Utrecht University
Departement of Mathematics

Master thesis Mathematical Sciences

On proving the Casselman-Wallach globalization theorem

Author:
M.A. Lip
Student number 4107276

Supervisor:
prof. dr. E.P. van den Ban
Second reader:
dr. J.W. van de Leur

July 5, 2018

Abstract

The recent proof of the Casselman-Wallach globalization theorem by J. Bernstein and B. Krötz is studied. Some parts are emphasized, while others are not treated in full detail. For certain parts where the original article [4] was not completely clear to us, more details are provided. In particular, more attention is given to proving that minimal principal series representations are good, and emphasis is placed on our new definition of these Harish-Chandra modules being of $\mathcal{D}$-type.

Contents

1 Introduction

Representation theory is a field of study that has applications in various fields, such as particle physics, chemistry, and of course several fields in mathematics. If we have a reductive Lie group G with maximal compact subgroup K, then the representation theory of G can be studied more algebraically by looking at so-called Harish-Chandra modules. These Harish-Chandra modules are $(\mathfrak{g}, K)$-modules that have finite K-multiplicities, and that are finitely generated. If one has a smooth representation E of G, taking the K-finite vectors yields a $(\mathfrak{g}, K)$-module, and if this yields a Harish-Chandra module isomorphic to V, we say that E is a globalization of V.

The Casselman-Wallach globalization theorem ([5]) states that if G is a linear real reductive Lie group, then every Harish-Chandra module V has a globalization in the category of smooth admissible moderate growth Fréchet representations of G, which is unique up to isomorphism. This theorem has been stated and proven by W. Casselman and N. Wallach in 1989, in [5]. While the result is rather straightforward and useful (as will be shown in an example below), the proof itself was very technical in nature. Because of this ([3]), in 2014 J. Bernstein and B. Krötz have published a different proof of the theorem in [4]. The objective of this thesis is to study this new proof, give an overview of the techniques used, and try to fill in some of the parts that were not fully worked out in the original article. As such, the research question for the thesis is "How is the Casselman-Wallach globalization theorem proved?".

To answer this question, the structure of this work will be as follows: below, we will motivate the importance of the theorem by looking at Helgason's conjecture. In Section 2, we will explain all the terminology used, and the necessary results to understand the rest of the work. We will then give an overview of the structure of the proof in Section 3, going into full detail on some steps, but referring to the original article, [4], for the proofs of other steps. Sections 4 and 5 are dedicated to proving the core result that is needed in the article, Theorem 46. The first of these two sections reduces the general case to a more manageable special case, while the second section proves the theorem for this case. In the Discussion, Section 6, we will discuss one particular step in the proof, Lemma 64, that we have not been able to solve yet, and detail some of the work that has been done in trying to prove this lemma. Finally, in Section 7, the Conclusion, we will give a summary of the proof and the used strategies.

Throughout this work, for every definition, lemma or theorem that is taken directly or slightly modified from some other work, we will indicate the source directly after the number of the respective definition, lemma or theorem. If only a name or page number is given, the source will always be the article by Krötz and Bernstein, [4]. Any definition or result that does not have a source listed is either original, or a standard result that we have re-proved for the sake of understanding the argument better. Since most of the work will be based on the article [4], at the end of each of the main sections we will include a brief reflection on which parts are simply explaining or outlining the work of Bernstein and Krötz, and which parts are original work needed to complete our understanding.

1.1 The Helgason conjecture

We will treat here briefly an example of representation theory to which the Casselman-Wallach globalization theorem can be applied, namely the Helgason conjecture.

Let G be a connected semisimple real Lie group, with maximal compact subgroup K and Iwasawa decomposition $G = KAN$. Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and let L_λ denote the line bundle $G \times_P \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is a one-dimensional complex linear space, equipped with the P-module structure $(man)z = a^{\lambda-\rho}z$. The space of smooth sections of this line bundle is given by

$$C^\infty(G/P, L_\lambda) = \{f \in C^\infty(G) \mid f(xman) = a^{\lambda-\rho}f(x)\}.$$

Let $\mathcal{P}_\lambda : C^\infty(G/P, L_\lambda) \to C^\infty(G/K)$ be the map sending a function f to the function

$$\mathcal{P}_\lambda f(x) := \int_K f(xk)\mathrm{d}k \in C^\infty(G/K).$$

This map is known as the Poisson transform, and it is known ([7]) that this maps $C^\infty(G/P, L_\lambda)$ to the space of joint eigenfunctions for $\mathbb{D}(G/K)$, which denotes the space of left G-invariant differential operators. In particular, S. Helgason has proved in 1977 ([7]) that if Re λ is dominant, then taking the K-finite vectors on both sides yields an isomorphism of $(\mathfrak{g}, K)$-modules

$$\mathcal{P}_\lambda : C^\infty(G/P; L_\lambda)_K \to \mathcal{E}_\lambda(G/K)_K.$$

