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Faculteit Bètawetenschappen

Differential Geometry in Physics

Bachelor Thesis

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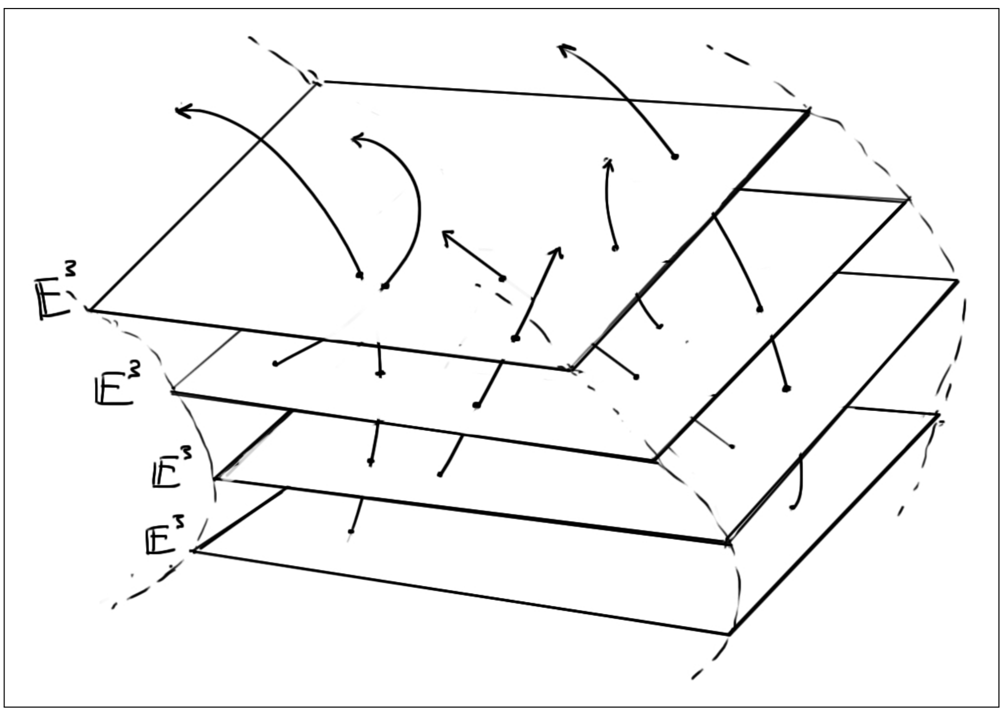

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Abstract

Differential geometry is a mathematical field which lies at the foundation of many theories in physics, such as general relativity, cosmology and string theory. A good understanding of these physical theories therefore requires one to be familiar with numerous techniques and mathematical constructs within this field, as well as their implementation. In this project we study this implementation in detail, by first developing the relevant mathematical tools and then applying them to the three fields named above. The main results include the derivation of Einstein's equation in vacuum and its extension to non-empty universes, the FLRW cosmological model and the Bianchi classification of anisotropic universes. We also discuss the ADE classification of semi-simple Lie algebras and their appearance in string theory, which is manifested by the ADE-ALE correspondence.

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Chapter 1

Introduction

During the times of the Old Greeks, when Aristotle had written his book *Physica* explaining the foundations of nature, large emphasis was placed on describing the observations we make of the world around us. Nowadays, physicists work together with mathematicians in order to find the underlying mechanisms behind these observations. A proper physical theory requires a solid mathematical foundation. One of the most important of such foundations is the theory of differential geometry, which describes the general properties of spaces in which most of our physics takes place. The goal of this thesis is to show how differential geometry underlies various physical theories. In particular, we will cover general relativity, cosmology and take a peek into the world of string theory. We also spend significant effort in describing the theory of Lie groups, a subject which is relevant for almost all theoretically involved areas of physics.

We start by introducing the necessary tools in a mathematical fashion in chapter 2. However, emphasis is placed on the physical relevance and interpretation of the various concepts such as manifolds, tensor fields and curvature.

As a direct application of these tools, we discuss the theory of general relativity in chapter 3. Starting with a historical overview of the picture of spacetime as a description of our universe, we then move into the derivation of the Einstein equation in vacuum. After discussing the stress-energy tensor we then give the full Einstein equation in non-empty space.

To appreciate the power of general relativity, we move into the FLRW cosmological model of a homogeneous and isotropic universe in chapter 4. From the Einstein equation and the perfect fluid assumption, we derive the relativistic Friedmann equations and discuss their interpretation. We find that they imply an expanding universe which originated from the Big Bang. We also touch on the popular debate on the cosmological constant and dark energy to explain the observed accelerated expansion of the universe.

Motivated by the FLRW model, we enter a more theoretical investigation of anisotropic universes, or Bianchi universes, in chapter 5. We again profit from our knowledge of differential geometry developed in chapter 2, by using the theory of Lie groups and Lie algebras in this particular setting to describe the isometries of an anisotropic universe. We introduce and derive general properties of Lie groups regarding invariant vector fields, Killing fields and then discuss a particular application to flat spacetimes. We provide the necessary classification of all 3 dimensional Lie groups. The resulting metric turns out to be distinctly different fromt the FLRW model, allowing for a universe that expands in one dimension, but contracts in another.

In chapter 6 we push our knowledge of Lie algebras further, discussing the classification of all semi-simple Lie algebras. We go into the theory of roots, representations and weights of Lie algebras. As an example we explore the classification of the Lie algebras of SU(2) and SU(3), both of which are important in the fields of quantum mechanics, regarding spin, and particle physics, regarding the strong force. It turns out that the semi-simple Lie algebras are completely determined by their Dynkin diagrams. Three of these, the ADE diagrams, are of great importance.

In the final chapter we combine our knowledge obtained in all preceding chapters, by discussing how the aforementioned ADE classification arises in the context of string theory. After a very generic introduction to type IIA string theory and M-theory, we study special configurations of D6 branes uplifted to M-theory, which results in a certain space called a KK-monopole. We state the surprising result that there is a connection between the minimal resolution of the KK-monopoles and the ADE Dynkin diagrams. As a result, depending on the exact nature of the situation, it is found that the corresponding gauge theory which describes the D6-branes has an SU(N) or SO(2N) symmetry. This is of great importance for the aforementioned fields of physics in which these symmetries are studied in great detail.

We end by reviewing the usage of differential geometry in these different fields and summarizing the most important findings. Finally, we point the interested reader to the currently active field of F-theory and elliptic fibrations, in which the combination of physics and differential geometry becomes even more beautiful.

Chapter 2

Differential Geometry: The Basis of Spacetime

'Imagine you are sitting on a train riding on a track along the x-axis', 'Consider a ball rolling off an inclined slope', or 'Let us investigate the motion of the Earth around the Sun'. These are all theoretical set-ups that are discussed in any classical mechanics course. In their early undergraduate years, physics students first make themselves comfortable with working in Cartesian coordinate systems. Then one often learns the advantages of using polar or spherical coordinates. Finally, the Lagrangian and Hamilton formalisms are introduced and general coordinates become the lifesavers of many students. Still, one might be inclined to believe that our universe is some kind of 3- or 4-dimensional Euclidean space, that is, the space $\mathbb{R}^3$ or $\mathbb{R}^4$ with the usual Euclidean metric.[1] At first sight it is not obvious whether this is indeed the case and it would be a big assumption to make. Fortunately, we do not *need* to assume it. Consider for example the ball rolling off an inclined slope. Compared to the size of the universe, this event takes place in a minuscule part of our world. Moreover, describing this event as if it took place in a Euclidean space seems to work just fine. Why not, then, assume that our universe *locally* looks like $\mathbb{R}^d$, for some integer d? The mathematical term for such an object is a *smooth manifold*. The strength of viewing our universe as some abstract manifold is that we do not need to consider its ambient space. What would there even be beyond our universe?

In this chapter we will discuss the relevant mathematical structures required to define *spacetime* and explain the physical intuition behind them. We begin with the definition of manifolds, their tangent spaces and tensor fields. Next we introduce the notion of covariant derivatives and the curvature of a manifold. Finally we look at integration on manifolds, which we require for a Lagrangian formulation of the dynamics of spacetime.

2.1 Manifolds and Tangent Spaces

We start by discussing the notion of a manifold. As described above, a smooth manifold M is a (toplogical) space that *locally* looks like $\mathbb{R}^d$. More precisely, at each point $p \in M$ there exists a *chart* (U, x) of M at p such that $U \subseteq M$, containing p, is an open subset and $x : U \to x(U) \subseteq \mathbb{R}^d$ is a homeomorphism[2]. Moreover, whenever we have two overlapping charts (U, x) and (V, y) we require that the *transition maps*

$$y \circ x^{-1} : x(U \cap V) \to y(U \cap V), \quad x \circ y^{-1} : y(U \cap V) \to x(U \cap V) \tag{2.2}$$

are smooth maps, as maps from $\mathbb{R}^d$ to $\mathbb{R}^d$. A good example of a smooth manifold that we will refer to frequently is the sphere $S^2 \subseteq \mathbb{R}^3$, our Earth if you will. Indeed, it is not surprising that our ancestors

[1] By the usual Euclidean metric on $\mathbb{R}^n$ we mean the map

$$d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}, \tag{2.1}$$

where x_i denotes the i-th component of the vector x.

[2] A map $f : X \to Y$ between two topological spaces X and Y is called a *homeomorphism* if and only if it is bijective and both f and its inverse f^{-1} are continuous.

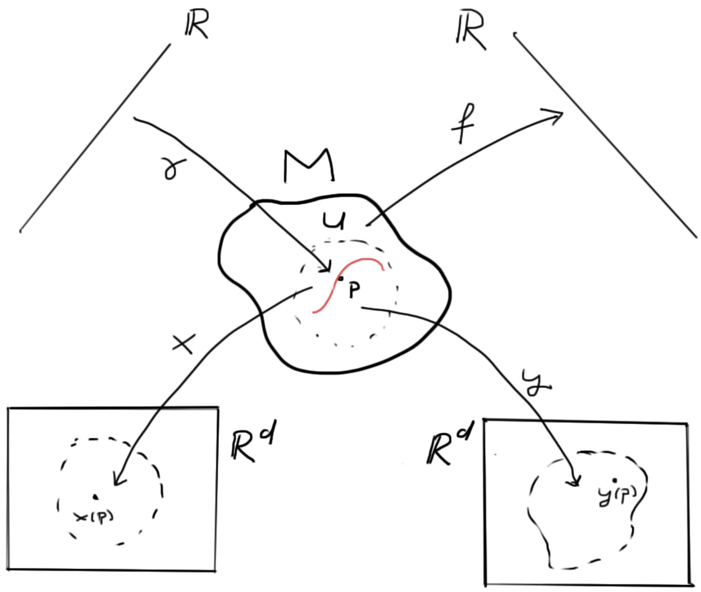

Figure 2.1: Shown is a manifold M, a point $p \in M$ and a curve γ, drawn in red, passing through p. Two charts, with the same domain, x and y are depicted and f is function from M to $\mathbb{R}$.

believed the Earth to be flat, since it does look, at least locally, like the plane $\mathbb{R}^2$. Charts are extremely useful objects, since they allow us to speak about a part of the manifold as if it is a Euclidean space. Given a curve $\gamma : I \to M$, with $I \subseteq \mathbb{R}$ an open interval, for instance a particle trajectory, we can call γ *smooth* if its representation in all charts (U, x) is smooth, i.e. the map

$$x \circ \gamma : I \to \mathbb{R}^d \tag{2.3}$$

should be smooth. A similar construction allows us to talk about smooth functions $f : M \to \mathbb{R}$, for example a temperature distribution on Earth. We denote the set of smooth functions on M by $C^\infty(M)$. See figure 2.1 for a picture depicting all the domains and images of the various maps involved.

Now suppose you are standing on the North Pole and want to move to some warmer place. How will you know in which direction to walk? For this we require the notion of a *tangent vector*. Given a point $p \in M$, say the North Pole, we imagine placing a plane tangent to M at p as shown in figure 2.2. This is known as the *tangent space*. We define a tangent vector v_p at p to be a linear map $v_p : C^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz rule (or product rule), i.e.

$$v_p(fg) = f(p)v_p(g) + v_p(f)g(p), \quad \forall f, g \in C^\infty(M), \tag{2.4}$$

where ∂_μ denotes the partial derivative with respect to the μ-th variable. The set of all such vectors is denoted by T_pM and can be made into a vector space. This may seem like an abstract algebraic definition, but returning to our example, a reasonable guess for tangent vectors is the partial derivatives of f. Partial derivatives are linear and satisfy the Leibniz rule, we merely need to be careful that we can only take the derivative of the representation of f in a chart. Indeed, we can construct an explicit basis of T_pM, called the *chart-induced basis*, by choosing a chart (U, x) at p and considering the partial derivative of the representation function:

$$\left(\frac{\partial}{\partial x^\mu}\right)_p (f) := \partial_\mu (f \circ x^{-1})|_{x(p)}. \tag{2.5}$$

In other words, vectors in tangent spaces can be described by the directional derivatives of smooth functions on the manifold. This relation becomes even more apparent by defining the *derivative* of f as follows:

$$(df)_p : T_pM \to \mathbb{R}, \quad (df)_p(v_p) := v_p(f). \tag{2.6}$$

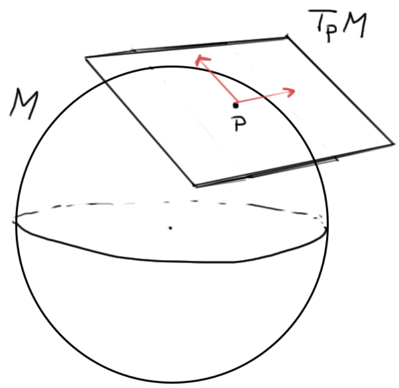

Figure 2.2: Shown is the 2-dimensional tangent space to a point p on the 2-sphere. In red two tangent vectors in T_pM are depicted.

Let us investigate what kind of object $(df)_p$ is, for this we expand a given tangent vector v_p in the chart-induced basis:

$$v_p = v_p^\mu \left(\frac{\partial}{\partial x^\mu}\right)_p, \tag{2.7}$$

from which it follows that

$$(df)_p(v_p) = v_p^\mu \left(\frac{\partial f}{\partial x^\mu}\right)_p \tag{2.8}$$

$$= v_p^\mu \partial_\mu (f \circ x^{-1})|_{x(p)}. \tag{2.9}$$

Hence, by applying $(df)_p$ to a tangent vector we find the change of the representation function along that vector. This is precisely what we set out to do and shows the role of tangent vectors. Note that one might be tempted to read equation (2.9) as the 'inner product' between a vector $\mathbf{v}$ and the gradient ∇f. However, we have not supplied such an inner product. Later on this role will be played by the so-called *metric*.

2.2 Tensor Fields and Integral Curves

Having found a way to attach vectors to a point on a manifold, the natural next step is to attach vectors at *every* point in such a way that the resulting *vector field* is smooth. One can think of, for instance, specifying a wind field on Earth which at every point points in a certain direction. More precisely, given a collection of tangent vectors at each point in M we can consider, for fixed $f \in C^\infty(M)$, the object

$$v(f) : M \to \mathbb{R}, \quad p \mapsto v_p(f). \tag{2.10}$$

Then, if $v(f)$ is a smooth map for all $f \in C^\infty(M)$, we call the resulting map $v : C^\infty(M) \to C^\infty(M)$ a *smooth vector field*.

Now we can ask the question: What happens when we let ourselves be 'pushed along' by the wind? More precisely, can we find a curve whose tangent field is precisely the given vector field, also known as an *integral curve*? The answer is yes, and we start by considering a tangent field to a curve, which is defined by:

$$T_\gamma : C^\infty(M) \to C^\infty(M), \quad T_\gamma(f) := (f \circ \gamma)', \tag{2.11}$$

were the prime denotes differentiation. By choosing a chart and applying the chain rule, we find that the components of T_γ are given by

$$T_\gamma^\mu = (x^\mu \circ \gamma)'. \tag{2.12}$$

Given a vector field v on M with components v^μ, the problem of finding an integral curve comes down to solving the differential equation:

$$\frac{d}{d\lambda}(x^\mu \circ \gamma(\lambda)) = v^\mu(\gamma(\lambda)). \tag{2.13}$$

The Picard-Lindelöf theorem of differential equations tells us that such an equation indeed has a (unique) solution defined on some open interval in $\mathbb{R}$ [1].

It turns out that in the theory of general relativity, a crucial object is a *tensor field*, which is in some sense a generalisation of a vector field. For example, the already mentioned metric is a type (0,2) tensor field. Let us first consider another simple type of tensor field, namely a *covector field.* Since T_pM is a vector space, we can consider its dual space T_p^*M, i.e. the set of linear maps from T_pM to $\mathbb{R}$. Having chosen a chart-induced basis for T_pM, we can make a basis for T_p^*M, with elements denoted by $(dx^\mu)_p$, by the requirement that

$$(dx^\mu)_p\left(\frac{\partial}{\partial x^\nu}\right)_p = \delta^\mu{}_\nu. \tag{2.14}$$

Similar to vectors, we can then expand an arbitrary covector in terms of this basis and also consider covector fields. Returning to our example of the derivative of a function $(df)_p$, this is, by definition, a covector with components given by

$$(df)_p\left(\frac{\partial}{\partial x^\mu}\right)_p = \partial_\mu(f\circ x^{-1})|_{x(p)}. \tag{2.15}$$

It should be mentioned that we have worked with components vector and covector fields in *one* specific chart. This choice, however, is rather arbitrary and it is important to understand what happens when we choose a different chart. Suppose we have two overlapping charts (U,x) and $(\tilde{U},\tilde{x})$, by using the chain rule it can be shown that the components of a vector v_p and a covector field ω_p are related by

$$\tilde{v}_p^\nu = v_p^\mu\left(\frac{\partial \tilde{x}^\nu}{\partial x^\mu}\right)_p, \quad \tilde{\omega}_{\nu,p} = \omega_{\mu,p}\left(\frac{\partial x^\nu}{\partial \tilde{x}^\mu}\right)_p. \tag{2.16}$$

Let us look at equation (2.16) in more detail. We see that the transformation between the components are the exact inverses of each other, since

$$\left(\frac{\partial x^\nu}{\partial \tilde{x}^\mu}\right)_p = \left(\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right)_p^{-1}. \tag{2.17}$$

Where does this relation come from? Well, the objects ω_p and v_p are abstract (co)vectors, which should not depend on the choice of basis. Therefore $\omega_p(v_p)$, which is a real number, should not change either. Let us compute what this is exactly:

$$\omega_p(v_p) = \sum_{\mu,\nu}\omega_{\mu,p}v_p^\nu(dx^\mu)_p\left(\frac{\partial}{\partial x^\nu}\right)_p \tag{2.18}$$

$$= \sum_{\mu,\nu}\omega_{\mu,p}v_p^\nu\delta_\nu^\mu \tag{2.19}$$

$$= \omega_{\mu,p}v_p^\mu. \tag{2.20}$$

But then we must have

$$\omega_{\mu,p}v_p^\mu = \tilde{\omega}_{\mu,p}\tilde{v}_p^\mu, \tag{2.21}$$

so if the components of v change, then the components of ω must change in the 'inverse' way, such that this equation holds. This also explains the names 'vector' vs. 'covector'.

Having defined covectors as maps from T_pM to $\mathbb{R}$, a natural generalisation is the following. We define a *tensor* T_p of type (k,l) at the point $p \in M$ to be a multi-linear map

$$T: \underbrace{T_p^*M\times\ldots\times T_p^*M}_{k \text{ times}}\times\underbrace{T_pM\times\ldots T_pM}_{l \text{ times}}\to\mathbb{R}.$$

Naturally, as we did for vectors and covectors, we can define a tensor at each point in a smooth way, resulting in a *tensor field*. We denote by $\mathcal{T}(k,l)$ the set of all (k,l) tensor fields, this is naturally a vector space. Let us discuss some important examples of tensors. A (0,1) tensor is precisely a covector. A (1,0) tensor is a map from T_p^*M to $\mathbb{R}$, i.e. an element of $T_p^{**}M$. However, since $T_p^{**}M$ is isomorphic to T_pM, we can identify a (1,0) tensor with a vector[3]. Lastly, let us consider a (1,1) tensor T. For some fixed $v_p \in T_pM$, the map $T(\cdot, v_p) : T_p^*M \to \mathbb{R}$ is a (1,0) tensor, i.e. a vector. Therefore we may interpret T as a linear map from T_pM to T_pM. Having these basic examples in mind is very useful, since we can construct higher order tensors from vectors and covectors using tensor products, see Appendix A.2.

We will see plenty of more advanced examples of tensor fields once we start discussing general relativity. We can, however, already define one of the most important objects, namely a *metric*. We have already mentioned that the metric should act as some kind of inner product, so naturally it will be defined as a (0,2) tensor field g_{ab} that is symmetric and non-degenerate.[4] More precisely, we have

1. (symmetric) $g_{ab}(v^c, w^d) = g_{ab}(w^d, v^c), \quad \forall v^c, w^d$.
2. (non-degenerate) If $g_{ab}(v^c, w^d) = 0$ for all v^c then $w^d = 0$.

Similarly to inner products, one can always *locally* find an orthonormal basis $\{v_1, \ldots, v_d\}$ of T_pM such that g_p takes the form $g_p = \text{diag}(+1, +1, \ldots, -1, -1, \ldots)$ [2]. Moreover, the number of $+$ signs and $-$ signs is independent of the basis. It is often called the *signature* of the metric. A familiar example is the Minkowski metric of special relativity, which has signature $+ - - -$.

One can use a metric to transition between vectors and covectors. Indeed, as a map $g(\cdot, v_p)$ it is a (0,1) tensor, i.e. a covector. Hence we can identify a metric as a map $g : T_pM \to T_p^*M$. Moreover, non-degeneracy ensures this map is a bijection. Let us write the metric in components:

$$g = g_{\mu\nu} dx^\mu dx^\nu.$$

This notation conveys the heuristic notion of 'infinitesimally squared distance'[5]. One can also view this equation as a natural generalisation of the Pythagorean theorem, in which case $g_{\mu\nu} = \delta_{\mu\nu}$.

