The Hodge Decomposition Theorem: An Alternative Approach

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Contents

Introduction

The notion of a manifold in mathematics is one of great importance. Intuitively, a manifold is a topological space that is locally topologically indistinguishable from the Euclidean space $\mathbb{R}^n$, but may be vastly different from a global perspective. In differential geometry one considers the so-called smooth manifolds, which is a class of manifolds that are locally the same as Euclidean space in a smooth way, allowing the smooth structure of $\mathbb{R}^n$ to carry over to the manifold. Usually, the manifolds of interest are or can be equipped with an additional structure, for example a Riemannian metric or a complex structure. Still, the global topology of a smooth manifold endowed with an additional structure is usually quite unrestrained. An important question can now be formulated: what effect does the existence of certain structure on a smooth manifold have on its topology? In many cases, an answer is not easily found, but the importance of finding such an answer can be easily understood: it can be crucial in the search of examples of a certain type of manifolds, or it can be a powerful tool for studying the topology of a particular manifold.

Some particular structures on smooth manifolds are the so-called complex structures, which arise naturally in many fields of both mathematics and physics. Intuitively, a complex structure restrains the local structure of a manifold by declaring the manifold to be locally indistinguishable from the *complex* Euclidean space $\mathbb{C}^n$. One may wonder whether imposing the existence of a complex structure on a manifold has any influence on the topology of the manifold. Unfortunately, the answer is not satisfactory. Apart from the obvious restriction on the dimension of the manifold, very little can be said about its global topological properties.

When a complex manifold is in addition endowed with a symplectic structure that is compatible with the complex structure in a certain way, we get the notion of Kähler manifolds. Although there are many manifolds that do not admit Kähler structures, this class is still large enough to contain important spaces. For example, all algebraic varieties, which are important objects in algebraic geometry, are Kähler, as well as all Riemann surfaces. Finally, if we also assume a Kähler manifold to be compact, a lot can be said about its topology.

One of the first restrictions on the topology of compact Kähler manifolds is due to Hodge in [9] (notice that this text was published already in 1941) and is a consequence of a theorem that nowadays is famously called the Hodge decomposition theorem on compact Kähler manifolds. It states that the de Rham cohomology groups of a compact Kähler manifold can be decomposed into the so-called Dolbeault cohomology groups. The former is actually a topological invariant, while the latter relates to the complex structure.

The heart of the proof of the Hodge decomposition theorem consists of two facts: an analytic result and a geometric one. The analytic result relates cohomology to so called harmonic forms, which are forms whose Laplacian vanish. The geometric result then relates the harmonics to the Kähler structure via the so-called *Kähler identities*. We now have arrived at a position to formulate the main topic of this thesis: we will prove the Kähler identities and discuss how the Hodge decomposition theorem is implied by these famous identities.

Aims

In most modern texts (e.g. [5, 10, 14, 16, 18]) the proof of the Kähler identities is more computational than conceptual and involves ad hoc computations with different operators relating to the Kähler structure that somehow magically act together in a nice way. However, more recently a new theory has been introduced by Nigel Hitchin and further developed by his students Marco Gualtieri and Gil Cavalcanti in [7] and [3], respectively. It is referred to as generalized complex geometry, and it is a generalization of both symplectic and complex geometry. Within its framework, the (generalized) Kähler identities on the so-called generalized Kähler manifolds take a more elegant form, and their proof is more natural. As the name suggests, the class of generalized Kähler manifolds contains the class of 'normal' Kähler manifolds in a natural way. Finally, we can formulate the main goal of this thesis: we aim to give a proof of the Kähler identities that is guided by insights from generalized complex geometry without making explicit use the general theory, so that it is self-contained. Put differently, we aim to construct an alternative proof that makes the following diagram 'commute'.

$$\begin{tikzcd}
\text{Generalized Kähler manifolds} \arrow[r, "\textit{Generalized proof}"] & \text{Generalized Kähler identities} \arrow[d] \\
\text{Kähler manifolds} \arrow[u, hook] \arrow[r, dotted, "\textit{Aim of this thesis}"] & \text{Kähler identities}
\end{tikzcd}$$

Additionally, this thesis serves as a brief introduction to the fields of complex geometry, symplectic geometry and finally Kähler geometry. It is assumed that the reader has knowledge of differentiable manifolds treated in an introductory course on this topic, as well as some familiarity with complex analysis, in particular holomorphic functions of one variable. Apart from this, it is completely self-contained.

Structure

The thesis consists of three chapters, all of about equal length. The first chapter serves as an introduction to complex geometry, while the second discusses the basics of symplectic geometry. The last sections of both of these chapters prepare the essential framework needed for our alternative approach. The reader may notice that these last sections are almost direct analogues of each other. This is no coincidence, for in generalized complex geometry, complex and symplectic structures are of the same kind. The third and final chapter deals with Kähler manifolds, and includes, in addition to an introduction to Kähler geometry, the proof of the (generalized) Kähler identities. The chapter concludes with a discussion of the Hodge decomposition theorem on compact Kähler manifolds.

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Chapter 1

Complex Geometry

The notion of a complex manifold is very similar to that of a smooth or analytic manifold: interchanging 'smooth' and 'holomorphic' in many of the definitions will be sufficient to obtain the correct definition in one category from the other. However, complex manifolds have some interesting properties that have no analogue in the real case. For example, there exist no compact complex submanifolds of $\mathbb{C}^n$ of positive dimension, whereas in the smooth case any smooth manifold can be embedded in some $\mathbb{R}^N$.

After defining complex manifolds from the position of holomorphic coordinates, we will introduce almost complex structures on real manifolds in Section 1.2. The almost complex structure gives rise to the ∂- and $\bar{\partial}$-operators, which we will study in more detail in Section 1.3. Finally we will decompose the complex-valued differential forms on an almost complex manifold into the eigenspaces of an action on forms induced by the almost complex structure. This action arises naturally in generalized complex geometry. Although we will not study any generalized complex geometry, it will be important in the description of many results in the coming chapters. Moreover, the decomposition is crucial for our main result of this thesis. It is for those reasons that a reader familiar with this area of geometry can skip the first three sections, but may be less acquainted with the last one.

Although there is lot of interesting theory on holomorphic functions and complex manifolds, we will only discuss the main features that we need to understand to prove the main result of this thesis. The interested reader can be referred to [10] and [18], the two texts on which the first three sections are based. If the reader wishes to learn more about generalized complex geometry, the standard reference is [7].

1.1 Complex manifolds

1.1.1 Holomorphic maps

Before we are able to define and work with the notion of complex manifolds, we first must understand what holomorphic maps are. We will start by discussing holomorphic functions of one variable, of which we will find a definition that we can generalize to more dimensions.

Recall from complex analysis that a function $f : U \to \mathbb{C}$, with U an open subset of $\mathbb{C}$, is called holomorphic if it is complex differentiable in any point in U, i.e. the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for any $z_0 \in U$, whose value will be denoted by $f'(z_0)$. Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we can translate this to the language of real functions. After the identification, a holomorphic f is real differentiable and its real derivative $df_{z_0} : \mathbb{R}^2 \to \mathbb{R}^2$ is given by the real Jacobian of f in matrix form. This derivative has to be the same as $f'(z_0)$ as a linear map, which means that $df_{z_0}(v) = f'(z_0)v$ for all $v \in \mathbb{R}^2 = \mathbb{C}$, where the left-hand side is just multiplication of complex numbers. In particular, we must have $df_{z_0}(iv) = idf_{z_0}(v)$ for all $v \in \mathbb{R}^2 = \mathbb{C}$, i.e. df_{z_0} is complex linear. Denoting $z = x + iy$ and $f = u + iv$ we can put the above in matrix form:

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Notice that the equations above are precisely the Cauchy-Riemann equations. In conclusion, f is holomorphic if and only if f is continuously differentiable in the real sense and its derivative commutes with multiplication by i, which just means that the derivative is complex linear. The latter definition is the one that we will generalize.

Definition 1.1. Let $m, n \in \mathbb{N}_0$ and let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open sets. We call a map $f : U \to V$ *holomorphic on* U if it is continuously differentiable in the real sense and at each point $z \in U$ the (real) derivative $df_z : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$ is complex linear. ▲

In this definition we make the identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ to make sense of the real derivative. Any map $f : U \to \mathbb{C}^m$ can be written in terms of its components $f = (f^1, \ldots, f^m)$ and the components can be decomposed into the real and imaginary parts, i.e. $f^j = u^j + iv^j$. When writing the complex coordinates $z = (z^1, \ldots, z^n)$ as $z^j = x^j + iy^j$, the condition that the real derivative of f commutes with i is equivalent to each of the components of f satisfying the Cauchy-Riemann equations with respect to each complex variable, i.e. $\partial_{x^j} u^k = \partial_{y^j} v^k$ and $\partial_{y^j} u^k = -\partial_{x^j} v^k$ for all j and k. If f is holomorphic on U, all the partial derivatives are continuous and we can conclude that all the components f^j are holomorphic in each variable z^i around any point in U. Conversely, if the components of f^j are all holomorphic to each variable z^i around any point in U, then all the partial derivatives must be continuous which implies that f is continuously (real-)differentiable on U. Since all the components satisfy the Cauchy-Riemann equations in each variable, the real derivative is complex linear and we conclude that f is holomorphic. Summarizing, a map of several complex variables is holomorphic if and only if its components are holomorphic functions with respect to each complex variable.

It should be clear that holomorphic functions are well-behaved in the sense that the composition of two holomorphic functions is again holomorphic.

The Cauchy-integral formula generalizes naturally to several dimensions. In the following, we let the polydisc centered at $z_0 \in \mathbb{C}^n$ be the set $B_\varepsilon(z_0) := \{z \in \mathbb{C}^n : |z^i - z_0^i| < \varepsilon^i\}$ with $\varepsilon = \{\varepsilon^1, \ldots, \varepsilon^n\}$.

Proposition 1.2. *Let $f : \overline{B_\varepsilon(z_0)} \to \mathbb{C}$ be a continuous function that is holomorphic on $B_\varepsilon(z_0)$. Then for $z \in B_\varepsilon(z_0)$ the following formula holds:*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\xi^i - z_0^i| = \varepsilon^i} \frac{f(\xi^1, \ldots, \xi^n)}{(\xi^1 - z^1) \cdots (\xi^n - z^n)} d\xi^1 \cdots d\xi^n.$$

Proof. The function f is by the discussion above holomorphic to every variable z^i around each point in $B_\epsilon(z_0)$. Applying the Cauchy integral formula for one variable iteratively yields the following result

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\xi^n - z_0^n| = \varepsilon^n} \cdots \int_{|\xi^1 - z_0^1| = \varepsilon^1} \frac{f(\xi^1, \ldots, \xi^n)}{(\xi^1 - z^1) \cdots (\xi^n - z^n)} \mathrm{d}\xi^1 \cdots \mathrm{d}\xi^n.$$

By continuity of f on $\overline{B_\varepsilon(z_0)}$ we can rewrite the repeated integral as one integral over the boundary, which gives the desired formula. □

A direct consequence of the Cauchy integral formula is that any holomorphic function $f : U \to \mathbb{C}$ can be written around any point $z_0 \in U$ as the convergent power series

$$f(z) = \sum_{k_1, \ldots, k_n = 0}^{\infty} a_{k_1, \ldots, k_n} (z^1 - z_0^1)^{k_1} \cdots (z^n - z_0^n)^{k_n}$$

with a formula for the coefficients given by

$$\begin{aligned} a_{k_1, \ldots, k_n} &= \frac{1}{k_1! \cdots k_n!} \frac{\partial^{k_1 + \cdots + k_n} f}{(\partial z^1)^{k_1} \cdots (\partial z^n)^{k_n}} \bigg|_{z = z_0} \\ &= \frac{1}{(2\pi i)^n} \int_{|\xi^i - z_0^i| = \varepsilon^i} \frac{f(\xi^1, \ldots, \xi^n)}{(\xi^1 - z^1)^{k_1 + 1} \cdots (\xi^n - z^n)^{k_n + 1}} \mathrm{d}\xi^1 \cdots \mathrm{d}\xi^n. \end{aligned}$$

In particular, any holomorphic map is smooth.

When $f : U \to V$ is a holomorphic map between two opens $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$, we call f a *biholomorphism* (or a *biholomorphic map*) if it is bijective and its inverse is also holomorphic. We now have developed the necessary theory of holomorphic maps to define complex structures on manifolds.

1.1.2 Definition

Recall that a topological manifold of dimension $m \in \mathbb{N}_0$ is a second-countable, Hausdorff topological space M which is locally homeomorphic to (some open subset of) $\mathbb{R}^m$. A coordinate chart (or just chart) for M is a pair (U, φ) where U is some open subset of M and $\varphi : U \to U'$ is a homeomorphism from U to some open subset $U' \subseteq \mathbb{R}^m$. An atlas for M is a collection of charts covering M.

Similar to the case of smooth manifolds, we want to define a holomorphic structure on the topological manifold in terms of a maximal holomorphic atlas on M. To make this precise, we introduce some terminology. Suppose that M is a $2n$-dimensional topological manifold. Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$, we call two charts (U, φ) and (V, ψ) *holomorphically compatible* if the transition map $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is a biholomorphism. A *holomorphic atlas* $\mathcal{A}$ on M is a collection of charts covering M such that any two charts in $\mathcal{A}$ are holomorphically compatible. A holomorphic atlas $\mathcal{A}$ called *maximal* if any chart that is compatible with all the charts in $\mathcal{A}$ is already in $\mathcal{A}$.

Definition 1.3. A *holomorphic structure* on a $2n$-dimensional topological manifold M is a maximal holomorphic atlas $\mathcal{A}$. We then call the pair $(M, \mathcal{A})$ a complex manifold of (complex) dimension n. ▲

When the context is clear, we will usually denote a complex manifold by M without referring to the atlas; the manifold comes automatically with an atlas and we will refer to charts in that atlas by holomorphic charts. Similarly, when we talk about a smooth manifold M, it implicitly comes with a smooth atlas and we will refer to its elements by smooth charts.

Usually, it is hard or even impossible to work with explicit maximal atlases. However, given any holomorphic atlas $\mathcal{A}$ on M, there is a unique maximal holomorphic atlas containing $\mathcal{A}$, and thus $\mathcal{A}$ determines a holomorphic structure on M. The proof of this is statement identical to the proof of the analogous statement in the smooth case, e.g. [11], proposition 1.17. It thus makes sense to call the pair $(M, \mathcal{A})$ a complex manifold for any holomorphic atlas $\mathcal{A}$.

The holomorphic structure on a complex manifold M can be used to push-forward and pullback maps on the manifold and maps on opens in $\mathbb{C}^n$. Since holomorphic charts should carry the structure of $\mathbb{C}^n$ to the manifold, it seems natural that a function $f : M \to \mathbb{C}^m$ is holomorphic if and only if $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}^m$ is holomorphic for each chart (U, φ) on M, as well as a similar statement for a function $f : \mathbb{C}^m \to M$. This leads to the following definition.

Definition 1.4. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be complex manifolds, a continuous map $f : M \to N$ is called *holomorphic* if for any two charts $(U, \varphi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ with $f(U) \subseteq V$ the function $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is holomorphic. The map f is called a *biholomorphism* if it is holomorphic, bijective and its inverse is also holomorphic. ▲

By this definition, holomorphic charts on M are actual biholomorphisms between the corresponding opens. Taking $N = \mathbb{C}$ (with its canonical atlas), we obtain the notion of complex-valued holomorphic maps on M, or simply *holomorphic functions* on M. If A is any subset of M, then we call a function on A holomorphic if it is the restriction of a holomorphic function defined on an open neighborhood of A. We denote by $\mathcal{O}(A)$ the set of holomorphic functions on A.

Now that we have discussed the basics of complex manifolds from the perspective of holomorphic charts, we will devote ourselves to examples for the rest of this section.

Example 1.5. Let U be an open subset of $\mathbb{C}^n$. We denote by Id_U the identity map on U. Then $\{\mathrm{Id}_U\}$ is a holomorphic atlas making $(U, \{\mathrm{Id}_U\})$ into a complex manifold. Holomorphic maps on $(U, \mathcal{A})$ are exactly the same as holomorphic maps on U as a subset of $\mathbb{C}^n$. More generally, any open subset U of a complex manifold $(M, \mathcal{A})$ is again a complex manifold of the same dimension. The holomorphic charts on U are induced by the holomorphic charts on M via restriction. As expected, holomorphic maps on U as a complex manifold are exactly the same as holomorphic maps on U as a subset of M. △

Example 1.6. The 2-*sphere* S^2 can be given a complex structure in the following way. Let $\mathbb{C}_1$ and $\mathbb{C}_2$ be two copies of $\mathbb{C}$ and let $f : \mathbb{C}_1 \setminus \{0\} \to \mathbb{C}_2 \setminus \{0\}$ be defined by $f(z) = z^{-1}$. Define an equivalence relation $\sim$ on $X = \mathbb{C}_1 \coprod \mathbb{C}_2$, the disjoint union of $\mathbb{C}_1$ and $\mathbb{C}_2$, by calling $z_1 \in \mathbb{C}_1$ equivalent to $z_2 \in \mathbb{C}_2$ if and only if $f(z_1) = z_2$. First of all, observe that S^2 is homeomorphic to $X/{\sim}$ as a topological space, as both are one-point compactifications of $\mathbb{C}$, and therefore $X/{\sim}$ is a topological manifold. Secondly, if we denote by $q : X \to S^2$ the quotient map, q restricted to $\mathbb{C}_{1/2}$ is actually a homeomorphism onto its image. Setting $U_1 = q(\mathbb{C}_1)$ and $U_2 = q(\mathbb{C}_2)$, we obtain two charts on S^2 given by $(U_1, (q\big|_{\mathbb{C}_1})^{-1})$ and $(U_2, (q\big|_{\mathbb{C}_2})^{-1})$. Because the transition map $(q\big|_{\mathbb{C}_2})^{-1} \circ (q\big|_{\mathbb{C}_1})$ is exactly the biholomorphism f, these charts are holomorphically compatible and thus define a holomorphic structure on S^2. The 2-sphere together with this complex structure is called the *Riemann sphere*. In general, a one-dimensional complex manifold is called a *Riemann surface*. △

Example 1.7. Let $\{w_1, \ldots, w_{2n}\}$ be (real)-linear independent vectors in $\mathbb{C}^n$ and let Γ denote the lattice over $\mathbb{Z}$ generated by these vectors, i.e. $\Gamma = \{j_1 w_1 + \cdots + j_{2n} w_{2n} : j_1, \ldots, j_{2n} \in \mathbb{Z}\}$. The quotient space $X = \mathbb{C}^n/\Gamma = \{z + \Gamma : \ z \in \mathbb{C}\}$ is called a *complex torus* and, denoting by $q : \mathbb{C}^n \to X$ the quotient map, can be given the structure of a complex manifold via q in a natural way. Around any point $z \in \mathbb{C}^n$ we find an open U_z around z such that $(U_z + (j_1 w_1 + \cdots + j_{2n} w_{2n}) \cap U_z = \emptyset$ whenever $j_1, \ldots, j_{2n}$ are not all equal to zero. The quotient map restricts to a homeomorphism on U_z, and thus, setting $V_{q(z)} = q(U_z)$, induces coordinates charts $\left(V_{q(z)}, (q\big|_{U_z})^{-1}\right)$ on X. These coordinate charts are easily seen to be holomorphically compatible, and thus induce a natural complex structure on X. Notice that X is diffeomorphic to the $2n$-torus $\mathbb{T}^{2n} = S^1 \times \overset{2n}{\cdots} \times S^1$.

However, not all complex tori are isomorphic as complex manifolds. The reason for this is that the lattice structure is encoded in the complex structure of the torus.

To see this, we restrict ourselves to the case $n = 1$. If $\varphi : \mathbb{C}/\Gamma_1 \to \mathbb{C}/\Gamma_2$ is a non-constant holomorphic map between two complex tori with lattices Γ_1 and Γ_2, then we necessarily have that $m\Gamma_1 \subset \Gamma_2$ for some nonzero $m \in \mathbb{C}$. In particular, there are complex tori of complex dimension one that are not isomorphic as complex manifolds. We will only give a brief sketch of how one can prove this claim. First, one lifts the map φ to a holomorphic map $\tilde{\varphi} : \mathbb{C} \to \mathbb{C}$. Next, one can prove that the derivative of $\tilde{\varphi}$ is periodic in Γ_1 and therefore constant by Liouville's theorem, so that $\tilde{\varphi}(z) = mz + b$ for some complex numbers m, b, where notably m is non-zero because $\tilde{\varphi}$ is not constant. Finally, as $\tilde{\varphi}$ is also a lift over the quotient by Γ_2, one can conclude that $m\Gamma_1 \subset \Gamma_2$. $\triangle$

Example 1.8. The *n-dimensional complex projective space* $\mathbb{CP}^n$ is defined as the space of 1-dimensional complex linear subspaces of $\mathbb{C}^{n+1}$. This can be constructed as a topological space as follows. We define an equivalence relation $\sim$ on $\mathbb{C}^{n+1} \setminus \{0\}$ by declaring two elements to be equivalent if and only if they both are element of the same 1-dimensional complex subspace. In other words, x is equivalent to y if and only if there exists a number $\lambda \in \mathbb{C}$ such that $\lambda x = y$. The space $\mathbb{CP}^n$ is then defined as the quotient $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\!\sim$ and we denote the quotient map by $q : \mathbb{C}^{n+1} \setminus \{0\}$. Endowed with this topology, $\mathbb{CP}^n$ is a second-countable Hausdorff space. We define the holomorphic structure as follows. For $i = 1, \ldots, n+1$, denote V_i as the open subset of $\mathbb{C}^n \setminus \{0\}$ where $z^i \neq 0$ and define $U_i := q(V_i)$. These are open subsets of $\mathbb{CP}^n$ because $q^{-1}(U_i) = V_i$ is open in $\mathbb{C}^n \setminus \{0\}$. Define $\tilde{\varphi}_i : V_i \to \mathbb{C}^n$ as

$$\tilde{\varphi}_i(z^1, \ldots, z^{n+1}) = \left(\frac{z^1}{z^i}, \ldots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \ldots, \frac{z^{n+1}}{z^i} \right).$$

First of all, z^i is nonzero on V^i, thus the formula makes sense. Secondly, it is obvious that $\tilde{\varphi}_i$ is holomorphic. Finally, a quick calculation verifies that $\tilde{\varphi}_i(z) = \tilde{\varphi}_i(w)$ if and only if $q(z) = q(w)$. Therefore, $\tilde{\varphi}_i$ descends to a (unique) continuous map $\varphi_i : U_i \to \mathbb{C}^n$ with the property that $\tilde{\varphi}_i = \varphi_i \circ q$. This is actually a homeomorphism since the inverse φ_i^{-1}, mapping $w \in \mathbb{C}^n$ to $q(w^1, \ldots, w^{i-1}, 1, w^i, \ldots, w^n)$, is continuous. Finally, we check that these maps are holomorphically compatible and therefore the charts (U_i, φ_i) define a holomorphic structure on $\mathbb{CP}^n$. Indeed, letting $i < j$, the transition maps are given by

$$\varphi_i \circ \varphi_j^{-1}(z^1, \ldots, z^n) = \left(\frac{z^1}{z^i}, \ldots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \cdots \frac{z^{j-1}}{z^i}, \frac{1}{z^i}, \frac{z^j}{z^i}, \ldots, \frac{z^n}{z^i} \right),$$

$$\varphi_j \circ \varphi_i^{-1}(z^1, \ldots, z^n) = \left(\frac{z^1}{z^j}, \ldots, \frac{z^{i-1}}{z^j}, \frac{1}{z^j}, \frac{z^i}{z^j}, \ldots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \ldots, \frac{z^n}{z^j} \right)$$

which are clearly holomorphic. Notice that $\mathbb{CP}^1$ is actually the Riemann sphere from Example 1.6. $\triangle$

Example 1.9. The *Hopf surface* H is a complex manifold of (complex) dimension 2 and will be an important counterexample in our discussion about Kähler manifolds. It can be constructed as follows. We consider 2-dimensional complex space without the origin $\mathbb{C}^2 \setminus \{0\}$ and we let $\mathbb{Z}$ act on $\mathbb{C}^2 \setminus \{0\}$ by multiplication by 2, i.e. given $z \in \mathbb{C}^2 \setminus \{0\}$

and $n \in \mathbb{Z}$ we have $n \cdot z = 2^n z$. The Hopf surface is then defined as $H = (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z}$. Let $q : \mathbb{C}^2 \setminus \{0\} \to H$ denote the quotient map. We will endow the Hopf surface with the structure of a complex manifold as follows. Around each $z \in \mathbb{C}^2 \setminus \{0\}$ we define an open around z by $U_z = \{w \in \mathbb{C}^2 \setminus \{0\} : \frac{3}{4}\|z\| < \|w\| < \frac{3}{2}\|z\|\}$. By construction, the quotient map is injective on U_z and therefore, setting $V_{q(z)} = q(U_z)$, we have coordinate charts $\left(V_{q(z)}, (q|_{U_z})^{-1}\right)$, which are easily checked to be holomorphically compatible. Moreover, these coordinate charts guarantee that H is second-countable and Hausdorff, making H into a complex manifold.

Interestingly, the Hopf surface H is diffeomorphic to the product of the 3-sphere and the 1-sphere $S^3 \times S^1$. To see this, we first define a diffeomorphism $\Phi : \mathbb{C}^2 \setminus \{0\} \to S^3 \times \mathbb{R}$ as $\Phi(z) = \left(\frac{z}{\|z\|}, \log\|z\|\right)$. The map Φ pushes the action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{0\}$ forward to a corresponding action on $S^3 \times \mathbb{R}$, given by $n \cdot (z, x) = (z, x + n \log 2)$ where $n \in \mathbb{Z}$ and $(z, x) \in S^3 \times \mathbb{R}$. We then notice that $S^3 \times \mathbb{R}$ modulo this action is exactly $S^3 \times S^1$, and the map Φ descends to a diffeomorphism $\Psi : H \to S^3 \times S^1$. △

1.1.3 Submanifolds

For any $k \leq n$, there is a standard embedding of $\mathbb{C}^k$ into $\mathbb{C}^n$ which simply sends $(z^1, \ldots, z^k)$ to $(z^1, \ldots, z^k, 0, \ldots, 0)$ in $\mathbb{C}^n$. Therefore, we can regard $\mathbb{C}^k$ as a subset of $\mathbb{C}^n$ via this embedding.

Definition 1.10. A subset N of a complex n-manifold M is called a *complex submanifold* of M if at each point $p \in N$ there exists a holomorphic chart (U, φ) on M around p such that $\varphi(N \cap U) = \varphi(U) \cap \mathbb{C}^k \subset \mathbb{C}^n$ for some $0 \leq k \leq n$. The integer $n - k$ is called the (complex) *codimension* of N. The chart (U, φ) is called a *flat chart* for S. ▲

Notice that such charts on M induce charts on N by composing them with the projection of $\mathbb{C}^n$ onto $\mathbb{C}^k$. The notion of submanifolds gives rise to the notion of a *holomorphic embedding*, which is a map $f : N \to M$ between two complex manifolds N and M such that the image of f is a submanifold of M and $f : N \to f(N)$ is a biholomorphism.

Up until this point, any definition related to complex manifolds we encountered has been a direct analogue of a definition in the smooth case. The next results will show that their behaviour is very different. We leave it to the interested reader to deduce that the following statement have no smooth analogue.

Theorem 1.11. *Let M be a compact and connected complex manifold and $f \in \mathcal{O}(M)$, then f is constant.*

Proof. Since $|f|$ is continuous and M is compact, $|f|$ attains its maximum at some point $p_0 \in M$. Define the subset $S = \{p \in M : f(p) = f(p_0)\}$. Clearly, S is closed. We will show that S is open. Let $p \in S$ and pick a holomorphic chart (U, φ) around p such that

$\varphi(p)$=0. Denote $\tilde{f} = f \circ \varphi^{-1}$. Find an open ball $B \subset \varphi(U)$ around zero and let $z \in B$. Then define $g(\lambda) := \tilde{f}(\lambda z)$ whenever λz is in B. Clearly, g is a holomorphic function of one variable which attains its maximum modulus at $\lambda = 0$. By the maximum modulus principle, g must be constant. In particular we have $g(0) = g(1 \cdot z) = \tilde{f}(z)$. This proves that $\tilde{f}$ is constant on B and thus that f is constant in some open neighborhood of p. We conclude that S must be open. As S is non-empty, connectedness of M implies that $S = M$ and therefore that f is constant. □

Corollary 1.12. *There are no compact complex submanifolds of* $\mathbb{C}^n$ *of positive dimension.*

Proof. Denote by $\pi^j : (z^1, \ldots, z^n) \mapsto z^j$ the projection onto the j-th coordinate. If N is a compact complex submanifold of $\mathbb{C}^n$, then each of the coordinate projections π^j must be constant when restricted to a connected component of N, which is enough to prove that N has dimension zero. □

1.1.4 Holomorphic vector bundles

Vector bundles play an important role in the study of manifolds. Basically, they are a generalization of linear algebra on manifolds, and allow us to apply of its machinery on manifolds. Moreover, they provide to us a way to carry over Euclidean analysis to analysis on manifolds. In this section we will define the notion of holomorphic vector bundles and give some examples, of which the holomorphic tangent bundle may be the most important one. Most (if not all) material treated in this section is completely analogous to smooth vector bundles. Therefore, many details are omitted.

Definition 1.13. Let M be a complex manifold. A *holomorphic vector bundle* of rank k is a complex manifold E together with a holomorphic map $\pi : E \to M$ satisfying the following conditions:

- For each $p \in M$, the fibre $E_p := \pi^{-1}(p)$ is a k-dimensional complex vector space.
- For each $p_0 \in M$, there exists an open neighborhood of U of p_0 and a biholomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ with the properties that $\pi_U \circ \psi = \pi$ (here $\pi_U : U \times \mathbb{C}^k \to U$ is the projection onto U) and that the induced map $\psi^p : E_p \to \mathbb{C}^k$ is complex linear. The pair (U, ψ) is called a *local (holomorphic) trivialization.* ▲

It is important to keep in mind that a holomorphic vector bundle is different from a complex vector bundle, as by the latter we mean a vector bundle over a smooth manifold whose fibres are complex vector spaces. As a side note, complex vector bundles actually will play a larger role than holomorphic vector bundles throughout this thesis, as complex structures over real bundles, defined and studied in detail in the next section, are an essential ingredient of Kähler geometry.

