S^1 \times SU(2)$, Lemma 3.4.4(i) implies that the map $\delta : S^1 \rightarrow S^1 : z \mapsto \dim(V)^{-1} \chi_{\tilde{\pi}}(z, 1) = \dim(V)^{-1} \chi_{\pi}(z1)$ sits inside the dual of S^1 . We now get from Example 3.4.6 that there exists an integer $k \in \mathbb{Z}$ such that $\delta(z) = z^k$ for all $z \in S^1$. Using Lemma 3.4.4(i) once again, we obtain

$$\tilde{\pi}(z, A) = \tilde{\pi}(z, 1)\tilde{\pi}(1, A) = \delta(z)\tilde{\pi}(1, A) = z^k \tilde{\pi}(1, A) = z^k \pi(A)$$

for all $z \in S^1$ and $A \in SU(2)$. From this it follows that $\pi|_{SU(2)} : A \mapsto \pi(A)$ is an irreducible representation of $SU(2)$ because $\tilde{\pi}$ is irreducible. Hence $\pi|_{SU(2)} \cong \pi_n$ for some $n \in \mathbb{N}$. Let $T : V \rightarrow P_n$ be the corresponding intertwining isomorphism. Then we conclude that $\pi(\phi(z, A)) = \tilde{\pi}(z, A) = z^k T^{-1} \pi_n(A) T = T \rho_{k,n}(z, A) T^{-1}$. Hence we have $(k, n) \in S$ and $\pi \cong \tilde{\rho}_{k,n}$. Thus $\{\tilde{\rho}_{k,n}\}_{(k,n) \in S}$ are all finite-dimensional irreducible representations of $U(2)$ up to equivalence. From the above we can also deduce that $\tilde{\rho}_{k,n} \not\cong \tilde{\rho}_{k',n'}$ whenever $(k, n) \neq (k', n') \in S$ (compare how the two characters acts on $S^1 \cdot 1 \subset U(2)$ and then compare the restrictions of the two representations to $SU(2)$). \square

A. Topics from functional analysis

This appendix contains some preliminary material on functional analysis. The theory presented in the first two sections of this appendix can be found in [DE14].

A.1 Compact operators

Recall that an linear map $T : X \rightarrow Y$ between two normed spaces X, Y over a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is called *compact operator* if the image $T(B(0; 1))$ is relatively compact in Y . In particular, this implies that $T(B(0; 1))$ is bounded, hence $T \in B(X, Y)$.

If $T : X \rightarrow Y$ is a linear map such that $\dim(\text{im } T) < \infty$, then we say that T is *finite rank*. As this implies that $\text{im } T$ is Banach space, this implies that T is a compact operator.

Lemma A.1.1. *Let H, H' be two Hilbert spaces. Suppose that $T : H \rightarrow H'$ is a bounded operator. Then the following are equivalent.*

- (i) *The operator T is compact.*
- (ii) *If $(e_n)_{n \in \mathbb{N}}$ is an orthonormal sequence in H , the image $(Te_n)_{n \in \mathbb{N}}$ has a convergent subsequence.*
- (iii) *If $(e_n)_{n \in \mathbb{N}}$ is an orthonormal sequence in H then $Te_n \rightarrow 0$ as $n \rightarrow \infty$.*
- (iv) *The operator T can be approximated by operator of finite rank. I.e., there exists a sequence $(F_n)_{n \in \mathbb{N}}$ of finite rank operators $H \rightarrow H'$ such that $F_n \rightarrow T$ as $n \rightarrow \infty$ (w.r.t. the operator norm $\|\cdot\|$).*

Proof. Note that (i) \Rightarrow (ii) follows immediately from the definition (first-countable compact spaces are in particular sequentially compact). We prove the implication (ii) \Rightarrow (iii). Let (e_n) be an orthonormal sequence in H . Assume to the contrary that $\|Te_n\|$ does not converge to 0 as $n \rightarrow \infty$. Then we extract a subsequence (Te_{n_k}) such that

$$\|Te_{n_k}\| \geq \varepsilon \tag{A.1}$$

for all $k \in \mathbb{N}$ for some $\varepsilon > 0$. By assumption, there exists a subsequence $(Te_{n_{k_i}})$ such that $Te_{n_{k_i}} \rightarrow v$ as $i \rightarrow \infty$ for some $v \in H'$. To simplify notation, set $v_i := e_{n_{k_i}}$. As the v_i 's are orthonormal, we have $\sum_{i=0}^{\infty} |\langle T^*v, v_i \rangle|^2 \leq \|T^*v\|^2 < \infty$ hence in particular $\langle T^*v, v_i \rangle \rightarrow 0$ as $i \rightarrow \infty$. Thus there exists some index $i \in \mathbb{N}$ such that $\|Tv_i - v\| < \varepsilon^2/(2\|T^*\| + 2)$ and $|\langle T^*v, v_i \rangle| < \varepsilon^2/2$. It follows that

$$\|Tv_i\|^2 \leq |\langle T^*(Tv_i - v), v_i \rangle| + |\langle T^*v, v_i \rangle| < \varepsilon^2,$$

which contradicts (A.1). This proves (ii) \Rightarrow (iii).