2.3 Covariant Derivatives and Curvature

An important observation is that tangent spaces at different points have *nothing* to do with each other, at least up till now. Without any further structure there is no natural way to compare tangent vectors at different points to each other. Our next goal is to introduce the notion of a *covariant derivative*, which allows us to do exactly this: transport vectors along curves to different points. A derivative operator or *covariant derivative*, ∇_c is a map

$$\nabla_c : \mathcal{T}(k,l) \to \mathcal{T}(k, l+1), \quad T^{a_1,\ldots,a_k}{}_{b_1,\ldots,b_k} \mapsto \nabla_c T^{a_1,\ldots,a_k}{}_{b_1,\ldots,b_k},$$

such that

1. (Linearity) For all $A, B \in \mathcal{T}(k,l)$ and $\alpha, \beta \in \mathbb{R}$
$$\nabla_c \left(\alpha A^{a_1\ldots a_k}{}_{b_1\ldots b_l} + \beta B^{a_1\ldots a_k}{}_{b_1\ldots b_l}\right) = \alpha \nabla_c A^{a_1\ldots a_k}{}_{b_1\ldots b_l} + \beta \nabla_c B^{a_1\ldots a_k}{}_{b_1\ldots b_l}. \tag{2.23}$$
2. (Leibniz) For all $A \in \mathcal{T}(k,l)$ and $B \in \mathcal{T}(k', l')$
$$\nabla_e \left(A^{a_1\ldots a_k}{}_{b_1\ldots b_l} B^{c_1\ldots c_{k'}}{}_{d_1\ldots d_{l'}}\right) = \left(\nabla_e A^{a_1\ldots a_k}{}_{b_1\ldots b_l}\right) B^{c_1\ldots c_{k'}}{}_{d_1\ldots d_{l'}} + A^{a_1\ldots a_k}{}_{b_1\ldots b_l} \left(\nabla_e B^{c_1\ldots c_{k'}}{}_{d_1\ldots d_{l'}}\right). \tag{2.24}$$

[3]Indeed, let V be a finite dimensional vector space, the consider the map

$$i : V \to V^{**}, \quad i(v)(\varphi) = \varphi(v), \tag{2.22}$$

then it is easily shown that i is an isomorphism between V and V^{**}

[4]From here on I will switch to *abstract index notation* for all tensor fields, see also Appendix A.1

[5]In section 2.4 this will become clearer once we use the metric to define an 'infinitesimal volume element', i.e. a volume form.

3. (Commutativity with contraction[6]) For all $A \in \mathcal{T}(k,l)$

$$\nabla_d \left(A^{a_1 \ldots c \ldots a_k}{}_{b_1 \ldots c \ldots b_l} \right) = (\nabla_d A)^{a_1 \ldots c \ldots a_k}{}_{b_1 \ldots c \ldots b_l}. \tag{2.25}$$

4. (Standard derivative on functions) For all $f \in C^\infty(M)$ and $t^a \in T_pM$

$$t(f) = t^a \nabla_a f. \tag{2.26}$$

5. (Torsion free) For all $f \in C^\infty(M)$

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f. \tag{2.27}$$

Intuitively, a covariant derivative is the natural generalisation of the derivative of a function to tensor fields, which is further approved by property (4). Indeed, by contraction with a vector, the covariant derivative answers the question of how a given *tensor field* changes in the direction of that vector.

Since the definition of covariant derivatives has such an algebraic nature, one might wonder whether every smooth manifold admits such structure. However, we can always consider the *ordinary derivative*, denoted by ∂_a, defined such that the components of $\partial_c T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}$ are just the derivatives of the components of $T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}$. The next question is whether covariant derivatives are unique. Let ∇_a and $\tilde{\nabla}_a$ be two distinct covariant derivatives, then it can be shown that there exists a symmetric[7] (1,2) tensor $C^c{}_{ab}$ such that

$$\nabla_a t^b = \tilde{\nabla}_a t^b + C^b{}_{ac} t^c, \tag{2.28}$$

and

$$\nabla_a \omega_b = \tilde{\nabla}_a \omega_b + C^c{}_{ab} \omega_b, \tag{2.29}$$

where v^a and ω_a are vector and covector fields respectively[3]. Similar equations hold for higher rank tensors. An important application is the case where $\tilde{\nabla}_a = \partial_a$, then we denote $C^a{}_{bc}$ as $\Gamma^a{}_{bc}$, which is know as the *Christoffel symbol*, i.e.

$$\nabla_a t^b = \partial_a t^b + \Gamma^b{}_{ac} t^c. \tag{2.30}$$

As remarked above, the covariant derivative tells us how a given tensor fields changes along a vector field. Now suppose we want to compare vectors in different tangent spaces, say at points p and q. Given a curve γ from p to q with tangent field t^a, we can transport a vector v^a along the curve by considering $t^a \nabla_a v^b$. However, to give an accurate representation of how v^a looked at p when arriving at q, we do not want v^a to change when being transported, i.e. we require

$$t^a \nabla_a v^b = 0, \tag{2.31}$$

and say that v^a is *parallely transported along* γ.

Let us examine equation (2.31) more closely. Choosing a chart and looking at the components, we find

$$\begin{aligned} t^\mu \partial_\mu v^\nu &= (x^\mu \circ \gamma)' \partial_\mu (v^\nu \circ x^{-1}) && (2.32)\\ &= (v^\nu \circ x^{-1} \circ x \circ \gamma)' && (2.33)\\ &= (v^\nu \circ \gamma)' && (2.34)\\ &= \frac{dv^\nu}{dt}. && (2.35) \end{aligned}$$

In total, then, using equation (2.30) we find the *parallel transport equation*

$$\frac{dv^\nu}{dt} + t^\mu \Gamma^\nu{}_{\mu\lambda} v^\lambda = 0. \tag{2.36}$$

An important feature of equation (2.36) is that when $\Gamma^a{}_{bc} = 0$ we find that

$$(v^\nu \circ \gamma)' = 0.$$

[6] See Appendix A.2 for the definition of contraction.

[7] Symmetric, in this case, means $C^c{}_{ab} = C^c{}_{ba}$.

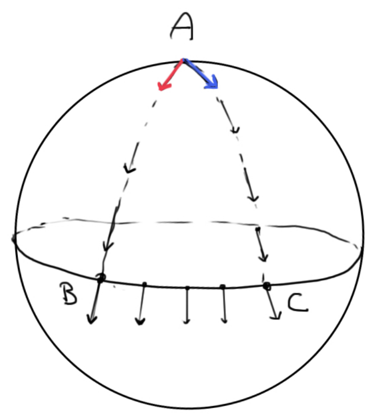

Figure 2.3: Shown is a tangent vector at the point A, depicted in red, which is paralelly transported from A to B to C and back to A. The result is the blue tangent vector. The fact that the red and blue vectors differ means that the 2-sphere is curved.

In other words, the vector field is constant on the curve. This is precisely what we would expect in a 'flat space'.

Interestingly, but perhaps unsurprisingly, the notion of parallelly transporting a vector along a curve is closely related to finding *geodesics*, or 'straight curves', on a manifold between two points. Indeed, one can consider the length l of a path with endpoints a and b, which is given by

$$l = \int_a^b \sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}dt, \tag{2.37}$$

and then use the variational principle to find the equation for the path. The result is exactly equation (2.36). [3]. Hence we can interpret a geodesics as a curve which parallel-transports its own tangent vector.

Given a metric g_{ab} on the manifold, a logical requirement is that the number $g_{ab}v^a w^b$ remains unchanged when we parallel transport the vector fields v^a and w^b along any curve with tangent field t^a, i.e.

$$t^a \nabla_a (g_{bc} v^b w^c) = 0. \tag{2.38}$$

Using the Leibniz rule and the fact that v^a and w^b are parallelly transported, we find

$$\nabla_a g_{bc} = 0. \tag{2.39}$$

It turns out that this condition uniquely determines ∇_a, i.e. a metric naturally induces a covariant derivative on a manifold, i.e. it fixes ${\Gamma^a}_{bc}$. In the rest of the text we will always use this covariant derivative.

Let us now move to the notion of curvature, by means of a simple example. Suppose you are standing on the North Pole, holding out a stick towards the equator, parallel to the ground. You are then instructed to parallelly transport the stick towards the equator. Having arrived there, you then parallelly transport the stick around a quarter of the circumference of the Earth. Finally, you parallelly transport the stick back to the North Pole. If all went well, you will find that the stick is rotated by 90 degrees with respect to its initial orientation. This is also depicted in figure 2.3. On the other hand, had you walked on a perfectly straight plane, the stick would always return to its starting position with exactly the same orientation. What is the difference? Well, the Earth is *curved*, whereas a flat plane is not. More precisely, we characterise the curvature of a manifold by the failure of successive applications of derivative operators to commute. In other words, we define the (1,3) *Riemann curvature tensor* ${R_{abc}}^d$ by

$$\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = {R_{abc}}^d \omega_d. \tag{2.40}$$

At first sight, the Riemann tensor might not necessarily be related to the changing of vectors as described in the example. However, it can be shown that for an infinitesimally small closed curve, the change in the vector is directly proportional to the Riemann tensor, i.e. [3]

$$\delta v^a \propto R_{cbd}{}^a. \tag{2.41}$$

From the Riemann tensor, one can define the *Ricci tensor* as

$$R_{ac} = R_{abc}{}^b. \tag{2.42}$$

and the *scalar curvature* R as

$$R = R_a{}^a. \tag{2.43}$$

In Appendix A.3 some properties of the Riemann and Ricci tensors are listed, moreover, the following very important equation is derived:

$$\nabla^a G_{ab} = 0, \tag{2.44}$$

where

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} \tag{2.45}$$

is the *Einstein tensor*. This tensor will play an important role in the *Einstein equation*, as will be discussed in detail in the next chapter. It will turn out that the Einstein tensor is related to the energy and momentum density of spacetime, then equation (B.10) essentially states the conservation of energy and momentum.

2.4 Differential Forms and Integration on Manifolds

When analysing the dynamical laws that govern the trajectories of objects in spacetime under for instance the influence of electromagnetic forces we will need to integrate a Lagrangian density over a manifold. To define this, we need the notion of differential (volume) forms. A *differential p-form* $\boldsymbol{\omega}$[8] is a totally anti-symmetric tensor field of type $(0,p)$. By introducing the *wedge product* defined as

$$(\omega \wedge \eta)_{a_1,\ldots,a_p,b_1,\ldots,b_q} = \frac{(p+q)!}{p!q!} w_{[a_1,\ldots,a_p}\eta_{b_1,\ldots,b_q]}$$

we can create higher order forms from 1-forms, such as the chart-induced covector fields dx^μ.

Next, let us try to integrate such forms on a manifold. Let us consider the following example. Suppose we have two overlapping charts (U,x) and (V,y) and let $\boldsymbol{\omega}$ be a differential n-form. Then we can expand $\boldsymbol{\omega}$ using the chart induced basis as follows:

$$\boldsymbol{\omega} = \omega_{(x)} dx^1 \wedge \ldots \wedge dx^n.$$

Now the idea would be to define the integral over $\boldsymbol{\omega}$ as an 'ordinary' integral over its representative function $\omega_{(x)} \circ x^{-1}$, i.e.

$$\int_{U\cap V} \boldsymbol{\omega} := \int_{x(U\cap V)} \omega_{(x)} \circ x^{-1} dx^1 \ldots dx^n, \tag{2.46}$$

where the right-hand side is a (Riemann) integral in $\mathbb{R}^n$. Unfortunately, a problem arises when when we look at another chart (V,y). Clearly the integral over $\boldsymbol{\omega}$ should not depend on the choice of charts. It can be shown that under a change of chart, we have

$$\omega_{(x)} = \omega_{(y)} \mathrm{Jac}(y \circ x^{-1}), \tag{2.47}$$

where $\mathrm{Jac}(y\circ x^{-1})$ denotes the Jacobian of the transformation $y \circ x^{-1}$. Note that this transformation is the same as in equation (2.17) Combining equations (2.46) and (2.47) and using the change of variables theorem from multidimensional analysis [4], we find that

$$\int_{x(U\cap V)} \omega_{(x)} \circ x^{-1} dx^1 \ldots dx^n = \int_{y(U\cap V)} \omega_{(y)} \circ x^{-1} \circ x \circ y^{-1} |\mathrm{Jac}(x\circ y^{-1})| \mathrm{Jac}(y \circ x^{-1}) dy^1 \ldots dy^n \tag{2.48}$$

$$= \pm \int_{y(U\cap V)} \omega_{(y)} \circ y^{-1} dy^1 \ldots dy^n. \tag{2.49}$$

[8] Since p-forms have a trivial index structure $\omega_{a_1\ldots a_p}$ we will often denote them in bold as $\boldsymbol{\omega}$

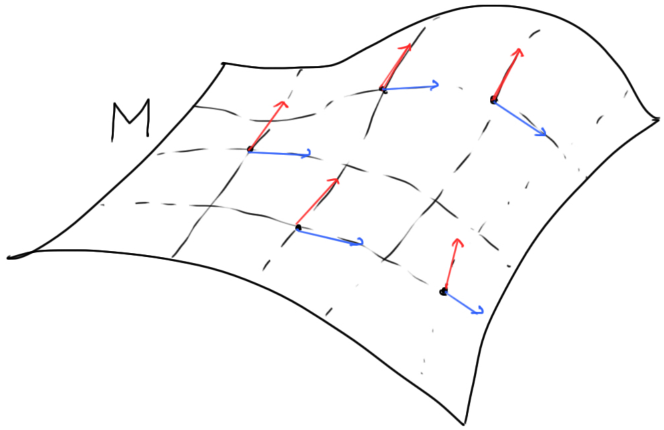

Figure 2.4: A 2-dimensional manifold M, together with several choices of ordered bases at different points is shown. An orientation on M requires that the ordering of the red and blue vectors stays the same, as is depicted.

We see that the integral almost transforms correctly under change of charts, up to a pesky minus sign. We can solve this by requiring that all our charts are such that $\mathrm{Jac}(y \circ x^{-1})$ is positive. This is equivalent to providing an *orientation* of the manifold. It should be noted that this is not always possible, the Möbius band, for instance, is not orientable.

Fortunately, given a metric g_{ab} there is a naturally induced orientation, up to a sign. We define a *volume form* $\boldsymbol{\epsilon}$, i.e. a continuous, nowhere vanishing n-form, by the requirement that

$$\epsilon^{a_1 \ldots a_n} \epsilon_{a_1 \ldots a_n} = (-1)^s n! \tag{2.50}$$

where s is the number of minus-signs in g_{ab}. It can be shown that this uniquely defines such a volume form. [3] More specifically, one finds

$$\boldsymbol{\epsilon} = \sqrt{\det(g_{\mu\nu})} dx^1 \wedge \ldots \wedge dx^n. \tag{2.51}$$

Intuitively, we can apply $\boldsymbol{\epsilon}$ to an ordered basis of chart-induced basis vectors, which will yield a positive number at every point in the manifold. This ensures, in a sense, that the coordinate frames are oriented the same along the entire manifold. This is also shown in figure 2.4.

We have now covered all the tools we require to start defining spacetime. Tensor fields, covariant derivatives, metrics and the Riemann and Einstein tensors will play a huge role in conveying all the properties we will require from spacetime.

Chapter 3

General Relativity and Einstein's Equation

One of the most beautiful applications of differential geometry is in the theory of general relativity. All the mathematical concepts of tensor algebra, tangent spaces, curvature and geodesics that were discussed in the previous chapter play extremely important roles. We will start with more of a historical discussion to understand where the underlying ideas of general relativity come from and how these ideas were incorporated into the structure of spacetime. We will find that spacetime is described by a manifold structure together with a metric. The form of this metric will determine what our universe looks like, i.e. it will describe the curvature of spacetime and its causality structure. It is, in turn, connected to the constituents of our universe, the matter distribution. The connection is given by the Einstein equation, which we will first derive in vacuum. Then, after a thorough introduction to the stress-energy-momentum tensor, we will postulate the full equation in non-empty space and shortly discuss how it is connected to the theory of Newtonian gravity.

3.1 The History of General Relativity

During the last centuries our description of the world around us has vastly changed. Ideas come and ideas go, but right now there are three core principles upon which general relativity is built. These are the *principle of relativity*, the *principle of equivalence* and the postulate that nothing can have a speed greater than that of light. [5] Let us discuss these principles in more detail and investigate their effect on our description of space(time).

In the very early days of Aristotle, it was believed that any object on which no force acts is in a state of rest. Hence the existence of a preferred state, namely rest, was postulated. Spacetime was viewed as the direct product of $\mathbb{E}^3$ (representing space) with $\mathbb{E}$ (representing time).[1] See figure 3.1 for a depiction of Aristotelian spacetime. Aristotle-inertial motions are motions that are at rest. As a result, there is a notion of both absolute time and absolute space. Spatial distance, at the same point in time, is given by the usual Euclidean metric

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{3.1}$$

Then, in the year 1638, Galileo stated what is known as the *principle of Galilean relativity*. He stated that observers that move with uniform velocity with respect to each other are equivalent. To use his original words:

> "...have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still."[6]

In other words, Galilean-inertial motions are motions with constant velocity. Later on Newton described his famous Newtonian mechanics in which such inertial motions become precisely the motions of particles

[1] Here $\mathbb{E}$ denotes the affine real line, i.e. the Euclidean space of dimension 1.

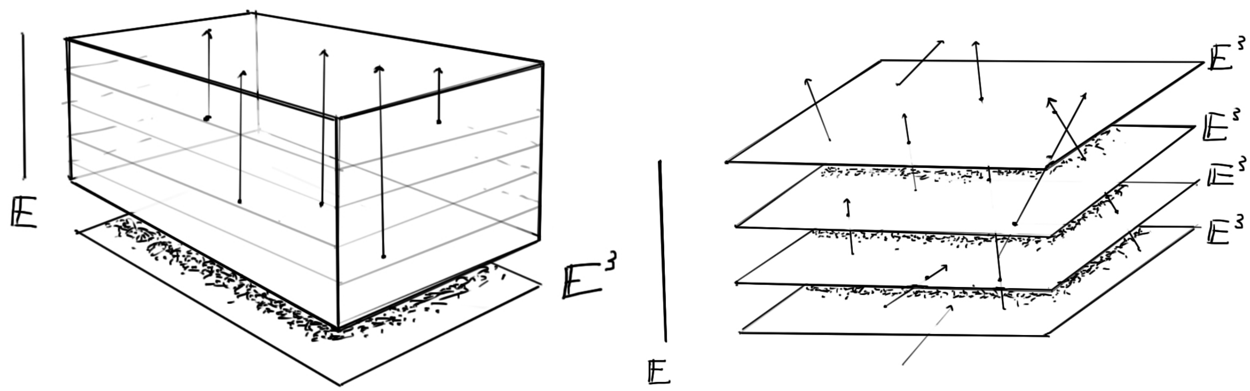

Figure 3.1: Left: Aristotle spacetime, the product space $\mathbb{E} \times \mathbb{E}^3$. Inertial motions are only those that are at rest, i.e. straight vertical lines. Right: Galilean spacetime, a flat fibre bundle of $\mathbb{E}^3$ over $\mathbb{E}$. Inertial motions are motions with uniform velocity, i.e. straight lines.

on which no force acts. The consequence of this equivalence between observers is the following: given an observer A moving with non-zero velocity with respect to an observer B who is at rest, we can always 'tilt' spacetime such that A appears at rest. Mathematically, we consider spacetime as a (topologically trivial) *fibre bundle* of $\mathbb{E}$ over $\mathbb{E}^3$. In a sense, we are allowed to 'shift' different layers with respect to each other, as is depicted in figure 3.1. As a result, there is no longer a notion of absolute space. It should be noted that Galilean spacetime is still flat.

Next, we consider the equivalence principle, which is related to the different descriptions of mass. If we wish to find the mass of a particle, we could do this in the following two ways:

- Use Newton's second law: $F = m_i a$, by applying a constant force on an object and measuring its acceleration. This is called the *inertial mass*.
- Use Newton's law of gravitation: $F_g = G\frac{m_g M_g}{r^2}$, by considering the interaction of the object with mass m_g with another object of mass M_g. This is called the *gravitational mass*.

It is an experimental result that the two descriptions are, in fact, equivalent, i.e. $m_i = m_g$[7]. This is a particular property of the gravitational force, indeed, the Coulomb force, for instance, does not have such a property. One can now perform the following thought experiment: We consider an observer A inside a rocket with the windows closed. Due to his clumsiness, A drops the book he was holding and he observes it falling towards the floor with an acceleration g. Therefore A believes that the rocket has not yet taken flight yet and opens the window curtains, only to realize that the rocket had indeed taken flight and flying with acceleration g. The conclusion is that *gravity cannot be distinguished from acceleration.* It was Einstein who took this equivalence and postulated that (Einstein)-inertial motions are the motions that particles take when the total non-gravitational force acting on them is zero. In other words, they are free-falling in a gravitational field. This alteration of inertial motion and its inclusion into the description of spacetime resulted in the Newton-Cartan theory, or Newtonian spacetime. Here spacetime is still considered as a (topologically trivial) fibre bundle of $\mathbb{E}$ over $\mathbb{E}^3$, but now it is equipped with a connection which has non-zero curvature. The metric is defined such that the geodesic motions are precisely Einstein-inertial motions. In short, we replace curved lines in flat space, with 'straight' lines in curved space.

Finally, we discuss the finiteness of the speed of light. One can consider a general 4-dimensional metric manifold, but to keep our intuition intact, let us start by considering $\mathbb{R}^4$ together with the Minkowski metric. Hence we enter the realm of special relativity. Using the metric, we can compute the 'length' of tangent vectors by

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2. \tag{3.2}$$

For a given curve through spacetime passing through some point, we distinguish three cases:

- $ds^2 > 0$: *Space-like*,

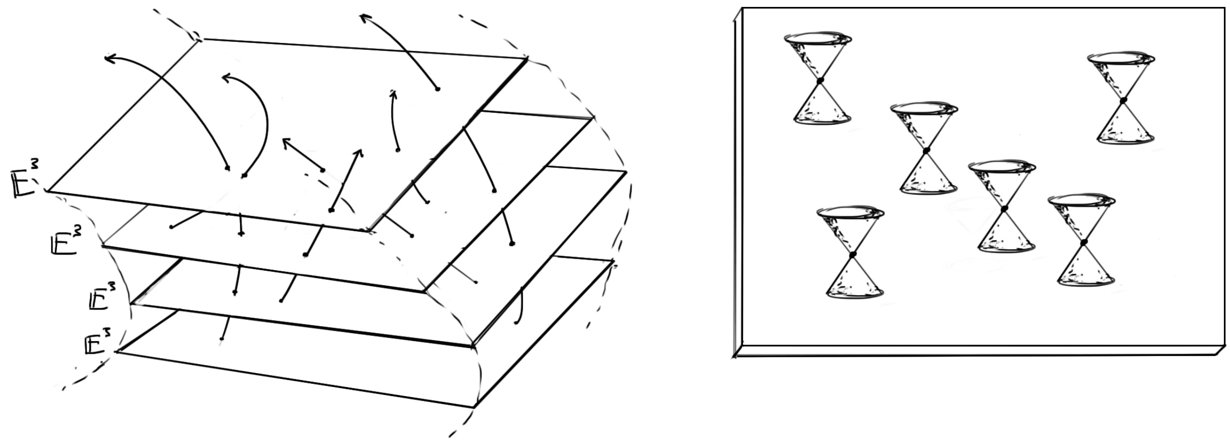

Figure 3.2: Left: Newtonian spacetime, a curved fibre bundle of $\mathbb{E}^3$ over $\mathbb{E}$. Inertial motions are motions that are free-falling in a gravitational field. Right: Minkowski spacetime, the flat manifold $\mathbb{R}^4$ (here drawn as $\mathbb{R}^2$) with the Minkowski metric. Light-cones are paralelly aligned and determine the causality structure of spacetime.