Any two local trivializations (U_α, ψ_α) and (U_β, ψ_β) of a holomorphic k-vector bundle $E \xrightarrow{\pi} M$ define a holomorphic map $g_{\alpha\beta} : U_{\alpha\beta} \to GL(k, \mathbb{C})$ where $U_{\alpha\beta} = U_\alpha \cap U_\beta$, called the *transition function* via:

$$g_{\alpha\beta}(p) = \psi_\alpha^p \circ (\psi_\beta^p)^{-1}.$$

The transition functions satisfy the following cocycle data:

$$\begin{aligned} g_{\alpha\beta}(p) \circ g_{\beta\gamma}(p) \circ g_{\gamma\alpha}(p) &= \mathrm{id}_{\mathbb{C}^k} \quad \text{for every } p \in U_{\alpha\beta\gamma}; \\ g_{\alpha\alpha}(p) &= \mathrm{id}_{\mathbb{C}^k} \quad \text{for every } p \in U_\alpha. \end{aligned}$$

Conversely, given an open covering $\{U_\alpha\}$ of a complex manifold M together with a family of holomorphic maps $g_{\alpha\beta} : U_{\alpha\beta} \to GL(k, \mathbb{C})$ satisfying the cocycle data above, one can construct a holomorphic vector bundle of M that has local trivializations defined on opens U_α with $g_{\alpha\beta}$ as transition maps. This vector bundle is constructed in the exact same way as the in smooth case. We will recall how this is done. First, one lets

$$\tilde{E} := \coprod_\alpha U_\alpha \times \mathbb{C}^k$$

be the disjoint union of the product spaces $U_\alpha \times \mathbb{C}^k$ equipped with the natural disjoint union topology. One then defines an equivalence relation $\sim$ on $\tilde{E}$ by declaring $(p, z) \in U_\alpha \times \mathbb{C}^k$ to be equivalent to $(q, w) \in U_\beta \times \mathbb{C}^k$ if and only if $p = q$ and $z = g_{\alpha\beta}(p)w$. Finally, one defines the quotient $E := \tilde{E}/\sim$ (endowed with the quotient topology) and the map $\pi : E \to M$ sending the equivalence class of $(p, z) \in U_\alpha \times \mathbb{C}^k$ to $p \in U_\alpha$. The topological space E can be given a holomorphic structure by means of the maps $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^k$ sending the equivalence class of $(p, z) \in U_\alpha \times \mathbb{C}^k$ to (p, z) itself, which are precisely local trivializations inducing the transition maps we already had. This makes $E \xrightarrow{\pi} M$ a holomorphic vector bundle. The final step is that one lets the committed reader fill in the details.

Canonical constructions in linear algebra carry over to vector bundles. Some examples are given below.

Examples 1.14. Let $E \xrightarrow{\pi_E} M$ and $F \xrightarrow{\pi_F} M$ be two holomorphic vector bundles over M. Then one can define the following holomorphic vector bundles:

(i) The *direct sum* $E \oplus F$ is the vector bundle whose fibres are equal to $E_p \oplus F_p$ for $p \in M$;

(ii) The *tensor product* $E \otimes F$ is the vector bundle whose fibres are equal to $E_p \otimes F_p$ for $p \in M$;

(iii) The *k-th exterior power* $\wedge^k E$ is the vector bundle whose fibres are equal to $\wedge^k E_p$ for $p \in M$;

(iv) The *k-th symmetric power* $\Sigma^k E$ is the vector bundle whose fibres are equal to $\Sigma^k E_p$ for $p \in M$;

(v) The *dual bundle* E^* is the vector bundle whose fibres are equal to the dual of E_p for $p \in M$. △

To be able to distinguish between different vector bundles, we need to know when they are isomorphic. We have the following definition.

Definition 1.15. Given two holomorphic vector bundles $E_i \xrightarrow{\pi_i} M_i$, $i = 1, 2$. A *homomorphism of holomorphic vector bundles* is a holomorphic map $F : E_1 \to E_2$ with the following properties.

- The map F is a $\mathbb{C}$-linear map of vector spaces when restricted to each fibre.
- There exists a holomorphic map $f : M_1 \to M_2$ with $\pi_2 \circ F = f \circ \pi_1$, i.e. the following diagram commutes.

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "f"] & M_2 \end{tikzcd}$$

If both F and f are biholomorphisms, then F is called an *isomorphism of holomorphic vector bundles.*

When $M_1 = M_2 = M$ and f is the identity on M, then F is called a *(holomorphic) vector bundle homomorphism.* When in this case F is also a biholomorphism, then F is called a *(holomorphic) vector bundle isomorphism.* Equivalently, we have the following commutative diagram.

$$\begin{tikzcd} E_1 \arrow[rr, "F"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd}$$

▲

To conclude this section, we will treat some examples of holomorphic vector bundles.

Example 1.16. If M is a complex manifold, then $M \times \mathbb{C}^k$ together with the projection map $\pi : M \times \mathbb{C}^k \to M$ is a holomorphic vector bundle, called the *trivial bundle* of rank k over M. The identity map on $M \times \mathbb{C}^k$ is actually a global trivialization. In general, vector bundles that admit a global trivialization are called *trivializable.* Since the global trivialization is a vector bundle isomorphism, any trivializable rank k bundle over a manifold M is isomorphic to the trivial bundle of rank k over M. △

Example 1.17. The *(complex) tautological line bundle* is the holomorphic vector bundle over $\mathbb{CP}^n$ whose fibre at $p \in \mathbb{CP}^n$ is actually p itself as a one-dimensional complex subspace of $\mathbb{C}^{n+1}$, hence the name. Precisely, the tautological line bundle is the space $T = \{(\ell, v) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} |\ v \in \ell\}$ together with the map $\pi : T \to \mathbb{CP}^n$ sending $(q(z), v) \in T$ to $q(z) \in \mathbb{CP}^n$. Adopting the notation of Example 1.8, it can be given the structure of a holomorphic vector bundle via local trivializations $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ sending $(q(z^1, \dots, 1, \dots, z^{n+1}), \lambda(z^1, \dots, 1, \dots, z^{n+1})) \in$

T to $(q(z^1, \dots, 1, \dots, z^{n+1}), \lambda) \in U_i \times \mathbb{C}$. The transition functions are given by $g_{ij}(q(z^1, \dots, z^{n+1})) = \frac{z^i}{z^j}$. Notice that this is independent of the choice of the representative $(z^1, \dots, z^{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}$, as it should be. △

Example 1.18. Let $(U_\alpha, \varphi_\alpha)$ be a holomorphic atlas covering a complex n-manifold M. Let $J_{\mathbb{C}}(\varphi_\alpha \circ \varphi_\beta^{-1})(p)$ denote the (complex) Jacobian of the transition map $\varphi_\alpha \circ \varphi_\beta^{-1}$ at a point $p \in U_{\alpha\beta}$. Define $g_{\alpha\beta} : U_{\alpha\beta} \to GL(n, \mathbb{C})$ by $g_{\alpha\beta}(p) = J_{\mathbb{C}}(\varphi_\alpha \circ \varphi_\beta^{-1})(p)$. The *holomorphic tangent bundle* TM is the rank n holomorphic vector bundle defined by the transition maps $g_{\alpha\beta}$. Similar to the smooth situation, the fibre T_pM at a point $p \in M$ can be described as the set of (complex) linear derivations on the set of holomorphic functions at p (recall that a function is holomorphic at p if and only if it is holomorphic on an open neighborhood of p). In local coordinates, a basis of T_pM is given by the partial derivatives $\{\frac{\partial}{\partial z^1}\big|_p, \dots, \frac{\partial}{\partial z^n}\big|_p\}$.[†] The local trivializations inducing the transition maps are precisely the ones that send a vector in T_pM to its coordinates with respect to this basis. △

Example 1.19. The *complex differential* $df : TM \to TN$ of a holomorphic map $f : M \to N$ between two complex manifolds M and N is a vector bundle homomorphism. We will give a definition, but as it is identically defined as the smooth differential, we will omit many details. Given a point $p \in M$, the map f induces a complex linear homomorphism $df_p : T_pM \to T_{f(p)}N$ as follows: given a derivation $D \in T_pM$, the derivation $df_p(D) \in T_{f(p)}M$ is defined as $df_p(D)(g) = D(g \circ f)$ for any holomorphic function g defined on a neighborhood of $f(p)$. If (U, φ) and (V, ψ) are charts on M and N such that $f(U) \subset V$, and if we denote by Φ and Ψ the induced local trivializations by (U, φ) and (V, ψ), then we have for $p \in U$ the following commutative diagram:

$$
\begin{tikzcd}
\mathbb{C}^m & T_{\varphi(p)}\mathbb{C}^m \arrow[l, "\text{coord}"'] \arrow[r, "d(\psi\circ f\circ\varphi^{-1})_{\varphi(p)}"] & T_{\psi(f(p))}\mathbb{C}^n \arrow[r, "\text{coord}"] & \mathbb{C}^n \\
& T_pM \arrow[ul, "\Phi_p"] \arrow[u, "d\varphi_p"'] \arrow[r, "df_p"] & T_{f(p)}N \arrow[u, "d\psi_{f(p)}"] \arrow[ur, "\Psi_{f(p)}"'] &
\end{tikzcd}
$$

where $m = dim(M)$ and $n = dim(N)$ and 'coord' denotes the coordinate map with respect to the basis of partial derivatives. We will use the diagram to show that df is a holomorphic vector bundle homomorphism. First, we see that $d\varphi$ and $d\psi$ are both isomorphisms of holomorphic vector bundles bundle maps (on $TU = \pi_M^{-1}(U)$ and $TV = \pi_N^{-1}(V)$, respectively). Furthermore, the map $d(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}$ is in coordinates just the Jacobian matrix, which is holomorphic with respect to p, making $d(\psi \circ f \circ \varphi^{-1}) : TU' \to TV'$ into a homomorphism of holomorphic vector bundles. We conclude that df must be holomorphic. One readily verifies that $\pi_N \circ df = f \circ \pi_M$, so that df is a homomorphism of holomorphic vector bundles. △

[†] Technically, the partial derivatives form a basis of the tangent space $T_{\varphi(p)}\mathbb{C}^n$, and the correct basis would be the inverse image of the partials by the isomorphism $d\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{C}^n$ induced by φ. The details are so similar to the smooth case that we don't attend to them here.

1.2 Almost complex structures

Almost complex structures give rise to an alternative way of describing complex manifolds. A main advantage of (almost) complex structures to holomorphic structures is that they are more concrete, as they are algebraic objects. A holomorphic structure induces a natural complex structure, and it turns out that this complex structure provides a more fruitful framework for complex analysis on manifolds.

We first introduce complex structures on vector spaces, and then we will generalize it to manifolds.

1.2.1 Linear complex structures

Let V be an m-dimensional real vector space. A *complex structure* on V is an automorphism J of V such that $J^2 = -\mathrm{id}_V$. The vector space V can be made into a complex vector space V_J by defining complex scalar multiplication as

$$(a+bi)\cdot v := av + bJ(v) \quad \text{for } a,b\in\mathbb{R} \text{ and } v\in V.$$

If $\{e_1,\dots,e_n\}$ is a complex basis of V, then $\{e_1, J(e_1),\dots,e_n,J(e_n)\}$ is a basis of V over the reals. In particular, we see that any real vector space with a complex structure necessarily must be even dimensional. The complex structure induces a standard orientation on the real vector space by declaring the basis $\{e_1, J(e_1),\dots,e_n,J(e_n)\}$ to be positively oriented for any complex basis $\{e_1,\dots,e_n\}$ (notice that this is indeed independent of the choice of the complex basis).

Example 1.20. Any $2n$-dimensional real vector space V can be given a complex structure as follows. Pick a basis $\{v_1,w_1,\dots,v_n,w_n\}$ and define $J: V\to V$ by $J(v_i) = w_i$ and $J(w_i) = -v_i$. The matrix of J w.r.t. this basis consists of n blocks of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$ on the diagonal, which is the same as the matrix representation of multiplication by i on $\mathbb{C}^n$. In general, however, there is no canonical way to endow an even dimensional real vector space with a complex structure. △

Another description, which we will introduce now, of a vector space with a complex structure will be slightly more useful. The *complexification* of the real vector space V is the vector space $V_\mathbb{C} := V\otimes_\mathbb{R}\mathbb{C}$ where the tensor product is the real tensor product. One can think of $V_\mathbb{C}$ as an extension of V on which complex scalar multiplication is made possible by creating a new element 'iv' for $v\in V$. Alternatively, given a basis $\{v_1,\dots,v_m\}$ of V, one can think of the complexification of V as the complex linear span of this basis. Conjugation on $V_\mathbb{C}$ is defined by $\overline{v\otimes z} = v\otimes\bar{z}$. Given a complex structure J on V, one can extend J to $V_\mathbb{C}$ by $J(v\otimes z) = J(v)\otimes z$ (which is just the complex linear extension). The extension still satisfies $J^2 = -\mathrm{id}_{V_\mathbb{C}}$, and since $\mathbb{C}$ is algebraically closed, J is guaranteed to have eigenvalues $\pm i$ on $V_\mathbb{C}$. We denote the $+i$-eigenspace of J by $V^{1,0}$ and the $-i$-eigenspace by $V^{0,1}$. Observe that $V_\mathbb{C} = V^{1,0}\oplus V^{0,1}$ and that the eigenspaces are explicitly given by $V^{1,0} = \{v\otimes 1 - J(v)\otimes i : v\in V\}$ and $V^{0,1} = \{v\otimes 1 + J(v)\otimes i : v\in V\}$, hence $\overline{V^{1,0}} = V^{0,1}$. Notice that the vector space

V_J is isomorphic to $V^{1,0}$ as complex vector spaces, and from now on we will make this identification implicitly. As a final remark, notice that $V^{1,0}$ is real isomorphic to $V^{0,1}$ via conjugation.

The exterior algebras $\wedge^\bullet V^{1,0}$ and $\wedge^\bullet V^{0,1}$ inject into $\wedge^\bullet V_\mathbb{C}$ in a natural way. Define $\wedge^{p,q}V$ as the subspace of $\wedge^\bullet V_\mathbb{C}$ generated by elements in the form $v \wedge w$ with $v \in \wedge^p V^{1,0}$ and $w \in \wedge^q V^{0,1}$. This yields the following decomposition:

$$\wedge^k V_\mathbb{C} = \bigoplus_{p+q=k} \wedge^{p,q} V.$$

If $\{v_1, \ldots, v_n\}$ is a complex basis of $V^{1,0}$, then $\{\overline{v}_1, \ldots, \overline{v}_n\}$ is a basis of $V^{0,1}$ and any element $\omega \in \wedge^{p,q}V$ can be written uniquely in the form

$$\omega = \sideset{}{'}\sum_{\substack{|I|=p\\|J|=q}} a_{I,J} v_{i_1} \wedge \cdots \wedge v_{i_p} \wedge \overline{v}_{j_1} \wedge \cdots \wedge \overline{v}_{j_q}$$

where the prime indicates a sum over strictly increasing multi-indices I and J and the coefficients $a_{I,J}$ are complex numbers.[†] We now can see clearly that we have a natural identification $\wedge^{p,q}V = \wedge^p V^{1,0} \otimes_\mathbb{C} \wedge^q V^{0,1}$. Furthermore, we have $\overline{\wedge^{p,q}V} = \wedge^{q,p}V$.

The complex structure J also induces a complex structure J^* on the dual V^*, which gives rise to a decomposition of the complexified dual $(V^*)_\mathbb{C} = (V^*)^{1,0} \oplus (V^*)^{0,1}$. The space $(V^*)^{1,0}$ consists of all the complex linear maps that vanish on $V^{0,1}$ and therefore can be identified with $(V^{1,0})^*$ (similar for $(V^*)^{0,1}$). Therefore, it makes sense to denote $V_\mathbb{C}^*$ as both the complexification of the dual and the dual of the complexification.

1.2.2 Almost complex manifolds

The linear algebra above generalizes to smooth manifolds, starting with the following definition.

Definition 1.21. Let M be a smooth manifold. An *almost complex structure* on M is a smooth vector bundle homomorphism $J : TM \to TM$ such that $J^2 = -\mathrm{id}$. A pair (M, J) of a smooth manifold and an almost complex structure is called an *almost complex manifold.* ▲

The almost complex structure induces a complex structure on each fibre T_pM and therefore turns T_pM into a complex vector space. Hence, by the discussion above, the almost complex structure makes TM a smooth complex vector bundle, that is, a vector bundle with fibres isomorphic to $\mathbb{C}^n$ as a complex vector space.

[†]From now on, we will omit using the subscripts $|I| = p$ and $|J| = q$; the prime alone should suffice that it is a sum over strictly increasing multi-indices.

Given a smooth manifold M and an almost complex structure J we want to consider the complexified tangent bundle $TM_{\mathbb{C}} := TM \otimes_{\mathbb{R}} \mathbb{C}$, as we did in our discussion of the linear algebra above. The almost complex structure induces a pointwise decomposition $T_pM_{\mathbb{C}} = T_p^{1,0}M \oplus T_p^{0,1}M$ in the $\pm i$-eigenspaces of J_p. The $\pm i$-eigenbundles $T^{1,0}M$ and $T^{0,1}M$ formed this way are actually smooth bundles over M since the bundle maps $J \mp i : TM_{\mathbb{C}} \to TM_{\mathbb{C}}$ have constant rank and therefore their kernels are smooth vector bundles over M, yielding the decomposition $TM_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$. Similarly, for the (complexified) cotangent bundle we have $T^*M_{\mathbb{C}} = (T^{1,0})^*M \oplus (T^{0,1})^*M$. As we did earlier with vector spaces, we can form the bundles $\wedge^{p,q}T^*M = \wedge^p(T^{1,0})^*M \otimes_{\mathbb{C}} \wedge^q(T^{0,1})^*M$, so that we obtain the following decomposition:

$$\wedge^k T^*M_{\mathbb{C}} = \bigoplus_{p+q=k} \wedge^{p,q}T^*M.$$

The space of *complex-valued differential forms* of degree k, denoted by $\Omega^k(M;\mathbb{C})$, is the space of (smooth) sections of the bundle $\wedge^k T^*M_{\mathbb{C}}$. Similarly, the complex-valued differentiable forms of type (p,q) are elements of $\Omega^{p,q}(M)$, which is the space of sections of the bundle $\wedge^{p,q}T^*M$. The decomposition above naturally extends to a decomposition of the space of differential forms:

$$\Omega^k(M;\mathbb{C}) = \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

Example 1.22. As expected, complex manifolds admit a natural almost complex structure. Let M be a complex manifold and denote M_s as its underlying smooth manifold. If $z^1, \dots, z^n$ are complex local coordinates around a point $p \in M$, taking the real and imaginary parts of $z^j = x^j + iy^j$ gives us a set of real local coordinates around $p \in M_s$. Multiplication by i on T_pM induces a complex structure J on T_pM_s by

$$J\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \frac{\partial}{\partial y^j}\bigg|_p \quad \text{and} \quad J\left(\frac{\partial}{\partial y^j}\bigg|_p\right) = -\frac{\partial}{\partial x^j}\bigg|_p.$$

This map is actually well-defined, since any two bases of partial derivatives from two different charts are related by the Jacobian of the transition map, which is complex linear. Since the partials $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial y^j}$ form a smooth local frame of TM_s, one readily verifies that J is a smooth vector bundle homomorphism.

Applying the linear algebra above, the induced complex structure J gives rise to a decomposition of the complexified (smooth) tangent bundle $TM_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$ and given local coordinates $z^j = x^j + iy^j$, the sections $\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right)$ and $\frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right)$ form local frames of the bundles $T^{1,0}M$ and $T^{0,1}M$, respectively. Dualizing, we obtain an induced complex structure J^* on the cotangent bundle defined together with local frames $dz^j = dx^j + idy^j$ and $d\bar{z}^j = dx^j - idy^j$ for the bundles $(T^{1,0})^*M$ and $(T^{0,1})^*M$. Therefore, any form $\omega \in \Omega^{p,q}(M)$ can be locally written as

$$\omega = \sum\nolimits' f_{I,J} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$$

for some smooth functions $f_{I,J}$. △

Within the framework we just constructed, we are able to generalize Definition 1.1 to complex manifolds, so that we obtain a more elegant characterization of holomorphic maps in terms of complex structures, without any explicit reference to charts.

Proposition 1.23. *Let $f : M \to N$ be a smooth map between two complex manifolds M and N and denote their induced almost complex structures as I_M and I_N, respectively. Then f is holomorphic if and only if $df \circ I_M = I_N \circ df$.*

Proof. Let $f : M \to N$ be a smooth map and denote df as its real derivative. Then from the definitions it follows that df commuting with the induced almost complex structure is equivalent with df being complex linear in local coordinates. The observation that df being complex linear in local coordinates is equivalent with f being holomorphic concludes the proof. □

The exterior derivative $d : \Omega^\bullet(M) \to \Omega^\bullet(M)$ on an almost complex manifold (M, J) can be extended complex linearly to $\Omega^\bullet(M;\mathbb{C})$. In general, the exterior derivative of a form of type (k,l) can land anywhere in $\Omega^{k+l+1}(M;\mathbb{C})$, and thus may have many components with respect to the decomposition of $\Omega^{k+l+1}(M;\mathbb{C})$ into forms of type (p,q). However, in the case that the almost complex structure is induced by a holomorphic structure, the exterior derivative splits into only two components. We can see this by computing the exterior derivative of the form ω above directly:

$$\begin{aligned}
d\omega =& \sum_{k=1}^{n} \sum{}' \frac{\partial f_{I,J}}{\partial x^k} dx^k \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \\
&+ \sum_{k=1}^{n} \sum{}' \frac{\partial f_{I,J}}{\partial y^k} dy^k \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \\
=& \sum_{k=1}^{n} \sum{}' \left(\frac{\partial f_{I,J}}{\partial z^k} + \frac{\partial f_{I,J}}{\partial \bar{z}^k} \right) \cdot \frac{1}{2} \left(dz^k + d\bar{z}^k \right) \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \\
&+ \sum_{k=1}^{n} \sum{}' \left(\frac{\partial f_{I,J}}{\partial z^k} - \frac{\partial f_{I,J}}{\partial \bar{z}^k} \right) \cdot \frac{1}{2} \left(dz^k - d\bar{z}^k \right) \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \\
=& \sum_{k=1}^{n} \sum{}' \frac{\partial f_{I,J}}{\partial z^k} dz^k \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \\
&+ \sum_{k=1}^{n} \sum{}' \frac{\partial f_{I,J}}{\partial \bar{z}^k} d\bar{z}^k \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}
\end{aligned}$$

The first component is in $\Omega^{p+1,q}(M)$, while the second is in $\Omega^{p,q+1}(M)$. In principle, an almost complex structure could have the same property, so that it behaves as it is induced by a holomorphic structure. Therefore, they have been given a special name. Denoting $\pi^{p,q}$ as the projection of $\Omega^\bullet(M;\mathbb{C})$ onto the (p,q)-component in $\Omega^{p,q}(M)$, we can define the ∂- and $\bar{\partial}$-operators, as well as the notion of integrability of almost complex structures.

Definition 1.24. On an almost complex manifold (M, J), the operators

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \quad \text{and} \quad \bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$$

are defined by $\partial := \pi^{p+1,q} \circ d$ and $\bar{\partial} := \pi^{p,q+1} \circ d$. The almost complex structure J is called *integrable* if $d = \partial + \bar{\partial}$ when restricted to $\Omega^{p,q}(M)$. Almost complex structures that are integrable are called complex structures. ▲

We can use the $\bar{\partial}$-operator to describe the holomorphic functions on a complex manifold in yet another way.

Proposition 1.25. *Let M be a complex manifold. A smooth (complex-valued) function f on M is holomorphic if and only if $\bar{\partial} f = 0$.*

Proof. Let f be any smooth function on the complex manifold M. Observe that in local coordinates (z^j) the form $\bar{\partial} f$ can be written as

$$\bar{\partial} f = \sum_j \frac{\partial f}{\partial \bar{z}^j} d\bar{z}^j.$$

Writing $f = u + iv$ and $\frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + \frac{\partial}{\partial y^j}\right)$, we see that $\frac{\partial f}{\partial \bar{z}^j} = 0$ if and only if f satisfies the Cauchy-Riemann equations with respect to the variable z^j. Therefore, $\bar{\partial} f = 0$ if and only if around any point f satisfies the Cauchy-Riemann equations w.r.t. each variable z^j, finishing the proof. □

We saw before that the almost complex structure induced by a holomorphic structure is integrable. Remarkably, the converse is also true and this intriguing and deep result was first proven by Newlander and Nirenberg in 1959.

Theorem 1.26 (**Newlander-Nirenberg**, [15])**.** *Let (M, J) be an almost complex integrable manifold. Then there exists a unique holomorphic structure on M inducing the almost complex structure J.*

We will not prove this theorem here because the proof is too involved and we will not need it later. For a proof in a modern text, the reader can be referred to [5]. This theorem tells us that there are two equivalent ways of describing complex manifolds. One can look at a complex manifold as a manifold together with coordinates that are compatible in some way, or one can regard it as a smooth manifold together with an algebraic structure satisfying an integrability condition. In the next chapter, we will see that via the Darboux theorem we obtain two similar perspectives for symplectic manifolds.

1.3 Integrability

In this section we will study the ∂- and $\bar{\partial}$-operators in more detail. We start with some straightforward results.

Proposition 1.27. *Let (M, J) be an almost complex manifold. The ∂- and $\bar{\partial}$-operators have the following properties.*

1. *(Leibniz rule) For any* $\alpha \in \Omega^{p,q}(M)$ *and* $\beta \in \Omega^{r,s}(M)$ *we have*

$$\partial(\alpha \wedge \beta) = (\partial\alpha) \wedge \beta + (-1)^{p+q}\alpha \wedge (\partial\beta);$$
$$\bar{\partial}(\alpha \wedge \beta) = (\bar{\partial}\alpha) \wedge \beta + (-1)^{p+q}\alpha \wedge (\bar{\partial}\beta).$$

2. *For any* $\alpha \in \Omega^{p,q}(M)$ *we have* $\overline{(\partial\bar{\alpha})} = \bar{\partial}\alpha$*, and therefore* $\partial^2 = 0 \iff \bar{\partial}^2 = 0$.

3. *If the almost complex structure is integrable, then*

$$\partial^2 = \partial\bar{\partial} + \bar{\partial}\partial = \bar{\partial}^2 = 0.$$

Proof. The first statement follows from the Leibniz rule for the exterior derivative. The proofs for both operators are very similar, so we will only write down the one for the ∂-operator. Given two forms $\alpha \in \Omega^{p,q}(M)$ and $\beta \in \Omega^{r,s}(M)$, we compute:

$$\begin{aligned}\partial(\alpha \wedge \beta) &= \pi^{p+r+1,r+s}(d(\alpha \wedge \beta)) \\ &= \pi^{p+r+1,q+s}((d\alpha) \wedge \beta) + (-1)^{p+q}\pi^{p+r+1,q+s}(\alpha \wedge (d\beta)) \\ &= (\pi^{p+1,q}(d\alpha)) \wedge \beta + (-1)^{p+q}\alpha \wedge (\pi^{r+1,s}(d\beta)) \\ &= (\partial\alpha) \wedge \beta + (-1)^{p+q}\alpha \wedge (\partial\beta),\end{aligned}$$

where in the penultimate equality we used the observation that since β is of type (r, s), the $(p + r + 1, q + s)$-component of $(d\alpha) \wedge \beta$ must be $(\pi^{p+1,q}(d\alpha)) \wedge \beta$, as well as a similar observation for the other term.

For the second statement, a quick observation verifies that $\pi^{p,q}(\bar{\omega}) = \overline{\pi^{q,p}(\omega)}$ for any $\omega \in \Omega^{\bullet}(M;\mathbb{C})$. Then, given $\alpha \in \Omega^{p,q}(M)$, we have

$$\overline{\partial\bar{\alpha}} = \overline{\pi^{q+1,p}(d\bar{\alpha})} = \pi^{p,q+1}(d\alpha) = \bar{\partial}\alpha.$$

The final statements follows from the fact that on $\Omega^{p,q}(M)$ we have

$$d^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2 = 0.$$

Because the operators ∂^2, $\bar{\partial}^2$ and $\partial\bar{\partial}+\bar{\partial}\partial$ all land in different components of $\Omega^{p+q+2}(M;\mathbb{C})$, they all must vanish. □

The given definition of integrability is usually hard to work with. Below we will find some more practical characterizations of integrability. We start with the following lemma.

Lemma 1.28. *Let* (M, J) *be an almost complex manifold. Then* J *is integrable if and only if* $d = \partial + \bar{\partial}$ *on* $\Omega^{1,0}(M)$.

Proof. One implication is trivial. The converse we only have to check locally, because d is a locally defined operator. Let $\omega \in \Omega^{p,q}(M)$ be a form. Locally, ω can be written as

$$\omega = \sum{}' f_{I,J} v_{i_1} \wedge \cdots \wedge v_{i_p} \wedge \overline{v}_{j_1} \wedge \cdots \wedge \overline{v}_{j_q}$$

for a local frame $\{v_1, \dots, v_m\}$ of the bundle $(T^{1,0})^*M$ and smooth (complex-valued) functions $f_{I,J}$. Taking the exterior derivative yields

$$\begin{aligned} d\omega &= \sum\nolimits' df_{I,J} \wedge v_{i_1} \wedge \cdots \wedge v_{i_p} \wedge \overline{v}_{j_1} \wedge \cdots \wedge \overline{v}_{j_q} \\ &+ \sum\nolimits' \sum_{k=1}^{p} (-1)^{k-1} f_{I,J} v_{i_1} \wedge \cdots \wedge dv_{i_k} \wedge \cdots \wedge v_{i_p} \wedge \overline{v}_{j_1} \wedge \cdots \wedge \overline{v}_{j_q} \\ &+ \sum\nolimits' \sum_{k=1}^{q} (-1)^{p} (-1)^{k-1} f_{I,J} v_{i_1} \wedge \cdots \wedge v_{i_p} \wedge \overline{v}_{j_1} \wedge \cdots \wedge d\overline{v}_{j_k} \wedge \cdots \wedge \overline{v}_{j_q}. \end{aligned}$$

Notice that $df_{I,J} \in \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$. Moreover, by assumption $dv_j \in \Omega^{2,0}(M) \oplus \Omega^{1,1}(M)$ and as a consequence $d\bar{v}_j = \overline{dv_j} \in \Omega^{1,1}(M) \oplus \Omega^{0,2}(M)$. We conclude that $d\omega \in \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$, which implies that $d\omega = \partial\omega + \bar{\partial}\omega$, finishing the proof. $\square$

As we did with the exterior derivative, one can extend the Lie bracket $[\cdot,\cdot]$ on the tangent bundle complex linearly to the complexified tangent bundle. Integrability can be expressed in terms of the Lie bracket, in the following theorem, providing a very practical characterization of integrability. Additionally, the next theorem will be used to find yet another characterization, which will conclude this section.

Theorem 1.29. *Let (M, J) be an almost complex manifold. Then J is integrable if and only if its $(-i)$-eigenbundle is involutive with respect to the Lie bracket, i.e. $[\Gamma(T^{0,1}M), \Gamma(T^{0,1}M)] \subset \Gamma(T^{0,1}M)$, with $\Gamma(T^{0,1}M)$ denoting the space of sections of the bundle $T^{0,1}M$.*

Proof. By Lemma 1.28, it is enough to show that $d = \partial + \bar{\partial}$ on $\Omega^{1,0}(M)$ is equivalent to the $(-i)$-eigenbundle being involutive. To start, we make the observation that $(T^{0,1})^*M$ annihilates $T^{1,0}M$ and vice versa. In particular, the $(0,2)$-component of a 2-form $\psi \in \Omega^2(M;\mathbb{C})$ vanishes if and only if for all sections $X, Y \in \Gamma(T^{0,1}M)$ we have $\psi(X,Y) = 0$. We will make use of this fact directly.