Here $\mathcal{E}_\lambda(G/K)$ denotes the aforementioned space of joint eigenfunctions. Helgason also conjectured that in order to reach all the eigenfunctions on the right-hand side, and not just the K-finite functions, the map would have to be extended to the space of hyperfunctions $\mathcal{B}(G/P, L_\lambda)$. This was indeed correct, and M. Kashiwara et al. have proved Helgason's conjecture in [8]. On the other hand, one could wonder what the image of the space $C^\infty(G/P; L_\lambda)$ is exactly under this isomorphism. This has already been investigated by Wallach and Casselman ([11]), and by Van den Ban and Schlichtkrull ([2]), but we will recover the result here by applying the Casselman-Wallach globalization theorem. We have that $\mathcal{P}_\lambda$ is a G-equivariant map, that is an isomorphism when restricted to the Harish-Chandra module $C^\infty(G/P; L_\lambda)_K$. So, according to the Casselman-Wallach globalization theorem, we can take a smooth admissible moderate growth Fréchet globalization on both sides, and the results will be isomorphic as G-modules, so by the G-equivariance this isomorphism is given by $\mathcal{P}_\lambda$. The space $C^\infty(G/P; L_\lambda)$ is a smooth admissible moderate growth Fréchet globalization of the left-hand side, and on the right-hand side such a globalization is $\mathcal{E}_\lambda^*(G/K)$, the space joint eigenfunctions of strong moderate growth, i.e. those joint eigenfunctions satisfying inequalities of the form

$$\|L_u\phi(g)\| \leq C_u\|g\|^r$$

for some fixed $r > 0$ and for all $u \in U(\mathfrak{g})$. Here $\|g\|$ is the norm of g, see Definition 14. So, it follows that

$$\mathcal{P}_\lambda : C^\infty(G/P, L_\lambda) \to \mathcal{E}_\lambda^*(G/K)$$

is a topological linear isomorphism, which is the same result that had been found before, found here by the Casselman-Wallach globalization theorem.

2 Representation theory and Harish-Chandra modules

In this section, we will treat the necessary subjects to understand the rest of this work. Readers that are already familiar with the subject can safely skip this section.

2.1 The Iwasawa decomposition

In this subsection we treat standard theories on the structure of Lie groups, such as Cartan involutions and the Iwasawa decomposition. For this subsection, we will initially work with a semisimple Lie group G, which is more restrictive than requiring it to be reductive. However, as part of the definition of a real reductive group, we will require the existence of very similar objects to the ones we will treat here, which we will recall at the end of this subsection.

We recall that an involution of a space is an automorphism σ, such that $\sigma^2 = I$. In particular, we will be looking at involutions of the Lie algebra $\mathfrak{g}$, and we can split this space into the plus and minus one eigenspaces, $\mathfrak{g}_\pm$.

Definition 1 ([1], Definition 15.1)**.** A *Cartan involution* on $\mathfrak{g}$ is a Lie algebra involution $\sigma : \mathfrak{g} \to \mathfrak{g}$, such that the Killing form B is positive definite on $\mathfrak{g}_+$ and negative definite on $\mathfrak{g}_-$. By tradition, a Cartan involution is denoted by θ, and the eigenspaces are denoted by $\mathfrak{k} := \mathfrak{g}_+$ and $\mathfrak{p} := \mathfrak{g}_-$.

Given a semisimple Lie algebra, such a Cartan involution will always exist, see for instance [10], Cor. 6.18. We will denote by K the connected Lie subgroup of G with Lie algebra $\mathfrak{k}$. If G is connected and has a finite center, it turns out ([1], Corollary 15.15) that K is a maximal compact subgroup of G. Furthermore, any maximal compact subgroup can be constructed in such a way. We also have the following result, referred to as the Cartan decomposition of G:

Theorem 2 ([1], Theorem 15.12)**.** *In the above notation, the map*

$$\phi : K \times \mathfrak{p} \to G, \quad (k, X) \mapsto k \exp(X)$$

is a diffeomorphism.

Given a Cartan involution θ, with corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, for any $H \in \mathfrak{p}$ we have that $\mathrm{ad}(H) : \mathfrak{g} \to \mathfrak{g}$ is symmetric with respect to the inner product

$$\langle X, Y \rangle = -B(X, \theta Y).$$

This implies that the map has real eigenvalues, so it can be diagonalized with respect to some basis. For any abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, we can simultaneously diagonalize all elements of $\mathfrak{a}$. We fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$, and write

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X, \forall H \in \mathfrak{a}\},$$

for any $\lambda \in \mathfrak{a}^*$.

Definition 3 ([1],Definition 16.3)**.** Any non-zero $\alpha \in \mathfrak{a}^*$ such that $g_\alpha \neq 0$ is called a *root* of $\mathfrak{a}$. The set of these roots is denoted by $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$.

Using these roots, we get the following standard root space decomposition:

Theorem 4 ([1], Lemma 16.4)**.** *The set of roots* Σ *is finite, and*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha,$$

as a direct sum of real linear spaces.