- $ds^2 = 0$: *Light-like*,
- $ds^2 < 0$: *Time-like*.

Next, we can consider all the possible time-like curves passing through that point. This will trace out a double cone called the *light-cone*, and it represents the possible futures and pasts one can reach from that point. In the case of special relativity these are precisely the curves whose speed is less than c. Moreover, one can retrieve the Newtonian picture by letting c go to infinity, i.e. allowing arbitrarily large velocities. In this case the width of the double cone will be infinite, resulting in an entire double half-plane. Since the Minkowski metric is constant, the resulting Christoffel symbols vanish and hence the Riemann tensor vanishes. In other words, spacetime in special relativity is flat. As a result the light-cones will be parallel to each other at all spacetime points, see figure 3.2

Having discussed all three principles that build general relativity, it is time to put everything together. First, one might ask whether we can incorporate the finiteness of the speed of light into the Galilean spacetime description. The answer is no, due to the existence of *absolute time*. The $\mathbb{E}^3$ slices of spacetime give us a plane in each tangent space. If we also impose the existence of light cones, we can use the metric to construct a preferred direction at each point, which is in contradiction with the principle of Galilean relativity. Hence we must abolish the concept of absolute time!

Indeed, *Einsteinnian spacetime* is defined as a 4-dimensional real manifold with a Lorentzian metric. Similarly to Newtonian spacetime, we consider the geodesics to be the inertial motions of particles. Only now we do not supply the metric by hand. In fact, the metric will be determined from the matter distribution by Einstein's equation, which will be discussed in the upcoming sections. Note that the matter distribution will encode information about the gravitational fields. Moreover, the metric still allows us to produce light cones, but due to the possible curvature of spacetime, these will no longer be parallel, as was the case in special relativity. One can interpret the light cones as being an additional *causality structure*. Since no observer can travel faster then the speed of light, wordlines are confined to the interior of the cones (or, in the case of light, to the boundary of the cones) hence certain spacetime events cannot be in causal contact with other spacetime events. See figure 3.3 for this final picture of spacetime.

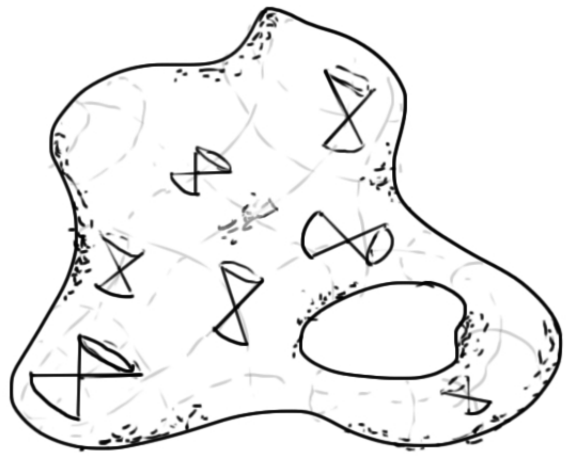

Figure 3.3: Einstein spacetime, the generalisation of Newtonian spacetime over a Lorentzian manifold, together with the causality structure introduced by the light cones, which are now no longer parallely alligned.

3.2 The Principles of General and Special Covariance

In the theory of special relativity, different inertial systems are related by the Lorentz transformations. Under such Lorentz transformations, the equations of motion remain the same. This reflects the fact that different observers measure the same physical quantities. This is known as the *principle of special covariance*. Note however, that the Lorentz transformations form a very special collection of transformations: they are linear. Indeed, if one chose to do special relativity in polar coordinates, the equations of motion would look very different (familiar terms such as the centrifugal and Coriolis force are introduced). General relativity takes this concept to a new level. The equivalence principle implies that *any* accelerated observer transformed into an observer which is free-falling in a gravitational field. As argued in the previous section, in general relativity we want to allow such observers to be inertial observers as well. As a result, we need to allow much more general transformations between observers than just the Lorentz transformations. We therefore postulate that the physical laws remain the same after *any* transformation applied to the manifold of spacetime as a whole. One can restate this by saying that all laws of physics are *tensor equations* and any intrinsic properties of spacetime itself are reflected by the metric g_{ab} and its derived quantities, such as the Riemann and Ricci tensors.

3.3 Einstein's Equation in Vacuum

We can summarize the entirety of the content of general relativity in the following statement:

> "Spacetime is a 4 dimensional manifold M on which there is defined a Lorentz metric g_{ab}. The curvature of g_{ab} is related to the matter distribution by the Einstein equation."[3]

Here the Einstein equation is given by:

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}, \tag{3.3}$$

with T_{ab} the so called *stress-energy-momentum tensor*, or *stress-energy tensor* for short. The stress-energy tensor will describe the energy and momentum density within spacetime. Let us first consider the case where

$T_{ab} = 0$, i.e. the universe is empty. Our goal is to find an equation for the metric g_{ab} and we will achieve this by considering the Lagrangian formalism. Hamilton's principle states that the motion of a system is such that the action

$$S = \int \mathcal{L}\, dt \tag{3.4}$$

is stationary with respect to variations in the path. The same principle applies now as well, the difference is that we are not solving for a path, but instead for the metric g_{ab} itself. Hence we must find a Lagrangian that depends on the metric and somehow conveys the dynamical laws of an empty universe. The *principle of general covariance* states that the only quantities that can appear in the laws of physics are the metric itself and quantities derivable from the metric. Therefore the simplest Lagrangian one could write down is the following:[2]

$$S[g^{ab}] = \int \sqrt{g} R\, d^4x, \tag{3.6}$$

which is known as the *Hilbert action*. Our first step is to consider the variation of S under a change $g^{ab} \to g^{ab} + \delta g^{ab}$. Since

$$R = R_a{}^a = g^{ab} R_{ab}, \tag{3.7}$$

we find for the variation of S:

$$\delta S[g^{ab}] = \int \delta(\sqrt{g} g^{ab} R_{ab})\, d^4x \tag{3.8}$$

$$= \int \delta(\sqrt{g}) g^{ab} R_{ab} + \sqrt{g}\delta(g^{ab}) R_{ab} + \sqrt{g} g^{ab} \delta(R_{ab})\, d^4x. \tag{3.9}$$

We consider the various objects in the integral piece by piece. First, we remark that

$$g^{ab}\delta g_{ab} = \delta(g^{ab} g_{ab}) - g_{ab}\delta g^{ab} \tag{3.10}$$

$$g^{ab}\delta g_{ab} = -g_{ab}\delta g^{ab}, \tag{3.11}$$

since $g^{ab} g_{ab}$ is constant. Next, we consider

$$\delta(\sqrt{g}) = \frac{1}{2\sqrt{g}} \delta(\det(g_{ab})) \tag{3.12}$$

$$= \frac{1}{2\sqrt{g}} g g^{ab} \delta g_{ab} \tag{3.13}$$

$$= -\frac{1}{2}\sqrt{g} g_{ab} \delta g^{ab}. \tag{3.14}$$

Finally, we consider the variation in the Ricci tensor. In Appendix A.3 it is shown that we can write the Ricci tensor in terms of the Christoffel symbols as follows:

$$R_{ab} = -\partial_a \Gamma^c{}_{cb} + \partial_c \Gamma^c{}_{ab} - \Gamma^c{}_{ad}\Gamma^d{}_{cb} + \Gamma^c{}_{cd}\Gamma^d{}_{ab}, \tag{3.15}$$

hence the variation is given by

$$\delta R_{ab} = -\partial_a \delta\Gamma^c{}_{cb} + \partial_c \delta\Gamma^c{}_{ab} - \Gamma^c{}_{ad}\delta\Gamma^d{}_{cb} - \Gamma^d{}_{cb}\delta\Gamma^c{}_{ad} + \Gamma^c{}_{cd}\delta\Gamma^d{}_{ab} + \Gamma^d{}_{ab}\delta\Gamma^c{}_{cd}. \tag{3.16}$$

Moreover, we notice that since variations of Christoffel symbols are again tensors, we have

$$\nabla_a \delta\Gamma^c{}_{cb} = -\partial_a \delta\Gamma^c{}_{cb} - \Gamma^c{}_{ad}\partial\Gamma^d{}_{cb} + \Gamma^d{}_{ac}\delta\Gamma^c{}_{db} + \Gamma^d{}_{ab}\partial\Gamma^c{}_{cd} \tag{3.17}$$

[2] In the theory of *unimodular gravity* (UG) one considers the even simpler action

$$S[g^{ab}] = \int R\, d^4x. \tag{3.5}$$

That is, a coordinate system is chosen such that $\sqrt{g} = 1$. It turns out that the resulting classical equations of motion in vacuum are the same as those obtained from the Hilbert action, i.e. the universe is Ricci-flat, see also equation 3.29. [8]

and

$$\nabla_c \delta\Gamma^c{}_{ab} = -\partial_a \delta\Gamma^c{}_{ab} - \Gamma^c{}_{cd}\delta\Gamma^d{}_{ab} + \Gamma^d{}_{ca}\delta\Gamma^c{}_{db} + \Gamma^d{}_{cb}\delta\Gamma^c{}_{ad}. \tag{3.18}$$

By renaming indices we see that the terms $\Gamma^c{}_{ad}\partial\Gamma^d{}_{cb}$ in equation (3.17) and $\Gamma^d{}_{ca}\delta\Gamma^c{}_{db}$ in equation (3.18) are equal, hence we find

$$\delta R_{ab} = -\nabla_a \delta\Gamma^c{}_{cb} + \nabla_c \delta\Gamma^c{}_{ab}. \tag{3.19}$$

It follows, since $\nabla_a g^{bc} = 0$, that

$$\begin{aligned} g^{ab}\delta R_{ab} &= -\nabla_a g^{ab}\delta\Gamma^c{}_{cb} + \nabla_c g^{ab}\delta\Gamma^c{}_{ab} && (3.20)\\ &= \nabla_c \left(-g^{cb}\delta\Gamma^c{}_{cb} + g^{ab}\delta\Gamma^c{}_{ab}\right) && (3.21)\\ &=: \nabla_c v^c. && (3.22)\end{aligned}$$

Therefore the term

$$\int \sqrt{g} g^{ab}\delta R_{ab}\; d^4x = \int \sqrt{g}\; \nabla_a v^a\; d^4x \tag{3.23}$$

is an integral over the divergence of some vector field v^a. By Stokes's theorem this is equivalent to a boundary term, which we will take to be zero. One can show that by adding a surface term to the Hilbert action the unwanted boundary term vanishes.[3]

Putting everything together, we find for the variation of the action:

$$\begin{aligned} \delta S[g^{ab}] &= \int \left(-\frac{1}{2}\sqrt{g} g_{ab}(\delta g^{ab}) g^{ab} R_{ab} + \sqrt{g}(\delta g^{ab}) R_{ab}\right)\; d^4x && (3.24)\\ &= \int \sqrt{g}\left(R_{ab} - \frac{1}{2} g_{ab} R\right)\delta g^{ab} d^4x. && (3.25)\end{aligned}$$

Since this holds for all variations δg^{ab} we find Einstein's equation in vacuum:

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 0. \tag{3.26}$$

Note that the equation $\nabla^a G_{ab} = 0$ is trivially satisfied, as it should. By multiplying equation (3.26) by g^{ab}, we find

$$\begin{aligned} g^{ab} R_{ab} - \frac{1}{2} g^{ab} g_{ab} R &= 0 && (3.27)\\ R - 2R &= 0 && (3.28)\\ R &= 0. && (3.29)\end{aligned}$$

Using equation(3.26) once more we see that $R_{ab} = 0$, i.e. spacetime vacuum is *Ricci-flat*.

The observant reader may have noticed that we could have considered an even simpler action, namely:

$$S[g^{ab}] = -2\int \sqrt{g}\; \Lambda\; d^4x \tag{3.30}$$

where Λ is some constant, usually called the *cosmological constant*[3]. By definition it is the value of energy density in vacuum space. Adding this action to the Hilbert action, we find the adjusted equation of motion:

$$R_{ab} - \frac{1}{2} g_{ab} R + g_{ab}\Lambda = 0. \tag{3.31}$$

The cosmological constant was introduced by Einstein in 1917 and postulated to be negative to make sure the universe was static, i.e. non-expanding [10]. However, as discovered by Hubble in 1929 by observing the red shift of light from distant stars, the universe is indeed expanding. Moreover, Einstein had not realized that a negative Λ would result in an unstable universe. Hence most cosmologists assumed the constant to be zero. From the 1990's onward, after it was discovered that the universe is expanding in an accelerated way. Therefore cosmological the constant is assumed to be non-zero and *positive*. The exact nature of the cosmological constant is yet to be understood and is closely related to the topic of *dark energy*. We will come back to this in section 4.2.

[3]The cosmological constant is the main reason for the recent interest in UG, due to its different nature in the theory. Problems concerning the cosmological constant thus appear differently in the two theories, which is especially interesting when quantum mechanical effects are taken into account, see for instance [9]

3.4 The Stress-Energy Tensor in Special Relativity

Having found that Einstein's equation in vacuum has the form

$$G_{ab} = 0, \tag{3.32}$$

it is now time to consider a non-empty universe, for which we expect the RHS of this equation to be non-zero and related to the distribution of matter in the universe. Let us first investigate what happens in the case of special relativity. The theory of special relativity asserts the following:

> "Spacetime is the manifold $\mathbb{R}^4$ with a flat metric η_{ab} of Lorentz signature defined on it." [3]

Explicitly, in the standard basis the metric is given by

$$\eta_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.33}$$

We can parameterize timelike curves by their *proper time* τ, defined as the length of the path between two parameter values, i.e.

$$\tau = \int_{\lambda_1}^{\lambda_2} \sqrt{-\eta_{ab}T^aT^b}\, d\lambda. \tag{3.34}$$

Intuitively, the proper time is the time that has passed for the observer while moving from $\gamma(\lambda_1)$ to $\gamma(\lambda_2)$. We can now define the *4-velocity* u^a as the tangent vector to a curve that is parameterized by τ. For a point-particle with rest mass m, we then define its *4-momentum* as

$$p^a = mu^a. \tag{3.35}$$

Finally, we can define the *energy* of the particle, as measured by an observer moving with 4-velocity v^a as

$$E = -p_a v^a. \tag{3.36}$$

Considering the observers frame, where $v^0 = c$ and $v^\mu = 0$, for $\mu = 1, 2, 3$, we recognize the energy as the 'time-component' of the 4-momentum. Next, we can consider a continuous distribution of matter, which will be described by the symmetric (0,2) stress-energy tensor T^{ab}. It is defined such that the component $T^{\mu\nu}$ describes the outward flux of the μ-th component of the 4-momentum in the direction of the ν-th component of the 4-velocity of some observer. It is best to examine these components piece by piece:

- T^{00}: The flux of *mass-energy* in the direction of *time*, in other words, the *spatial mass-energy density.*
- $T^{i0} = T^{0i}$, $i = 1, 2, 3$: The flux of the i-th component of the *spatial momentum* in the direction of *time*, in other words, the *spatial momentum density.*
- T^{ij}, $i, j = 1, 2, 3$: The flux of the i-th component of the *spatial momentum* in the j-th direction of *space*. In other words, the *pressure* for $i = j$ and *shear stress* for $i \neq j$.[4]

For an observer with 4-velocity v^a we have that

$$T_{ab}v^b = T_{a0}, \tag{3.37}$$

with respect to the observer itself. Hence we interpret $-T_{ab}v^b$ as the density of mass-energy (and momentum) as measured by the observer.

To summarize, the stress-energy tensor is a combination of the energy and momentum densities, together with the pre-relativity stress tensor, which is used to describe the deformations of, for instance, viscous fluids.

[4] Note that energy per volume equals force per area, which is precisely how pressure and shear stress are defined.

3.5 The Stress-Energy Tensor of a Perfect Fluid

An important example of a stress-energy tensor is that of a perfect fluid. That is because in many cosmological models the universe itself is modelled as a perfect fluid, i.e. an idealised model in which there are no shear stresses, viscosity or heat conduction. Such a fluid is completely characterised by its density ρ and pressure P. Since there are no shear stresses nor momentum fluxes, in the rest frame of the fluid the stress-energy tensor, in matrix form, must be:

$$T^{ab} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \tag{3.38}$$

It follows that in general coordinates, with u^a the 4-velocity of the fluid, we must have

$$T_{ab} = \rho u_a u_b + P(\eta_{ab} + u_a u_b). \tag{3.39}$$

Assuming no external forces, the equation of motion for the fluid will be given by

$$\partial^a T_{ab} = 0. \tag{3.40}$$

Writing this out and using that η_{ab} is constant, we find

$$(\rho + P)\left(u_a \partial^a u_b + u_b \partial^a u_a\right) + u_a u_b \partial^a \rho + (\eta_{ab} + u_a u_b)\partial^a P = 0. \tag{3.41}$$

We can separate this equation by considering the parts that are parallel and perpendicular to u^b, which respectively give the equations:[5]

$$u_a \partial^a \rho + (\rho + P)\partial^a u_a = 0, \tag{3.42}$$

$$(\rho + P)u^a \partial_a u_b + (\eta_{ab} + u_a u_b)\partial^a P = 0. \tag{3.43}$$

Let us now consider the non-relativistic limit, i.e. $P \ll \rho$, $u^\mu = (1, \mathbf{v})$ and $v\frac{dP}{dt} \ll |\nabla P|$, then for equation (3.42) we find:

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0, \tag{3.44}$$

hence

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{3.45}$$

which is the well-known *continuity equation*, which is equivalent to the conservation of mass. For equation (3.43) we find:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right) + \nabla P + \mathbf{v}\frac{\partial P}{\partial t} + \mathbf{v}(\mathbf{v} \cdot \nabla)P = 0, \tag{3.46}$$

hence

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P, \tag{3.47}$$

which is the *Euler equation*. We conclude that in the non-relativistic regime, the stress-energy tensor as given in equation (3.39) indeed satisfies the appropriate equations which describe a perfect fluid.

[5]Indeed, by contracting with u^b we have

$$u^b(\eta_{ab} + u_a u_b)\partial^a P = (u_a - u_a)\partial^a P = 0,$$

since $u_b u^b = -1$. Moreover, we have

$$\begin{aligned} u^b u^a \partial_a ub &= u^a \partial_a (u^b u_b) - u^a u_b \partial_a u^b \\ &= -u^a u^b \partial_a u_b, \end{aligned}$$

from which it follows that $u^b u^a \partial_a ub = 0$. Therefore equation (3.43) is indeed orthogonal to u^b.

Let us look at equation 3.40 in even more detail, by considering an observer v^a such that $\partial_a v^b = 0$. As described in section 3.4, we interpret the quantity

$$J_a = -T_{ab}v^b$$

as the mass-energy density of the fluid as measured by the observer. It follows, then, from equation (3.40) that

$$\partial^a J_a = 0. \tag{3.48}$$

Gauss' theorem then implies that the flux of energy through any 3-dimensional surface of a 4-dimensional volume is equal to zero. That is, we have *conservation of energy.* Conversely, one can show that conservation of energy requires the divergence of the stress tensor to be zero.

3.6 The Stress-Energy Tensor in General Relativity: Einstein's Equation

As in special relativity, the stress-energy tensor plays the role of measuring the energy-momentum densities and fluxes in spacetime. Actually, a simple transition would be to make a substitution $\eta_{ab} \to g_{ab}$ in all equations, which is called the 'minimal substitution rule'. It should be noted, though, that this rule is not entirely free of ambiguity. The stress-energy tensor in general relativity is defined entirely analogous to the special relativity, with its components having the same physical interpretation. However, since spacetime is no longer flat in general relativity, we should be careful since some aspects do not transfer over entirely. Returning to the perfect fluid, the stress tensor is given by

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b), \tag{3.49}$$

which satisfies the equation of motion

$$\nabla^a T_{ab} = 0. \tag{3.50}$$

In curved spacetime it is no longer possible to find observers for whom $\nabla_a v^b = 0$ everywhere. However, over *small* regions of spacetime, it *is* possible to find such observers. Hence, in the context of general relativity, equation (**??** constitutes *local energy conservation.* It is postulated that all stress-energy tensors, not just that of the perfect fluid, obey equation (3.50). Moreover, for the same reason it is impossible to parallelly transport a vector along any curve (as was the case in special relativity), hence a given observer cannot, in general, define the energy of a particle at a different point in spacetime.

Let us turn back to our initial goal: finding the metric of our universe. Instead of prescribing a metric by hand, as in special relativity, we consider the metric as a dynamical variable. It should depend on the matter distribution of spacetime, which reflects the fact that we incorporate gravity into the curvature of spacetime. A clue for the right equations comes from the theory of Newtonian gravity. Given a gravitational potential φ, Poisson's equation tells us that[6]

$$\nabla^2 \varphi = 4\pi\rho, \tag{3.51}$$

where ρ is the matter density. On the RHS, the matter density is related to the stress-energy tensor by:

$$T_{ab}v^a v^b \leftrightarrow \rho. \tag{3.52}$$

On the LHS, we see a term resembling the tidal acceleration, which in general relativity is related to the Ricci tensor by:

$$R_{ab}v^a v^b \leftrightarrow \partial_c \partial^c \varphi. \tag{3.53}$$

These correspondences led Einstein to postulate the following equation:

$$R_{ab} = 4\pi T_{ab}. \tag{3.54}$$

[6]Note that we use natural units here, i.e. $G = 1$, $c = 1$.