We turn to the implication (iii) \Rightarrow (iv). If T has finite rank, we are done. Hence assume that $\text{im } T$ has infinite dimension. We construct a orthonormal sequence (e_n) inductively. First, choose $e_0 \in \partial B(0; 1)$ such that $\|Te_0\| > \|T\|/2 = 1/2 \cdot \sup_{v \in \partial B(0; 1)} \|Tv\|$. Assume $\{e_0, \dots, e_n\}$ have been constructed. Denote P_n for the projection onto $V_n := \text{span}\{e_0, \dots, e_n\}$. Note that $T(\text{id}_H - P_n) \neq 0$. Indeed, otherwise we would have $\text{im } T = T(V_n \oplus V_n^\perp) = T(V_n)$, i.e. T has finite rank. Hence there exists a $v \in \partial B(0; 1)$ such that $\|T(\text{id}_H - P_n)v\| > \|T(\text{id}_H - P_n)\|/2$. Consider $w := (\text{id}_H - P_n)v$. Then $\|w\| \neq 0$. We now set $e_{n+1} := w/\|w\|$. It is readily verified that $\|w\| \geq 1$, hence $\|Te_{n+1}\| \geq \|Tw\| > \|T(\text{id}_H - P_n)\|/2$. Setting $F_n := TP_n$, we get our desired sequence of finite rank operators since

$$\|T - F_n\| = \|T(\text{id}_H - P_n)\| < 2\|Te_{n+1}\|,$$

and hence $\|T - F_n\| \rightarrow 0$ as $n \rightarrow \infty$ by assumption.

Finally we prove (iv) \Rightarrow (i). It suffices to show that for a bounded sequence (x_n) in H , the sequence (Tx_n) has a converging subsequence. Let (F_n) be a sequence of finite rank operators such that $F_n \rightarrow T$ as $n \rightarrow \infty$. Inductively, we can find a collection of sequences of subsequences $(x_{n_{i,k}})_{k \in \mathbb{N}}$ of (x_n) , $i \in \mathbb{N}$, such that $(F_i(x_{n_{i,k}}))$ converges and $(x_{n_{i+1,k}})$ is a subsequence of $(x_{n_{i,k}})$ for all i . We show that $(Tx_{n_{k,k}})_{k \in \mathbb{N}}$ is Cauchy. By completeness of H' , this implies that we have found a desired converging subsequence. Let $\varepsilon > 0$ and $i \in \mathbb{N}$ such that $\|T - F_i\| < \varepsilon$. Let $N \in \mathbb{N}$ be such that $\|F_i(x_{n_{i,k}} - x_{n_{i,j}})\|$ whenever $k, j \geq N$. Then for every $k, j \geq \max\{N, i\}$ we have

$$\begin{aligned} \|T(x_{n_{k,k}} - x_{n_{j,j}})\| &\leq \|T - F_i\|(\|x_{n_{k,k}}\| + \|x_{n_{j,j}}\|) + \|F_i(x_{n_{k,k}} - x_{n_{j,j}})\| \\ &< (\|x_{n_{k,k}}\| + \|x_{n_{j,j}}\| + 1)\varepsilon. \end{aligned}$$

As (x_n) is bounded, this completes the proof. \square

Recall the famous spectral theorem from functional analysis. Most introductory texts on functional analysis will contain a proof of this theorem. For instance, a proof may be found [Tao14, Theorem 1.4.11] or [DE14, Theorem 5.2.2].

Theorem A.1.2 (Spectral theorem). *Let T be a compact self-adjoint operator on a Hilbert space H . Then there exists a countable sequence of non-zero eigenvalues (λ_n) which tends to zero, yielding an orthogonal decomposition*

$$H = \ker(T) \oplus \overline{\bigoplus_n E_n}$$

where $E_n := \ker(T - \lambda_n \text{id}_H)$ is the eigenspace associated to the eigenvalue λ_n for all n . Furthermore, all eigenspaces E_n are finite dimensional.