We set $\mathfrak{m}$ to be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, i.e.

$$\mathfrak{m} := \mathfrak{k} \cap \mathfrak{g}_0.$$

Then it follows that $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$, since $\mathfrak{g}_0$ is θ-stable. Now, we fix a positive system Σ^+ for Σ, and denote the sum of positive root spaces and the sum of negative root spaces respectively by

$$\mathfrak{n} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha, \qquad \overline{\mathfrak{n}} := \bigoplus_{\alpha \in -\Sigma^+} \mathfrak{g}_\alpha.$$

Both of these spaces are subalgebras, and it follows that we can decompose

$$\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

We can also change this decomposition to yield the infinitesimal Iwasawa decomposition:

Lemma 5 ([1], Lemma 17.3)**.** *The Lie algebra* $\mathfrak{g}$ *allows the following decomposition as a direct sum of real linear spaces:*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

If we then denote by A the connected subgroup of G with Lie algebra $\mathfrak{a}$, and by N the connected subgroup of G with Lie algebra $\mathfrak{n}$, we get the following global Iwasawa decomposition:

Theorem 6 ([1],Theorem 17.6)**.** *The map*

$$\varphi : (k, a, n) \mapsto kan, K \times A \times N \to G$$

is a diffeomorphism.

Now, all the above theory holds for semisimple groups, but we are interested in the case of reductive groups. We take the definition of a reductive Lie group from [10]:

Definition 7 ([10], p. 384)**.** A *reductive Lie group* is a 4-tuple (g, K, θ, B) consisting of a Lie group G, a compact subgroup K of G, a Lie algebra involution θ of the Lie algebra $\mathfrak{g}$ of G, and a non-degenerate, $\mathrm{Ad}(G)$-invariant, θ-invariant, bilinear form B on $\mathfrak{g}$, such that

- $\mathfrak{g}$ is a reductive Lie algebra,
- the decomposition of $\mathfrak{g}$ into $+1$ and -1 eigenspaces is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K,
- $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal under B, and B is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$,
- multiplication, as a map from $K \times \exp \mathfrak{p}$ into G is a diffeomorphism onto, and
- every automorphism $\mathrm{Ad}(g)$ of $\mathfrak{g}_{\mathbb{C}}$ is inner for $g \in G$, i.e. is given by some ϕ in Int $\mathfrak{g}_{\mathbb{C}}$.

Under this definition, one can show that K is a maximal compact subgroup of G ([10], Prop. 7.19a), the fourth property immediately gives us the Cartan decomposition, and by defining roots similar to before, we also get the Iwasawa decomposition again ([10], Prop. 7.31). So we see that all the results from this section extend from the semisimple Lie groups to the reductive Lie groups. From here on out we will assume that our Lie group G is a linear reductive real Lie group.

2.2 Harish-Chandra modules

To understand the main theorem, we first look at what Harish-Chandra modules are, and what constitutes a globalization of such a module.

Definition 8 (p.65)**.** A *locally finite K-module V* is a complex continuous K-representation π on V, such that for every vector $v \in V$ the space $\mathrm{span}\{\pi(k)v | k \in K\}$ is finite-dimensional.

For any smooth representation (π, E) of K, or of G, we denote by $E^{K-\mathrm{fin}}$ the K-module consisting of K-finite vectors. Here, v being K-finite means exactly that $\mathrm{span}\{\pi(k)v|k \in K\}$ is finite-dimensional.

Definition 9 (p.65)**.** By a *$(\mathfrak{g}, K)$-module V* we mean a space V with a locally finite action of K and an action of the Lie algebra $\mathfrak{g}$, such that:

- The derived action of K coincides with the action of $\mathfrak{g}$, when restricted to $\mathfrak{k} = \mathrm{Lie}\ K$.
- The actions are compatible, i.e. for all $k \in K, X \in \mathfrak{g}, v \in V$ we have

$$k \cdot (X \cdot v) = \mathrm{Ad}(k)X \cdot (k \cdot v).$$

Given a smooth G-representation E, it now follows that the K-finite vectors $E^{K-\mathrm{fin}}$ form a $(\mathfrak{g}, K)$-module. Here the action of $\mathfrak{g}$ is the derived action that follows from the G-representation. A Harish-Chandra module will be a special case of a $(\mathfrak{g}, K)$-module, so we introduce the following terminology:

Definition 10 (p.65)**.** A K-module E will be called *admissible*, if for all finite-dimensional representations (τ, W) of K the multiplicity space $\mathrm{Hom}_K(W, E)$ is finite-dimensional. In other words, E is admissible if every isotypical component $E[\tau]$ is finite-dimensional.

Definition 11 (p.66)**.** A *Harish-Chandra module* is an admissible $(\mathfrak{g}, K)$-module, that is finitely generated as a $\mathfrak{g}$-module. We will call a smooth Fréchet representation (π, E) (i.e. a smooth representation, where E is a Fréchet space) *admissible* if the underlying $(\mathfrak{g}, K)$-module $E^{K-\text{fin}}$ is admissible.

By definition, taking the K-finite vectors of an admissible G-representation yields a Harish-Chandra module. The process of going in the opposite direction is called globalization:

Definition 12 (p.68)**.** Given a Harish-Chandra module V, a *globalization* of V is a G-representation (π, E), such that the space of K-finite vectors $E^{K-\text{fin}}$ is isomorphic to V as a $(\mathfrak{g}, K)$-module.