Unfortunately, $\nabla^a R_{ab} \neq 0$, in general, whereas $\nabla^a T_{ab} = 0$ due to (local) energy conservation. However, we have already come across a different tensor whose divergence *is* zero, namely the Einstein tensor G_{ab}. Indeed, in the vacuum we found the Einstein tensor as the main object of the laws of motion. For this reason we consider the equation:[7]

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}. \tag{3.55}$$

This equation was written down by Einstein in 1915 and is known as the *Einstein field equation*. Indeed, one can show that, for weak fields, this equation reduces to Poisson's equation [11]. It is an equation that is solved for the metric of spacetime itself, which is dependent on the matter distribution in the universe. However, the stress-energy tensor can itself depend on the metric, as we have seen in the perfect fluid. As a result the Einstein equation is difficult to solve. Nevertheless, exact solutions have been found and limit cases have been extensively studied. In the next chapter we will discuss this subject in detail.

[7]By convention, we divide the stress-energy tensor by an extra factor of 2.

Chapter 4

Models of our Universe

In the previous chapter we encountered the evolution of our description of spacetime and the theory of general relativity. Moreover, the Einstein equation was discussed in detail. By itself, the Einstein equation is already quite a remarkable object. If solved, one can obtain the entire structure of the universe, quite a powerful feat. Unfortunately, there are not many models which can be solved exactly. In this chapter we will discuss one of these, namely the Robertson-Walker model, in which the universe is assumed to be spatially homogeneous and isotropic. We will find that these assumptions already significantly reduce the complexity of the Einstein tensor, basically reducing to the cases of a flat, spherical and hyperbolic universe, each having constant curvature. By then considering the universe to be a perfect fluid, exact solutions to the Einstein equation are found. These solutions have the remarkable property that they imply the expansion of the universe and therefore the existence of the Big Bang. Lastly, a short discussion is included about the accelerated expansion of the Universe, which is in turn related to the cosmological constant and the presence of dark energy.

4.1 Homogeneity and Isotropy

Simply put, spatial homogeneity and isotropy mean that space looks the same everywhere, and it looks the same in every direction. In 2001, the CMB space mission WMAP was launched to measure the temperature of the Cosmic Microwave Background, the results are shown in figure 4.1. It was found was the fractional deviation in temperature is of the order 10^{-4}, implying that the universe is isotropic at large scales (of the order of 100 Mpc) [12]. A reasonable starting point for studying cosmological models of our own universe is therefore the study of homogeneous and isotropic spacetimes.

We start by giving precise definitions of what it means for spacetime to be spatially homogeneous and isotropic. We say spacetime is *spatially homogeneous* if there exists hyper-surfaces Σ_t, parametrised by a 'time' t that 'foliate' spacetime, such that for each t and any two points p and q in Σ_t there exists an isometry of the spatial metric that carries p into q. This is also shown in figure 4.2. Intuitively, we think of isometries as translations and rotations. Hence we require that at a given time, space *looks exactly the same everywhere*, i.e. it is invariant under rotations and translations.

We say spacetime is *spatially isotropic at a point* p if for any timelike observer with tangent u^a and any two 'spatial tangent vectors' s_1 and s_2 at p, there exists a rotation that carries s_1^a into s_2^a but leaves p and u^a fixed[1], see also figure 4.2. This implies that there exists no preferred choice of spatial tangent vector at p. In other words, for any observer *all directions look the same*. Suppose that the hyper-surfaces of homogeneity are *not* orthogonal to timelike wordlines. Then an observer can determine its relative velocity with respect to such a hyper-surface, by measurement of physical quantities, and hence choose a preferred spatial direction. Therefore, isotropy everywhere implies that all hyper-surfaces of homogeneity are orthogonal to timelike wordlines.

Given a metric g_{ab} on the whole of spacetime, we can consider the metric h_{ab} induced onto a specific hyper-surface. Let us first consider the consequences of spatial isotropy. We consider the Riemann tensor

[1]By spatial tangent vectors, we mean vectors orthogonal to the tangent u^a of the observer.

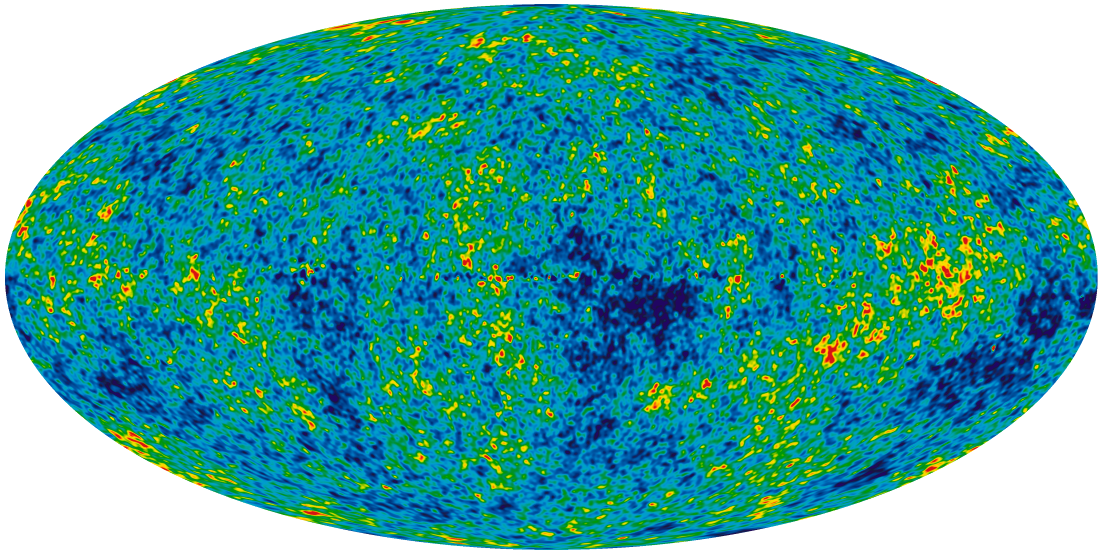

Figure 4.1: A map of the temperature of the CMB from WMAP [13].

${}^{(3)}R_{ab}{}^{cd}$, induced by the three-dimensional metric h_{ab}, as a linear map which sends two-forms to two-forms. Then it is symmetric, hence it has an orthonormal basis of eigenvectors. By isotropy, these eigenvectors must have different eigenvalues, otherwise we could distinguish between different directions. Therefore ${}^{(3)}R_{ab}{}^{cd}$, interpreted as a linear map, must be proportional to the identity, i.e.[2]

$$ {}^{(3)}R_{ab}{}^{cd} = K\delta^{c}{}_{[a}\delta^{d}{}_{b]}, \tag{4.1} $$

where K is a scalar function on the hyper-surface. Lowering the indices with the metric we find for the Riemann tensor:

$$ {}^{(3)}R_{abcd} = Kh_{c[a}h_{b]d}. \tag{4.2} $$

However, homogeneity implies that K is constant on the whole hyper-surface. It follows that the Ricci tensor is given by:

$$ R_{ab} = Kh_{ab}. \tag{4.3} $$

Finally, the scalar curvature is equal to $R = 3K$. As a result, there are only three distinct cases, namely K being positive, negative and zero. Hence, to find the geometry of this spacetime, we must find all isotropic and homogeneous 3 dimensional spaces with positive, negative and zero curvature. Clearly, in the case of $K = 0$ we return to the Euclidean plane $\mathbb{R}^3$, with a metric given by

$$ ds^2 = dx^2 + dy^2 + dz^2, \tag{4.4} $$

or in spherical coordinates

$$ ds^2 = d\psi^2 + \psi(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{4.5} $$

For $K > 0$, we find the 3-spheres whose metric is given by

$$ ds^2 = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{4.6} $$

[2] For a given two-form ω_{ab}, we have

$$ \begin{aligned} \delta^{c}{}_{[a}\delta^{d}{}_{b]}\omega_{ab} &= \frac{1}{2}\left(\delta^{c}{}_{a}\delta^{d}{}_{b}\omega_{ab} - \delta^{c}{}_{b}\delta^{d}{}_{a}\omega_{ab}\right) \\ &= \frac{1}{2}\left(\omega_{cd} - \omega dc\right) \\ &= \omega_{cd}, \end{aligned} $$

where the last equality follows from the fact that ω_{ab} is anti-symmetric. Hence $\delta^{c}{}_{[a}\delta^{d}{}_{b]}$ is indeed the identity map.

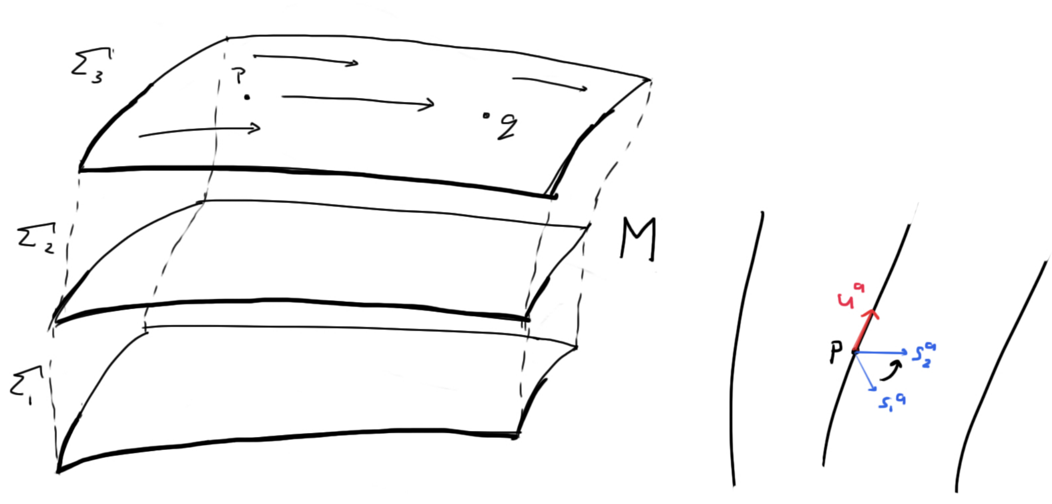

Figure 4.2: Left: Three hyper-surfaces Σ_1, Σ_2, Σ_3 of a 3-dimensional manifold M are shown. In each hyper-surface, we require that the points p and q differ only by translations and rotations. Right: An observer with tangent u^a at a point p is shown, together with two 'spatial' tangent vectors s_1^a and s_2^a. For isotropy we require that s_1^a can be rotated into s_2^a.

Finally, for $K < 0$ we find the hyperboloids, which might be familiar from special relativity where one considers Minkowski spacetime. The metric of the unit hyperboloid is given by

$$ds^2 = d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{4.7}$$

Intuitively, spheres have positive curvature since lines that start out parallel tend to converge towards each other. Similarly, on a hyperboloid parallel lines tend to diverge from each other. Of course, on the flat plane parallel lines do not deviate from each other, hence they have zero curvature. Moreover, spheres and hyperboloids are the simplest geometrical objects that are homogeneous and isotropic everywhere.

The question that remains now is what the metric is on the entire spacetime. Since the surfaces of homogeneity are orthogonal to observer wordlines, the metric g_{ab} cannot contain 'cross-terms' such as $d\tau dx$. Moreover, since observers having 4-velocities of unit length -1 we must have

$$g_{00} = g_{ab}v^a v^b = -1. \tag{4.8}$$

Therefore the full metric is given by

$$ds^2 = -d\tau^2 + a^2(\tau)\begin{cases} d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\varphi^2) & K > 0, \\ dx^2 + dy^2 + dz^2 & K = 0, \\ d\psi^2 + \sinh^2\psi(d\theta^2 + \sin^2\theta\, d\varphi^2) & K < 0. \end{cases} \tag{4.9}$$

Here $a(\tau)$ is a proportionality factor which will be determined by the Einstein equation. This is known as the *Robertson-Walker* cosmological model.

Having assumed the universe to be homogeneous and isotropic, we were already able to reduce the possible geometries of spacetime to three distinct cases. Moreover, we already have the metric and therefore the Einstein tensor G_{ab}. What remains is to specify a stress-energy tensor T_{ab} and solve the Einstein equation. This will be done in the next section.

But before that, is serves to shortly interpret the three geometrical cases in more detail. The metric gives us information on the local structure of spacetime. In particular, the Robertson-Walker metric tells us that, locally, our universe looks like part of a sphere, a flat plane, or a hyperboloid. However, it should be emphasized that we cannot make any statements about the global topology of our universe.

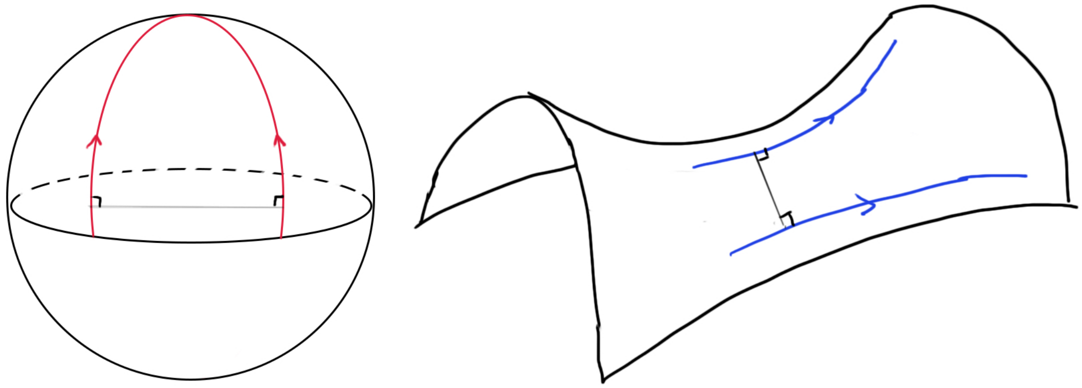

Figure 4.3: On the left the sphere, which has positive curvature, and on the right a saddle (part of a hyperboloid), which has negative curvature, are shown. Initially parallel lines converge towards each other in positively curved spaces, shown in red, and diverges from each other in negatively curved spaces, shown in blue.

4.2 The Universe as a Homogeneous and Isotropic Perfect Fluid

Having discussed the Robertson-Walker model the next step is to specify a stress-tensor and solve Einstein's equation. We will use the stress tensor for a perfect fluid given by

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b). \tag{4.10}$$

Considering the matrix form of T_{ab}, it is easily seen that all spatial directions are equivalent, this implies that all the diagonal 'space-space' components of G_{ab} are equal. Moreover, since an observer must be at rest with respect to a hyper-surface of homogeneity, the term $G_{ab}u^a$ cannot have a spatial component. Therefore all the 'time-space' components of G_{ab} are zero. Finally, a similar argument shows that the off-diagonal 'space-space' terms of G_{ab} zero. We therefore need to solve only two independent equations, namely[3]:

$$G_{\tau\tau} = 8\pi T_{\tau\tau} = 8\pi\rho, \tag{4.11}$$
$$G_{xx} = 8\pi T_{xx} = 8\pi P g_{xx}. \tag{4.12}$$

It remains to compute the elements $G_{\tau\tau}$ and G_{xx}, using the Robertson-Walker metric. In appendix A.3 it is shown that for the Levi-Civita connection the Christoffel symbols are related to the metric by:

$$\Gamma^{\sigma}{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right). \tag{4.13}$$

It follows that

$$\begin{aligned}
\Gamma^{\tau}{}_{xx} &= \frac{1}{2} g^{\tau\rho} \left(\partial_x g_{x\rho} + \partial_x g_{\rho x} - \partial_\rho g_{xx} \right) && (4.14)\\
&= -\frac{1}{2} g^{\tau\tau} \partial_\tau g_{xx} && (4.15)\\
&= a\dot{a}, && (4.16)
\end{aligned}$$

where $\dot{a} = \frac{da(\tau)}{d\tau}$. Similarly, we calculate

$$\begin{aligned}
\Gamma^{x}{}_{x\tau} &= \frac{1}{2} g^{x\rho} \left(\partial_x g_{x\rho} + \partial_x g_{\rho x} - \partial_\rho g_{xx} \right) && (4.17)\\
&= \frac{1}{2} g^{xx} \partial_\tau g_{xx} && (4.18)\\
&= \frac{\dot{a}}{a}. && (4.19)
\end{aligned}$$

[3]Note that one could just as well have chosen the components G_{yy} or G_{zz} instead of G_{xx}.

Finally, it is easily checked that all other terms vanish. Now we can write the Ricci tensor in terms of the Christoffel symbols as:[4]

$$R_{ab} = \partial_c \Gamma^c{}_{ab} - \partial_a \Gamma^c{}_{cb} + \Gamma^d{}_{ab}\Gamma^c{}_{cd} - \Gamma^d{}_{cb}\Gamma^c{}_{ad}. \tag{4.20}$$

We will consider the case $K = 0$, the other cases are similar. First, we compute using the fact that no quantities depend on x:

$$\begin{aligned}
R_{xx} &= \partial_\mu \Gamma^\mu{}_{xx} - \partial_x \Gamma^\mu{}_{\mu x} + \Gamma^\nu{}_{xx}\Gamma^\mu{}_{\mu\nu} - \Gamma^\nu{}_{\mu x}\Gamma^\mu{}_{x\nu} && (4.21)\\
&= \partial_\tau \Gamma^\tau{}_{xx} + 3\Gamma^\tau{}_{xx}\Gamma^x{}_{x\tau} - 2\Gamma^x{}_{\tau x}\Gamma^\tau{}_{xx} && (4.22)\\
&= a\ddot{a} + \dot{a}^2 + a\dot{a}\frac{\dot{a}}{a} && (4.23)\\
&= a\ddot{a} + 2\dot{a}^2. && (4.24)
\end{aligned}$$

Second, we compute

$$\begin{aligned}
R_{\tau\tau} &= \partial_\mu \Gamma^\mu{}_{\tau\tau} - \partial_\tau \Gamma^\mu{}_{\mu\tau} + \Gamma^\nu{}_{\tau\tau}\Gamma^\mu{}_{\mu\nu} - \Gamma^\nu{}_{\mu\tau}\Gamma^\mu{}_{\tau\nu} && (4.25)\\
&= -3\partial_x \Gamma^x{}_{x\tau} - 3\Gamma^x{}_{x\tau}\Gamma^x{}_{x\tau} && (4.26)\\
&= -3\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) - 3\frac{\dot{a}^2}{a^2} && (4.27)\\
&= -\frac{3\ddot{a}}{a}. && (4.28)
\end{aligned}$$

Lastly, we find for the Ricci scalar:

$$\begin{aligned}
R = g^{ab}R_{ab} &= g^{\tau\tau}R_{\tau\tau} + 3g^{xx}R_{xx} && (4.29)\\
&= \frac{3\ddot{a}}{a} + \frac{3}{a^2}\left(2\dot{a}^2 + a\ddot{a}\right) && (4.30)\\
&= 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). && (4.31)
\end{aligned}$$

Hence, the two-nonzero components of the Einstein tensor are given by:

$$\begin{aligned}
G_{xx} &= R_{xx} - \frac{1}{2}g_{xx}R && (4.32)\\
&= a\ddot{a} + 2\dot{a}^2 - 3a^2\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) && (4.33)\\
&= -2a\ddot{a} - \dot{a}^2. && (4.34)
\end{aligned}$$

and

$$\begin{aligned}
G_{\tau\tau} &= R_{\tau\tau} - \frac{1}{2}g_{\tau\tau}R && (4.35)\\
&= -\frac{3\ddot{a}}{a} + 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) && (4.36)\\
&= \frac{3\dot{a}^2}{a^2}. && (4.37)
\end{aligned}$$

We arrive, finally, at the corresponding equations:

$$G_{\tau\tau} = \frac{3\dot{a}^2}{a^2} = 8\pi\rho = 8\pi T_{\tau\tau}, \tag{4.38}$$

$$G_{xx} = -2a\ddot{a} - \dot{a}^2 = a^2 8\pi P = 8\pi T_{xx}. \tag{4.39}$$

[4] Again, see appendix A.3.

Combining the two equations yields the more revealing equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho, \tag{4.40}$$

$$\frac{\ddot{a}}{a} = -4\pi\left(\frac{\rho}{3} + P\right). \tag{4.41}$$

Strikingly, the equations we find in general for the sphere, flat plane and hyperbola are given by:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2}, \tag{4.42}$$

$$\frac{\ddot{a}}{a} = -4\pi\left(\frac{\rho}{3} + P\right). \tag{4.43}$$

where $k = +1$ for the sphere, $k = 0$ for the plane and $k = -1$ for the hyperbola. These equations are known as the *Friedmann equations*. Next, we will discuss these equations in detail.

The early universe was dominated by radiation and matter, both of which have a positive pressure P. Then it follows from equation 4.43 that $\ddot{a} < 0$. In other words, since a is a measure for the spatial distance between events, the universe is either expanding ($\dot{a} > 0$) or contracting ($\dot{a} < 0$)[5]. Moreover, this expansion or contraction happens *everywhere*, hence there is no preferred centre of expansion or contraction. If the distance between two events is given by R, then we define the *Hubble constant*[6] as $H(\tau) = \dot{a}/a$. It follows that

$$v := \frac{dR}{d\tau} = \frac{R}{a}\frac{da}{d\tau} = HR. \tag{4.44}$$

Intuitively, H is a measure for the relative expansion of the universe, at a given time τ. The measurement of the red-shift of the light of distant starts was a confirmation that the universe is expanding. Indeed, this discovery was a major reason why the theory of general relativity was accepted. Given that the universe is expanding it follows from equation (4.43) that it had been expanding at a faster rate earlier in time. If the expansion was uniform in time, then at a time H^{-1} we would have $a = 0$. Therefore, at a time less than H^{-1}, the universe was in a *singular state*, which is commonly called the *Big Bang*. One might think that the existence of the Big Bang is due to our initial assumption of homogeneity and isotropy. This is not the case. Such singularities turn out to be very general features of cosmological solutions, as is described by the *singularity theorems.*[7]

Let us now turn to the state of the current universe. One of the greatest mysteries in modern cosmology is the measurement that our universe is not only expanding, but doing so in an accelerated way. Compare this to the early universe whose expansion was slowing down. The first evidence for this accelerated expansion came from observations of the radiation emitted by type Ia supernovae, which was dimmer than expected.[15] More recent evidence comes from weak lensing tomography with the Hubble Space Telescope Cosmic Evolution Survey (COSMOS).[16] However, looking at equation (4.43), we see that for matter which has a positive pressure the expansion will never be accelerated (since $\ddot{a} < 0$). Therefore, it is currently accounted for by the addition of a positive cosmological constant Λ, which has an equation of state given by $P = -\rho$ which results in $\ddot{a} > 0$. Indeed, one can study a spatially flat universe which contains only a cosmological constant, also known as a *De Sitter universe*, and one finds an exponential dependence of the scale factor on τ:[17]

$$a(\tau) = e^{H\tau}. \tag{4.45}$$

The presence of the cosmological constant constitutes additional mass-energy called *dark energy*, which makes up 70% of the known mass-energy in the universe. Moreover, models of a *flat* universe show excellent agreement with the observations. This is further confirmed by measurements of the Cosmic Microwave Background (CMB), from which information about the cosmological constant and, in particular, the energy density and curvature of the Universe can be extracted. These results are also consistent with current flat models.[18] In short, the fact that our observable universe appears to be flat points to the existence of dark

[5] Apart from the moment the universe transitions between expansion and contraction.