Let $\varphi \in \Omega^{1,0}(M)$ and $X, Y \in \Gamma(T^{0,1}M)$. Computing the exterior derivative yields

$$(d\varphi)(X,Y) = X(\varphi(Y)) - Y(\varphi(X)) - \varphi([X,Y]) = -\varphi([X,Y]).$$

Hence, $d\varphi \in \Omega^{2,0}(M) \oplus \Omega^{1,1}(M)$ if and only if $\varphi([X,Y]) = 0$ for all sections $X, Y \in \Gamma(T^{0,1}M)$. Since $\varphi \in \Omega^{1,0}(M)$ was arbitrary, we conclude that $d = \partial + \bar{\partial}$ on $\Omega^{0,1}(M)$ if and only if $[X,Y] \in \Gamma(T^{0,1}M)$ for all $X, Y \in \Gamma(T^{0,1}M)$. This finishes the proof. $\square$

Proposition 1.30. *Let (M, J) be an almost complex manifold. Then J is integrable if and only if $\bar{\partial}^2 = 0$.*

Proof. One implication is part of Proposition 1.27. For the converse, suppose that $\bar{\partial}^2 = 0$. Let $X, Y \in \Gamma(T^{0,1}M)$ be two sections. We will show that $[X,Y] \in \Gamma(T^{0,1}M)$ and therefore, by the theorem above, that J is integrable. Let $f \in \Omega^{0,0}(M)$ be a smooth function

$$\begin{aligned}
0 = \bar{\partial}^2 f(X,Y) &= (d(\bar{\partial} f))(X,Y) \\
&= X(\bar{\partial} f(Y)) - Y(\bar{\partial} f(X)) - \bar{\partial} f([X,Y]) \\
&= X(df(Y)) - Y(df(X)) - \bar{\partial} f([X,Y]) \\
&= d^2 f(X,Y) + df([X,Y]) - \bar{\partial} f([X,Y]) \\
&= \partial f([X,Y])
\end{aligned}$$

where in the third line we used that $(T^{1,0})^* M$ annihilates $T^{0,1}M$ and in the last line that $d = \partial + \bar{\partial}$ on $\Omega^{0,0}(M)$. Since f is an arbitrary function and locally forms of type (1,0) are generated by elements in the form $\bar{\partial} f$, we are left with no other choice but to conclude that $[X,Y] \in \Gamma(T^{0,1}M)$, finishing the proof. □

1.4 The complex decomposition

We saw before that the forms of degree k decompose naturally into forms of type (p,q). There is a less immediate decomposition of the space of forms of an almost complex manifold that, despite losing track of the degree, will be one of the building blocks of the framework in chapter 3. We start by a discussion of linear algebra to obtain the decomposition, after which we will turn to manifolds. In the end, we will see that integrability of an almost complex structure can be expressed in terms of this decomposition. This section is an adaptation of the basic material treated in [7].

The linear complex decomposition

Although we will not introduce generalized complex geometry, we need just one aspect of its language in order to formulate our definitions and proofs in the rest of this thesis. Let V be a real vector space of dimension $2n$. In generalized complex geometry, one considers the space $\mathbb{V} = V \oplus V^*$. The vector space V acts naturally on the exterior algebra $\wedge^\bullet V^*$ by interior contraction, i.e. for a vector $X \in V$ and an element $\varphi \in \wedge^\bullet V^*$ this action is given by $X \cdot \varphi = \iota_X \varphi$. Furthermore, the dual V^* acts on the exterior algebra by the wedge product, i.e. for $\xi \in V^*$ and $\varphi \in \wedge^\bullet V^*$ this action is given by $\xi \cdot \varphi = \xi \wedge \varphi$. As one can guess, we obtain a natural action of $\mathbb{V}$ on $\wedge^\bullet V^*$ given by $(X + \xi) \cdot \varphi = \iota_X \varphi + \xi \wedge \varphi$. This action is compatible with the canonical dual pairing, given by $(X, \xi) = \xi(X)$ for $X \in V$ and $\xi \in V^*$, in the following way:

$$\begin{aligned}
(X+\xi)\cdot((X+\xi)\cdot\varphi) &= \iota_X(\iota_X\varphi + \xi\wedge\varphi) + \xi\wedge(\iota_X\varphi + \xi\wedge\varphi) \\
&= \xi(X)\varphi = (X,\xi)\varphi.
\end{aligned}$$

This is all we will discuss of the generalized complex theory.

Suppose now that the vector space V is endowed with a complex structure J. This complex structure induces a complex structure $\mathcal{J}_J : \mathbb{V} \to \mathbb{V}$ given by $\mathcal{J}_J(X + \xi) =$

$-J(X) + J^*(\xi)$, which has the following matrix form

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}.$$

There is an action of this complex structure that is compatible with the action of $\mathbb{V}$ on the exterior algebra $\wedge^\bullet V^*$. This action is defined as follows. Let $\{e_1, \ldots, e_{2n}\}$ be a basis of V and denote by $\{e^1, \ldots, e^{2n}\}$ its dual basis. We can write the coefficients of J as $J_i^j = e^j(J(e_i))$. The action is then given by

$$\mathcal{J}_J \cdot \varphi = J_i^j e^i \cdot e_j \cdot \varphi, \quad \text{for all } \varphi \in \wedge^\bullet V^*.$$

One can verify that this is indeed independent of the choice of basis. Compatibility is guaranteed by the following Leibniz rule.

Lemma 1.31. *The action of $\mathcal{J}_J$ satisfies the following identity for all $v \in \mathbb{V}$ and $\varphi \in \wedge^\bullet V^*$:*

$$\mathcal{J}_J \cdot v \cdot \varphi = \mathcal{J}_J(v) \cdot \varphi + v \cdot \mathcal{J}_J \cdot \varphi.$$

Proof. By linearity, we can check this identity on $X \in V$ and $\xi \in V^*$ separately. We then compute:

$$\begin{aligned} \mathcal{J}_J(X) \cdot \varphi + X \cdot \mathcal{J}_J \cdot \varphi &= -J_i^j e^i(X)(\iota_{e_j}\varphi) + \iota_X \left(J_i^j e^i \wedge (\iota_{e_j}\varphi)\right) \\ &= J_i^j e^i \wedge (\iota_{e_j}\iota_X\varphi) = \mathcal{J}_J \cdot X \cdot \varphi; \end{aligned}$$

$$\begin{aligned} \mathcal{J}_J \cdot \xi \cdot \varphi &= J_i^j e^i \wedge (\iota_{e_j}(\xi \wedge \varphi)) = J_i^j \xi(e_j) e^i \wedge \varphi + J_i^j \xi \wedge e^i \wedge (\iota_{e_j}\varphi) \\ &= \mathcal{J}_J(\xi) \cdot \varphi + \xi \cdot \mathcal{J}_J \cdot \varphi. \end{aligned}$$

This finishes the proof. $\square$

We extend the theory above complex linearly to $\mathbb{V}_{\mathbb{C}}$. The action of $\mathcal{J}_J$ on a covector $\xi \in V_{\mathbb{C}}^*$ is easily calculated: it is simply given by the dual of the complex structure. In other words, we have $\mathcal{J}_J \cdot \xi = J^*(\xi)$. Therefore, given a $\varphi \in \wedge^{p,q} V^*$, we have by the Leibniz rule that $\mathcal{J}_J \cdot \varphi = J^* \cdot \varphi = i(p-q)\varphi$. We deduce that the action of $\mathcal{J}_J$ has eigenvalues $\{-in, -i(n-1), \ldots, in\}$. Denoting U^k as the ik-eigenspace of the action of $\mathcal{J}_J$, we have the following decompositions:

$$U^k = \bigoplus_{p-q=k} \wedge^{p,q} V^*$$

$$\wedge^\bullet V_{\mathbb{C}}^* = \bigoplus_{k=-n}^{n} U^k.$$

The complex decomposition on manifolds

The linear algebra above straightforwardly generalizes to manifolds. Given an almost complex manifold (M, J) of real dimension $2n$, the almost complex structure induces a bundle map $\mathcal{J}_J : \mathbb{T}M \to \mathbb{T}M$ where $\mathbb{T}M = TM \oplus T^*M$ is called the double tangent bundle. The corresponding action of $\mathcal{J}_J$ on $\wedge^\bullet T^*M$ has eigenvalues $-in, i(n-1), \ldots, in$ and the ik-eigenbundle is given by

$$U^k = \bigoplus_{p-q=k} \wedge^{p,q} T^*M.$$

Smoothness of these eigenbundles is then guaranteed by smoothness of the bundles $\wedge^{p,q}T^*M$. Denoting $\mathcal{U}^k$ as the space of sections of the bundle U^k, we arrive at the decomposition of complex valued forms in the ik-eigenspaces of $\mathcal{J}_J$:

$$\mathcal{U}^k = \bigoplus_{p-q=k} \Omega^{p,q}(M);$$
$$\Omega^\bullet(M;\mathbb{C}) = \bigoplus_{k=-n}^{n} \mathcal{U}^k.$$

Integrability of the almost complex structure can also be described in terms of these eigenspaces. In the following theorem the map $\pi^k : \wedge^\bullet T^*M_{\mathbb{C}} \to U^k$ denotes the projection onto the ik-eigenspace of the action of $\mathcal{J}_J$.

Theorem 1.32. *Let (M, J) be an almost complex manifold. Then J is integrable if and only if*

$$d(\mathcal{U}^k) \subset \mathcal{U}^{k+1} \oplus \mathcal{U}^{k-1}$$

for all k. In fact, J is integrable if and only if

$$d(\mathcal{U}^n) \subset \mathcal{U}^{n-1}$$

Additionally, the components $\partial = \pi^{k+1} \circ d$ and $\bar{\partial} = \pi^{k-1} \circ d$ correspond to the usual ∂- and $\bar{\partial}$-operators.

Proof. We start by ordering the (p, q)-spaces into the following diamond.

$$\begin{tikzcd}
 & & \Omega^{p+1,q+1}(M) & & \\
 & \Omega^{p,q+1}(M) & & \Omega^{p+1,q}(M) & \\
\Omega^{p-1,q+1}(M) & & \Omega^{p,q}(M) \arrow[ul, "\bar{\partial}"'] \arrow[ur, "\partial"] & & \Omega^{p+1,q-1}(M) \\
 & \Omega^{p-1,q}(M) & & \Omega^{p,q-1}(M) & \\
 & & \Omega^{p-1,q-1}(M) & &
\end{tikzcd}$$

The columns have constant eigenvalue while the rows have constant degree. It is clear that if $d = \partial + \bar{\partial}$ on $\Omega^{p,q}(M)$ for all p and q, then $d(\mathcal{U}^k) \subset \mathcal{U}^{k+1} \oplus \mathcal{U}^{k-1}$ for all k. Conversely, if $d(\mathcal{U}^k) \subset \mathcal{U}^{k+1} \oplus \mathcal{U}^{k-1}$ for all k, then given $\omega \in \Omega^{p,q}(M)$, its exterior derivative $d\omega$ can only land in $\Omega^{p,q+1}(M) \oplus \Omega^{p+1,q}(M)$, because the exterior derivative must increase the degree by one.

For the last statement of the theorem, assume that $d(\mathcal{U}^n) \subset \mathcal{U}^{n-1}$. Let $X_1, X_2 \in \Gamma(T^{0,1}M)$ be two sections. Pick sections $e_1, \ldots, e_n \in \Gamma(T^{1,0}M)$ that locally define a frame of $T^{1,0}M$ and let $e^1, \ldots e^n$ be its dual. Our assumption implies that the $(n-2, 2)$-component of $d(e^1 \wedge \cdots \wedge e^n)$ vanishes. Thus we have for all $i = 1, \ldots, n$

$$d(e^1 \wedge \cdots \wedge e^n)(X_1, X_2, e_1, \ldots \hat{e}_i \ldots, e_n) = 0.$$

On the other hand, recalling $(T^{1,0})^*M$ is the annihilator of $T^{0,1}M$, a direct computation yields:

$$d(e^1 \wedge \cdots \wedge e^n)(X_1, X_2, e_1, \ldots \hat{e}_i \ldots, e_n) = (-1)^i e^i([X_1, X_2])$$

We observe that $[X_1, X_2]$ is annihilated by $(T^{1,0})^*M$ and thus that $[X_1, X_2] \in \Gamma(T^{0,1}M)$. Hence, $T^{0,1}M$ is involutive w.r.t. the Lie bracket and thus J is integrable by Theorem 1.29. $\square$

Chapter 2

Symplectic Geometry

This chapter provides a basic introduction to symplectic geometry. Symplectic structures are, next to the complex structures, one of the main ingredients of Kähler manifolds. In addition, the symplectic decomposition treated in Section 2.4 will be an other cornerstone for the proof of the main result of this thesis.

We will start with a short discussion of the basics of symplectic geometry. Then we will treat a bit of Poisson geometry so that we can use its results in the final section: the symplectic decomposition.

The first two sections are based on [16] and chapter 22 of [11].

2.1 Symplectic linear algebra

Let V be an n-dimensional real vector space (unless otherwise specified, all the vector spaces are real and finite throughout this section). Any 2-covector $\omega \in \wedge^2 V^*$ induces a corresponding linear map $\hat{\omega} : V \to V^*$ sending $v \in V$ to $\iota_v \omega \in V^*$. We call a 2-covector ω *non-degenerate* when its corresponding map is an isomorphism of vector spaces.

Definition 2.1. Let V be a vector space. A 2-vector $\omega \in \wedge^2 V^*$ is called *symplectic* if it is non-degenerate. The pair (V, ω) is called a *symplectic vector space* and ω is called the *symplectic structure.* ▲

Example 2.2. Let $\{v_1, w_1, \ldots, v_n, w_n\}$ be a basis of a $2n$-dimensional vector space V and let $\{v^1, w^1, \ldots, v^n, w^n\}$ be its dual. Then the 2-covector

$$\omega = \sum_{i=1}^{n} v^i \wedge w^i$$

is symplectic. It acts on the basis vectors as

$$\omega(v_i, v_j) = \omega(w_i, w_j) = 0, \qquad \omega(v_i, w_j) = \delta_{ij}.$$

Notice that ω is uniquely determined by these equations. △

As we will see later, any symplectic vector space has a basis in which the symplectic structure has the form of the structure in Example 2.2 above. First we need to introduce some terminology.

Definition 2.3. Let (V, ω) be a symplectic vector space and let $S \subseteq V$ be a subspace. We define the *symplectic complement* of S as

$$S^\perp = \{w \in V : \omega(v, w) = 0 \ \forall v \in S\}.$$

We call S

- *symplectic* when $S \cap S^\perp = \{0\}$,
- *isotropic* when $S \subset S^\perp$,
- *maximally isotropic* or *Lagrangian* when $S = S^\perp$. ▲

The condition $S \cap S^\perp = \{0\}$ precisely means that ω restricted to S is non-degenerate. This makes $(S, \omega|_S)$ into a symplectic vector space, hence the name.

The orthogonal complement of a subspace of an inner product space is always complementary to the subspace itself. This is not true for symplectic complements. However, it does when one only looks at the dimensions, as stated in the following lemma.

Lemma 2.4. *Let (V, ω) be a symplectic vector space and $S \subseteq V$ a linear subspace. Then*

$$\dim S + \dim S^\perp = \dim V.$$

Proof. Define $\Phi : V \to S^*$ by $\Phi(v) = \hat{\omega}(v)|_S$. The non-degeneracy condition on ω implies that Φ is surjective. Indeed, given any $s \in S^*$, pick $\tilde{s} \in V^*$ such that $\tilde{s}$ extends s. Then $\Phi(\omega^{-1}(\tilde{s})) = s$. Furthermore, one easily checks that the kernel of Φ is exactly the symplectic complement of S. Therefore we have

$$\dim S^\perp + \dim\, S = \dim \mathrm{Ker}\ \Phi + \dim \mathrm{Im}\ \phi = \dim V$$

finishing the proof. □

One small consequence of this lemma is that in general $(S^\perp)^\perp = S$. Indeed, first one checks that $S \subset (S^\perp)^\perp$ and then, after applying the Lemma 2.4 to S and $S^\perp$, one concludes that the dimensions are the same. A larger consequence is that any symplectic vector space has a basis in which the symplectic structure is standard, as stated in the following theorem.

Theorem 2.5. *Let (V, ω) be an m-dimensional symplectic vector space. Then V has even dimension $m = 2n$ and there exists a basis $\{v_1, w_1, \ldots, v_n, w_n\}$ such that $\omega(v_i, v_j) = \omega(w_i, w_j) = 0$ and $\omega(v_i, w_j) = \delta_{ij}$.*

Proof. We will use induction to the dimension of V. The case that $m = 0$ is trivial. Now let $m \geq 1$ and suppose the theorem holds for all symplectic vector spaces with dimension less than m. Let v_1 be any nonzero vector in V. The non-degeneracy condition then implies that $\iota_{v_1}\omega$ is also nonzero and therefore one can find a vector $w_1 \in V$ such that $(\iota_{v_1}\omega)(w_1) = \omega(v_1, w_1) \neq 0$. Scaling w_1 if necessary, we may assume that $\omega(v_1, w_1) = 1$. Notice, because of the skew-symmetry of ω, that v_1 and w_1 are linearly independent. The subspace $S = \operatorname{span}\{v_1, w_2\}$ is symplectic by construction and therefore $S^\perp$ is also symplectic. By Lemma 2.4 the dimension of $S^\perp$ is $m-2$. Applying the induction hypothesis, we find natural number n and a basis $\{v_2, w_2, \ldots, v_n, w_n\}$ of $S^\perp$ satisfying the equations of the theorem. It follows that $\{v_1, w_1, \ldots, v_n, w_n\}$ is the desired basis. □

We will end this section with a small application of this theorem.

Proposition 2.6. *Let V be a $2n$-dimensional vector space and let $\omega \in \wedge^2 V^*$. Then ω is symplectic if and only if $\omega^n = \omega \wedge \cdots \wedge \omega \neq 0$.*

Proof. Suppose that ω is symplectic. Find a basis $\{v_1, w_1 \ldots, v_n, w_n\}$ such that ω can be written as

$$\omega = \sum_{i=1}^n v^i \wedge w^i.$$

Then

$$\omega^n = \sum_{i_1,\ldots,i_n=1}^n v^{i_1} \wedge w^{i_1} \wedge \cdots \wedge v^{i_n} \wedge w^{i_n}.$$

Notice that one can change any two indices i_j and i_k in the summand without changing the sign. Additionally, whenever $i_j = i_k$ for different j and k, the summand is zero. Thus, we have

$$\omega^n = n!(v^1 \wedge w^1 \wedge \cdots \wedge v^n \wedge w^n) \neq 0.$$

Conversely, suppose ω is degenerate. Then there exists a nonzero vector $v \in V$ such that $\hat{\omega}(v) = \iota_v \omega = 0$. This implies that $\iota_v(\omega^n) = n(\iota_v\omega) \wedge \omega^{n-1} = 0$. Since v is nonzero and ω^n is a top-degree covector, we must conclude that $\omega^n = 0$. □

2.2 Symplectic manifolds

Generalizing the linear algebra above, any 2-form $\omega \in \Omega^2(M)$ on a manifold M induces a corresponding vector bundle homomorphism $\hat{\omega} : TM \to T^*M$ defined by $\hat{\omega}_p(v) = \iota_v\omega_p \in T_p^*M$ for $p \in M$ and $v \in T_pM$. Naturally, the 2-form ω is called non-degenerate if $\hat{\omega}$ is a vector bundle isomorphism. Equivalently, one can say that ω is non-degenerate when $\omega_p \in \wedge^2 T_p^*M$ is non-degenerate for all $p \in M$. Now let us define symplectic forms on manifolds.

Definition 2.7. Let M be a smooth manifold and $\omega \in \Omega^2(M)$. Then ω is called *symplectic* if it closed and non-degenerate. The pair (M, ω) is called a *symplectic manifold* and ω is called the *symplectic structure.* ▲

Let (M, ω) be a symplectic manifold. Then for any point $p \in M$, the vector space (T_pM, ω_p) is also symplectic. By Theorem 2.5, M must be even dimensional. Additionally, Proposition 2.6 tells us that ω^n is a nowhere vanishing top-degree form. Thus, any symplectic manifold is automatically orientable and carries a canonical orientation determined by ω^n.

Before concerning ourselves with examples, we define, for completeness, morphisms between symplectic manifolds.

Definition 2.8. Let (M_1, ω_1) and (M_2, ω_2). A diffeomorphism $\varphi : M_1 \to M_2$ is called a *symplectomorphism* when $\varphi^*\omega_2 = \omega_1$. If such a map exists, we call (M_1, ω_1) and (M_2, ω_2) *symplectomorphic.* ▲

Example 2.9. On $\mathbb{R}^{2n}$ with coordinates $(x^1, y^1, \ldots, x^n, y^n)$ one can define the *standard symplectic form* by

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i.$$

This seemingly trivial example is still rather important, since any symplectic manifold is locally indistinguishable from this standard example, as we will see later in the Darboux theorem. △

Example 2.10. On $\mathbb{C}^n$, with coordinates (z^j), one can define the symplectic form

$$\omega = \frac{i}{2} \sum_{j=1}^{n} dz^j \wedge d\bar{z}^j.$$

Notice that, after identifying $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via $z^j = x^j + iy^j$, this example is actually the same as the previous example. △

Example 2.11. Any oriented surface Σ is a symplectic manifold. The orientation gives us a nowhere vanishing 2-form ω on Σ which is automatically closed since the degree of $d\omega$ exceeds the dimension of the surface. Non-degeneracy follows directly from Proposition 2.6. △

Example 2.12. Let X be a smooth n-manifold and let $M = T^*X$ be its cotangent bundle regarded as a manifold. Recall that the bundle map $\pi : M \to X$ sends an element $p = (x, \xi) \in M$ to $\pi(p) = x \in X$. One defines the *tautological 1-form* α pointwise on M as

$$\tau_p = d\pi_p^* \xi \in T_p^* M \quad \text{for } p = (x, \xi) \in M.$$

Here $d\pi_p^*$ denotes the dual of the linear map $d\pi_p : T_pM \to T_xX$. To check that it is smooth, let $x^1, \ldots, x^n$ be local coordinates of X defined on an open subset U. Recall that these coordinates induce a chart on M in the following way: given a point (x, ξ) in $\pi^{-1}(U)$, one can write $\xi = \sum_i \xi_i (dx^i)_x$. The functions ξ_i depend smoothly on (x, ξ) and thus the functions $x^1, \ldots, x^n, \xi_1, \ldots, \xi_n$ define a chart on M. Given a point $p = (x, \xi) \in \pi^{-1}(U)$ and a tangent vector $v \in T_pM$ we can write

$$v = \sum_{i=1}^{n} dx_p^i(v) \frac{\partial}{\partial x^i}\bigg|_p + (d\xi_i)_p(v) \frac{\partial}{\partial \xi_i}\bigg|_p.$$

After making the observation that $d\pi_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial}{\partial x^i}\Big|_x$ and $d\pi_p\left(\frac{\partial}{\partial \xi_i}\Big|_p\right) = 0$, we can write the tautological 1-form τ as follows:

$$\begin{aligned}\tau_p(v) = d\pi_p^*\xi &= \xi\left(d\pi_p(v)\right) \\ &= \sum_{i=1}^n \xi_i dx_p^i(v).\end{aligned}$$

Thus, τ can be expressed on $\pi^{-1}(U)$ as

$$\tau = \sum_{i=1}^n \xi_i dx^i$$

which is clearly smooth. Notice that this also justifies the name of the tautological one-form. The *canonical symplectic form* on $M = T^*X$ is then defined as

$$\omega = -d\tau.$$

Clearly, ω is closed because it is exact. In the coordinates x^i, ξ^i, the canonical symplectic form ω takes the form of the standard symplectic form

$$\omega = \sum_{i=1}^n dx^i \wedge d\xi_i$$

and therefore ω is indeed symplectic. △

The different properties of subspaces of symplectic vector spaces, mentioned in the previous section, carry over to submanifolds of symplectic manifolds in a natural way.

Definition 2.13. Let (M, ω) be a symplectic manifold and $S \subset M$ a submanifold. Then S is called a *symplectic*, *isotropic* or *Lagrangian* submanifold of M if T_pS has the same property as a subspace of (T_pM, ω_p) for all $p \in S$. ▲

Example 2.14. Given a manifold X, one can regard a one-form α on X as a map $\alpha : X \to M$ where $M = T^*X$. This map is actually an embedding of X into M. We equip M with the canonical symplectic form. When α is the zero-section, the tangent space of $\alpha(M)$ is given in local coordinates by $T_x\alpha(X) = \text{span}(dx^i_{(x,0)}) \subset T_{(x,0)}M$. Clearly, for any two vectors $v, w \in T_x\alpha(X)$ we have $\omega_{(x,0)}(v, w) = 0$. Thus, $\alpha(X)$ is an isotropic submanifold of M. Since its dimension is half of the dimension of M, it is in fact a Lagrangian submanifold of M.

Suppose now that α is arbitrary. We would like to find out when $\alpha(X)$ is a Lagrangian submanifold of M. First of all, notice that the dimension of X is exactly half of the dimension of M. Therefore, $\alpha(X)$ is Lagrangian if and only if it is isotropic. Secondly, we make the observation that in general a submanifold $S \overset{i}{\hookrightarrow} N$ of a symplectic manifold (N, ω) is isotropic if and only if $i^*\omega = 0$. Finally, to check when $\alpha(X)$ is isotropic, we pick local coordinates (x^i) on X and denote by (x^i, ξ_i) its corresponding coordinates on M. Then we can write

$$\alpha(x) = (x^i, \alpha_i(x))$$

where $\alpha = \alpha_i dx^i$ in the local coordinates on X. If τ is the tautological one-form on M, then its pullback under α is

$$\alpha^*\tau = \alpha^*(\xi_i dx^i) = \alpha_i dx^i = \alpha.$$

Therefore, the pullback of the canonical symplectic form ω under α becomes:

$$\alpha^*\omega = -\alpha^* d\tau = -d(\alpha^*\tau) = -d\alpha$$

We conclude that $\alpha(X)$ is a Lagrangian submanifold of M if and only if $d\alpha = 0$. $\triangle$

Our discussion of the basics of symplectic geometry would not be complete without mentioning one of its most fundamental results: the Darboux theorem. It tells us that locally there is no way to distinguish between different symplectic manifolds.

Theorem 2.15 (Darboux). *Let (M, ω) be a $2n$-dimensional symplectic manifolds. Then around any point $p \in M$ there are smooth coordinates $x^1, \dots, x^n, y^1, \dots, y^n$ in which the symplectic form ω takes the form of the standard symplectic form, i.e.*

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i.$$

This theorem is somewhat analogous to the Newlander-Nirenberg theorem for complex manifolds in the sense that it tells us that there are two equivalent ways of describing symplectic manifolds: the first one is by an algebraic condition (a non-degenerate 2-form) together with an integrability condition (closedness), while the second one is to look at the certain coordinates with symplectomorphisms as transition functions. These coordinates are usually referred to as Darboux coordinates. We will give a proof of this theorem after we introduced the Poisson bracket.

2.3 Towards Poisson geometry and back

Basically, this section mainly serves to prove that two specific operators, which will be introduced later, commute. One can choose to prove this via a brute-force calculation, but that is exactly what we tended to avoid in this thesis. Instead, there is a much deeper reason that this specific identity is true: it is a rather trivial result in a related field of mathematics that goes by the name of Poisson geometry. In this section we will see that symplectic geometry and its application to Hamiltonian mechanics leads to the Poisson bracket, the main ingredient of Poisson manifolds. Within this framework, the commutator of these specific operates arise naturally as the interior product of Schouten-Nijenhuis bracket of a bivector and we will see that this naturally vanishes for our case. Additionally, taking a small side-track allows us to give a proof of the Darboux theorem. This section is based on [6], [11] chapter 22, [12] and [17], where notably in [12] the result relating interior contraction to the Schouten-Nijenhuis bracket can be found.

2.3.1 Hamiltonian vector fields

Recall that on a symplectic manifold (M, ω) the bundle map corresponding to ω is denoted by $\hat{\omega} : TM \to T^*M$ sending $X \in T_pM$ to $\omega_p(X, \cdot) \in T_p^*M$.

Definition 2.16. Let (M, ω) be a symplectic manifold. For a function $f \in C^\infty(M)$ we define the *Hamiltonian vector field of* f as $X_f = \hat{\omega}^{-1}(df)$. Equivalently, the Hamiltonian vector field of f is the unique vector field X_f such that $\iota_{X_f}\omega = df$. ▲

The motivation of this definition comes from classical mechanics. Formally, a *Hamiltonian system* is a symplectic manifold (M, ω) together with a smooth function $H \in C^\infty(M)$ called the *Hamiltonian.* In practice (that means, in physics), the manifold M is usually the cotangent bundle $M = T^*X$, called the *phase space*, of some manifold X with the x- and ξ-coordinates as in Example 2.12 corresponding to the position- and momentum coordinates, respectively. Integral curves of the Hamiltonian vector field are called *trajectories* of the system and are solutions to the classical equations of motion. To see this, we compute the Hamiltonian vector field of H explicitly in Darboux coordinates. Letting $2n$ be the dimension of M, we start by writing

$$X_H = \sum_{i=1}^{n} \left(v^i \frac{\partial}{\partial x^i} + w^i \frac{\partial}{\partial y^i} \right),$$

where v^i and w^i are coefficients. On the other hand, for the differential of H we have

$$dH = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial x^i} dx^i + \frac{\partial H}{\partial y^i} dy^i \right).$$

By definition, we have $\iota_{X_H}\omega = dH$, filling this in yields

$$\begin{aligned}
\iota_{X_H}\omega &= \iota_{X_H} \sum_{i=1}^{n} dx^i \wedge dy^i \\
&= \left(\sum_{j=1}^{n} \left(v^j \iota_{\partial_{x^j}} + w^j \iota_{\partial_{y^j}} \right) \right) \left(\sum_{i=1}^{n} dx^i \wedge dy^i \right) \\
&= \sum_{i=1}^{n} (v^i dy^i - w^i dx^i) \\
&= \sum_{i=1}^{n} \left(\frac{\partial H}{\partial x^i} dx^i + \frac{\partial H}{\partial y^i} dy^i \right).
\end{aligned}$$

We conclude that $v^i = \frac{\partial H}{\partial y^i}$ and $w^i = -\frac{\partial H}{\partial x^i}$. The integral curves of X_H therefore satisfy the set of equations

$$\dot{x}^i(t) = \frac{\partial H}{\partial y^i}(x(t), y(t)) \quad \text{and} \quad \dot{y}^i(t) = -\frac{\partial H}{\partial x^i}(x(t), y(t)).$$

When $M = T^*X$ is the cotangent bundle of some manifold X, and the x- and y-coordinates correspond to the x- and ξ-coordinates from Example 2.12, these equations are precisely the equations of motion from classical mechanics.

Physical quantities, like momentum or energy, can be regarded as functions on the phase space of a physical system in the sense that at every point in the phase space that quantity takes a certain value. Additionally, it is assumed that this value depends smoothly on the points in the phase space. Therefore, physical quantities can be regarded as smooth functions on the phase space manifold. A physical quantity is conserved when it is constant along all solutions to the equations of motion of that system. Formally, this means that in a Hamiltonian system (M, ω, H) a physical quantity $f \in C^\infty(M)$ is conserved if it is left invariant by the flow of X_H. In equations, this is precisely the case when $X_H(f) = df(X_H) = \omega(X_f, X_H) = 0$. Moreover, when f is not conserved, the quantity $X_H(f) = df(X_H) = \omega(X_f, X_H)$ describes the rate of change of f when the system follows the flow of X_H. Therefore, the pairing of smooth functions $(f, g) \mapsto \omega(X_f, X_g)$ seems natural from the perspective of classical mechanics. This leads to the following definition.

