The 1:3:4 resonance

Dynamics near a resonant equilibrium

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Abstract

We study the dynamics near a local extremum of a Hamiltonian function, for which the frequencies of the linearization are in 1:3:4 resonance. The 1:3:4 resonance is a genuine second order resonance, for which the truncated second normal form is expected to no longer be integrable, as opposed to the integrable truncated first normal form.

The dynamics of the first normal form are studied using singular reduction. We consider a detuning and the addition of the quartic self-interaction terms; Hamiltonian Hopf bifurcations are observed under variation of internal parameters. We present stability results on the normal modes for the first normal forms of various families of resonances. We consider the singular reduction, with respect to the periodic flow of the quadratic Hamiltonian, and we analyse the dynamics of the second normal form. Lastly, we study the dynamics of the indefinite 1:3:-4 resonance.

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1 Introduction

The topic of resonant equilibria may look like an exotic one amid the theory of dynamical systems: within the non-generic class of Hamiltonian systems, the appearance of equilibria with lower order resonances is non-generic as well. However, the amount of studies devoted to resonances is quite considerate, and the topic surfaces in many of the larger works on dynamical systems. In some cases, resonances are noted simply as a hindrance to integrability of the normal form approximations, but otherwise they are appreciated for the interesting underlying dynamics. In particular, one may be interested in the existence of integrals for the lower order normal forms, in families of periodic orbits or in bifurcations due to a change of the values of internal or natural parameters. A notable example which shows all three of these is the 1:-1 resonance, which leads to the Hamiltonian Hopf bifurcation (see [11] or [21]).

To set the stage, we provide a bit of information on Poisson manifolds and Hamiltonian vector fields. Then, using actions generated by complete flows, we move on to an informal introduction to normal form theory and lastly we explain the reasons for studying equilibria with lower order resonances and the 1:3:4 resonance in particular.

1.1 Poisson manifolds

As a reference for Poisson geometry one can consult the lecture notes from the course on Poisson geometry in Utrecht of fall 2017, [4]. Alternatively, for a focus on Hamiltonian dynamics and theory on normal forms and the Hamiltonian Hopf bifurcation, see [12].

Let M denote a smooth manifold. A Poisson bracket on M is a skew-symmetric, $\mathbb{R}$-bilinear binairy operation on the space of smooth functions $\{.,.\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying (for any $f, g, h \in C^\infty(M)$)

(i) the Jacobi identity: $\{\{f, g\}, h\} = \{f, \{g, h\}\} + \{\{f, h\}, g\}$,

(ii) the Leibniz identity: $\{f \cdot g, h\} = f \cdot \{g, h\} + \{f, h\} \cdot g$.

The pair $(M, \{.,.\})$ is called a Poisson manifold. A canonical example is $M = \mathbb{R}^{2n}$ with coordinates $(q_1, ..., q_n, p_1, ..., p_n)$ where we define the canonical Poisson bracket $\{.,.\} = \{.,.\}_{can}$ by

$$\{q_i, p_j\} = \delta_{ij}, \{q_i, q_j\} = 0, \{p_i, p_j\} = 0.$$

We use the skew-symmetry requirement to find $\{p_i, q_j\} = -\delta_{ij}$. From the Leibniz identity and bilinearity we can deduce how the bracket acts on functions, using the following lemma from lecture notes [4] or [12].

Lemma 1.1. *We have for any (local) coordinates $(U, x_1, ..., x_n)$ and smooth functions $f, g \in C^\infty(U)$ that*

$$\{f, g\}\big|_U = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \{x_i, x_j\}. \tag{1.1}$$

Now using the lemma we find that the canonical bracket acts on functions in $C^\infty(\mathbb{R}^{2n})$ as

$$\{f, g\}_{can} = \sum_{i}^{n} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}. \tag{1.2}$$

From the Leibniz identity we can furthermore conclude that if we are given a Poisson bracket on M and any smooth function H, we have an induced derivation X_H defined by $X_H(f) = \{f, H\}$. As we know derivations arise from (smooth) vector fields (see e.g. [17]), we call X_H the *Hamiltonian vector field* of H. Note that the Hamiltonian vector field of H preserves H, as by the skew-symmetry we have

$$X_H(H) = \{H, H\} = -\{H, H\} = 0.$$

It should be noted that the spaces we encounter in this thesis may not be smooth manifolds; some of them will have singular points where the tangent space is not well-defined. They will have Poisson algebras defined on them however, the structure matrix of which drops in rank at the singularities. The singularities are caused by isotropies of the symmetries which are reduced. The reduced spaces we find are semi-algebraic varieties, i.e. a set of solutions of polynomial equations and inequalities. In one degree of freedom the singular points are always equilibria as here the rank of the structure matrix drops to zero; in two degrees of freedom the picture is more intricate.

We encounter this most severely in chapter 5 when we deal with the reduction of the quadratic Hamiltonian H_0^0, which has 15 algebraic invariants. The actual dimension of the phase space is much lower however, due to algebraic relations, called *syzygies*, between them. It is interesting that the singularity at the origin is so degenerate that we have to go to $\mathbb{R}^{15}$ to understand it. We refer to [6] for a rigorous treatment of these reduced spaces and plentiful examples.

1.2 $\mathbb{R}$-actions generated by Hamiltonians

If we take any smooth function F we can consider the flow of the induced Hamiltonian vector field X_F. If this flow is complete, we have an $\mathbb{R}$-action on the manifold M, namely $(t, x) \mapsto \varphi_F^t(x)$. If furthermore the flow is periodic with a fixed period $T > 0$, we have an $\mathbb{S}^1$-action. Before we proceed further let us define what a Casimir is.

Definition 1.2. *A Casimir of $(M, \{.,.\})$ is a function $F \in C^\infty(M)$ such that its bracket with any other function vanishes, i.e.*

$$\{f, F\} = 0 \quad \forall f \in C^\infty(M).$$

We can now prove the following proposition, which we use several times in the next chapter.

Proposition 1.3. *Let F be a smooth function on $(M, \{.,.\})$ whose Hamiltonian flow is complete and consider the resulting $\mathbb{R}$-action on M. We have an induced bracket on the subalgebra of $\mathbb{R}$-invariant functions $C^\infty(M)^{\mathbb{R}}$, and the function F is a Casimir for the induced bracket.*

Proof. We show that for two functions $f, g \in C^\infty(M)^{\mathbb{R}}$, their bracket is also invariant; all required properties are inherited from the original bracket. We know that

$$f \in C^\infty(M)^{\mathbb{R}} \iff f \circ \varphi_F^t = f \iff \{f, F\} \equiv 0 \tag{1.3}$$

where the last equivalence follows by considering the derivative with respect to t. Indeed, we have for any $x \in M$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=s} f \circ \varphi_F^t(x) = d_{\varphi_F^s(x)} f(X_{F,\varphi_F^s(x)}) = X_F(f)(\varphi_F^s(x)) = \{f, F\}(\varphi_F^s(x))$$

and we use the fact that $\varphi_F^0 = \mathrm{Id}_M$. Since the flow is complete, the right hand side is well-defined for all times s. Now by the Jacobi identity, we know that

$$\{\{f,g\},F\} = \{\{f,F\},g\} + \{f,\{g,F\}\} = 0$$

and thus we see that $\{f,g\}$ is indeed invariant under the action. The fact that F is a Casimir on the induced bracket is immediate from (1.3). □

1.3 The quadratic Hamiltonian

Let $M = \mathbb{R}^{2n}$ be equipped with the canonical bracket. We wish to describe dynamics around an elliptic equilibrium of a Hamiltonian vector field, that is a point p in the phase space where the vector field vanishes, and for which the eigenvalues of the linearization of the vector field at p are all on the imaginary axis.

Let $\mathcal{H}$ be an analytic Hamiltonian whose Hamiltonian vector field has an equilibrium. Without loss of generality we take the equilibrium to be at the origin, so 0 is a critical point of the Hamiltonian. As the value of the Hamiltonian at the equilibrium does not influence the dynamics, we choose it to be 0 and can expand $\mathcal{H}$ around 0 as

$$\mathcal{H} = H_0^0 + H_1^0 + H_2^0 + ... \tag{1.4}$$

where H_k^0 is a homogeneous polynomial of degree $k+2$. If the equilibrium is elliptic, and the linearization has no nilpotent parts, we can, through a canonical change of coordinates, bring the quadratic part H_0^0 into the form

$$H_0^0 = \tfrac{1}{2}\sum_{j=1}^{n} \omega_j (q_j^2 + p_j^2), \tag{1.5}$$

where $\pm \mathrm{i}\omega_j$ are the eigenvalues of the equilibrium. We call $\omega := (\omega_1, ..., \omega_n)$ the *frequency vector* (of H_0^0), and its components the frequencies. Note that the relative signs of the frequencies, which depend on the original Hamiltonian, are important.

Terminology on resonances

As the terminology on resonances can get quite confusing, we supply the most relevant definitions here. This will come in handy in chapter 4, where we study the stability of normal modes.

Firstly, if all of the frequencies are non-zero, we say ω is *non-degenerate*. On the other hand, if at least one of the components of ω vanishes, we call it *degenerate*. A *resonance vector* or *annihilator* of ω is a non-zero, integer-valued vector perpendicular to the frequency vector, i.e. $k \in \mathbb{Z}^n \setminus \{0\}$ is a resonance vector of ω if

$$\langle k, \omega \rangle = \sum_{j=1}^{n} k_j \omega_j = 0. \tag{1.6}$$

For the resonance vectors of ω we have the following:

- ⋄ A resonance vector k is called *primitive* if, in case another annihilator k' satisfies $k' = mk$ for $m \in \mathbb{Z}$, then we must have $m = \pm 1$.
- ⋄ We have a norm on the resonance vectors given by $|k| := \sum_{j=1}^{n} |k_j|$.

◇ The *resonance lattice* (of ω) is the $\mathbb{Z}$-span of the resonance vectors, which equals

$$\Gamma_\omega := \{k \in \mathbb{Z}^n : \langle k, \omega \rangle = 0\}. \tag{1.7}$$

If there are exactly $n-1$ linearly independent resonance vectors of $\omega \in \mathbb{R}^n \setminus \{0\}$, then we say ω is *fully resonant*. This is equivalent to the requirement that the resonance lattice of ω has co-dimension 1. (Note that if there are n independent resonance vectors, then $\omega = 0$.) In the fully resonant case we speak of the $\omega_1 : \omega_2 : ... : \omega_n$ resonance.

Lastly, we say the non-degenerate resonance $\omega_1 : \omega_2 : ... : \omega_n$ is *definite* if all the frequencies ω_j have the same symplectic sign in 1.5, and *indefinite* otherwise. Note that the resonance is (in)definite if the quadratic Hamiltonian 1.5 is (in)definite. In chapter 6 we compare the indefinite resonance 1:3:-4 to the definite 1:3:4 resonance.

In case of the 1:3:4 resonance, we see immediately that it is definite. The frequency vector $(1, 3, 4)$ is non-degenerate and fully resonant, as we have e.g. the annihilators $(1, 1, -1)$ and $(3, -1, 0)$, which are both primitive and have respective norms 3 and 4.

The flow of the quadratic Hamiltonian

Consider the flow of the Hamiltonian vector field $X_{H_0^0}$ induced by the quadratic part H_0^0 as above. From the canonical coordinates on $\mathbb{R}^{2n}$ we construct complex coordinates on the new phase space $\mathbb{C}^n$ through $x_j = p_j + \mathrm{i}q_j, y_j = \overline{x_j}$. Then we can deduce that

$$\begin{aligned} \{x_i, y_j\} &= -\mathrm{i}\{p_i, q_j\} + \mathrm{i}\{q_i, p_j\} = 2\mathrm{i}\delta_{ij}, \\ \{x_i, x_j\} &= \{y_i, y_j\} = 0. \end{aligned} \tag{1.8}$$

In these coordinates the quadratic Hamiltonian is given by

$$H_0^0 = \tfrac{1}{2} \sum_{j=1}^{n} \omega_j x_j y_j. \tag{1.9}$$

The vector field $X_{H_0^0}$ takes a rather simple form in these coordinates, namely we have

$$\begin{aligned} \dot{x}_j &= \{x_j, H_0^0\} = \mathrm{i}\omega_j x_j, \\ \dot{y}_j &= \overline{\dot{x}_j} = -\mathrm{i}\omega_j y_j. \end{aligned}$$

For the latter equation we make use of the the fact that the quadratic Hamiltonian (and any other Hamiltonian we encounter in this thesis) is real-valued. We can integrate this easily. We find

$$\begin{aligned} x_j(t) &= e^{\mathrm{i}\omega_j t} x_j(0), \\ y_j(t) &= e^{-\mathrm{i}\omega_j t} y_j(0). \end{aligned}$$

We see that the orbits lie in n-tori parametrized by the value of the *actions* $\tau_j := \frac{1}{2} x_j y_j = \frac{1}{2}(q_j^2 + p_j^2)$ which are conserved quantities of the vector field $X_{H_0^0}$.

For non-degenerate ω, almost all of the phase space $\mathbb{C}^n$ is occupied by the family of these invariant n-tori; in the subspaces defined by setting at least one of the τ_j equal to zero, we find lower dimensional tori. Dropping down to a single $\tau_i \neq 0$, we find a family of periodic orbits, the so called *normal modes*, in this case the i-modes.

As the flow above is complete, it gives rise to an $\mathbb{R}$-action on the phase space $\mathbb{R}^{2n}$, which leaves the orbits of $X_{H_0^0}$ invariant. If the frequency vector is fully resonant then the flow of $X_{H_0^0}$ is periodic, which follows from the following proposition. In that case we have an $\mathbb{S}^1$-action on the phase space.

Proposition 1.4. *Let $\omega = (\omega_1, ..., \omega_n) \in \mathbb{R}^n \setminus \{0\}$ be a (possibly degenerate) frequency vector. Then ω is fully resonant if, and only if, there exists $c > 0$ such that $c\omega \in \mathbb{Z}^n$.*

Proof. First assume there is $c > 0$ such that $c\omega \in \mathbb{Z}^n$. Note that $\Gamma_\omega = \Gamma_{c\omega}$, by the bilinearity of the inner product. Without loss of generality, we can permute the indices of the coordinates such that if there are ω_j equal to 0, they are the first, say m, indices. As our first m resonance vectors we choose $e_1, ..., e_m$. Then the resonance lattice $\Gamma_{c\omega}$ is at least $(n-1)$-dimensional, as it also contains the annihilators ℓ^q, $q \in \{m+1, ..., n-1\}$, defined by

$$\ell^q_j = \begin{cases} c\omega_{q+1} & \text{if } j = q \\ -c\omega_q & \text{if } j = q+1 \\ 0 & \text{else.} \end{cases}$$

As $\omega_j \neq 0$ for $j \in \{m+1, ..., n\}$, these are independent. Thus there are at least $n-1$ independent resonance vectors. On the other hand,

$$\Gamma_{c\omega} \subset \{x \in \mathbb{R}^n : \langle x, c\omega \rangle = 0\} \tag{1.10}$$

and thus the dimension of $\Gamma_{c\omega}$ is at most $n-1$. Thus there are exactly $n-1$ independent resonance vectors and therefore ω is fully resonant.

For the other implication, assume we have exactly $n-1$ independent annihilators, label them k^i, $i \in \{1, ..., n-1\}$. Then the matrix K defined by $(K)_{i,j} = k^i_j$ induces a linear map $\tilde{K} : \mathbb{R}^n \to \mathbb{R}^{n-1}$, which restricts to a linear map $\mathbb{Z}^n \to \mathbb{Z}^{n-1}$, as the coefficients of K are all integers. Note that by independence of the vectors k^i, the matrix K has rank $n-1$ and so the linear subspace $\ker(\tilde{K})$ must have dimension 1. As $\omega \in \ker(\tilde{K})$, we can then write

$$\ker(\tilde{K}) = \{t\omega : t \in \mathbb{R}\}.$$

Observe that $\{t\omega : t \in \mathbb{R}\}$ contains the one-dimensional lattice $\ker(K|_{\mathbb{Z}^n})$. If we take any non-zero element of the one-dimensional lattice, say ℓ, we find c through

$$c = |\langle \ell, \omega \rangle| / \|\omega\|^2 > 0.$$

This completes the proof as $c\omega = \pm\ell \in \mathbb{Z}^n \setminus \{0\}$. □

Note that the existence of such a $c > 0$ (such that $c\omega \in \mathbb{Z}^n$) means that the non-zero components of the frequency vectors have integer-valued ratios. Then we can rescale ω, by c, such that all the frequencies are integers. The following corollary is then an easy consequence.

Corollary 1.5. *If a (possibly degenerate) frequency vector ω is fully resonant, then the flow of the associated quadratic Hamiltonian as given by (1.5), is periodic.*

Proof. We know there is a constant $c > 0$ such that $c\omega \in \mathbb{Z}^n$. Now using the complex coordinates as given by (1.8), we can write the flow acting on the coordinate x_j as $\varphi^t_{H^0_0}(x_j) = e^{i\omega_j t} x_j$, which implies $\varphi^{2\pi c}_{H^0_0}(x_j) = x_j$, as $c\omega_j$ is an integer. By conjugation all sides of the equation, the same holds for y_j. As x_j, y_j fully determine the coordinates q_j, p_j, the flow is periodic with period less than or equal to $2\pi c$. (Note that the flow is stationairy on the set $\{x \in \mathbb{C}^n : x_k = 0 \text{ for } k : \omega_k \neq 0\}$.) □

In particular, we have that in two degrees of freedom a single resonance guarantees the flow to be periodic (assuming the quadratic Hamiltonian has no nilpotent parts). In three degrees of freedom this fails to be true. Indeed, consider the frequency vector $(1, \sqrt{5}, 1+\sqrt{5})$. This has the resonance vector $(1, 1, -1)$, however the flow of the quadratic Hamiltonian is not periodic, only quasi-periodic.

1.4 Normal forms

Recall the expansion

$$\mathcal{H} = H_0^0 + H_1^0 + H_2^0 + ...$$

around an elliptic equilibrium, where again H_j^0 denotes a homogeneous polynomial of degree $j+2$. We say that the Hamiltonian $\mathcal{H}$ is in *normal form up to order* ℓ with respect to H_0^0 if

$$\{H_j^0, H_0^0\} \text{ for } 1 \leq j \leq \ell. \tag{1.11}$$

Generically, Hamiltonians near elliptic equilibria are not in normal form with respect to their quadratic parts. However, using e.g. the theory of Lie series, one can bring the Hamiltonian $\mathcal{H}$ into normal form (up to arbitrairy order) by choosing a series of coordinate transformations[1]. The non-symmetric terms are transformed to higher order.

The normalized (up to order ℓ) version of (1.4), is written as

$$\mathcal{H} \circ \psi_\ell = H_0^0 + H_0^1 + H_0^2 + ... + H_0^\ell + \mathcal{O}(\ell+3) \tag{1.12}$$

where with ψ_ℓ we denote the composition of normalizing canonical transformations, and H_0^k is a homogenous polynomial of degree $k+2$, which Poisson commutes with H_0^0. Note that generically the sequence of compositions of these canonical transformation does not converge. The terms $\mathcal{O}(\ell+3)$ are (usually) not in normal form.

One can then use the *(truncated) normal form of order* ℓ, defined as

$$\bar{H}^\ell := H_0^0 + H_0^1 + ... + H_0^\ell, \tag{1.13}$$

to obtain a polynomial Hamiltonian that functions as an approximation to the original Hamiltonian $\mathcal{H}$, or, by the same token, $\mathcal{H}$ is a perturbation of the normal form $\bar{H}^\ell$. (We refer to the truncated normal form of order 1 (or 2) simply as the *first (or second) normal form.*) The truncated Hamiltonian is invariant with respect to the flow generated by H_0^0. The presence of the additional integral H_0^0 allows to make a better study of the dynamics. In the case that the frequency vector of H_0^0 is fully resonant, we know from corollary 1.5 that the flow $X_{H_0^0}$ is periodic, and in particular we have an $\mathbb{S}^1$-symmetry.

By staying close to the elliptic equilibrium, one can reduce the influence of the higher order terms $\mathcal{O}(\ell+3)$, and so close to the equilibrium the Hamiltonian $\bar{H}^\ell$ approximates the original Hamiltonian $\mathcal{H}$. However, note that it may not be the case that the dynamics of the Hamiltonian in normal form approximate that of the original Hamiltonian.

In two degrees of freedom, Hamiltonians in normal form are integrable. Aside of the Hamiltonian itself, the quadratic part is a second integral. However, integrability is not automatic in three degrees of freedom, as that would require a third integral. The lower order normal forms might have additional integrals, however there is no guarantee that also higher order truncated normal forms have these integrals.

[1] These are chosen to be time-1 flows of Hamiltonian vector fields, so the transformations are canonical.

For the theory of computing normal forms we refer to [11] (in particular appendix C and references therein) and [23] (sections 10.7 and 11 and references therein). For an algorithm in Mathematica to do the (Lie triangle) computations for you, see [7] and its addendum [8].

Terms appearing in the normal form

Let us consider what possible terms can be present in the normal forms with respect to $H_0^0 = \sum_{j=1}^n \omega_j \tau_j$. We firstly note that the operator $m \mapsto \{m, H_0^0\}$ takes a diagonal form in the coordinates x_j, y_j, as

$$\{x_j^{a_j} y_j^{b_j}, H_0^0\} = \omega_j \{x_j^{a_j} y_j^{b_j}, \tau_j\} = \mathrm{i}\omega_j (a_j - b_j) x_j^{a_j} y_j^{b_j}. \tag{1.14}$$

So for a monomial $m = x_1^{a_1} ... x_n^{a_n} y_1^{b_1} ... y_n^{b_n}$ to commute with H_0^0, we must have

$$\{m, H_0^0\} = \left(\mathrm{i} \sum_{j=1}^n (a_j - b_j)\omega_j \right) m = \mathrm{i}\langle a - b, \omega \rangle m. \tag{1.15}$$

In particular, $a - b = (a_1 - b_1, ..., a_n - b_n) \in \mathbb{Z}^n$ must be either zero or an annihilator. Considering the fact that our Hamiltonian $\mathcal{H}$ is real-valued, the only terms that can be in the normal form $\bar{H}^\ell$ are either polynomials in the invariants τ_j – which we call *self-interaction terms*– or terms of the form $\mathrm{Re}(P(\tau)x^k)$, where $P(\tau)$ is a complex polynomial in the invariants τ_j, the integer-valued vector k an annihilator and we use the notation

$$x^k := \prod_{j:k_j>0} x_j^{k_j} \prod_{j:k_j<0} y_j^{-k_j}. \tag{1.16}$$

Terms of the latter form are called *resonant terms*. Here we keep in mind that the total degree of these polynomials may not exceed ℓ.

1.5 Higher order resonances

If the frequency vector ω has only resonance vectors k s.t. $|k| \geq \ell + 2$, then the truncated normal form $\bar{H}^\ell$ consists only of self-interaction terms. Higher order resonances (i.e. with only large annihilators) require higher order normal forms to study the dynamics due to the resonant terms, and the self-interaction terms have a larger influence on the dynamics. In that light we would like to mention a general treatment of higher order resonances by Sanders [22].

In the case that $\bar{H}^\ell$ consists only of self-interaction terms, it is integrable, taking simply the τ_j as integrals. Consider in particular the case that $\ell \geq 4$, then the normal form starts as

$$\bar{H}^\ell = \sum_{j=1}^n \omega_j \tau_j + \sum_{i,j=1}^n c_{ij} \tau_i \tau_j + ... \tag{1.17}$$

whose dynamics close to the origin are very similar to those of the quadratic Hamiltonian H_0^0. The only influence of the quartic terms is a variation in the frequencies. However, this is more important than it may seem at first sight. Indeed, we can show the persistence of a large part of the family of maximal tori for generic values of the constants c_{ij}, following [11]. This is shown using KAM-theory (which we do note elaborate on here but for which we refer to the same book).

The gist of the story is that under certain non-degeneracy conditions (Cantor)-families of maximal tori, those that satisfy *Diophantine conditions* on the frequencies, persist under small perturbations of the Hamiltonian. A (frequency) vector $\tilde{\omega} \in \mathbb{R}^n$ satisfies the (γ, σ)-Diophantine condition, with $\gamma > 0$ and $\sigma > n - 1$, if

$$\forall k \in \mathbb{Z}^n \setminus \{0\} : \big|\langle k, \omega \rangle\big| \geq \frac{\gamma}{|h|^\sigma}. \tag{1.18}$$

The set of all vectors satisfying the (γ, σ)-Diophantine condition goes to full measure if we choose σ large enough and let $\gamma \to 0$. It should be immediate now that the frequencies on the tori of the linear system (that is, of H_0^0) do not satisfy Diophantine conditions, precisely due the presence of resonances.

This is of importance for the normal forms approximations, as the original Hamiltonian $\mathcal{H}$ which was normalized can be seen as such a small perturbation to the truncated normal forms, by staying as close to the origin as needed.

Now to continue our analysis of $\bar{H}^\ell$, take, without loss of generality, the c_{ij} to be symmetric. The frequencies $\tilde{\omega}_j$ on these n-tori are given by Hamilton's equations, in particular

$$\tilde{\omega}_j(\tau) = \frac{\partial \bar{H}^\ell}{\partial \tau_j}(\tau) = \omega_j + 2\sum_{i=1}^{n} c_{ij}\tau_j + ... \tag{1.19}$$

An example of a sufficient non-degeneracy is *Kolmogorov non-degeneracy*, for which we require that the frequency mapping is a local diffeomorphism. By the inverse function theorem, this is satsified in a neighbourhood of the origin in the case that

$$\det\big(D^2\bar{H}^\ell(0)\big) = \det((2c_{ij})_{i,j}) \neq 0. \tag{1.20}$$

Under this (generic) condition close to the origin most of the family of maximal tori with Diophantine frequencies survives small perturbations of the Hamiltonian. Similar results hold true for the lower dimensional tori; for the normal j-mode the Diophantine condition 1.17 becomes $\omega_j \neq 0$.

1.6 Lower order resonances

For resonances with annihilators of length 3 (or 4) we can study the first (or second) normal form, and due to the presence of resonant terms the dynamics can differ largely from the non-resonant case. In particular, the invariants τ_j are no longer integrals and no longer parametrize families of invariant tori.

From proposition 1.4 we know that if we have exactly $n - 1$ independent resonance vectors, the components of the frequency vector have integer relations and the flow of the quadratic Hamiltonian is periodic. In that light, we recall a definition from [23].

Definition 1.6. *If the resonance vectors of the frequency vector $\omega = (\omega_1, ..., \omega_n)$ with norm less than or equal to $\nu + 2$ span a codimension 1 sublattice of $\mathbb{Z}^n$, and $\nu \in \mathbb{N}$ is minimal, then we say that $\omega_1 : ... : \omega_n$ is a* genuine ν-th order resonance.

In particular, by corollary 1.5, genuine resonances of all orders have periodic flows for their quadratic Hamiltonians. We see that the 1:3:4 resonance is a genuine second order resonance in three degrees of freedom. Indeed, we already remarked that we have resonance vectors $(1, 1, -1)$ and $(3, -1, 0)$ of length 3 and 4 respectively.

The significance of the 1:3:4 resonance being of genuine second order is that already in the second normal form we have two resonant terms. While Hamiltonians in normal form with just a single resonant term are still integrable, this is not guaranteed in the case of two resonant terms. In chapter 5 we take a look at the influence of the second resonant term.

A well-known exception to this, however, is the 1:2:2 resonance, see e.g. [1], although the third integral is lost as one passes on to higher order truncated normal forms. For other definite resonances of genuine first order the existence of integrals for the second normal form has been disproven, see e.g. [14] for the 1:2:3 resonance. Moreover, this result has recently even been improved to non-integrability of the first normal form, by Christov [3]. The (non-)existence of integrals is of central importance in the study of the dynamics of resonant equilibria, although there is no conclusive answer to what the non-existence of integrals means in terms of dynamics.