[6] Note that the Hubble 'constant' depends on the proper time τ. Usually one denotes by the Hubble constant the present value of $\dot{a}/a$, and by the Hubble parameter the actual function.

[7] An excellent discussion on singularity theorems is provided by [14].

energy. The question, then, is what exactly this dark energy is. There are multiple theories where dark energy is described by a scalar field, similar to the Higgs field in particle physics, which include quintessence, phantoms, K-essence, tachyons, ghost condensates and dilatonic dark energy to name a few.[19] In string theory there are numerous spacetimes which could describe the vacuum spacetime, of which a few contain a cosmological constant. Putting all these tine spacetimes together will, on average, result in a vacuum with a tiny cosmological constant, which would explain why the universe is still very flat. In short, it is still completely uncertain what dark energy is and it is one of the prime research candidates for discovering new theories in physics.

The Friedmann equations arise from our starting assumption of spatial homogeneity and isotropy. We saw that isotropy, in particular, resulted in a large reduction in the complexity of the Riemann tensor. It will be informative to relax this condition and to consider spaces which are only homogeneous. Such spaces are known as Bianchi universes. Their study will require knowledge of Lie groups and Lie derivatives, which will be discussed in detail in the next chapter.

Chapter 5

Bianchi Universes

In the previous chapter we discussed the implications of a spatially homogeneous and isotropic universe. As a result we found three possible configurations for the spacetime metric, and the additional perfect fluid assumption allowed us to explicitly solve the Einstein equation. We will take this a step further, by considering a universe which is only spatially homogeneous, but not isotropic. The resulting models are known as *Bianchi universes.* Although they are not of great physical relevance, as the universe is, to a good approximation, indeed isotropic, the tools that are required to study these spaces are of great importance in many fields of physics. These include the concepts of Lie groups and Lie algebra's, which play important roles in quantum mechanics and particle physics, in particular in the formulation of the Standard Model.

We will start by discussing *isometries*, which require the important notion of a *Lie derivative* and the *pull-back* of forms. Then we introduce Lie groups and consider spatially homogeneous universes as the orbits of a one-parameter group of diffeomorphisms, which are related to the Killing vector fields. Finally we investigate the implications on the general form of the spacetime metric and classify all the possible Bianchi universes.

5.1 Isometries

The goal of this section is to develop a precise, and intuitive, formulation of what it means for a metric to have a certain symmetry. As an example, it is quite clear that the round sphere S^2, embedded in $\mathbb{R}^3$, possesses rotational symmetry. The question, then, it to make clear what is meant by such a rotational symmetry. For this the concept of *pull-backs* and *push-forwards* is necessary. Let us be very general at first. Let M and N be two smooth manifolds, and $\varphi : M \to N$ be a smooth map.[1] Let $f : N \to \mathbb{R}$ be a smooth map. We define the *pull-back* $\varphi_* f$ of f as a map

$$\varphi_* f : M \to \mathbb{R}, \quad \varphi_* f = f \circ \varphi. \tag{5.1}$$

Note that, unless φ is invertible, there is no natural way to 'push a function on M forward to N'. However, due to the nature of vectors being defined as maps on functions, it is possible to define the *push-forward* $\varphi^* v_p$ of a vector v_p at a point $p \in M$ by

$$\varphi^* v_p : C^\infty(N) \to \mathbb{R}, \quad (\varphi^* v_p) f = v_p(f \circ \varphi). \tag{5.2}$$

Similarly, to functions, there is not necessarily a natural way to pull back vectors on N to M. However, since covectors are dual to vectors, one might expect this to be possible for covectors. Indeed, we define the pull-back $\varphi_* \omega$ of a covector $\omega_\varphi(p)$ on N by

$$\varphi_* \omega_{\varphi(p)} : T_p M \to \mathbb{R}, \quad (\varphi_* \omega_{\varphi(p)}) v_p = \omega_{\varphi(p)}(\varphi^* v_p). \tag{5.3}$$

In a similar way we can define the push-forward of a $(k, 0)$ tensor, and the pull-back of a $(0, l)$ tensor. Moreover, when φ is a invertible, which will be the case in the rest of the chapter, we can define the pull-back

[1] Here smoothness means that the representative map

$$y \circ \varphi \circ x^{-1}$$

is smooth for all charts x and y of M and N respectively.

of an arbitrary (k,l) tensor as:

$$(\varphi^*T)^{a_1\dots a_k}{}_{b_1\dots b_l}(v_1)^{a_1}\dots(v_k)^{a_k}(\omega_1)_{b_1}\dots(\omega_l)_{b_l} = T^{a_1\dots a_k}{}_{b_1\dots b_l}([\varphi^{-1}]^*v_1)^{a_1}\dots([\varphi^{-1}]^*v_k)^{a_k}(\varphi_*\omega_1)_{b_1}\dots(\varphi_*\omega_l)_{b_l}. \tag{5.4}$$

It is instructive to consider the pull-back and push-forward maps in charts. Let (U,x) be a chart of $p\in M$, and (V,y) a chart of $\varphi(p)\in N$. Then we have

$$(\varphi^*)^\mu{}_\nu = dy^\mu_{\varphi(p)}\left[\varphi^*\left(\frac{\partial}{\partial x^\nu}\right)_p\right] \tag{5.5}$$

$$= \left[\varphi^*\left(\frac{\partial}{\partial x^\nu}\right)_p\right]y^\mu \tag{5.6}$$

$$= \left(\frac{\partial(y^\mu\circ\varphi)}{\partial x^\nu}\right)_p. \tag{5.7}$$

Hence, considering the component functions of φ^*, we find

$$(\varphi^*)^\mu{}_\nu = \frac{\partial y^\mu}{\partial x^\nu}. \tag{5.8}$$

Similarly, for the push-forward we find almost the exact same formula, namely

$$(\varphi_*)_\mu{}^\nu = \frac{\partial y^\mu}{\partial x^\nu}. \tag{5.9}$$

Finally, for an arbitrary tensor one can show that[20]

$$(\varphi^*T)^{\mu_1\dots\mu_k}{}_{\nu_1\dots\nu_l} = \frac{\partial y^{\mu_1}}{\partial x^{\alpha_1}}\cdots\frac{\partial y^{\mu_k}}{\partial x^{\alpha_k}}\frac{\partial x^{\beta_1}}{\partial y^{\nu_1}}\cdots\frac{\partial x^{\beta_l}}{\partial y^{\nu_l}}T^{\alpha_1\dots\alpha_k}{}_{\beta_1\dots\beta_l}. \tag{5.10}$$

Note that this is precisely the transformation law for tensors we discussed in chapter 2, specifically in equation (2.16). The reason is that taking the pull-back of a tensor under a diffeomorphism $\varphi: M\to M$ and changing coordinates are really the same thing. Given a manifold M with coordinates x we can either *passively* change our coordinates to y, or we can *actively* change the manifold under a diffeomorphism, such that the resulting coordinates φ^*x are the same.

By taking the difference between φ^*T and T we can see whether a certain tensor field T has changed under a diffeomorphism T. Hence we say that φ is a *symmetry* of T if $\varphi^*T=T$. In particular, if $\varphi^*g=g$, where g is a metric, we call φ an *isometry*. As an example, we can consider the round sphere S^2, described by polar coordinates (θ,ϕ). We can take a transformation $\varphi_t(\theta,\phi)=(\theta,\phi+t)$, which amounts to a rotation of the sphere around the azimuth. This allows us to make sense of 'rotational symmetry' of the sphere. We can, however, do more. In this example we have not just one diffeomorphism, but an entire class of diffeomorphisms, parametrized by t. In particular, this one-parameter group of diffeomorphisms is generated by some vector field, and vice-versa. Given a vector field, we can construct a one-parameter group of diffeomorphisms from the integral curves. This will be explored further in the next section.

5.2 Flows and the Lie Derivative

Given a vector field v^a on a compact[2] manifold M, we saw in section 2.2 that v^a induces an integral curve. Let us then define the map

$$\varphi:\mathbb{R}\times M\to M,\quad \varphi(t,p)=\gamma_p(t), \tag{5.11}$$

where $\gamma_p:\mathbb{R}\to M$ is the unique integral curve such that $\gamma(0)=p$. One usually calls φ the *flow* of v^a. In the following we denote, for $t\in\mathbb{R}$, $\varphi_t:=\varphi(t,\cdot):M\to M$. Then the set $\{\varphi_t: t\in\mathbb{R}\}$ forms a one-parameter group of diffeomorphisms, with multiplication given by composition, hence we have $\varphi_t\circ\varphi_s=\varphi_{t+s}$ and $\varphi_t^{-1}=\varphi_{-t}$.

[2]It turns out that any smooth vector field on a compact manifold is *complete*, which means that we can follow its integral curve for all times. Otherwise we would have to consider some open interval $I\subseteq\mathbb{R}$ instead of the whole of $\mathbb{R}$.

Given a tensor field T, we can now consider the difference $\varphi_t^* T - T$, for all times t. We therefore define the *Lie derivative of g along the flow of v^a* as

$$\pounds_v T := \lim_{t\to 0} \frac{\varphi_t^* T - T}{t}. \tag{5.12}$$

It follows that φ is an isometry if $\pounds_v g = 0$ everywhere. Unfortunately, the Lie derivative is not the most practical tool to find isometries. However, it can be shown, by considering charts that are adapted to the vector field v^a, that the action of the Lie derivative on some vector field w^a is exactly the *commutator* [3]:

$$\pounds_v w^a = [v, w]^a, \tag{5.13}$$

where

$$[v, w]f := v(w(f)) - w(v(f)). \tag{5.14}$$

Moreover, using the Leibniz rule and the fact that

$$\pounds_v w^a = [v, w]^a = v^b \nabla_b w^a - w^b \nabla_b v^a, \tag{5.15}$$

for all covariant derivatives ∇_a, one can show that the action of the Lie derivative on some dual vector field ω_a is given by:

$$\pounds_v \omega_a = v^b \nabla_b \omega_a + \omega_b \nabla_a v^b. \tag{5.16}$$

In particular, it follows that for the metric g_{ab}, we have

$$\pounds_v g_{ab} = v^c \nabla_c g_{ab} + g_{cb} \nabla_a v^c + g_{ac} \nabla_b v^c \tag{5.17}$$

$$= \nabla_a v_b + \nabla_b v_a. \tag{5.18}$$

If $(\varphi_t^* g)_{ab} = g_{ab}$, for all t, then we call the vector field that generates the flow φ a *Killing vector field*, usually denoted by ξ^a. From the remark above, for a vector field to be a Killing vector field, it suffices that the *Killing equation*:

$$\nabla_a \xi_b + \nabla_b \xi_a = 0, \tag{5.19}$$

is satisfied. To summarize, given a metric g_{ab} with induced covariant derivative ∇_a, we can look for Killing vectors ξ^a via the above equation and then the metric is invariant along the integral curve of ξ^a. Unfortunately, we do not know the metric of our spacetime. Instead, we wish to impose a symmetry on spacetime, namely that of spatial homogeneity, and investigate the possible metrics that obey this symmetry. Hence, we should start with the flow under which we want the metric to be invariant. The question, then, is what this flow should be, and where it comes from. For this we require the notion of Lie groups and left/right-invariant tensor fields.

5.3 Lie Groups and Right/Left-Invariant Tensor Fields

In this section we consider general properties of Lie groups, which are useful not only in the context of studying cosmological models, but also for understanding the quantum mechanical effect of spin and the structure of the Standard Model. First, a Lie group G of dimension n is defined as a group which is also a manifold, such that the multiplication and inverse maps

$$m : G \times G \to G,\ (g, h) \mapsto gh, \quad i : G \to G,\ g \mapsto g^{-1} \tag{5.20}$$

are smooth. We have already seen an example of a group, namely the one-parameter group of diffeomorphisms φ_t induced by a vector field. However, this group is, in a sense, to large to carry a manifold structure, i.e. we cannot locally describe these diffeomorphisms with just m parameters. However, the group of isometries, generated by the Killing fields, is small enough. In particular, there are at most $n(n+1)/2$ linearly independent Killing fields. This can be seen from the following: In $\mathbb{R}^n$, the isometries are given by translations and rotations. There are n independent directions one can translate into, and $n(n-1)/2$ independent rotations, since for n directions there are $n-1$ directions to rotate into, and we divide by 2 to avoid double counting. Hence there are $n(n+1)/2$ independent Killing fields in $\mathbb{R}^n$. One can show that the same holds for a general

n-dimensional Lie group [3]. Hence, the one-parameter group of diffeomorphisms induced by the (finitely many) Killing fields indeed forms a Lie group, which will be heavily used later on. It will denoted by the Lie group of isometries. Let us first establish some general properties of Lie groups. Of great importance will be the so-called *left translation* map, given by:

$$L_h : G \to G, \quad L_h(g) = hg, \tag{5.21}$$

where $h \in G$. We say that a vector field v^a is *left-invariant* if it is invariant under left translations, i.e.

$$(L_h^* v)^a = v^a. \tag{5.22}$$

It is clear that the set of left-invariant vector fields can be made into a vector space. Moreover, a left-invariant vector field v^a is completely determined by its value at the identity $e \in G$, since at any other point $h \in G$, we have

$$v_h^a = (L_h^* v_e)^a. \tag{5.23}$$

Next, we can consider whether the commutator of two left-invariant vector field v^a and w^a is again left-invariant. More generally, for any diffeomorphism $\varphi : M \to M$, equation (5.2) in terms of fields is given by:

$$\{(\varphi^* v)f\} \circ \varphi = v(f \circ \varphi). \tag{5.24}$$

Hence, for the commutator we find

$$\begin{aligned}
\{(\varphi^*[v,w])f\} \circ \varphi &= [v,w](f \circ \varphi) && (5.25)\\
&= v(w(f \circ \varphi)) - w(v(f \circ \varphi)) && (5.26)\\
&= v(\{(\varphi^* w)f\} \circ \varphi) - w(\{(\varphi^* v)f\} \circ \varphi) && (5.27)\\
&= ((\varphi^* v)\{(\varphi^* w)\}f) \circ \varphi - ((\varphi^* w)\{(\varphi^* v)\}f) \circ \varphi && (5.28)\\
&= \{[\varphi^* v, \varphi^* w]f\} \circ \varphi. && (5.29)
\end{aligned}$$

Since this holds for all f, we conclude that

$$\varphi^*[v,w] = [\varphi^* v, \varphi^* w]. \tag{5.30}$$

If follows that if v^a and w^a are left-invariant, then $[v,w]^a$ is also left-invariant. Since the commutator depends linearly on v^a and w^a, it follows that there exists a $(1,2)$ tensor $c^a{}_{bc}$, called the *structure constant tensor*, such that

$$[v,w]^a = c^a{}_{bc} v^b w^c. \tag{5.31}$$

From its definition it is clear that

$$c^a{}_{bc} = -c^a{}_{cb}. \tag{5.32}$$

Moreover, from the Jacobi identity for the commutator

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0, \tag{5.33}$$

it follows that

$$c^e{}_{d[a} c^d{}_{bc]} = 0. \tag{5.34}$$

A finite dimensional vector space, together with a $(1,2)$ tensor $c^a{}_{bc}$ that satisfies equations (5.32) and 5.34 is called a (*Lie algebra*). Hence the above results imply that the space of all left-invariant vector fields on a Lie group form a Lie algebra. The advantage is that Lie algebra's are simpler objects to work with than Lie groups. It can be compared to approximating a function by its derivative. Indeed, the Lie algebra we have found is exactly the tangent space at the identity T_eG, together with the commutator bracket, where a left-invariant vector field is created from each vector in T_eG. Hence we can view a Lie algebra as the linear approximation of a Lie group. Fortunately, compared to the example of the derivative of a function, we lose very little information in doing so. Indeed, one can show that for every Lie algebra $\mathfrak{g}$ there exists a unique simply connected Lie group G whose Lie algebra is again $\mathfrak{g}$ [21].

Completely analogously, we can define the *right-translation* map, given by:

$$R_h : G \to G, \quad R_h(g) = gh. \tag{5.35}$$

Now let x^a be a right-invariant vector field, and denote by φ_t its one-parameter group, then for any $f \in C^\infty(G)$ we have

$$\frac{d}{dt}(f \circ \varphi_t) = x^a(f \circ \varphi_t). \tag{5.36}$$

Now consider the following:

$$\frac{d}{dt}\left(f \circ R_h^{-1} \circ \varphi_t \circ R_h\right) = x^a(f \circ R_h^{-1} \circ \varphi_t) \circ R_h, \tag{5.37}$$

since R_h does not depend on t. Since x^a is right-invariant, it follows from equation (5.24) that

$$x^a(f \circ R_h^{-1} \circ \varphi_t) \circ R_h = x^a(f \circ R_h^{-1} \circ \varphi_t \circ R_h). \tag{5.38}$$

In other words, we find

$$\frac{d}{dt}\left(f \circ R_h^{-1} \circ \varphi_t \circ R_h\right) = x^a(f \circ R_h^{-1} \circ \varphi_t \circ R_h). \tag{5.39}$$

Hence the map $R_h^{-1} \circ \varphi_t \circ R_h$ is an integral curve of x^a. By uniqueness of the integral curve, we conclude that

$$\varphi_t = R_h^{-1} \circ \varphi_t \circ R_h, \tag{5.40}$$

or

$$\varphi_t \circ R_h = R_h \circ \varphi_t, \quad \forall h \in G. \tag{5.41}$$

Therefore the flow of x^a commutes with right-translation. Using this fact and defining $h(t) = \varphi_t(e)$, we can compute

$$L_{h(t)}(g) = h(t)g = R_g(h(t)) = R_g \circ \varphi_t(e) = \varphi_t \circ R_g(e) = \varphi_t(g). \tag{5.42}$$

In other words,

$$\varphi_t = L_{h(t)}. \tag{5.43}$$

We may interpret this as follows: given an element $h(t) \in G$, for each t, we can consider this element as the flow $\varphi_t(e)$, for some one-parameter group induced by a right-invariant vector field x^a. Then we can describe left translation by h using the flow φ_t. In short, right-invariant vector fields *generate* left translations.

If v^a is a left-invariant vector field, it must be invariant under the action of $L_{\varphi_t(e)}$, for all φ_t, i.e. for all right-invariant vector fields x^a. Hence it follows from equation (5.43) that it is invariant under the flow x^a, which implies that

$$\pounds_x v^a = [x, v]^a = 0. \tag{5.44}$$

From equation (5.44) and a similar relation for dual vector fields, one can show that for any two right-invariant vector fields x^a and y^a we have

$$[x, y]^c = -c^c{}_{ab} x^a y^b, \tag{5.45}$$

which is the same as equation (5.31) for left-invariant vector fields, but with a minus sign. One final relation that will prove to be quite important is the following [3]

$$2\nabla_{[a}\alpha_{b]} = -\alpha_c c^c{}_{ab}, \tag{5.46}$$

where α_a is any left-invariant dual vector field on G.

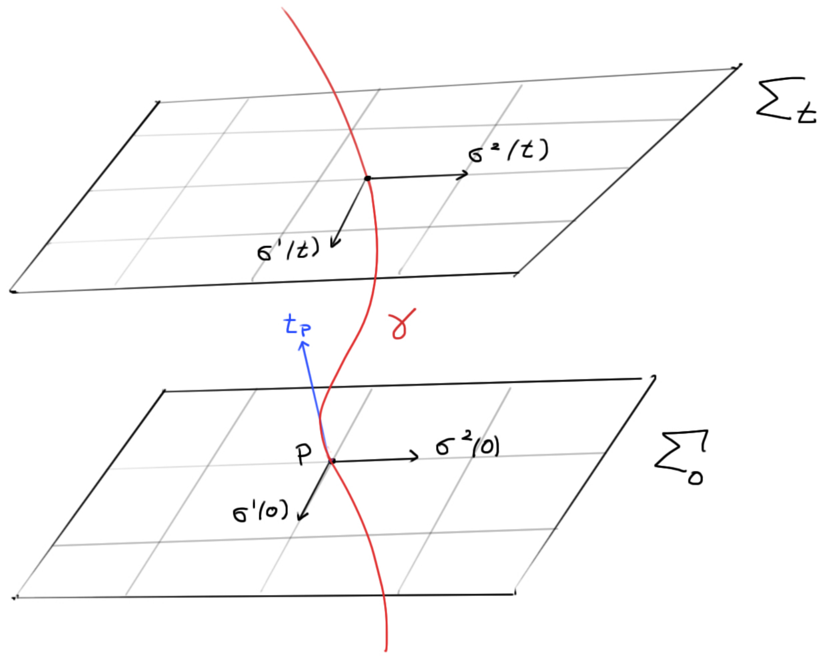

Figure 5.1: A depiction of the construction of a dual basis field on each hypersurface Σ_t. Due to homogeneity of each hypersurface the same construction can be applied from any point in $p \in \Sigma_0$, from which we construct a vector field t^a which allows us to transfer structure from Σ_0 to other hypersurfaces.

5.4 Homogeneous Spacetime

Having discussed the most important properties of Lie groups, we are now ready to consider homogeneous spacetime. By definition, a spacetime (M, g_{ab}) is called *homogeneous* if there exists a family of spacelike hypersurfaces Σ_t such that, for any two points $p, q \in \Sigma_t$, there exists a unique isometry $g : M \to M$ in the Lie group of isometries such that $g(p) = q$. A few remarks are in order. By demanding uniqueness of such isometries, we can make a correspondence between each Σ_t and G as follows. We (arbitrarly) pick a point $p_0 \in G$, to which we assign an element $g_0 \in G$. Then, for any other point p, we define the corresponding group element by the unique $g \in G$ such that $g(p_0) = p$. It follows that, for $h \in G$, we have

$$h(p) = h(g(p_0)) = h \circ g(p_0) = (hg)(p_0) = [L_h(g)]p_0. \tag{5.47}$$

Therefore we can describe the action of any element h of G on Σ_t by left translation by h in G. In a similar fashion we can identify tensor fields on Σ_t with tensor fields on G, from which it follows that the tensor fields on Σ_t which are invariant under an isometry g, are exactly the tensor fields on G which are left-invariant, under multiplication with g. In particular, such vector fields satisfy the relation (5.31). Moreover, since the Killing vector fields induce the isometries on Σ_t they must correspond to the right-invariant vector fields on G, hence they satisfy the relation (5.45).