Definition 2.17. Let (M, ω) be a symplectic manifold. We define the *Poisson bracket*

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

by any of the following equivalent formulas $\{f, g\} = \omega(X_f, X_g) = df(X_g) = X_g(f)$. ▲

Recall that in Darboux coordinates x^i, y^i, letting $2n$ be the dimension of M, we can write X_g as $X_g = \sum_{i=1}^n \left(\frac{\partial g}{\partial y^i} \frac{\partial}{\partial x^i} - \frac{\partial g}{\partial x^i} \frac{\partial}{\partial y^i} \right)$ and therefore the Poisson bracket takes the following form:

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial y^i} - \frac{\partial f}{\partial y^i} \frac{\partial g}{\partial x^i} \right).$$

The equations of motion for a Hamiltonian system can now be expressed elegantly as

$$\dot{x}^i = \{x^i, H\} \text{ and } \dot{y}^i = \{y^i, H\}, \text{ or in general } \dot{f} = \{f, H\} \text{ for any physical quantity } f.$$

Proposition 2.18. *The Poisson bracket on a symplectic manifold (M, ω) satisfies for all $f, g, h \in C^\infty(M)$:*

(i) Skew-symmetry: $\{f, g\} = -\{g, f\}$;
(ii) Bilinearity: $\{f, \lambda g + \mu h\} = \lambda\{f, g\} + \mu\{f, h\}$ *for all* $\lambda, \mu \in \mathbb{R}$;
(iii) Jacobi identity: $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$;
(iv) Leibniz rule: $\{f, gh\} = g\{f, h\} + \{f, g\}h$;
(v) Compatibility with Lie bracket: $X_{\{f,g\}} = -[X_f, X_g]$.

Proof. Properties (i) and (ii) are immediate from the definitions $\{f, g\} = \omega(X_f, X_g)$ and $X_f = \hat{\omega}^{-1}(df)$. The Leibniz rule follows directly from the characterization $\{f, gh\} = -\{gh, f\} = -X_f(gh)$. Next, property (iii) follows from property (v) because the Lie bracket satisfies the Jacobi identity, so we are left with proving (v). Instead of proving the equality directly, we will prove that

$$\omega(X_{\{f,g\}}, Y) = -\omega([X_f, X_g], Y)$$

for all vector fields Y. Then, by non-degeneracy of ω, property (v) follows. The right-hand side equals $\omega(X_{\{f,g\}}, Y) = Y(\{f,g\}) = Y(X_g(f))$. To compute the left-hand side, we first make the observation that ω is invariant under the flow of a Hamiltonian vector field. Indeed, by Cartan's magic formula

$$\mathcal{L}_{X_g}(\omega) = d\iota_{X_f}\omega + \iota_{X_f} d\omega = d(df) + 0 = 0.$$

Next, observe that

$$\begin{aligned} 0 &= \left(\mathcal{L}_{X_g}\omega\right)(X_f, Y) \\ &= X_g\left(\omega(X_f, Y)\right) - \omega\left([X_g, X_f], Y\right) - \omega\left(X_f, [X_g, Y]\right) \end{aligned}$$

The first term reduces to $X_g(\omega(X_f,Y)) = X_g(df(Y)) = X_g(Y(f))$, while the second can be rewritten as $\omega(X_f, [X_g, Y]) = df([X_g, Y]) = X_g(Y(f)) - Y(X_g(f))$. Substituting these results in the previous equation yields

$$\begin{aligned} 0 &= -\omega([X_g, X_f], Y) + X_g(Y(f)) - X_g(Y(f)) + Y(X_g(f)) \\ &= -\omega([X_g, X_f], Y) + \omega(X_{\{f,g\}}, Y) \\ &= \omega([X_f, X_g], Y) + \omega(X_{\{f,g\}}, Y), \end{aligned}$$

finishing the proof. □

We will end this section with a proof of the Darboux Theorem (Theorem 2.15). The proof relies on the following two lemmas.

Lemma 2.19. *Let (M, ω) be a $2n$-dimensional symplectic manifold. Then smooth coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ are Darboux if and only if their Poisson brackets satisfy*

$$\{x^i, x^j\} = \{y^i, y^j\} = 0 \quad \text{and} \quad \{x^i, y^i\} = \delta_{ij}.$$

Proof. Suppose $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ are Darboux, then their Poisson brackets can be calculated directly:

$$\begin{aligned} \{x^i, x^j\} &= \sum_{k=1}^{n} \left(\frac{\partial x^i}{\partial x^k}\frac{\partial x^j}{\partial y^k} - \frac{\partial x^i}{\partial y^k}\frac{\partial x^j}{\partial x^k} \right) = 0, \\ \{y^i, y^j\} &= \sum_{k=1}^{n} \left(\frac{\partial y^i}{\partial x^k}\frac{\partial y^j}{\partial y^k} - \frac{\partial y^i}{\partial y^k}\frac{\partial y^j}{\partial x^k} \right) = 0, \\ \{x^i, y^j\} &= \sum_{k=1}^{n} \left(\frac{\partial x^i}{\partial x^k}\frac{\partial y^j}{\partial y^k} - \frac{\partial x^i}{\partial y^k}\frac{\partial y^j}{\partial x^k} \right) = \sum_{k=1}^{n} \delta_{ik}\delta_{kj} = \delta_{ij}. \end{aligned}$$

Conversely, suppose as set of coordinates (x^i, y^i) satisfies the equations from the lemma. Notice that

$$\begin{aligned} X_{x^i}(x^j) &= \omega(X_{x^j}, X_{x^i}) = \{x^j, x^i\} = 0, \\ X_{x^i}(y^j) &= \omega(X_{y^j}, X_{x^i}) = \{y^j, x^i\} = -\delta^{ij}. \end{aligned}$$

From this we conclude that $X_{x^i} = -\frac{\partial}{\partial y^i}$. Making a similar observation for X_{y^i} we find that $X_{y^i} = \frac{\partial}{\partial x^i}$. Therefore, we have $\omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \omega\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0$ and $\omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = \delta^{ij}$, which can only happen when ω is standard with respect to these coordinates. □

Lemma 2.20. *Let M be an n-dimensional smooth manifold and let $X_1, \ldots, X_k$ be linear independent commuting vector fields on some open $U \subset M$. Then for each point $p \in U$ there exist smooth coordinates (x^i) centered at p such that $X_i = \frac{\partial}{\partial x^i}$ for $i = 1, \ldots, k$.*

This lemma is usually covered in a course on smooth manifolds so we will not prove it here. This lemma is stated as part of theorem 9.46 in [11] where a proof is given. We are now ready to give a proof of the Darboux theorem.

Proof of Theorem 2.15. Let (M, ω) be a symplectic manifold of dimension $2n$ and $p \in M$ a point. We will prove the following by induction: for each $k = 0, \ldots, n$ there are smooth functions $(x^1, \ldots, x^k, y^1, \ldots, y^k)$ vanishing at p satisfying $\{x^i, x^j\} = \{y^i, y^j\} = 0$ and $\{x^i, y^j\} = \delta_{ij}$ such that the 1-forms $dx^1, dy^1, \ldots, dx^k, dy^k$ form a linearly independent set. Letting $k = n$, combined with Lemma 2.19 then proves the Darboux theorem.

For $k = 0$, there is nothing to prove. Suppose the statement is true for some $k \in \{0, \ldots, n-1\}$. Notice that, by non-degeneracy of ω, the Hamiltonian vector field $X_f = \hat{\omega}^{-1}(df)$ of a smooth function f vanishes if and only if $df = 0$. Therefore, linear independence of $\{dx^i, dy^i\}$ translates to linear independence of $\{X_{x^i}, X_{y^i}\}$. These vector fields actually commute. Indeed, using property (v) of Proposition 2.18, we observe that $[X_{x^i}, X_{x^j}] = -X_{\{x^i,x^j\}} = 0 = -X_{\{y^i,y^j\}} = [X_{y^i}, X_{y^j}]$ and $[X_{x^i}, X_{y^j}] = -X_{\{x^i,y^j\}} = \hat{\omega}^{-1}(d\{x^i, y^j\}) = \hat{\omega}^{-1}(d\delta_{ij}) = 0$. We now apply Lemma 2.20 to find coordinates $u^1, \ldots, u^{2n}$ centered at p such that $\frac{\partial}{\partial u^i} = X_{x^i}$ and $\frac{\partial}{\partial u^{i+k}} = X_{y^i}$ for $i = 1, \ldots, k$. Set $y^{k+1} = u^{2k+1}$. By construction, we have

$$\{y^{k+1}, x^i\} = dy^{k+1}(X_{x^i}) = du^{2k+1}\left(\frac{\partial}{\partial u^i}\right) = 0$$

$$dy^{k+1}(X_{y^i}) = \{y^{k+1}, y^i\} = du^{2k+1}\left(\frac{\partial}{\partial u^{i+k}}\right) = 0$$

for all $i = 1, \ldots, k$. Now observe in the same way as before that $X_{x^1}, \ldots, X_{x^k}, X_{y^1}, \ldots X_{y^{k+1}}$ form a linearly independent set of commuting vector fields. Applying Lemma 2.20 again, we find coordinates $v^1, \ldots, v^{2n}$ centered at p such that $\frac{\partial}{\partial v^i} = X_{x^i}$ for $i = 1, \ldots, k$ and $\frac{\partial}{\partial v^i} = X_{y^i}$ for $i = 1, \ldots, k+1$. Set $x^{k+1} = v^{2k+1}$. To check that we have found our desired functions for $k+1$, we first notice that $dx^1, \ldots, dx^{k+1}, dy^1, \ldots, dy^{k+1}$ are linearly independent because their Hamiltonian vector fields are. Moreover, one easily checks that $\{x^{k+1}, x^i\} = 0$ and $\{x^{k+1}, y^i\} = \delta_{i(k+1)}$, as desired. □

2.3.2 A bit of Poisson geometry

In many cases, it is the notion of a Poisson bracket instead of a symplectic structure that plays a crucial role for describing physical systems. Therefore, an axiomatic approach to the Poisson bracket, without an underlying symplectic structure, seems reasonable. As it turns out, properties (i) to (iv) of Proposition 2.18 are the defining properties of the Poisson bracket, leading to the notion of a Poisson geometry.

Definition 2.21. Let M be a smooth manifold. A *Poisson bracket* on M is a Lie bracket

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

on the space of smooth functions on M, such that for each $f \in C^\infty(M)$ the linear map $\{f,\cdot\}$ is a derivation on $C^\infty(M)$. The pair $(M, \{\cdot,\cdot\})$ is called a *Poisson manifold.* ▲

The fact that $\{f,\cdot\}$ is a derivation for all $f \in C^\infty(M)$ allows us to describe the Poisson bracket with a bivector field $\pi \in \Gamma(\wedge^2 TM)$ via $\{f,g\} = \pi(df,dg)$, where we made the canonical identification $(T^*)^*M = TM$ to regard π as a bilinear map on T^*M. Conversely, any bivector field $\pi \in \Gamma(\wedge^2 TM)$ induces a bilinear map $\{\cdot,\cdot\}$ on $C^\infty(M)$ by $\{f,g\} := \pi(df,dg)$. However, in general, this bilinear map will not satisfy the Jacobi identity, and therefore fails to be a Poisson bracket. To describe Poisson brackets in terms of bivector fields, we need an extra condition on the bivector field. This condition is described by the so-called Schouten-Nijenhuis bracket, which is an extension of the Lie bracket to multi-vector fields, defined later in this section. First we need some more theory.

Multi-vector fields

With *multi-vector fields* we mean elements of the space of smooth sections of the bundle $\wedge^\bullet TM$, denoted by $\Gamma(\wedge^\bullet TM)$. The grading on multi-vector fields is induced by the grading on $\wedge^\bullet TM$, that is, a multi-vector field of degree p (or a p-vector field) is an element of $\Gamma(\wedge^p TM)$. A multi-vector field A is called *homogeneous of degree* p if A is a p-vector field for some p. If this is the case, we denote $|A| = p$. Clearly, multi-vector fields of degree zero are smooth functions, and multi-vector fields of degree one are the usual vector fields. Just as one can contract differential forms with vector fields, one can contract multi-vector fields with one-forms in the exact same way. This interior multiplication with a one-form α is also denoted by ι_α. Dually, one can contract differential forms by multi-vector fields by $\iota_{X_1\wedge\cdots\wedge X_n}\alpha = \iota_{X_n}\cdots\iota_{X_1}\alpha$ where $X_1,\ldots,X_n$ are vector fields and $\alpha \in \Omega^\bullet(M)$ a differential form.

Supercommutators

Many identities involving the exterior derivative and the Lie derivative are actually a supercommutator of endomorphisms of the exterior algebra $\Omega^\bullet(M)$. Formally, an endomorphism A in $\mathrm{End}(\Omega^\bullet(M))$, the space of endomorphisms of the bundle $\Omega^\bullet(M)$, has degree $|A| = p$ when it changes the degree of forms by p. If this is the case, then A is called *homogeneous of degree* p. For example, the exterior derivative d has degree 1 and interior multiplication ι_X by a vector field X has degree -1, and more generally, the interior multiplication by a p-vector field has degree $-p$. We define the *supercommutator* of two homogeneous endomorphisms $A, B \in \mathrm{End}(\Omega^\bullet(M))$ to be

$$\langle A, B\rangle = AB - (-1)^{|A||B|}BA.$$

Many identities can be expressed as supercommutators, for example:

- The identities $d^2 = 0$ and $\iota_X\iota_Y = -\iota_Y\iota_X$ can be obtained from $\langle d, d\rangle = 0$ and $\langle \iota_X, \iota_Y\rangle = 0$;
- The Lie derivative along a vector field X: $\mathcal{L}_X = \langle \iota_X, d\rangle$ by Cartan's magic formula;
- Interior contraction of the Lie bracket of two vector fields X, Y: $\iota_{[X,Y]} = \langle \mathcal{L}_X, \iota_Y\rangle = \langle\langle \iota_X, d\rangle, \iota_Y\rangle = -\langle\langle \iota_Y, d\rangle, \iota_X\rangle$;
- The Lie derivative of the Lie bracket of two vector fields X, Y: $\mathcal{L}_{[X,Y]} = \langle \mathcal{L}_X, \mathcal{L}_Y\rangle$, and so forth...

The Schouten-Nijenhuis bracket

To extend the Lie bracket to multi-vector fields to the Schouten-Nijenhuis bracket, it seems natural to extend the identity $\iota_{[X,Y]} = \langle\langle \iota_X, d\rangle, \iota_Y\rangle$ or $\iota_{[X,Y]} = -\langle\langle \iota_Y, d\rangle, \iota_X\rangle$. As it turns out, the Schouten-Nijenhuis bracket satisfies $\iota_{[P,Q]_{SN}} = -\langle\langle \iota_Q, d\rangle, \iota_P\rangle$ for multi-vector fields P and Q. However, we cannot use this directly as a definition because it is not clear that this defines a well-defined bracket. We will introduce the Schouten-Nijenhuis bracket in the normal way, and then show that it indeed has this property, and therefore is a very natural extension of the Lie bracket.

Definition 2.22. Let M be a smooth manifold. The *Schouten-Nijenhuis bracket (SN-bracket)* is the bilinear map on multivector fields

$$[\cdot,\cdot]_{SN} : \Gamma(\wedge^\bullet TM) \times \Gamma(\wedge^\bullet TM) \to \Gamma(\wedge^\bullet TM)$$

defined on homogeneous elements $X_1 \wedge \cdots \wedge X_p$ and $Y_1 \wedge \cdots \wedge Y_q$ as

$$[X_1\wedge\cdots\wedge X_p, Y_1\wedge\cdots\wedge Y_q]_{SN} = \sum_{i,j}(-1)^{i+j}[X_i, Y_j]\wedge X_1\wedge\cdots\hat{X}_i\cdots\wedge X_p\wedge Y_1\cdots\hat{Y}_j\cdots\wedge Y_q$$

for vector fields X_i and Y_j and

$$[f, X_1\wedge\cdots\wedge X_p]_{SN} = -\iota_{df}(X_1\wedge\cdots\wedge X_p);\ [X_1\wedge\cdots\wedge X_p, f]_{SN} = (-)^p\iota_{df}(X_1\wedge\cdots\wedge X_p)$$

for a smooth function f. We will omit using the subtext SN when the context is clear. ▲

It is hard to do any calculations with this definition. The next lemma will give us some properties of the SN-bracket so that we are able to compute it more efficiently, after which we are able to write the interior multiplication by this bracket in terms of the supercommutator as described above.

Lemma 2.23. *The Schouten-Nijenhuis bracket on a manifold M satisfies for all homogeneous multi-vector fields P, Q and R:*

(i) Graded skew-symmetry: $[P, Q]_{SN} = -(-1)^{(|P|-1)(|Q|-1)}[Q, P]_{SN}$;

(ii) Poisson identity: $[P, Q\wedge R]_{SN} = [P, Q]_{SN}\wedge R + (-1)^{(|P|-1)|Q|}Q\wedge[P, R]_{SN}$.

Proof. The lemma straightforwardly follows from the definition and the skew-symmetry of the wedge product. □

Theorem 2.24. *Let P and Q be multi-vector fields on a manifold M. Then the following identity holds:*

$$\iota_{[P,Q]_{SN}} = -\langle\langle \iota_Q, d\rangle, \iota_P\rangle.$$

Proof. Notice that the identity holds on degree zero and degree 1 vector fields. We argue by induction to the degrees of both arguments. Let P, Q, X be homogeneous multi-vector fields such that X has degree 1 and suppose the identity is true up to the degrees of P and Q. Then, using the lemma above, we find:

$$\begin{aligned}
\iota_{[P\wedge X,Q]} &= -(-1)^{|P|(|Q|-1)}\iota_{[Q,P\wedge X]} \\
&= -(-1)^{|P|(|Q|-1)}\iota_{[Q,P]\wedge X} - \iota_{P\wedge[Q,X]} \\
&= -(-1)^{|P|(|Q|-1)}\iota_X\iota_{[Q,P]} + \iota_{[X,Q]}\iota_P \\
&= +(-1)^{|P|(|Q|-1)}(-1)^{(|P|-1)(|Q|-1)}\iota_X\iota_{[P,Q]} + \iota_{[X,Q]}\iota_P \\
&= -(-1)^{(|Q|-1)}\iota_X\langle\langle \iota_Q, d\rangle, \iota_P\rangle - \langle\langle \iota_Q, d\rangle, \iota_X\rangle\iota_P \\
&= -(-1)^{(|Q|-1)}\iota_X\left(\langle \iota_Q, d\rangle\iota_P - (-1)^{|P|(|Q|-1)}\iota_P\langle \iota_Q, d\rangle\right) \\
&\quad - \left(\langle \iota_Q, d\rangle\iota_X - (-1)^{(|Q|-1)}\iota_X\langle \iota_Q, d\rangle\right)\iota_P \\
&=(-1)^{(|P|+1)(|Q|-1)}\iota_{P\wedge X}\langle \iota_Q, d\rangle - \langle \iota_Q, d\rangle\iota_{P\wedge X} \\
&= -\langle\langle \iota_Q, d\rangle, \iota_{P\wedge X}\rangle.
\end{aligned}$$

A computation very similar to the one we just did will show that $\iota_{[P,Q\wedge X]_{SN}} = -\langle\langle \iota_{Q\wedge X}, d\rangle, \iota_P\rangle$, finishing the proof. □

Corollary 2.25. *The Schouten-Nijenhuis bracket on a manifold M satisfies the graded Jacobi identity. That is:*

$$(-1)^{(|P|-1)(|R|-1)}[P,[Q,R]]+(-1)^{(|Q|-1)(|P|-1)}[Q,[R,P]]+(-1)^{(|R|-1)(|Q|-1)}[R,[P,Q]] = 0,$$

for homogeneous multi-vector fields P, Q and R.

Proof. After the observation that for any multi-vector field we have $\iota_P = 0$ if and only if $P = 0$, we can use Theorem 2.24 and the fact that $d^2 = 0$ to see that all the terms, except the ones we need, cancel out. □

Back to Poisson manifolds

We finally have developed the necessary theory to characterize Poisson manifolds in terms of a bivector, given by the following proposition.

Proposition 2.26. *Let M be a manifold and $\pi \in \Gamma(\wedge^2 TM)$. Then the associated bilinear map $\{\cdot,\cdot\}$ on $C^\infty(M)$ defined by $\{f,g\} := \pi(df,dg)$ satisfies the Jacobi identity if and only if $[\pi,\pi]_{SN} = 0$.*

Proof. In this proof we will just write $[\cdot,\cdot]$ for the SN-bracket. Let $f,g,h \in C^\infty(M)$ be given. First, notice that $\{f,g\} = [g,[f,\pi]]$. Using the graded Jacobi identity of the SN-bracket, we have the following equalities:

$$\begin{aligned}
[f,[\pi,\pi]] &= -2[\pi,[f,\pi]];\\
[g,[\pi,[f,\pi]]] &= [\pi,[[f,\pi],g]] - [[f,\pi],[g,\pi]];\\
[h,[[f,\pi],[g,\pi]]] &= -[[f,\pi],[[g,\pi],h]] - [[g,\pi],[h,[f,\pi]]].
\end{aligned}$$

Using these equalities to compute $[\pi,\pi]$, we obtain:

$$\begin{aligned}
[\pi,\pi](df,dg,dh) &= -[h,[g,[f,[\pi,\pi]]]] = 2[h,[g,[\pi,[f,\pi]]]]\\
&= 2[h,[\pi,[[f,\pi],g]]] - 2[h,[[f,\pi],[g,\pi]]]\\
&= -2[h,[[[g,[f,\pi]],\pi]]] + 2[[f,\pi],[[g,\pi],h]] + 2[[g,\pi],[h,[f,\pi]]]\\
&= -2\{\{f,g\},h\} + 2\{f,\{g,h\}\} + 2\{g,\{h,f\}\}\\
&= 2(\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}).
\end{aligned}$$

Because exact 1-forms generate $\Omega^1(M)$, we can conclude that $[\pi,\pi]$ must be zero if and only if the associated bilinear map satisfies the Jacobi identity, finishing the proof. □

This classification of Poisson manifolds in terms of the associated bivector field leads to the modern definition of Poisson manifolds, which is again an algebraic structure on a manifold satisfying an integrability condition.

Definition 2.27. A *Poisson manifold* is a manifold M equipped with a bivector field $\pi \in \Gamma(\wedge^2 TM)$ satisfying $[\pi,\pi]_{SN} = 0$. The bivector field π is called the *Poisson structure.* ▲

Analogous to the dual case, a bivector field $\pi \in \Gamma(\wedge^2 TM)$ on a manifold induces a bundle map $\hat{\pi} : T^*M \to TM$ sending $\alpha \in T_p^*M$ to $\iota_\alpha \pi_p \in T_pM$. Again, the bivector field π is called *non-degenerate* when its associated bundle map $\hat{\pi}$ is a bundle isomorphism, i.e. for each $p \in M$ the map $\hat{\pi}_p : T_p^*M \to T_pM$ is an isomorphism of vector spaces. This way, each non-degenerate bivector field π corresponds to a non-degenerate 2-form $\omega \in \Omega^2(M)$ and vice versa via $\hat{\omega} = -\hat{\pi}^{-1}$. The minus sign is chosen so that the corresponding Poisson brackets coincide. Indeed, given two smooth functions $f,g \in C^\infty(M)$, we have $\{f,g\} = df(\hat{\omega}^{-1}(dg)) = -df(\hat{\pi}(dg)) = -\pi(dg,df) = \pi(df,dg)$. Interestingly, integrability of a non-degenerate bivector π directly translates to integrability of its corresponding 2-form ω (and vice versa), establishing a one-to-one correspondence between non-degenerate Poisson geometry and symplectic geometry. For notational purposes, given a 2 form ω, we sometimes denote its corresponding bivector field π by $-\omega^{-1}$.

Theorem 2.28. *Let M be a manifold. Let $\pi \in \Gamma(\wedge^2 TM)$ be a non-degenerate bivector field and let $\omega \in \Omega^2(M)$ be its corresponding 2-form. Then ω is symplectic if and only if π is Poisson.*

Proof. Because ω is non-degenerate by assumption, we only have to check that $d\omega = 0$ if and only if $[\pi, \pi]_{SN} = 0$. First we make the observation that $[\pi, \pi]_{SN} = 0$ if and only if it vanishes on exact one-forms. Then we find for $f, g, h \in C^\infty(M)$, recalling our calculation in the proof of Proposition 2.26:

$$[\pi, \pi]_{SN}(df, dg, dh) = 2(\{f, \{g, h\}\} + \text{c.p.})$$

where c.p. stands for cyclic permutations of f, g, h. On the other hand, denoting $X = \hat{\pi}(df)$, $Y = \hat{\pi}(df)$ and $Z = \hat{\pi}(dh)$:

$$\begin{aligned} d\omega(X, Y, Z) &= (X(\omega(Y, Z)) + \text{c.p.}) - (\omega([X, Y], Z) + \text{c.p.}) \\ &= (\{f, \{g, h\}\} + \text{c.p.}) - (\{f, \{g, h\}\} + \{g, \{h, f\}\} + \text{c.p.}) \\ &= -(\{f, \{g, h\}\} + \text{c.p.}) \end{aligned}$$

where in the second equality we substituted the following identities:

$$\begin{aligned} X(\omega(Y, Z)) &= \hat{\pi}(df)(\omega(\hat{\pi}(dg), \hat{\pi}(dh))) = \{f, \{g, h\}\} \\ \omega([\hat{\pi}(df), \hat{\pi}(dg)], \hat{\pi}(dh)) &= -\omega(\hat{\pi}(dh), [\hat{\pi}(df), \hat{\pi}(dg)]) \\ &= dh([\hat{\pi}(df), \hat{\pi}(dg)] \\ &= \{f, \{g, h\}\} - \{g, \{f, h\}\} \end{aligned}$$

We conclude that $[\pi, \pi]_{SN}(\alpha, \beta, \gamma) = -2d\omega(\hat{\pi}(\alpha), \hat{\pi}(\beta), \hat{\pi}(\gamma))$ for all one-forms α, β, γ. Non-degeneracy of π then ensures that $d\omega = 0$ if and only if $[\pi, \pi]_{SN} = 0$. $\square$

Let (M, ω) be a symplectic manifold. An operator that will be important in the next chapter is the interior contraction by the bivector $-\omega^{-1}$, and we will denote this operator by Λ. We now have developed the necessary theory to give an elegant proof of the following lemma.

Lemma 2.29. *Let (M, ω) be a symplectic manifold. The operators $[\Lambda, d] = \Lambda d - d\Lambda$ and Λ commute.*

Proof. By Theorem 2.28 the bivector field $\pi = -\omega^{-1}$ is Poisson. Applying Theorem 2.24 yields:

$$0 = -\iota_{[\pi,\pi]_{SN}} = \langle\langle\Lambda, d\rangle, \Lambda\rangle.$$

Because Λ has degree -2 and d has degree 1 as endomorphisms on $\Omega^\bullet(M)$, both super-commutators correspond with the normal commutator, finishing the proof. $\square$

2.4 The symplectic decomposition

As we did in the previous chapter on complex geometry, we finish this chapter with a theorem that relates integrability of a symplectic structure to the eigenspaces of an action by the symplectic structure on the exterior algebra of the cotangent bundle. This decomposition was discovered by Cavalcanti in [3], on which this section is based.

It is not a coincidence that many of the objects and results discussed here have direct analogues in the material from Section 1.4. The reason for this is that both of these discussions are special cases that arise in generalized complex geometry. The general theory has enabled us to draw similarities between complex and symplectic geometry that would not have been easily found otherwise.

2.4.1 The linear symplectic decomposition

Let V be a vector space of dimension $2n$ and denote $\mathbb{V} = V \oplus V^*$. Recall from Section 1.4 that $\mathbb{V}$ acts on the exterior algebra of V^* by

$$(X+\xi)\cdot\varphi = \iota_X\varphi + \xi\wedge\varphi, \quad \text{where } X+\xi \in \mathbb{V},\ \varphi \in \wedge^\bullet V^*.$$

This is compatible with the dual pairing via

$$(X+\xi)\cdot(X+\xi)\cdot\varphi = \xi(X)\varphi, \quad \text{for all } X+\xi \in \mathbb{V},\ \varphi \in \wedge^\bullet V^*.$$

Suppose now that V is endowed with a symplectic structure $\omega \in \wedge^2 V^*$. The symplectic structure induces a complex structure on $\mathbb{V}$ via

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\hat{\omega}^{-1} \\ \hat{\omega} & 0 \end{pmatrix}.$$

With this complex structure, we can again act on the exterior algebra $\wedge^\bullet V^*$ in such a way that it is compatible with the action described above. This action is defined as

$$\mathcal{J}_\omega \cdot \varphi = -\omega\wedge\varphi + \Lambda\varphi, \quad \text{for all } \varphi \in \wedge^\bullet V^*.$$

where Λ denotes the interior multiplication with the bivector $-\omega^{-1}$. Similar to Section 1.4 we want to find an eigenspace decomposition of the (complexified) exterior algebra of V^*. We start, after extending the actions above complex linearly, with a lemma containing some identities involving the Λ-operator, after which we are able to find the eigenspaces of the action of $\mathcal{J}_\omega$. In what comes next we have the following notation:

$$e^B\varphi = \left(1 + B\wedge + \frac{1}{2}B\wedge B\wedge + \dots\right)\varphi \qquad \text{for a 2-covector } B \in \wedge^2 V^*_{\mathbb{C}};$$
$$e^{\iota_\beta}\varphi = \left(1 + \iota_\beta + \frac{1}{2}\iota_\beta^2 + \dots\right)\varphi \qquad \text{for a bivector } \beta \in \wedge^2 V_{\mathbb{C}}.$$

Sometimes we abbreviate $e^B = e^B 1$.