2 Dynamics and integrability of the first normal form of the 1:3:4 resonance

For the 1:3:4 resonance we consider the first order normal form, which contains a single resonant term. We work with the complex coordinates $x_j, y_j = \overline{x_j}$ defined in (1.8), so our phase space is $\mathbb{C}^3$. Note, however, that the Hamiltonians we work with are real valued. Aside from the quadratic part

$$H_0^0(x,y) = \tfrac{1}{2}\big(x_1y_1 + 3x_2y_2 + 4x_3y_3\big) \tag{2.1}$$

we also get monomials of degree three in the normal form $\bar{H}^1 = H_0^0 + H_0^1$. Indeed, one easily calculates that for example $K_1 := x_1x_2y_3$ satisfies

$$\{K_1, H_0^0\} = 0.$$

The same holds for the conjugate $\overline{K_1} = y_1y_2x_3$. In fact, K_1 and $\overline{K_1}$ are the only monomials of degree three in (x,y) commuting with H_0^0. We know this because the only annihilators k of the frequency vector $(1,3,4)$ satisfying $|k| = 3$ are $(1,1,-1)$ and $(-1,-1,1)$. Therefore we obtain as a general expression for the first normal form $\bar{H}^1 = H_0^0 + \mathrm{Re}(AK_1)$, where $A \in \mathbb{C}$.

However, with a change of phase in (x_3, y_3), we can ensure that the constant A is a non-negative (real) number. Indeed, if $A = |A|\, e^{\mathrm{i}\delta}$, then by mapping $(x_3, y_3) \mapsto (e^{\mathrm{i}\delta}x_3, e^{-\mathrm{i}\delta}y_3)$, we find

$$AK_1 = |A|\, e^{i\delta}x_1x_2y_3 \mapsto |A|\, x_1x_2y_3 = |A|\, K_1.$$

Therefore, without loss of generality, a general expression for the first normal form is given by

$$\bar{H}^1 = H_0^0 + A\,\mathrm{Re}(K_1) \tag{2.2}$$

where now $A \geq 0$.

2.1 Existence of a third integral, invariants and syzygies

To understand the dynamics of $\bar{H}^1$, defined by the vector field $X_{\bar{H}^1}(f) = \{f, \bar{H}^1\}$, an important question is the existence of functionally independent integrals. Through our normalization we immediately obtain – aside from the Hamiltonian function itself – the integral H_0^0. This is by design, as in the normalization procedure only those terms, whose bracket which H_0^0 vanishes, remain. Now, let us consider the brackets of $\tau_i = \frac{1}{2}x_iy_i$ with the invariant K_1

$$\begin{aligned}
\{\tau_1, K_1\} &= \tfrac{1}{2}x_1x_2y_3\{y_1, x_1\} = -\mathrm{i}K_1\\
\{\tau_2, K_1\} &= -\mathrm{i}K_1\\
\{\tau_3, K_1\} &= \mathrm{i}K_1.
\end{aligned}$$

The expressions for the conjugate are obtained by conjugating both sides, for example $\{\tau_1, \overline{K_1}\} = \mathrm{i}\overline{K_1}$. From the above and the fact that the τ_i commute, we find that the expressions $T_{12} = \tau_1 - \tau_2, T_{13} = \tau_1 + \tau_3$ and $T_{23} = \tau_2 + \tau_3$ are all conserved under the flow of $\bar{H}^1$. Note that if we have chosen one integral, then the other two can be obtained as a linear combination of the chosen one and H_0^0, which in terms of the τ_i is given by

$$H_0^0 = \tau_1 + 3\tau_2 + 4\tau_3. \tag{2.3}$$

For example, choosing T_{12}, we have

$$T_{13} = \frac{H_0^0 + 3T_{12}}{4}$$
$$T_{23} = \frac{H_0^0 - T_{12}}{4}.$$

We thus have a range of choices of integrals. Instead of untying the Gordian knot of the reduction using the flow of H_0^0 – a much trickier problem, which we tackle in chapter 5 – we use T_{12}, T_{23} and $\bar{H}^1$ as integrals. We start by examining the $\mathbb{S}^1$-action induced by the flow of T_{12}. Indeed, we have the action

$$\begin{aligned} g : \mathbb{S}^1 \times \mathbb{C}^3 &\to \mathbb{C}^3 \\ (s, x_1, x_2, x_3) &\mapsto (e^{\mathrm{i}s} x_1, e^{-\mathrm{i}s} x_2, x_3). \end{aligned} \tag{2.4}$$

We omit here the action on the variables y_i, which can be obtained by conjugation, i.e. $g(s, y) = \overline{g(s, x)}$. The action g has algebraic invariants $\tau_1, \tau_2, \sigma_1 := x_1 x_2, \sigma_2 := y_1 y_2$, and lastly x_3 and y_3. These give a complete description of the subalgebra of smooth g-invariant functions, as the following lemma shows.

Lemma 2.1. *All smooth g-invariant functions can be written in terms of the invariants $\tau_i, \sigma_i, x_3, y_3$.*

Proof. During the proof we ignore the third degree of freedom (x_3, y_3), as it is invariant under the flow. Let $x_1^k y_1^l x_2^m y_2^n$ be an invariant monomial. It must satisfy $k - l = m - n$. In the case that $k \geq l$ we know $m \geq n$ and we can then write

$$\begin{aligned} x_1^k y_1^l x_2^m y_2^n &= (x_1 y_1)^l x_1^{k-l} (x_2 y_2)^n x_2^{m-n} \\ &= 2^{l+n} \tau_1^l \tau_2^n (x_1 x_2)^{k-l} \\ &= 2^{l+n} \tau_1^l \tau_2^n \sigma_1^{k-l}. \end{aligned}$$

In the case that $k \leq l$, we also know $m \leq n$ and we calculate

$$\begin{aligned} x_1^k y_1^l x_2^m y_2^n &= (x_1 y_1)^k y_1^{l-k} (x_2 y_2)^m y_2^{n-m} \\ &= 2^{k+m} \tau_1^k \tau_2^m \sigma_2^{l-k}. \end{aligned}$$

Every g-invariant polynomial in the coordinates (x, y) can be written as a linear combination of these g-invariant monomials, because the action g is diagonal in these coordinates. From polynomials this property extends, by compactness of the group $\mathbb{S}^1$, to all smooth functions, see [18]. □

The integral T_{12} becomes a Casimir on the reduced space, as stated in proposition 1.3, as the algebraic invariants are elements of the space of invariant functions. We also find the syzygy, an algebraic relation, namely $P(\tau, \sigma) = 4\tau_1\tau_2 - \sigma_1\sigma_2 \equiv 0$ and the condition $\tau_i \geq 0$. Before we delve into the dynamics, let us reduce once more, using this time the action generated by the flow of the integral T_{23}. On the coordinates x_i it acts like

$$\begin{aligned} h : \mathbb{S}^1 \times \mathbb{C}^3 &\to \mathbb{C}^3 \\ (t, x_1, x_2, x_3) &\mapsto (x_1, e^{\mathrm{i}t} x_2, e^{\mathrm{i}t} x_3). \end{aligned} \tag{2.5}$$

First we study what h does to the previously found invariants. Apart from $x_3 \mapsto e^{it} x_3, y_3 \mapsto e^{-it} y_3$, we find that $\tau_i \mapsto \tau_i, \sigma_1 \mapsto e^{it}\sigma_1$ and $\sigma_2 \mapsto e^{-it}\sigma_2$. Thus we see that $\tau_j, j = 1, 2, 3$ and $\rho := \sigma_1\sigma_2, K_1 = \sigma_1 y_3, \overline{K_1} = \sigma_2 x_3$ all form new invariants. The question arises whether these suffice.

Lemma 2.2. *Any h-invariant monomial of $\tau_j, \sigma_j, j = 1, 2$ and x_3, y_3, can be written as a product of the invariants τ_i, ρ, K_1 and $\overline{K_1}$.*

Proof. Take any monomial of the form $\sigma_1^k \sigma_2^l x_3^m y_3^n$. If the monomial is h-invariant, we know that $k - l = n - m$. In the case that $k \geq l$, the inequality $n \geq m$ holds as well. Thus we write

$$\begin{aligned} \sigma_1^k \sigma_2^l x_3^m y_3^n &= (\sigma_1 \sigma_2)^l \sigma_1^{k-l} (x_3 y_3)^m y_3^{n-m} \\ &= 2^m \rho^l \tau_3^m K_1^{k-l}. \end{aligned}$$

And again, when $k \leq l$, we get

$$\begin{aligned} \sigma_1^k \sigma_2^l x_3^m y_3^n &= (\sigma_1 \sigma_2)^k \sigma_2^{l-k} (x_3 y_3)^n x_3^{m-n} \\ &= 2^n \rho_1^k \tau_3^n \overline{K_1}^{l-k}. \end{aligned}$$

We need not include τ_1, τ_2, as these are already invariant under h. □

Again from proposition 1.3, we know that T_{23} is a Casimir on the reduced space. Instead of K_1 and its conjugate, we prefer to work with the real valued $u := \mathrm{Re}(K_1)$ and $v := \mathrm{Im}(K_1)$. These simplify the upcoming calculations. As ρ can be identified with $4\tau_1\tau_2$ through the syzygy $P(\tau, \rho) = 4\tau_1\tau_2 - \rho \equiv 0$, we do not consider it separately. We also get a new syzygy $Q(\tau, u, v) = \frac{1}{2}u^2 + \frac{1}{2}v^2 - 4\tau_1\tau_2\tau_3 \equiv 0$.

To sum up: we have coordinates $(\tau_1, \tau_2, \tau_3, u, v)$, satisfying conditions $\tau_i \geq 0$; on the reduced space we have Casimirs T_{12} and T_{23} – the generators of the $\mathbb{S}^1$-actions – and on the reduced space the syzygy $Q(\tau, u, v) \equiv 0$ holds.

2.2 Coordinates and the structure of the phase space

Because we have three independent integrals, we should be able to give a full description of the dynamics. We can write our Hamiltonian as

$$\bar{H}^1(\tau_1, \tau_2, \tau_3, u, v) = T_{12} + 4T_{23} + Au. \tag{2.6}$$

Here we take $A > 0$, as he case $A = 0$ is that of the well-understood quadratic Hamiltonian. In table 1 we have written down the structure matrix of the Poisson algebra.

$\{a, b\}$	τ_1	τ_2	τ_3	u	v	b
τ_1	0	0	0	v	$-u$	
τ_2	0	0	0	v	$-u$	
τ_3	0	0	0	$-v$	u	
u	$-v$	$-v$	v	0	$4(\tau_1\tau_2 - \tau_1\tau_3 - \tau_2\tau_3)$	
v	u	u	$-u$	$-4(\tau_1\tau_2 - \tau_1\tau_3 - \tau_2\tau_3)$	0	
a						

Table 1: The Poisson algebra in the invariants $\tau_1, \tau_2, \tau_3, u$ and v.

Using these expressions we calculate the vector field of $X_{\bar{H}^1}$, which is

$$X_{\bar{H}^1}\begin{pmatrix}\tau_1\\ \tau_2\\ \tau_3\\ u\\ v\end{pmatrix} = A\begin{pmatrix} v\\ v\\ -v\\ 0\\ -4(\tau_1\tau_2-\tau_1\tau_3-\tau_2\tau_3)\end{pmatrix} \tag{2.7}$$

and we have in particular $\dot{u}=0$, which we can also see directly from the form of our Hamiltonian. We are now free to fix the Casimirs $T_{12}=\tau_1-\tau_2=\lambda$, and $T_{23}=\tau_2+\tau_3=\mu$, allowing us to eliminate variables. We find that $\tau_1=\tau_2+\lambda, \tau_3=-\tau_2+\mu$. Let us rename $w:=\tau_2$. We can now focus our attention on the space

$$\mathcal{P}_{\lambda,\mu}=\{(u,v,w)\in\mathbb{R}^3 : \mu\geq w\geq \max(0,-\lambda) \text{ and } Q_{\lambda,\mu}\equiv 0\} \tag{2.8}$$

which we identify with the reduced space. The restriction on w comes from the restrictions $\tau_i\geq 0$. If we let

$$q_{\lambda,\mu}(w):=8w(\lambda+w)(\mu-w), \tag{2.9}$$

then we can define the function $Q_{\mu,\lambda}$, which is the syzygy under the idenfitications we made, by

$$Q_{\lambda,\mu}(u,v,w):=\tfrac{1}{2}\big(u^2+v^2-q_{\lambda,\mu}(w)\big). \tag{2.10}$$

We can write the Poisson bracket for two functions f,g on $\mathcal{P}_{\lambda,\mu}$ as

$$\{f,g\}=\langle\nabla f\times\nabla g,\nabla Q_{\lambda,\mu}\rangle \tag{2.11}$$

where $\nabla=(\partial_u,\partial_v,\partial_w)^T$. For ease we also show the Poisson algebra in (u,v,w) in table 2. In terms of (u,v,w) the vector field of $X_{\bar{H}^1}$ takes the form

$$X_{\bar{H}^1}\begin{pmatrix}u\\ v\\ w\end{pmatrix} = A\begin{pmatrix}0\\ \frac{1}{2}q'_{\lambda,\mu}(w)\\ v\end{pmatrix}. \tag{2.12}$$

$\{a,b\}$	u	v	w	b
u	0	$-\frac{1}{2}q'_{\lambda,\mu}(w)$	$-v$	
v	$\frac{1}{2}q'_{\lambda,\mu}(w)$	0	u	
w	v	$-u$	0	
a				

Table 2: The Poisson algebra in the invariants u,v,w.

2.3 The surface $\mathcal{P}_{\lambda,\mu}$ and isotropies of the torus action

The zero-level set of the Casimir $Q_{\lambda,\mu}$ defines a surface of revolution in $\mathbb{R}^3$; we are interested in the restriction to $\mathbb{R}^2 \times [\max(-\lambda, 0), \mu]$. As in this domain the product $w(\lambda + w)(\mu - w)$ is bounded, we know that the semi-algebraic variety $\mathcal{P}_{\lambda,\mu}$ is compact. The presence of isotropy groups of the torus action

$$\begin{aligned}(g, h) :\ & \mathbb{T}^2 \times \mathbb{C}^3 \to \mathbb{C}^3 \\ & (s, t, x_1, x_2, x_3) \mapsto (e^{\mathrm{i}s} x_1, e^{\mathrm{i}(t-s)} x_2, e^{\mathrm{i}t} x_3)\end{aligned}$$

causes the surface to degenerate. Let us first examine the isotropies.

Proposition 2.3. *Consider the 2-torus action induced by the integrals T_{12} and T_{23} on the phase space $\mathbb{C}^3$. Away from the origin, which has the entire 2-torus as isotropy group, we only find the following non-trivial isotropy groups, isomorphic to $\mathbb{S}^1$:*

$$\begin{aligned}\Gamma_1 &= \{(s, 0) \in \mathbb{T}^2\} \quad \textit{for} \ \ \tau_1 = \tau_2 = 0, \\ \Gamma_2 &= \{(0, s) \in \mathbb{T}^2\} \quad \textit{for} \ \ \tau_2 = \tau_3 = 0, \\ \Delta &= \{(s, s) \in \mathbb{T}^2\} \quad \textit{for} \ \ \tau_1 = \tau_3 = 0.\end{aligned} \tag{2.13}$$

In particular, the presence of isotropy coincides exactly with the normal modes of $\bar{H}^1$.

Proof. Consider the restricted $\mathbb{T}^2$-action

$$(g, h)\big|_{\mathbb{T}^2 \times \{0\} \times \{0\} \times \mathbb{C}}(s, t, 0, 0, x_3) = (0, 0, e^{\mathrm{i}t} x_3).$$

The $\mathbb{S}^1$-action g acts as the identity here and h acts freely, so we encounter the isotropy group $\Gamma_1 \cong \mathbb{S}^1$ here. Restricting to $\mathbb{T}^2 \times \mathbb{C} \times \{0\} \times \{0\}$ one obtains Γ_2. If $\tau_1 = \tau_3 = 0$ the restricted action is

$$(g, h)\big|_{\mathbb{T}^2 \times \{0\} \times \mathbb{C} \times \{0\}}(s, t, 0, x_2, 0) = (0, e^{\mathrm{i}(t-s)} x_2, 0).$$

So for $s = t$, the restricted action is trivial, and we get the isotropy group Δ isomorphic to $\mathbb{S}^1$. At the origin we have $\tau_1 = \tau_2 = \tau_3 = 0$ and then the whole action is trivial. □

The reduced phase space is singular exactly at the normal modes and the origin, which by the proposition, coincides with isotropy of the torus action. Let us summarize the connection.

Only points in the phase space with $\lambda = 0$ have isotropy group Γ_1, but the condition is not sufficient. For the surface $\mathcal{P}_{0,\mu}$ we find a singular equilibrium at the origin, with a homoclinic orbit attached – see figure 1. The origin of the reduced phase space, the singular equilibrium, is exactly identified with the 3-mode, as when $\lambda = 0$ and $w = \tau_2 = 0$, we must have $\tau_1 = \lambda + \tau_2 = 0$.

Setting $\mu = 0$ coincides exactly with the 1-mode. Indeed, when $\mu = 0$, the phase space consists only of the origin, due to the restrictions on the coordinate w. Then the origin in this case corresponds to the 1-normal mode $\tau_1 = \lambda, \tau_2 = \tau_3 = 0$.

When $\mu = -\lambda > 0$, the interval $[\max(0, -\lambda), \mu]$ is now just the point $\{\mu\}$, where $q_{\lambda,\mu}$ vanishes. Then by the syzygy $Q_{\mu,\lambda}$ we must have $u^2 + v^2 = 0$ Thus the phase space is again a single point $\mathcal{P}_{-\mu,\mu} = \{(0, 0, \mu)\}$. This corresponds to the 2-mode, as $\lambda + \mu = \tau_1 + \tau_3 = 0, \tau_2 = \mu > 0$.

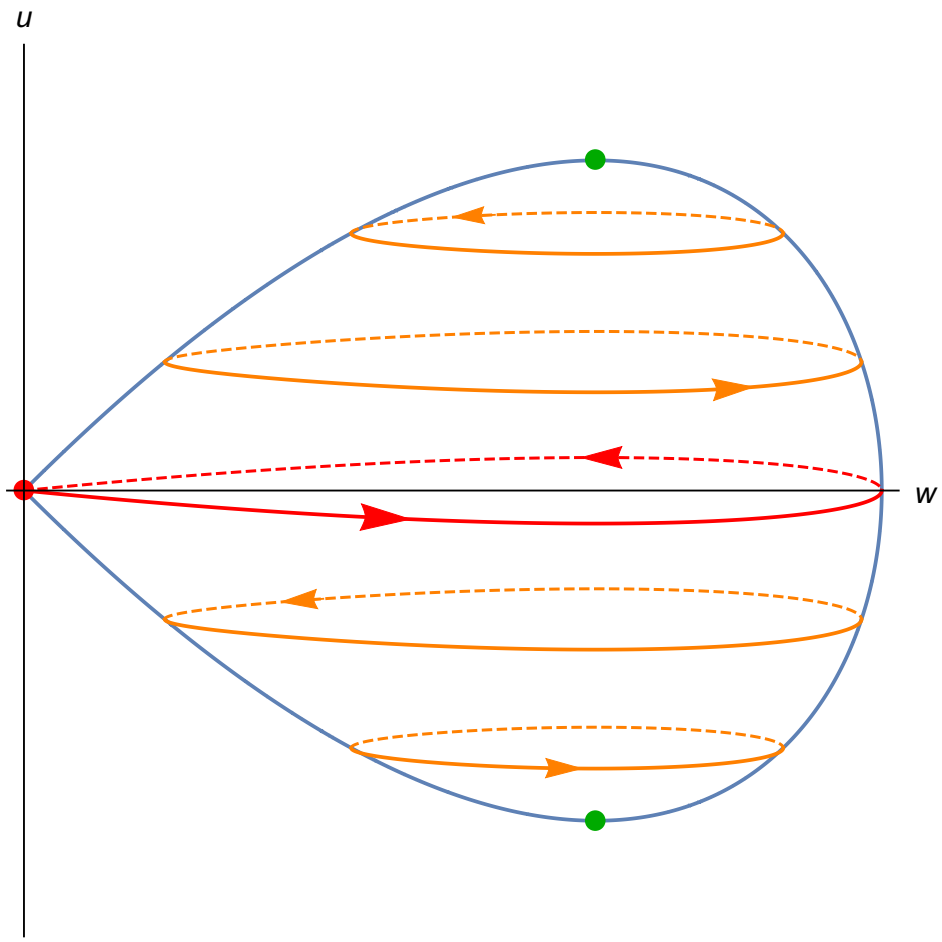

Figure 1: In blue the slice $\mathcal{P}_{\lambda,\mu} \cap \{v = 0\}$, for the values $\lambda = 0, \mu = 1$. The orange curves are periodic orbits in the (v, w)-planes; the red orbit in the $\{u = 0\}$ plane is homoclinic to the singular equilibrium at the origin. The green points at the top and bottom are ellipitic equilibria, where the tangent spaces of $\mathcal{P}_{\lambda,\mu}$ and the energy surface coincide.

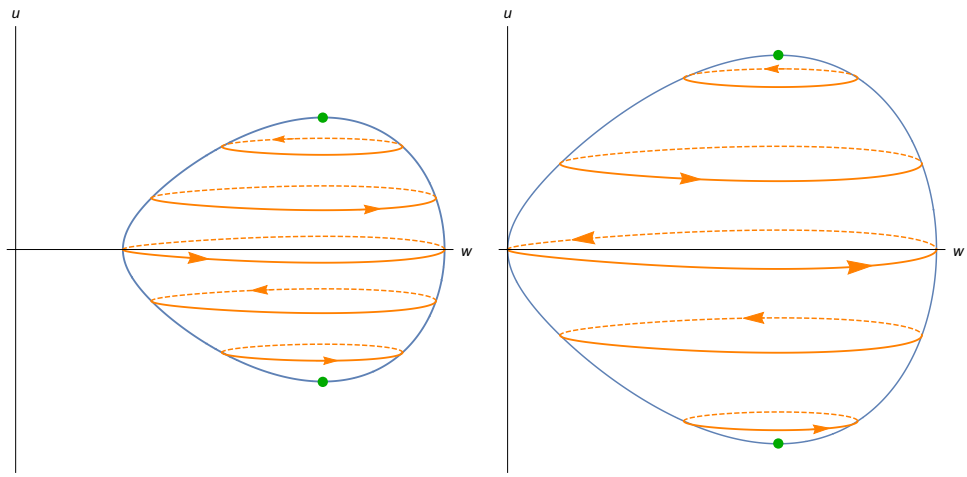

Figure 2: The same pictures as above, but now for the (regular) cases $\lambda = -1/4, \mu = 1$ and $\lambda = 1/4, \mu = 1$. Apart from the elliptic equilibria in green, all orbits (in orange) are periodic.

2.4 Dynamics in the reduced spaces

The energy surfaces $\{\bar{H}^1 = h\}$ are the planes $\{(u, v, w) : u = \frac{h-\lambda-4\mu}{A}\}$ parallel to the (v, w)-plane. The orbits of our system are contained in the intersection of this plane with the phase space $\mathcal{P}_{\lambda,\mu}$, i.e. a slice of our surface. We know this because the orbits of the Hamiltonian vector field $X_{\bar{H}^1}$ stay in the same level set of $\bar{H}^1$.

We note that except in the case of the singular equilibrium, the intersections are in fact the orbits – almost all of them periodic and precisely one for each energy level as we can see in figures 1 and 2.

In the case of the singular equilibrium, the intersection is the union of two orbits: the unstable equilibrium and its homoclinic orbit – the red dot and curve in figure 1. There are also two regular equilibria. Indeed, we note that the vector field of $\bar{H}^1$ is given by

$$X_{\bar{H}^1}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = A\begin{pmatrix} 0 \\ -12w^2 + 8(\mu-\lambda)w + 4\lambda\mu \\ v \end{pmatrix}. \tag{2.14}$$

The vector field vanishes if $v = 0$ and $w = w_\pm = \frac{1}{3}\big(\mu - \lambda \pm \sqrt{\lambda^2 + \lambda\mu + \mu^2}\big)$, which means we have equilibria there. The solution w_+ leads us to the two tangential intersections, which are the green points in figures 1 and 2. Their coordinates are

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \pm\sqrt{8}\sqrt{w_+(\lambda + w_+)(\mu - w_+)} \\ 0 \\ w_+ \end{pmatrix}. \tag{2.15}$$

These two cases happen exactly at the extremal values $u = \pm\sqrt{8}\sqrt{w_+(\lambda + w_+)(\mu - w_+)}$ of u within $\mathcal{P}_{\lambda,\mu}$, and in terms of dynamics both these points are elliptic equilibria, with eigenvalues $\pm i\sqrt{8}A\sqrt[4]{\lambda^2 + \lambda\mu + \mu^2}$.

The solution w_- is only interesting in the case that $\lambda = 0$, and leads to the unstable singular equilibrium at the origin from figure 1. For other values of λ, the solution w_- is negative and thus not a part of the phase space.

2.5 The energy-momentum mapping $\mathcal{EM}$

We are now ready to study the energy-momentum mapping for $\bar{H}^1$,

$$\mathcal{EM} = (\bar{H}^1, T_{12}, T_{23}) : \mathbb{C}^3 \to \mathbb{R}^3. \tag{2.16}$$

The reason we study this mapping is that, as $\bar{H}^1$ commutes with all components of the mapping, the fibers $\mathcal{EM}^{-1}(h, \lambda, \mu)$ are invariant sets of $X_{\bar{H}^1}$.

To calculate the singular values we calculate the critical points, i.e. points in phase space where the differentials $d\bar{H}^1, dT_{12}, dT_{23}$ are linearly dependent (over $\mathbb{R}$). With respect

to the basis $(dx_1, dy_1, ..., dy_3)$ the differentials are

$$d\bar{H}^1 = \frac{1}{2}\begin{pmatrix} y_1 + Ax_2y_3 \\ x_1 + Ay_2x_3 \\ 3y_2 + Ax_1y_3 \\ 3x_2 + Ay_1x_3 \\ 4y_3 + Ay_1y_2 \\ 4x_3 + Ax_1x_2 \end{pmatrix}, \; dT_{12} = \frac{1}{2}\begin{pmatrix} y_1 \\ x_1 \\ -y_2 \\ -x_2 \\ 0 \\ 0 \end{pmatrix}, \; dT_{23} = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ y_2 \\ x_2 \\ y_3 \\ x_3 \end{pmatrix}. \tag{2.17}$$

Firstly we note that the normal modes (which are indeed invariant sets) consist of critical points. Indeed, at the 1-mode the differential of T_{23} vanishes, at the 2-mode we have $dT_{12} = -dT_{23}$ and at the 3-mode the differential of T_{12} vanishes. This covers all possible linear dependencies between these two differentials, aside from the origin where all differentials vanish. Thus any other linear dependencies must involve $d\bar{H}^1$, which we consider now. We have the following proposition.

Proposition 2.4. *Let $z = (z_1, z_2, z_3) \in \mathbb{C}^3$, and consider the energy-momentum mapping $\mathcal{EM}$ as in (2.16). Denote $(h, \lambda, \mu) := \mathcal{EM}(z)$, and $K_1(z) = z_1z_2\overline{z_3}$ and $\tau_j = \frac{1}{2}z_j\overline{z_j}$. The following are equivalent:*

(i) z is a critical point of $\mathcal{EM}$.

(ii) The functions $\mathrm{Im}(K_1)$ and $\tau_1\tau_2 - \tau_1\tau_3 - \tau_2\tau_3$ vanish at z.

(iii) The reduced Hamiltonian (2.6) on $\mathcal{P}_{\lambda,\mu}$ has an equilibrium for energy h.

Proof. The equivalence of (ii) and (iii) follows directly from observing that $\mathrm{Im}(K_1) = \{w, u\}$ and $\tau_1\tau_2 - \tau_1\tau_3 - \tau_2\tau_3 = -\frac{1}{4}\{v, u\}$, and the vector field (4.17) vanishes if and only if both of these functions vanish.

For the equivalence between (i) and (ii) we check different cases. If $z = (0, 0, 0)$ the proposition is trivial. Next, if exactly two of the components z_i vanish, then the orbit of z is a normal mode. We already saw above that in that case z is a critical point. It can be easily checked that (ii) also holds.