Our goal will be to use the properties of left and right-invariant vector fields on G to find the form of the induced metric h_{ab} on Σ_t. Let us start in one particular hypersurface Σ_0, on which we can define a dual basis $(\sigma^1)_a$, $(\sigma^2)_a$, $(\sigma^3)_a$ which is preserved under the isometries.[3] Since we assume the spatial metric to be

[3] This can be done in the following way: We choose an arbitrary basis at some point $p \in \Sigma_0$, and define the basis at any other point by requiring the resulting vector field to be left-invariant. This then implies that the entire dual basis field is preserved under the isometries by an earlier comment.

preserved under isometries, we must be able to decompose it into this basis as:

$$h_{ab} = \sum_{\alpha,\beta=1}^{3} h_{\alpha\beta}(\sigma^\alpha)_a(\sigma^\beta)_b, \tag{5.48}$$

for some functions $h_{\alpha\beta}$ on Σ_0. Note however, that since neither h_{ab} nor the dual basis elements can change under isometries, i.e. left translations, these functions must have the same value at each point in Σ_0, i.e. they are constant. There are now 2 things that need to be done, (1) extend the basis on Σ_0 to one on each Σ_t and (2) find a relation that fixes the form of these basis dual vectors. We start with some point $p \in \Sigma_0$ and define t_p^a to be the unit normal to Σ_0 at p, see also figure 5.1. Let γ be the geodesic determined by (p, t_p^a). This geodesic will allow us to pass our information to other hypersurfaces, in the following way. Let u^a be the tangent field to γ. Since γ is initially orthogonal to Σ_0, it follows from the Killing equation that

$$u^b\nabla_b(\xi_a u^a) = 0, \tag{5.49}$$

for any Killing field ξ^a defined on any hypersurface Σ_t. Hence γ will be orthogonal to every hypersurface. We now label the hypersurfaces by the eigentime t at which γ passing through it. By construction, the vector field $t^a = -\nabla^a t$ will be orthogonal to every Σ_t and its integral curves are all geodesics, since spatial homogeneity ensures that this construction (which relied on the point $p \in \Sigma_0$) works everywhere on Σ_0. We can now use this vector field t^a to transport our basis of Σ_0 to one of Σ_t by requiring that

$$(\sigma^\alpha)_a(t) = [\varphi_t^*\sigma^\alpha(0)]_a, \tag{5.50}$$

where φ_t denotes the flow of t^a. By definition of the Lie derivative, this is equivalent to

$$\pounds_t(\sigma^\alpha)_a = 0. \tag{5.51}$$

With this we have achieved our first goal, since we can now write the metric g_{ab} as

$$g_{ab} = -\nabla_a t\nabla_b t + \sum_{\alpha,\beta=1}^{3} h_{\alpha\beta}(t)(\sigma^\alpha)_a(\sigma^\beta)_b, \tag{5.52}$$

where $h_{\alpha\beta}$ can only depend on t, as it is constant on each hypersurface, and the vector fields $(\sigma^\alpha)_a$ are defined by (5.50). Lastly, we need to find an equation that these $(\sigma^\alpha)_a$ should satisfy. Since each Σ_t is in correspondence with G, we know from equation (5.46) that the component of $2\nabla_{[a}(\sigma^\alpha)b]$ is related to the structure constant $c^c{}_{ab}$. However, one can show that [3]

$$t^a\nabla_{[a}(\sigma^\alpha)_{b]} = 0, \tag{5.53}$$

in other words, $\nabla_{[a}(\sigma^\alpha)_{b]}$ has no component orthogonal to Σ_t. We have therefore found our relation that fixes the dual basis elements, and therefore g_{ab}, namely:

$$2\nabla_{[a}(\sigma^\alpha)_{b]} = -(\sigma^\alpha)_c c^c{}_{ab}. \tag{5.54}$$

To summarize: By using the function t as defined above, the topological structure of the spacetime manifold is $M = \mathbb{R} \times G$, since each hypersurface is in correspondence with the Lie group. Moreover, the metric on M is determined by equations (5.52) and (5.54). In other words, a homogeneous spacetime is completely fixed by specifying a Lie group G, which in turn specifies the structure constants of its corresponding Lie algebra. Hence our next tasks are to determine all the possible 3-dimensional Lie groups, to find the induced dual basis fields and to solve Einstein's equation for the functions $h_{\alpha\beta}$. We will not discuss these steps in detail, but it will be instructive to outline the general procedure behind them and to discuss the results.

We start by specifying all the three dimensional Lie groups. As said before, up to global topological structure, a Lie group is uniquely determined by a Lie algebra. It can be shown that a three dimensional Lie algebra is, in turn, uniquely determined by a dual vector A_a and a symmetric tensor M^{ab} satisfying the following equation:

$$M^{ab}A_b = 0. \tag{5.55}$$

Let us consider the various possibilities. We distinguish two different classes. In class A, we have that $A_a = 0$, in which case equation (5.55) is trivially satisfied. The resulting Lie algebras are then determined, up to isomorphisms, by the signature and rank of M^{ab}, which results in the following cases:

- rank(M^{ab}) = 0, i.e. $M^{ab} = 0$
- rank(M^{ab}) = 1, signature +
- rank(M^{ab}) = 2, signature ++ or +-
- rank(M^{ab}) = 3, signature +++ or ++-

In class B, we have $A_a \neq 0$. As a result M^{ab} cannot have rank greater than 2, otherwise equation (5.55) could not be satisfied. Again, we have the cases

- rank(M^{ab}) = 0, i.e. $M^{ab} = 0$
- rank(M^{ab}) = 1, signature +
- rank(M^{ab}) = 2, signature ++ or +-

However, one can show that when M^{ab} has rank two there must exist a scalar α which satisfies [3]:

$$A_e A_f = \alpha M^{ac} M^{bd} \epsilon_{cde} \epsilon_{abf}. \tag{5.56}$$

As a result, we find two one-parameter families of Lie algebras classified by α.

From here one could consider the Einstein equation and analyse its form in terms of the basis $(\sigma^\alpha)_a$ using, for instance, the *tetrad method*. We will not pursue this derivation in detail but merely give the result in a very special case.[4] The case we will consider is when $G = \mathbb{R}^3$, with the group multiplication given by addition. This case is called "Type I" in the Bianchi classification of homogeneous universes. Since this group is abelian, the structure constant tensor ${c^c}_{ab}$ vanishes. This simplifies the entire discussion greatly, since we may take our dual basis fields as the canonical basis fields $(dx)_a$, $(dy)_a$ and $(dz)_a$. One can then show that Einstein's equation in vacuum implies the following form for the metric:

$$ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{3p_3} dz^2, \tag{5.57}$$

where p_1, p_2, p_3 are constants which satisfy:

$$\sum_\alpha p_\alpha = 1, \tag{5.58}$$

and

$$\left(\sum_\alpha p_\alpha \right)^2 = \sum_\alpha p_\alpha^2. \tag{5.59}$$

Equation (5.57) is known as the *Kasner* solution. Note that unless $p_1 = 1$ and $p_2 = p_3 = 0$, equation (5.59) implies that two of the p_α's are positive, and one negative. Hence the Kasner solution describes a homogeneous universe which expands in two directions and contract in one direction. Compare this with the Robertson-Walker solution, which necessarily showed the same expansion (or contraction) in all directions. An interesting theoretical applications of this particular metric is found in string theory. The idea is that the spacetime dimensions are described by the expanding directions, whereas the contracting ones are the additional compact dimensions. Due to inflation, these contracting dimensions have become very tiny, which explains why we do not observe them nowadays. In chapter 7 we discuss these string theoretical concepts such as compact dimensions in more detail. Before that, we will push our knowledge of Lie algebras further by classifying them and, in particular, considering the ADE classification.

[4] The derivation is shown partly in [3], and more completely in [22].

Chapter 6

The Classification of Semi-Simple Lie Algebras

One of the most important steps in the classification of the Bianchi universes that was discussed in the previous chapter was finding all the 3-dimensional Lie algebras. This was done by considering equation (5.55) for various possibilities of M^{ab} and A_a. This, however, is a special property of the three-dimensional case which does not necessarily generalize to higher-dimensional Lie algebras. Understanding, for instance, the 8-dimensional lie algebra $\mathfrak{su}(3)$ is of utmost importance in the context of particle physics. It turns out that many of the interesting Lie algebras fall under the category of semi-simple Lie algebras, whose classification is well known. In this chapter we present the theory of classifying semi-simple Lie algebras using roots and Dynkin diagrams. Of special interest will be the ADE classification, which plays an important role in string theory and is the central object of the next chapter. We apply the general theory to the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$, which are of particular interest in the context of quantum mechanics and particle physics. Representations are also briefly touched upon.

6.1 Semi-Simple Lie algebras and the Cartan Subalgebra

We start by discussing the definition of a semi-simple Lie algebra. Let $(\mathfrak{g}, [\cdot,\cdot])$ be a finite-dimensional Lie algebra. We say that $\mathfrak{g}$ is *simple* if its only *ideals* are trivial. Here an *ideal* is a Lie subalgebra $\mathfrak{i} \subseteq \mathfrak{g}$ satisfying $[\mathfrak{i}, \mathfrak{g}] \subseteq \mathfrak{i}$. In a sense a simple Lie algebra cannot be made 'any simpler', i.e. smaller, by taking quotients with ideals. A *semi-simple* Lie algebra, then, is any Lie algebra that can be written as the direct sum of simple Lie algebras. One might worry that semi-simple Lie algebras are a very special case of Lie algebras, and that their classification does little good. However, Levi's decomposition theorem states that *any* finite-dimensional Lie algebra can be written as semi-direct product of a solvable ideal and a semi-simple Lie subalgebra. It turns out that solvable ideals are not easy to classify, but semi-simple Lie algebras are! Hence the classification of semi-simple Lie algebras will at least give us some insight into the form of general Lie algebras.

From a more physical point of view, Lie algebras consist of operators, such as the spin operators. To make this more precise, we require the notion of the *adjoint representation*, which is defined as a map[1]

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \quad X \mapsto \mathrm{ad}_X \tag{6.1}$$

where ad_X is a map

$$\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}, \quad Y \mapsto [X, Y]. \tag{6.2}$$

In short, the map ad assigns to each element of $\mathfrak{g}$ a linear map, which physicist call an *operator*.[2] Moreover, in physics one often wants to diagonalize as many of these operators as possible to simplify their action on states. For this reason we define the *Cartan subalgebra* of $\mathfrak{g}$ to be the maximal subspace $\mathfrak{h} \subseteq \mathfrak{g}$ such that

[1] Here $\mathrm{End}(\mathfrak{g})$ denotes the set of endomorphisms of $\mathfrak{g}$, i.e. the set of linear maps from $\mathfrak{g}$ to $\mathfrak{g}$.

[2] Actually, in the context of physics a more general notion of a representation is used. This is discussed in detail in section 6.4.

all its generators commute and ad_H is diagonalizable, for all $H \in \mathfrak{h}$. It can be shown that such a Cartan subalgebra always exists, and that all Cartan subalgebras are isomorphic to each other [21].

6.2 Roots and Dynkin Diagrams [3]

In the spirit of diagonalizing as many operators as possible, we define the *roots* of a Cartan subalgebra $\mathfrak{h}$ as follows: A non-zero linear functional λ on $\mathfrak{h}$ is called a *root* if there exists an $X \in \mathfrak{g}$ such that

$$\mathrm{ad}_H(X) = [H, X] = \lambda(H)X, \quad \forall H \in \mathfrak{h}. \tag{6.3}$$

In other words, a root is an eigenvalue of ad_H for each H simultaneously, whose dependence is linear. We denote the set of roots by Φ. Next, we define the following bilinear symmetric form (i.e. a pseudo inner product)

$$K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \quad K(X, Y) = \mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y), \tag{6.4}$$

known as the *Killing form*. Cartan's criterion states that a Lie algebra is semi-simple if and only if the Killing form is invertible. We consider the components of the Killing form. Let $\{X_1, \ldots, X_N\}$ be the generators of $\mathfrak{g}$, it follows that

$$\begin{aligned} \mathrm{ad}_{X_i} \circ \mathrm{ad}_{X_j}(X_k) &= [X_i, [X_j, X_k]] && (6.5)\\ &= c^m{}_{jk}[X_i, X_m] && (6.6)\\ &= c^n{}_{im}c^m{}_{jk}X_n. && (6.7) \end{aligned}$$

Taking the trace then amounts to taking $k = n$, hence we find

$$K_{ij} = c^m{}_{in}c^n{}_{jm}. \tag{6.8}$$

It turns out that to describe a Lie algebra we do not require all roots, but merely the *fundamental* roots. Indeed, since ad_H is anti-symmetric it is clear that if λ is a root, then so is $-\lambda$, hence there is a sense of degeneracy in Φ. Therefore we define $\Pi \subseteq \Phi$ to be a set of fundamental roots if $\Pi = \{\pi_1, \ldots, \pi_k\}$ is linearly independent and any root $\lambda \in \Phi$ can be written as

$$\lambda = \epsilon \sum_{i=1}^{k} n_i \pi_i, \tag{6.9}$$

where $n_i \in \mathbb{N}$ and $\epsilon \in \{+1, -1\}$.[4] Hence by definition we have that $\Phi \subseteq \mathrm{span}_{\pm\mathbb{N}}\Pi$, where $\mathrm{span}_{\pm\mathbb{N}}$ denotes the 'span' using equation (6.9). Moreover, it can be shown that the complex span of Π recovers all the linear functionals on $\mathfrak{h}$, i.e. $\mathrm{span}_{\mathbb{C}}\Pi = \mathfrak{h}^*$ [21].

A natural question, then, is whether we can reconstruct the set of roots from the set of fundamental roots. The answer is yes. We can consider the real span of Π, which we denote by

$$\mathfrak{h}^*_{\mathbb{R}} := \mathrm{span}_{\mathbb{R}}\Pi. \tag{6.10}$$

Cleary $\mathfrak{h}^*_{\mathbb{R}} \subseteq \mathfrak{h}^*$. More generally, the space $\mathfrak{h}^*$ naturally inherits a symmetric bilinear form from the Killing form on $\mathfrak{g}$ as follows. First, we can consider K only on the space $\mathfrak{h}$ using the restriction $K|_{\mathfrak{h}}$. Then we use the fact that there is a natural map from $\mathfrak{h}$ to $\mathfrak{h}^*$, namely

$$i : \mathfrak{h} \to \mathfrak{h}^*, \quad i(H) := K(H, \cdot). \tag{6.11}$$

Since we restrict our attention to semi-simple Lie algebras the Killing form is invertible and hence the map i has an inverse. Therefore we can define

$$K^* : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}, \quad K^*(\lambda, \mu) := K(i^{-1}(\lambda), i^{-1}(\mu)). \tag{6.12}$$

[3] This section is largely based on a set of lectures on "Geometrical Anatomy of Theoretical Physics" by Frederic Schuller.[23]

[4] Here $\mathbb{N} = \{0, 1, 2, \ldots\}$, hence the coefficients may be zero.

Let us return to the space $\mathfrak{h}_{\mathbb{R}}^*$ and the question whether we can reconstruct all the roots from just the fundamental roots. Using the Killing form K^* restricted to $\mathfrak{h}_{\mathbb{R}}^*$ we can consider the action of the *Weyl group* on this space, which is defined as:

$$W := \{w_\lambda : \lambda \in \Phi\}, \tag{6.13}$$

where

$$w_\lambda : \mathfrak{h}_{\mathbb{R}}^* \to \mathfrak{h}_{\mathbb{R}}^*, \quad \mu \mapsto \mu - 2\left(\frac{K^*(\mu,\lambda)}{K^*(\lambda,\lambda)}\right)\lambda. \tag{6.14}$$

are the *Weyl transformations*. It is these transformations that allow us to consider only fundamental roots. This is because the fundamental roots generate all Weyl transformations and *any* root can be reconstructed by repeatedly applying Weyl transformations to fundamental roots. More precisely, for any $w \in W$ we have

$$w_\lambda = w_{\pi_1} \circ \ldots \circ w_{\pi_l} \tag{6.15}$$

for some $\pi_1, \ldots, \pi_l \in \Pi$. Moreover, for any root $\lambda \in \Phi$ there exists a $w \in W$ and $\pi \in \Pi$ such that $\lambda = w(\pi)$. Also, it can be shown that the Killing form $K^*|_{\mathfrak{h}_{\mathbb{R}}^*}$ is an *inner product*, i.e. it is positive-definite, which we from here on will denote by $\langle\cdot,\cdot\rangle$. This allows us to not only measure lengths of roots, but also consider the angle between them. If $H_1, \ldots, H_k$ are the generators of $\mathfrak{h}$, then we may interpret each functional λ as a vector $(\lambda_1, \ldots, \lambda_k)$. We can then draw the roots in $\mathbb{R}^k$ resulting in the so-called *root diagram*.

We now derive a remarkable result that greatly simplifies the classification of semi-simple Lie groups. For $\pi_i, \pi_j \in \Pi$, consider the Weyl transformation

$$w_{\pi_i}(\pi_j) = \pi_j - 2\frac{\langle \pi_j, \pi_i\rangle}{\langle \pi_i, \pi_i\rangle}\pi_j. \tag{6.16}$$

We have stated that the result must be a root, i.e. $w_{\pi_i}(\pi_j) \in \Phi$. However, since $\Phi \subseteq \mathrm{span}_{\pm\mathbb{N}}\Pi$, this implies that the factor

$$C_{ij} := 2\frac{\langle \pi_j, \pi_i\rangle}{\langle \pi_i, \pi_i\rangle} \tag{6.17}$$

must be some non-positive integer if $i \neq j$ (clearly $C_{ii} = 2$ for all i). The matrix formed by the components C_{ij} is known as the *Cartan matrix*. Finally, we define the *bond number* n_{ij} as

$$n_{ij} := C_{ij}C_{ji}. \tag{6.18}$$

It follows from its definition and the fact that $\langle\cdot,\cdot\rangle$ is symmetric that

$$n_{ij} = 4\frac{\langle \pi_j, \pi_i\rangle}{\langle \pi_i, \pi_i\rangle}\frac{\langle \pi_i, \pi_j\rangle}{\langle \pi_j, \pi_j\rangle} = 4\cos^2(\sphericalangle(\pi_i, \pi_j)), \tag{6.19}$$

where $\sphericalangle(\pi_i, \pi_j)$ denotes the angle between π_i and π_j. It follows that $0 \leq n_{ij} \leq 4$, and since C_{ij} is an integer, so is n_{ij}. As a result, there are only 4 possible values for n_{ij}, namely $0, 1, 2, 3$. This restriction allows us to consider the so-called *Dynkin diagrams*. The rules are as follows:

1. Draw a circle for every fundamental root.

2. If π_i and π_j are two fundamental roots, draw n_{ij} lines between them.

3. If there are 2 or 3 lines between π_i and π_j, draw an arrow along the lines pointing from the root with bigger length to the root with smaller length.

In short, we have found a one-to-one correspondence between semi-simple Lie algebras and their Dynkin diagrams. Figure 6.1 shows all the possible Dynkin diagrams. Of particular interest are the so-called *simply laced* Dynkin diagrams, i.e. those where the bond number is 1 everywhere. The classification of these Lie algebras is known as the *ADE-classification*. It turns out that the ADE-classification has a deep connection with string theory, as is discussed in the final chapter. In the next sections we discuss some Dynkin diagrams in more detail, namely those of $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$.

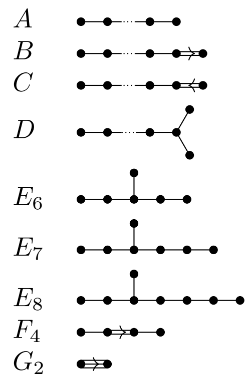

Figure 6.1: A list of all the Dynkin diagrams.

6.3 Example: $\mathfrak{su}(2)$

Abstractly, the Lie algebra $\mathfrak{su}(2)$ is a 3-dimensional Lie algebra with generators X_1, X_2, X_3 which satisfy the following commutation relations:

$$[X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1, \quad [X_3, X_1] = 2X_2. \tag{6.20}$$

Therefore, the only non-zero components of the structure constant tensor are:

- $c^3{}_{12} = c^1{}_{23} = c^2{}_{31} = 2.$
- $c^3{}_{21} = c^1{}_{32} = c^2{}_{13} = -2$

Using equation (6.8) one find for the Killing form:

$$K_{ij} = -8\delta_{ij}. \tag{6.21}$$

Since K is invertible we can conclude that $\mathfrak{su}(2)$ is indeed semi-simple.

Next, we need to find the Cartan subalgebra. Since none of the generators commute, we can simply pick $\mathfrak{h} = \{X_3\}$. Physically, we choose to diagonalize the spin-operator in the z-direction, which is the standard convention. Next, we need to find the roots, so consider an arbitrary element $X \in \mathfrak{g}$, then we can write X in terms of the generators as

$$X = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3. \tag{6.22}$$

It follows that

$$[X_3, X] = \alpha_1[X_3, X_1] + \alpha_2[X_3, X_2] + \alpha_3[X_3, X_3] \tag{6.23}$$
$$= 2\alpha_1 X_2 - 2\alpha_2 X_1. \tag{6.24}$$

We then solve $[X_3, X] = \lambda X$, we find two solutions:

$$\alpha_1 = 1, \quad \alpha_2 = \pm i, \quad \alpha_3 = 0 \tag{6.25}$$

and $\lambda = \mp 2i$ correspondingly. The root space is therefore $\Phi = \{2i, -2i\}$, indeed we see that if α is a root, then so is $-\alpha$. The root space is one-dimensional, hence its roots diagram will simply be the line $\mathbb{R}$ with two dots at the points ± 2. We can choose $\Pi = \{2i\}$ as a set of fundamental roots. As a result our discussion of $\mathfrak{su}(2)$ is especially simple, since the Cartan matrix is just the 1×1 matrix (2) and the Dynkin diagram is given simply by one circle:

Let us consider the root vectors in more detail:

$$X_+ = X_1 + iX_2, \quad X_- = X_1 - iX_2. \tag{6.26}$$

$$A_1 \quad \bullet$$

Figure 6.2: The Dynkin diagram A_1 of the Lie algebra $\mathfrak{su}(2)$.

They can be recognized as the *raising* and *lowering* operators which are used to describe spin in quantum mechanics. Indeed, the procedures involving the so-called *ladder operators* are derived precisely from the roots and root vectors of the underlying Lie algebra. Moreover, this is not just a property of $\mathfrak{su}(2)$. In the discussion of $\mathfrak{su}(3)$ similar objects will arise, as is explained in section 6.5.