Lemma 2.30. *Let (V,ω) be a symplectic vector space. For any $X \in V_{\mathbb{C}}$ and any $\varphi \in \wedge^\bullet V^*_{\mathbb{C}}$ the following identities hold:*

$$\Lambda\left((\iota_X\omega)\wedge\varphi\right) = (\iota_X\omega)\wedge\Lambda\varphi - \iota_X\varphi;$$
$$2ie^{\frac{i\Lambda}{2}}\left((\iota_X\omega)\wedge\varphi\right) = 2i(\iota_X\omega)\wedge e^{\frac{i\Lambda}{2}}\varphi + e^{\frac{i\Lambda}{2}}\iota_X\varphi.$$

Proof. We choose a basis $\{x_i, y_i\}$ of $V_{\mathbb{C}}$ in which ω is standard, i.e. $\omega = \sum_i x^i \wedge y^i$. Notice that in this basis the bivector $-\omega^{-1}$ takes the form $-\omega^{-1} = \sum_i x_i \wedge y_i$. By linearity, it is enough to check the first identity on $X = x_i$ and $X = y_i$. Because both cases are so similar, we only check the case that $X = x_i$. Let $\varphi \in \wedge^\bullet V_{\mathbb{C}}^*$. Then we have:

$$\begin{aligned}\Lambda\left((\iota_{x_i}\omega) \wedge \varphi\right) = \Lambda\left(y^i \wedge \varphi\right) &= \sum_j \iota_{y_j}\left(\iota_{x_j}\left(y^i \wedge \varphi\right)\right)\\ &= -\sum_j \iota_{y_j}\left(y^i \wedge (\iota_{x_i}\varphi)\right) = -\iota_{x_i}\varphi + y^i \wedge (\Lambda\varphi),\end{aligned}$$

which proves the first identity. Applying the first identity iteratively, and using that Λ commutes with ι_X, we obtain:

$$\Lambda^k((\iota_X\omega) \wedge \varphi) = (\iota_X\omega) \wedge \Lambda^k\varphi - k\Lambda^{k-1}\iota_X\varphi.$$

Exponentiation yields:

$$\begin{aligned}2ie^{\frac{i\Lambda}{2}}((\iota_X\omega) \wedge \varphi) &= 2i\sum_k \frac{i^k}{2^k k!}\Lambda^k((\iota_X\omega) \wedge \varphi)\\ &= 2i\sum_k \frac{i^k}{2^k k!}(\iota_X\omega) \wedge \Lambda^k\varphi + \sum_k \frac{i^{k-1}}{2^{k-1}(k-1)!}\Lambda^{k-1}(\iota_X\varphi)\\ &= 2i(\iota_X\omega) \wedge e^{\frac{i\Lambda}{2}}\varphi + e^{\frac{i\Lambda}{2}}\iota_X\varphi,\end{aligned}$$

which is the desired result. □

A first application of this lemma is to prove that, just like the complex case, the action of $\mathcal{J}_\omega$ satisfies some kind of Leibniz rule and thereby justifies its compatibility with the action of $\mathbb{V}_{\mathbb{C}}$ on the exterior algebra $\wedge^\bullet V_{\mathbb{C}}^*$.

Lemma 2.31. *For any $v \in \mathbb{V}_{\mathbb{C}}$ and $\varphi \in \wedge^\bullet V_{\mathbb{C}}^*$, the action of $\mathcal{J}_\omega$ satisfies the following equation:*

$$\mathcal{J}_\omega \cdot v \cdot \varphi = \mathcal{J}_\omega(v) \cdot \varphi + v \cdot \mathcal{J}_\omega \cdot \varphi.$$

Proof. Decomposing $v = X + \xi$, we can check this identity for X and ξ separately. For X the calculation is straightforward:

$$\begin{aligned}\mathcal{J}_\omega(X) \cdot \varphi + X \cdot \mathcal{J}_\omega \cdot \varphi &= (\iota_X\omega) \wedge \varphi + \iota_X(-\omega \wedge \varphi + \Lambda(\varphi))\\ &= -\omega \wedge (\iota_X\varphi) + \Lambda(\iota_X\varphi)\\ &= \mathcal{J}_\omega \cdot X \cdot \varphi.\end{aligned}$$

Because ω is non-degenerate, we can find a vector $Y \in V$ such that $\iota_Y\omega = \xi$. Noticing that $\mathcal{J}_\omega(\xi) = -Y$, we obtain using Lemma 2.30:

$$\begin{aligned}\mathcal{J}_\omega \cdot \xi \cdot \varphi &= -\omega \wedge \xi \wedge \varphi + \Lambda((\iota_Y\omega) \wedge \varphi)\\ &= -\xi \wedge \omega \wedge \varphi + \xi \wedge \Lambda\varphi - \iota_Y\varphi\\ &= \xi \cdot \mathcal{J}_\omega \cdot \varphi + \mathcal{J}_\omega(\xi) \cdot \varphi,\end{aligned}$$

finishing the proof. □

We now have done the necessary preparation to give and proof the eigenspace decomposition of the action of $\mathcal{J}_\omega$.

Theorem 2.32. *Let (V, ω) be a symplectic vector space of dimension $2n$. The action of $\mathcal{J}_\omega$ on $\wedge^\bullet V_{\mathbb{C}}^*$ has the eigenvalues $\{-in, -i(n-1), \dots, i(n-1), in\}$ and the eigenspaces are given by*

$$U^{n-k} = e^{i\omega} e^{\frac{i\Lambda}{2}} \wedge^k V_{\mathbb{C}}^*.$$

Moreover, $\wedge^\bullet V_{\mathbb{C}}^*$ decomposes into these eigenspaces, i.e.

$$\wedge^\bullet V_{\mathbb{C}}^* = \bigoplus_{k=-n}^{n} U^k.$$

Proof. We start with the case $k = 0$. First of all, notice that $\Lambda(\omega) = n$ and $\Lambda(\omega^j) = j(n-(j-1))\omega^{j-1}$. Indeed, after choosing a basis $\{x_i, y_i\}$ in which ω is standard, a direct computation gives:

$$\begin{aligned}
\Lambda(\omega^j) &= \sum_i \iota_{y_i}\iota_{x_i}\omega^j = j\sum_i \iota_{y_i}((\iota_{x_i}\omega)\wedge\omega^{j-1}) \\
&= j\sum_i (\iota_{y_i}\iota_{x_i}\omega)\wedge\omega^{j-1} + (j-1)y^i\wedge x^i\wedge\omega^{j-2} \\
&= j(n-(j-1))\omega^{j-1}.
\end{aligned}$$

Next, we let $\mathcal{J}_\omega$ act on $e^{i\omega}$:

$$\begin{aligned}
\mathcal{J}_\omega \cdot e^{i\omega} &= -\omega\wedge e^{i\omega} + \Lambda(e^{i\omega}) = -\omega\wedge e^{i\omega} + \sum_j \frac{i^j}{j!}\Lambda(\omega^j) \\
&= -\omega\wedge e^{i\omega} + \sum_j \frac{i^j}{(j-1)!}(n-(j-1))\omega^{j-1} \\
&= -\omega\wedge e^{i\omega} + ine^{i\omega} - \sum_j \frac{i^j}{(j-2)!}\omega^{j-1} \\
&= -\omega\wedge e^{i\omega} + ine^{i\omega} + \omega\wedge e^{i\omega} = ine^{i\omega}
\end{aligned}$$

This proves that U^n is a subset of the in-eigenspace, at least. We proceed by induction. Suppose that $\mathcal{J}_\omega$ acts on U^{n-k} by multiplication by $i(n-k)$. Given $X \in V_{\mathbb{C}}$ and $\varphi \in \wedge^k V_{\mathbb{C}}^*$ we consider the element $(X + i\iota_X\omega)\cdot\psi$ with $\psi = e^{i\omega}(e^{\frac{i\Lambda}{2}}\varphi)$.† Then on one hand we have:

$$\begin{aligned}
J_\omega \cdot (X + i\iota_X\omega)\cdot\psi &= \mathcal{J}_\omega(X + i\iota_X\omega)\cdot\psi + (X + i\iota_X\omega)\cdot\mathcal{J}_\omega\cdot\psi \\
&= -i(X + i\iota_X\omega)\cdot\psi + i(n-k)(X + i\iota_X\omega)\cdot\psi \\
&= i(n-(k+1))(X + i\iota_X\omega)\cdot\psi.
\end{aligned}$$

†The reason why we consider this particular element is that in the context of generalized complex geometry, the action of the $(+i)$-eigenspace L of $\mathcal{J}_\omega$ takes U^k to U^{k+1}, while the action of the $(-i)$-eigenspace takes U^k to U^{k-1}. Unfortunately, we haven't studied enough generalized complex geometry to make use of the full power of this theory, so we cannot use this result. We can, however, use it to guide our steps.

On the other hand, we can rewrite $(X + i\iota_X\omega)\cdot\psi$ as:

$$\begin{aligned}(X + i\iota_X\omega)\cdot e^{i\omega}(e^{\frac{i\Lambda}{2}}\varphi) &= \iota_X(e^{i\omega}(e^{\frac{i\Lambda}{2}}\varphi) + i(\iota_X\omega)\wedge e^{i\omega}(e^{\frac{i\Lambda}{2}}\varphi))\\ &= e^{i\omega}\left(2i(\iota_X\omega)\wedge e^{\frac{i\Lambda}{2}}\varphi + e^{\frac{i\Lambda}{2}}(\iota_X\varphi)\right)\\ &= e^{i\omega}e^{\frac{i\Lambda}{2}}(2i(\iota_X\omega)\wedge\varphi)\end{aligned}$$

where in the third equality we used the second identity of Lemma 2.30. Since ω is non-degenerate, any element in $\wedge^{k+1}V^*_{\mathbb{C}}$ can be written in the form $2i(\iota_X\omega)\wedge\varphi$ for some $\varphi\in\wedge^k V^*_{\mathbb{C}}$ and $X\in V$. Therefore, we can conclude that $U^{n-(k+1)}$ is at least a subspace of the $i(n-(k+1))$.

Finally, we notice that $e^{i\omega}e^{\frac{i\Lambda}{2}}$ is a (linear) automorphism of $\wedge^\bullet V^*_{\mathbb{C}}$ (indeed, one can quickly verify that it is injective) and therefore preserves the dimensions of $\wedge^k V^*_{\mathbb{C}}$. Counting dimensions, we see that

$$\wedge^\bullet V^*_{\mathbb{C}} = \bigoplus_{k=-n}^{n} U^k,$$

and thus U^k is actually equal to the ik-eigenspace. □

2.4.2 The symplectic decomposition on manifolds

Like before, we can generalize the linear algebra to manifolds. Given a manifold M equipped with a non-degenerate 2-from $\omega\in\Omega^2 M$, we obtain a bundle map $\mathcal{J}_\omega : \mathbb{T}M\to\mathbb{T}M$ via $\mathcal{J}_\omega = \left(\begin{smallmatrix}0 & -\hat{\omega}^{-1}\\ \hat{\omega} & 0\end{smallmatrix}\right)$. The action of this bundle map on $\wedge^\bullet T^*M_{\mathbb{C}}$ produces a decomposition into the ik-eigenbundles $U^k = e^{i\omega}e^{\frac{i\Lambda}{2}}\wedge^{n-k}T^*M_{\mathbb{C}}$:

$$\wedge^\bullet T^*M_{\mathbb{C}} = \bigoplus_{k=-n}^{n} U^k.$$

Smoothness of the ik-eigenbundles is guaranteed by smoothness of $e^{i\omega}$ and $e^{\frac{i\Lambda}{2}}$ as bundle maps. The space of smooth sections of the bundle U^k is denoted by $\mathcal{U}^k$ and results in a decomposition of complex valued forms on M:

$$\mathcal{U}^{n-k} = e^{i\omega}e^{\frac{i\Lambda}{2}}\left(\Omega^k(M;\mathbb{C})\right);$$
$$\Omega^\bullet(M;\mathbb{C}) = \bigoplus_{k=-n}^{n}\mathcal{U}^k.$$

Similar to the final result of Section 1.4, integrability of ω can be described by these eigenbundles. In the following, we denote $\partial = \pi^{k+1}\circ d$ and $\bar\partial = \pi^{k-1}\circ d$, where d is the exterior derivative and $\pi^k : \wedge^\bullet T^*M_{\mathbb{C}}\to U^k$ the projection map. Additionally, we let $\Psi : \wedge^\bullet T^*_{\mathbb{C}}M\to\wedge^\bullet T^*_{\mathbb{C}}M$ be given by $\Psi = e^{i\omega}e^{\frac{i\Lambda}{2}}$.

Theorem 2.33. *Let M be a manifold equipped with a non-degenerate 2-form $\omega\in\Omega^2(M)$. Then ω is symplectic if and only of $d = \partial + \bar\partial$ on $\mathcal{U}^k$ for all k. In fact, ω is symplectic if and only if $d = \partial+\bar\partial$ on $\mathcal{U}^n$.*

Additionally, the ∂- and $\bar\partial$-operators are given by $\partial(\Psi(\varphi)) = \Psi\left(\frac{1}{2i}[\Lambda, d]\varphi\right)$ and $\bar\partial(\Psi(\varphi)) = \Psi(d\varphi)$.

Proof. Suppose that ω is symplectic. By definition we have that $d\Lambda = \Lambda d - [\Lambda, d]$. Then, because by Lemma 2.29 the operators $[\Lambda, d]$ and Λ commute, we obtain by induction that $d\Lambda^j = \Lambda^j d - j\Lambda^{j-1}[\Lambda, d]$. We apply this to our computation of the exterior derivative d on a form $e^{i\omega}e^{\frac{i\Lambda}{2}}\varphi \in \mathcal{U}^k$:

$$
\begin{aligned}
d(e^{i\omega}e^{\frac{i\Lambda}{2}}\varphi) &= id\omega \wedge e^{i\omega}e^{\frac{i\Lambda}{2}}\varphi + e^{i\omega}d(e^{\frac{i\Lambda}{2}}\varphi) = e^{i\omega}\sum_j \frac{i^j}{2^j j!} d\Lambda^j\varphi \\
&= e^{i\omega}\left(\sum_j \frac{i^j}{2^j j!}\Lambda^j d\varphi + \frac{1}{2i}\sum_j \frac{i^{j-1}}{2^{j-1}(j-1)!}\Lambda^{j-1}[\Lambda, d]\varphi\right) \\
&= e^{i\omega}e^{\frac{i\Lambda}{2}}d\varphi + e^{i\omega}e^{\frac{i\Lambda}{2}}\frac{1}{2i}[\Lambda, d]\varphi \\
&= \bar{\partial} + \partial,
\end{aligned}
$$

where in the last step we used the observation that $d\varphi \in \Omega^{n-k+1}(M; \mathbb{C})$ and $[\Lambda, d]\varphi \in \Omega^{n-k-1}(M; \mathbb{C})$, so that $\partial(e^{i\omega}e^{\frac{i\Lambda}{2}}\varphi) = e^{i\omega}e^{\frac{i\Lambda}{2}}\frac{1}{2i}[\Lambda, d]\varphi$ and $\bar{\partial}(e^{i\omega}e^{\frac{i\Lambda}{2}}\varphi) = e^{i\omega}e^{\frac{i\Lambda}{2}}d\varphi$.

Conversely, suppose that d sends $\mathcal{U}^n$ to $\mathcal{U}^{n-1}$. First, we notice that $e^{\frac{i\Lambda}{2}}$ is the identity when restricted to $\Omega^1(M)$. Thus, an element $e^{i\omega}e^{\frac{i\Lambda}{2}}\alpha = e^{i\omega}\alpha \in \mathcal{U}^{n-1}$ is zero if and only if its degree-1 component is zero. Then, given $e^{i\omega}f \in \mathcal{U}^n$, we have:

$$
d(e^{i\omega}f) = id\omega \wedge e^{i\omega}f + e^{i\omega}df.
$$

By assumption, $d(e^{i\omega}f) \in \mathcal{U}^{n-1}$. Additionally, $e^{i\omega}df \in \mathcal{U}^{n-1}$ and therefore $id\omega \wedge e^{i\omega}f \in \mathcal{U}^{n-1}$. The degree-1 component of this last element is clearly zero and therefore $d\omega \wedge e^{i\omega}f = 0$. Since f is an arbitrary function, we deduce that $d\omega = 0$, so that ω is integrable. □

Corollary 2.34. *On a symplectic manifold (M, ω) the operators ∂^2, $\partial\bar{\partial} + \bar{\partial}\partial$ and $\bar{\partial}^2$ are all equal to zero.*

Proof. We use Theorem 2.33 to write $d^2 = \partial^2 + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\partial}^2 = 0$ and observe that each of the mentioned operators land in a different $\mathcal{U}^k$. □

Chapter 3

Kähler Geometry

In this concluding chapter, we will state and proof the main result of this thesis: the Hodge decomposition on compact Kähler manifolds. To arrive at that point, we first define the notion of Kähler manifolds. After that, we will concern ourselves with (generalized) Hodge theory, where we will make heavy use of the results from the previous two chapters. It should be pointed out the Hodge theory we introduce here slightly differs from the classical Hodge theory. The (generalized) Hodge star we introduce in this chapter has great advantages over the classical Hodge star. For instance, the former diagonalizes along the common eigenspaces of forms from the previous chapters (as stated in Theorem 3.23), while for the latter the compatibility with the decomposition by degree is only slight. Although proving Theorem 3.23 requires some technical proofs of important lemmas, we hope that this serves as an argument to justify the naturality of our choices. We will finish our discussion of Hodge theory with the Kähler identities, one of the two pillars supporting the proof of the Hodge decomposition on compact Kähler manifolds. In our framework, these famous identities take a form that is slightly different than usual, but much more elegant. Finally, we will turn towards cohomology. We will state without proof the other pillar on which the Hodge decomposition theorem on compact Kähler manifolds rests, after which we finally get to its proof. We will conclude the chapter with some applications of this result.

3.1 Riemannian and Hermitian metrics

Metrics on manifolds are the last ingredient we need to define Kähler manifolds. We start by introducing Riemannian metrics on a smooth manifold. Then we endow the smooth manifold with an almost complex structure that is compatible with the Riemannian metric in such a way that it leads to the notion of almost Hermitian manifolds. Finally, we will see that any Hermitian structure automatically comes with a non-degenerate 2-form. After this last observation we arrive at the end of the road to Kähler manifolds. This section, as well as the next, is based on [10], [11] and [14].

3.1.1 Riemannian manifolds

Let us start with the definition.

Definition 3.1. Let M be a smooth manifold. A *Riemannian metric* is a smooth symmetric rank-2 tensor $g \in \Gamma(\Sigma^2 T^* M)$ such that at each point $p \in M$ the bilinear map g_p is an inner product on $T_p M$. A manifold endowed with a Riemannian metric is called a *Riemannian manifold.* ▲

In local coordinates (x^i), we can write any Riemannian metric g on a manifold M as

$$g = g_{ij} dx^i \otimes dx^j.$$

The matrix g_{ij} is symmetric and positive definite. Using the symmetry $g_{ij} = g_{ji}$, we can write the metric in terms of the symmetric product as follows.

$$\begin{aligned} g &= g_{ij} dx^i \otimes dx^j \\ &= \frac{1}{2}\left(g_{ij} dx^i \otimes dx^j + g_{ji} dx^i \otimes dx^j\right) \\ &= \frac{1}{2} g_{ij}\left(dx^i \otimes dx^j + dx^j \otimes dx^i\right) \\ &= g_{ij} dx^i \cdot dx^j, \end{aligned}$$

where the dot $\cdot$ denotes the symmetric product.

Example 3.2. The standard Euclidean metric on $\mathbb{R}^n$ (as a manifold, not as a vector space) is given, in coordinates (x^i), by

$$g = \delta_{ij} dx^i \cdot dx^j.$$

This is related to the standard inner product on $T_x\mathbb{R}^n$ as follows. For $v, w \in T_x\mathbb{R}^n$ we have

$$g_x(v, w) = \delta_{ij}(dx^i_x \cdot dx^j_x)(v, w) = \sum_i dx^i(v) dx^i(w) = \sum_i v^i w^i. \qquad \triangle$$

Example 3.3. Every smooth n-manifold M can be endowed with a Riemannian metric. First, one chooses a collection of smooth charts $(U_\alpha, \varphi_\alpha)$ covering M. Then, one defines a Riemannian metric on each of the opens U_α by pulling back the standard Euclidean metric on $\mathbb{R}^n$ to U_α, i.e. one defines a Riemannian metric on U_α by $g_\alpha = \varphi_\alpha^* g_0$, where g_0 is the standard Riemannian metric on $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$. Then one chooses a partition of unity $\{\rho_\alpha\}$ subordinated to the open cover $\{U_\alpha\}$ and defines a Riemannian metric g on M by $g = \sum_\alpha \rho_\alpha g_\alpha$. △

As orthonormality is an important and very useful concept in the study of inner product spaces, it is very useful for Riemannian manifolds. In particular, when studying inner product spaces, one cannot go around orthonormal bases. These have a natural analogue on manifolds, namely orthonormal frames. If (M, g) is a Riemannian n-manifold, then a *local orthonormal frame* over an open subset U is a local frame

$\{X^1, \ldots, X^n\}$ over U such that at each point $p \in U$ the vectors $\{X^1_p, \ldots, X^n_p\}$ form an orthonormal basis of T_pM. Fortunately, the existence of an orthonormal basis on any inner product space extends to Riemannian manifolds, as shown by the next proposition.

Proposition 3.4. *Let (M, g) be a Riemannian manifold. Then there exists a smooth local orthonormal frame around any point in M.*

Proof. Pick local coordinates (x^i) and apply the Gram-Schmidt procedure to the local vector fields $\{\frac{\partial}{\partial x^i}\}$. Then we notice that every step in this procedure can be done smoothly. $\square$

The Riemannian metric g induces a bundle isomorphism $\hat{g} : TM \to T^*M$ sending $v \in T_pM$ to $g_p(v, \cdot) \in T^*_pM$. This way, we can define, for each $p \in$M, an inner product on $\wedge^k T^*_pM$ as

$$\langle v^1 \wedge \cdots \wedge v^k, w^1 \wedge \cdots \wedge w^k\rangle_{g_p} = \det(g_p(\hat{g}^{-1}(v^i), \hat{g}^{-1}(w^j)), \quad v^i, w^j \in T^*_pM.$$

This extends to a smooth symmetric bilinear map on $\wedge^k T^*M$ and we will denote its corresponding section in $\Gamma(\Sigma^2(\wedge^k T^*M))$ also by g. Therefore, we can talk about the length of a form at a point $p \in M$.

Definition 3.5. Let (M, g) be an oriented Riemannian manifold. We call the unique positive top-degree form of unit length the *Riemannian volume form* and denote it by vol_g. ▲

One may wonder whether this is Riemannian volume form is actually well-defined. Existence is immediate. Indeed, from the orientation on M we obtain a positive top-degree form ω. Then we simply divide ω by its length to obtain the Riemannian volume form, i.e. $\mathrm{vol}_g = \omega/\sqrt{g(\omega, \omega)}$. For uniqueness, it is enough to notice that, because $\wedge^{top} T^*_pM$ is one dimensional for all $p \in M$, to each positive scalar λ there is a unique positive vector in $\wedge^{top} T_pM$ with length λ. Given a local positive orthonormal frame $X^1, \ldots, X^n$, we can write vol_g as

$$\mathrm{vol}_g = X^1 \wedge \cdots \wedge X^n.$$

3.1.2 Hermitian manifolds

Recall from linear algebra that a Hermitian product on a complex vector space V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ satisfying the following conditions:

(i) (Sesquilinearity) The map $\langle \cdot, \cdot \rangle$ is complex linear in the first argument and conjugate linear in the second;

(ii) (Conjugate symmetry) For all $v, w \in V$ we have $\langle v, w \rangle = \overline{\langle w, v \rangle}$;

(iii) (Positive-definiteness) For all $v \in V$ we have $\langle v, v \rangle \geq 0$, where the equality holds if and only if $v = 0$.

Any Hermitian product $\langle\cdot,\cdot\rangle$ on a complex vector space V satisfies $\langle iv, iw\rangle = \langle v, w\rangle$ for all $v, w \in V$. Therefore, if V is a real vector space with an inner product $\langle\cdot,\cdot\rangle$ and a complex structure J, it seems natural to call these two structures compatible when $\langle J(v), J(w)\rangle = \langle v, w\rangle$ for all $v, w \in V$. This leads to the following definition on manifolds.

Definition 3.6. Let (M, J) be an almost complex manifold. A Riemannian metric h on M is called a *Hermitian* metric if $h_p(v, w) = h_p(J_p(v), J_p(w))$ for all $p \in M$ and $v, w \in T_pM$. ▲

A Hermitian metric h on an almost complex manifold (M, J) comes with a 2-form, called the *fundamental 2-form*, defined by $\omega_p(v, w) = h_p(J_p(v), w)$. Because J and $\hat{h}$ are both isomorphisms, the fundamental 2-form ω is always non-degenerate.

Example 3.7. Every almost complex n-manifold (M, J) can be endowed with a Hermitian metric. First one chooses a Riemannian metric g on M and then defines the Hermitian metric as $h_p(v, w) = g_p(v, w) + g_p(J_p(v), J_p(w))$ for $p \in M$ and $v, w \in T_pM$. As a corollary, any almost complex manifold is orientable. Indeed, after picking a Hermitian metric h on M, then we obtain a nowhere-vanishing top-degree form ω^n (a consequence of Lemma 2.6), with ω the fundamental 2-form. This orientation coincides with the orientation induced by almost complex structures in Section 1.2. △

Let h be a Hermitian metric on an almost complex manifold (M, J). As usual, we can extend the Hermitian metric complex linearly to the complexified tangent bundle. If the almost complex structure J comes from a holomorphic structure, we can actually compute the fundamental 2-from ω of the Hermitian metric h explicitly. First, observe that if $v, w \in T_p^{1,0}M$ or $v, w \in T_p^{0,1}M$, then $h_p(v, w) = h_p(J_p(v), J_p(w)) = h_p(iv, iw) = -h_p(v, w)$. We deduce, as J preserves its own eigenspaces, that ω is of type $(1,1)$. Choosing local holomorphic coordinates (z^i), we can write, by symmetry of h:

$$h = h_{ij}(dz^i \otimes d\bar{z}^j + d\bar{z}^j \otimes dz^i),$$

where $h_{ij} = h\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right)$ is a positive definite Hermitian matrix (Hermitian in the sense that $h_{ij} = \bar{h}_{ji}$). A simple computation shows that $\omega(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^k}) = ih_{jk}$. Therefore, ω can be written as

$$\omega = ih_{jk} dz^j \wedge d\bar{z}^k.$$

Conversely, a positive definite Hermitian matrix h_{ij} defines a local Hermitian metric h via the formula above.

Finally, we define, for the sake of completeness, the notion of Hermitian manifolds. After we have done that, the end of the road to Kähler is in plain sight.

Definition 3.8. A triple (M, g, J) of a smooth manifold, a Riemannian metric and an almost complex structure is called an *almost Hermitian manifold* if g is a Hermitian metric with respect to the almost complex structure J. The triple is called a *Hermitian manifold* when in addition the almost complex structure J is integrable. ▲

3.2 Kähler manifolds

Kähler geometry is where all the previous material comes together. It is the combination of three important structures in geometry: a Riemannian metric, a complex structure and a symplectic structure.

Definition 3.9. Let M be a smooth manifold. A *Kähler structure* on M is a triple (g, J, ω) consisting of a Riemannian metric g, a complex structure J and a symplectic structure ω related by the commutative diagram below.

$$\begin{tikzcd} TM \arrow[rr, "\hat{g}"] & & T^*M \\ & TM \arrow[ul, "J"] \arrow[ur, "\hat{\omega}"'] & \end{tikzcd}$$

If there exists a Kähler structure on a manifold M, then M is said to be of *Kähler type*. The quadruple (M, g, J, ω) is called a *Kähler manifold* and the metric g is called a *Kähler metric*. ▲

There are a lot of different (equivalent) definitions of Kähler manifolds, and the most useful definition would depend on the context. For example, when encountering Kähler manifold in the context of complex geometry, a definition of a Kähler manifold as a Hermitian manifold with closed fundamental form may be more natural, while in the context of symplectic geometry laying emphasis on integrability of the complex structure may be preferable, since the symplectic form is already integrable by definition. The reason for choosing our definition is that, in it, each of the structures is equally important, so that it allows multiple viewpoints. We will dedicate the rest of this section to examples.

Example 3.10. Complex Euclidean space $\mathbb{C}^n$ admits a standard Kähler structure. Let g, J and ω denote the standard Euclidean metric, the standard complex structure and the standard symplectic structure on $\mathbb{C}^n$, respectively. First, in coordinates ($z^j = x^j + iy^j$) we can write the metric g as

$$g = \delta_{jk} dx^j \cdot dx^k + \delta_{jk} dy^j \cdot dy^k.$$

To see that the structures are compatible, we quickly verify that

$$g\left(J\left(\frac{\partial}{\partial x^j}\right), \frac{\partial}{\partial y^k}\right) = -g\left(J\left(\frac{\partial}{\partial y^j}\right), \frac{\partial}{\partial x^k}\right) = \delta_{jk};$$
$$g\left(J\left(\frac{\partial}{\partial y^i}\right), \frac{\partial}{\partial y^k}\right) = g\left(J\left(\frac{\partial}{\partial x^i}\right), \frac{\partial}{\partial x^k}\right) = 0.$$

Therefore $\hat{g} \circ J = \hat{\omega}$, making $(\mathbb{C}^n, g, J, \omega)$ into a Kähler manifold. △

Example 3.11. Any Riemann surface Σ is a manifold of Kähler type. Indeed, letting J denote the complex structure on Σ, we pick a Hermitian metric g compatible with J. Then its fundamental 2-form ω is automatically closed, because the degree of $d\omega$ exceeds the dimension of Σ. △

Example 3.12. The complex torus from Example 1.7 admits natural Kähler structure. Given a lattice Γ in $\mathbb{C}^n$, we see that translation by elements in this lattice leaves the standard Kähler structure of $\mathbb{C}^n$ invariant. Therefore, adopting the notation from Example 1.7, the charts in the form $\left(V_{q(z)}, (q\big|_{U_z})^{-1}\right)$ pull back the standard Kähler structure to a Kähler structure on the complex torus $\mathbb{C}^n/\Gamma$. △

Example 3.13. The complex projective space $\mathbb{CP}^n$ admits a canonical Kähler metric, called the *Fubini-Study metric*, that we will construct in this example.

Let (U_i, φ_i) denote the standard charts on $\mathbb{CP}^n$ as in Example 1.8. Define smooth functions $f_i : U_i \to \mathbb{R}$ by

$$f_i(q(z^1, \dots, z^{n+1})) = \log\left(\sum_k \left|\frac{z^k}{z^i}\right|^2\right).$$

Next, we define a 2-form of type $(1,1)$ on U_i by

$$\omega_i = \frac{i}{2\pi}\partial\bar{\partial} f_i \in \Omega^{1,1}(U_i).$$

To verify that this defines a global 2-form ω_{FS} on $\mathbb{CP}^n$, we check that $\omega_i\big|_{U_i\cap U_j} = \omega_j\big|_{U_i\cap U_j}$. First of all, we make the following observation: for $q(z^k) \in U_i \cap U_j$ we have

$$\begin{aligned} f_i(q(z^k)) &= \log\left(\sum_k \left|\frac{z^k}{z^i}\right|^2\right) = \log\left(\left|\frac{z^j}{z^i}\right|^2 \sum_k \left|\frac{z^k}{z^j}\right|^2\right) \\ &= \log\left(\left|\frac{z^j}{z^i}\right|^2\right) + f_j(q(z^k)). \end{aligned}$$

Secondly, we notice that $\frac{z^j}{z^i}$ is precisely the component of the chart φ_i, which is holomorphic. In general, frequently applying Proposition 1.25), the following identity holds for any holomorphic function g on the domain where it is non-zero:

$$\partial\bar{\partial}\log(g\bar{g}) = \partial\left(\frac{1}{g\bar{g}}\bar{\partial}(g\bar{g})\right) = \partial\left(\frac{g\bar{\partial}\bar{g}}{g\bar{g}}\right) = \partial\left(\frac{\bar{\partial}\bar{g}}{\bar{g}}\right) = \partial\left(\frac{1}{\bar{g}}\right)\wedge\bar{\partial}\bar{g} + \frac{1}{\bar{g}}\partial\bar{\partial}\bar{g} = 0$$

where in the last equality we also used that $\partial\bar{\partial} = -\bar{\partial}\partial$ on a complex manifold (see Proposition 1.27). We conclude that $\partial\bar{\partial}f_i = \partial\bar{\partial}f_j$ on $U_i \cap U_j$ and thus ω_i and ω_j agree on their overlap. Hence, ω_{FS} is a well-defined smooth global $(1,1)$-form.