On the other hand, if just one component of z vanishes, both (i) and (ii) do not hold. Indeed, for this case linear dependence of the differentials implies that for some $m_1, m_2 \in \mathbb{R}$ we have

$$\begin{pmatrix} (1+m_1)\overline{z_1} + Az_2\overline{z_3} \\ (1+m_1)z_1 + A\overline{z_2}z_3 \\ (3-m_1+m_2)\overline{z_2} + Az_1\overline{z_3} \\ (3-m_1+m_2)z_2 + A\overline{z_1}z_3 \\ (4+m_2)\overline{z_3} + A\overline{z_1z_2} \\ (4+m_2)z_3 + Az_1z_2 \end{pmatrix} = 0. \tag{2.18}$$

Here we scaled away the coefficient of $d\bar{H}^1$, which must be nonzero, as remarked above. Now if we set, for example, p_1 to be zero, we find that $Ap_2\overline{p_3}$ must vanish, which isn't possible as we assumed these are all nonzero. A similar argument holds for $p_2 = 0$ or $p_3 = 0$. Thus there are no linear dependencies in this case, ergo (i) does not hold. On the other hand, precisely one of $\tau_1\tau_2, \tau_2\tau_3, \tau_1\tau_3$ is nonzero, thus (ii) holds neither.

Lastly, we take a look at the case that all components of z are nonzero. First assume that z is a critical point, so there is linear dependence. We can again obtain the previous set of equations. From those equations, by multiplying each line of (2.18) by the conjugate of the second term of the same line, and dividing by A, we obtain the following

$$\begin{pmatrix} (1+m_1)\overline{K_1} + 4A\tau_2\tau_3 \\ (1+m_1)K_1 + 4A\tau_2\tau_3 \\ (3-m_1+m_2)\overline{K_1} + 4A\tau_1\tau_3 \\ (3-m_1+m_2)K_1 + 4A\tau_1\tau_3 \\ (4+m_2)\overline{K_1} + 4A\tau_1\tau_2 \\ (4+m_2)K_1 + 4A\tau_1\tau_2 \end{pmatrix} = 0. \tag{2.19}$$

From this we see right away that $\mathrm{Im}(K_1) = 0$, by substracting the first equation from the second one. (Note that $m_1 \neq -1$ and $m_2 \neq -4$ because $\tau_j \neq 0$ for $j = 1, 2, 3$.) The fact that $\tau_2\tau_3 + \tau_1\tau_3 = \tau_1\tau_2$ follows by adding the first and third equalities and substracting the fifth.

For the other implication, we choose simply

$$m_1 = -1 - A\frac{z_2\overline{z_3}}{\overline{z_1}}, \qquad m_2 = -4 - A\frac{z_1 z_2}{z_3},$$

which immediately solves the first and last equations of (2.18). Note that as $K_1 \in \mathbb{R}$ we have

$$\frac{z_2\overline{z_3}}{\overline{z_1}} = \frac{K_1}{z_1\overline{z_1}} = \frac{\overline{K_1}}{z_1\overline{z_1}} = \frac{\overline{z_2}z_3}{z_1}$$

and

$$\frac{z_1 z_2}{z_3} = \frac{K_1}{z_3\overline{z_3}} = \frac{\overline{K_1}}{z_3\overline{z_3}} = \frac{\overline{z_1 z_2}}{\overline{z_3}}$$

and therefore m_1 and m_2 are both real numbers. We also see that m_1 and m_2 solve the second and second-last equations using the expressions on the right hand side. Plugging in the expressions for m_1 and m_2 in the third equation we obtain

$$A(\overline{z_1}^{-1} z_2\overline{z_2}\overline{z_3} - z_1 z_2\overline{z_2}z_3^{-1} + z_1\overline{z_3} = \frac{4A}{\overline{z_1}z_3}(\tau_2\tau_3 - \tau_1\tau_2 + \tau_1\tau_3) = 0.$$

As the fourth equation of (2.18) is conjugate of the third one, it follows easily that this also this expression vanishes. Thus we have established linear dependence, which completes the last equivalence, and therefore the proof. □

It is an easy calculation now that the set of critical points in $C_p \subset \mathbb{C}^3$ is given by

$$C_p = \{(0,0,z_3) \in \mathbb{C}^3\} \cup \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_3 = \frac{\pm z_1 z_2}{\sqrt{|z_1|^2 + |z_2|^2}}\}. \tag{2.20}$$

(Note that the 1- and 2- modes are included in the second set on the right hand side.) The more important takeaway however is that we can simply look at the equilibria in the reduced spaces to determine all critical values of $\mathcal{EM}$. We already know all the equilibria in the reduced space, which we summarize in table 3 together with the corresponding singular values of $\mathcal{EM}$. In the figure 3 we can see how the singular values lie in $\mathbb{R}^3$.

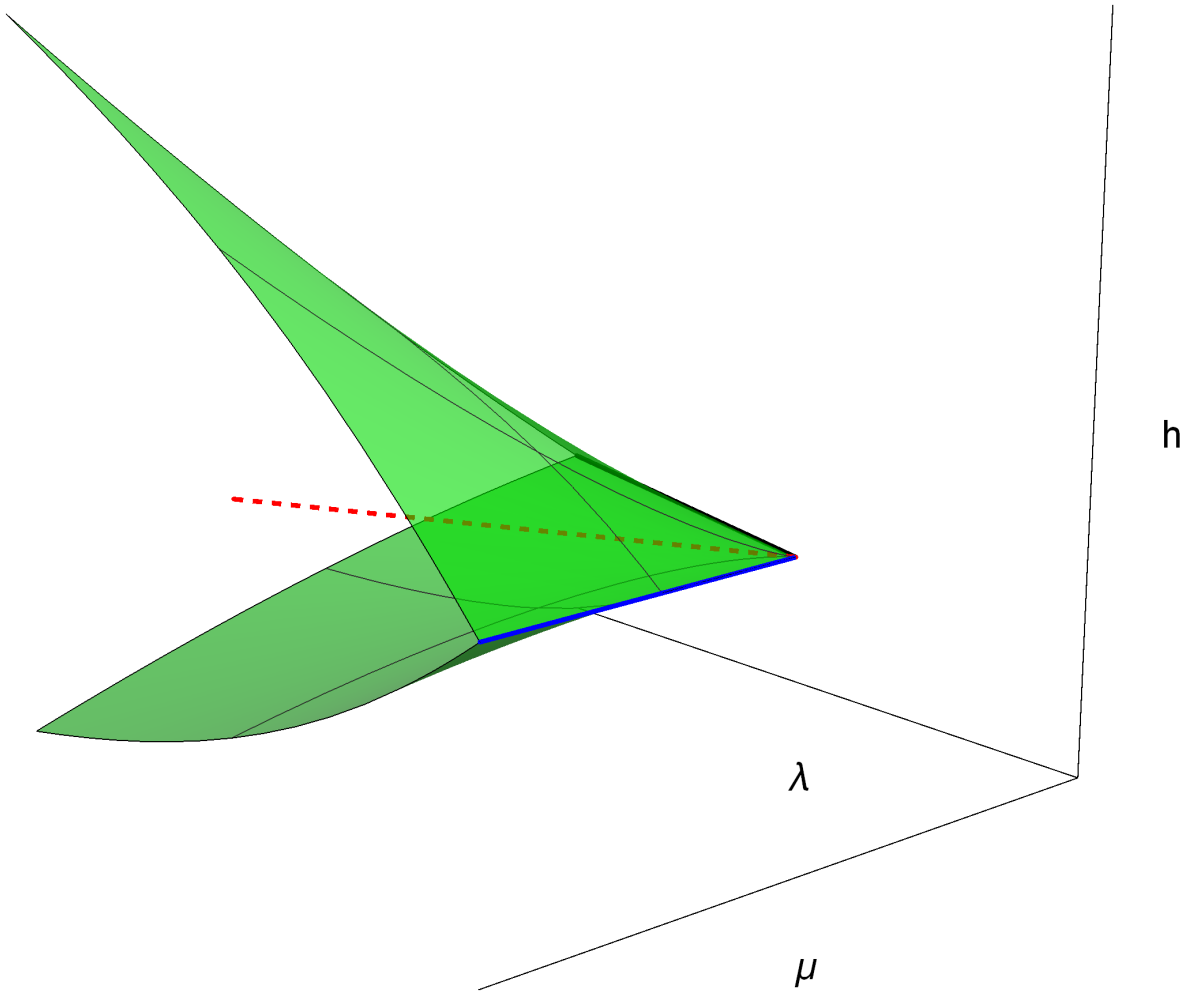

Figure 3: The singular values of the energy-momentum mapping in (h, λ, μ)-space, for $A = 1$. The green surfaces are the image of the regular equilibria of the reduced Hamiltonian. The red dotted line is the image of the 3-mode, and the blue and black lines at the boundary of the surfaces are the images of respectively the 1- and 2-mode. The normal modes meet at the origin. Note that the axes are off for better visibily.

Description	Singular values	Colorcode
origin	$(0, 0, 0)$	-
1-mode	$(h, \lambda, \mu) = (\lambda, \lambda, 0)$ for $\lambda > 0$	black
2-mode	$(h, \lambda, \mu) = (3\mu, \mu, -\mu)$ for $\mu > 0$	blue
3-mode and homoclinic orbits	$(h, \lambda, \mu) = (4\mu, 0, \mu)$ for $\mu > 0$	red
relative equilibria	$(h, \lambda, \mu) = (\lambda + 4\mu \pm A\sqrt{q_{\lambda,\mu}(w_+)}, \lambda, \mu)$ for $0 < \mu \neq -\lambda$	green

Table 3: A list of the singular values of the energy-momentum mapping $\mathcal{EM} = (\bar{H}^1, T_{12}, T_{23})$, together with their color codes in figure 3.

2.6 Fibers of $\mathcal{EM}$ and reconstruction of the dynamics of $\bar{H}^1$

As the fibers of $\mathcal{EM}$ are invariant sets for the flow of $X_{\bar{H}^1}$, they are a useful tool in the study of the dynamics induced by $\bar{H}^1$. The fiber of a regular value of $\mathcal{EM}$ is either empty (if it does not lie in the space enclosed by the surfaces seen in figure 3) or a 3-torus. These 3-tori are foliated by conditionally periodic orbits, which are the superposition of a periodic orbit in the reduced phase space with the motion along the two $\mathbb{S}^1$-orbits of the actions generated by T_{12} and T_{23}.

The frequencies of the motion along the two $\mathbb{S}^1$-orbits of T_{12}, T_{23} are – noting the dependency of u on λ through the syzygy $Q_{\mu,\lambda}(u,v,w)$ – respectively given by

$$\begin{aligned} \beta_\lambda(w) &:= \frac{\partial \bar{H}^1}{\partial \lambda} = 1 + A\frac{\mathrm{d}u}{\mathrm{d}\lambda}, \\ \beta_\mu(w) &:= \frac{\partial \bar{H}^1}{\partial \mu} = 4 + A\frac{\mathrm{d}u}{\mathrm{d}\mu}. \end{aligned} \tag{2.21}$$

The regular equilibria at $(u,v,w) = (\pm\sqrt{q_{\lambda,\mu}(w_+)}, 0, w_+)$ reconstruct to conditionally periodic motion orbits on an invariant 2-torus in $\mathbb{C}^3$, which is the fiber for the energy-momentum mapping above the critical values in green in figure 3. It is periodic if β_λ and β_μ have integer relations, and quasi-periodic if not. An interesting case occurs when $\partial_\mu u = 4\partial_\lambda u$, as then $\beta_\mu(w_+) = 4\beta_\lambda(w_+)$. Below, we show that this happens for a one-parameter family parametrized by the energy of the quadratic Hamiltonian. The orbits close on such a *resonant torus* and form circles which foliate the invariant 2-torus; the motion is periodic.

The periodic orbits in the reduced space reconstruct to conditionally periodic motion on invariant 3-tori in the original phase space $\mathbb{C}^3$, which is the fiber for regular points of $\mathcal{EM}$. The motion is periodic if the three frequencies have integer relations, or alternatively, the frequency vector on that torus is fully resonant.

The intermediate case is when two of the frequencies have integer relations, then the orbits are dense on 2-tori. but not on the maximal 3-torus.

If the frequencies do not satisfy any integer relations, i.e. the frequency vector on that torus has no annihilators, then the orbits on the 3-torus are dense. This is expected to happen for almost all 3-tori. By KAM theory, the family of frequency vectors satisfying Diophantine conditions can survive small perturbations of the dynamics if the frequency mapping is a local diffeomorphism so as to satisfy Kolmogorov non-degeneracy.

The fibers corresponding to the 1- and 2-modes are circles. These fibers are pre-images of the points in boundaries of the (green) surfaces of figure 3, except for the origin, whose fiber is only a point. For the normal modes the orbit of $X_{\bar{H}^1}$ is contained in the orbit of the 2-torus action generated by T_{12} and T_{23}. Lastly, the fibers containing the 3-mode, i.e. those above $(4\mu, 0, \mu)$ for $\mu > 0$, are the product of a pinched 2-torus $\mathcal{T}$ and a 1-torus. The pinch $\{p\} \times \mathbb{S}^1 \subset \mathcal{T} \times \mathbb{S}^1$ is precisely the 3-mode, while the orbits inside the rest of the pinched torus

$$\big(\mathcal{T} \times \mathbb{S}^1\big) \setminus \big(\{p\} \times \mathbb{S}^1\big) \cong \mathbb{R} \times \mathbb{T}^2$$

are the homoclinic orbit superposed with the two $\mathbb{S}^1$-orbits.

2.7 Frequencies on the invariant tori

To calculate the frequencies of the two $\mathbb{S}^1$ orbits of the actions we reduced, we need to find suitable angles ϕ_λ, ϕ_μ, such that

$$\begin{aligned} \{\phi_\lambda, \lambda\} &= \{\phi_\mu, \mu\} = 1, \\ \{\phi_\lambda, \mu\} &= \{\phi_\mu, \lambda\} = 0. \end{aligned} \tag{2.22}$$

Indeed, then we find the expressions of equation 2.21, as we can recover e.g. $\beta_\lambda = \dot{\phi}_\lambda$ as

$$\begin{aligned} \dot{\phi}_\lambda &= \{\phi_\lambda, \bar{H}^1\} = \{\phi_\lambda, \lambda\} + 4\{\phi_\lambda, \mu\} + A\{\phi_\lambda, u\} \\ &= 1 + A\{\phi_\lambda, \lambda\}\frac{\mathrm{d}u}{\mathrm{d}\lambda}. \end{aligned}$$

If we find such angles, then the frequencies for the two regular equilibria, at $w = w_+, v = 0, u = \pm\sqrt{q_{\lambda,\mu}(w_+)}$, can be explicitly calculated. Using the expression $w_+ = \frac{1}{3}\big(\mu - \lambda + \sqrt{\mu^2 + \lambda\mu + \lambda^2}\big)$, we find that

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}\lambda} &= \pm\frac{\partial\sqrt{q_{\lambda,\mu}(w_+)}}{\partial\lambda} \\ &= \pm\frac{q'_{\lambda,\mu}(w_+)\frac{\partial w_+}{\partial\lambda} + 8w_+(\mu - w_+)}{2\sqrt{q_{\lambda,\mu}(w_+)}} \\ &= 4\,\frac{w_+(\mu - w_+)}{u(w_+)} \end{aligned} \tag{2.23}$$

and similarly

$$\frac{\mathrm{d}u}{\mathrm{d}\mu} = 4\,\frac{w_+(\lambda + w_+)}{u(w_+)}. \tag{2.24}$$

Here we use that $q'_{\lambda,\mu}(w_+) = 0$. The requirement

$$\tau_1 = \lambda + w_+ = 4(\mu - w_+) = 4\tau_3, \tag{2.25}$$

together with

$$q'_{\lambda,\mu}(w) = 4(\tau_1\tau_2 - \tau_1\tau_3 + \tau_2\tau_3) = 0, \tag{2.26}$$

implies that $\tau_1 = 3\tau_2$. Working this out gives us that the two families (one for $u > 0$ and one for $u < 0$) satisfy $\mu = 7\lambda/8$. Alternatively, we note that $H_0^0 = \tau_1 + 3\tau_2 + 4\tau_3 = 3\tau_1 = \eta$, so the two families are parametrized by $(\tau_1, \tau_2, \tau_3) = (\eta/3, \eta/9, \eta/12)$.

Amplitude-phase coordinates

To understand the existence and dynamics of these angles, let us consider our Hamiltonian system in terms of *amplitude-phase coordinates*, as suggested in [23]. This boils down to using as coordinates the invariants τ_j and the corresponding angle, defined as $\phi_j := \arg(x_j)$. Indeed, we take

$$\begin{aligned} x_j &= \sqrt{2\tau_j}e^{-\mathrm{i}\phi_j}, \\ y_j &= \sqrt{2\tau_j}e^{\mathrm{i}\phi_j}. \end{aligned}$$

While $w = \tau_2$, the invariants u and v are written as

$$\begin{aligned} u &= \sqrt{8}\sqrt{\tau_1\tau_2\tau_3}\cos(\phi_1+\phi_2-\phi_3), \\ v &= -\sqrt{8}\sqrt{\tau_1\tau_2\tau_3}\sin(\phi_1+\phi_2-\phi_3). \end{aligned}$$

These coordinates are canonical, (i.e. $\{\phi_i, \tau_j\} = \delta_{ij}$ and other brackets vanish,) but are singular when one of the invariants τ_j vanish. For the invariant 2-tori of $\bar{H}^1$, i.e. the regular equilibria in the reduced spaces, this is no problem as we are away from the sets $\{\tau_j = 0\}$.

The Hamiltonian is written in terms of these coordinates as

$$\bar{H}^1(\tau,\phi) = \tau_1 + 3\tau_2 + 4\tau_3 + \sqrt{8}A\sqrt{\tau_1\tau_2\tau_3}\cos(\phi_1+\phi_2-\phi_3) \tag{2.27}$$

and thus we have the dynamics

$$X_{\bar{H}^1}\begin{pmatrix}\tau_1\\ \tau_2\\ \tau_3\\ \phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \begin{pmatrix} -\sqrt{8}A\sqrt{\tau_1\tau_2\tau_3}\sin(\phi_1+\phi_2-\phi_3)\\ -\sqrt{8}A\sqrt{\tau_1\tau_2\tau_3}\sin(\phi_1+\phi_2-\phi_3)\\ \sqrt{8}A\sqrt{\tau_1\tau_2\tau_3}\sin(\phi_1+\phi_2-\phi_3)\\ 1+\sqrt{2}A\sqrt{\frac{\tau_2\tau_3}{\tau_1}}\cos(\phi_1+\phi_2-\phi_3)\\ 3+\sqrt{2}A\sqrt{\frac{\tau_1\tau_3}{\tau_2}}\cos(\phi_1+\phi_2-\phi_3)\\ 4+\sqrt{2}A\sqrt{\frac{\tau_1\tau_2}{\tau_3}}\cos(\phi_1+\phi_2-\phi_3)\end{pmatrix}. \tag{2.28}$$

Notice that in the Hamiltonian and its vector field we only find the *combination angle*

$$\psi_1 := \phi_1+\phi_2-\phi_3 = \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}\cdot\begin{pmatrix}1\\ 1\\ -1\end{pmatrix}, \tag{2.29}$$

where by a combination angle we mean the inner product of $\phi = (\phi_1, \phi_2, \phi_3)$ with an annihilator of the frequency vector $(1, 3, 4)$. We can find the angles ϕ_λ and ϕ_μ as the angles

$$\begin{aligned} \phi_\lambda &= \phi_1 + \frac{\partial w}{\partial \lambda}\psi_1, \\ \phi_\mu &= \phi_3 + \frac{\partial w}{\partial \mu}\psi_1. \end{aligned} \tag{2.30}$$

It is easy to check that these angles satisfy (2.22), while they are consistent with $\{\phi_\lambda, w\} = \frac{\partial w}{\partial \lambda}$ and $\{\phi_\mu, w\} = \frac{\partial w}{\partial \mu}$. The angles ϕ_1 and ϕ_3 together form the combination angle

$$\psi_2 := 4\phi_1 - \phi_3 = \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}\cdot\begin{pmatrix}4\\ 0\\ -1\end{pmatrix}. \tag{2.31}$$

Any other combination angle can be found as a linear combination of ψ_1 and ψ_2, as the resonance lattice $\Gamma_{(1,3,4)}$ is two-dimensional. The dynamics of ψ_1 and ψ_2 under $\bar{H}^1$ are

given by

$$\begin{aligned}\dot{\psi}_1 &= \sqrt{2}A\frac{\tau_1\tau_3+\tau_2\tau_3-\tau_1\tau_2}{\sqrt{\tau_1\tau_2\tau_3}}\cos(\psi_1),\\ \dot{\psi}_2 &= \sqrt{2}A\sqrt{\tau_1\tau_2\tau_3}\left(\frac{4\tau_3-\tau_1}{\tau_1\tau_3}\right)\cos(\psi_1).\end{aligned} \tag{2.32}$$

If we can solve $\dot{\psi}_1 = \dot{\psi}_2 = 0$ for $\sin(\psi_1) = 0$, while not all $\dot{\phi}_j$ vanish, then we have found periodic orbits. Indeed, the time derivatives of the τ_j all vanish and the frequencies $\dot{\phi}_j$ have precisely two resonance vectors, implying the flow there is periodic.

Now firstly we have that, as the invariants τ_j are nonzero,

$$\sin(\psi_1) = 0 \quad \Longleftrightarrow \quad \mathrm{Im}(K_1) = v = 0. \tag{2.33}$$

Then, under the assumption that $\sin(\psi_1) = 0$, and we have

$$\dot{\psi}_1 = 0 \quad \Longleftrightarrow \quad \tau_1\tau_2 = \tau_1\tau_3 + \tau_2\tau_3. \tag{2.34}$$

and also

$$\dot{\psi}_2 = 0 \quad \Longleftrightarrow \quad \tau_1 = 4\tau_3. \tag{2.35}$$

Thus we have again found the requirements $q'_{\lambda,\mu}(w) = 0$ and $v = 0$ for the equilibrium of $X_{\bar{H}^1}$, and we have also recovered the necessary condition (2.25) for the motion on the invariant 2-torus to be periodic.

To translate this back to the frequencies β_λ and β_μ, note that if $\dot{\psi}_1 = 0$, then we have

$$4\beta_\lambda - \beta_\mu = 4\dot{\phi}_1 - \dot{\phi}_3 + \left(4\frac{\partial w}{\partial\lambda} - \frac{\partial w}{\partial\mu}\right)\dot{\psi}_1 = 4\dot{\phi}_1 - \dot{\phi}_3 = \dot{\psi}_2. \tag{2.36}$$

In other words, if $\dot{\psi}_1 = 0$, then $\beta_\mu = 4\beta_\lambda$ is equivalent with $\dot{\psi}_2 = 0$.

We would like to point out that there is an even more degenerate case. The remaining periodic motion can be analysed by considering the time derivatives of any of the angles ϕ_j. We have e.g.

$$\dot{\phi}_1 = 1 \pm \sqrt{2}A\sqrt{\tau_2\tau_3}\tau_1 = 1 \pm \frac{\sqrt{6}}{2}A\sqrt{\tau_3}.$$

If this vanishes the invariant 2-torus is filled entirely with fixed points. This happens for $\tau_3 = 2/(3A^2)$ for the family with $\psi_1 = \pi$. The value of the energy of the quadratic Hamiltonian for this degenerate case is $\eta = 8/A^2$.

3 Detuning the resonance

One might wonder what happens to the dynamics of $\bar{H}^1$ in case the resonance is not exact, but approximate. To study this scenario, we perturb the first part of the frequencies slightly, while retaining the same resonant terms. Here we present new results on the unfolding of the resonance through a detuning parameter.

The so called *detuned* first normal form, has the form

$$\bar{H}^1_\alpha(x,y) = (1+\alpha_1)\tau_1 + (3+\alpha_2)\tau_2 + (4+\alpha_3)\tau_3 + A\,\mathrm{Re}(K_1). \tag{3.1}$$

We note immediately that this Hamiltonian has the same integrals T_{12} and T_{23}, because these are polynomials in the τ_j. Thus after fixing our integrals at λ and μ, we apply the same reduction and get the following Hamiltonian on the reduced space:

$$\bar{H}^1_\alpha(u,v,w) = (1+\alpha_1)\lambda + (4+\alpha_3)\mu + (\alpha_1+\alpha_2-\alpha_3)w + Au.$$

Without loss of generality[2] we can set $\alpha_1 = \alpha_3 = 0$ as constant terms can be absorbed in the Hamiltonian. We continue working with $\alpha := \alpha_2$. This parameter unfolds the resonance (at least for the first normal form, for higher order normal forms we need another parameter). The reduced Hamiltonian then has the form

$$\bar{H}^1_\alpha(u,v,w) = \lambda + 4\mu + \alpha w + Au. \tag{3.2}$$

The detuned energy shells are now slightly tilted versions of the resonant ones:

$$\{\bar{H}^1_\alpha = h\} = \{(u,v,w) \in \mathbb{R}^3 : Au + \alpha w = h - \lambda - 4\mu\}. \tag{3.3}$$

We can see this in figure 4 for the case $\lambda = 0$, which we study in closer detail.

The stabilization of the singular equilibrium

We consider (for now) α to be an external parameter and μ fixed. For small α there is no qualitative change in the dynamics in all cases. However, recall that as $\lambda = 0$ we have a homoclinic orbit attached to the singular equilibrium at the origin. As we increase $|\alpha|$, the energy plane $\{\bar{H}^1_\alpha = h\}$ tilts more and more and one of the elliptic equilibria aproaches the singular equilibrium; when the equilibria meet the homoclinic orbit vanishes, and the singular equilibrium has stabilized.

To find the bifurcating value, we note that when one of the curves

$$u^\pm_{\lambda,\mu}(v,w) = \pm\sqrt{q_{\lambda,\mu}(w) - v^2}$$

satisfies the equation

$$\frac{\partial u^\pm_{0,\mu}(0,w)}{\partial w} = -\frac{\alpha}{A}$$

for $w \neq 0$, then the tangent plane of the phase space $\mathcal{P}_{0,\mu}$ and the energy plane $\{\bar{H}^1_\alpha = h\}$ coincide. In turn, this implies that the vector field $X_{\bar{H}^1_\alpha}$ vanishes there, and we have a regular equilibrium there. Thus for the collision to occur we have the requirement that

$$\lim_{w\downarrow 0} \frac{\partial u^\pm_{0,\mu}(0,w)}{\partial w} = -\frac{\alpha}{A}. \tag{3.4}$$

[2] We note that in the reconstruction this may influence whether the motions are periodic or only quasi-periodic.

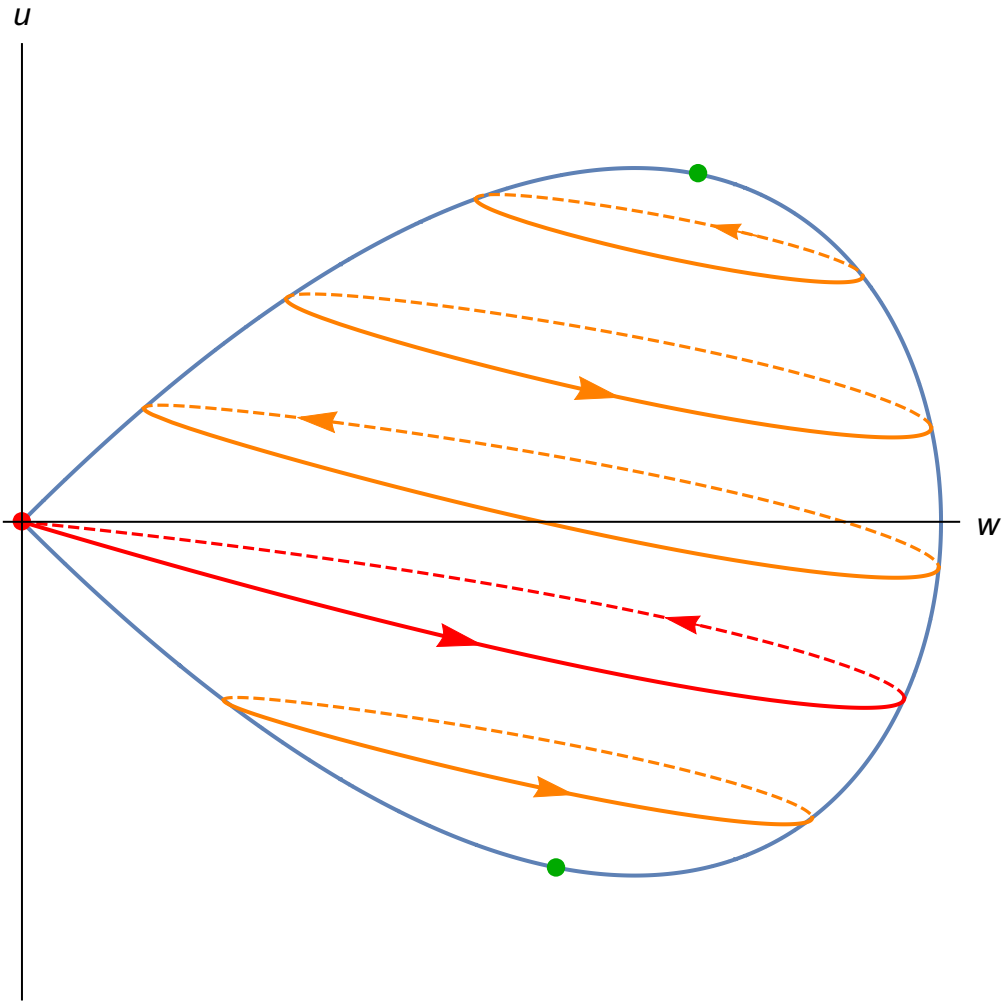

Figure 4: This is the detuned version of figure 1, for the value $\alpha/A = \frac{1}{5}$. The orange curves are the periodic orbits in the $\{\bar{H}^1_\alpha = h\}$-planes; the red curve is homoclinic to the singular equilibrium at the origin. The green points at the top and bottom are still ellipitic equilibria; these are the points where the tangent spaces of $\mathcal{P}_{\lambda,\mu}$ and the energy surface coincide.