The preceding procedure turned out to be surprisingly simple. However, this is mainly due to the fact that $\mathfrak{su}(2)$ only has 3 generators, with rather simple commutation relations. Things become more difficult in the case of, for instance, $\mathfrak{su}(3)$, where there are 8 generators. Handling the commutation relations and finding roots becomes an intensive act of algebraic labour. To simplify things, the next section considers representations of Lie algebras. As a result, the problem of finding roots really just comes down to solving matrix equations, which modern software can do very easily.

6.4 Representations of Lie Algebras

The theory of representations of Lie algebras (and Lie groups) is extremely important in both mathematics and in physics. An example of which is *Wigner's theorem*, which states that any symmetry transformation of the projective Hilbert space (i.e. the space of physical states in quantum mechanics) can be described by a unitary representation. However, in the context of this thesis, we are mainly interested in simplifying our description of Lie algebras.

Let $(\mathfrak{g}, [\cdot,\cdot])$ be a finite-dimensional Lie algebra. A *representation* of $\mathfrak{g}$ is a vector space V, together with a map[5]

$$\rho : \mathfrak{g} \to \mathrm{End}(V), \tag{6.27}$$

such that ρ is a Lie algebra homomorphism, i.e.

$$\rho([X,Y]) = X \circ Y - Y \circ X, \quad \forall X, Y \in \mathfrak{g}. \tag{6.28}$$

In other words, we represent each element in $\mathfrak{g}$ by a matrix, such that the resulting matrices satisfy the same commutation relations as the algebra elements. This might seem like an abstract definition at first, but it is really quite explicit. A representation of $\mathfrak{su}(2)$, for instance, is given by $V = \mathbb{C}^2$ together with the map:[6]

$$\rho : \mathfrak{su}(2) \to \mathrm{GL}(2,\mathbb{C}), \quad X_i \mapsto -i\sigma_i \tag{6.29}$$

where σ_i denotes the i-th *Pauli matrix*, given by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.30}$$

In fact, we have already encountered a representation before, namely the *adjoint representation*, where $V = \mathrm{End}(\mathfrak{g})$ and $\rho = \mathrm{ad}$. Indeed, the entire theory that was developed in the previous sections is applicable to any representation. The only difference is that the roots of a general representation are called the *weights*. More precisely: given a representation (V, ρ) of a semi-simple Lie group $\mathfrak{g}$, with Cartan subalgebra $\mathfrak{h}$, we say that a non-zero functional λ is a *weight* if there exists a $v \in V$ such that

$$\rho(H)v = \lambda(H)v, \quad \forall H \in \mathfrak{h}. \tag{6.31}$$

In the case where $V = \mathbb{C}^n$, this is really just an eigenvalue equation for the matrix that represents H. The resulting root diagram are is called a *weight lattice* and simply contains the eigenvectors v. Moreover, the Killing form only changes, with respect to the adjoint representation, by a multiplicative factor, namely the

[5]Here we canonically make $\mathrm{End}(V)$ into a Lie algebra by equipping it with the standard commutator.

[6]Since ρ is a Lie group homomorphism, it is, in particular, a linear map. Hence it suffices to say how ρ acts on the generators of $\mathfrak{su}(2)$.

Dynkin index of the representation. An important result is that the resulting Dynkin diagram of a semi-simple Lie algebra is the same for every representation. In other words, when classifying semi-simple Lie algebras, we might as well pick a representation that makes our lives the easiest.

An important insight that we gain from using representations in the classification of semi-simple Lie algebras is the following. Let (V, ρ) be a representation of $\mathfrak{g}$, with Cartan subalgebra $\mathfrak{h}$, and let v be a weight vector with weight λ, and finally let X be a root vector with root α. Then consider, for $H \in \mathfrak{h}$:

$$\begin{aligned} \rho(H)(\rho(X)v) &= \rho(HX)v && (6.32)\\ &= \rho([H,X] + XH)v && (6.33)\\ &= \rho([H,X])v + \rho(X)\rho(H)v && (6.34)\\ &= \alpha\rho(X)v + \lambda\rho(X)v && (6.35)\\ &= (\alpha+\lambda)\rho(X)v. && (6.36) \end{aligned}$$

In other words, $\rho(X)v$ is again a weight vector with weight $\alpha + \lambda$.[7] As a result, by considering the difference between weights we can find the roots of a Lie algebra. This is shown in more detail in the next section.

6.5 Example: $\mathfrak{su}(3)$

The Lie algebra $\mathfrak{su}(3)$ has 8 generators $X_1, \dots, X_8$, which can be represented by the 3×3 complex Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2 \end{pmatrix},$$

where traditionally $X_i = \frac{1}{2}\lambda_i$. Explicit calculation of the commutators between all the generators gives the following structure constant tensor:

$$[X_i, X_j] = i{c^k}_{ij} X_k, \tag{6.37}$$

where ${c^k}_{ij}$ is completely anti-symmetric, with its only non-zero components given by (up to permutations)

- ${c^1}_{23} = 1$.
- ${c^1}_{47} = -{c^1}_{56} = {c^2}_{46} = {c^2}_{57} = {c^3}_{45} = -{c^3}_{67} = \frac{1}{2}$.
- ${c^4}_{58} = {c^6}_{78} = \frac{1}{2}\sqrt{3}$.

Since the representations of X_3 and X_8 are already diagonalized we can take the Cartan subalgebra $\mathfrak{h}$ to be generated by X_3 and X_8. We now find the roots and root vectors of $\mathfrak{h}$. Let $X \in \mathfrak{su}(3)$ be arbitrary, then we can decompose X in terms of the generators as

$$X = \sum_{n=1}^{8} \alpha_n X_n. \tag{6.38}$$

We require X to be a root vector of X_3 and X_8 for some linear functional λ, hence we put:

$$[X_3, X] = \lambda(X_3)X, \quad [X_8, X] = \lambda(X_8)X. \tag{6.39}$$

We can write λ in terms of the dual basis of $\{X_3, X_8\}$ as 2-dimensional row vector:

$$\lambda = (\lambda^{(3)}, \lambda^{(8)}), \tag{6.40}$$

[7] Or $\rho(X)v$ is the zero vector.

where $\lambda^{(3)} = \lambda(X_3)$ and $\lambda^{(8)} = \lambda(X_8)$. Equation (6.39) is easily solved for $\alpha_1, \ldots, \alpha_8$ and λ, using for instance Mathematica. Taking $a_1 = a_4 = a_6 = 1$, a possible set of solutions is:

$$Y_1 = X_6 + iX_7, \quad \lambda_1 = \left(-\frac{1}{2}, \frac{1}{2}\sqrt{3}\right), \tag{6.41}$$

$$Y_2 = X_6 - iX_7, \quad \lambda_2 = \left(\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right), \tag{6.42}$$

$$Y_3 = X_4 - iX_5, \quad \lambda_3 = \left(-\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right), \tag{6.43}$$

$$Y_4 = X_4 + iX_5, \quad \lambda_4 = \left(\frac{1}{2}, \frac{1}{2}\sqrt{3}\right), \tag{6.44}$$

$$Y_5 = X_1 - iX_2, \quad \lambda_5 = (-1, 0), \tag{6.45}$$

$$Y_6 = X_1 + iX_2, \quad \lambda_6 = (1, 0). \tag{6.46}$$

Hence the root space is given by $\Phi = \{\lambda_1 \ldots, \lambda_6\}$, with corresponding root vectors $Y_1, \ldots, Y_6$. A possible choice of fundamental roots is then $\Pi = \{\lambda_1, \lambda_6\}$. Explicit calculation of the Killing form yields

$$K_{ij} = -3\delta_{ij}. \tag{6.47}$$

Since K is invertible, we conclude that $\mathfrak{su}(3)$ is indeed semi-simple. Finally, we need to calculate the Cartan matrix. Clearly $C_{11} = C_{22} = 2$. For the other 2 elements, we need to explicitly calculate what the components of K^* are.

$$\langle \lambda_1, \lambda_6 \rangle = K(i^{-1}(\lambda_1), i^{-1}(\lambda_6)). \tag{6.48}$$

Since K is diagonal, it is easily seen that

$$i\left(\frac{1}{3}\left(-\frac{1}{2}X_3 + \frac{1}{2}\sqrt{3}X_8\right)\right)(X_3) = \frac{1}{3}K(-\frac{1}{2}X_3 + \frac{1}{2}\sqrt{3}X_8, X_3) \tag{6.49}$$

$$= -\frac{1}{6}K_{33} - \frac{1}{6}\sqrt{3}K_{83} \tag{6.50}$$

$$= -\frac{1}{2} \tag{6.51}$$

$$= \lambda_1(X_3), \tag{6.52}$$

and a similar result holds when applied to X_8. Therefore we find

$$i^{-1}(\lambda_1) = -\frac{1}{3}\left(-\frac{1}{2}X_3 + \frac{1}{2}\sqrt{3}X_8\right). \tag{6.53}$$

Similarly, we have

$$i^{-1}(\lambda_6) = -\frac{1}{3}X_3. \tag{6.54}$$

It follows that

$$C_{21} = 2\frac{\langle \lambda_1, \lambda_6 \rangle}{\langle \lambda_6, \lambda_6 \rangle} \tag{6.55}$$

$$= 2\frac{K(-\frac{1}{2}X_3 + \frac{1}{2}\sqrt{3}X_8, X_3)(-\frac{1}{3})(-\frac{1}{3})}{K(X_3, X_3)(-\frac{1}{3})(-\frac{1}{3})} \tag{6.56}$$

$$= 2\frac{K(X_3, X_3)(-\frac{1}{3})(-\frac{1}{3})(-\frac{1}{2})}{K(X_3, X_3)(-\frac{1}{3})(-\frac{1}{3})} \tag{6.57}$$

$$= -1. \tag{6.58}$$

In a similar fashion one can calculate that $C_{12} = -1$, hence the Cartan matrix is given by:

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{6.59}$$

More generally, one can show that for $\mathfrak{su}(n)$, the Cartan matrix is given by [21]

$$C = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}. \tag{6.60}$$

Returning to $\mathfrak{su}(3)$, it follows from the Cartan matrix that the angle between the two roots λ_1 and λ_6 is equal to 120°, and the bond number n_{12} is equal to 1. The resulting Dynkin diagram is shown in figure 6.3

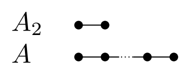

Figure 6.3: The Dykin diagrams A_2 and A_n corresponding to the Lie algebras $\mathfrak{su}(3)$ and $\mathfrak{su}(n+1)$.

We can also draw the root diagram of $\mathfrak{su}(3)$ using the whole set of roots and calculating their lengths and the angles between them. The result is shown in figure 6.4, which also shows the resulting weight lattice for the representation using the Gell-Mann matrices, i.e. the fundamental representation. Indeed, as remarked in section 6.4, the difference between two weights is a root.

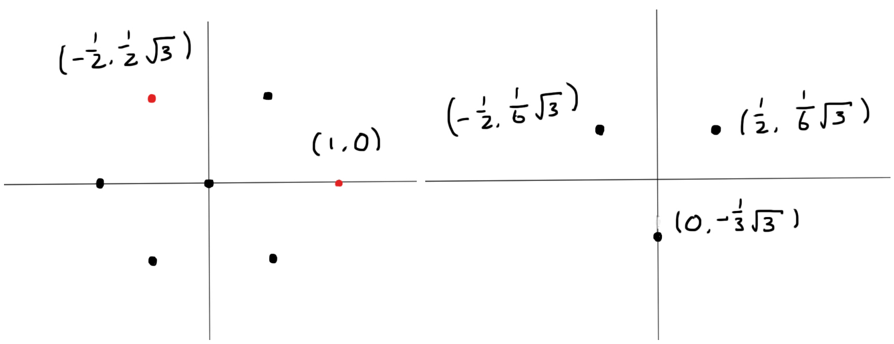

Figure 6.4: Left: The root diagram of $\mathfrak{su}(3)$, a choice of fundamental root vectors is drawn in red. Right: The weight lattice of $\mathfrak{su}(3)$ in the fundamental representation.

One final remark is in order. Similarly to the discussion of $\mathfrak{su}(2)$, we recognize three ladder operators:

$$Y_1 = X_6 + iX_7, \quad Y_2 = X_6 - iX_7, \tag{6.61}$$
$$Y_3 = X_4 - iX_5, \quad Y_4 = X_4 + iX_5, \tag{6.62}$$
$$Y_5 = X_1 - iX_2, \quad Y_6 = X_1 + iX_2. \tag{6.63}$$

Moreover, upon closer inspection we see that the roots of a pair of ladder operators lie on one line. This is, in fact, a general property of the root diagram of any semi-simple Lie algebra. The root vectors themselves constitute multiple $\mathfrak{su}(2)$ subalgebras [24]. Therefore the understanding of $\mathfrak{su}(2)$ is vital for our description of more general Lie algebras.

Chapter 7

The ADE Classification in String Theory

In this final chapter we tread (lightly) into the realm of *string theory* to investigate an interesting relation between the ADE-classification, which was the central topic of the previous chapter, and the geometric structure of branes in type IIa string theory. Everything we have discussed so far will play an important role here. We require our notion of manifolds to understand the more general spaces of orbifolds and orientifolds. Moreover, such spaces must still be solutions to Einstein's equation in vacuum. Finally, we require our understanding of Lie algebras and Lie groups to make the connection with the ADE-classification and interpret the results in terms of symmetries of gauge theories. It should be noted that, since this is a first contact with the topic, this chapter serves as a rough overview of how the various concepts discussed in this project come together. As a result, many details are omitted and the emphasis lies on pictorial and intuitive explanations.

We start by introducing D-branes in type IIA string theory, in particular the D6 brane. It turns out that the D6 brane, when 'uplifted' to M-theory, is related to a geometry called the Kaluza-Klein monopole or Taub-NUT space. Stacking multiple N KK-monopoles onto each other, either freely or confined to an orientifold plain, creates interesting behaviour near singularities. In fact, the intersection matrix of two-cycles near a singularity is precisely the Cartan matrix of $\mathfrak{su}(N)$ or $\mathfrak{so}(2N)$, depending on whether the D6 branes are confined to an orientifold plane. As a result, these singularities admit an ADE-classification. We also mention the more general result, namely the ADE-ALE correspondence, of which the above is an explicit example. Finally, we discuss the physical consequence of these results, namely the fact that we can relate the membranes wrapped around the two cycles to massless particles (near the singularity), which then become the gauge bosons of a gauge theory. Then this gauge theory will have an $SU(N)$ or $SO(2N)$ symmetry.

7.1 D-Branes in type IIa string theory and M-theory

In string theory one is interested in the behaviour of so-called *extended objects*, which are generalizations of point particles or 0-branes. The most basic example is a *string* or 1-brane, which is any object which has a 1-dimensional spatial extent. A *membrane* or 2-brane, then, is has a 2-dimensional spatial extent and can be envisioned a a sheet which is allowed to oscillate, bend, etc... Of great importance is a brane's *tension* or energy per unit volume, which for a 0-brane is just its mass. Branes are studied in high dimensional spacetimes, which are usually solutions to Einstein's equation in vacuum. Their dynamics are governed by *gauge theories*.[25] In type IIA string theory one studies the so-called D-branes. Such branes can be seen as walls to which the ends of one dimensional strings can attach themselves. These strings will have Dirichlet conditions on the coordinates which are transverses to the brane, and Neumann conditions on the coordinates which lie along the brane. Hence the string is allowed to move along the brane, but not into or out of it. The branes that are studied are the D0, D2, D4 and D6-branes. However, the D6-brane turns out to be particularly interesting, as will be discussed in the next section, where we consider the lift of D branes from type IIA string theory to M-theory.

For now, let us stick to D6 branes in type IIA string theory. It turns out that there are two particular

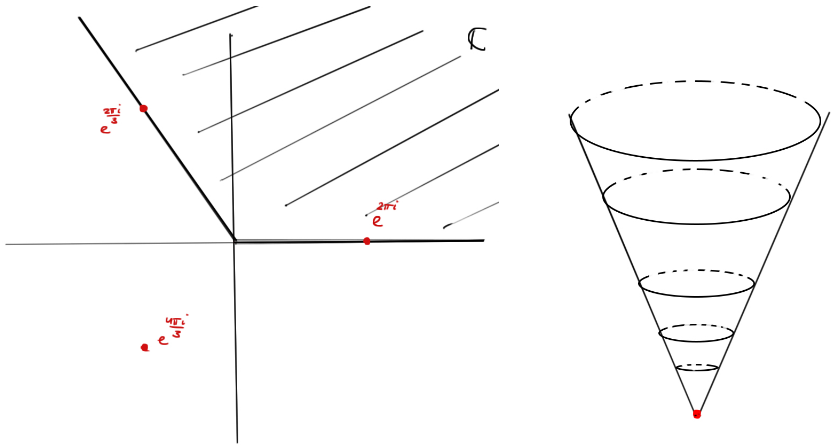

Figure 7.1: Left: The complex plane $\mathbb{C}$, one third of the plane is shaded. Right: The resulting cone which represents $\mathbb{C}/\Gamma$, where Γ is the group $\mathbb{Z}_3$.

configurations of interest. In the first case one considers a stack of N $D6$-branes lying on top of each other. In the second case we again consider N $D6$-branes, but they are now placed on top of an *orientifold* plane, which is a generalization of an *orbifold*. Let us discuss these spaces in more detail. Recall that a manifold is a space which is locally diffeomorphic to an open in $\mathbb{R}^n$. An orbifold is a generalization of a manifold, as it allows the presence of points whose neighbourhoods are diffeomorphic to a quotient $\mathbb{R}^n/\Gamma$ by some finite group Γ. Let us discuss the simplest possible 2d orbifold. Consider $\mathbb{C}$ together with the group action

$$z \mapsto z e^{\frac{2i\pi}{3}n}, \quad n = 1, 2, 3 \tag{7.1}$$

Intuitively, the complex plane is divided in three equal segments which, in the quotient space, are glued together to form a *cone*, with the origin forming the tip of the cone. In particular, the point $z = 0$ forms a *singularity* of the resulting orbifold. Note that the point $z = 0$ is precisely a *fixed point* of the group action. Indeed, for singularities to occur one requires an action that is not *free*, i.e. it should have at least one fixed point. Hence in the context of physics, orbifolds are often defined as quotients by non-free groups. For example, one considers a space M/G, with G a non-free subgroup of the group of isometries. One can sometimes 'remove' such singularities, a process which is called the *resolution of singularities.* In particular, in the following we will be interested in the so-called *minimal resolution.* For the previous example, a resolution of the singularity would be to make the tip of the cone smooth as shown in figure 7.1.

Finally, in the context of string theory, strings can also have an orientation. An *orientifold*, then, is a generalization of an orbifold in which also the opposite orientations of strings are identified. Orientifolding therefore produces strings which have no orientation anymore. We will see that free D6 branes behave differently than D6 branes which are confined to an orientifold.

7.2 M-Theory and ADE Singularities

In 1995 it was discovered that considering type IIA string theory in which the string coupling goes to infinity results in a new 11-dimensional theory, now known as *M-theory.*[26] One can 'lift' the D-branes of type IIa string theory to M-theory, for example the D0, D2 and D4 branes are uplifted to the M2 and M5 branes, which are very hard to study. However, the D0 and D6 branes do not uplift to an M-brane, instead they uplift to geometry, in the sense that they appear as a change in the metric. Therefore, considering these branes in this new theory where the are manifested into the geometrical structure of spacetime might lead to new insights. In particular, the D6 brane becomes the 11d Kaluza-Klein (KK) monopole. The KK-monopole is a solution to the equations of motion of 11d supergravity. As a space it is given by the product of the 4d Taub-NUT spacetime N_4 with the 7d Minkowski spacetime $\mathbb{R}^{6,1}$. The Taub-NUT space is actually the minimal resolution of a very complex orbifold containing multiple singularities. One can write down an

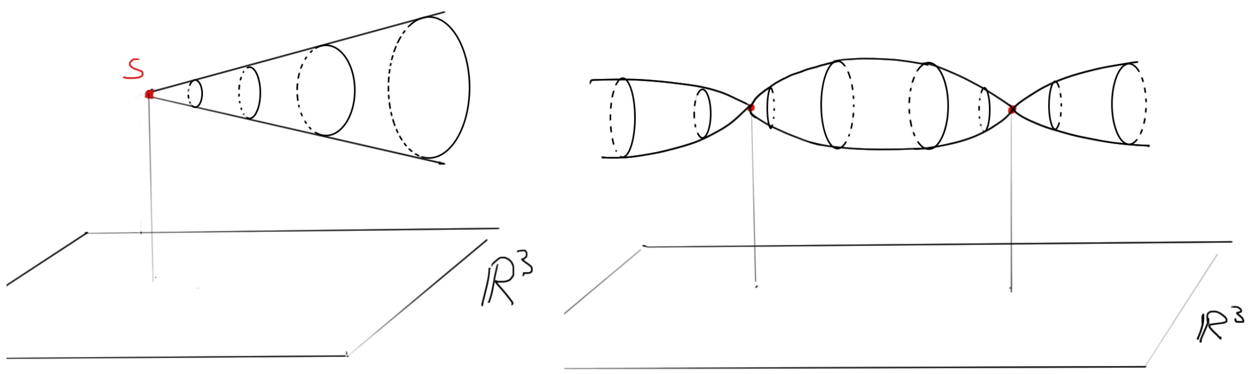

Figure 7.2: Left: The Taub-NUT space for a single free KK-monopole, the point S is an apparent singularity. Right: The Taub-NUT space for multiple free KK-monopoles.

explicit metric for the Taub-NUT space as follows. Consider the following 1-forms on the 3-sphere, written in terms of the Euler angles:

$$\sigma_1 = \sin\psi d\theta - \cos\psi \sin\theta d\varphi, \tag{7.2}$$

$$\sigma_2 = \cos\psi d\theta + \sin\psi \sin\theta d\varphi, \tag{7.3}$$

$$\sigma_3 = d\psi + \cos\theta d\varphi. \tag{7.4}$$

Then the metric of the Taub-NUT space is given by [27]:

$$ds^2 = \frac{1r+n}{4r-n}dr^2 + \frac{r-n}{r+n}n^2\sigma_3^2 + \frac{1}{4}(r^2-n^2)(\sigma_1^2+\sigma_2^2), \tag{7.5}$$

where n is related to the number of branes that are uplifted. One can view the Taub-NUT space as a *circle bundle* over $\mathbb{R}^3$, i.e. it is a fibre bundle for which each fibre is the circle S^1. This is shown in figure 7.2, where a single KK monopole has been drawn. To be more precise, the resulting figure shows what happens for the M-theory uplift of a stack of N D6-branes which are not confined to an additional surface, like an orientifold plane.