Furthermore, the form ω_{FS} is real. Indeed, from $\overline{\partial\bar{\partial}} = \bar{\partial}\partial = -\partial\bar{\partial}$ we see that $\omega_i = \bar{\omega}_i$. Integrability of ω_{FS} follows directly form the integrability of the complex structure:

$$d\omega_i = \frac{i}{2\pi}(\partial + \bar{\partial})\partial\bar{\partial}f_i = \frac{i}{2\pi}(\partial^2\bar{\partial} - \bar{\partial}^2\partial)f_i = 0.$$

We then define the *Fubini-Study metric* as the tensor g_{FS} defined by

$$(g_{FS})_p(v, w) = (\omega_{FS})_p(v, J_p(w)), \quad \text{for all } p \in \mathbb{CP}^n \text{ and } v, w \in T_p\mathbb{CP}^n_{\mathbb{C}}.$$

Because ω_{FS} is a real 2-form of type $(1,1)$, the tensor g_{FS} is compatible with the complex structure and the matrix $(g_{FS})_{ij}$ w.r.t. the coordinates φ_i is Hermitian. It remains to show that this matrix is positive definite. To calculate the entries $(g_{FS})_{ij}$, we first make the observation that

$$f_i \circ \varphi_i^{-1}(w^k) = \log\left(1 + \sum_k |w^k|^2\right).$$

Then in local coordinates, we can compute the entries via the form ω_{FS}:

$$\begin{aligned}\omega_i &= \frac{i}{2\pi}\partial\bar{\partial}\log\left(1+\sum_k |w^k|^2\right)\\ &= \frac{i}{2\pi}\frac{\sum_k dw^k \wedge d\bar{w}^k}{1+\sum_k |w^k|^2} - \frac{\left(\sum_k \bar{w}^i dw^i\right)\wedge\left(\sum_k w^k d\bar{w}^k\right)}{1+\sum_k |w^k|^2}\\ &= i\frac{1}{2\pi\left(1+\sum_k |w^k|^2\right)}\sum_{j,k} h_{jk} dw^j \wedge d\bar{w}^k,\end{aligned}$$

where $h_{ij} = \left(1 + \sum |w^k|^2\right)\delta_{ij} - \bar{w}^i w^j$. Reading off the coefficients $(g_{FS})_{ij}$, we only need to show that the matrix h_{ij} is positive definite. This follows from the Cauchy-Schwarz inequality for the standard Hermitian product $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^n$: if $v \in \mathbb{C}^n$ is a non-zero vector, then we have

$$\begin{aligned}v^T(h_{ij})\bar{v} &= \langle v, v\rangle + \langle w, w\rangle\langle v, v\rangle - v^T \bar{w} w^T \bar{v}\\ &= \langle v, v\rangle + \langle w, w\rangle\langle v, v\rangle - \langle u, w\rangle\langle w, u\rangle\\ &= \langle v, v\rangle + \langle w, w\rangle\langle v, v\rangle - |\langle w, v\rangle|^2 > 0.\end{aligned}$$

We conclude that g_{FS} is a Hermitian metric with a closed fundamental form ω_{FS}, making $(\mathbb{CP}^n, g_{FS}, J, \omega_{FS})$ into a Kähler manifold. △

Example 3.14. Let (M, g, J, ω) be a Kähler manifold. Then each complex submanifold $S \overset{i}{\hookrightarrow} M$ is again a Kähler manifold when restricting the structures on M to S, i.e. $(S, i^*g, i^*J, i^*\omega)$ is a Kähler manifold. To prove this, first we notice that for $p \in S$, the restriction of g_p to T_pS is again an inner product, thus i^*g is a Riemannian metric on S. Next, since there exists local flat charts for S, the complex structure J restricts to the complex structure on S. Finally, one readily verifies that $\hat{g}|_S \circ J|_S = \hat{\omega}_S$. We obtain integrability of $i^*\omega$ immediately: $di^*\omega = i^*d\omega = 0$. △

Example 3.15. The Hopf surface H from Example 1.9 is an example of a complex manifold that does *not* admit a Kähler structure. The reason for this is that, calling on our knowledge of differentiable manifolds, the second de Rham cohomology group $H^2_{dR}(H) = H^2_{dR}(S^3 \times S^1)$ is zero, and therefore any closed 2-form in $\Omega^2(H)$ is exact. If the H were of Kähler type, then after choosing a Kähler metric, we have $\int_H \omega\wedge\omega \neq 0$ with ω the fundamental 2-form. But since ω is closed, it is exact and can be written as $\omega = d\alpha$ for some one-form α. But then $\int_H \omega\wedge\omega = \int_H (d\alpha)\wedge(d\alpha) = \int_H d(\alpha\wedge d\alpha) = 0$ by Stokes' theorem. We conclude that H is not of Kähler type. △

3.3 Linear Hodge theory

Generalized Hodge theory is in principle more or less the same as the classical Hodge theory. The main difference is that the latter heavily makes use of the framework provided by generalized complex geometry. However, this only leads to minor sign differences. Nevertheless, generalized Hodge theory is much more elegant in its own framework, and compatible with the eigenspace decompositions discussed in the previous chapters. Since in this thesis we will only make use of generalized Hodge theory, we call it just Hodge theory. When a reference is made to classical Hodge theory, it will be done so explicitly. This section, as well as the next, is based on the papers [2], [4] and [8], although the concepts and proofs presented in these papers had to be adapted or even replaced for this thesis.

3.3.1 The Hodge star

Let us start by defining two central features of Hodge theory, and discuss them in more detail.

Definition 3.16. Let V be an oriented real m-dimensional vector space equipped with an inner product g. Let $\{e_1, \dots, e_m\}$ be a positive orthonormal basis of V. We define the *(generalized) Hodge star* operator $\star$ by

$$\star : \wedge^\bullet V^* \to \wedge^\bullet V^*, \quad \star\varphi = (e_m + \hat{g}(e_m)) \cdot \ldots \cdot (e_1 + \hat{g}(e_1)) \cdot \varphi. \qquad \blacktriangle$$

Definition 3.17. For a vector space V, the *Chevalley pairing* $(\cdot,\cdot)_{Ch}$ on $\wedge^\bullet V^*$is defined by

$$(\varphi, \psi)_{Ch} = (\varphi \wedge \psi^t)_{top}, \quad \text{for all} \quad \varphi, \psi \in \wedge^\bullet V^*,$$

where the subscript *top* indicates taking the top-degree component, and the superscript t is the transposition defined on homogeneous multi-covectors $(v^1 \wedge \cdots \wedge v^k)^t = v^k \wedge \cdots \wedge v^1 = (-1)^{\frac{k(k-1)}{2}} v^1 \wedge \cdots \wedge v^k$. ▲

Give two multi-covectors $\varphi, \psi \in \wedge^\bullet V^*$, we can decompose them by degree as $\varphi = \sum_j \varphi_j$ and $\psi = \sum_j \psi_j$. We then have a formula for the Chevalley pairing:

$$(\varphi, \psi)_{Ch} = \sum_j (-1)^{\frac{(m-j)(m-j-1)}{2}} \varphi_j \wedge \psi_{m-j}.$$

Let $\{e_i\}$ be a basis of the vector space V and let $\{e^i\}$ be its dual. An important property of the Chevalley pairing that we will use a lot in the future is that the degree-k covector $e^{i_1} \wedge \cdots \wedge e^{i_k}$ pairs non-trivially to $e^{j_1} \wedge \cdots \wedge e^{j_l}$ if and only if the multi-indices $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_l\}$ are complementary, i.e. $I \cup J = \{1, \dots, n\}$ and $l = m - k$.

Before we move on to the properties of the Hodge star, we will first shortly discuss some of its features to get a feeling for how it works.

We start the discussion with the following useful identities. Recall from Section 1.4 that for any $X+\xi \in \mathbb{V}$ and $\varphi \in \wedge^\bullet V^*$ we have $(X+\xi)\cdot(X+\xi)\cdot\varphi = \xi(X)\varphi$. From this we can obtain the following identities: for $X, Y \in V$ and $\xi, \eta \in V^*$, we have

$$\begin{aligned} X\cdot Y\cdot\varphi &= -Y\cdot X\cdot\varphi;\\ \xi\cdot\eta\cdot\varphi &= -\eta\cdot\xi\cdot\varphi;\\ X\cdot\xi\cdot\varphi &= \xi(X)\varphi - \xi\cdot X\cdot\varphi. \end{aligned}$$

Now let $\{e_1,\dots,e_m\}$ be an orthonormal basis of V and denote $e^i = \hat{g}(e_i)$ and $E_i = e_i + e^i$. Notice that we can always anti-commute the actions of E_i and E_j whenever $i \neq j$, while if $i = j$, we have $E_i \cdot E_i \cdot \varphi = \varphi$ for all multi-covectors φ.

Let us compute the Hodge star on some specific elements. First of all, one easily sees that

$$\star(1) = e^m \wedge \cdots \wedge e^1 = (-1)^{\frac{m(m-1)}{2}}\mathrm{vol}_g \quad \text{and} \quad \star(\mathrm{vol}_g) = 1.$$

Now it gets more complicated. Let $I = \{i_1,\dots,i_k\}$ be a strictly increasing subset of $\{1,\dots,m\}$ and let $J = \{j_1,\dots,j_{m-k}\}$ denote its complement, also ordered so that it is strictly increasing. We want to compute the Hodge star on the element $e^{i_1}\wedge\cdots\wedge e^{i_k}$. First of all, notice that $E_l \cdot e^{i_1}\wedge\cdots\wedge e^{i_k} = \iota_{e_l}(e^{i_1}\wedge\cdots\wedge e^{i_k})$ when $l = i_1,\dots,i_k$ and $E_l \cdot e^{i_1}\wedge\cdots\wedge e^{i_k} = e^l \wedge e^{i_1}\wedge\cdots\wedge e^{i_k}$ when $l = j_1,\dots,j_{m-k}$. Therefore, if we let σ_{JI} denote the permutation sending $(1,\dots,m)$ to $(j_1,\dots,j_{m-k},i_1,\dots,i_k)$ we have

$$\begin{aligned} \star(e^{i_1}\wedge\cdots\wedge e^{i_k}) &= E_m\cdot\ldots\cdot E_1\cdot(e^{i_1}\wedge\cdots\wedge e^{i_k})\\ &= (-1)^{\frac{m(m-1)}{2}}\mathrm{sign}(\sigma_{JI})E_{j_1}\cdot\ldots\cdot E_{j_{m-k}}\cdot E_{i_1}\cdot\ldots\cdot E_{i_k}\cdot(e^{i_1}\wedge\cdots\wedge e^{i_k})\\ &= (-1)^{\frac{m(m-1)}{2}}(-1)^{\frac{k(k-1)}{2}}\mathrm{sign}(\sigma_{JI})E_{j_1}\cdot\ldots\cdot E_{j_{m-k}}\cdot E_{i_k}\cdot\ldots\cdot E_{i_1}\cdot(e^{i_1}\wedge\cdots\wedge e^{i_k})\\ &= (-1)^{\frac{m(m-1)}{2}}(-1)^{\frac{k(k-1)}{2}}\mathrm{sign}(\sigma_{JI})e^{j_1}\wedge\cdots\wedge e^{j_{m-k}}. \end{aligned}$$

Unfortunately, we cannot reduce this any further because the sign of the permutation σ_{IJ} highly depends on I and J. However, this result is already enough to prove the basic properties of the Hodge star in the proposition below.

Proposition 3.18. *Let V be an oriented real m-dimensional vector space equipped with an inner product g. Then the following statements hold.*

(i) The Hodge star $\star$ is independent of the choice of positive orthonormal basis.

(ii) We have $\star^2 = (-1)^{\frac{m(m-1)}{2}}$.

(iii) For any nonzero $\varphi \in \wedge^\bullet V^*$, we have $(\varphi, \star\varphi)_{Ch} > 0$.

(iv) For all $\varphi, \psi \in \wedge^\bullet V^*$ we have $(\varphi, \star\psi)_{Ch} = (\psi, \star\varphi)_{Ch}$.

(v) The generalized Hodge star $\star$ is related to the classical Hodge star $*$ via $(\varphi, \star\psi)_{Ch} = (\varphi \wedge *\psi)_{top}$.

Proof. Statement (i) follows from the fact that the determinant of a linear map sending a positive orthonormal basis V to another positive orthonormal basis of V is always 1, then applied to our formula of the Hodge star we found in the discussion above.

Property (ii) follows directly:

$$\star^2\varphi = E_m\cdot\ldots\cdot E_1\cdot E_m\cdot\ldots\cdot E_1\cdot\varphi = (-1)^{\frac{m(m-1)}{2}}E_n\cdot\ldots\cdot E_1\cdot E_1\cdot\ldots\cdot E_m\cdot\varphi = (-1)^{\frac{m(m-1)}{2}}\varphi.$$

For (iii), notice that it reduces to proving the inequality for elements in the form $\lambda e^{i_1}\wedge\cdots\wedge e^{i_k}$ with $\lambda\neq 0$, because $e^{i_1}\wedge\cdots\wedge e^{i_k}$ pairs trivially with $e^{j_1}\wedge\cdots\wedge e^{j_l}$ whenever I and J are not complementary. We compute:

$$\begin{aligned}
(\lambda e^{i_1}\wedge\cdots\wedge e^{i_k}, \star\,\lambda e^{i_1}\wedge\cdots\wedge e^{i_k})_{Ch}& \\
&= (\lambda e^{i_1}\wedge\cdots\wedge e^{i_k}, \star\lambda e^{i_1}\wedge\cdots\wedge e^{i_k})_{Ch}\\
&= \lambda^2(-1)^{\frac{m(m-1)}{2}}(-1)^{\frac{k(k-1)}{2}}\operatorname{sign}(\sigma_{JI})e^{i_1}\wedge\cdots\wedge e^{i_k}\wedge e^{j_{n-k}}\wedge\cdots\wedge e^{j_1}\\
&= \lambda^2\operatorname{sign}(\sigma_{JI})(-1)^{\frac{k(k-1)}{2}}e^{j_1}\wedge\cdots\wedge e^{j_{m-k}}\wedge e^{i_k}\wedge\cdots\wedge e^{i_1}\\
&= \lambda^2\operatorname{sign}(\sigma_{JI})e^{j_1}\wedge\cdots\wedge e^{j_{m-k}}\wedge e^{i_1}\wedge\cdots\wedge e^{i_k}\\
&= \lambda^2\mathrm{vol}_g > 0.
\end{aligned}$$

To prove (iv), it again reduces to prove the symmetry for $\varphi = e^{i_1}\wedge\cdots\wedge e^{i_k}$. But then the result is straightforward because $\star\varphi$ only pairs non-trivially with φ (up to scalar multiplication).

As for statement (v) there is nothing to prove, since we have not defined the classical Hodge star. It is stated here so that the reader who is already familiar with classical Hodge theory can relate the new theory to familiar material. Classical Hodge theory is introduced in for example [10] and [18]. □

Notice that (iii) and (iv) imply that the pairing $(\cdot,\star\cdot)_{Ch}$ is an inner product on $\wedge^\bullet V^*$. This inner product is, in fact, equal to the usual inner product induced by g on $\wedge^\bullet V^*$ and induces a Hermitian product $\wedge^\bullet V^*_{\mathbb{C}}$ via $(\varphi,\star\bar{\psi})_{Ch}$.

3.3.2 Hodge theory on Kähler vector spaces

We have already defined the notion of a Kähler manifold. In this section, however, we will only discuss linear algebra, and thus, for completeness, we start by defining the linear analogue of Kähler manifolds.

Definition 3.19. A *Kähler vector space* is a quadruple (V, g, J, ω) of a real vector space V, an inner product g, a complex structure J and a symplectic 2-vector ω related by the commutative diagram below.

$$\begin{tikzcd}
V \arrow[rr, "\hat{g}"] & & V^* \\
& V \arrow[ul, "J"] \arrow[ur, "\hat{\omega}"'] &
\end{tikzcd}$$

▲

Although Hodge theory can be done on any finite dimensional (oriented) inner product space, we are particularly interested in its relation to the other structures on a Kähler vector spaces. Recall from Section 1.4 and Section 2.4 that J and ω induce complex structures $\mathcal{J}_J$ and $\mathcal{J}_\omega$ on $\mathbb{V}$, and that those complex structures give rise to a decomposition of $\wedge^\bullet V^*_\mathbb{C}$ into eigenspaces of the action of $\mathcal{J}_J$ and $\mathcal{J}_\omega$ on $\wedge^\bullet V^*_\mathbb{C}$. We denote by $U^p_\mathcal{J}$ and U^q_ω the eigenspaces of the actions of $\mathcal{J}_J$ and $\mathcal{J}_\omega$, respectively.

The rest of this section consists of important results about the compatibility of these structures with the Hodge theory. Although the proofs can be quite technical, these results will make the Hodge star and the Chevalley pairing incredibly easy to work with, and will hopefully convince the reader of the naturality of our choices. For instance, at the end of this section we will know that the Hodge star diagonalizes on the common eigenspaces of the complex action and the symplectic action described in the previous chapters!

We start with the following proposition.

Proposition 3.20. *Let (V, g, J, ω) be Kähler vector of dimension $2n$. Then the action of $\mathcal{J}_J$ and $\mathcal{J}_\omega$ on $\wedge^\bullet V^*_\mathbb{C}$ commute, i.e. $\mathcal{J}_J \cdot \mathcal{J}_\omega \cdot \varphi = \mathcal{J}_\omega \cdot \mathcal{J}_J \cdot \varphi$ for all $\varphi \in \wedge^\bullet V^*_\mathbb{C}$. Therefore, we have a decomposition of common eigenspaces*

$$\wedge^\bullet V^*_\mathbb{C} = \bigoplus_{p,q} U^{p,q},$$

where $U^{p,q} = U^p_J \cap U^q_\omega$.

Proof. We argue by induction. First observe, for degree zero we have $\mathcal{J}_\omega \cdot \mathcal{J}_J \cdot 1 = 0$ as $\mathcal{J}_J \cdot 1 = 0$. On the other hand, we have, writing $\omega = \frac{1}{2}\omega_{ij} e^i \wedge e^j$ (where $\omega_{ij} = -\omega_{ji}$) and $J = J^j_i e^i \otimes e_j$:

$$\begin{aligned}
\mathcal{J}_J \cdot \mathcal{J}_\omega \cdot 1 &= \frac{1}{2} J^j_i e^i \cdot e_j \cdot \omega_{kl} e^k \wedge e^l \\
&= \frac{1}{2} J^j_i \omega_{kl} e^i \wedge (\iota_{e_j}(e^k \wedge e^l)) \\
&= J^j_i \omega_{jk} e^i \wedge e^k.
\end{aligned}$$

Next, we observe that the coefficient $J^j_i \omega_{jk}$ is symmetric in i and k. Indeed, g is symmetric and

$$-g(e_i, e_k) = \omega(J(e_i), e_k) = J^j_i \omega(e_j, e_k) = J^j_i \omega_{jk}.$$

Since $e^i \wedge e^k$ is antisymmetric in i and k, we have $\mathcal{J}_J \cdot \mathcal{J}_\omega \cdot 1 = 0$.

Now suppose that the actions of $\mathcal{J}_J$ and $\mathcal{J}_\omega$ commute up to degree k. Then, using that both actions satisfy a certain Leibniz rule (Lemma 1.31 and Lemma 2.31) and that $\mathcal{J}_J \mathcal{J}_\omega = \mathcal{J}_\omega \mathcal{J}_J$ as automorphism of $\mathbb{V}$, we verify for $\xi \in V^*_\mathbb{C}$ and $\varphi \in \wedge^k V^*_\mathbb{C}$:

$$\begin{aligned}
\mathcal{J}_J \cdot \mathcal{J}_\omega \cdot (\xi \wedge \varphi) &= \mathcal{J}_J \cdot \mathcal{J}_\omega \cdot \xi \cdot \varphi = \mathcal{J}_J \cdot \mathcal{J}_\omega(\xi) \cdot \varphi + \mathcal{J}_J \cdot \xi \cdot \mathcal{J}_\omega \cdot \varphi \\
&= \mathcal{J}_J(\mathcal{J}_\omega(\xi)) \cdot \varphi + \mathcal{J}_\omega(\xi) \cdot \mathcal{J}_J \cdot \varphi + \mathcal{J}_J(\xi) \cdot \mathcal{J}_\omega \cdot \varphi + \xi \cdot \mathcal{J}_J \cdot \mathcal{J}_\omega \cdot \varphi \\
&= \mathcal{J}_\omega \cdot \mathcal{J}_J \cdot (\xi \wedge \varphi).
\end{aligned}$$

□

Lemma 3.21. *On a Kähler vector space (V, g, J, ω) with (real) dimension $2n$, the common eigenspace $U^{p,q}$ is trivial when $p + q \not\equiv n \pmod 2$ or when $|p + q| > n$. If $U^{p,q}$ is not trivial, then it is given by $U^{p,q} = e^{i\omega} e^{\frac{i\Lambda}{2}} \wedge^{k,l} V^*$ where $k = \frac{n+p-q}{2}$ and $l = \frac{n-(p+q)}{2}$.*

Proof. First of all, recall that from the complex structure J we obtained the decomposition $\wedge^m V_{\mathbb{C}}^* = \bigoplus_{k+l=m} \wedge^{k,l} V^*$. Recalling our expression for the iq-eigenspace of the action of $\mathcal{J}_\omega$ from Theorem 2.32 we obtain:

$$U_\omega^q = e^{i\omega} e^{\frac{i\Lambda}{2}} \bigoplus_{k+l=n-q} \wedge^{k,l} V^* = \bigoplus_{k+l=n-q} e^{i\omega} e^{\frac{i\Lambda}{2}} \wedge^{k,l} V^*,$$

where in the last equality we used that $e^{i\omega} e^{\frac{i\Lambda}{2}}$ is an isomorphism. Let $\varphi \in U_\omega^q$ and decompose φ with respect to the direct sum above, i.e. $\varphi = \sum_{k+l=n-q} \varphi^{k,l}$ with $\varphi^{k,l} \in e^{i\omega} e^{\frac{i\Lambda}{2}} \wedge^{k,l} V^*$.

Next, to compute the action of $\mathcal{J}_J$ on φ, we first notice that, if $\psi \in \wedge^{k,l} V^*$, then $\omega \wedge \psi \in \wedge^{k+1,l+1} V^*$, because ω is of type $(1,1)$. Similarly, we have that $\Lambda\psi \in \wedge^{k-1,l-1}$. Therefore we can conclude that $\mathcal{J}_J \cdot \varphi^{k,l} = i(k-l)\varphi^{k,l}$ so that $\mathcal{J}_J \cdot \varphi = \sum_{k+l=n-q} i(k-l)\varphi^{k,l}$. We now have put ourselves in the right position to prove the lemma.

Let $\varphi \in U_\omega^q$ and decompose φ as $\varphi = \sum_{k+l=n-q} \varphi^{k,l}$. Suppose now that φ is also an element of U_J^p. Since the decomposition of U_ω^q is direct, we have that φ is zero or $\varphi = \varphi^{k,l}$ for some integers $0 \le k, l \le n$ such $k - l = p$ and $k + l = n - q$. Solving for k and l yields $k = \frac{n+p-q}{2}$ and $l = \frac{n-(p+q)}{2}$. If $p + q \not\equiv n \pmod 2$, then k and l are not integers, and therefore φ must be zero. If $|p+q| > n$, then l is either negative or greater than n, again implying that φ is zero. Finally, if $p + q \equiv n \pmod 2$ and $|p+q| \le n$, then we see that these integers k and l indeed exist. We conclude that in that case $U^{p,q} = e^{i\omega} e^{\frac{i\Lambda}{2}} \wedge^{k,l} V^*$. $\square$

With this lemma in mind, we can visualize the decomposition of Proposition 3.20 by ordering the non-trivial components into the diamond below. The subscript (k,l) below $U^{p,q}$ is the corresponding $k = \frac{n+p-q}{2}$ and $l = \frac{n-(p+q)}{2}$ from the previous lemma.

$$\begin{array}{ccccccccc}
 & & & & U^{0,n}_{(0,0)} & & & & \\
 & & U^{-1,n-1}_{(0,1)} & & & & U^{1,n-1}_{(1,0)} & & \\
 & \cdots & & & U^{0,n-2}_{(1,1)} & & & \cdots & \\
U^{-n,0}_{(0,n)} & & \cdots & & \cdots & & \cdots & & U^{n,0}_{(n,0)} \\
 & \cdots & & & U^{0,n-2}_{(n-1,n-1)} & & & \cdots & \\
 & & U^{-1,-n+1}_{(n-1,n)} & & & & U^{1,-n+1}_{(n,n-1)} & & \\
 & & & & U^{0,-n}_{(n,n)} & & & &
\end{array}$$

The following lemma relates the Chevalley pairing to the eigenspace decomposition. The proof is rather long and tedious, but the result is crucial for the proofs and calculations in the next section.

Lemma 3.22. *On a symplectic vector space (V, ω), the Chevalley pairing is invariant under the isomorphisms $e^{i\omega}$ and $e^{\frac{i\Lambda}{2}}$, i.e. for all $\varphi \in \wedge^\bullet V_{\mathbb{C}}^*$ we have $(e^{i\omega}\varphi, e^{i\omega}\psi)_{Ch} = (\varphi, \psi)_{Ch} = (e^{\frac{i\Lambda}{2}}\varphi, e^{\frac{i\Lambda}{2}}\psi)_{Ch}$. As a corollary, on a Kähler vector space, the Chevalley pairing pairs $U^{p,q}$ trivially to $U^{r,s}$ whenever $(p,q) \neq (-r,-s)$.*

Proof. We start with the isomorphism $e^{i\omega}$. Let $\varphi, \psi \in \wedge^\bullet V_{\mathbb{C}}^*$, then by a direct computation we obtain

$$\begin{aligned}
\left(e^{i\omega}\varphi, e^{i\omega}\psi\right)_{Ch} &= \left(\sum_k \frac{i^k}{k!}\omega^k \wedge \varphi, \sum_l \frac{i^l}{l!}\omega^l \wedge \psi\right)_{Ch} \\
&= \left(\sum_{k,l} \frac{i^k i^l}{k! l!}\omega^k \wedge \varphi \wedge (\omega^l)^t \wedge \psi^t\right)_{top} \\
&= \left(\sum_{k,l} \frac{i^k i^l}{k! l!}\omega^k \wedge (-1)^l \omega^l \wedge \varphi \wedge \psi^t\right)_{top} \\
&= \left(e^{i\omega}e^{-i\omega}\varphi \wedge \psi^t\right)_{top} = (\varphi, \psi)_{Ch}.
\end{aligned}$$

For the isomorphism $e^{\frac{i\Lambda}{2}}$, we start by choosing a basis $\{x_k, y_k\}$ in which ω is standard and let $\{x^k, y^k\}$ denote its dual. Then, as we saw before, the bivector $-\omega^{-1}$ takes the form $-\omega^{-1} = \sum_k x_k \wedge y_k$ and thus Λ can be written as $\Lambda = \sum_k \iota_{x_k \wedge y_k}$. Since all the interior contractions $\iota_{x_k \wedge y_k}$ commute with each other, we can write the operator $e^{\frac{i\Lambda}{2}}$ as

$$e^{\frac{i\Lambda}{2}} = \prod_k \exp\left(\frac{i\iota_{(x_j \wedge y_j)}}{2}\right).$$

We will show that each of the operators $\exp\left(-\frac{\iota_{(x_j \wedge y_j)}}{2i}\right)$ leaves the Chevalley pairing invariant separately. Fix an index j. In the following, we will abuse the notation an denote by x (resp. y) both the basis vector x_j (resp. y_j) and the dual basis vector x^j (resp. y^j). It will be clear from the context which is meant. Additionally, we will denote the operator $\exp\left(-\frac{\iota_{(x_j \wedge y_j)}}{2i}\right)$ by A to make the notations less expansive. Let $\varphi \in \wedge^k V_{\mathbb{C}}^*$ and $\psi \in \wedge^l V_{\mathbb{C}}^*$. We can write φ and ψ as

$$\varphi = \varphi_0 + x \wedge \varphi_1 + y \wedge \varphi_2 + x \wedge y \wedge \varphi_3, \quad \psi = \psi_0 + x \wedge \psi_1 + y \wedge \psi_2 + x \wedge y \wedge \psi_3,$$

where $\iota_x \varphi_i = \iota_y \varphi_i = \iota_x \psi_i = \iota_y \psi_i = 0$. Notice that $A\varphi = \varphi + \frac{i}{2}\varphi_3$, so that we have

$$(A\varphi, A\psi)_{Ch} = (\varphi, \psi)_{Ch} + \frac{i}{2}(\varphi, \psi_3)_{Ch} + \frac{i}{2}(\varphi_3, \psi)_{Ch} - \frac{1}{4}(\varphi_3, \psi_3)_{Ch}.$$

Obviously, the degree of φ_3 is $k-2$ and the degree of ψ_3 is $l-2$. In the case $k+l = 2n$ (where $2n$ is the dimension of V), we see clearly that $(A\varphi, A\psi)_{Ch} = (\varphi, \psi)_{Ch}$, as the other terms vanish.

In the case $k+l-2 = 2n$ we have $(\varphi, \psi)_{Ch} = (\varphi_3, \psi_3)_{Ch} = 0$. We will show that the remaining terms sum to zero. First of all, notice that $\iota_y(\varphi_0 \wedge \psi_3^t) = \iota_y(x \wedge \varphi_1 \wedge \psi_3^t) = 0$. Because y is non-zero and both $\varphi_0 \wedge \psi_3^t$ and $x \wedge \varphi_1 \wedge \psi_3^t$ are top-degree covectors, they must vanish. Similarly, we can infer from $\iota_x(y \wedge \varphi_2 \wedge \psi_3^t) = 0$ that $y \wedge \varphi_2 \wedge \psi_3^t = 0$. Therefore, we can write

$$(\varphi, \psi_3)_{Ch} = x \wedge y \wedge \varphi_3 \wedge \psi_3.$$

Using a similar argument, we have that

$$(\varphi_3, \psi)_{Ch} = \varphi_3 \wedge (x \wedge y \wedge \psi_3)^t = -x \wedge y \wedge \varphi_3 \wedge \psi_3 = -(\varphi, \psi_3)_{Ch}.$$

We conclude that $(A\varphi, A\psi)_{Ch} = (\varphi, \psi)_{Ch} = 0$ in this case.

Next, the case that $k+l-4 = 2n$ is easily checked. Indeed, again from $\iota_y(\varphi_3 \wedge \psi_3^t) = 0$ we can conclude that $\varphi_3 \wedge \psi_3^t = 0$. Therefore, we have $(\varphi_3, \psi_3)_{Ch} = 0$. Again, it is verified that $(A\varphi, A\psi)_{Ch} = 0 = (\varphi, \psi)_{Ch}$.

Finally, in all the other cases, any two of the covectors $\varphi, \psi, \varphi_3, \psi_3$ pair to zero, so that it is indeed true that $(A\varphi, A\psi)_{Ch} = 0 = (\varphi, \psi)_{Ch}$. This proofs the first statement of the lemma.