Note that this limit makes exists only for $\lambda = 0$, as then $u^{\pm}_{0,\mu}(0, w) = \sqrt{8}|w|\sqrt{\mu - w}$. Solving (3.4) leads to the values $\alpha_- = -\sqrt{8}A\sqrt{\mu}$ for the collsion of the equilibrium with $u > 0$, and $\alpha_+ = \sqrt{8}A\sqrt{\mu}$ for the equilibrium with $u < 0$.

Thus, at $\alpha_\pm = \pm\sqrt{8}A\sqrt{\mu}$, one of the regular equilibria has collided with the singular equilibrium; as we increase $|\alpha|$ the singular equilibrium remains, however it is now stable. The regular equilibrium has disappeared completely. Note that in all cases the singular equilibrium still corresponds to the normal 3-mode $\tau_1 = \tau_2 = 0, \tau_3 = \mu$.

3.1 In two degrees of freedom

To better understand the bifurcation going on here, we take a step back. One may suspect that the stabilisation of the 3-mode happens through a Hamiltonian Hopf bifurcation, due to the conical singularity. Therefore we go back to two degrees of freedom. We first reduce the action of T_{23}, as the 3-normal mode then becomes a (regular) equilibrium. We

recall the action generated by T_{23}, which is given by

$$\begin{aligned} h : \mathbb{S}^1 \times \mathbb{C}^3 &\to \mathbb{C}^3 \\ (t, x_1, x_2, x_3) &\mapsto (x_1, e^{\mathrm{i}t}x_2, e^{\mathrm{i}t}x_3), \end{aligned} \tag{3.5}$$

which has invariants y_2x_3, x_2y_3, and x_1, y_1, τ_2, τ_3. Now instead of these invariants, we prefer this time to work with real coordinates for the normal form computations. We choose

$$\mu := \tau_2 + \tau_3, \qquad \nu := \tau_2 - \tau_3, \qquad \eta := \mathrm{Re}(x_2y_3), \qquad \zeta := \mathrm{Im}(x_2y_3).$$

There is again a syzygy, namely $S_\mu := \nu^2+\eta^2+\zeta^2-\mu^2 \equiv 0$. In particular ν, η, ζ live on the sphere $\mathbb{S}^2_\mu$ of radius μ. Also instead of x_1, y_1 we use the canonical coordinates q_1, p_1, which we recall from the introduction. Let us quickly calculate the Poisson algebra. Firstly

$$\begin{aligned} \{\nu, \eta\} &= \tfrac{1}{4}\{x_2y_2 - x_3y_3, x_2y_3 + y_2x_3\} \\ &= \tfrac{1}{4}\big(x_2y_3\{y_2, x_2\} - x_2y_3\{x_3, y_3\} + y_2x_3\{x_2, y_2\} - x_2y_3\{y_3, x_3\}\big) \\ &= \mathrm{i}(-x_2y_3 + y_2x_3) = 2\zeta \\ \{\zeta, \nu\} &= 2\eta \\ \{\eta, \zeta\} &= 2\nu \end{aligned}$$

or simply

$$\{f, g\} = \langle \nabla f \times \nabla g, \nabla S_\mu \rangle \tag{3.6}$$

with respect to $\nabla = (\partial_\nu, \partial_\eta, \partial_\zeta)$. Lastly μ is a Casimir as it generates the $\mathbb{S}^1$-action h, by proposition 1.3. The Hamiltonian then becomes

$$\bar{H}^1_\alpha(x_1, y_1, \mu, \nu, \eta, \zeta) = \frac{q_1^2 + p_1^2}{2} - \frac{1-\alpha}{2}\nu + \frac{7+\alpha}{2}\mu + A(p_1\eta - q_1\zeta). \tag{3.7}$$

The Hamiltonian vector field of $\bar{H}^1_\alpha$, given by $X_{\bar{H}^1_\alpha} = \{-, \bar{H}^1_\alpha\}$, is

$$X_{\bar{H}^1_\alpha} \begin{pmatrix} q_1 \\ p_1 \\ \mu \\ \nu \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} p_1 + A\eta \\ -q_1 - A\zeta \\ 0 \\ 2A(p_1\zeta - q_1\eta) \\ (1-\alpha)\zeta + 2Aq_1\nu \\ -(1-\alpha)\eta - 2Ap_1\nu \end{pmatrix}. \tag{3.8}$$

The 3-normal mode $\tau_1 = \tau_2 = 0, \tau_3 = \mu$ now corresponds to the point $p = (0, 0, \mu, -\mu, 0, 0)$ as $\eta^2 + \zeta^2 = \mu^2 - \nu^2$, by the syzygy. As expected, this point is an equilibrium for (3.8).

3.2 Recovering the linear centraliser unfolding

The pictures in one degree of freedom indicate that the equilibrium at p should stabilize as we increase α. To study the local behaviour near p, we use cylindrical coordinates on the sphere $\mathbb{S}^2_\mu$. Let

$$\begin{aligned} \kappa &= \eta/\sqrt{2\mu}, \\ \nu &= -\sqrt{\mu^2 - 2\mu\kappa^2}\cos\Big(\sqrt{2}\varphi/\sqrt{\mu}\Big), \\ \zeta &= \sqrt{\mu^2 - 2\mu\kappa^2}\sin\Big(\sqrt{2}\varphi/\sqrt{\mu}\Big), \end{aligned} \tag{3.9}$$

so we obtain $\varphi = \sqrt{\mu/2}\arctan\big(\frac{\zeta}{-\nu}\big)$. Note that these are canonical, in the sense that

$$\begin{aligned}\{\varphi,\kappa\} &= \frac{\sqrt{\mu/2}}{\sqrt{2\mu}}(\{\zeta,\eta\}\frac{-\nu}{\zeta^2+\nu^2} + \{\nu,\eta\}\frac{\zeta}{\zeta^2+\nu^2}) \\ &= \frac{1}{2}\frac{2\nu^2+2\zeta^2}{\zeta^2+\nu^2} = 1.\end{aligned} \tag{3.10}$$

Expanding the expressions for ν and ζ around p (i.e. around $\varphi = \kappa = 0$) as a power series yields the Taylor approximations

$$\begin{aligned}\nu &= -\mu + \varphi^2 + \kappa^2 + \frac{3\kappa^4 - 6\kappa^2\varphi^2 - \varphi^4}{6\mu} + \mathcal{O}(6) \\ \zeta &= \sqrt{2\mu}\varphi - \frac{\sqrt{2}}{3\sqrt{\mu}}(\varphi^3 + 3\kappa^2\varphi) + \mathcal{O}(5)\end{aligned} \tag{3.11}$$

where $\mathcal{O}(n)$ denotes terms in φ and κ (or q_1, p_1) of order n or higher. Using these expressions we expand our Hamiltonian as

$$\begin{aligned}\bar{H}^1_\alpha(q_1,p_1,\varphi,\kappa) = 4\mu &+ \frac{q_1^2+p_1^2}{2} - \frac{1-\alpha}{2}\Big(\varphi^2+\kappa^2+\frac{3\kappa^4-6\kappa^2\varphi^2-\varphi^4}{6\mu}\Big) \\ &+ \sqrt{2}A\sqrt{\mu}\Big(p_1\kappa - q_1\varphi + q_1\frac{\varphi^3+3\kappa^2\varphi}{3\mu}\Big) + \mathcal{O}(6).\end{aligned} \tag{3.12}$$

We find as expected that the quadratic part of this Hamiltonian

$$\tfrac{1}{2}(q_1^2+p_1^2) - \tfrac{1-\alpha}{2}(\varphi^2+\kappa^2) + \sqrt{2}A\sqrt{\mu}(p_1\kappa - q_1\varphi) \tag{3.13}$$

undergoes the Krein collision, i.e. the collision of two pairs of eigenvalues on the imaginairy axis, which split off to form a complex quartet. This becomes apparent after a linear change of coordinates; the new coordinates are given by

$$\begin{pmatrix} c_1 \\ d_1 \\ c_2 \\ d_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} q_1 \\ p_1 \\ \varphi \\ \kappa \end{pmatrix} \tag{3.14}$$

and in particular we have

$$\begin{aligned}\{c_1,d_1\} &= \tfrac{1}{2}(-\{p_1,q_1\} - \{\kappa,\varphi\}) = 1, \\ \{c_2,d_2\} &= \tfrac{1}{2}(\{q_1,p_1\} + \{\varphi,\kappa\}) = 1,\end{aligned}$$

so the transformation is canonical. Note that if we let

$$\begin{aligned}N(c,d) &= \tfrac{1}{2}(c_1^2+c_2^2), \\ M(c,d) &= \tfrac{1}{2}(d_1^2+d_2^2), \\ S(c,d) &= c_1d_2 - d_1c_2,\end{aligned} \tag{3.15}$$

(following the description as in [11], p. 66), then we have

$$\begin{aligned}q_1^2+p_1^2 &= N(c,d) + M(c,d) + S(c,d), \\ \varphi^2+\kappa^2 &= N(c,d) + M(c,d) - S(c,d), \\ p_1\kappa - q_1\varphi &= N(c,d) - M(c,d).\end{aligned} \tag{3.16}$$

These are completed by $P(c,d) = c_1 d_1 + c_2 d_2$ to describe the invariants of the action generated by S. The quadratic Hamiltonian part of the Hamiltonian in (3.12) is written in terms of S, N, M as

$$\tfrac{1}{2}\big((\alpha + \sqrt{8}A\sqrt{\mu})N + (\alpha - \sqrt{8}A\sqrt{\mu})M + (2-\alpha)S\big) \tag{3.17}$$

and we thus recover the bifurcating value $\alpha_+ = \sqrt{8}A\sqrt{\mu}$; the other bifurcation at α_- can be recovered by switching the roles of N and M, as we show below.

3.2.1 Non-degeneracy of higher order terms

Unfortunately, the Hamiltonian truncated from (3.12) is not invariant with respect to

$$S(q_1, p_1, \varphi, \kappa) = \frac{q_1^2 + p_1^2}{2} - \frac{\varphi^2 + \kappa^2}{2}$$

which is the quadratic part of the integral T_{12}, in terms of $(q_1, p_1, \varphi, \kappa)$. To determine the non-degeneracy of the higher order terms of the Hamiltonian Hopf bifurcation we have to normalize with respect to S. The absence of cubic terms does help a lot, however, as does the fact that S is semi-simple.

The first term we encounter, $3\kappa^4 - 6\kappa^2\varphi^2 - \varphi^4$, resides in the image of X_S. Indeed, consider the quartic polynomial $-3\kappa^3\varphi - \kappa\varphi^3$ which is transformed under X_S to

$$\begin{aligned} X_S(-3\kappa^3\varphi - \kappa\varphi^3) &= \tfrac{3}{2}\{\kappa^3\varphi, \varphi^2 + \kappa^2\} + \tfrac{1}{2}\{\kappa\varphi^3, \varphi^2 + \kappa^2\} \\ &= 9\kappa^2\varphi^2\{\kappa, \varphi\} + 3\kappa^4\{\varphi, \kappa\} + \varphi^4\{\kappa, \varphi\} + 3\kappa^2\varphi^2\{\varphi, \kappa\} \\ &= 3\kappa^4 - 6\kappa^2\varphi^2 - \varphi^4 \end{aligned}$$

and in particular, this quartic term has no contribution to the quartic terms of the normalized Hamiltonian. Our hope resides in the next term we encounter: $q_1\varphi^3 + 3q_1\kappa^2\varphi$. First we note that the kernel of X_S is generated by the quadratic polynomials

$$q_1^2 + p_1^2, \quad \varphi^2 + \kappa^2, \quad p_1\kappa - q_1\varphi, \quad q_1\kappa + p_1\varphi$$

and the normalized Hamiltonian might thus contain something of the form

$$A(p_1\kappa - q_1\varphi)(\varphi^2 + \kappa^2) + B(q_1\kappa + p_1\varphi)(\varphi^2 + \kappa^2)$$

for $A, B \in \mathbb{R}$. Solving the homological equation (see [11], appendix C) yields $A = -3/4, B = 0$. In terms of N, M, S, P (as functions of (c,d)) we can write

$$(p_1\kappa - q_1\varphi)(\varphi^2 + \kappa^2) = N^2 - M^2 - NS + MS.$$

Thus after the normalization with respect to S, the Hamiltonian (3.12) becomes

$$\begin{aligned} \mathcal{H}_\alpha(S, N, M, P) = 4\mu &+ \tfrac{1}{2}\big((\alpha + \sqrt{8}A\sqrt{\mu})N + (\alpha - \sqrt{8}A\sqrt{\mu})M + (2-\alpha)S\big) \\ &+ \frac{\sqrt{2}A}{4\sqrt{\mu}}(M^2 - N^2 + NS - MS) + \mathcal{O}(3) \end{aligned} \tag{3.18}$$

(where we use $\mathcal{O}(n)$ for terms of order n or higher but now for the coordinates N, M, S, P). Formally, we still have to normalize with respect to N. This allows us to drop the terms N^2 and NS, and we are left with

$$\begin{aligned} \bar{\mathcal{H}}_\alpha(S, N, M, P) = 4\mu &+ \tfrac{1}{2}\big((\alpha + \sqrt{8}A\sqrt{\mu})N + (\alpha - \sqrt{8}A\sqrt{\mu})M + (2-\alpha)S\big) \\ &+ \frac{\sqrt{2}A}{4\sqrt{\mu}}(M^2 - MS) + \mathcal{O}(3). \end{aligned} \tag{3.19}$$

At $\alpha_+ = \sqrt{8}A\sqrt{\mu}$ the coefficients of N and that of M^2 are both positive. This allows us to conclude that this Hamiltonian Hopf bifurcation is supercritical (cf. def. 2.24 of [11]).

To determine the non-degeneracy of higher order terms for the other bifurcation, which occurs at $\alpha_- = -\alpha_+$, we can simply interchange the roles of N and M in equation (3.18). This is achieved by interchanging the roles of c and d. Indeed, let

$$(\tilde{c}_1, \tilde{d}_1, \tilde{c}_2, \tilde{d}_2) := (d_1, -c_1, d_2, -c_2). \tag{3.20}$$

Note that this transformation is canonical. Then we have

$$S(\tilde{c}, \tilde{d}) = S(c, d), \qquad M(\tilde{c}, \tilde{d}) = N(c, d), \qquad N(\tilde{c}, \tilde{d}) = M(c, d).$$

In particular, the Hamiltonian (3.18) is then

$$\begin{aligned} \widetilde{\mathcal{H}}_\alpha(S, N, M, P) = 4\mu &+ \tfrac{1}{2}\big((\alpha - \sqrt{8}A\sqrt{\mu})N + (\alpha + \sqrt{8}A\sqrt{\mu})M + (2-\alpha)S\big) \\ &+ \frac{\sqrt{2}A}{4\sqrt{\mu}}(N^2 - M^2 + MS - NS) + \mathcal{O}(3). \end{aligned} \tag{3.21}$$

Now a Krein collision occurs at α_-. We can conclude that this bifurcation is supercritical as well, as the coefficient of N is now negative, which also holds for the coefficient of M^2. We may conclude that the normal mode destabilizes through a Hamiltonian Hopf bifurcation as we increase the value of the integral μ, or alternatively, stabilizes as we increase the detuning parameter α.

We remark that if $\alpha_+ = 2$, the coefficient of S vanishes, which is problematic for the normal form approximation. This happens for $\mu = 1/(2A^2)$.

3.3 The detuned energy-momentum mapping

It is interesting to see what happens to the energy-momentum mapping as we shift the (external) parameter α and pass one of the bifurcation values $\alpha_\pm$. Alternatively we can trigger the bifurcation by fixing α and increasing the (internal) parameter μ. In any case, consider the 1-parameter family of energy-momentum maps defined by

$$\mathcal{EM}_\alpha = (\bar{H}^1_\alpha, T_{12}, T_{23}). \tag{3.22}$$

The analysis here is similar to the one we saw in section 2.5. We have the following generalization of proposition 2.4.

Proposition 3.1. *Let $z = (z_1, z_2, z_3) \in \mathbb{C}^3$, and consider the 1-parameter family of energy-momentum mappings $(\mathcal{EM}_\alpha)_\alpha$. Let $\mathcal{EM}_\alpha(z) = (h_\alpha, \lambda_\alpha, \mu_\alpha)$. The following are equivalent:*

(i) z is a critical point of $\mathcal{EM}_\alpha$.

(ii) The functions $\operatorname{Im}(K_1)$ and $\alpha \operatorname{Re}(K_1) - 4A(\tau_1\tau_2 - \tau_1\tau_3 - \tau_2\tau_3)$ vanish at z.

(iii) The reduced detuned Hamiltonian (3.2) on $\mathcal{P}_{\lambda_\alpha,\mu_\alpha}$ has an equilibrium for energy h_α.

The proof of the above is a very slight modification of the proof of proposition 2.4, and we omit it.

We note that we can describe the set of critical points $C_{p,\alpha}$ as

$$C_{p,\alpha} = \{(0,0,z_3) \in \mathbb{C}^3\} \cup \{(z_1,z_2,z_3) \in \mathbb{C}^3 : z_3 = \frac{z_1 z_2(-\frac{a}{8A} \pm \sqrt{\frac{a^2}{64A^2} + |z_1|^2 + |z_2|^2}}{|z_1|^2 + |z_2|^2}\}. \tag{3.23}$$

The set of critical values we are looking for is $\mathcal{EM}_\alpha(C_{p,\alpha})$. As before, it constists of the images of the origin, normal modes and the relative equilibria. One of the regular stable equilibria becomes singular and coincides with the 3-mode as we pass through one of the bifurcation values $\alpha_\pm$, but only for the value $\lambda = 0$.

To plot these figures, we use condition (ii) of 3.1 to obtain the equation (in terms of (u,v,w))

$$\alpha^2 u^2 = \frac{A^2}{4}\left(q'_{\lambda,\mu}(w)\right)^2 \tag{3.24}$$

and thus using the syzygy $q_{\lambda,\mu}(w) \equiv u^2 + v^2$ and the requirement that $v = 0$, we obtain the quartic equation

$$\begin{aligned} 18A^2w^4 + (12A^2(\mu-\lambda) + \alpha^2)w^3 + (8A^2(\mu^2 - \tfrac{1}{2}\lambda\mu + \lambda^2) \\ - \alpha^2(\mu-\lambda))w^2 + \lambda\mu(8A^2(\mu-\lambda) - \alpha^2)w + 2A^2\lambda^2\mu^2 = 0. \end{aligned} \tag{3.25}$$

The two solutions $w_j = w_j(\alpha,\lambda,\mu)$ which we are interested in (for which we choose $w_1 \leq w_2$) are precisely those that satisfy

$$w_j(0,\lambda,\mu) = \tfrac{1}{3}\left(\mu - \lambda + \sqrt{\mu^2 + \lambda\mu + \lambda^2}\right) = w_+$$

so in particular $w_j(0,\lambda,\mu)$ equals the solution w_+ of section 2.4. If we set

$$u_j = (-1)^{j+1}\,\mathrm{sign}(\alpha)\sqrt{8w_j(\lambda + w_j)(\mu - w_j)}$$

we recover the signs we lost through squaring, as now we find the correct energy levels to be

$$h_j(\alpha,\lambda,\mu) = \lambda + 4\mu + Au_j(\alpha,\lambda,\mu) + \alpha w_j(\alpha,\lambda,\mu).$$

For $\alpha = \pm 8$, the set of critical values in (h,λ,μ)-space is depicted in figure 5. Another interesting point of view is to fix μ, and vary the parameter α. This is depicted in figure 6. In both figures we can recognize parts of the swallowtail surface, characteristic for the Hamiltonian Hopf bifurcation. In particular the surface is not smooth (at the 3-mode) before the bifurcation, but smoothens as the 3-mode leaves the surface as a single thread. Compare to [11], fig. 2.14 (a).

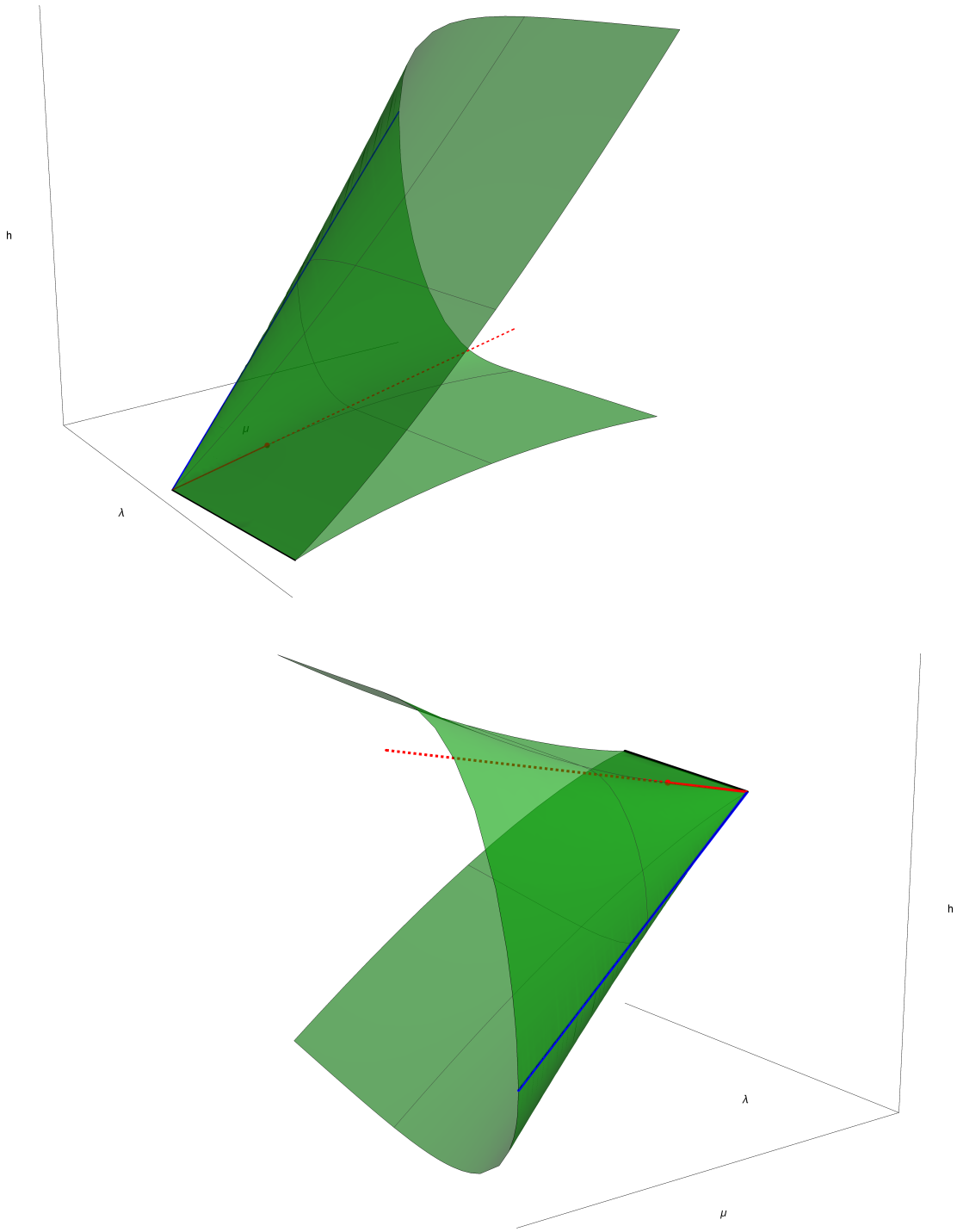

Figure 5: The singular values of the detuned energy-momentum mapping, for $A = 1, \alpha = 8$ (upper) and $\alpha = -8$ (lower). The green surfaces are the images of the regular equilibria of the reduced detuned Hamiltonian. The red line is the image of the 3-mode, dashed if it is not part of one of the surfaces. The red dot denotes the point of separation from the surface. The black and blue lines at the boundary of the surfaces are the images of respectively the 1- and 2-mode and all normal modes meet at the origin. The axes start away from the origin for visibility.

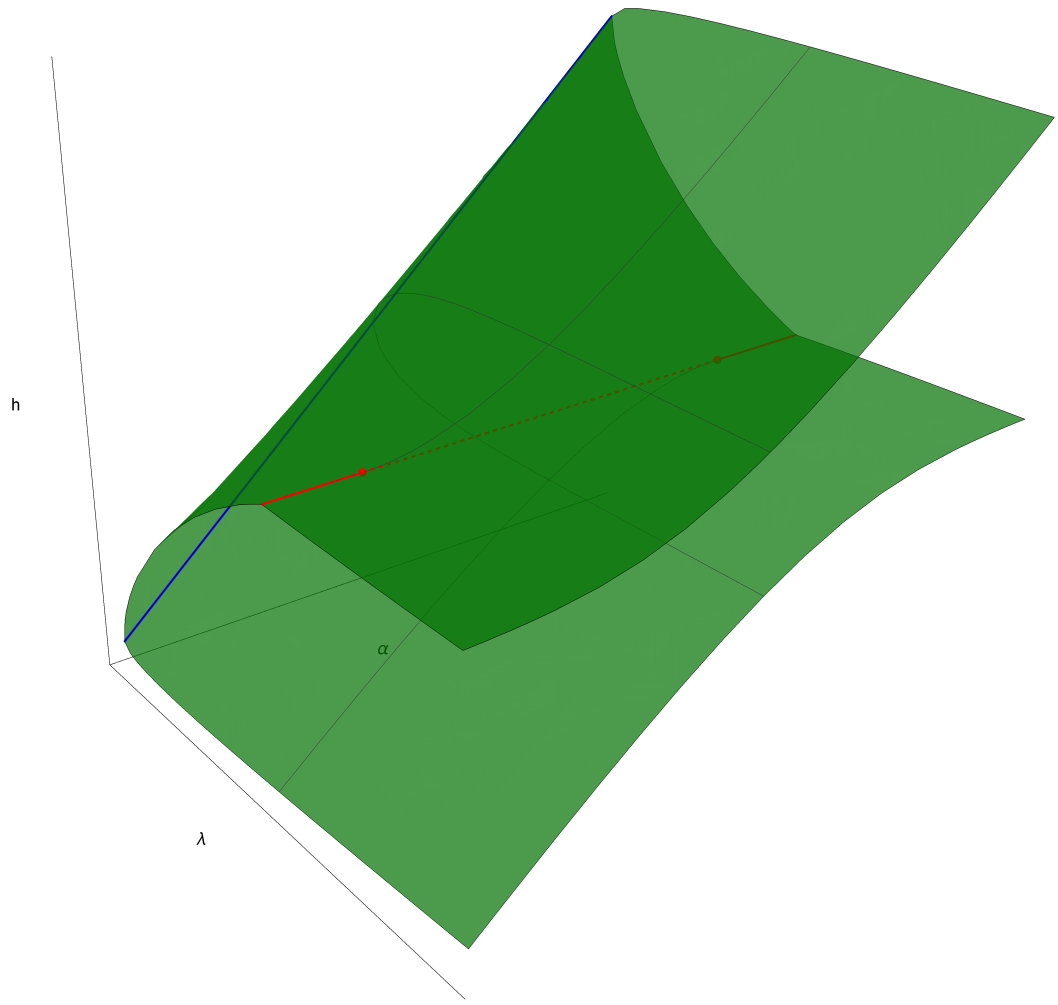

Figure 6: The singular values of the detuned energy-momentum mapping, for $A = 1, \mu = 18$, treating α as a parameter. The green surface are the images of the regular equilibria of the reduced detuned Hamiltonian. The red line is the image of the 3-mode, dashed if it is not part of one of the surfaces. The red dots denote the point of separation from the surface. The blue line, which forms the intersection of the boundaries of the two surfaces, is the image of the 2-mode (for $\lambda = -\mu = -18$).