Now, in the Casselman-Wallach globalization theorem, we are concerned with a specific type of globalizations, namely the smooth admissible Fréchet globalizations of moderate growth. In the next subsection, we look into this moderate growth.

2.3 Representations of moderate growth

To understand a notion of growth, we first need to introduce a scale structure on a Lie group:

Definition 13 (p.49)**.** We define a *scale* on a Lie group G to be a function $s : G \to \mathbb{R}^+$, such that:

- s and s^{-1} are locally bounded.
- s is sub-multiplicative, i.e. $s(gh) \leq s(g)s(h)$ for all $g, h \in G$.

We order scales by setting

$$s \preccurlyeq s' \iff (\exists C > 0, N \in \mathbb{N}), \forall g \in G : \quad s(g) \leq Cs'(g)^N.$$

This ordering induces a notion of equivalence on scales. We define a *scale structure* on G as the equivalence class $[s]$ of a scale function s.

Note that every equivalence class $[s]$ allows a continuous representative, so we will only consider continuous scale functions. One specific case that will be important to us is that of the maximal scale structure: we look at the case of a connected group G, and we fix a left-invariant Riemannian metric $\mathbf{g}$ on G. Then we can define a distance function $d(g, h)$, the infimum of lengths of piecewise smooth curves between g and h in G. This distance function is then also left G-invariant, so it can be recovered from $d(g) := d(g, e)$, where e is the unit of G. We then have that $d(g)$ is sub-additive, i.e. $d(gh) \leq d(g) + d(h)$ for all $g, h \in G$. But then $s_{\max}(g) := e^{d(g)}$ defines a scale function on G. It turns out that this scale is maximal in the sense that any other scale s satisifies $s \preccurlyeq s_{\max}$ ([6], Lemma 2). In particular, it follows that the the equivalence class $[s_{\max}]$ is independent of the choice of metric.

Definition 14 (p.49)**.** The above scale structure $[s_{\max}]$ is called the *maximal scale structure*. We will also denote $s_{\max}(g) = \|g\|$.

Throughout this section we will work with an arbitrary scale structure to motivate the definitions, but in the rest of this work we will assume that G is equipped with the maximal scale structure. We denote by $(G, [s])$ a Lie group G, equipped with a particular scale structure $[s]$. We record the following property:

Lemma 15. *Let G be a linear real reductive Lie group. Then for any $g \in G$ we have $\|g\| = \|g^{-1}\|$.*

Proof. Since the distance function is left invariant, we have

$$\|g\| = e^{d(g)} = e^{d(g,e)} = e^{d(e,g^{-1})} = e^{d(g^{-1},e)} = \|g^{-1}\|.$$

□

Now, we can define representations of moderate growth:

Definition 16 (p.53)**.** A representation of *moderate growth*, with respect to a scale structure $[s]$, is a Fréchet representation E of a Lie group G such that for every semi-norm p on E there exists a semi-norm q on E and an integer $N > 0$ such that

$$p(\pi(g)v) \leq s(g)^N q(v),$$

for all $g \in G$.

To more easily work with these kinds of representations, we will introduce a slightly different notion of representations. To do so, we first introduce the following terminology:

Definition 17 (p.51)**.** Let (π, E) be a representation of G. We call a semi-norm p on E a *G-continuous semi-norm* if the action of G on E is continuous with respect to the topology induced by p.

Definition 18 (p.51)**.** Let (π, E) be a continuous representation of G on a semi-normed space (E, p) (with the topology induced by p). Then the function $s_\pi : g \mapsto \|\pi(g)\|$ is a scale. Here $\|\cdot\|$ denotes the operator norm. We say that (π, E) is *$[s]$-bounded* if $s_\pi \preccurlyeq s$. Similarly, given a Fréchet representation (π, E), we will call a G-continuous seminorm p *$[s]$-bounded* if the representation is $[s]$-bounded with respect to the seminorm.

Definition 19 (Definition 2.6)**.** A Fréchet representation (π, E) of G, equipped with scale structure $[s]$, will be called an *F-representation* if the topology of E is induced by a countable family of G-continuous $[s]$-bounded semi-norms $(p_n)_{n\in\mathbb{N}}$.

Note that when we equip G with the maximal scale structure, then any representation is $[s_{\max}]$-bounded, so any G-continuous seminorm will also be $[s_{\max}]$-bounded. Furthermore, we have the following:

Lemma 20 (Lemma 2.9)**.** *Let (π, E) be an F-representation and $H \subset E$ a closed G-invariant subspace. Then the sub- and quotient representations on H and E/H are F-representations as well.*

Proof. The spaces are again Fréchet, and the semi-norms that induced the topology on E will induce semi-norms on H and E/H that also induce the relevant subspace- or quotient topology. The result follows. □

The following lemma will relate our two notions of representations:

Lemma 21 (Lemma 2.10)**.** *Let (π, E) be a Fréchet representation of the Lie group $(G, [s])$. Then the following are equivalent:*

- *(π, E) is of moderate growth.*
- *(π, E) is an F-representation.*

Proof. By definition, an F-representation is of moderate growth via the $[s]$-boundedness of the semi-norms.