A.2 Hilbert-Schmidt operators

Throughout this section, let H be a Hilbert space. For an orthonormal basis A of H , consider the map $\|\cdot\|_{\text{HS}, A} : \text{End}(H) \rightarrow [0, \infty]$ such that

$$\|T\|_{\text{HS}, A}^2 = \sum_{v \in A} \|Tv\|^2. \quad (T \in \text{End}(H))$$

We claim that this quantity does not depend on the chosen orthonormal basis. Let B be another orthonormal basis of H and $T \in \text{End}(H)$, then

$$\|T\|_{\text{HS},A}^2 = \sum_{v \in A} \langle Tv, Tv \rangle = \sum_{v \in A} \sum_{w \in B} \langle Tv, w \rangle \langle w, Tv \rangle = \sum_{w \in B} \sum_{v \in A} \langle Tv, w \rangle \langle w, Tv \rangle.$$

Here we used the theorem of Tonelli. It follows that

$$\|T\|_{\text{HS},A}^2 = \sum_{w \in B} \sum_{v \in A} \langle T^*w, v \rangle \langle v, T^*w \rangle = \sum_{w \in B} \|T^*w\|^2 = \|T^*\|_{\text{HS},B}^2.$$

It follows that $\|T\|_{\text{HS},A}^2 = \|T^*\|_{\text{HS},A}^2 = \|T^{**}\|_{\text{HS},B}^2 = \|T\|_{\text{HS},B}^2$. Hence, we denote

$$\|\cdot\|_{\text{HS}} := \|\cdot\|_{\text{HS},A}.$$

We say that T is a *Hilbert-Schmidt operator* whenever $\|T\|_{\text{HS}} < \infty$. Consider the subset of Hilbert-Schmidt operators

$$\text{HS}(H) := \{T \in \text{End}(H) \mid \|T\|_{\text{HS}} < \infty\}.$$

It is readily verified that this is a linear space and that $\|\cdot\|_{\text{HS}}$ restricted to this space is a norm (one can use the properties of the ℓ^2 -norm to show this).

Proposition A.2.1. *Every operator $T \in \text{HS}(H)$ is compact.*

Proof. On the strength of Lemma A.1.1, it suffices to show that for every orthonormal sequence $(e_n) \subset H$, we have $Te_n \rightarrow 0$ as $n \rightarrow \infty$. This is indeed the case. One can expand this orthonormal sequence to an orthonormal basis A of H . As

$$\sum_{i=0}^{\infty} \|Te_i\|^2 \leq \sum_{v \in A} \|Tv\|^2 = \|T\|_{\text{HS}}^2 < \infty,$$

this implies that $\|Te_i\| \rightarrow 0$ as $n \rightarrow \infty$, as desired. \square

A.3 Hilbert direct sums

Let I be an index set and $\{H_i\}_{i \in I}$ a collection of Hilbert spaces over a fixed field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then we can consider the (algebraic) direct sum $\bigoplus_{i \in I} H_i$. We endow this direct sum with the inner product

$$\langle v, w \rangle = \sum_{i \in I} \langle v_i, w_i \rangle. \quad (v, w \in \bigoplus_{i \in I} H_i)$$

This makes the direct sum an inner product space, as one readily checks. However, in general, this space might not be complete again. We will construct an explicit completion of this space.

Consider the set

$$H := \widehat{\bigoplus_{i \in I} H_i} := \left\{ v \in \prod_{i \in I} H_i \mid \sum_{i \in I} \|v_i\|^2 < \infty \right\}.$$

It is readily verified that this set is a linear space over \mathbb{K} . We now try to put an inner product on this space. For $v, w \in H$, we claim that $\sum_{i \in I} \langle v_i, w_i \rangle$ exists (i.e. we have unconditional convergence). Indeed, a fortiori, this sum converges absolutely as $\sum_{i \in I} |\langle v_i, w_i \rangle| \leq \sum_{i \in I} \|v_i\| \|w_i\| < \infty$. The latter follows from the fact that $(\|v_i\|)_{i \in I}, (\|w_i\|)_{i \in I} \in \ell^2(I)$.

It is now readily verified that

$$\langle v, w \rangle = \sum_{i \in I} \langle v_i, w_i \rangle, \quad (v, w \in H)$$

defines an inner product on H . Endowed with this inner product, $H = \widehat{\bigoplus_{i \in I} H_i}$ is called the *Hilbert direct sum* of $\{H_i\}_{i \in I}$. We claim that H is the completion of the algebraic direct sum $\bigoplus_{i \in I} H_i$.