Reconstruction of the dynamics of $\bar{H}^1_\alpha$

The reconstruction of the dynamics is generally the same as that for $\bar{H}^1$, although the frequencies are perturbed sligthly. As in section 2.7 we compute that the frequencies of respectively the $\mathbb{S}^1$-orbits of T_{12} and of T_{23} are

$$\beta_\lambda = 1 + A\frac{\mathrm{d}u}{\mathrm{d}\lambda} + \alpha\frac{\partial w}{\partial \lambda} \tag{3.26}$$

and

$$\beta_\mu = 4 + A\frac{\mathrm{d}u}{\mathrm{d}\mu} + \alpha\frac{\partial w}{\partial \mu}. \tag{3.27}$$

The family of periodic orbits we found for $\bar{H}^1$, remain intact, for $\tau_1 = 4\tau_3$. This is easy to see if we recall the amplitude-phase coordinates. We have that

$$4\beta_\lambda - \beta_\mu = \dot{\psi}_2 + \left(4\frac{\partial w}{\partial \lambda} - \frac{\partial w}{\partial \mu}\right)\dot{\psi}_1 \tag{3.28}$$

and the equilibrium in the reduced space occurs only if $\dot{\psi}_1 = 0$. The requirement $\tau_1 = 4\tau_3$ then implies that $\dot{\psi}_2 = 0$, so $4\beta_\lambda = \beta_\mu$ and the orbit closes

We note that for μ below the value of $\frac{\alpha^2}{8A^2}$, the 3-mode is stable and the fiber of $\mathcal{EM}_\alpha$ above these points is not a pinched torus, but simply a 2-torus. As we increase μ the 3-mode is destabilized through a supercritical periodic Hamiltonian Hopf bifurcation, and separates from the surface as a single thread.

3.4 The self-interaction terms

Before we study the second normal form in chapter 5, we look at the effect of adding only the self-interaction terms of fourth order. Their influence on the dynamics is not unlike the detuning, but more phenomena can occur. At this stage we have the benefit that the integrals T_{12} and T_{23} are not lost as we add these fourth order terms. For a general study of these kind of Hamiltonians and their reduction to one degree of freedom, see Kummer [15].

We follow a similar strategy in the analysis of the dynamics, although Kummer did not consider the case when the phase space had singularities, which we give special consideration. We add a quadratic polynomial in the τ_j's (so quartic in (x, y)) to the detuned Hamiltonian $\bar{H}^1_\alpha$, to obtain the Hamiltonian

$$L_\alpha := \bar{H}^1_\alpha + \sum_{j=1}^{n} c_{ij}\tau_i\tau_j. \tag{3.29}$$

Without loss of generality we take the matrix $(c_{ij})_{i,j}$ to be symmetric. Because the τ_i commute we know that T_{12} and T_{23} are integrals of L_α, and we can still work in the reduced phase space $\mathcal{P}_{\lambda,\mu}$, with invariants $u = \mathrm{Re}(K_1), v = \mathrm{Im}(K_1), w = \tau_2$. The reduced Hamiltonian is

$$\begin{aligned} L_\alpha(u, v, w) = \lambda + 4\mu + c_{11}\lambda^2 + 2c_{13}\lambda\mu + c_{33}\mu^2 \\ + Au + (\lambda C_2 + \mu C_3 + \alpha)w + C_1 w^2 \end{aligned} \tag{3.30}$$

where the constants C_i are defined as

$$\begin{aligned} C_1 &:= c_{11} + c_{22} + c_{33} + 2(c_{12} - c_{13} - c_{23}), \\ C_2 &:= 2(c_{11} + c_{12} - c_{13}), \\ C_3 &:= 2(c_{13} + c_{23} - c_{33}). \end{aligned} \tag{3.31}$$

We note that, much like the constant $A > 0$, the constants c_{ij} and therefore C_i depend only on the original Hamiltonian which is normalized, and the constants should not be treated as internal parameters. However, they could be seen as external parameters, for example in the case that we are reducing a parameter-dependent family of Hamiltonians.

Using the Poisson algebra for (u, v, w) as denoted in table 2, we can calculate the Hamiltonian vector field X_{L_α} on $\mathcal{P}_{\lambda,\mu}$. The vector field is given by

$$X_{L_\alpha}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -(2C_1 w + \lambda C_2 + \mu C_3 + \alpha)v \\ (2C_1 w + \lambda C_2 + \mu C_3 + \alpha)u - 4A(3w^2 - 2(\mu - \lambda)w - \lambda\mu) \\ Av \end{pmatrix}. \tag{3.32}$$

If we introduce the notation

$$C_{\lambda,\mu,\alpha} := \lambda C_2 + \mu C_3 + \alpha, \tag{3.33}$$

and recall the definition

$$q_{\lambda,\mu}(w) := 8w(\lambda + w)(\mu - w),$$

we can write this more compactly as

$$X_{L_\alpha}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -(2C_1 w + C_{\lambda,\mu,\alpha})v \\ (2C_1 w + C_{\lambda,\mu,\alpha})u + \frac{1}{2}Aq'_{\lambda,\mu}(w) \\ Av \end{pmatrix}. \tag{3.34}$$

Naturally we are interested in the critical points of this vector field. Firstly for $\dot{w} = 0$ we must have $v = 0$. This equality also gives us the equality $\dot{u} = 0$. Lastly solving $\dot{v} = 0$ is, through the syzygy $u^2 + v^2 = q_{\lambda,\mu}(w)$, equivalent to solving the following equation in w:

$$2C_1 w + C_{\lambda,\mu,\alpha} = \mp A\frac{q'_{\lambda,\mu}(w)}{2\sqrt{q_{\lambda,\mu}(w)}} = -A\frac{\mathrm{d}u}{\mathrm{d}w}\Big|_{v=0}(w) \tag{3.35}$$

which leads to the quintic equation

$$q_{\lambda,\mu}(w)(2C_1 w + C_{\lambda,\mu,\alpha})^2 = \frac{A^2}{4}(q'_{\lambda,\mu}(w))^2. \tag{3.36}$$

The number of viable (that is, "on the phase space", so $\mu \geq w \geq \max(0, -\lambda)$) solutions to this equations might vary to up to four; in the non-singular case one of the solutions is always "outside the phase space". Instead of looking at this equation directly we resort to considering the geometry at hand.

We can understand the dynamics qualitatively by considering the intersection of the reduced phase space $\mathcal{P}_{\lambda,\mu}$ with the energy surfaces $L_\alpha^{-1}(h)$. The level sets of L_α are parabolic cylinders: parabolas in the (w, u)-plane, and constant along the v-axis. Indeed, we have

$$L_\alpha^{-1}(h) = \{(u, v, w) : Au = -C_1 w^2 - C_{\lambda,\mu,\alpha} w + h - \lambda - 4\mu - c_{11}\lambda^2 - 2c_{13}\lambda\mu - c_{33}\mu^2\}. \tag{3.37}$$

Because a critical point of X_{L_α} requires $v = 0$, we ignore the v-direction and focus on the intersection of the parabolas in the (w, u)-plane with the curves $\{u^2 = q_{\lambda,\mu}(w)\}$.

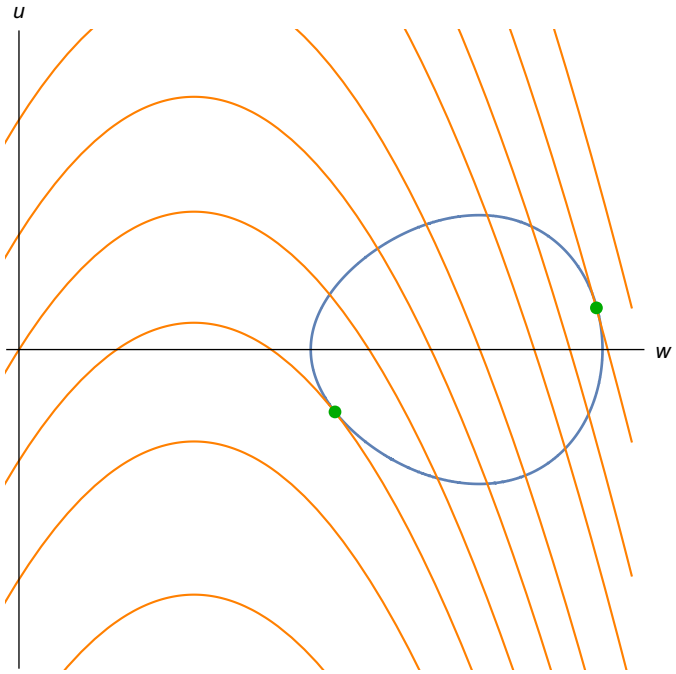

Figure 7: The ntersections of the energy surfaces $L_\alpha^{-1}(h)$ (in orange) and the phase space $\mathcal{P}_{\lambda,\mu}$ (in blue), plotted for $(\lambda, \mu, \alpha, C_1, C_2, C_3) = (-1, 2, -3, 5, -3, -3)$, in the (w, u)-plane. The green dots indicate the location of the tangent intersections (i.e. the elliptic equilibria).

3.4.1 The regular case

In the case that $\mu > -\lambda \neq 0$ the surface has no singularities. Qualitatively, we can distinguish four cases for the relative positions of the level sets of L_α to the reduced phase space $\mathcal{P}_{\lambda,\mu}$.

Firstly there is the case that out of all the intersections of $\mathcal{P}_{\lambda,\mu}$ and the level sets $L_\alpha^{-1}(h)$, only two are tangential. An example of this case is depicted in figure 7. We know that at a tangential intersection there is a critical point of X_{L_α}, as the w-coordinate of the critical points must be a solution to equation (3.36).

In terms of dynamics, this case is equivalent to the one of $\bar{H}_1$ for $\mu > -\lambda \neq 0$, with two elliptic equilibria and the rest of the reduced phase space $\mathcal{P}_{\lambda,\mu}$ filled with periodic orbits.

If λ and μ are such that

$$\frac{-C_{\lambda,\mu,\alpha}}{2C_1} \in (\max(-\lambda, 0), \mu) \tag{3.38}$$

then the w-coordinate of the peak of the parabolas lies "on the phase space". Then depending on the slope of the parabolas (i.e. on the relation of λ and μ with the constants C_i and α), the parabolas could touch from the inside. This second case is depicted in figure 8. We see that there are two homoclinic orbits attached to the equilibrium colored red and therefore it is unstable; it is in fact hyperbolic. The equilbrium and its homoclinic orbits divide the reduced phase space $\mathcal{P}_{\lambda,\mu}$ in three parts, each containing an elliptic equilibrium.

Note that we only vary λ between figures 7 and 8; the values of the constants and of $\mu = 2$ are the same. Under variation of the value λ of the integral T_{12}, we see

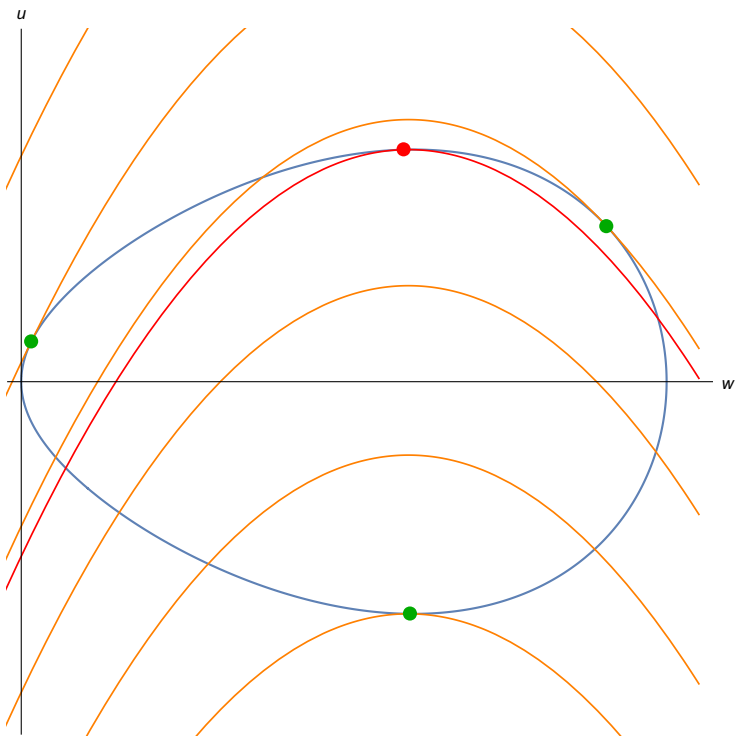

Figure 8: The intersections of energy surfaces $L_\alpha^{-1}(h)$ (orange and red) with the reduced phase space $\mathcal{P}_{\lambda,\mu}$ (blue), plotted for $(\lambda, \mu, \alpha, C_1, C_2, C_3) = (1, 2, -3, 5, -3, -3)$. The green and red dots indicate the location of the tangent intersections; at the green dots there are elliptic equilibria, at the red dot there is a hyperbolic one.

a bifurcation occur, which leads to the third case. At $\lambda \approx -0.29, \mu = 2$, we have second order contact between a parabola and the phase space in the (w, u)-plane. A new, parabolic, equilibrium emerges, for which all the eigenvalues of the equilibrium vanish, but the linearization of the vector field does not vanish. Dynamically speaking we have a centre-saddle bifurcation: an elliptic and a hyperbolic equilibrium meet at the parabolic equilibrium and vanish. As we increase λ, the parabolic equilibrium splits off into an elliptic (centre) and a hyperbolic equilibrium (saddle) with two homoclinic orbits. Furthermore, as we go through the bifurcation, energy values emerge which can correspond to two different periodic orbits.

Indeed, the matrix of the linearization at an equilibrium $(u_0, 0, w_0)$ is

$$\begin{pmatrix} 0 & -2C_1 w_0 - C_{\mu,\lambda,\alpha} & 0 \\ 2C_1 w_0 + C_{\mu,\lambda,\alpha} & 0 & 2C_1 u_0 + \frac{1}{2} A q''_{\lambda,\mu}(w_0) \\ 0 & A & 0 \end{pmatrix} \tag{3.39}$$

which has the characteristic polynomial

$$-\lambda \left(\lambda^2 - 2AC_1 u_0 - \tfrac{1}{2} A^2 q''_{\lambda,\mu}(w_0) + (2C_1 w_0 + C_{\mu,\lambda,\alpha})^2 \right). \tag{3.40}$$

Note firstly that, in general we have, away from the roots of $q_{\lambda,w}$,

$$q''_{\lambda,\mu}(w) = \frac{\left(q'_{\lambda,\mu}(w)\right)^2}{2q_{\lambda,\mu}(w)} + 2\sqrt{q_{\lambda,\mu}(w)} \frac{\partial^2}{\partial w^2} \left(\sqrt{q_{\lambda,\mu}(w)} \right). \tag{3.41}$$

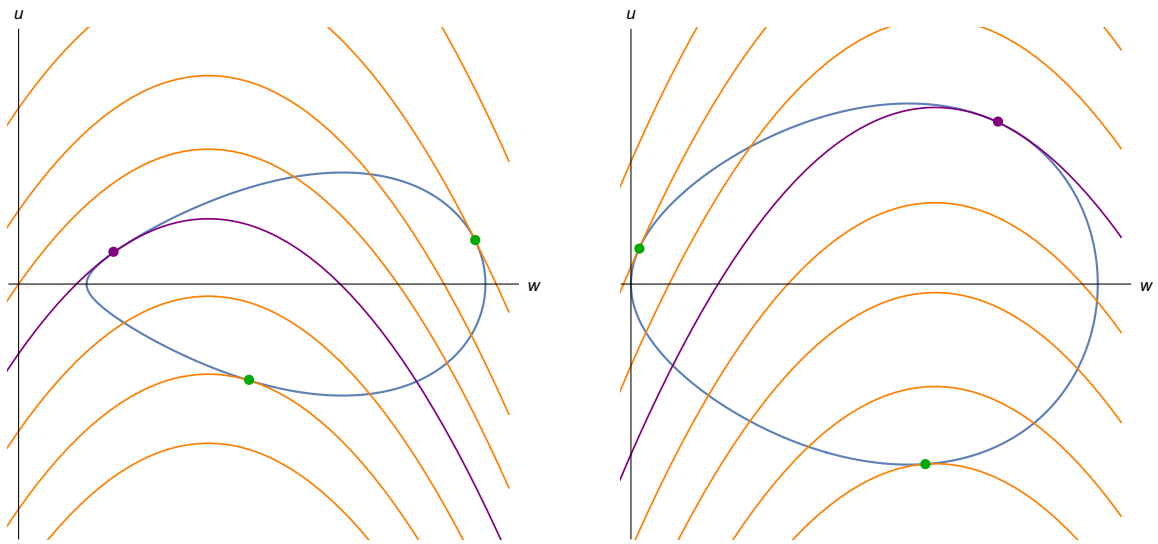

Figure 9: The intersections of the energy shells $L_\alpha^{-1}(h)$ (orange and purple) with the reduced phase space $\mathcal{P}_{\lambda,\mu}$ (blue), for $\lambda \approx -0.29$ (left) and $\lambda \approx 1.34$ (right) and $(\mu, \alpha, C_1, C_2, C_3) = (2, -3, 5, -3, -3)$. The green dots indicate the location of elliptic equilibria, with each two non-zero eigenvalues. At the purple dots there are parabolic equlibria, where the energy shells and the reduced phase space have second order contact in the (w, u)-plane. Both these parabolic equilibria are dynamically unstable, as there are homoclinic orbits to them.

In case of second order contact, i.e. a double root of the quintic equation (3.36), we have the equalities (3.35) and

$$u_0 \frac{\partial^2 u}{\partial w^2}(w_0) = \sqrt{q_{\lambda,\mu}(w_0)} \frac{\partial^2}{\partial w^2} \sqrt{q_{\lambda,\mu}(w)}(w_0) = -2C_1/A, \tag{3.42}$$

which allow us to write equation (3.41) as

$$\tfrac{1}{2} A^2 q''_{\lambda,\mu}(w_0) = (2C_1 w_0 + C_{\mu,\lambda,\alpha})^2 - 2AC_1 u_0. \tag{3.43}$$

This turns the characteristic polynomial (3.40) into $-\lambda^3$.

If we increase λ even more compared to the value in figure 8, then the hyperbolic equilibrium meets a different elliptic equlibrium and disappears through another centre-saddle bifurcation, at $\lambda \approx 1.34$. Both the examples $\lambda \approx -0.29, 1.34$ are shown in figure 9. In both examples the parabolic equilibria are unstable, as they have homoclinic orbits to them. The two parabolic equilibria are each part of a family, each forming a curve in (λ, μ)-space. As these curves meet, at say (λ_0, μ_0), there is a degenerate case of third order contact between the parabola and the reduced phase space in the (w, u)-plane. This last case is depicted in figure 10, and for our choice of constants $(\alpha, C_1, C_2, C_3) = (-3, 5, -3, -3)$ lies at $(\lambda_0, \mu_0) \approx (-0.19, 0.52)$. We find this point by finding a third order root of equation (3.36), which alternatively can be described as a fourth order root of

$$A^2 q_{\lambda,\mu}(w) = \left(C_1 w^2 + C_{\lambda,\mu,\alpha} w + \lambda + 4\mu + c_{11}\lambda^2 + 2c_{13}\lambda\mu + c_{33}\mu^2 - h\right)^2. \tag{3.44}$$

Differentiating thrice, we find a requirement for the parabolic equilibrium at (λ_0, μ_0), namely the w-coordinate of the equilibrium must lie at

$$w_0 = -\frac{C_1 C_{\lambda_0,\mu_0,\alpha} + 4A^2}{2C_1^2}. \tag{3.45}$$

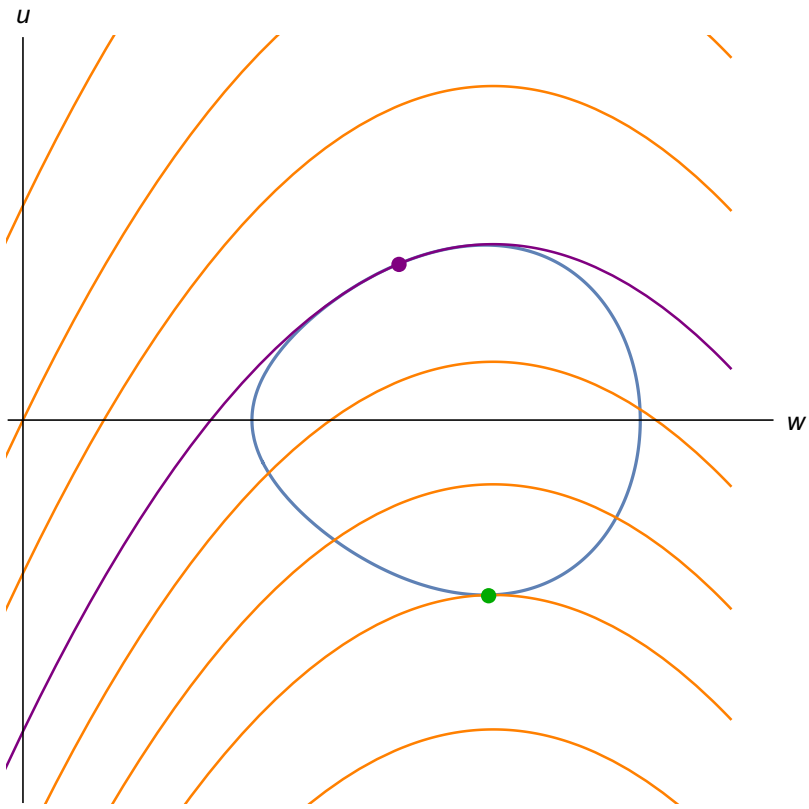

Figure 10: The intersections of the energy shells $L_\alpha^{-1}(h)$ (orange and purple) with the reduced phase space $\mathcal{P}_{\lambda,\mu}$ (blue), for $(\lambda, \mu) = (\lambda_0, \mu_0) \approx (-0.19, 0.52)$ and again $(\alpha, C_1, C_2, C_3) = (-3, 5, -3, -3)$. The green dots indicate the location of stable equilibria, with each two non-zero eigenvalues. At the purple dot there is a parabolic equlibrium, where the energy shell and the reduced phase space have third order contact. The parabolic equilibrium is dynamically stable, as its corresponding energy shell is touching the phase space from the outside.

This bifurcation at (λ_0, μ_0) is a cusp bifurcation, as described in e.g. [16]. Indeed, using the syzygy $Q_{\lambda,\mu}$ we can write the vector field equation for v in terms of w and the parameters λ and μ, i.e. $\dot{v} = f(w, \lambda, \mu)$. Then at (λ_0, μ_0) we recover (numerically) all the required degeneracy and non-degeneracy conditions for the cusp bifurcation for the equilibrium at w_0. In particular, we find that

$$f(w_0, \lambda_0, \mu_0) = f_w(w_0, \lambda_0, \mu_0) = f_{ww}(w_0, \lambda_0, \mu_0) = 0, \tag{3.46}$$

while

$$f_{www}(w_0, \lambda_0, \mu_0) \approx -300 \tag{3.47}$$

and

$$(f_\lambda f_{w\mu} - f_\mu f_{w\lambda})(w_0, \lambda_0, \mu_0) \approx -17.9 \tag{3.48}$$

The linearization of the equilibrium at (λ_0, μ_0) is again nonzero, although it is stable, unlike the parabolic equilibria of the two one-parameter families. This can be seen from figure 10, as the energy shells hit from the outside.

3.4.2 The singular case

In the case that $\mu > \lambda = 0$, we have a singularity at the origin, where the structure matrix has rank 0. This is a double root of (3.36). Excluding this solution, equation (3.35) simplifies to

$$2C_1 w + C_{0,\mu,\alpha} = \pm\sqrt{2}A\frac{(3w - 2\mu)}{\sqrt{\mu - w}}. \tag{3.49}$$

Equilibria outside of the origin arise as solutions to this equation. If the equation has a double root in $(0, \mu)$, we have again a parabolic equilibrium there. Note that this equation has at most three solutions, which implies that there can only be a third (unstable) solution if the singular equilibrium at the origin is stable.

One of the regular equilibria might collide with (or arise out of) the singular equilibrium as we change the value of μ; we studied this earlier in the chapter for the Hamiltonian $\bar{H}^1_\alpha$. Indeed, for $\bar{H}^1_\alpha$ we explained that as we increase the detuning parameter α (or, alternatively, decrease μ), a supercritical Hamiltonian Hopf bifurcation takes place at the origin. (And thus, in $\mathbb{C}^3$, at the 3-mode). The singular equilibrium at the origin switches stability as it collides with one of the regular equilibria.

Now let us consider that bifurcation when we include the self-interaction terms. For a good geometric understanding of the situation, we use the approach of Kummer, [15]. Instead of deforming the energy shells, we deform the phase space to be "sausage-shaped".

We use the transformations

$$\begin{aligned} \tilde{u} &:= Au + C_1 w^2 + C_{\lambda,\mu,\alpha} w, \\ \tilde{v} &:= Av, \end{aligned} \tag{3.50}$$

which allows us to write the syzygy as

$$\tilde{Q}_{\mu,\lambda}(\tilde{u}, \tilde{v}, w) := (\tilde{u} - C_1 w^2 - C_{\lambda,\mu,\alpha} w)^2 + \tilde{v}^2 - A^2 q_{\lambda,\mu}(w) \equiv 0 \tag{3.51}$$

and the phase space becomes $\tilde{\mathcal{P}}_{\lambda,\mu} = \tilde{Q}^{-1}_{\mu,\lambda}(0) \cap \{\mu \geq w \geq 0\}$. The advantage of this approach is that the energy shells are now planes parallel to the (v, w)-plane. An example of such a space and the dynamics induced by the accompanying Hamiltonian L_α is shown in figure 11. We can analyse when the singular equilibrium at the origin changes stability. To that purpose we check the derivatives of the functions

$$p^\pm_\mu(w) := C_1 w^2 + C_{0,\mu,\alpha} w \pm \sqrt{8}Aw\sqrt{\mu - w} \tag{3.52}$$

at the origin. These functions together parametrize the curve $\tilde{\mathcal{P}}_{\lambda,\mu} \cap \{v = 0\}$. From figure 11, it should be clear that if both of these derivatives have the same sign, the equilibrium is stable, while if they have different signs the equilibrium is unstable. Now we calculate that

$$\left.\frac{\mathrm{d}p^\pm_\mu}{\mathrm{d}w}\right|_{w=0} = C_{0,\mu,\alpha} \pm \sqrt{8}A\sqrt{\mu}. \tag{3.53}$$

We can recognize the bifurcating value of $\pm\sqrt{8}A\sqrt{\mu}$ which we saw earlier in the chapter for the detuning. For α and C_3 nonzero, we find either two or no solutions in terms of $\sqrt{\mu}$, taking into account the resctrictions $\mu > 0$. For the constants as we chose them in the figures above no bifurcations occur.