To prove the corollary, notice that if two multi-covectors $\varphi \in \wedge^{k_1,l_1} V^*$ and $\psi \in \wedge^{k_2,l_1}$ pair non-trivially, then we must have that $k_1 + k_2 = l_1 + l_2 = n$. The assertion then follows from combining this observation, the result we just proved and Lemma 3.21. $\square$

We now arrive at the relation of the Hodge star in terms of the actions of $\mathcal{J}_J$ and $\mathcal{J}_\omega$. First of all, recall that the complex structure induces an orientation on V, and the Hodge star on a Kähler vector space will be defined with respect to this orientation.

In the following, we define the maps $\mathbb{J}_J$ and $\mathbb{J}_\omega$ on $\wedge^\bullet V_\mathbb{C}^*$ as $\mathbb{J}_J^{-1}\varphi = \exp\left(-\frac{\pi}{2}\mathcal{J}_J\right)\varphi = \sum_k \frac{(-1)^k}{k!}\left(\frac{\pi}{2}\mathcal{J}_J\right) \cdot \overset{k}{\dots} \cdot \left(\frac{\pi}{2}\mathcal{J}_J\right) \cdot \varphi$ and $\mathbb{J}_\omega = \exp\left(\frac{\pi}{2}\mathcal{J}_\omega\right)\varphi$.

Theorem 3.23 ([2], Lemma 2.9)**.** *On a Kähler vector space (V, g, J, ω) we have $\star = \mathbb{J}_J^{-1}\mathbb{J}_\omega$. Equivalently, $\star\big|_{U^{p,q}} = i^{q-p}$.*

Proof. Notice that the statement is equivalent to proving that $\star\big|_{U^{p,q}_{(k,l)}} = i^{n-2k}$. To calculate the Hodge star, we fix an orthonormal positive basis in the form $\{x_1, J(x_1), \dots, x_n, J(x_n)\}$ (where $2n$ is the dimension of V). For notational purposes, we denote $y_j = J(x_j)$. We make a few observations.

First of all, it is indeed possible to find such a basis. One starts by picking an orthonormal basis $\{x_1, \dots, x_n\}$ for V as a complex vector space. Then, since J is orthogonal with respect to the metric, the basis described above indeed is an orthonormal real basis and positive with respect to the orientation induced by J. Secondly, as the basis $\{x_1, y_1, \dots, x_n, y_n\}$ is orthonormal with respect to the metric, the usual dual basis $\{x^1, y^1, \dots, x^n, y^n\}$ coincides with the dual basis with respect to the metric, i.e. $\hat{g}(x_j) = x^j$ and $\hat{g}(y_j) = y^j$. Finally, the symplectic structure ω is actually standard with respect to this basis. Indeed, we have by compatibility of the structures that $\omega(x_i, x_j) = g(y_i, x_j) = -g(x_i, y_j) = -\omega(y_i, y_j) = 0$ and $\omega(x_i, y_j) = g(y_i, y_j) = \delta_{ij}$.

To reduce the notation even more, we set $X_i = x_i + x^i \in \mathbb{V}_\mathbb{C}$ and $Y_i = y_i + y^i \in \mathbb{V}_\mathbb{C}$. Notice that the Hodge star then takes the form $\star = Y_n \cdot X_n \cdot \dots \cdot Y_1 \cdot X_1 \cdot$.

We will prove the theorem by induction on k and l. First, for $k = l = 0$, have $U^{0,n}_{(0,0)} = e^{i\omega} e^{\frac{i\Lambda}{2}} \wedge^{0,0} V^* = e^{i\omega} \wedge^{0,0} V^*$. As $\wedge^{0,0} V^* \cong \mathbb{C}$, it suffices to compute the Hodge star of $e^{i\omega}$. As ω is standard, we have $\iota_{x_j} e^{i\omega} = iy_j \wedge e^{i\omega}$ and $\iota_{y_j} e^{i\omega} = -ix_j \wedge e^{i\omega}$. Then a straightforward computation yields $Y_j \cdot X_j \cdot e^{i\omega} = ie^{i\omega}$. It follows that $\star e^{i\omega} = i^n e^{i\omega}$.

Next, suppose $\star\big|_{U^{p,q}_{(k,l)}} = i^{n-2k}$ for some (k,l). We will verify that the Hodge star on $U^{p+1,q-1}_{(k+1,l)}$ is multiplication by $i^{n-2(k+1)}$. We start with the observation that ω is of type $(1,1)$ and therefore we have that $\iota_X \omega \in (V^*)^{1,0}$ for all $X \in V^{0,1}$. In particular, the space $\wedge^{k+1,l} V^*$ is generated by elements in the form $2i(\iota_X \omega) \wedge \varphi$ with $\varphi \in \wedge^{k,l} V^*$ and $X \in V^{0,1}$. Furthermore, a basis of $V^{0,1}$ is given by $\{\bar{z}_1, \ldots, \bar{z}_n\}$ where $\bar{z}_j = x_j + iy_j$.[†] Therefore, it is enough to compute the Hodge star on elements in the form $e^{i\omega} e^{\frac{i\Lambda}{2}} \left(2i\iota_{\bar{z}_j} \omega \wedge \varphi\right)$ with $\varphi \in \wedge^{k,l} V^*$. Recalling our computation in the proof of Theorem 2.32, we have

$$e^{i\omega} e^{\frac{i\Lambda}{2}} \left(2i(\iota_{\bar{z}_j} \omega) \wedge \varphi\right) = (\bar{z}_j + i\iota_{\bar{z}_j} \omega) \cdot e^{i\omega} e^{\frac{i\Lambda}{2}} \varphi = (X_j + iY_j) \cdot e^{i\omega} e^{\frac{i\Lambda}{2}} \varphi.$$

To calculate the Hodge star, notice that the actions of X_i and Y_i anti-commute with the actions of X_j Y_j whenever $i \neq j$, so that $X_i \cdot Y_i \cdot (X_j + iY_j) \cdot \psi = (X_j + iY_j) \cdot Y_j \cdot X_j \cdot \psi$ for all $\psi \in \wedge^\bullet V^*_{\mathbb{C}}$. Furthermore, when $i = j$, we have $X_j \cdot Y_j \cdot \psi = -Y_j \cdot X_j \cdot \psi$ and $X_j \cdot X_j \cdot \psi = Y_j \cdot Y_j \cdot \psi = \psi$. We use these observations to obtain

$$Y_j \cdot X_j \cdot (X_j + iY_j) \cdot \psi = -(X_j + iY_j) \cdot Y_j \cdot X_j \cdot \psi, \quad \text{for all } \psi \in \wedge^\bullet V^*_{\mathbb{C}}.$$

We deduce that

$$\begin{aligned} \star e^{i\omega} e^{\frac{i\Lambda}{2}} \left(2i\iota_{\bar{z}_j} \omega \wedge \varphi\right) &= \star(X_j + iY_j) \cdot e^{i\omega} e^{\frac{i\Lambda}{2}} \varphi = -(X_j + iY_j) \cdot \star e^{i\omega} e^{\frac{i\Lambda}{2}} \varphi \\ &= i^{n-2(k+1)} e^{i\omega} e^{\frac{i\Lambda}{2}} \left(2i\iota_{\bar{z}_j} \omega \wedge \varphi\right). \end{aligned}$$

In a similar fashion, one can show that the Hodge star on $U^{p-1,q+1}_{(k,l+1)}$ is multiplication by i^{n-2k}. This finishes the proof. $\square$

3.4 Hodge theory on manifolds

As usual after a discussion of linear algebra, we generalize it to manifolds.

3.4.1 Hodge theory on compact Riemannian manifolds

Let (M, g) be an oriented Riemannian n-manifold. To check that our linear Hodge star actually extends to a smooth operator $\star : \wedge^\bullet T^*M \to \wedge^\bullet T^*M$, we pick around each point $p \in M$ a smooth positive orthonormal frame on an open neighborhood U_p of p and use it to define a smooth Hodge star $\star_p$ on U_p. To check that it defines a global smooth operator, we notice, as the Hodge star is independent of the choice of the (positive) basis, that $\star_p\big|_{U_p \cap U_q} = \star_q\big|_{U_p \cap U_q}$. The generalization of the rest of the linear algebra in Subsection 3.3.1 is straightforward.

[†]Usually, the convention is that the basis of $V^{0,1}$ is given by $\bar{z}_j = \frac{1}{2}(x_j + iy_j)$. However, the factor $\frac{1}{2}$ will only make our computations more complicated, which is why it is omitted here.

If M is compact, we can define an inner product on $\Omega^\bullet(M)$ as

$$\mathbb{G}(\varphi,\psi) = \int_M (\varphi, \star\psi)_{Ch}, \quad \varphi,\psi\in\Omega^\bullet(M).$$

By Proposition 3.18, this is indeed an inner product. The inner product can lead to the notion of adjoint operators. Recall that in general the *adjoint (or dual)* of an automorphism A on an inner product space $(V, \langle\cdot,\cdot\rangle)$ (can be infinite dimensional) is the unique linear map A^* satisfying $\langle A^*x, y\rangle = \langle x, Ay\rangle$ for all $x, y \in V$. In our case we are especially interested in the adjoint of the exterior derivative. We start with a basic, but important result.

Lemma 3.24. *On a compact Riemannian m-manifold (M, g), we have the following equality:*

$$\int_M (\varphi, d\psi)_{Ch} = (-1)^m \int_M (d\varphi,\psi)_{Ch} \quad \textit{for all } \varphi,\psi\in\Omega^\bullet(M).$$

Therefore, the adjoint of the exterior derivative d with respect to the inner product $\mathbb{G}$ is the operator $d^* = (-1)^{\frac{m(m+1)}{2}} \star d\star$.

Proof. We reduce it to the case $\varphi\in\Omega^k(M)$. Then it is enough to check the equality for $\psi\in\Omega^{m-k-1}(M)$, since forms of other degree will pair trivially to φ. We compute

$$\begin{aligned}\int_M (\varphi, d\psi) &= -\int_M \varphi\wedge(d\psi)^t = -(-1)^{\frac{(m-k)(m-k-1)}{2}}\int_M \varphi\wedge d\psi\\ &= -(-1)^{\frac{(m-k)(m-k-1)}{2}+k+1}\int_M d\varphi\wedge\psi = -(-1)^m\int_M d\varphi\wedge\psi^t\\ &= (-1)^m\int_M (d\varphi,\psi)_{Ch},\end{aligned}$$

where in the third equality we used Stokes' theorem to say that

$$0 = \int_{\partial M}\varphi\wedge\psi = \int_M d(\varphi\wedge\psi) = \int_M d\varphi\wedge\psi + (-1)^k\int_M\varphi\wedge d\psi. \qquad\square$$

By the properties of the Chevalley pairing, we see that the adjoint of the exterior derivative sends forms of degree k to forms of degree $k-1$. Notice that $d^2 = 0$ implies $(d^*)^2 = 0$ and therefore we have the complex

$$\cdots \xleftarrow{d^*} \Omega^{k-1}(M) \xleftarrow{d^*} \Omega^k(M) \xleftarrow{d^*} \Omega^{k+1}(M) \xleftarrow{d^*} \cdots.$$

Definition 3.25. On a compact Riemannian manifold, a differential form φ is called *coclosed* when $d^*\varphi = 0$ and *coexact* when there exists a form ψ such that $d^*\psi = \varphi$. ▲

Together with the exterior derivative and its adjoint, we can form a self-adjoint operator.

Definition 3.26. On a compact Riemannian manifold, we define the *d-Laplacian* as $\Delta_d = dd^* + d^*d$. ▲

Because the adjoint d^* decreases the degree of forms by 1, the Laplacian is a degree preserving operator. Special forms which are important for our final result are the so-called harmonic forms: forms whose Laplacian vanishes.

Definition 3.27. Let (M,g) be a compact Riemannian manifold and $\varphi \in \Omega^\bullet(M)$. We call φ *d-harmonic* if $\Delta_d \varphi = 0$. If $\mathcal{U} \subset \Omega^\bullet(M)$ is a linear subspace, we denote the space of d-harmonic forms of in U by

$$\mathcal{H}_d(\mathcal{U}) = \{\varphi \in \mathcal{U} : \ \Delta_d \varphi = 0\}.$$

In the special cases that $\mathcal{U} = \Omega^k(M)$ and $\mathcal{U} = \Omega^\bullet(M)$ we write $\mathcal{H}_d^k(M) = \mathcal{H}_d(\Omega^k(M))$ and $\mathcal{H}_d^\bullet(M) = \mathcal{H}_d(\Omega^\bullet(M))$, respectively. ▲

Proposition 3.28. *Let (M,g) be a compact Riemannian manifold and $\varphi \in \Omega^\bullet(M)$. Then a differential form is d-harmonic if and only if it is closed and coclosed.*

Proof. First, if a form is closed and coclosed, then it is certainly harmonic. The other implication follows from the observation that for any d-harmonic $\varphi \in \Omega^\bullet(M)$ we have

$$0 = \mathbb{G}(\Delta_d \varphi, \varphi) = \mathbb{G}(d\varphi, d\varphi) + \mathbb{G}(d^*\varphi, d^*\varphi).$$

Then, since $\mathbb{G}$ is an inner product, both $d\varphi$ and $d^*\varphi$ must be zero. □

3.4.2 Hodge theory on compact Kähler manifolds

Next, we endow the compact Riemannian manifold (M,g) with an almost complex structure J so that it becomes almost Hermitian and denote its fundamental 2-form by ω. The actions of $\mathcal{J}_J$ and $\mathcal{J}_\omega$ on the bundle $\wedge^\bullet T^*M_\mathbb{C}$ commute by Lemma 3.20 and give rise to the decompositions

$$\wedge^\bullet T^*M_\mathbb{C} = \bigoplus_{\substack{|p+q| \leq n \\ p+q \equiv n \\ (mod\ 2)}} U^{p,q},$$

$$\Omega^\bullet(M;\mathbb{C}) = \bigoplus_{\substack{|p+q| \leq n \\ p+q \equiv n \\ (mod\ 2)}} \mathcal{U}^{p,q},$$

where $U^{p,q} = U_J^p \cap U_\omega^q$ and $\mathcal{U}^{p,q} = \mathcal{U}_J^p \cap \mathcal{U}_\omega^q$ denotes the space of smooth sections of the bundle $U^{p,q}$. The bundles can be written as $U^{p,q} = e^{i\omega} e^{\frac{i\Lambda}{2}} \wedge^{k,l} T^*M_\mathbb{C}$ (with the appropriate k and l), guaranteeing its smoothness. Furthermore, the space of smooth sections of the bundle $U^{p,q}$ can be written as $\mathcal{U}^{p,q} = e^{i\omega} e^{\frac{i\Lambda}{2}} \Omega^{k,l}(M)$. The Hodge star $\star$ restricts to multiplication by i^{q-p} on both $U^{p,q}$ and $\mathcal{U}^{p,q}$. Finally, the Chevalley pairing $(\cdot,\cdot)_{Ch}$ extends to differential forms and notably pairs $\mathcal{U}^{p,q}$ trivially to $\mathcal{U}^{r,s}$ whenever $(p,q) \neq (-r,-s)$. At last, one can extend the metric $\mathbb{G}$ to $\Omega^\bullet(M;\mathbb{C})$ so that it becomes a Hermitian inner product as

$$\mathbb{G}(\varphi,\psi) = \int_M (\varphi, \star\bar{\psi})_{Ch}, \quad \varphi,\psi \in \Omega^\bullet(M;\mathbb{C}).$$

Proposition 3.29. *On a compact almost Hermitian manifold (M, g, J), the decomposition of $\Omega^\bullet(M;\mathbb{C})$ into the spaces $\mathcal{U}^{p,q}$ is orthogonal with respect to the Hermitian product $\mathbb{G}$.*

Proof. We will check that for $\varphi \in \mathcal{U}^{p,q}$ and $\psi \in \mathcal{U}^{r,s}$ we have that $\mathbb{G}(\varphi, \psi) = 0$ when $(p, q) \neq (r, s)$. First of all, we claim that $\overline{U^{r,s}} = U^{-r,-s}$. This follows directly from the fact that the actions of $\mathcal{J}_J$ and $\mathcal{J}_\omega$ are real and thus commute with conjugation. Therefore, we have that $\bar{\psi} \in \mathcal{U}^{-r,-s}$. Because the Hodge star $\star$ preserves the spaces $\mathcal{U}^{r,s}$, we infer from Lemma 3.22 that $(\varphi, \star\bar{\psi})_{Ch} = 0$, implying that $\mathbb{G}(\varphi, \psi) = 0$. □

Finally, we consider the exterior derivative. For a general almost Hermitian manifold, it is not clear what the exterior derivative on $\mathcal{U}^{p,q}$ looks like, as it may have a lot of different components. We would like to specify a few components of the exterior derivative that will be of much interest to us later. In the definition below, we denote $\pi^{p,q}$ as the projection $\pi^{p,q} : \wedge^\bullet T^*M_{\mathbb{C}} \to U^{p,q}$.

Definition 3.30. Let (M, g, J) be an almost Hermitian manifold and let ω be its fundamental 2-form. We define the operators

$$\delta_+ : \mathcal{U}^{p,q} \to \mathcal{U}^{p+1,q+1}, \qquad \delta_- : \mathcal{U}^{p,q} \to \mathcal{U}^{p+1,q-1},$$
$$\bar{\delta}_+ : \mathcal{U}^{p,q} \to \mathcal{U}^{p-1,q-1}, \qquad \bar{\delta}_- : \mathcal{U}^{p,q} \to \mathcal{U}^{p-1,q+1},$$

as $\delta_+ = \pi^{p+1,q+1} \circ d$, $\delta_- = \pi^{p+1,q-1} \circ d$, $\bar{\delta}_+ = \pi^{p-1,q-1} \circ d$ and $\bar{\delta}_- = \pi^{p-1,q+1} \circ d$. ▲

Let ∂_J, $\bar{\partial}_J$ and ∂_ω, $\bar{\partial}_\omega$ denote components of the exterior derivative associated with the almost complex structure J and the non-degenerate 2-form ω, respectively (in other words, $\partial_J : \mathcal{U}^p_J \to \mathcal{U}^{p+1}_J$, $\partial_\omega : \mathcal{U}^q_\omega \to \mathcal{U}^{q+1}_\omega$ etc.). Let δ denote any of the operators d, $\partial_{J/\omega}$, $\bar{\partial}_{J/\omega}$, $\delta_\pm$, $\bar{\delta}_\pm$. Associated to δ we have its adjoint δ^*. By virtue of the Chevalley pairing, δ^*, acts in the opposite direction of δ. That is to say, the operators $\delta\delta^*$ and $\delta^*\delta$ respect the decomposition that δ is associated with. For example, if $\delta = \delta_+$, then δ^* maps a form in $\mathcal{U}^{p,q}$ to $\mathcal{U}^{p-1,q-1}$, while if $\delta = \partial_J$, then δ^* sends $\Omega^{k,l}(M)$ into $\Omega^{k-1,l}(M)$. We introduce some terminology that will be useful later.

Definition 3.31. Let (M, g, J) be an almost Hermitian manifold and let δ denote any of the operators d, $\partial_{J/\omega}$, $\bar{\partial}_{J/\omega}$, $\delta_\pm$, $\bar{\delta}_\pm$. We call a form φ *δ-closed* (*δ-coclosed*) if $\delta\varphi = 0$ ($\delta^*\varphi = 0$) and *δ-exact* (*δ-coexact*) if there exists a form ψ such that $\varphi = \delta\psi$ ($\varphi = \delta^*\psi$). ▲

As we did with the exterior derivative, one can form Laplacians for each of the specified components of the exterior derivative, which leads to its associated harmonic forms.

Definition 3.32. Let (M, g, J) be a compact almost Hermitian manifold with fundamental 2-form ω. Let δ be any of the operators d, $\partial_{J/\omega}$, $\bar{\partial}_{J/\omega}$, $\delta_\pm$, $\bar{\delta}_\pm$. We define the *δ-Laplacian* as $\Delta_\delta = \delta\delta^* + \delta^*\delta$. ▲

Definition 3.33. Let (M, g, J) be a compact almost Hermitian manifold with fundamental 2-form ω. Let δ be any of the operators $\partial_{J/\omega}$, $\bar{\partial}_{J/\omega}$, $\delta_\pm$, $\bar{\delta}_\pm$. A form $\varphi \in \Omega(M;\mathbb{C})$ is called *δ-harmonic* when $\Delta_\delta \varphi = 0$. For a subspace $\mathcal{U} \subset \Omega^\bullet(M;\mathbb{C})$, we denote space of δ-harmonic forms in $\mathcal{U}$ as:

$$\mathcal{H}_\delta(\mathcal{U}) = \{\varphi \in \mathcal{U} : \ \Delta_\delta \varphi = 0\}. \qquad \blacktriangle$$

Proposition 3.34. *Let (M, J, g) be an almost Hermitian manifold with fundamental 2-form ω. Let δ be any of the operators $\partial_{J/\omega}$, $\bar{\partial}_{J/\omega}$, $\delta_\pm$, $\bar{\delta}_\pm$. Then a form $\varphi \in \Omega^\bullet(M;\mathbb{C})$ is δ-harmonic if and only if it is δ-closed and δ-coclosed.*

Proof. The proof is the same as the proof of Proposition 3.28. $\square$

From this moment, we let (M, g, J, ω) be a $2n$-dimensional compact Kähler manifold. The integrability conditions on the structures J and ω put heavy restrictions on the exterior derivative d. Indeed, Theorem 1.32 and Theorem 2.33 decompose the exterior derivative in $d = \partial_J + \bar{\partial}_J$ on U^p_J and $d = \partial_\omega + \bar{\partial}_\omega$ on U^q_ω. Hence, the integrability conditions on J and ω are equivalent to the exterior derivative splitting into the four components $d = \delta_+ + \delta_- + \bar{\delta}_+ + \bar{\delta}_-$ on $\mathcal{U}^{p,q}$ as shown in the diagram below.

$$\begin{tikzcd}
\mathcal{U}^{p-1,q+1} & {} & \mathcal{U}^{p+1,q+1} \\
{} & \mathcal{U}^{p,q} \arrow[ul, "\bar{\delta}_-"'] \arrow[ur, "\delta_+"] \arrow[l, dotted, "\bar{\partial}_J"'] \arrow[r, dotted, "\partial_J"] \arrow[u, dotted, "\partial_\omega"'] \arrow[d, dotted, "\bar{\partial}_\omega"] \arrow[dl, "\bar{\delta}_+"] \arrow[dr, "\delta_-"'] & {} \\
\mathcal{U}^{p-1,q-1} & {} & \mathcal{U}^{p+1,q-1}
\end{tikzcd}$$

We read off that $\partial_J = \delta_+ + \delta_-$ and $\partial_\omega = \delta_+ + \bar{\delta}_-$. Remarkably, because of the imposed integrability conditions, the adjoints of these components are closely related. These relations are called the *(generalized) Kähler identities*, and imply, as a corollary, the equality of the Laplacians of all available differential operators.

Theorem 3.35 ((Generalized) Kähler identities, Gualtieri [8])**.** *On a compact Kähler manifold we have the identities*

$$\delta_+^* = \bar{\delta}_+ \quad \textit{and} \quad \delta_-^* = -\bar{\delta}_-.$$

Proof. Let (M, g, J, ω) be a compact Kähler $2n$-manifold. As the proofs are practically the same for both operators, we will only discuss it for δ_+. First of all, we claim that $\int_M (\varphi, \delta_+\psi)_{Ch} = \int_M (\delta_+\varphi, \psi)_{Ch}$. By Proposition 3.29 it is enough to verify this claim for $\varphi \in \mathcal{U}^{p,q}$ and $\psi \in \mathcal{U}^{-p-1,-q-1}$, but then the claim directly follows from Lemma 3.24. Indeed, using that the other components of d do not send ψ to $U^{-p,-q}$, we compute:

$$\int_M (\varphi, \delta_+\psi)_{Ch} = \int_M (\varphi, d\psi)_{Ch} = \int_M (d\varphi, \psi) = \int_M (\delta_+\varphi, \psi).$$

Secondly, we claim that $\overline{U^{p,q}} = U^{-p,-q}$. This follows directly from the fact that the actions of $\mathcal{J}_J$ and $\mathcal{J}_\omega$ commute with conjugation.

Finally, we prove the identity $\mathbb{G}(\delta_+\varphi, \psi) = -\mathbb{G}(\varphi, \bar{\delta}_+\psi)$. Using again that it is enough to check it on $\varphi \in \mathcal{U}^{p,q}$ and $\psi \in \mathcal{U}^{p+1,q+1}$, and applying repeatedly that $\star = i^{q-p}$ on $U^{p,q}$, we calculate:

$$\begin{aligned}\mathbb{G}(\delta_+\varphi,\psi) &= \int_M (\delta_+\varphi, \star\bar{\psi})_{Ch} = i^{-q+p}\int_M (\varphi, \delta_+\bar{\psi})_{Ch}\\ &= i^{-q+p}(-1)^n\int_M (\varphi, \star\star\,\delta_+\bar{\psi})_{Ch} = i^{-q+p}(-1)^n\int_M (\varphi, \star\star\,\overline{\bar{\delta}_+\psi})_{Ch}\\ &= i^{-2q+2p}(-1)^n\int_M (\varphi, \star\overline{\bar{\delta}_+\psi})_{Ch} = \mathbb{G}(\varphi, \bar{\delta}_+\psi),\end{aligned}$$

where in the second line we used that $\star^2 = (-1)^{\frac{2n(2n-1)}{2}} = (-1)^n$ and in the last equality we used that $p - q \equiv p + q \equiv n \pmod 2$. $\square$

Corollary 3.36. *On a compact Kähler manifold (M, g, J, ω), we have the following equalities of the Laplacians:*

$$\Delta_d = 2\Delta_{\partial_{J/\omega}} = 2\Delta_{\bar{\partial}_{J/\omega}} = 4\Delta_{\delta_\pm} = 4\Delta_{\delta_\pm}.$$

Proof. From the Kähler identities we immediately get $\Delta_{\delta_\pm} = \Delta_{\bar{\delta}_\pm}$. Then, we observe that the component of d^2 that lands in $U^{p,q}$ is equal to $\delta_+\bar{\delta}_+ + \bar{\delta}_+\delta_+ + \delta_-\bar{\delta}_- + \bar{\delta}_-\delta_- = \Delta_{\delta_+} - \Delta_{\delta_-}$. Since $d^2 = 0$, we get $\Delta_{\delta_+} = \Delta_{\delta_-}$.

Next, we have for ∂_J:

$$\begin{aligned}\Delta_{\partial_J} &= \partial_J\partial_J^* + \partial_J^*\partial_J = (\delta_+ + \delta_-)(\bar{\delta}_+ - \bar{\delta}_-) + (\bar{\delta}_+ - \bar{\delta}_-)(\delta_+ + \delta_-)\\ &= \Delta_{\delta_+} + \Delta_{\delta_-} + \delta_-\bar{\delta}_+ + \bar{\delta}_+\delta_- - \delta_+\bar{\delta}_- - \bar{\delta}_-\delta_+ = 2\Delta_{\delta_+},\end{aligned}$$

where in the last equality we used $d^2 = 0$ again to obtain $\delta_+\bar{\delta}_- + \bar{\delta}_-\delta_+ = \delta_-\bar{\delta}_+ + \bar{\delta}_+\delta_- = 0$. We repeat the same computation for $\bar{\partial}_{J/\omega}$ and ∂_ω to conclude that each of the Laplacians are equal for these operators.

Finally, a last computation, very similar to the ones we just did, will verify that $\Delta_d = 2\Delta_{\partial_J}$, from which all the desired equalities follow. $\square$

Corollary 3.37. *On a compact Kähler manifold of dimension $2n$, we have the following decompositions.*

(i) $$\mathcal{H}_d^\bullet(M;\mathbb{C}) = \bigoplus_{q=-n}^{n} \mathcal{H}_d(\mathcal{U}_\omega^q) \quad \text{and} \quad \mathcal{H}_d(\mathcal{U}_\omega^q) = \bigoplus_{\substack{|p+q|\le n\\ p+q\equiv n\\ (mod\ 2)}} \mathcal{H}_d(\mathcal{U}^{p,q});$$

(ii) $$\mathcal{H}_d^m(M;\mathbb{C}) = \bigoplus_{k+l=m} \mathcal{H}_d(\Omega^{k,l}(M)).$$

Proof. Because of the equality, $\Delta_d = 2\Delta_{\bar{\partial}_\omega} = 4\Delta_{\bar{\delta}_+}$, we know that Δ_d preserves the spaces $\mathcal{U}_\omega^q$ and $\mathcal{U}^{p,q}$. As the decomposition into the $\mathcal{U}^{p,q}$-spaces is direct, the first result follows. The second result follows by a similar argument involving the equality $\Delta_d = 2\Delta_{\bar{\partial}_J}$. $\square$

Corollary 3.38. *On a Kähler manifold, a form $\varphi \in \mathcal{U}^{p,q}$ is d-closed if and only if it is d-coclosed. Equivalently, a form $\varphi \in \mathcal{U}^{p,q}$ is d-closed if and only if it is harmonic.*

Proof. We start with a form $\varphi \in \mathcal{U}^{p,q}$. The exterior derivative $d\varphi$ vanishes if and only each of the components $\delta_\pm\varphi$, $\bar{\delta}_\pm\varphi$ are zero, which in turn is equivalent to the vanishing of the adjoint $d^*\varphi$. □

3.5 The Hodge decomposition theorem

Finally, in the last section of this thesis, we will discuss the implications of the Kähler identities for the topology of a Kähler manifold. The connection between the Kähler identities and the topology of the manifold is not obvious at first sight. It is established via two big theorems: the Hodge-de Rham theorem relates harmonic theory to the de Rham cohomology, and the de Rham theorem states that the de Rham cohomology is actually the same as the singular cohomology, which is a purely topological object (for the latter, see [11], chapter 18). In the end we will see that for compact Kähler manifolds, information of the Kähler structure is actually encoded in the de Rham cohomology.

In principle, since we have proven the equality of the Laplacians, we can let go of our alternative approach and follow the usual way to arrive at the Hodge decomposition outlined in [10] and [18]. Although these last sections are roughly based on these texts, we will keep track of how our alternative approach propagates parallel to the more standard viewpoint.

3.5.1 Cohomology

We will first recall what cohomology is and then define some cohomologies related to the structures on a Kähler manifold.

Definition 3.39. A *cochain complex* $(\mathcal{C}^\bullet, d^\bullet)$ is a family of Abelian groups $(\mathcal{C}^i)^{i\in\mathbb{Z}}$ together with homomorphisms $d^n : \mathcal{C}^n \to \mathcal{C}^{n+1}$ satisfying $d^{n+1} \circ d^n = 0$. Often, we will write down the cochain complex $(\mathcal{C}^\bullet, d^\bullet)$ as

$$\cdots \xrightarrow{d^{k-2}} \mathcal{C}^{k-1} \xrightarrow{d^{k-1}} \mathcal{C}^k \xrightarrow{d^k} \mathcal{C}^{k+1} \xrightarrow{d^{k+1}} \cdots .$$

An element $c \in \mathcal{C}^n$ is called *closed* when it is in the kernel of d^n and is called *exact* when it is in the image of d^{n-1}. ▲

In our situation, the Abelian groups are always subspaces of differential forms and the homomorphisms are components of the exterior derivative. We are already familiar with the de Rham complex on a manifold M, given by

$$\cdots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \cdots .$$

Associated to the de Rham complex, we have the de Rham cohomology, defined as

$$H^k_{dR}(M) = \frac{\ker\, d : \Omega^k(M) \to \Omega^{k+1}(M)}{\mathrm{im}\, d : \Omega^{k-1}(M) \to \Omega^k(M)}.$$

In general, one can define the cohomology of any cochain complex as follows.