Conversely, assume that (π, E) is of moderate growth, and take p, q and $N > 0$ as in the definition above. Then,

$$\tilde{p}(v) := \sup_{g \in G} \frac{p(\pi(g)v)}{s(g)^N}$$

defines a semi-norm on E such that $p \leq \tilde{p} \leq q$, and $\tilde{p}(\pi(g)v) \leq s(g)^N \tilde{p}(v)$ for all $g \in G$. The first part means that the $\tilde{p}$ define the topology on E, and the second part means that $\tilde{p}$ is G-continuous and $[s]$-bounded. Since the topology of a Fréchet space can be induced by a countable family of semi-norms, it suffices to use countably many $\tilde{p}$ to define the topology, and it follows that (π, E) is an F-representation. □

Definition 22 (p.55)**.** We call an F-representation (π, E) that is smooth (i.e. for every vector $v \in E$ the map $g \mapsto \pi(g)v$ is smooth) an *SF-representation*.

By Lemma 21, we see that an SF-representation is a smooth moderate growth Fréchet representation, so by the definition of admissible, we see that the SF-globalizations of a Harish-Chandra module are precisely the smooth admissible moderate growth Fréchet globalizations. Throughout the rest of this work we will therefore be talking mainly about SF-globalizations.

2.4 Schwartz space

In this subsection, we introduce the Schwartz space which we will need in proving the Casselman-Wallach globalization theorem.

We denote by $\mathcal{M}(G)$ the Banach space of complex and bounded Borel measures on the Lie group G. We use here the norm of total variation. Let $\mathcal{M}_c(G) \subset \mathcal{M}(G)$ denote the compactly supported complex measures. The left action of G on itself induces an action of G on $\mathcal{M}(G)$. However, this action is not continuous everywhere. We call a measure μ continuous if the orbit map

$$G \to \mathcal{M}(G), \qquad g \mapsto (\lambda_g)_*\mu$$

is continuous, where λ_g denotes left multiplication. We denote by $\widetilde{\mathcal{M}}(G)$ the space of continuous complex measures. By fixing a left Haar measure $\mathrm{d}g$, we get an isomorphism

$$L^1(G) \to \widetilde{\mathcal{M}}(G), \qquad f \mapsto f \cdot \mathrm{d}g.$$

If we have a representation (π, E) of G on a complete locally convex vector space, such as when we have a Fréchet representation, we denote by Π the corresponding action of $\mathcal{M}_c(G)$ on E:

$$\Pi(\mu)v = \int_G \pi(g)v\mathrm{d}\mu(g).$$

For a function $f \in C_c^\infty(G)$, we will similarly denote

$$\Pi(f)v = \int_G f(g)\pi(g)v\mathrm{d}g.$$

These integrals converge since E is complete and locally convex. Depending on the specific type of representation, other measures might also define an action on E, and in the case of F-representations, we will look specifically at rapidly decreasing measures:

Definition 23 (p.56)**.** The space of *rapidly decreasing continuous complex measures* on G, equipped with a scale structure $[s]$, is defined as

$$\mathcal{R}(G) := \{\mu \in \widetilde{M}(G) \mid \forall n \in \mathbb{N} : s(\cdot)^n \in L^1(G, |\mu|)\}.$$

We denote by $L \times R$ the regular representation of $G \times G$ on functions on G, i.e.

$$(L \times R)(g_1, g_2)f(g) := f(g_1^{-1}gg_2),$$

for any continuous function f. This representation extends to measures, and it yields an F-representation of $G \times G$, namely $(L \times R, \mathcal{R}(G))$. Now, for any $u \in \mathfrak{g}$ we abbreviate $L_u := \mathrm{d}L(u)$ and similarly R_u for the derived representations, and extend this notation to $u \in U(\mathfrak{g})$. With that notation, we define:

Definition 24 (p.56)**.** The smooth vectors of $(L \times R, \mathcal{R}(G))$ constitute the *Schwartz space*

$$\mathcal{S}(G) := \{f \cdot \mathrm{d}g \mid f \in C^\infty(G); \forall u, v \in U(\mathfrak{g}), \forall n \in \mathbb{N} : s(\cdot)^n L_u R_v f \in L^1(G)\}.$$

We will use this Schwartz space later to construct minimal globalizations, and to prove properties of these globalizations, but to motivate their definition at this point, we mention the following result, without proving it:

Theorem 25 (Proposition 2.20)**.** *Let G be a Lie group. Then the following categories are isomorphic:*

- *The category of SF-representations of G.*
- *The category of non-degenerate continuous algebra representations of $\mathcal{S}(G)$ on Fréchet spaces.*

Here a non-degenerate algebra representation is a representation of an algebra $\mathcal{A}$ without 1, on a space M, such that $\mathcal{A}M = M$.

2.5 G-continuous norms and the Sobolev ordering

We have already talked about G-continuous semi-norms on G-representations, but we will also need the concept of G-continuous norms on Harish-Chandra modules.

Definition 26 (p.70)**.** Let V be a Harish-Chandra module, and p a norm on V. We say that p is a *G-continuous* norm on V if the completion of V with respect to p gives rise to a continuous Banach representation of G.