It is now not unreasonable to suspect that also here a Hamiltonian Hopf bifurcation is taking place as $|C_{0,\mu,\alpha}| = |\mu C_3 + \alpha|$ passes through the value $\sqrt{8}A\sqrt{\mu}$. Then the

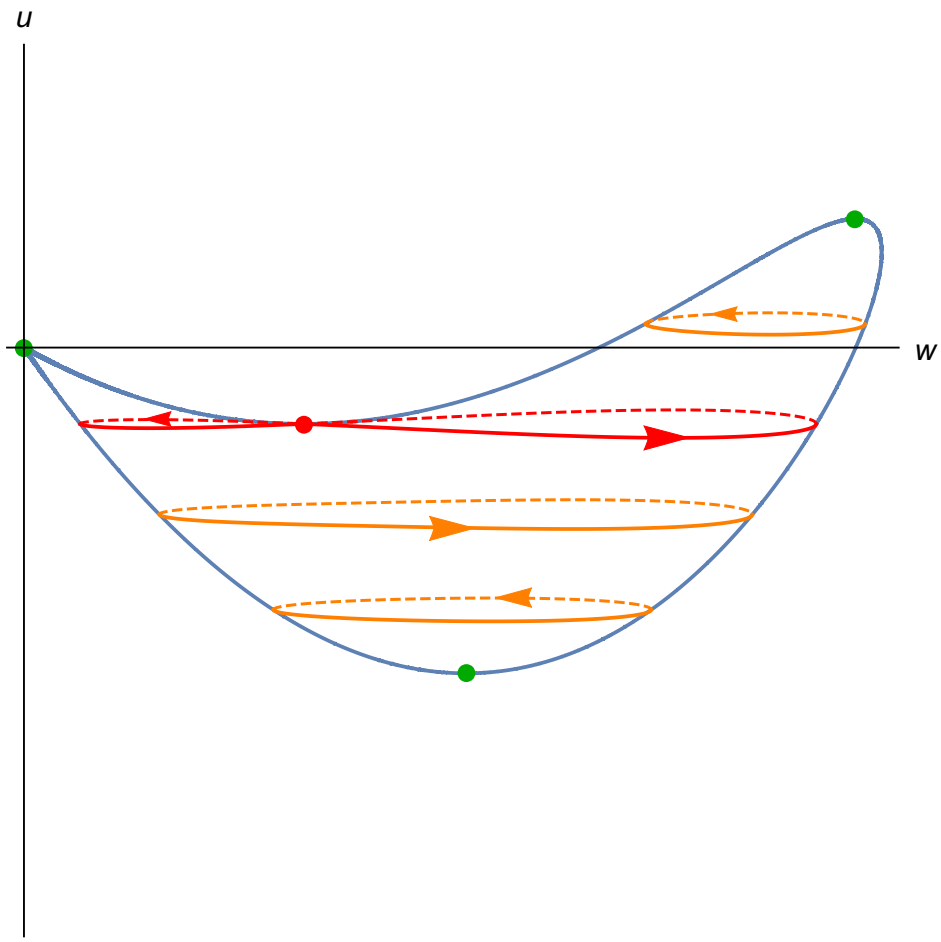

Figure 11: In this picture we see the reduced phase space and the dynamics of L_α in $(\tilde{u}, \tilde{v}, w)$-space after the transformations in (3.50), for the values $(\lambda, \mu, \alpha, C_1, C_2, C_3) = (0, 2, -3, 5, -3, -3)$. The green dots denote stable equilibria, the red dot an unstable one, with two homoclinic orbits to it.

derivatives of one of the functions $p_\mu^\pm$ vanishes at the origin. However, depending on the constants C_1, C_2, C_3, the bifurcation could now also be subcritical, depending on the higher order terms. We can determine this by checking the second order derivative of the function whose first derivative vanishes. In more geometric terms we are checking whether at the bifurcation the energy shells hit the phase space from outside (supercritical) or the inside (subcritical). We have

$$\left.\frac{\mathrm{d}^2 p_\mu^\pm}{\mathrm{d}w^2}\right|_{w=0} = 2C_1 \mp \frac{\sqrt{8}A}{\sqrt{\mu}}. \tag{3.54}$$

If we consider e.g. the bifurcation for $C_{0,\mu,\alpha} = \sqrt{8}A\sqrt{\mu}$ then it is the derivative of the function p_μ^+ which vanishes at the origin. As this curve parametrizes the upper half of the curve $\tilde{\mathcal{P}}_{\lambda,\mu} \cap \{v = 0\}$, we consider whether the second derivative is negative (supercritical) or positive (subcritical). We find that the bifurcation is supercritical if $\sqrt{\mu} < -\frac{\sqrt{2}A}{C_1}$ – which is consistent with the detuning – and it is subcritical if $\sqrt{\mu} > -\frac{\sqrt{2}A}{C_1}$. If we have equality then the Hamiltonian Hopf bifurcation is degenerate and we need to consider even higher order derivatives, see e.g. [20].

3.5 The subcritical bifurcation

To formalize our argument, we again return to two degrees of freedom. After reduction of μ with the same invariants and Poisson algebra as we saw earlier in the chapter in section 3.1, namely $q_1, p_1, \mu, \nu, \eta, \zeta$, we obtain the reduced Hamiltonian

$$\begin{aligned} L_\alpha = &\tfrac{1}{2}(7+\alpha)\mu + \tfrac{1}{4}(c_{22}+2c_{23}+c_{33})\mu^2 + (1+(c_{12}+c_{13})\mu)\tau_1 + c_{11}\tau_1^2 \\ &+ \left(\tfrac{1}{2}(\alpha - 1 + (c_{22}+c_{33})\mu + (c_{12}-c_{13})\tau_1\right)\nu \\ &+ \tfrac{1}{4}(c_{22}-2c_{23}+c_{33})\nu^2 + A(p_1\eta - q_1\zeta). \end{aligned} \tag{3.55}$$

The Hamiltonian vector field of L_α induces the dynamics

$$X_{L_\alpha}\begin{pmatrix} q_1 \\ q_2 \\ \mu \\ \nu \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} (1+(c_{12}-c_{13})\mu + 2c_{11}\tau_1)p_1 + A\eta \\ -(1+(c_{12}-c_{13})\mu + 2c_{11}\tau_1)q_1 - A\zeta \\ 0 \\ 2A(p_1\zeta - q_1\eta) \\ -(\alpha - 1 + (c_{22}+c_{33})\mu + 2(c_{12}-c_{13})\tau_1 + (c_{22}-2c_{23}+c_{33})\nu)\zeta + 2Aq_1\nu \\ (\alpha - 1 + (c_{22}+c_{33})\mu + 2(c_{12}-c_{13})\tau_1 + (c_{22}-2c_{23}+c_{33})\nu)\eta - 2Ap_1\nu. \end{pmatrix} \tag{3.56}$$

Now the point $(q_1, p, \mu, \nu, \eta, \zeta) = (0, 0, \mu, -\mu, 0, 0)$, which is the image of the 3-mode, is still an equilibrium.

We continue as in section 3.2 to find a local description. In the coordinates κ, φ, as given in (3.9), the Hamiltonian is

$$\begin{aligned} L_\alpha = &4\mu + c_{23}\mu^2 + \big(1 + 2c_{13}\mu\big)\tau_1 + c_{11}\tau_1^2 \\ &+ \big(\alpha - 1 + 2(c_{23}-c_{33})\mu + 2(c_{12}-c_{13})\tau_1\big)\frac{\varphi^2+\kappa^2}{2} \\ &+ \big(\alpha - 1 + 2(c_{23}-c_{33})\mu\big)\frac{3\kappa^4 - 6\kappa^2\varphi^2 - \varphi^2}{12\mu} \\ &+ \big(c_{22} - 2c_{23} + c_{33}\big)\left(\frac{\varphi^2+\kappa^2}{2}\right)^2 \\ &+ \sqrt{2\mu}A\left(p_1\kappa - q_1\varphi - q_1\frac{\varphi^3 + 3\kappa^2\varphi}{3\mu}\right) + \mathcal{O}(6). \end{aligned} \tag{3.57}$$

As expected, the quadratic part of L_α commutes with the quadratic Hamiltonian

$$S(q_1, p_1, \varphi, \kappa) = \frac{q_1^2 + p_1^2}{2} - \frac{\varphi^2 + \kappa^2}{2}. \tag{3.58}$$

We normalize all of L_α with respect to S and N. Recall that N is one of the invariants of the action generated by S. In terms of $q_1, p_1, \varphi, \kappa$, it equals

$$N = \tfrac{1}{2}\left(q_1^2 + p_1^2 + \varphi^2 + \kappa^2\right) + p_1\kappa - q_1\varphi \tag{3.59}$$

This is essentially the same calculation as in section 3.2. We can then write the normalized Hamiltonian in terms of the invariants $S(c,d), N(c,d)$ and $M(c,d)$ from equation (3.16).

The normalized Hamiltonian takes the form

$$\begin{aligned}L'_\alpha &= 4\mu + c_{23}\mu^2 + \tfrac{1}{2}\big((C_{0,\mu,\alpha} + \sqrt{8}A)N + (C_{0,\mu,\alpha} - \sqrt{8}A)M + (2 + (c_{13} - c_{23} + c_{33} - \alpha)S\big) \\ &+ \left(\frac{\sqrt{2}A}{4\sqrt{\mu}} + \frac{C_1}{4}\right) M^2 + \left(\frac{\sqrt{2}A}{4\sqrt{\mu}} + \tfrac{1}{2}\big(c_{11} - c_{22} - c_{33} + 2c_{23}\big)\right) MS \\ &+ \tfrac{1}{4}\big(c_{11} + c_{22} + c_{33} - 2c_{12} + 2c_{13} - 2c_{23}\big)S^2 + \mathcal{O}(3).\end{aligned} \tag{3.60}$$

We then conclude that, as $C_{0,\mu,\alpha} = \alpha + \mu C_3$ goes through $\sqrt{8}A\sqrt{\mu}$, the Hamiltonian L_α undergoes a Hamiltonian Hopf bifurcation, if the higher order terms don't vanish. As the sign of $N = \sqrt{8}A\sqrt{\mu}$ is always positive, we have the supercritical case if $C_1 = 0$ and else if

$$\frac{\sqrt{2}A}{4\sqrt{\mu}} > -\frac{C_1}{4}$$

or equivalently

$$\sqrt{\mu} < -\frac{\sqrt{2}A}{C_1}. \tag{3.61}$$

The bifurcation is subcritical if $C_1 \neq 0$ and

$$\sqrt{\mu} > -\frac{\sqrt{2}A}{C_1}. \tag{3.62}$$

In case of equality, the coefficent of M^2 vanishes and the bifurcation is degenerate. The other possible bifurcation, if $C_{0,\mu,\alpha} = -\sqrt{8}A\sqrt{\mu}$ is treated by switching the role N and M (before we normalize with respect to N). This can be achieved with the transformation (3.20). In that case the bifurcation is subcritical if $C_1 \neq 0$ and

$$\sqrt{\mu} > \frac{\sqrt{2}A}{C_1}. \tag{3.63}$$

while it is supercritical if the inequality is reversed or if $C_1 = 0$, and degenerate if we have equality here.

To relate back to our findings in one degree of freedom, we are checking how the paraboloids look locally with respect to the phase space. In the supercritical case, when we the energy shells touch from the outside of the phase space, the parabola at the bifurcation "looks like a straight line", it's not curved enough to be noticable. In the subcritical case, when the energy shells touch from the inside, the parabola is curved enough to cut through a part of phase space.

3.6 Reconstruction of the dynamics of L_α

Let us consider the reconstruction of our findings in one degree of freedom back to the original phase space $\mathbb{C}^3$. Firstly, the 1- and 2-modes are, aside of the frequency of the oscillation, unaffected by the quartic self-interaction terms. This holds in more generality, see corollary 4.5 in chapter 4. However, the 3-mode may destabilize and restabilize now through a subcritical or even degenerate periodic Hamiltonian Hopf bifurcation, in addition to the supercritical case we found earlier, for the detuning, by an increase in the value of μ (or alternatively an increase in the value of the quadratic Hamiltonian H_0^0).

The two-parameter (in (λ, μ)) families of elliptic equilibria in the reduced spaces reconstruct to families of elliptic invariant 2-tori. These 2-tori are foliated by conditionally periodic orbits, which is the stationairy orbit of the elliptic equilibria superposed with the two $\mathbb{S}^1$-orbits of T_{12} and T_{23}. For generic values of the constants c_{ij}, we no longer recover the family of resonant 2-tori. Recalling the calculations of 2.7, the frequencies of on the invariant 2-tori are given by

$$\beta_\lambda = 1 + 2c_{11}\lambda + 2c_{13}\mu + A\frac{\mathrm{d}u}{\mathrm{d}\lambda} + C_2 w + \big(2C_1 w + C_{\lambda,\mu,\alpha}\big)\frac{\partial w}{\partial \lambda} \tag{3.64}$$

and

$$\beta_\mu = 1 + 2c_{33}\mu + 2c_{13}\lambda + A\frac{\mathrm{d}u}{\mathrm{d}\mu} + C_1 w + \big(2C_1 w + C_{\lambda,\mu,\alpha}\big)\frac{\partial w}{\partial \mu}. \tag{3.65}$$

To reconstruct the orbits in $\mathbb{C}^3$ for the two-parameter families of hyperbolic equilibria and their two homoclinic orbits, we superpose to each point of the "figure 8" the two $\mathbb{S}^1$-orbits of T_{12} and T_{23}. This figure can be understood as glueing together two 3-tori along an (invariant) 2-torus, the 2-torus corresponding to the hyperbolic equilibrium. The orbits inside these 3-tori converge towards the hyperbolic invariant 2-torus, while displaying conditionally periodic dynamics in the remaining two directions.

These families of hyperbolic invariant 2-tori are the result of two one-parameter families, of quasi-periodic centre-saddle bifurcations. For these families, we find unstable parabolic invariant 2-tori, which are part of pinched 3-tori. These split into the aforementioned elliptic invariant 2-tori and hyperbolic invariant 2-tori.

If it exists, at the common boundary (λ_0, μ_0) of these one parameter families, we find a single stable parabolic invariant 2-torus. This parabolic 2-torus arises as an unstable 2-torus meets two elliptic 2-tori.

Persistence

As in the case of higher resonances we saw in section 1.5, the addition of the self-interaction terms gives a better chance of persistence of the families of Diophantine (maximal) 3-tori. This is because the frequencies of the reduced $\mathbb{T}^2$-action becomes less degenerate. Therefore the Kolmogorov condition is expected to be satisfied for generic values of the constants c_{ij}.

The persistence of the periodic Hamiltonian Hopf bifurcation and the quasi-periodic centre-saddle bifurcations are adressed in [11], and the latter more specifically in [13], but are beyond the scope of this thesis.

A critical note regarding persistence with respect to higher order normal forms of the 1:3:4 resonance is in place; the second normal form includes the resonant term $\mathrm{Re}(Bx_1^3 y_2)$ for $B \in \mathbb{C}$. This resonant term is of the same order as the quartic self-interaction terms. Therefore generally we cannot consider the second normal form as a small perturbation of the Hamiltonian L_α, even close to the origin.

4 Intermezzo: Stability of normal modes

Among the orbits of the first normal form of the 1:3:4 resonance we have encountered families of normal modes, of which one was unstable. The existence (or, more accurately, definition) of normal modes goes back to Lyapunov; we recall a theorem from [2] which we slightly restrict to fit our setting.

Theorem 4.1 (Lyapunov subcenter theorem)**.** *Let H be a smooth function on $\mathbb{R}^{2n}$, whose Hamiltonian vector field X_H has a critical point at p. Denote the characteristic exponents of the linearization of X_H at p by $\nu_1, ..., \nu_n, -\nu_1, ..., -\nu_n$.*

Assume that $\nu_1 = \mathrm{i}\beta_1$ for $\beta_1 > 0$, and assume (the non-resonance condition) that no ν_j, $j = 2, ..., n$, is an integer multiple of ν_1.

Then there is a one parameter family γ_ε defined for $\varepsilon \in (0, \epsilon_0]$ of closed orbits of X_H that approach p as $\varepsilon \to 0$ and whose periods approach $2\pi/\beta_1$.

We remark that the theorem of Lyapunov doesn't work to prove the existence of the 1-mode for the 1:3:4 resonance, although it does for the 2- and 3-modes. We shall see in chapter 5 that indeed the 1-mode need not be present.

A related statement without a resonance condition is due to Weinstein. In [25], Weinstein showed that at least n short-periodic orbits[3] reside on each energy surface close to an elliptic equilibrium, as long as the quadratic part of the Hamiltonian is definite. In the resonant case in particular the resonance must be definite. However these short-periodic orbits might not be part of families emerging from of the equilibrium.

The results of Lyapunov and Weinstein do not tell us about the stability of these periodic orbits. For a restricted class of resonant Hamiltonians, namely those in normal form with a single resonant term, we present here a new result on the (in)stability of their normal modes. The result can be directly applied to some families of resonances, ordered by their annihilators, however just a small number of these are of higher interest. Although restricted, this class of Hamiltonians forms an important step in the understanding of certain resonant equilibria.

We start our analysis with the following lemma, generalizing the integrals we found for the first normal form of the 1:3:4-resonance. It is written in more generality than we need below, but the proof is essentially the same anyway.

Lemma 4.2. *Let ω be a (possibly degenerate) frequency vector and assume there exists an annihilator k of ω. Let $A > 0$ and consider the Hamiltonian*

$$\mathcal{H} = \sum_{i=1}^{n} \omega_i \tau_i + A \operatorname{Re}(x^k) \tag{4.1}$$

where we use the notation from equation (1.16). For every $\psi \in \Gamma_k$, i.e. for an annihilator to the annihilator k, the linear combination

$$I_\psi := \sum_{j=1}^{n} \psi_j \tau_j \tag{4.2}$$

is an integral of $\mathcal{H}$. This remains true if we add any polynomial in the τ_i to $\mathcal{H}$.

[3]These are sometimes also referred to as normal modes in the literature, but we reserve this for the periodic orbits in a single degree of freedom.

Proof. Let $\psi \in \Gamma_k$. Recall that $\tau_j := \frac{1}{2}x_j y_j$. Then we note that

$$\{I_\psi, x^k\} = \sum_{j=1}^{n} \psi_j \{\tau_j, x^k\} = \sum_{j=1}^{n} \psi_j(-\mathrm{i}k_j)x^k = -\mathrm{i}\langle \psi, k\rangle x^k = 0 \tag{4.3}$$

where the last equality is a result of $\psi \in \Gamma_k$. The same holds for the conjugate $y^k = x^{-k}$ by conjugating both sides of the equation, so $\{I_\psi, \mathrm{Re}(x^k)\} = 0$. Lastly as the τ_j's commute, we conclude that $\{I_\psi, \mathcal{H}\} = 0$. □

4.1 Reduction of phase space

To continue our analysis, we look at the reduction of the action of the integrals we obtain for $\mathcal{H}$ from the lemma above. Henceforth we assume the frequency vector ω is fully resonant. In particular, ω has exactly $n-1$ independent annihilators; assume k is one of them and is primitive.

Consider $\mathcal{H}$ given by (4.1). Now because the τ_j's commute, the $I_\psi,\ \psi \in \Gamma_k$ commute with one another as well, and this leads to an integrable system. Indeed, there are exactly $n-1$ linearly independent elements ψ^ℓ of Γ_k. This gives us $n-1$ linearly independent combinations I_{ψ^ℓ}, which are $n-1$ functionally independent integrals. Together with the Hamiltonian $\mathcal{H}$, we thus have n integrals.

The integrals I_{ψ^ℓ} yield an $(n-1)$-torus action given by (per coordinate x_m)

$$(t_1, ..., t_{n-1}, x_m) \mapsto \exp\left(\mathrm{i}\sum_{\ell=1}^{n-1} \psi_m^\ell t_\ell\right) x_m. \tag{4.4}$$

The invariants of this action include $\tau_j, 1 \leq j \leq n$ and $x^k, y^k = x^{-k}$. Actually, an invariant monomial, say, $x_1^{a_1}...x_n^{a_n}y_1^{b_1}...y_n^{b_n}$ with $a_j, b_j \in \mathbb{Z}$ for $j = 1, ...n$, can always be written as

$$x_1^{a_1}...x_n^{a_n}y_1^{b_1}...y_n^{b_n} = 2^{|c|}\tau^c x^d$$

where $c, d \in \mathbb{Z}^n$, and x^d uses the notation defined in equation (1.16). Then, as the τ_j are invariant, so must x^d be, and this leads to the requirement that, for $1 \leq \ell \leq n-1$ and any $t \in \mathbb{R}$,

$$\exp\left(\mathrm{i}\sum_{m=1}^{n} d_m \psi_m^\ell t\right) = \exp\big(\mathrm{i}\langle d, \psi^\ell\rangle t\big) = 1$$

and thus we have the requirement $\langle d, \psi^\ell\rangle = 0$ for $1 \leq \ell \leq n-1$. Therefore $\psi^\ell \in \Gamma_d$. In particular we find that, as k is primitive, $d = m_d k$ for some $m_d \in \mathbb{Z}$ and therefore $x_1^{a_1}...y_n^{b_n}$ can be written in terms of the τ_j's and x^k, y^k.

We note that the space of invariants does not depend on the representatives ψ^ℓ, and thus we can speak of *the* reduced space (under Γ_k). The reduced space can be described using the invariants as above, with I_{ψ^ℓ} as Casimirs and the syzygy

$$x^k y^k - 2^{|k|}\prod_{j=1}^{n} \tau_j^{|k_j|} \equiv 0. \tag{4.5}$$

For the proof of the theorem below we need a more explicit description of the reduced phase space, which we now give. For simplicity we continue our analysis with $n = 3$,

but we conjecture that a result similar to the proposition below can be proven in higher degrees of freedom as well.

For simplicity, assume $k_1k_2 \neq 0$ and $k_2 > 0$. If the frequency vector is nond-degenerate this is not a restriction, as the former can be achieved by permuting the coordinate indices, and the latter by mapping $k \mapsto -k$, which leaves the Hamiltonian $\mathcal{H}$ invariant.

Now choose

$$\psi^1 = (k_2, -k_1, 0), \qquad \psi^2 = (0, -k_3, k_2), \tag{4.6}$$

set $w = \tau_2/k_2$, and fix the integrals,

$$I_{\psi^1} = k_2\tau_1 - k_1k_2w = \lambda, \qquad I_{\psi^2} = k_2\tau_3 - k_2k_3w = \mu. \tag{4.7}$$

For the 1:3:4 resonance we have $k = (1, 1, -1)$ and then these integrals coincide with T_{12} and T_{23} of chapter 2. If we let

$$q_{k,\lambda,\mu}(w) := \frac{2^{|k|}}{k_2^{|k_1|+|k_3|-|k_2|}} \left(\lambda + k_1k_2w\right)^{|k_1|} w^{|k_2|} \left(\mu + k_2k_3w\right)^{|k_3|}, \tag{4.8}$$

which is the more complicated, generalized version of $q_{\lambda,\mu}(w)$, and as usual, write $u = \mathrm{Re}(x^k), v = \mathrm{Im}(x^k)$, then the syzygy (4.5) can be written as

$$Q_{k,\lambda,\mu}(u, v, w) := \tfrac{1}{2}(u^2 + v^2 - q_{k,\lambda,\mu})(w) \equiv 0. \tag{4.9}$$

We can now identify the reduced phase space with the zero-level set of $Q_{k,\lambda,\mu}$, together with restrictions on w. To be precise, this is the set

$$\mathcal{P}_{k,\lambda,\mu} := Q^{-1}_{k,\lambda,\mu}(0) \cap R(k_1, \lambda) \cap R(k_3, \mu) \tag{4.10}$$

where $R(q, \nu)$ gives the restrictions on w:

$$R(q, \nu) := \begin{cases} \{\frac{\nu}{(-q)k_2} \geq w \geq 0\} & \text{if } \ q < 0, \\ \{w \geq \max(0, \frac{-\nu}{qk_2})\} & \text{if } \ q > 0, \\ \{w \geq 0\} & \text{if } \ q = 0. \end{cases} \tag{4.11}$$

We can now easily write down the reduced Poisson structure on $\mathcal{P}_{k,\lambda,\mu}$:

$$\{f, g\} = \langle \nabla f \times \nabla g, \nabla Q_{k,\lambda,\mu}\rangle. \tag{4.12}$$

We note that all of the above also works for $k_3 = 0$ (therefore for any ω_3), only then τ_3 is simply an integral of $\mathcal{H}$ and plays no role in the dynamics. With the reduction in hand, we can continue to our theorem on the stability of normal modes.

Theorem 4.3. *Let $n = 3$, let ω be a fully resonant and non-degenerate frequency vector and k one of its primitive annihilators, satisfying $k_1k_2 \neq 0$. Let $\mathcal{H}$ be the Hamiltonian from equation (4.1).*

Then among the orbits of $X_{\mathcal{H}}$ there is a 1-parameter family of 3-modes. The 3-modes are (Lyapunov-)stable if, and only if, $k_1k_2 < 0$.

Proof. Without loss of generality, assume $k_2 > 0$, by mapping $k \mapsto -k$ if necessary. From lemma 4.2 we know that I_{ψ^1} and I_{ψ^2} are integrals of our Hamiltonian, where the annihilators ψ^i of k are chosen as in equation (4.6),. Fixing the values of our integrals

$$I_{\psi^1} = k_2\tau_1 - k_1k_2w = \lambda, \qquad I_{\psi^2} = k_2\tau_3 - k_2k_3w = \mu,$$

we can identify the reduced phase space with $\mathcal{P}_{k,\lambda,\mu}$ from equation (4.10); it has the Poisson structure as given in equation (4.12).

We first show that there is indeed a 1-parameter family of 3-modes. For $\lambda = 0, \mu > 0$, the origin $(u, v, w) = (0, 0, 0)$ is part of the reduced phase space $\mathcal{P}_{k,\lambda,\mu}$, as it is included in $Q^{-1}_{k,\lambda,\mu}(0)$, $R(k_1, 0)$ and $R(k_3, \mu)$.

The pre-image of the origin, under our identifications, is precisely the 3-mode:

$$M_\mu := \left\{ (x_1, x_2, x_3) \in \mathbb{C}^3 : \tau_1 = \tau_2 = 0, \tau_3 = \frac{\mu}{k_2} \right\}. \tag{4.13}$$

We know this because if $w = 0$ and $\lambda = 0$ then both τ_1 and τ_2 vanish. For these sets to be normal modes we check that they are preserved under the flow of $X_{\mathcal{H}}$ and do not contain any equilibria. (We do this up in $\mathbb{C}^3$.)

We have periodic motion for $\mu > 0$, because on M_μ the dynamics of $X_{\mathcal{H}}$ are that of a harmonic oscillator in the third degree of freedom. Indeed, we have

$$X_{\mathcal{H}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Bigg|_{M_\mu} = \begin{pmatrix} \mathrm{i}\omega_1 x_1 + 2\mathrm{i}|k_1|\, A\ x^{\mathrm{sign}(k_1)k+e_1} \\ \mathrm{i}\omega_2 x_2 + 2\mathrm{i}|k_2|\, A\ x^{\mathrm{sign}(k_2)k+e_2} \\ \mathrm{i}\omega_3 x_3 + 2\mathrm{i}|k_3|\, A\ x^{\mathrm{sign}(k_3)k+e_3} \end{pmatrix} \Bigg|_{M_\mu} = \begin{pmatrix} 0 \\ 0 \\ \mathrm{i}\omega_3 x_3 \end{pmatrix} \tag{4.14}$$

where the terms $x^{\mathrm{sign}(k_j)k+e_j}$ all vanish on M_μ as both $k_1 \neq 0$ and $k_2 \neq 0$. By the non-degeneracy of ω, the vector field does not vanish so we have no equilibria. We conclude that we have found a family of 3-modes, parametrized by μ. (Note that equivalently we can parametrize the family by the energy of the quadratic Hamiltonian H^0_0, as it can be written as in terms of the integrals I_{ψ^1} and I_{ψ^2}.)

Now regarding the stability, first assume $k_1k_2 < 0$, and therefore $k_1 < 0$. Taking $\lambda = 0, \mu > 0$, we find that the reduced phase space $\mathcal{P}_{k,\lambda,\mu}$ consists only of the origin, i.e. the image of the 3-mode. This is because now $R(k_1, \lambda) = \{w = 0\}$ and the equality $w = 0$ implies that $u = v = 0$ by the syzygy.