It may look as if this configuration has a singularity, similarly to the cone from the previous section, but this is not the case. Here the apparent singularity is a result of our choice of charts. However, once multiple KK-monopoles are placed on top of each other, these apparent singularities overlap in such a way to create an actual singularity. One can then consider the minimal resolution, which amounts to pulling the different 'pinches' away from each other, the resulting space is drawn in figure 7.2.

It turns out that these singularities are very special, they admit an ADE classification! After performing the minimal resolution, one can study what kind of pattern the 'pinches' in figure 7.2 follow. It turns out that the pattern is exactly the A_{N-1} Dynkin diagram. Here we have explicitly considered the uplift of D6 branes, resulting in the KK monopoles, but there are also more general configurations, obtained with the same ingredients (like confining the branes to an orientifold plane), for which the above procedure results in the D and E Dynkin diagrams. One can also consider the so-called *intersection matrix*. This matrix describes how the different 2-cycles (the potato-shaped objects bounded by the red points in the left of figure 7.2) intersect each other. The result is that this intersection matrix is precisely one of the Cartan matrices of A_{N-1}, D_N or E_6, E_7, E_8, depending on the particular configuration [28]. As a result, such singularities are also called *ADE-singularities*. This incredible result has some important implications, both from the physical and mathematical points of view. We will discuss these implications in the next section.

7.3 The ADE-ALE Correspondence and Gauge Symmetries

In the previous section we found that the singularities of the KK-monopoles are, in fact, ADE-singularities. After performing the minimal resolution, the resulting space is an example of an *ALE-space*, which stands

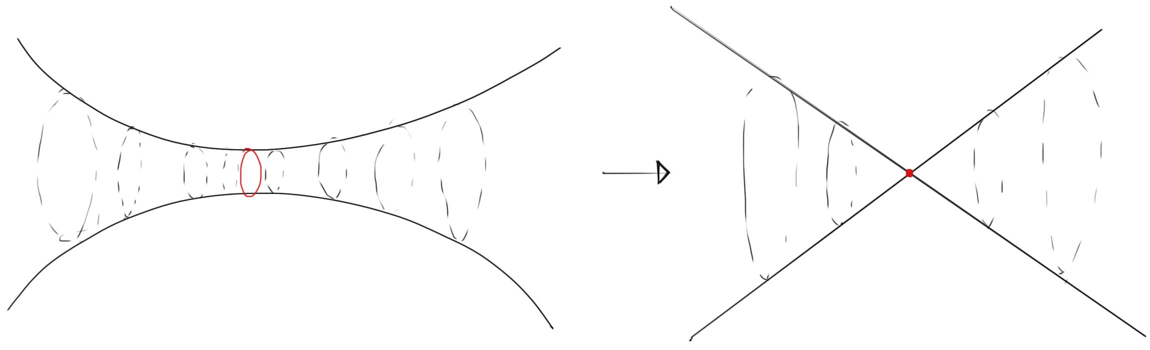

Figure 7.3: Left: A string wrapped around a two dimensional surface, to minimize its tension it moves towards the middle of the configuration. Right: The same configuration, but with a singularity in the middle. Once the string reaches this singularity its tension goes to zero, resulting in a massless particle.

for an *Almost Locally Euclidean space*. As the name suggests, an ALE-space is a Riemannian 4-manifold which at infinity resembles the quotient $\mathbb{R}^4/\Gamma$, for some finite group Γ. It turns out that *any* 4-dimensional ALE space is diffeomorphic to the minimal resolution of $\mathbb{C}^2/\Gamma$ for some $\Gamma \subseteq \mathrm{SU}(2)$.[29] Furthermore, the finite subgroups of SU(2) also admit an ADE-classification, which is also called the *McKay correspondence*. In other words, there is a relation between 4-dimensional ALE spaces and the simply laced Dynkin diagrams. This relation is called the *ADE-ALE correspondence*. In fact, from the point of view of string theory, these ALE spaces arise as the minimal resolutions of the uplifts of configurations of D6 branes to M-theory. We can push this relation even further, by considering how, in string theory, branes are connected to *gauge theories*.

Turning to a more physical perspective, we again consider the singularities of the KK-monopoles. But before that, we go further back to type IIa string theory. It is a general fact that, given d-dimensional surface, one can 'wrap' a $d+1$-dimensional brane around it, in the intuitive sense of the word. Consider the situation drawn in figure 7.3. Intuitively, one expects the 1-dimensional brane (or string) to move towards the middle of the configuration, since the string is stretched the least (i.e. its tension is minimal). Next, consider the same configuration, but with a singularity in the middle of the double cone. Once the string reaches the singularity, its tension, or energy, goes to zero. It turns out that, in type IIa string theory, this corresponds to a *massless particle*. Uplifting our discussion to M-theory, our picture becomes the following. One can wrap an M2 brane (the uplift of an D2 brane) around the 2-cycles shown in the right of figure 7.2. Then, upon reducing the area of the 2-cycle, by bringing the two red points closer together, we again create a situation where the area of an M2 brane goes to zero. A massless particles is thus created. Moreover, since the intersection of the 2-cycles is governed by the intersection matrix, which in turn is equal to the Cartan matrix, there is a strong relation between this massless particle and the root vectors of the corresponding Lie algebra. In particular, one can show that the resulting particle is a *charged vector boson*, with its charge being related to the roots of the Lie algebra. A massless charged vector boson is precisely a *gauge boson*. Hence the above procedure describes how a gauge theory arises in the context of M-theory. Moreover, from the ADE classification it follows that this gauge theory has an $\mathrm{SU}(N)$ or $\mathrm{SO}(2N)$ symmetry, depending on whether the singularity is classified as A, D or E. It should be noted that one can perform explicit calculations on for instance the Taub-NUT space, in the case of the A singularity, using the explicit metric as shown above. However, this becomes extremely difficult and even impossible for the D and E cases. [30]

Combining the ADE-ALE correspondence together with this theory of gauge symmetries, we arrive at the following conclusion: An ALE space, classified by an ADE element, can be seen as the M-theory lift of a certain configuration of branes whose gauge group is described by that same ADE element. In other words, M-theory gives us a new, and physical, background as to why ALE spaces admit an ADE classification which relates to gauge theory. Moreover, the resulting gauge theories one finds in this manner conform to the results found in other theories. Hence one can view this as a consistency check for M-theory.

Chapter 8

Conclusion

The goal of this project was to develop an understanding of how differential geometry underlies various fields of physics. To this end, we first prepared the necessary tools by considering manifolds, tangent spaces and tensor algebra. We also found that the key object to describe physical spaces is the metric, which allowed us to discuss curvature. Naturally, this lead to the theory of general relativity, in which we replace gravitational force by the curvature of spacetime, and allow spacetime itself to be a very general Lorentzian 4-manifold. From the Hilbert action we derived the Einstein equation in vacuum, for which our acquired knowledge of tensor algebra and integration on manifolds was essential. Turning to a more physical side of things, we introduced the energy-stress tensor as the primary object which describes the content of our universe in terms of energy and momentum densities. Based on the theory of Newtonian gravity we posed the full Einstein equation and discussed the implementation of a cosmological constant to create a static universe, which turned out to be unstable.

Having discussed the Einstein equation in detail, the next step was to consider the simplest possible solutions which may describe our universe. Again we profited from a mathematical formulation of homogeneity and isotropy, allowing us to simplify the Riemann tensor considerably and reduce the possible metrics of our universe to three cases: flat, spherical and hyperbolic. Adding a scale factor and considering the full 4-dimensional metric we arrived at the well-known FLRW cosmological model. To fully solve the Einstein equation we assumed the universe to be a perfect fluid and hence arrived at the Friedmann equations which describe the expansion of the universe. A short discussion on dark energy as an explanation for the measured accelerated expansion of the universe was included. We then moved towards a more theoretically interesting situation by investigating the anisotropic Bianchi universes. To this end we introduced Lie groups and Lie algebras as the main objects to describe the symmetries in a system, the spatial homogeneity in this case. We described the procedure to arrive at an equation which fixes the metric and stated the solution in a flat universe. Interestingly, we found a universe which necessarily expands in one direction, but contracts in another, as opposed to the FLRW model.

Intrigued by the power of Lie algebras, we then moved to a mathematical discussion on the classification of all semi-simple Lie algebras. By considering a special subalgebra, the Cartan subalgebra, and finding the roots and root vectors, which greatly resembled eigenvalues and eigenvectors, we found that we can describe the entire algebra in terms of just the fundamental roots. This massive simplification led to the ADE classification of simply laced Dynkin diagrams. It turns out that this ADE classification pops up in many different fields, but one field was of particular interest to me. In the last chapter we discussed how this ADE classification arises in the context of string theory. Our acquired knowledge of geometric structures, e.g. manifolds and fibre bundles, was vital to understanding the necessary objects such as branes and orbifolds. Moreover, all the objects used to describe the semi-simple Lie algebras such as the Cartan matrix and Dynkin diagrams returned once we considered a configuration of KK-monopoles in M-theory. This provided beautiful connection between the physical geometric description provided by M-theory and the Lie algebras which describe the underlying gauge symmetry of the theory's gauge bosons. Finally, this framework also provides an understanding for why ALE spaces also admit an ADE classification, as these ALE spaces arise precisely as the uplifts of configuration of D-branes to M-theory.

This is, however, not the end of the story. Similar mathematics appears in the field of *F-theory*, one of the branches of string theory, which may provide new models for Grand Unified Theories. In particular, here one can study the so-called *elliptic fibrations* which are, in a sense, generalizations of the ALE spaces discussed in chapter 7. The hope of F-theory is to provide a geometric answer to many of the questions that arise in particle physics (such as e.g. Yukawa couplings). Such spaces are extremely complex and, as a result, fascinating to investigate. It is in these theories that differential geometry and physics combine to full fruition to provide us with new models of our world. I hope this thesis has provided a gateway to this wonderful, undiscovered and inspiring field of research.

Acknowledgements

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Appendix A

Definitions

A.1 Abstract Index Notation

A tensor T of type (k,l) will be denoted by k upper Latin indices $a_1,\ldots,a_k$, denoting the covariants slots, and l lower Latin indices $b_1,\ldots,b_l$, denoting the contravariant slots. For example, $T^{abcd}{}_{ef}$ denotes a $(4,2)$ tensor. Note that the indices are merely an indication of the type of object that T is, and has nothing to do with the components of T.

Certain operations are easily expressed in terms of abstract index notation. For example, we denote the contraction with respect to the i-th and j-th slots of a tensor T by repeating an index over the contracted slots. For example $T^{abc}{}_{bd}$ denotes the tensor $T^{abc}{}_{de}$ of type $(2,1)$ which is contracted with respect to the second covariant and first contravariant slots. The tensor product of two tensors is written by simply writing the tensors after one another, for example $T^{ab}{}_c S^d{}_{efg}$ denotes the tensor of type $(3,4)$ which is obtained from the tensor product of the $(2,1)$ tensor $T^{ab}{}_c$ and the $(1,3)$ tensor $S^a{}_{bcd}$.

Whenever we want to specify the *components* of a tensor, for instance in an equation which shows the transformation behaviour of components under a change of basis, we use Greek indices. For example $T^\mu{}_\nu$ denotes a basis component of the tensor $T^a{}_b$. Moreover, given a metric g_{ab} we denote the inverse metric g^{-1}_{ab} by g^{ab}. Hence by definition $g^{ab}b_{bc} = \delta^a{}_c$. As a result we can consistently use the metric and its inverse to lower and raise indices of tensors. For example $g_{ab}v^a = v_b$ and $g^{ab}v_a = v^b$.

A.2 Operations on Tensors

In the following $\mathcal{T}(k,l)$ denotes set of all (k,l) tensors on a vector space V.

Definition A.2.1 (Contraction).
Let $T \in \mathcal{T}(k,l)$, the *contraction* of T with respect to the i-th and j-th slot is a map $C : \mathcal{T}(k,l) \to \mathcal{T}(k-1,l-1)$ with

$$CT = \sum_{\sigma=1}^{d} T(\ldots, v^{\sigma^*}, \ldots; \ldots, v_\sigma, \ldots)$$

where $\{v_\sigma\}$ is a basis of V, $\{v^{\sigma^*}\}$ is its dual basis and the vectors are inserted in the i-th and j-th slots respectively.

Definition A.2.2 (Tensor product).
Let V be a finite-dimensional vector space, and let T be a (k,l) tensor and S an (r,s) tensor. We define the *tensor product* $T \otimes S$ by

$$T \otimes S(\omega^1,\ldots,\omega^k,\omega^{k+1},\ldots,\omega^{k+r}; v_1,\ldots,v_l,v_{l+1},\ldots,v_{l+s}) :=$$
$$T(\omega^1,\ldots,\omega^k; v_1,\ldots,v_l) S(\omega^{k+1},\ldots,\omega^{k+r};, v_{l+1},\ldots,v_{l+s}).$$

Then $T \otimes S$ is a tensor of type $(k+r, l+s)$. **Note**: We often drop the '$\otimes$' for clarity.

Definition A.2.3 (Symmetrization and anti-symmetrization).
Let $T_{a_1 \dots a_k}$ be a type $(0,k)$ tensor, then we define

$$T_{(a_1 \dots a_k)} := \frac{1}{k!} \sum_{\pi} T_{a_{\pi(1)} \dots a_{\pi(k)}}$$
$$T_{[a_1 \dots a_k]} := \frac{1}{k!} \sum_{\pi} \delta_\pi T_{a_{\pi(1)} \dots a_{\pi(k)}}$$

as the symmetrization and anti-symmetrization of $T_{a_1 \dots a_k}$ respectively. Here the sum runs over all permutations of $1, \dots, k$ and δ_π is $+1$ for even permutations and -1 for odd permutations.

A.3 The Riemann Tensor

For a given covariant derivative ∇_a, the Riemann tensor is defined by

$$\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = R_{abc}{}^d \omega_d. \tag{A.1}$$

First, we list some properties of the Riemann tensor:

1. $R_{abc}{}^d = -R_{bac}{}^d$.
2. $R_{[abc]}{}^d = 0$.
3. If $\nabla_a g_{bc} = 0$, then $R_{abcd} = -R_{abdc}$.
4. (Bianchi identity) $\nabla_{[a} R_{bc]d}{}^e = 0$.

We define the *Ricci tensor* R_{ac} as

$$R_{ac} = R_{abc}{}^b. \tag{A.2}$$

It follows from 1, 2 and 3 that the Ricci tensor is symmetric. We also define the *scalar curvature* R as

$$R = R_a{}^a. \tag{A.3}$$

Appendix B

Derivations

B.1 Identities involving the Riemann Tensor

B.1.1 Divergence of the Einstein Tensor

Let us derive an important equation that the Ricci tensor and scalar curvature satisfy. We start by contracting the Bianchi identity and explicitly writing out the anti-symmetrization:

$$\nabla_{[a} R_{bc]d}{}^{a} = 0 \tag{B.1}$$

$$\nabla_a R_{bcd}{}^{a} + \nabla_b R_{cad}{}^{a} + \nabla_c R_{abd}{}^{a} - \nabla_a R_{cbd}{}^{a} - \nabla_b R_{acd}{}^{a} - \nabla_c R_{bad}{}^{a} = 0. \tag{B.2}$$

By 1 we have, for instance, $\nabla_a R_{cbd}{}^{a} = -\nabla_a R_{bcd}{}^{a}$, so we find:

$$\nabla_a R_{bcd}{}^{a} + \nabla_b R_{cad}{}^{a} + \nabla_c R_{abd}{}^{a} = 0 \tag{B.3}$$

$$\nabla_a R_{bcd}{}^{a} + \nabla_b R_{cd} - \nabla_c R_{bd} = 0 \tag{B.4}$$

Raising the index d with the metric and contracting b and d we find

$$\nabla_a R_{bc}{}^{ba} + \nabla_b R_c{}^{b} - \nabla_c R_b{}^{b} = 0 \tag{B.5}$$

$$\nabla_a R_c{}^{a} + \nabla_b R_c{}^{b} - \nabla_c R = 0 \tag{B.6}$$

$$2\nabla_a R_c{}^{a} - \nabla_c R = 0 \tag{B.7}$$

$$\nabla_a \left(g_{bc} R^{ab} \right) - \frac{1}{2} g_{bc} g^{ab} \nabla_a R = 0 \tag{B.8}$$

Using again the fact that $\nabla_a g_{bc} = 0$, we find

$$\nabla_a \left(R^{ab} - \frac{1}{2} R g^{ab} \right) = 0, \tag{B.9}$$

or

$$\nabla^a G_{ab} = 0, \tag{B.10}$$

where

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} \tag{B.11}$$

is the *Einstein tensor.*

B.1.2 Riemann Tensor and Christoffel Symbols

We will explicitly write the Riemann tensor in terms of the Christoffel symbols. Starting from the definition

$$R_{abc}{}^{d} \omega_d = \left(\nabla_a \nabla_b - \nabla_b \nabla_a \right) \omega_c \tag{B.12}$$

and using

$$\nabla_a \omega_c = \partial_a \omega_c - \Gamma^d{}_{ac}\omega_d \tag{B.13}$$

we can write

$$R_{abc}{}^d \omega_d = \nabla_a(\partial_b \omega_c - \Gamma^d{}_{bc}\omega_d) - \nabla_b(\partial_a \omega_c - \Gamma^d{}_{ac}\omega_d). \tag{B.14}$$

We calculate the two covariant derivatives separately, first:

$$\nabla_a(\partial_b \omega_c - \Gamma^e{}_{bc}\omega_d) = \partial_a(\partial_b \omega_c - \Gamma^d{}_{bc}\omega_d) - \Gamma^e{}_{ab}(\partial_e \omega_c - \Gamma^d{}_{ec}\omega_d) - \Gamma^e{}_{ac}(\partial_b \omega_e - \Gamma^d{}_{be}\omega_d) \tag{B.15}$$

and secondly:

$$\nabla_b(\partial_a \omega_c - \Gamma^d{}_{ac}\omega_d) = \partial_b(\partial_a \omega_c - \Gamma^d{}_{ac}\omega_d) - \Gamma^e{}_{ba}(\partial_e \omega_c - \Gamma^d{}_{ec}\omega_d) - \Gamma^e{}_{bc}(\partial_a \omega_e - \Gamma^d{}_{ae}\omega_d). \tag{B.16}$$

Subtracting these two equations and using the symmetry property of the Christoffel symbol we find

$$R_{abc}{}^d = \partial_b \Gamma^d{}_{ac} - \partial_a \Gamma^d{}_{bc} + \Gamma^e{}_{ac}\Gamma^d{}_{be} - \Gamma^e{}_{bc}\Gamma^d{}_{ae}. \tag{B.17}$$

It follows that the Ricci tensor can be written as:

$$R_{ab} = R_{acb}{}^c = \partial_c \Gamma^c{}_{ab} - \partial_a \Gamma^c{}_{cb} + \Gamma^d{}_{ab}\Gamma^c{}_{cd} - \Gamma^d{}_{cb}\Gamma^c{}_{ad} \tag{B.18}$$

B.2 A Characterisation of 3D Lie Algebras

In terms of the structure constant tensor $c^c{}_{ab}$ the Jacobi identity is given by:

$$c^e{}_{d[a}c^d{}_{bc]} = 0. \tag{B.19}$$

It can be shown that for a 3-dimensional Lie algebra $c^c{}_{ab}$ can be written in terms of an anti-symmetric tensor M^{ab} and a dual vector A_a as

$$c^c{}_{ab} = M^{cd}\epsilon_{dab} + \delta^c{}_{[a}A_{b]}. \tag{B.20}$$

Our goal is to insert this equation into the Jacobi identity and find an relation that M^{ab} and A_a should obey. We can simplify the calculations by contracting the Jacobi identity with the anti-symmetric tensor ϵ^{abc}. We first consider the term quadratic in M^{ab}:

$$M^{ef}\epsilon_{fda}M^{df}\epsilon_{fbc}\epsilon^{abc} = 2M^{ef}\epsilon_{fda}M^{df}\delta_f{}^a = 0. \tag{B.21}$$

Next, the term quadratic in A_a:

$$\begin{aligned}
\delta^e{}_{[d}A_{a]}\delta^d{}_{[b}A_{c]}\epsilon^{abc} &= \frac{1}{4}\left(\delta^e{}_d\delta^d{}_b A_a A_c + \delta^e{}_a\delta^d{}_c A_d A_b - \delta^e{}_a\delta^d{}_b A_d A_c - \delta^e{}_d\delta^d{}_c A_a A_b\right)\epsilon^{abc} && \text{(B.22)}\\
&= \frac{1}{4}\left(A_a A_c \epsilon^{aec} + A_b A_c \epsilon^{ebc} - A_b A_c \epsilon^{ebc} - A_a A_b \epsilon^{abe}\right) && \text{(B.23)}\\
&= \frac{1}{4}A_a\left(A_c\epsilon^{aec} - A_b\epsilon^{abe}\right) && \text{(B.24)}\\
&= \frac{1}{2}A_a A_b \epsilon^{aeb} && \text{(B.25)}\\
&= 0. && \text{(B.26)}
\end{aligned}$$

Finally, we consider the cross terms:

$$\begin{aligned}
M^{ef}\epsilon_{fda}\delta^d{}_{[b}A_{c]}\epsilon^{abc} + M^{df}\epsilon_{fbc}\delta^e{}_{[d}A_{a]}\epsilon^{abc} &= M^{ef}\delta^d{}_{[b}A_{c]}\left(\delta_f{}^b\delta_d{}^c - \delta_f{}^c\delta_d{}^b\right) + 2M^{df}\delta^e{}_{[d}A_{a]}\delta_f{}^a && \text{(B.27)}\\
&= M^{eb}\delta^c{}_{[b}A_{c]} - M^{ec}\delta^b{}_{[b}A_{c]} + 2M^{da}\delta^e{}_{[d}A_{a]} && \text{(B.28)}\\
&= 2M^{eb}\delta^c{}_{[b}A_{c]} + 2M^{[da]}\delta^e{}_d A_a && \text{(B.29)}\\
&= 2M^{eb}\left(A_b - 3A_b\right) && \text{(B.30)}\\
&= -4M^{eb}A_b. && \text{(B.31)}
\end{aligned}$$

Putting everything together, we arrive at the desired equation:

$$M^{ab}A_b = 0. \tag{B.32}$$

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