We will put a pre-order on the set of G-continuous norms on a given module V. To do so, we need the following:

Definition 27 (p.54)**.** Given a (semi-)norm p on a G-representation or Harish-Chandra module, we fix a basis $X_1, \dots, X_n$ of $\mathfrak{g}$, and denote the action of $\mathfrak{g}$ by $\mathrm{d}\pi$. We define the *k-th Sobolev (semi-)norm* p_k as follows:

$$p_k(v) := \left(\sum_{m_1+\ldots+m_n \leq k} p(\mathrm{d}\pi(X_1^{m_1} \cdots X_n^{m_n})v)^2 \right)^{\frac{1}{2}}.$$

Of course the specific norm depends on the choice of basis, but a different choice of basis will yield an equivalent semi-norm. Now, using the Sobolev norms, we can introduce a pre-order on the set of G-continuous norms:

Definition 28 (p.70)**.** The *Sobolev ordering*, denoted by $\preccurlyeq$, is a pre-order defined by

$$p \preccurlyeq q \iff (\exists k \in \mathbb{N}_0, C > 0) : p(v) \leq C q_k(v), \ \forall v \in V.$$

We say that p and q are *Sobolev-equivalent*, notated $p \asymp q$, provided $p \preccurlyeq q$ and $q \preccurlyeq p$.

We will mainly be interested in G-continuous norms up to Sobolev-equivalence. It turns out that in this case, we can assume stricter properties on our norms:

Theorem 29 (Theorem 5.5(i))**.** *Let V be a Harish-Chandra module, and p a G-continuous norm. Then there exists a G-continuous Hilbert norm q such that p is Sobolev equivalent to q.*

Proof. See [4], p.70-71. Here an explicit G-continuous norm is constructed, using theories from convex analysis, that is euivalent to p. □

Therefore, we can assume that any G-continuous norm is a Hilbert norm. In addition, we will assume that all norms are K-invariant.

2.6 Representation theory

In this section, we treat some standard constructions involving representations.

Given a Banach representation (π, E), we can look at the toplogical dual E^*. If we fix a norm p for E, then E^* is again a Banach space, with respect to the dual norm

$$p^*(\lambda) := \sup_{p(v)\leq 1} |\lambda(v)| \quad (\lambda \in E^*).$$

We would like to define a dual representation π^* on E^* by setting

$$\pi^*(g)(\lambda) := \lambda \circ \pi(g^{-1}).$$

However, this will not necessarily be continuous as a function of g for every λ. For example, if G is a compact Lie group, and we take $E = L^1(G)$, the dual is $E^* = L^\infty(G)$, but the action is only continuous on $C(G) \subset L^\infty(G)$. To solve this, we define the subspace $\widetilde{E} \subset E^*$ consisting of those $\lambda \in E^*$ for which the orbit map $g \mapsto \pi^*(g)(\lambda)$ is continuous as a function from G to E^*. This will be a closed, G-invariant subspace of E^*. If we restrict the action of G to this subspace, we therefore obtain a Banach representation $(\pi^*, \widetilde{E})$.

Definition 30 (p.58)**.** The above representation $(\pi^*, \widetilde{E})$, also denoted by $(\tilde{\pi}, \widetilde{E})$ is called the *contragredient representation* of (π, E).

Similarly to this, we would like to have a dual module for Harish-Chandra modules. If V is a Harish-Chandra module, we denote by V^* its algebraic dual. We define actions of $\mathfrak{g}$ and K on the dual again, similar to above, and again we run into the problem that this will not necessarily yield a Harish-Chandra module. This time, to solve this, we denote by $\widetilde{V} \subset V^*$ the K-finite vectors in V^*. We note that $\widetilde{V}$ is a $\mathfrak{g}$-submodule of V^*. It turns out that $\widetilde{V}$ will again be a Harish-Chandra module.

Definition 31 (p.67)**.** The above Harish-Chandra module $\widetilde{V}$ will be called the *Harish-Chandra module dual to V*.

Given a representation of G, we can restrict the action to a closed subgroup H of G to obtain an H-representation. The reverse process of obtaining a G-representation from an H-representation is as follows:

Definition 32 ([1],p.93-94)**.** Let (V, ξ) be a finite-dimensional representation of a closed subgroup $H \subset G$. We define the *induced representation* of G, denoted by $\mathrm{ind}_H^G(\xi)$ as the space of smooth functions $\phi : G \to V$, transforming according to

$$\phi(gh) = \xi(h)^{-1}\phi(g), \quad (g \in G, h \in H).$$

The action of G on this space is the restriction of the left regular representation, i.e.

$$[\pi(g)\phi](g') = \phi(g^{-1}g'), \quad (g, g' \in G).$$

We will mostly be interested in a specific case of this. Recall the subgroups A and N of G, used in the Iwasawa decomposition, and define M as the centralizer of $\mathfrak{a}$ in K. We then define the minimal parabolic subgroup $P_{\min}$ of G as

$$P_{\min} = MAN.$$

By the Iwasawa decomposition, this is a closed subgroup of G, and the map $(m, a, n) \mapsto man$ is a diffeomorphism of $M \times A \times N$ onto P. If we now have a finite-dimensional representation (W, ξ) of P, we can look at the representation induced by this. Following the notation of [4], we will denote this induced representation by

$$I^\infty(W) = \mathrm{ind}_P^G(\xi).$$

From this representation, we create the following Harish-Chandra module:

Definition 33 (p.67)**.** Let W be a finite-dimensional $P_{\min}$-representation. The K-finite vectors

$$I(W) := I^\infty(W)^{K-\mathrm{fin}}$$

of the induced representation form a Harish-Chandra module, which will be referred to as the *minimal principal series representation* belonging to W.