In this case orbits starting nearby a 3-mode stay close to it, due to the definitiveness of I_{ψ^1}. Indeed, let $p \in \mathbb{C}^3$ be a point ε-close to the set M_μ, i.e. $d(p, M_\mu) < \varepsilon$ for $\mu > 0$, for $\sqrt{\frac{2\mu}{k_2}} > \varepsilon > 0$. Then

$$\tau_1(p) = \tfrac{1}{2}|p_1|^{\,2} < \tfrac{1}{2}\varepsilon^2.$$

We get the same estimate for $\tau_2(p)$. We therefore get an estimate on our integral:

$$I_{\psi^1}(p) = k_2\tau_1(p) + (-k_1)\tau_2(p) < \tfrac{1}{2}\left(k_2 - k_1\right)\varepsilon^2.$$

Now as I_{ψ^1} Poisson-commutes with $\mathcal{H}$, it is invariant under the flow $\varphi^t_{\mathcal{H}}$ of the Hamiltonian vector field of $\mathcal{H}$, i.e.

$$(\varphi^t_{\mathcal{H}})^* I_{\psi^1} = I_{\psi^1}.$$

Remember that $k_1 < 0$, and in particular I_{ψ^1} dominates both $k_2\tau_1$ and $(-k_1)\tau_2$. Therefore, applying $(\varphi^t_{\mathcal{H}})^*$ to the equation above, we obtain the bounds

$$0 \le \tau_1(\varphi^t_{\mathcal{H}}(p)) \le \frac{I_{\psi^1}(p)}{k_2} < \frac{k_2 - k_1}{2k_2}\varepsilon^2,$$
$$0 \le \tau_2(\varphi^t_{\mathcal{H}}(p)) \le \frac{I_{\psi^1}(p)}{-k_1} < \frac{k_2 - k_1}{-2k_1}\varepsilon^2.$$

For $\tau_3(p)$ we first estimate that

$$|p_3| \le \inf_{(0,0,x_3)\in M_\mu} |p_3 - x_3| + |x_3| < \varepsilon + \sqrt{\frac{2\mu}{k_2}},$$

and reversely

$$\begin{aligned} |p_3| &> |p_3| - (\varepsilon - \inf_{(0,0,x_3)\in M_\mu} |p_3 - x_3|) \\ &\ge \inf_{(0,0,x_3)\in M_\mu} |x_3| - \varepsilon = \sqrt{\frac{2\mu}{k_2}} - \varepsilon > 0. \end{aligned}$$

This leads to the inequalities

$$\frac{\mu}{k_2} - \sqrt{\frac{2\mu}{k_2}}\varepsilon + \tfrac{1}{2}\varepsilon^2 < \tau_3(p) < \frac{\mu}{k_2} + \sqrt{\frac{2\mu}{k_2}}\varepsilon + \tfrac{1}{2}\varepsilon^2.$$

And so we can deduce that

$$I_{\psi^2}(p) = k_2\tau_3(p) - k_3\tau_2(p) < \mu + \sqrt{2\mu k_2}\varepsilon + \tfrac{1}{2}\big(k_2 + |k_3|\big)\varepsilon^2$$

and

$$I_{\psi^2}(p) = k_2\tau_3(p) - k_3\tau_2(p) > \mu - \sqrt{2\mu k_2}\varepsilon + \tfrac{1}{2}\big(k_2 - |k_3|\big)\varepsilon^2,$$

which allows us to conclude that

$$|I_{\psi^2}(p) - \mu| < \sqrt{2\mu k_2}\varepsilon + \tfrac{1}{2}\big(k_2 + |k_3|\big)\varepsilon^2$$

Now we can estimate that τ_3 stays close to the value μ/k_2, as

$$\begin{aligned} \left|\tau_3(\varphi^t_{\mathcal{H}}(p)) - \frac{\mu}{k_2}\right| &= \left|\frac{I_{\psi^2}(p) - \mu + k_3\tau_2(\varphi^t_{\mathcal{H}}(p))}{k_2}\right| \\ &\le \left|\frac{I_{\psi^2}(p) - \mu}{k_2}\right| + \left|\frac{k_3\tau_2(\varphi^t_{\mathcal{H}}(p))}{k_2}\right| \\ &< \sqrt{\frac{2\mu}{k_2}}\varepsilon + \left(\tfrac{1}{2} + \frac{|k_3|\,(k_2 + 2(-k_1))}{2(-k_1)k_2}\right)\varepsilon^2 \end{aligned}$$

We conclude that in this case the distance of the orbit of p to the 3-mode stays bounded.

Now consider the case $k_1k_2 > 0$, so under our assumption that $k_2 > 0$ this is the case that $k_1 > 0$. As before we set $\lambda = 0, \mu > 0$, and note that $R(k_1, 0) = \{w \ge 0\}$. The syzygy takes the form

$$\tfrac{1}{2}u^2 + \tfrac{1}{2}v^2 - \tfrac{1}{2}\frac{k_1^{|k_1|}2^{|k|}}{k_2^{|k_3|-|k_2|}} w^{|k_1|+|k_2|}\,(\mu + k_2k_3w)^{|k_3|} \equiv 0. \tag{4.15}$$

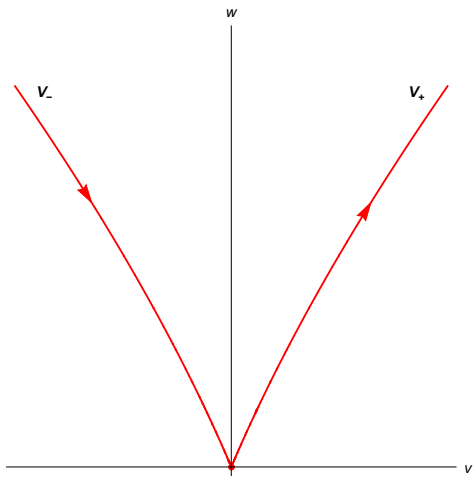

Figure 12: The intersection of the reduced phase space $\mathcal{P}_{k,0,\mu}$ with the plane $\{u = 0\}$, for $k = (1, 1, 1)$ and $\mu = 1$. We have unstable (V_+) and stable (V_-) manifolds to the equilibrium at the origin.

The reduced phase space is in this case a surface of revolution with a conical (if $|k_1| = |k_2| = 1$) or cusp singularity at the origin, which corresponds precisely to the 3-mode.

The reduced Hamiltonian is $\mathcal{H}(u, v, w) = H_0^0 + Au$, and therefore the energy surfaces are the planes $\{u = h\}$ parallel to the (v, w)-plane, where h is the value of the Hamiltonian $\mathcal{H}$. If we intersect the phase space with the energy surface going through the origin (so $u = 0$) we get the set

$$\{0\} \cup V_+ \cup V_-$$

where the curves $V_\pm$ are attached to the origin:

$$V_\pm := \left\{(0, v, w) \in \mathbb{R}^3 : w > 0, v = \pm\sqrt{q_{k,0,\mu}(w)}\right\} \cap R(k_3, \mu) \tag{4.16}$$

Using the Poisson structure (4.12) we can deduce that V_- is the the stable and V_+ the unstable manifold to the equilibrium at $(0, 0, 0)$. Indeed, in the reduced phase space the vector field is

$$X_{\mathcal{H}} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = A \begin{pmatrix} 0 \\ -\partial_w Q_{k,\lambda,\mu} \\ \partial_v Q_{k,\lambda,\mu} \end{pmatrix} = A \begin{pmatrix} 0 \\ \frac{1}{2} q'_{k,\lambda,\mu}(w) \\ v \end{pmatrix} \tag{4.17}$$

The derivative of $q_{k,\lambda,\mu}(w)$ is always positive for small, non-zero values of w. Indeed, we have

$$q'_{k,\lambda,\mu}(w) = q_{k,\lambda,\mu}(w) \left(\frac{|k_1| + |k_2|}{w} + \frac{k_2 k_3 |k_3|}{\mu + k_2 k_3 w} \right)$$

and as $q_{k,\lambda,\mu}(w) \geq 0$ (with equality only at 0 and, if $k_3 < 0$, at $w = \frac{\mu}{k_2(-k_3)}$), we only need to investigate the second factor. If $k_3 \geq 0$, it is always positive, and for $k_3 < 0$, the second factor vanishes for

$$w = \frac{(|k_1| + |k_2|)\mu}{k_2(-k_3)|k|}.$$

Now we see that starting on V_-, we have $v < 0$ and we converge towards the origin, while starting on V_+ we have $v > 0$ and we diverge away from it. This is illustrated in figure 12.

We conclude that the singular equilibrium at the origin in the reduced space is unstable, and therefore the 3-mode (up in $\mathbb{C}^3$) is unstable as well. $\square$

To fix thoughts, we stress that the main importance of the theorem below is for the case that $|k| = 3$ or 4, in three degrees of freedom. In the case of $|k| = 3$, the Hamiltonian $\mathcal{H}$ arises naturally as long as there are no other resonant terms of order 3, and we need not artifically ignore the quartic self-interaction terms. Moreover, due to the lower order of the resonant term (3 vs. 4), we can expect an unstable normal mode to remain unstable close to the origin as we move to higher order normal forms. In this light, we have a conjecture on the switching of stability of the normal mode in question, inspired by the behaviour we saw for the 1:3:4 resonance.

Conjecture 4.4. *Let $n = 3$, let ω be a fully resonant and non-degenerate frequency vector and assume $k = (1, 1, \pm 1)$ is an annihilator to ω. Let $\mathcal{H}'$ be the Hamiltonian*

$$\mathcal{H}' = \sum_{j=1}^{3} \omega_j \tau_j + A \operatorname{Re}(x^k) + \sum_{i,j=1}^{3} c_{ij} \tau_i \tau_j \tag{4.18}$$

where $A > 0$ and $c_{ij} \in \mathbb{R}$, $i, j = 1, ..., n$ are assumed to be generic.

If there are values of the quadratic part $\langle \omega, \tau \rangle$ of $\mathcal{H}'$ for which the 3-mode is unstable, then the 3-mode is stabilized by a Hamiltonian Hopf bifurcation, triggered by a change in the value of the quadratic part.

Note that the quadratic part $\langle \omega, \tau \rangle$ is an integral for $\mathcal{H}'$. The reason we exclude the case $k_3 = 0$, is because the singularity at the origin of the reduced space, which is the image of the 3-mode, is not conical, but cusp. This is precisely the case for the 1:-2 subresonances.

Additional to the quartic self-interaction terms one could consider the same conjecture while detuning the resonance or adding a resonant term $\operatorname{Re}(x^\ell)$ with $|\ell| \geq 5$, as long as the 3-mode are periodic orbits.

In the case that $|k| = 4$, the story becomes a bit more tricky because the self-interaction terms and the resonant term x^k have the same order. It is then a question of the relative magnitude of the constants whether the 3-mode is stable or unstable.

A closer investigation of the proof of our proposition already leads to interesting observations regarding the stable normal modes.

Corollary 4.5. *Let ω, k and $\mathcal{H}$ be as in theorem 4.3, and assume k satisfies $k_1 k_2 < 0$. Consider the Hamiltonian*

$$\mathcal{H}' = \mathcal{H} + P(\tau_1, ..., \tau_3) \tag{4.19}$$

where P is any polynomial with all terms of degree two or higher. Then among the orbits of $X_{\mathcal{H}'}$ there is a 1-parameter family of stable 3-modes (notwithstanding possible degeneracies).

Proof. The Hamiltonian $\mathcal{H}'$ admits the same integrals as $\mathcal{H}$, as the τ_j's commute. For the stability, the same proof applies, as it only uses properties of the reduced phase space and the integrals. One of the members of the family of 3-modes could consist fully of equilibria. For that to happen we need that

$$\begin{aligned} 0 = \dot{x_3} &= \tfrac{1}{2}\{x_3, x_3 y_3\}(\omega_3 + \frac{\partial P(0,0,\tau_3)}{\partial \tau_3}) \\ &= \mathrm{i} x_3(\omega_3(\omega_3 + \frac{\partial P(0,0,\tau_3)}{\partial \tau_3}) \end{aligned}$$

so outside of the values of μ for which $\frac{\partial P(0,0,k_2\mu)}{\partial \tau_3} = -\omega_3$, we conclude that we indeed have 3-modes. $\square$

To continue, we can give a better characterization of the stable and unstable manifolds $V_\pm$ of the singular equilibrium at the origin in the case that $k_1 k_2 > 0$. Firstly, in the case that $k_3 \geq 0$, the orbits are unbounded; there are no homo- nor heteroclinic orbits.

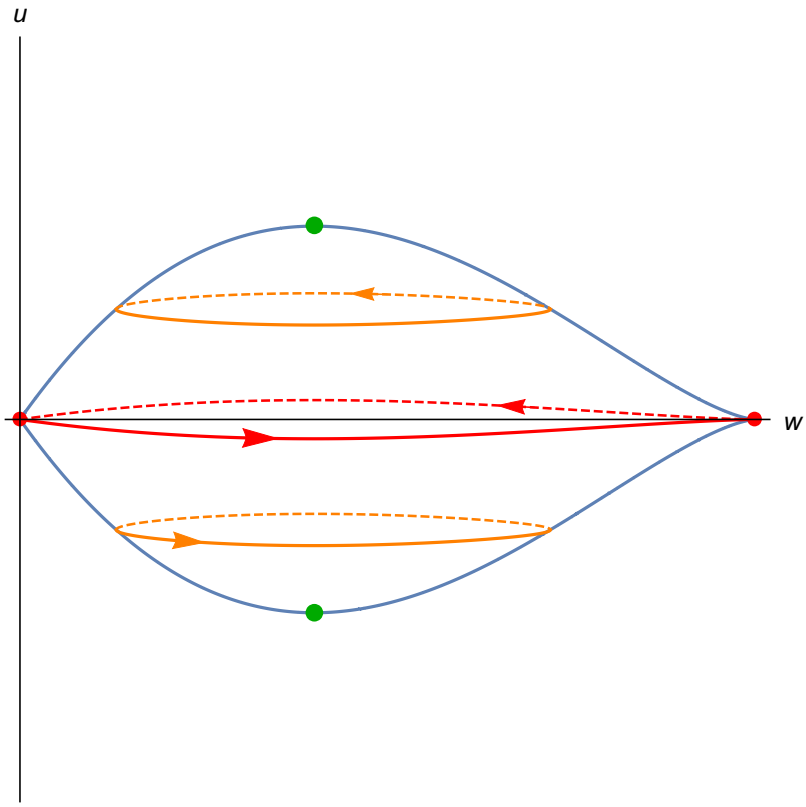

Figure 13: In blue the slice $\mathcal{P}_{k,0,\mu} \cap \{v = 0\}$, for the case $k = (1, 1, -3)$ and $\mu = 1$. The orange curves are periodic orbits in the planes parallel to (v, w)-plane; the red curves in the $\{u = 0\}$ plane are heteroclinic between the singular equilibria at the origin and at $(u, v, w) = (0, 0, \frac{\mu}{k_2(-k_3)})$. The green points at the top and bottom are ellipitic equilibria, where the tangent spaces of the phase space and the energy surface coincide.

However in the case that $k_3 = -1$, which we have seen in the case of the 1:3:4 resonance, the curves V_- and V_+ meet tangentially at $(u, v, w) = (0, 0, \frac{\mu}{k_2})$. The stable and unstable manifolds overlap and therefore there is a homoclinic orbit.

In the case that $k_3 \leq -2$, we have another equilibrium at $(u, v, w,) = (0, 0, \frac{\mu}{k_2(-k_3)})$. Indeed, the w-coordinate is a double root of $p_{k,0,\mu}$ and therefore the vector field as given in equation (4.17) vanishes. The interior of $V_- \cap R(k_3, \mu)$ is the stable manifold to the origin and the unstable manifold to p_0, and vice versa for the interior of $V_+ \cap R(k_3, \mu)$, and we have two heteroclinic orbits connecting them. This case is depicted in figure 13 for $k_3 = -3$.

In both cases, there are regular equilibria at

$$v = 0, \qquad w = w_+ := \frac{(|k_1| + |k_2|)\mu}{k_2(-k_3)|k|}, \qquad u = \pm\sqrt{p_{k,\lambda,\mu}(w_+)}.$$

Note that w_+ is a root of the derivative of $p_{k,\lambda,\mu}$, which together with $v = 0$, gives us a critical point for the vector field of $\mathcal{H}$.

4.2 Applicability

Of course an important question is whether for a given resonance the Hamiltonian $\mathcal{H}$ arises naturally. As remarked before, the most natural case is when the annihilator is of order 3. Also one would like that there are no other resonant terms present in the first normal form approximation, as their presence prohibits the reduction we performed to one degree of freedom. Hence, the answer is no for all of the genuine first order resonances, due to the large amount of resonant terms in the first normal forms. This could be remedied by discrete symmetry assumptions, which can appear naturally in certain applications, see [23] and references therein.

In the case of the genuine second order resonances, we already know an example, namely the 1:3:4 resonance. The theorem also applies to indefinite resonances, and therefore the normal modes of the indefinite 1:-3:4, 1:-3:-4 and 1:3:-4 resonances, with respective annihilators $(-1, 1, 1)$, $(-1, 1, -1)$ and $(1, 1, 1)$, are readily analysed. In particular, the 1:-3:4 resonance has an unstable 1-mode, the 1:-3:-4 resonance has an unstable 2-mode and the 1:3:-4 resonance has all normal modes unstable, globally for the first normal form and close to the origin for the higher order normal forms.

Other resonances of interest are the genuine second order resonances with a 1:2 subresonance (the 1:2:5, 1:2:6, 1:3:6, 2:3:4 and 2:3:6), and their indefinite siblings with a 1:-2 subresonance. These always have the annihilator (up to permutation) $(\pm 2, 1, 0)$, which means the degree of freedom not involved in the subresonance has a stable normal mode in the definite case, and an unstable normal mode in the indefinite case. The theorem unfortunately does not tell us about the dynamics of the other degrees of freedom. However, most systems with a 1:2-subresonance are quite well-understood, see e.g. [10].

If we also consider higher order resonances, there are whole families of resonances to which we could apply the theorem. The information for resonances with annihilator k of length $|k| = 3$ is summed up in table 4.

Resonance	Annihilator	Stable NM	Unstable NM
1:2 subresonance	(-2,1,0)	3	-
1:-2 subresonance	(2,1,0)	-	3
$l : m : l + m$	(1,1,-1)	1, 2	3
$l : -m : l + m$	(-1,1,1)	2,3	1
$l : -m : -(l + m)$	(-1,1,-1)	1,3	2
$l : m : -(l + m)$	(1,1,1)	-	1,2,3

Table 4: Stability of normal modes (NM) for the families of resonances with an annihilator k of length $|k| = 3$ and $k_2 > 0$. Note that these results holds only for the Hamiltonian $\mathcal{H}$ of equation (4.1).

5 Reducing the action generated by H_0^0.

The integrals T_{12} and T_{23} are lost as we pass on to the second normal form. Therefore we examine the reduction of the action generated by H_0^0, which remains valid. In this chapter we consider the reduced space and the Poisson algebra of the invariants, we recover the resonant 2-tori for the first normal form and present some new expressions of the vector field in two degrees of freedom, which is useful to obtain periodic orbits in general position.

The quadratic part H_0^0 of our Hamiltonian generates the flow

$$\varphi^t_{H_0^0}(x_1, x_2, x_3) = (e^{\mathrm{i}t}x_1, e^{3\mathrm{i}t}x_2, e^{4\mathrm{i}t}x_3) \tag{5.1}$$

which is an $\mathbb{S}^1$-action on the phase space $\mathbb{C}^3$. We are interested in the invariants of this action. Our first guess is to look at the annihilators $k \in \Gamma_{(1,3,4)}$. If we make a map L from the space of (complex, i.e. in x and its conjugate y) monomials in $\mathbb{C}^3$ to $\mathbb{Z}^3$ defined by

$$L : x_1^{k_1} y_1^{l_1} x_2^{k_2} y_2^{l_2} x_3^{k_3} y_3^{l_3} \mapsto (k_1 - l_1, k_2 - l_2, k_3 - l_3)$$

then such a monomial m is invariant under the action if and only if $L(m) \in \Gamma_{(1,3,4)}$.

The resonance lattice $\Gamma_{(1,3,4)}$ is two-dimensional. However, on the side of the monomials we have 15 invariants, which are – besides the τ_i – the monomials

$$\begin{array}{lll} J_1 = x_1^3 y_2 & J_2 = x_1^4 y_3, & J_3 = x_2^4 y_3^3, \\ K_1 = x_1 x_2 y_3, & K_2 = x_1^2 y_2^2 x_3, & K_3 = x_1 y_2^3 x_3^2 \end{array}$$

and their complex conjugates. We know these invariants suffice to generate any other invariants from [23], where we can find the generators of invariants for other genuine first and second order resonances as well. As before, the compactness of the group $\mathbb{S}^1$ acting on our phase space allows us to conclude that every smooth $\varphi^t_{H_0^0}$-invariant function can be written in terms of these invariants.

5.1 The Poisson algebra and syzygies of the invariants

The Poisson algebra in the invariants $\tau_i, J_i, \overline{J_i}, K_i, \overline{K_i},\ i = 1, 2, 3$, is given through tables 5, 6, 7 and 8. We omit the relations $\{\tau_i, \tau_j\}$ as these all vanish. Furthermore, we omit many brackets which can be deduced from the ones given, for example

$$\{J_1, \overline{K_1}\} = \overline{\{\overline{J_1}, K_1\}} = 6\mathrm{i}K_2.$$

After reduction of the $\mathbb{S}^1$-action, we expect a reduced phase space which is four-dimensional, and thus there should be relations, or syzygies, between these invariants; we should also take into account the Casimir H_0^0, the dynamics are restricted to its level sets. (We refer to this reduced phase space as the H_0^0*-reduced phase space*, to avoid confusion with the space we obtained in chapter 2.) Note that the syzygies describe singularities in the H_0^0-reduced phase space. These are due to the isotropy of the action (5.1); the action has isotropy group $\mathbb{Z}_3$ at the 2-mode and $\mathbb{Z}_4$ at the 3-mode.

From [24] we know that it is possible – although not necessarily optimal – to work locally with 7 invariants and a (possibly different) single monomial as divisor; here we do not take the Casimir H_0^0 into account.

$\{a,b\}$	J_1	J_2	J_3	K_1	K_2	K_3	b
τ_1	$-3\mathrm{i}J_1$	$-4\mathrm{i}J_2$	0	$-\mathrm{i}K_1$	$-2\mathrm{i}K_2$	$-\mathrm{i}K_3$	
τ_2	$\mathrm{i}J_1$	0	$-4\mathrm{i}J_3$	$-\mathrm{i}K_1$	$2\mathrm{i}K_2$	$3\mathrm{i}K_3$	
τ_3	0	$\mathrm{i}J_2$	$3\mathrm{i}J_3$	$\mathrm{i}K_1$	$-\mathrm{i}K_2$	$-2\mathrm{i}K_3$	
a							

Table 5: The Poisson brackets of the τ_i with the invariants J_j and K_j, $i,j=1,2,3$. The expressions for the brackets with the conjugates $\overline{J}_i, \overline{K}_i$ are obtained by conjugating the right entry.

$\{a,b\}$	J_1	J_2	J_3	b
J_1	0	0	$-8\mathrm{i}K_1^3$	
J_2	0	0	0	
J_3	$8\mathrm{i}K_1^3$	0	0	
$\overline{J_1}$	$16\mathrm{i}(\tau_1^3-9\tau_1^2\tau_2)$	$48\mathrm{i}\tau_1K_1$	0	
$\overline{J_2}$	$48\mathrm{i}\tau_1\overline{K_1}$	$32\mathrm{i}(\tau_1^4-16\tau_1^3\tau_3)$	$6\mathrm{i}\overline{K_2}^2$	
$\overline{J_3}$	0	$6\mathrm{i}K_2^2$	$128\mathrm{i}(9\tau_2^4\tau_3^2-16\tau_2^3\tau_3^3)$	
a				

Table 6: The Poisson brackets of the J_i's between themselves and their conjugates.

$\{a,b\}$	K_1	K_2	K_3	b
J_1	$-2\mathrm{i}J_2$	0	0	
J_2	0	$-2\mathrm{i}J_1^2$	$-2\mathrm{i}J_1K_2$	
J_3	0	$4\mathrm{i}(8\tau_2\tau_3-3\tau_2^2)K_1^2$	$192\mathrm{i}(2\tau_2^2\tau_3^2-\tau_2^3\tau_3)K_1$	
$\overline{J_1}$	$-6\mathrm{i}\overline{K_2}$	$16\mathrm{i}(\tau_1^2-3\tau_1\tau_2)\overline{K_1}$	$12\mathrm{i}(\tau_1-\tau_2)\overline{K_1}^2$	
$\overline{J_2}$	$4\mathrm{i}(\tau_1-4\tau_3)\overline{J_1}$	$-32\mathrm{i}\tau_1\overline{K_1}^2$	$-8\mathrm{i}\overline{K_1}^3$	
$\overline{J_3}$	$4\mathrm{i}(3\tau_2-4\tau_3)K_3$	0	0	
a				

Table 7: The Poisson brackets relations of the J_j's and their conjugates with the K_i's. We note that the expressions for the brackets with the conjugates $\overline{K_i}$'s are obtained by conjugating the right entry .

Apart from the generators, [23] also mentions the following syzygies:

$$K_2^2 \equiv J_1K_3, \qquad K_3^2 \equiv \overline{J_3}K_2 \qquad K_1^4 \equiv J_2J_3. \tag{5.2}$$

Of course the conjugate syzygies are also valid.

Using the map L, we can easily find more relations. Let us denote $\vec{\jmath}_i = L(J_i), \vec{k}_i = L(K_i)$, and observe that $2\tau_i \in \ker L$. Since the resonance lattice $\Gamma_{(1,3,4)}$ is two-dimensional, we only need 2 out of these 6 vectors to generate the lattice. This way we obtain relations between the monomials, if we keep track of the amount of factors $(2\tau_i)$ which get sent to 0.

$\{a,b\}$	K_1	K_2	K_3	b
K_1	0	$-4\mathrm{i}(\tau_2-2\tau_3)J_2$	$-4\mathrm{i}(2\tau_2-3\tau_3)K_2$	
K_2	$4\mathrm{i}(\tau_2-2\tau_3)J_1$	0	0	
K_3	$4\mathrm{i}(2\tau_2-3\tau_3)K_2$	0	0	
$\overline{K_1}$	$8\mathrm{i}(\tau_1\tau_2-\tau_1\tau_3-\tau_2\tau_3)$	$-4\mathrm{i}K_3$	$-2\mathrm{i}\overline{J_3}$	
$\overline{K_2}$	$-4\mathrm{i}\overline{K_3}$	$32\mathrm{i}(4\tau_1^2\tau_2\tau_3-4\tau_1^2\tau_2\tau_3-\tau_1^2\tau_2^2)$	$32\mathrm{i}(3\tau_1\tau_2\tau_3-\tau_2^2\tau_3-\tau_1\tau_2^2)\overline{K_1}$	
$\overline{K_3}$	$-2\mathrm{i}J_3$	$32\mathrm{i}(3\tau_1\tau_2\tau_3-\tau_2^2\tau_3-\tau_1\tau_2^2)K_1$	$64\mathrm{i}(9\tau_1\tau_2^2\tau_3^2-4\tau_1\tau_2^3\tau_3-\tau_2^3\tau_3^2)$	

Table 8: The Poisson brackets of the K_j's between themselves and their conjugates.

As an example we choose $\vec{k}_1, \vec{\jmath}_1$ and let us obtain an expression for $\vec{\jmath}_2$ in terms of the former two. From

$$\vec{k}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{j}_1 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

we obtain

$$\vec{\jmath}_2 = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} = \vec{\jmath}_1 + \vec{k}_1.$$

From this we get the syzygy $2\tau_2 J_2 \equiv J_1 K_1$, because we cancelled once in the direction $(0,1,0)$. Now we give the expressions for the other annihilators in terms of $\vec{k}_1$ and $\vec{\jmath}_1$. We have

$$\begin{aligned} \vec{\jmath}_3 &= (0,4,-3) = 3\vec{k}_1 - \vec{\jmath}_1, \\ \vec{k}_2 &= (2,-2,1) = -\vec{k}_1 + \vec{\jmath}_1, \\ \vec{k}_3 &= (1,-3,2) = -2\vec{k}_1 + \vec{\jmath}_1, \end{aligned}$$

from which we can recover the syzygies (keeping in mind that if $L(p) = \vec{l}$ then $L(\overline{p}) = -\vec{l}$)

$$\begin{aligned} 2\tau_2 J_2 &\equiv J_1 K_1, \\ (2\tau_1)^3 J_3 &\equiv K_1^3 \overline{J_1}, \\ 2\tau_1 K_2 &\equiv \overline{K_1} J_1, \\ (2\tau_1)^2 K_3 &\equiv \overline{K_1}^2 J_1, \end{aligned} \tag{5.3}$$

and by conjugating both sides, we get similar syzygies for the conjugates $\overline{J_i}$ and $\overline{K_i}$.