It is clear that $\bigoplus_{i \in I} H_i$ is dense in H . Indeed, if $v \in H$ then $J := \{i \in I \mid v_i \neq 0\}$ is countable. If J is finite, then we are done. Otherwise, we choose a bijection $\mathbb{N} \rightarrow J : k \mapsto i_k$. For $n \in \mathbb{N}$, let v^n be the unique element of $\bigoplus_{i \in I} H_i$ such that $v_{i_k}^n = v_{i_k}$ for $0 \leq k \leq n$ and $v_i^n = 0$ for the other indices i . Then we deduce that $\|v - v^n\|^2 = \sum_{k=n+1}^{\infty} \|v_{i_k}\|^2 \rightarrow 0$ as $n \rightarrow \infty$.

It remains to show that H is complete. Let $(v^n)_{n \in \mathbb{N}}$ be a Cauchy sequence in H . It follows that for every $i \in I$, $(v_i^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in H_i . Hence we find a $v \in \prod_{i \in I} H_i$ such that $v_i^n \rightarrow v_i$ as $n \rightarrow \infty$ for every $i \in I$. Now, let $0 < \varepsilon < 1$. Then we find a $N \in \mathbb{N}$ such that $\|v^n - v^m\|^2 < \varepsilon$ whenever $n/m \geq N$. We claim that $\sum_{i \in I} \|v_i^n - v_i\|^2 \leq 4\varepsilon$ for $n \geq N$. Indeed, let $F \subset I$ be a finite subset. Choose a $m \geq N$ such that $\|v_i^m - v_i\| < \varepsilon/(|F| + 1)$ for all $i \in F$. We now obtain

$$\begin{aligned} \sum_{i \in F} \|v_i^n - v_i\|^2 &\leq \sum_{i \in F} (\|v_i^n - v_i^m\| + \|v_i^m - v_i\|)^2 \\ &\leq \|v^n - v^m\|^2 + 2 \sum_{i \in F} \|v_i^n - v_i^m\| \|v_i^m - v_i\| + \sum_{i \in F} \|v_i^m - v_i\|^2 \\ &< \varepsilon + 2 \sum_{i \in F} \varepsilon/(|F| + 1) + \sum_{i \in F} \varepsilon/(|F| + 1) \leq 4\varepsilon. \end{aligned}$$

As this holds for any finite subset $F \subset I$, we get $\sum_{i \in I} \|v_i^n - v_i\|^2 \leq 4\varepsilon$. Using this, one now readily checks that $v \in H$ and $v^n \rightarrow v$ as $n \rightarrow \infty$.

We finish this discussion of Hilbert direct sums by stating the following properties; these results are readily verified.

Proposition A.3.1. *Let V be a vector space. Suppose that for every $i \in I$ we have a linear map $T_i : V \rightarrow H_i$ such that $\sum_{i \in I} \|T_i v\|^2 < \infty$ for all $v \in V$. Then there exists a unique linear map*

$$\widehat{\prod_{i \in I} T_i} : V \rightarrow \widehat{\bigoplus_{i \in I} H_i}$$

such that the following diagram commutes for all $j \in I$.

$$\begin{array}{ccc} V & \xrightarrow{\widehat{\prod_{i \in I} T_i}} & \widehat{\bigoplus_{i \in I} H_i} \\ & \searrow T_j & \downarrow P_j \\ & & H_j \end{array}$$

Here P_j denotes the projection $\widehat{\bigoplus_{i \in I} H_i} \rightarrow H_j$.

Proposition A.3.2. *Let $\{V_i\}_{i \in I}$ be another collection of Hilbert spaces. Suppose that for every $i \in I$ we have a unitary operator $T_i : V_i \rightarrow H_i$. Then there exists a unique unitary operator*

$$\widehat{\bigoplus_{i \in I} T_i} : \widehat{\bigoplus_{i \in I} V_i} \rightarrow \widehat{\bigoplus_{i \in I} H_i}$$

such that the following diagram commutes for all $j \in I$.

$$\begin{array}{ccc} \widehat{\bigoplus_{i \in I} V_i} & \xrightarrow{\widehat{\bigoplus_{i \in I} T_i}} & \widehat{\bigoplus_{i \in I} H_i} \\ \downarrow S_j & & \downarrow P_j \\ V_j & \xrightarrow{T_j} & H_j \end{array}$$

Here P_j, S_j denote, respectively, the projections $\widehat{\bigoplus_{i \in I} H_i} \rightarrow H_j$ and $\widehat{\bigoplus_{i \in I} V_i} \rightarrow V_j$.

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