Definition 3.40. Let $(\mathcal{C}^\bullet, d^\bullet)$ be a cochain complex. We define the *n-th cohomology group* of $(\mathcal{C}^\bullet, d^\bullet)$ as

$$H^n = \frac{\ker\, d^n}{\mathrm{im}\, d^{n-1}}$$

The cohomology groups together form the *cohomology* of $(\mathcal{C}^\bullet, d^\bullet)$. An element $\tilde{c}$ of a cohomology group H^n is called a *cohomology class* and is usually written as $\tilde{c} = [c]$, where $c \in \ker\, d^n$ is any representative. ▲

We have seen that the structures discussed in this thesis split the exterior derivative into different components, and each of these components squares to zero whenever the structures are integrable, and therefore gives rise to a different cochain complex. A clever reader may be able to guess what the following examples will be about.

Example 3.41. If M is a manifold, the complex-valued differential forms together with the exterior derivative form the cochain complex

$$\cdots \xrightarrow{d} \Omega^{k-1}(M;\mathbb{C}) \xrightarrow{d} \Omega^k(M;\mathbb{C}) \xrightarrow{d} \Omega^{k+1}(M;\mathbb{C}) \xrightarrow{d} \cdots.$$

The cohomology of this complex is the *(complex) de Rham cohomology* and its cohomology groups are denoted by $H^k_{dR}(M;\mathbb{C})$. △

Example 3.42. Let (M, J) be a complex manifold. We define the *k-th Dolbeault complex* as

$$\cdots \xrightarrow{\bar{\partial}_J} \Omega^{k,l-1}(M) \xrightarrow{\bar{\partial}_J} \Omega^{k,l}(M) \xrightarrow{\bar{\partial}_J} \Omega^{k,l+1}(M) \xrightarrow{\bar{\partial}_J} \cdots.$$

Associated to these complexes, we define the (k,l)-th Dolbeault-cohomology group to be

$$H^{k,l}_{Db}(M) = \frac{\ker\, \bar{\partial}_J : \Omega^{k,l}(M) \to \Omega^{k,l+1}(M)}{\mathrm{im}\, \bar{\partial}_J : \Omega^{k,l-1}(M) \to \Omega^{k,l}(M)}.$$

Of course, one can do the same with the operator ∂_J. △

Example 3.43. On a symplectic $2n$-manifold (M, ω), the exterior derivative also splits into two components $d = \partial_\omega + \bar{\partial}_\omega$. As both components satisfy $\partial_\omega^2 = \bar{\partial}_\omega^2 = 0$, we obtain the complexes

$$\cdots \underset{\bar{\partial}_\omega}{\overset{\partial_\omega}{\rightleftarrows}} \mathcal{U}^{q-1}_\omega \underset{\bar{\partial}_\omega}{\overset{\partial_\omega}{\rightleftarrows}} \mathcal{U}^q_\omega \underset{\bar{\partial}_\omega}{\overset{\partial_\omega}{\rightleftarrows}} \mathcal{U}^{q+1}_\omega \underset{\bar{\partial}_\omega}{\overset{\partial_\omega}{\rightleftarrows}} \cdots.$$

Interestingly, from Theorem 2.33, the cohomology groups of the $\bar{\partial}_\omega$-complex, given by

$$H^q_{\bar{\partial}_\omega}(M) = \frac{\ker\, \bar{\partial}_\omega : \mathcal{U}^q_\omega \to \mathcal{U}^{q-1}_\omega}{\mathrm{im}\, \bar{\partial}_\omega : \mathcal{U}^{q+1}_\omega \to \mathcal{U}^q_\omega},$$

are canonically isomorphic to the (complex) de Rham cohomology groups. To see this, recall that the expression for $\bar{\partial}_\omega$ is given by $\bar{\partial}_\omega(\Psi(\varphi)) = \Psi(d\varphi)$ where $\Psi = e^{i\omega}e^{\frac{i\Lambda}{2}}$ is the vector bundle isomorphism. Clearly, this isomorphism descends to an isomorphism of the cohomology groups, so that $H^{n-q}_{dR}(M;\mathbb{C})$ is isomorphic to $H^q_{\bar{\partial}_\omega}(M)$ in a natural way. △

Example 3.44. On a compact Kähler $2n$-manifold (M, g, J, ω), the exterior derivative d splits into the four components $\delta_\pm$, $\bar{\delta}_\pm$, each of which squares to zero, so that we obtain complexes and cohomologies associated to them. We will only discuss the $\bar{\delta}_+$-operator in this example because it relates, as we will see, to the previous two examples. For each $0 \leq k \leq n$ we define the complex

$$\cdots \xleftarrow{\bar{\delta}_+} \mathcal{U}^{p-1,q-1} \xleftarrow{\bar{\delta}_+} \mathcal{U}^{p,q} \xleftarrow{\bar{\delta}_+} \mathcal{U}^{p+1,q+1} \xleftarrow{\bar{\delta}_+} \cdots \xleftarrow{\bar{\delta}_+} \mathcal{U}^{k,n-k} \longleftarrow 0,$$

and its associated cohomology

$$H^{p,q}_{\bar{\delta}_+}(M) = \frac{\ker\, \bar{\delta}_+ : \mathcal{U}^{p,q} \to \mathcal{U}^{p-1,q-1}}{\mathrm{im}\, \bar{\delta}_+ : \mathcal{U}^{p+1,q+1} \to \mathcal{U}^{p,q}}.$$

Analogous to Example 3.43, the cohomology we just defined is isomorphic to the Dolbeault cohomology from Example 3.42 in a canonical way. To prove this, first notice that, given a form $\varphi \in \Omega^{k,l}(M)$, the operator $\bar{\delta}_+ = \pi^{p-1,q-1} \circ d = \pi^{p-1} \circ \pi^{q-1} \circ d$ (with π^{p-1} and π^{q-1} the obvious appropriate projections) acts on $\Psi(\varphi)$ as

$$\bar{\delta}_+(\Psi(\varphi)) = \pi^{k-(l+1)}(\Psi(d\varphi)) = \Psi\left(\bar{\partial}_J\varphi\right),$$

where in the last equality we recalled that the isomorphism $\Psi = e^{i\omega}e^{\frac{i\Lambda}{2}}$ preserves the eigenspaces of the action of $\mathcal{J}_J$. We infer that Ψ naturally descends to an isomorphism between the Dolbeault homology group $H^{k,l}_{Db}(M)$ and $H^{p,q}_{\bar{\delta}_+}(M)$ (with the corresponding $p = k - l$, $q = n - k - l$). △

3.5.2 The Hodge decomposition theorem on compact Kähler manifolds

A relation between harmonic forms and the cohomology of a manifold is something that is highly non-trivial. It is therefore even more astonishing that there is actually a one-to-one correspondence between the harmonic forms and the de Rham cohomology groups on (oriented) compact Riemannian manifolds, which is, fortunately to us, a large class of manifolds (remember that any manifold can be endowed with a Riemannian metric). This deep result is a consequence of one of the main theorems in elliptic operator theory. We will not prove it here, as it is more of an analytic result than a geometric one, and goes beyond the scope of this thesis. We will not even state the theorem in its full generality, but only specify it for the cases we are interested in. A treatment of elliptic operator theory can be found in [18], chapter 4, where also this result is proved.

Theorem 3.45 (Hodge-de Rham decomposition)**.** *On a compact Riemannian manifold (M,g) there are the following orthogonal decompositions.*

$$\begin{aligned}\Omega^k(M) &= d\Omega^{k-1}(M)\oplus\mathcal{H}^k_d(M)\oplus d^*\Omega^{k+1}(M);\\ \Omega^k(M;\mathbb{C}) &= d\Omega^{k-1}(M;\mathbb{C})\oplus\mathcal{H}^k_d(M;\mathbb{C})\oplus d^*\Omega^{k+1}(M;\mathbb{C}).\end{aligned}$$

On a compact Hermitian manifold (M,g,J) there are the following natural orthogonal decompositions.

$$\begin{aligned}\Omega^{k,l}(M) &= \bar\partial\Omega^{k-1,l}(M)\oplus\mathcal{H}_{\bar\partial}\left(\Omega^{k,l}(M)\right)\oplus\bar\partial^*\Omega^{k+1,l}(M);\\ \Omega^{k,l}(M) &= \partial\Omega^{k,l-1}(M)\oplus\mathcal{H}_{\partial}\left(\Omega^{k,l}(M)\right)\oplus\bar\partial^*\Omega^{k,l+1}(M).\end{aligned}$$

On a compact symplectic manifold (M,ω) equipped with a metric there are the following natural orthogonal decompositions.

$$\begin{aligned}\mathcal{U}^q_\omega &= \partial_\omega\mathcal{U}^{q-1}_\omega\oplus\mathcal{H}_{\partial_\omega}(\mathcal{U}^q_\omega)\oplus\partial^*_\omega\mathcal{U}^{q+1};\\ \mathcal{U}^q_\omega &= \bar\partial_\omega\mathcal{U}^{q+1}_\omega\oplus\mathcal{H}_{\bar\partial_\omega}(\mathcal{U}^q_\omega)\oplus\bar\partial^*_\omega\mathcal{U}^{q+1}.\end{aligned}$$

On a compact Kähler manifold (M,g,J,ω) there are the following natural orthogonal decompositions.

$$\begin{aligned}\mathcal{U}^{p,q} &= \delta_+\mathcal{U}^{p-1,q-1}\oplus\mathcal{H}_{\bar\delta_+}\left(\mathcal{U}^{p,q}\right)\oplus\bar\delta_+\mathcal{U}^{p+1,q+1};\\ \mathcal{U}^{p,q} &= \delta_-\mathcal{U}^{p-1,q+1}\oplus\mathcal{H}_{\delta_-}\left(\mathcal{U}^{p,q}\right)\oplus\bar\delta_-\mathcal{U}^{p+1,q-1}.\end{aligned}$$

The relation to cohomology now becomes clear. We make the observation that a δ-coexact form is δ-closed only when it is zero. Indeed, a δ-coexact form φ can be written as $\delta^*\psi$ for some other form ψ. If in addition φ is δ-closed, then we have $\mathbb{G}(\delta^*\psi,\delta^*\psi)=\mathbb{G}(\delta\varphi,\psi)=0$. Therefore, we can conclude the following corollaries.

Corollary 3.46. *Let (M,g) be a compact Riemannian manifold. Then the natural maps $\mathcal{H}^k_d(M)\hookrightarrow H^k_{dR}(M)$ and $\mathcal{H}^k_d(M;\mathbb{C})\hookrightarrow H^k_{dR}(M;\mathbb{C})$, mapping a harmonic form to its cohomology class, are isomorphisms.*

Let (M,g,J) be a compact Hermitian manifold. Then the natural map $\mathcal{H}_{\bar\partial_J}\left(\Omega^{k,l}(M)\right)\hookrightarrow H^{k,l}_{Db}(M)$, mapping a harmonic form to its cohomology class, is an isomorphism.

Let (M,ω) be a compact symplectic manifold equipped with a metric. Then the natural map $\mathcal{H}_{\bar\partial_\omega}(\mathcal{U}^q_\omega)\hookrightarrow H^q_{\bar\partial_\omega}(M)$, mapping a harmonic form to its cohomology class, is an isomorphism.

Let (M,g,J,ω) be compact Kähler manifold. Then the natural map $\mathcal{H}_{\bar\delta_+}\left(\mathcal{U}^{p,q}\right)\hookrightarrow H^{p,q}_{\bar\delta_+}(M)$, mapping a harmonic form to its cohomology class, is an isomorphism.

Proof. This follows directly from Theorem 3.45. □

Corollary 3.47 (The $\partial\bar\partial$-lemma)**.** *Let (M,J,g,ω) be a compact Kähler manifold and let $\varphi\in\Omega^{k,l}(M)$ be a d-closed form. Then the following statements are equivalent.*

(i) The form φ is d-exact.

(ii) The form φ is ∂_J-exact.
(iii) The form φ is $\bar{\partial}_J$-exact.
(iv) The form φ is $\partial_J\bar{\partial}_J$-exact, i.e. $\varphi = \partial_J\bar{\partial}_J\psi$ for some $\psi \in \Omega^{k-1,l-1}(M)$.

Proof. We start with the observation that (*iv*) implies (*i*), (*ii*) and (*iii*). Furthermore we see that any of the statements imply that φ is orthogonal to $\mathcal{H}(\Omega^{k,l}(M))$, the harmonic forms in $\Omega^{k,l}(M)$ with respect to any of the appropriate operators. We will show that this implies (*iv*).

Let φ be a d-closed form in $\Omega^{k,l}(M)$ that is orthogonal to $\mathcal{H}(\Omega^{k,l}(M))$. Then in particular it is ∂_J-closed and by Theorem 3.45 it is ∂_J-exact, i.e. $\varphi = \partial_J\alpha$ for some $\alpha \in \Omega^{k-1,l}(M)$. Next, we apply Theorem 3.45 again to write the form α as $\alpha = \bar{\partial}_J\beta_1 + \bar{\partial}_J^*\beta_2 + \beta_H$ where β_H is harmonic. This yields $\varphi = \partial_J\bar{\partial}_J\beta_1 + \partial_J\bar{\partial}_J^*\beta_2$. Because φ is also $\bar{\partial}_J$-closed, we have that $\bar{\partial}_J\partial_J\bar{\partial}_J^*\beta_2 = 0$. Using the Kähler identities, one readily verifies that $\partial_J\bar{\partial}_J^* = -\bar{\partial}_J^*\partial$ so that we have

$$\mathbb{G}(\partial_J\bar{\partial}_J^*\beta_2, \partial_J\bar{\partial}_J^*\beta_2) = -\mathbb{G}(\bar{\partial}_J\partial_J\bar{\partial}_J^*\beta_2, \partial_J\beta_2) = 0.$$

We conclude that $\varphi = \partial_J\bar{\partial}_J\beta_1$. □

The approach we have taken allows us to formulate a similar lemma in terms of the ∂_ω- and $\bar{\partial}_\omega$-operators. It should be pointed out that the following is not a new result: it is equivalent to the symplectic $d\delta$-lemma via the isomorphism in Example 3.43, see [8] for more details.

Corollary 3.48 (The $\partial_\omega\bar{\partial}_\omega$-lemma; alternative $\partial\bar{\partial}$-lemma)**.** *Let (M, J, g, ω) be a compact Kähler manifold and let $\varphi \in \mathcal{U}_\omega^q$ be a d-closed form. Then the following statements are equivalent.*

(i) The form φ is d-exact.
(ii) The form φ is ∂_ω-exact.
(iii) The form φ is $\bar{\partial}_\omega$-exact.
(iv) The form φ is $\partial_\omega\bar{\partial}_\omega$-exact, i.e. $\varphi = \partial_\omega\bar{\partial}_\omega\psi$ for some $\psi \in \mathcal{U}_\omega^q$.

Proof. Essentially the same as the proof of the $\partial\bar{\partial}$-lemma. □

Finally, we are in a position to prove the main result of our thesis: the famous Hodge decomposition on compact Kähler manifolds. From our approach we even obtain (related) decompositions encoding information of the Kähler structures into the topology of the manifold.

Corollary 3.49 (Hodge decomposition on compact Kähler manifolds)**.** *On a compact complex $2n$-manifold of Kähler type, we have the following decomposition.*

$$H^m_{dR}(M;\mathbb{C}) = \bigoplus_{k+l=m} H^{k,l}_{Db}(M).$$

Moreover, conjugation is an isomorphism between $H^{k,l}_{Db}(M)$ and $H^{l,k}_{Db}(M)$ as subspaces of $H^m_{dR}(M;\mathbb{C})$.

If a Kähler metric is given, then we have in addition the decompositions

$$H^{\bullet}_{dR}(M;\mathbb{C}) = \bigoplus_{m=0}^{2n} H^m_{dR}(M;\mathbb{C}) = \bigoplus_{q=-n}^{n} H^q_{\bar\partial_\omega}(M) \quad \textit{and} \quad H^q_{\bar\partial_\omega}(M) = \bigoplus_{\substack{|p+q|\le n \\ p+q\equiv n \\ (mod\ 2)}} H^{p,q}_{\bar\delta_-}(M).$$

Moreover, conjugation is an isomorphism between $H^{p,q}_{\bar\delta_+}(M)$ and $H^{-p,-q}_{\bar\delta_+}(M)$ as subspaces of $H^{\bullet}_{dR}(M;\mathbb{C})$.

Proof. To prove the first assertion, we choose a Kähler metric and use Corollary 3.36 and Corollary 3.37 to obtain the isomorphism

$$H^m_{dR}(M;\mathbb{C}) \cong \mathcal{H}^m_d(M;\mathbb{C}) = \bigoplus_{k+l=m} \mathcal{H}_{\bar\partial_J}\left(\Omega^{k,l}(M)\right) \cong \bigoplus_{k+l=m} H^{k,l}_{Db}(M).$$

To proof that this isomorphism is independent of the choice of metric, choose any other Kähler metric and denote the harmonics with respect to this other metric with as $\mathcal{H}'^{k,l}_{\bar\partial_J}\left(\Omega^{k,l}(M)\right)$. We obtain an isomorphism between $\mathcal{H}'^{k,l}_{\bar\partial_J}\left(\Omega^{k,l}(M)\right)$ and $\mathcal{H}^{k,l}_{\bar\partial_J}\left(\Omega^{k,l}(M)\right)$ via

$$\mathcal{H}'^{k,l}_{\bar\partial_J}\left(\Omega^{k,l}(M)\right) \cong H^{k,l}_{Db}(M) \cong \mathcal{H}^{k,l}_{\bar\partial_J}\left(\Omega^{k,l}(M)\right)$$

Let $\alpha \in \mathcal{H}^{k,l}_{\bar\partial_J}\left(\Omega^{k,l}(M)\right)$ and $\alpha' \in \mathcal{H}'^{k,l}_{\bar\partial_J}\left(\Omega^{k,l}(M)\right)$ be such that they define the same cohomology class in $H^{k,l}_{Db}(M)$. When we show that the cohomology classes $[\alpha], [\alpha'] \in H^m_{dR}(M;\mathbb{C})$ coincide, we are done.

First of all, since $[\alpha] = [\alpha']$ in $H^{k,l}_{Db}(M)$, we can write $\alpha - \alpha' = \bar\partial_J\beta$ for some $\beta \in \Omega^{k,l-1}(M)$. Next, we notice that both α and α' are closed, as both d-harmonic with respect to the appropriate metric. Therefore we have $d(\alpha - \alpha') = d\bar\partial_J\beta = 0$. In particular, $\bar\partial_J\beta$ is either zero (then we are done) or is not d^*-exact (with respect to the first metric). Finally, $\bar\partial_J\beta$ is orthogonal to $\mathcal{H}^k_d(M;\mathbb{C})$ with respect to the first metric. Indeed, given a form $\gamma \in \mathcal{H}^k_d(M;\mathbb{C})$, by Corollary 3.36 also it is $\bar\partial_J$-harmonic and thus satisfies $\bar\partial_J^*\gamma = 0$. This yields $\mathbb{G}(\gamma, \bar\partial_J\beta) = \mathbb{G}(\partial_J^*\gamma, \beta) = 0$. By the Hodge-de Rham decomposition theorem we find that $\bar\partial_J\beta \in d\Omega^{m-1}(M;\mathbb{C})$, implying that $[\alpha] = [\alpha']$ in $H^m_{dR}(M;\mathbb{C})$.

For the second statement, it should be clear that conjugation is a well-defined map between $H^{k,l}_{Db}(M)$ and $H^{l,k}_{Db}(M)$ as subspaces of $H^m_{dR}(M;\mathbb{C})$.[†] To see that this is actually an isomorphism, recall that conjugation is an isomorphism between $\Omega^{k,l}(M)$ and $\Omega^{l,k}(M)$.

Turning to the other decomposition, we use again Corollary 3.36 and Corollary

[†]In general, conjugation on $H^{k,l}_{Db}(M)$ is not well-defined, as ∂_J-exact forms need not to be $\bar\partial_J$-exact. In Kähler case, however, this is true for closed forms by the $\partial\bar\partial$-lemma. Therefore, under the identification of $H^{k,l}_{Db}(M)$ as a subspace of $H^m_{dR}(M;\mathbb{C})$ it actually is well-defined.

3.37 to obtain the isomorphisms via

$$H^\bullet_{dR}(M;\mathbb{C}) \cong \mathcal{H}^\bullet_d(M;\mathbb{C}) = \bigoplus_{q=-n}^{n} \mathcal{H}_{\bar\partial_\omega}(\mathcal{U}^q_\omega) \cong \bigoplus_{q=-n}^{n} H^q_{\bar\partial_\omega}(M);$$
$$H^q_{\bar\partial_\omega}(M) \cong \mathcal{H}_{\bar\partial_\omega}(\mathcal{U}^q_\omega) = \bigoplus_{\substack{|p+q|\le n\\ p+q\equiv n\\ (mod\ 2)}} \mathcal{H}_{\bar\delta_-}(\mathcal{U}^{p,q}) \cong \bigoplus_{\substack{|p+q|\le n\\ p+q\equiv n\\ (mod\ 2)}} H^{p,q}_{\bar\delta_-}(M).$$

Again, it should be clear that conjugation is a well-defined isomorphism between $\overline{H}^{p,q}_{\bar\delta_+}(M) = H^{-p,-q}_{\bar\delta_+}(M)$ as subspaces of $H^\bullet_{dR}(M;\mathbb{C})$, because the only closed forms on $\mathcal{U}^{p,q}$ are the harmonic ones. □

As we saw before in Example 3.44, the Dolbeault cohomology groups $H^{k,l}_{Db}(M)$ are isomorphic to the cohomology groups $H^{p,q}_{\bar\delta_-}(M)$ in a canonical way, so as long as the metric is specified, the two decompositions are equivalent. Usually the first one is used in applications, but in some cases the second may be preferable because on $\mathcal{U}^{p,q}$, the closed forms are exactly the same as the harmonic ones.

This theorem tells us that, remarkably, on a compact complex manifold of Kähler type, the decomposition of the de Rham cohomology is entirely dictated by the fact that it is of Kähler type, independent of any choice of metric! We will discuss some implications and applications of the theorem in the next section.

The compactness condition is really necessary. Indeed, take for example $\mathbb{C}$ with the standard Kähler structure. Then $H^1_{dR}(\mathbb{C};\mathbb{C}) = 0$, but for every non-constant holomorphic function $f : \mathbb{C} \to \mathbb{C}$ the $(1,0)$-form fdz satisfies $\bar\partial(fdz) = \frac{\partial f}{\partial \bar z} d\bar z \wedge dz = 0$, while obviously $[fdz] \in H^{1,0}_{Db}(\mathbb{C})$ is non-zero. Thus for $\mathbb{C}$ we have $H^1_{dR}(\mathbb{C};\mathbb{C}) \neq H^{1,0}_{Db}(\mathbb{C}) \oplus H^{0,1}_{Db}(\mathbb{C})$.

3.5.3 Applications

Most applications are quite involved and rely on theories that we didn't discuss in this thesis. For this reason, some of the applications below call on knowledge beyond the scope of this text.

The first implication that we are going to concern ourselves with is one regarding the dimension of the (de Rham) cohomology groups of a manifold of Kähler type. We start by introducing the following notion.

Definition 3.50. Let M be a smooth manifold. For any non-negative integer m, we define the *m-th Betti number* $b_k(M)$ to be

$$b_m(M) = \dim_{\mathbb{R}} H^m_{dR}(M) = \dim_{\mathbb{C}} H^m_{dR}(M;\mathbb{C}). \qquad \blacktriangle$$

By the de Rham theorem, the Betti numbers are actually topological invariants. We apply the Hodge theorem for compact Kähler manifolds to put restraints on these topological invariants.

Corollary 3.51. *Let M be a compact manifold of Kähler type. Then for all odd m the Betti numbers $b_m(M)$ are even.*

Proof. This follows directly from the first two assertions in Theorem 3.49. $\square$

This corollary can be used to proof that certain manifolds do not admit Kähler structures, an example is given below.

Example 3.52 ([13], Example 3.1.19)**.** The Kodaira-Thurston manifold KT may be one of the simplest examples of a manifold that admits both symplectic and complex structures, but is not of Kähler type. We will not go through the details, but we will give an outline of how one can construct this manifold and the structures.

Let $\Gamma = \mathbb{Z}^2 \times \mathbb{Z}^2$ be the (non-commutative) group with group operation

$$(j,k) \circ (j',k') = (j+j', k + A_j k') \quad \text{for } j, j', k, k' \in \mathbb{Z}^2,$$

where $A_j = \left(\begin{smallmatrix} 1 & j_1 \\ 0 & 1 \end{smallmatrix}\right)$ for $j = (j_1, j_2) \in \mathbb{Z}^2$. We let Γ act on $\mathbb{R}^4$ via

$$\Gamma \times \mathbb{R}^4 \to \mathbb{R}^4 : ((j,k),(x,y)) \mapsto (x + j, A_j y + k).$$

The *Kodaira-Thurston manifold* is then defined as $KT = \mathbb{R}^4/\Gamma$. This is actually a smooth manifold, as one can check that the quotient map $q : \mathbb{R}^4 \to KT$ carries the smooth structure of $\mathbb{R}^4$ over to KT. The symplectic form $\omega_0 = dx^1 \wedge dx^2 + dy^1 \wedge dy^2$ (where (x^1, x^2, y^1, y^2) are the coordinates on $\mathbb{R}^4$) is invariant under the action of Γ, and therefore descends to a symplectic form ω on KT.

To construct a complex structure on KT, one first checks that the diffeomorphism

$$\mathbb{R}^4 \to \mathbb{C}^2 : (x^1, x^2, y^1, y^2) \mapsto \left(x^1 + iy^1, y^1 + ix^2 - \frac{x^1 y^2}{2} - i\frac{(x^1)^2 + (y^2)^2}{4}\right)$$

pulls the standard complex structure on $\mathbb{C}^2$ back to an almost complex structure on $\mathbb{R}^4$ given by

$$J_{\mathbb{R}^4}(x^1, x^2, y^1, y^2) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -x^1 \\ x^1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This almost complex structure is actually integrable. Moreover, it is invariant under the action of Γ and thus descends to a complex structure on KT.

Nevertheless, the Kodaira-Thurston manifold does not admit any Kähler structures. Indeed, the fundamental group of KT is $\pi_1(KT) = \Gamma$, and the Abelianization of Γ is $\Gamma/[\Gamma,\Gamma] = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, so that that the first (simplicial) homology group $H_1(KT)$ is equal to $\mathbb{Z}^3$. By the universal coefficient theorem and the de Rham theorem, we can conclude that $b_1(KT) = 3$, and thus that KT does not admit any Kähler structures. $\triangle$

Finally, in the last application we will compute the Dolbeault cohomology of the complex projective space.

Corollary 3.53. *The Dolbeault cohomology groups of* $\mathbb{CP}^n$ *are*

$$H^{k,l}_{Db}(\mathbb{CP}^n) = \begin{cases} \mathbb{C} & \textit{when } k = l \textit{ and } 0 \leq k \leq n; \\ 0 & \textit{when } k \neq l. \end{cases}$$

Proof. We use our knowledge of differential geometry or CW-complexes to recall that the de Rham cohomology groups of $\mathbb{CP}^n$ are†

$$H^m_{dR}(\mathbb{CP}^n; \mathbb{C}) = \begin{cases} \mathbb{C} & \text{when } m \text{ is even and } 0 \leq m \leq 2n; \\ 0 & \text{in all other cases.} \end{cases}$$

Furthermore, since $\mathbb{CP}^n$ is Kähler, it admits a closed non-degenerate $(1,1)$-form ω. We will prove that the cohomology class $[\omega^k] \in H^{k,k}_{Db}(\mathbb{CP}^n)$ is non-zero. Suppose to the contrary that ω^k is $\bar{\partial}$-exact for some $k \geq 0$, then by the $\partial\bar{\partial}$-lemma it is also d-exact, i.e. $\omega^k = d\alpha$ for some $2k-1$-form α. Then we compute

$$0 < \int_M \omega^n = \int_M d\alpha \wedge \omega^{n-k} = \int_M \alpha \wedge d(\omega^{n-k}) = 0,$$

which is clearly a contradiction. Note that in the penultimate equality we applied Stokes' theorem. The corollary then follows from the Hodge decomposition theorem. □

†One can prove this by induction on n using a Mayer-Vietoris sequence, or via the cell-complex structure of $\mathbb{CP}^n$.

Afterword

Further developments

As we already discussed in the introduction, the Hodge decomposition theorem on compact Kähler manifolds was one of the first theorems restricting the topology of Kähler manifolds. A natural question arises: are there more?

The answer is positive. Another famous restriction is due to the *Hard Lefschetz theorem*. It states that on a compact Kähler $2n$-manifold (M, g, J, ω) the map

$$H^{n-k}_{dR}(M) \to H^{n+k}_{dR}(M) : \; [\varphi] \mapsto [\omega^k \wedge \varphi]$$

is always an isomorphism. In particular, this theorem implies that both the odd and the even Betti numbers must be non-decreasing up until the complex dimension of the manifold, e.g. when n is even, we have

$$\begin{aligned} b_0(M) \leq b_2(M) \leq \ldots \leq b_{n-2}(M) \leq b_n(M), \\ b_1(M) \leq b_3(M) \leq \ldots \leq b_{n-3}(M) \leq b_{n-1}(M). \end{aligned}$$

The interested reader can be referred to [18], where this theorem is discussed in the chapter of Hodge theory on compact Kähler manifolds.

A more recent development is due to Amorós (and others) and involves restrictions on the fundamental group of Kähler manifolds. This has been worked out in [1].

Most intriguingly, research on these topics is deep and intersects the fields of topology, differential and algebraic geometry, and complex analysis.

A few words on the proof

Finally, we would like to reflect on the alternative approach taken in this thesis. Has it been more insightful? Or has it only made things more magical?

We start by discussing some of its advantages. First of all, by the nature of our approach, the proof we gave in this thesis can be generalized directly, up to choice of signs, to generalized Kähler manifolds. In fact, a generalization to generalized Kähler manifolds will only make the proof less complicated, because many of the intermediate results in this thesis are more natural in framework of generalized complex geometry. Secondly, the alternative Hodge theory developed in Section 3.3 and Section 3.4 is more compatible with Kähler manifolds and therefore more elegant. Finally, in its alternative form, the Kähler identities, as well as its proof, are more insightful within its *own* framework and, not unimportant, are more aesthetically pleasing.

However, to arrive at the position to formulate and proof the alternative Kähler identities, we had to pay something in return. Setting up the framework has been quite a task, and fairly often involved choices that were only fully justified afterwards. For this reason, one can argue that in this approach the magic of the proof has shifted towards the linear algebra. Finally, one might say that the effort put in the set-up may be better spent on an introduction to generalized complex geometry, where the choices made actually are natural, but one needs to keep in mind that a reader of a self-contained discussion of generalized complex geometry needs to be acquainted with both complex and symplectic geometry.

It is left as an exercise to the reader to find out which approach is (personally) preferable.

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