Now, using these definitions, we can state the Casselman embedding theorem:

Theorem 34 (Theorem 4.4, proven in [12], Cor.4.2.4)**.** *For every Harish-Chandra module V, there exists a finite-dimensional P_{min}-representation W, and a $(\mathfrak{g}, K)$-embedding $V \to I(W)$.*

This means that if we look at submodules of minimal principal series representations, we already have all Harish-Chandra modules. From this observation, and a study of the minimal principal series representations, we can give a polynomial bound on the K-multiplicities of a Harish-Chandra module. We will often abuse our notation, and identify an equivalence class $[\tau] \in \hat{K}$ with a representative τ, which means we will regularly talk about $\tau \in \hat{K}$, τ being an irreducible representation. If we denote by $\mathfrak{t}$ the Lie algebra of a maximal torus in K, we will often identify τ with its highest weight in $i\mathfrak{t}^*$. In particular, $\|\tau\|$ will refer to the Cartan-Killing norm of this highest weight τ. From studying the dimensions of isotypical components in $I(W)$, we can now obtain the following, which is actually used in the proof of Theorem 29:

Theorem 35 (Theorem 4.5)**.** *Let V be a Harish-Chandra module. Then there exists a $C > 0$ such that*

$$\dim V[\tau] \leq C(1 + \|\tau\|)^{\dim K - \dim M} \quad (\tau \in \hat{K}).$$

This bound can be proved using the Casselman embedding theorem and Harish-Chandra's subquotient theorem, but using Frobenius reciprocity and the Weyl dimension formula a slightly weaker version can be proved already, which also suffices for our purposes.

2.7 Comparing with the article

Some parts of this section were based on Sections 2 through 5 of the article, [4], while others treated subject matter that the article already presumed known. We have re-ordered and restructured the needed definitions and results, and skipped many of the proofs. All definitions and results in this sections can be found in other works already, the only original work has been putting them together and structuring them in a way that fits this thesis.

3 The main proof

In this section, we go over the structure of the proof in the article. We will not treat all proofs in detail, for instance those proofs relying on techniques that we will not need elsewhere will only be outlined.

3.1 Globalizations

We adopt the terminology from the article, and call a Harish-Chandra module V *good* if it admits a unique SF-globalization. To show that the globalizations are unique, we look at two extreme cases: minimal and maximal SF-globalizations.

Definition 36 (p.75)**.** An SF-globalization V^∞ of a Harish-Chandra module V will be called *minimal* if it satisfies the following universal property: for any SF-globalization (π, E) of V, there exists a (necessarily unique) continuous G-equivariant map $V^\infty \to E$ which extends the identity morphism $V \to V$.

Throughout this thesis we adopt the notation from the article to write minimal globalizations as V^∞. From the definition it is clear that minimal globalizations are unique (up to isomorphism), if they exist. It turns out they do always exist, and can be constructed explicitly:

Lemma 37 (p.75)**.** *Let V be a Harish-Chandra module. Take any Banach globalization (π, E) of V, and let $\mathbf{v} = \{v_1, \dots, v_k\}$ be a set of generators of V. If we view $\mathcal{S}(G)^k$ as a G-module under the left regular representation, the linear map*

$$\mathcal{S}(G)^k \to E, \quad (f_1, \dots, f_k) \mapsto \sum_{j=1}^{k} \Pi(f_j) v_j$$

is linear, continuous, G-equivariant, and its kernel $\mathcal{S}(G)_{\boldsymbol{v}}$ is independent of the choice of (π, E). The quotient $\mathcal{S}(G)^k/\mathcal{S}(G)_{\mathbf{v}}$ then is the minimal globalization V^∞.

Proof. It is clear that the map is linear and continuous. Since we are using the left regular representation, it also readily follows that it is G-equivariant. That the kernel is independent of the choice of (π, E) follows from [5], Theorem 3.2. Now, if we denote by $\mathcal{S}(G)^{K\times K}$ the $K \times K$-finite functions of $\mathcal{S}(G)$, we have that

$$\Pi(\mathcal{S}(G)^{K\times K})V = V,$$

so it follows that $\mathcal{S}(G)^k/\mathcal{S}(G)_{\mathbf{v}}$ is an SF-globalization of V. By construction, this globalization embeds into any Banach globalization of V. If we have an SF-globalization E, we have that the topology on E is induced by a countable family of G-continuous semi-norms $\{p_i\}_{i\in\mathbb{N}}$. We denote by E_i the Banach completion of E with respect to p_i, and we see that $\mathcal{S}(G)^k/\mathcal{S}(G)_{\mathbf{v}}$ embeds into each of those. Since the topology of E is induced by the semi-norms, we have that

$$E = \bigcap_i E_i,$$

so it follows that we can embed $\mathcal{S}(G)^k/\mathcal{S}(G)_{\mathbf{v}}$ into E as we claimed. $\square$