In the reduction of the first normal form, in chapter 2, we encountered the syzygy Q which can now be written as

$$Q(\tau, K_1, \overline{K_1}) = 8\tau_1\tau_2\tau_3 - K_1\overline{K_1}. \tag{5.4}$$

For each pair of invariant and its conjugate we get a similar syzygy:

$$\begin{aligned} J_1\overline{J_1} &= 2^4\tau_1^3\tau_2, & K_1\overline{K_1} &= 2^3\tau_1\tau_2\tau_3, \\ J_2\overline{J_2} &= 2^5\tau_1^4\tau_3, & K_2\overline{K_2} &= 2^5\tau_1^2\tau_2^2\tau_3, \\ J_3\overline{J_3} &= 2^7\tau_2^4\tau_3^3, & K_3\overline{K_3} &= 2^6\tau_1\tau_2^3\tau_3^2. \end{aligned} \tag{5.5}$$

This list of syzygies is not exhaustive Lastly, because the flow is generated by H_0^0, H_0^0 is a Casimir in the H_0^0-reduced phase space. Therefore dynamics take place in the intersection of the H_0^0-reduced phase space with the surfaces

$$\{(\tau, J, K) \in \mathbb{R}^3 \times \mathbb{C}^6 : \tau_1 + 3\tau_2 + 4\tau_3 = \eta, \tau_i \geq 0\} \tag{5.6}$$

where η is the value of H_0^0.

5.2 The first normal form in two degrees of freedom

We start our analysis of the dynamics in two degrees of freedom by considering the first normal $\bar{H}^1$ in terms of the 15 invariants. The H_0^0-reduced Hamiltonian is given by

$$\bar{H}^1 = H_0^0 + A\,\mathrm{Re}(K_1) \tag{5.7}$$

and so the Hamiltonian vector field of $\bar{H}^1$ on the H_0^0-reduced space is

$$X_{\bar{H}^1} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ J_1 \\ J_2 \\ J_3 \\ K_1 \\ K_2 \\ K_3 \end{pmatrix} = A \begin{pmatrix} \mathrm{Im}(K_1) \\ \mathrm{Im}(K_1) \\ -\mathrm{Im}(K_1) \\ -\mathrm{i}J_2 + 3\mathrm{i}K_2 \\ 2\mathrm{i}(4\tau_3 - \tau_1)J_1 \\ 2\mathrm{i}(4\tau_3 - 3\tau_2)\overline{K_3} \\ 4\mathrm{i}(\tau_1\tau_3 + \tau_2\tau_3 - \tau_1\tau_2) \\ 2\mathrm{i}(\tau_2 - 2\tau_3)J_1 + 2\mathrm{i}K_3 \\ 2\mathrm{i}(2\tau_2 - 3\tau_3)K_2 + \mathrm{i}\overline{J_3} \end{pmatrix}. \tag{5.8}$$

Let us look for equilibria of this vector field. Firstly for $\dot{\tau}_j = 0$ we need $\mathrm{Im}(K_1) = 0$. As we saw for the reconstruction in section 2.7, requiring the equality

$$\tau_1\tau_2 = \tau_1\tau_3 + \tau_2\tau_3, \tag{5.9}$$

to hold suffices to keep $\mathrm{Im}(K_1)$ at 0, as then $\dot{K}_1$ vanishes.

However, if we require that at least one of the invariants τ_j, $j = 1, 2, 3$ vanishes, then to satisfy the above equation another τ_i, $i \neq j$ must vanish as well. In that case, through the syzygies shown in (5.5), all the invariants J_j, K_j, $j = 1, 2, 3$ and their conjugates vanish, and we have found an equilibrium for (5.8). These equilibria correspond to the normal modes. Analysis of the eigenvalues of the linearized system yields that the 1- and 2-modes are elliptic, while the 3-mode is hyperbolic.

If we require that $\tau_1, \tau_2, \tau_3 \neq 0$, then the equalities $\mathrm{Im}(K_1) = 0$ and (5.9) imply that the motion in two degrees of freedom is periodic. We can show this quite neatly making use of the syzygies of (5.3) to write

$$\begin{aligned} \dot{J}_1 &= \frac{-\mathrm{i}AJ_1K_1}{2\tau_2} + \frac{3\mathrm{i}AJ_1\overline{K_1}}{2\tau_1} \\ &= \frac{\mathrm{i}A}{2}\frac{\tau_2 - 3\tau_1}{\tau_1\tau_2}K_1J_1 \end{aligned} \tag{5.10}$$

where for the last equality we make use of the fact that $K_1 = \overline{K_1}$. Therefore we obtain the equation

$$\begin{aligned} \ddot{J}_1 &= \frac{-A^2}{4}\left(\frac{3\tau_2 - \tau_1}{\tau_1\tau_2}\right)^2 K_1^2 J_1 \\ &= -2A^2\frac{(3\tau_2 - \tau_1)^2}{\tau_1 + \tau_2} J_1. \end{aligned} \tag{5.11}$$

To obtain the last equality we made use of (5.9). Since the invariants τ_j, $j = 1, 2, 3$ are fixed, we see that we have oscillatory behaviour for J_1. By conjugation, the same holds for $\overline{J_1}$. The frequency of the oscillation is

$$\beta := \sqrt{2}A\frac{|3\tau_2 - \tau_1|}{\sqrt{\tau_1 + \tau_2}} \tag{5.12}$$

We can do a similar analysis for the other invariants, or we make use of the observation that we can write all the other invariants in terms of the invariants τ_j, J_1, K_1 and their conjugates, as we know the τ_j do not vanish. For example, we have

$$J_3(t) = \frac{\overline{J_1}(t)K_1^3}{8\tau_1^3} \tag{5.13}$$

which therefore has the same oscillatory behaviour, with the same frequency. As all the frequencies coincide, we conclude that the points satisfying $\mathrm{Im}(K_1) = 0$ and (5.9) form a circle of periodic points.

There is, moreover, a special case, if the equality $\tau_1 = 3\tau_2$ holds. (This, together with (5.9), implies that also $\tau_1 = 4\tau_3$ or vice versa.) In this case the frequency of the oscillation vanishes, and we can deduce that we have not periodic motion, but an equilibrium for the vector field above. In fact, for each initial value of J_1 we get an equilibrium and thus we get an entire circle of equilibria.

In three degrees of freedom, these are the resonant 2-tori we found in section 2.7 with the reconstruction of the dynamics of $\bar{H}^1$. One can verify the vanishing of other terms explicitly, but it is easier to note that in that case the invariants τ_j, K_1, J_1 and their conjugates are all conserved, and thus by the syzygies (5.3), all invariants are.

The family of these equilibria is parametrized by the energy of the quadratic Hamiltonian. We have

$$H_0^0 = \eta = \tau_1 + 3\tau_2 + 4\tau_3 = 3\tau_1, \tag{5.14}$$

so we obtain $\tau_1 = \eta/3, \tau_2 = \eta/9, \tau_3 = \eta/12$. This is precisely the case that $\lambda = 7/8\mu$ in the notation of chapter 2.

5.3 The second resonant term

Let us consider the second normal form

$$\bar{H}^2 = H_0^0 + A\,\mathrm{Re}(K_1) + \sum_{j=1}^{3} c_{ij}\tau_i\tau_j + B\,\mathrm{Re}(J_1) \tag{5.15}$$

where we note that we set B to be a positive real number through a rotation of the (q_2, p_2)-plane. We take again the matrix $(c_{ij})_{i,j}$ to be symmetric.

The vector field due to the second resonant term is

$$X_{B\,\mathrm{Re}(J_1)}\begin{pmatrix}\tau_1\\ \tau_2\\ \tau_3\\ J_1\\ J_2\\ J_3\\ K_1\\ K_2\\ K_3\end{pmatrix} = B\begin{pmatrix}3\,\mathrm{Im}(J_1)\\ -\mathrm{Im}(K_1)\\ 0\\ 8\mathrm{i}(9\tau_1^2\tau_2-\tau_1^3)\\ -24\mathrm{i}\tau_1K_1\\ -4\mathrm{i}K_1^3\\ \mathrm{i}J_2+3\mathrm{i}\overline{K_2}\\ 8\mathrm{i}(3\tau_1\tau_2-\tau_1^2)\overline{K_1}\\ -6\mathrm{i}(\tau_1-\tau_2)\overline{K_1}^2\end{pmatrix}. \tag{5.16}$$

It is interesting to note that, quite in accordance with Lyapunovs theorem 4.1, the 1-mode is not invariant for the vector field above. Setting $\mathrm{Im}(J_1)=0$ and $\tau_2=\tau_3=0$, we are left with $\dot{J}_1=-8\mathrm{i}\tau_1^3\neq 0$. We give the full vector field of $\bar{H}^2$ for the invariants τ_j, J_1, K_1, making use of the syzygies (5.3). We have

$$X_{\bar{H}^2}\begin{pmatrix}\tau_1\\ \tau_2\\ \tau_3\\ J_1\\ K_1\end{pmatrix} = \begin{pmatrix}A\,\mathrm{Im}(K_1)+3B\,\mathrm{Im}(J_1)\\ A\,\mathrm{Im}(K_1)-B\,\mathrm{Im}(J_1)\\ -A\,\mathrm{Im}(K_1)\\ \Big(\mathrm{i}A\Big(-\frac{K_1}{2\tau_2}+3\frac{\overline{K_1}}{2\tau_1}\Big)-2\mathrm{i}c_{J_1}(\tau)\Big)J_1+8\mathrm{i}B(9\tau_1^2\tau_2-\tau_1^3)\\ \Big(\mathrm{i}B\Big(\frac{J_1}{2\tau_2}+3\frac{\overline{J_1}}{2\tau_1}\Big)-2\mathrm{i}c_{K_1}(\tau)\Big)K_1+4\mathrm{i}A(\tau_1\tau_3+\tau_2\tau_3-\tau_1\tau_2)\end{pmatrix}, \tag{5.17}$$

where

$$\begin{aligned} c_{J_1}(\tau) &= (3c_{11}-c_{12})\tau_1+(3c_{12}-c_{22})\tau_2+(3c_{13}-c_{23})\tau_3,\\ c_{K_1}(\tau) &= (c_{11}+c_{12}-c_{13})\tau_1+(c_{12}+c_{22}-c_{23})\tau_2+(c_{13}+c_{23}-c_{33})\tau_3. \end{aligned} \tag{5.18}$$

Periodic solutions can be found by setting $\mathrm{Im}(K_1)=\mathrm{Im}(J_1)=0$ and solving $\dot{J}_1=0$ and $\dot{K}_1=0$, which using the syzygies (5.5) can be written as rational equations in the invariants τ_j. Note that one can make use of the integral $H_0^0=\eta$. These periodic solutions reconstruct to condtionally periodic orbits on invariant 2-tori in $\mathbb{C}^3$ by superposing the $\mathbb{S}^1$-orbit of H_0^0.

6 Indefinite resonances

If one encounters an elliptic equilibrium of a Hamiltonian vector field, chances are that the quadratic part of the Hamiltonian,

$$H_0^0 = \tfrac{1}{2}\sum_{j=1}^{n} \omega_j (q_j^2 + p_j^2),$$

is not definite. (For this expression we assume no nilpotent parts are present in the linearization.) Equivalently, the components of the frequency vector ω have different signs. In case ω is fully resonant and non-degenerate, we speak of an *indefinite resonance*.

The dynamics of normal forms for indefinite resonances can differ largely from those of their definite counterparts. Foremostly, the elliptic equilibrium need not be dynamically stable. Indeed, in the definite case the quadratic Hamiltonian serves as a Lyapunov function, but it does not fullfill this purpose for indefinite resonances. Therefore unbounded orbits may appear. Another example of possible differences was presented in table 4. There we saw that the stability of the normal modes for indefinite resonances differs from the definite counterparts.

The ±1:±3:±4 resonances.

The indefinite resonances -1:3:4, 1:-3:4 and 1:3:-4 are closely related to the definite 1:3:4 resonance. (Note that the resonances -1:-3:-4, -1:-3:4, -1:3:-4 and 1:-3:-4 can be obtained from the ones mentioned by reversing the time direction.) In particular, the annihilators are similar: they only differ in their relative signs, as we can see in table 9. Therefore the orders of the annihilators are the same, and we conclude that the order of appearance of their associated invariants (through $k \mapsto x^k$) in the normal forms is also the same. The invariants of the action of H_0^0 are similar as well. However, we already saw that the sign of the annihilators is very important for the dynamics in theorem 4.3, as relative signs determine the stability of normal modes.

Resonance	Annihilators	Invariants
1:3:4	$(1,1,-1),\ (3,-1,0),\ (4,0,-1),$ $(2,-2,1),\ (1,-3,2),(0,4,-3).$	$x_1x_2y_3,\ x_1^3y_2,\ x_1^4y_3,$ $x_1^2y_2^2x_3,\ x_1y_2^3x_3^2,\ x_2^4y_3^3.$
-1:3:4	$(1,-1,1),\ (3,1,0),\ (4,0,1),$ $(2,2,-1),\ (1,3,-2),(0,4,-3).$	$x_1y_2x_3,\ x_1^3x_2,\ x_1^4x_3,$ $x_1^2x_2^2y_3,\ x_1x_2^3y_3^2,\ x_2^4y_3^3.$
1:-3:4	$(1,-1,-1),\ (3,1,0),\ (4,0,-1),$ $(2,2,1),\ (1,3,2),(0,4,3).$	$x_1y_2y_3,\ x_1^3x_2,\ x_1^4y_3,$ $x_1^2x_2^2x_3,\ x_1x_2^3x_3^2,\ x_2^4x_3^3.$
1:3:-4	$(1,1,1),\ (3,-1,0),\ (4,0,1),$ $(2,-2,-1),\ (1,-3,-2),(0,4,3).$	$x_1x_2x_3,\ x_1^3y_2,\ x_1^4x_3,$ $x_1^2y_2^2y_3,\ x_1y_2^3y_3^2,\ x_2^4x_3^3.$

Table 9: The annihilators (up to sign) and invariants (up to conjugation) for the 1:3:4 resonance and its indefinite counterparts.

6.1 In one degree of freedom

In chapter 2, we found the integrals $T_{12} := \tau_1 - \tau_2$ and $T_{23} := \tau_2 + \tau_3$ for the first normal form of the 1:3:4 resonance. By reducing the two $\mathbb{S}^1$-actions generated by T_{12} and T_{23} we were able to study the dynamics in one degree of freedom. For the indefinite cases we can proceed similarly, although the integrals are slightly different. We give the integrals in table 10, together with the information needed to obtain the reduced phase spaces.

Resonance	Integrals	$q_{\lambda,\mu}(w)$	Restrictions R
1:3:4	$\lambda = \tau_1 - \tau_2, \quad \mu = \tau_2 + \tau_3$	$8(\lambda + w)w(\mu - w)$	$[\max(0, -\lambda), \mu]$
-1:3:4	$\lambda = \tau_1 + \tau_2, \quad \mu = \tau_2 + \tau_3$	$8(\lambda - w)w(\mu - w)$	$[0, \min(\lambda, \mu)]$
1:-3:4	$\lambda = \tau_1 + \tau_2, \quad \mu = -\tau_2 + \tau_3$	$8(\lambda - w)w(\mu + w)$	$[\max(0, -\mu), \lambda]$
1:3:-4	$\lambda = \tau_1 - \tau_2, \quad \mu = -\tau_2 + \tau_3$	$8(\lambda + w)w(\mu + w)$	$[\max(0, -\lambda, -\mu), \infty)$

Table 10: Here we see the integrals of the first normal form for the indefinite counterparts of the 1:3:4 resonance, together with the information needed to obtain the reduced phase spaces $\mathcal{P}_{\lambda,\mu}$.

Indeed, the reduced phase space $\mathcal{P}_{\lambda,\mu}$ can in all cases be obtained by

$$\mathcal{P}_{\lambda,\mu} = \{(u, v, w) \in \mathbb{R}^2 \times R : u^2 + v^2 = q_{\lambda,\mu}(w)\}. \tag{6.1}$$

The table 10 leads to an interesting observation about the dynamic stability of the origin. For the -1:3:4 and 1:-3:4 resonances, the reduced phase spaces are compact, due to the restrictions on the coordinate w. The length of the interval R increases linearly with the value of one of the integrals T_{12} or T_{23}. This makes us suspect that orbits close to the origin are bounded.

Indeed, in the case of the -1:3:4 resonance, the sum

$$T_{12} + T_{23} = \tau_1 + 2\tau_2 + \tau_3 = \lambda + \mu \tag{6.2}$$

is definite and an integral as well. This sum serves as a Lyapunov function to the equilibrium at the origin which is therefore stable. For the 1:-3:4 resonance, the sum $T_{12} + T_{13} = 2\tau_1 + \tau_2 + \tau_3 = \lambda - \mu$ does the trick. We conclude that for these two resonances, although they are both indefinite, the origin is dynamically stable for the first normal form.

Up to a permutation of the coordinate indices, the analysis of the first normal form for the -1:3:4 and 1:-3:4 resonances does not differ much from that of the 1:3:4 resonance. In each case we recover two stable normal modes and an unstable one, the latter with a homoclinic orbit. This is in line with table 4, as we have in both cases, up to permutation, the annihilator $k = (1, 1, -1)$. Furthermore, notwithstanding the cases corresponding to the stable normal modes, we find two regular equilibria; these correspond to conditionally periodic orbits on invariant 2-tori in the original phase space.

However, unlike the definite case, we do not find a family of resonant tori foliated by periodic motion. Indeed, recall that for the resonant tori of the 1:3:4 resonance we had the requirements

$$\tau_1\tau_2 = \tau_1\tau_3 + \tau_2\tau_3 \quad \text{and} \quad \tau_1 = 4\tau_3.$$

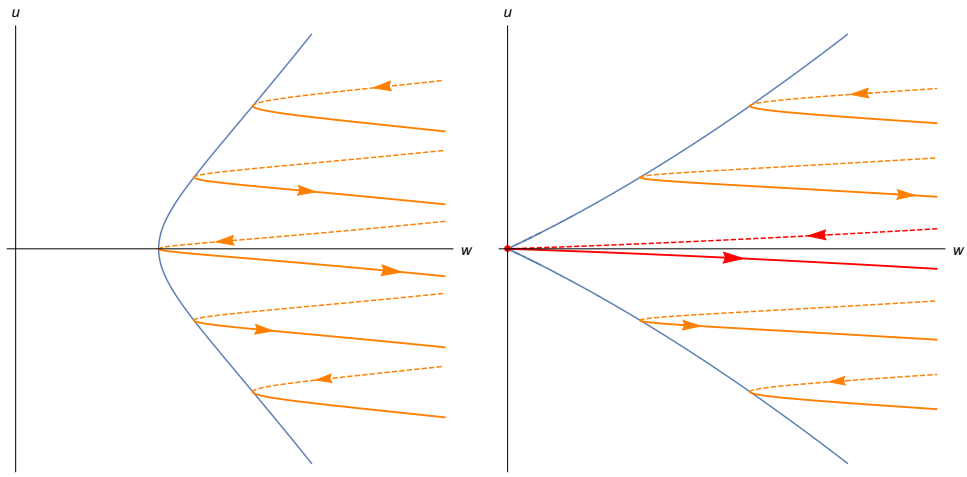

Figure 14: In blue the slice $\mathcal{P}_{\lambda,\mu} \cap \{v = 0\}$, for the indefinite 1:3:-4 resonance, for the cases $\lambda = 1, \mu = -1/2$ (left) and $\lambda = 1, \mu = 0$ (right). The orange curves are unbounded orbits in the (v, w)-planes; the red curve in the $\{u = 0\}$ plane is homoclinic to the singular equilibrium at the origin.

These could be satisfied by taking $\tau_1 = 3\tau_2 = 4\tau_3$. However, if we take for example the -1:3:4 resonance, we get a requirement of the form

$$\tau_2\tau_3 = \tau_1\tau_2 + \tau_1\tau_3 \quad \text{and} \quad \tau_1 = -4\tau_3$$

Due to the restrictions on the invariants τ_j, the latter requirement only leads to the trivial solution $\tau_1 = 0 = \tau_3$, which gives us back the 2-mode.

6.2 The 1:3:-4 resonance

This picture is quite different for the 1:3:-4 resonance. We already know that for the first normal form of this resonance all three normal modes are unstable, see table 4. The first normal form is given by

$$\bar{H}^1 = \tau_1 + 3\tau_2 - 4\tau_3 + A\,\mathrm{Re}(x_1x_2x_3). \tag{6.3}$$

We now have the integrals $T_{12} = \tau_1 - \tau_2$ and $T_{23} = \tau_2 - \tau_3$. Examples of the reduced phase spaces are shown in figures 14 and 15. Note the absence of the regular equilibria compared to the definite case, e.g. figure 1.

The reduced space is given by

$$\mathcal{P}_{\lambda,\mu} := \{(u, v, w) \in \mathbb{R}^2 \times [\max(0, -\lambda, -\mu), \infty) : u^2 + v^2 = 8(\lambda + w)w(\mu + w)\} \tag{6.4}$$

and is in particular not compact. For $\mu = 0, \lambda > 0$, $\lambda = 0, \mu > 0$ we have a conical singularity at the origin corresponding to respectively the 1-mode and the 3-mode. For $\mu = \lambda < 0$, we recover the 2-mode at $w = -\mu$. If both λ and μ vanish, the syzygy takes the form

$$u^2 + v^2 = 8w^3 \tag{6.5}$$

and we have a cusp singularity at the origin, which in this case also corresponds with the origin in $\mathbb{C}^3$. This is illustrated in figure 15. The origin is dynamically unstable; however a small perturbation (respecting the integrals T_{12} and T_{23}) of the Hamiltonian, such as a detuning or the addition of the quartic self-interaction terms, as in chapter 3, could stabilize it.

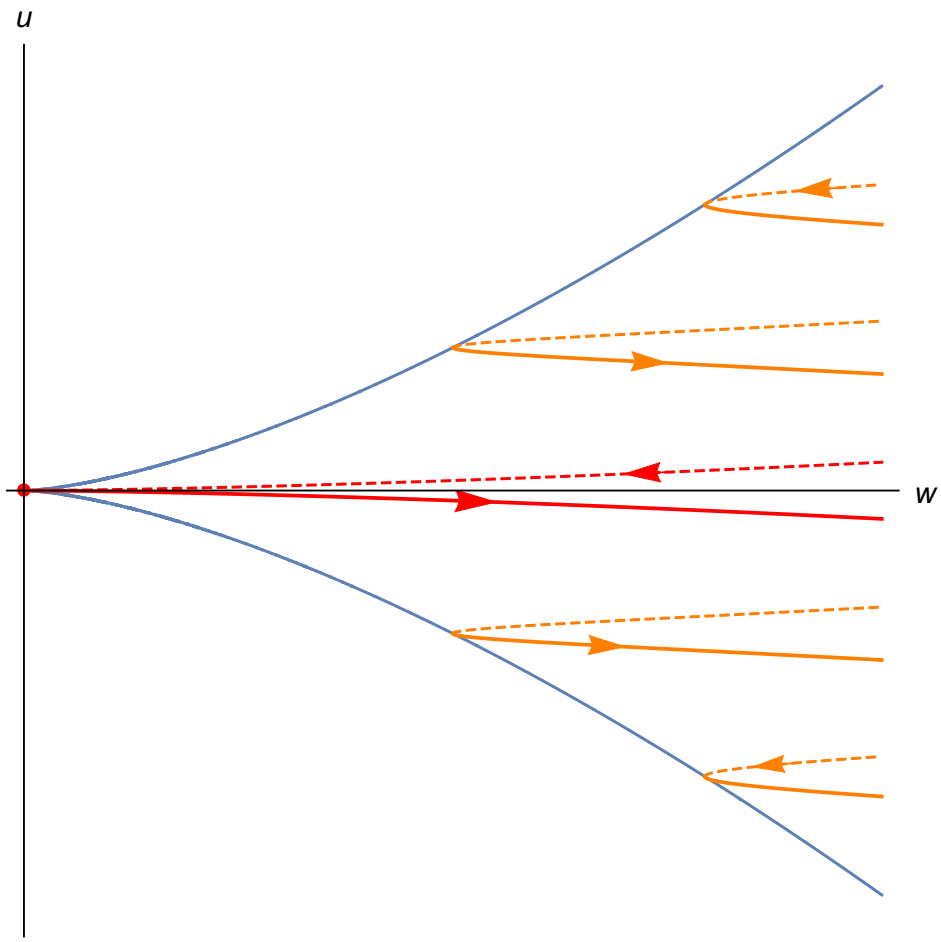

Figure 15: In blue the slice $\mathcal{P}_{\lambda,\mu} \cap \{v = 0\}$ for the indefinite 1:3:-4 resonance, for the case $\lambda = 0, \mu = 0$ The orange curves are unbounded orbits in the (v, w)-planes; the red curve in the $\{u = 0\}$ plane is homoclinic to the (cusply) singular equilibrium at the origin.

Stabilization of normal modes

In line with conjecture 4.4, we expect the normal modes of the first normal form of the 1:3:-4 resonance to change stability by Hamiltonian Hopf bifurcations if we detune or add the quartic self-interaction terms.

For simplicity's sake, we restrict to checking what happens as we detune the Hamiltonian. Unlike the definite case, we expect that the bifurcations due to detuning are subcritical instead of supercritical. This is due to the concavity of the reduced phase spaces, as opposed to the convex reduced spaces of the 1:3:4 resonance.

Let us consider the 2-modes, for which we need $\mu = \lambda < 0$. As in chapter 3, we check the derivatives of the curves at the singularity (at $w = -\mu = -\lambda > 0$)

$$q^{\pm}(w) = \alpha w \pm \sqrt{8}A(\mu + w)\sqrt{w}. \tag{6.6}$$

We find firstly that

$$\frac{\mathrm{d}q^{\pm}}{\mathrm{d}w}\Big|_{w=-\mu} = \alpha \pm \sqrt{8}A\sqrt{-\mu}. \tag{6.7}$$

For the bifurcation at $\alpha = -\sqrt{8}A\sqrt{-\mu}$, we calculate the second derivative of q^{+}. We have

$$\frac{\mathrm{d}^2 q^{\pm}}{\mathrm{d}w^2}\Big|_{w=-\mu} = \frac{1}{2\sqrt{-\mu}} > 0, \tag{6.8}$$

indicating that the bifurcation is indeed subcritical. A similar analysis holds for the other bifurcation for $\alpha = \sqrt{8}A\sqrt{-\mu}$.

The subcritical Hamiltonian Hopf bifurcation comes with subordinate centre-saddle bifurcations; as the normal modes stabilize, we find new stable and unstable regular equilibria close to it; the majority of the orbits remain unbounded.

Reconstruction and persistence

An important difference with the definite case is the presence of unbounded orbits in the dynamics of the normal form approximations. For the first normal form these orbits do not reconstruct to conditionally periodic motion on invariant 3-tori; their geometry is of the form $\mathbb{T}^2 \times \mathbb{R}$. Therefore the results of KAM-theory do not apply.

Due to detuning of the resonance and the addition of quartic self-interaction terms stable invariant periodic motion and 2-tori origin may appear close to the origin of $\mathbb{C}^3$; they "compactify" the energy shells. In a vicinity of these we find invariant 3-tori; the families of invariant 3-tori with Diophantine frequencies may persist if the frequency mapping is sufficiently non-degenerate. However, the same objections apply as in the definite case: the second normal form cannot be considered as a small perturbation of the first normal form with quartic self-interaction terms, due to the same order of the polynomials. For better persistence results, a full study of the second normal form is required.

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