Note that from this lemma it also follows that within any globalization of V, one can find V^∞ by applying $\Pi(\mathcal{S}(G))$. Using this, we can prove the following lemma:

Lemma 38 (Lemma 6.2)**.** *Let V be a Harish-Chandra module, and let V^∞ be its unique minimal SF-globalization. Let $W \subset V$ be a submodule, and $U := V/W$. If we denote by $\overline{W}$ the closure of W in V^∞, then $U^\infty = V^\infty/\overline{W}$.*

Proof. We write $(\pi_U, V^\infty/\overline{W})$ for the quotient representation obtained from (π, V^∞). Then, since $\Pi(\mathcal{S}(G))V = V^\infty$, we get that $\Pi_U(\mathcal{S}(G))U = V^\infty/\overline{W}$, and the result follows. $\square$

We also note that if V admits a maximal G-continuous norm p with respect to the Sobolev ordering, then V^∞ coincides with the smooth vectors of the Banach completion of (V, p). However, the existence of a minimal globalization does not yet imply the existence of such a maximal norm.

The second important case of globalizations is that of a maximal globalization:

Definition 39 (p.76)**.** An SF-globalization $V^\infty_{\max}$ of a Harish-Chandra module V is called *maximal* if for any SF-globalization (π, E) of V there exists a (necessarily unique) continuous linear G-equivariant map $E \to V^\infty_{\max}$ sitting above the identity morphism $V \to V$.

It is again clear from the definition that if a maximal globalization exists, it is unique up to isomorphism. However, it is not as easy to show the existence of maximal globalizations compared to minimal ones. As it turns out, a maximal globalization of V exists if and only if there exists a G-continuous Hilbert norm q that is minimal with respect to the Sobolev ordering. But then we have:

Lemma 40. *A Harish-Chandra module V is good if and only if all G-continuous norms on V are Sobolev-equivalent.*

Proof. Suppose that V is good. Then all SF-globalizations are isomorphic, so the minimal and maximal globalization coincide. The maximal globalization is induced by a minimal norm q with respect to the Sobolev ordering. But then, since q induces the minimal globalization, it must be a maximal norm too. Since the norm is both minimal and maximal, all G-continuous norms must be equivalent to q, hence all G-continuous norms are Sobolev-equivalent.

On the other hand, if all G-continuous norms are Sobolev-equivalent, we can take any such norm and look at the globalization it induces. Since the norm is minimal, this globalization is maximal, and since it is maximal, the globalization is minimal. Therefore, it is the unique SF-globalization, and V is good. $\square$

Now that we have related being good to Sobolev-equivalence of norms, we have:

Lemma 41 (Lemma 6.5)**.** *A Harish-Chandra module V is good if and only if its dual $\widetilde{V}$ is good.*

Proof. We have that $p \asymp q$ if and only if $p^* \asymp q^*$. Therefore all G-continuous norms on V are Sobolev-equivalent if and only if all G-continuous norms on $\widetilde{V}$ are Sobolev-equivalent. $\square$

As mentioned before, it is difficult to explicitly construct maximal globalizations. However, if we have a submodule of a good Harish-Chandra module, it is easier:

Lemma 42 (Lemma 6.6)**.** *Let U be a good Harish-Chandra module, and U^∞ its unique SF-globalization. Let $V \subset U$ be a submodule and let $\overline{V}$ be the closure of V in U^∞. Then $V^\infty_{\max} = \overline{V}$.*

Proof. We want to show that the induced norm on V from U^∞ is minimal with respect to the Sobolev ordering. So, take $\tilde{q}$ any G-continuous Hilbert norm on U (they are all equivalent), and define $q = \tilde{q}|_V$. If we let p be any G-continuous Hilbert norm on V, we now want to show that $q \preccurlyeq p$. To do this, let $\pi : \widetilde{U} \to \widetilde{V}$ be the map dual to the inclusion of V in U. Since U was good, so is $\widetilde{U}$, so for the dual norms we have that $p^* \circ \pi \preccurlyeq \tilde{q}^*$. By taking the dual again and restricting to V, we get that $q \preccurlyeq p$, as we wanted. $\square$

To close off this subsection, we record a lemma that is used in a proof later on:

Lemma 43 (Lemma 6.7)**.** *Let $V_1 \subset V_2 \subset V_3$ be an inclusion chain of Harish-Chandra modules. Suppose that V_2 and V_3/V_1 are good. Then V_2/V_1 is good.*

Proof. Let $\overline{V_3}$ be any SF-globalization of V_3. Let $\overline{V_1}, \overline{V_2}$ be the closures of V_1 and V_2 in $\overline{V_3}$. Since V_2 is good, all SF-globalizations are isomorphic, so we have that $\overline{V_2} = V_2^\infty$. By Lemma 38 we then have that $\overline{V_2}/\overline{V_1} = (V_2/V_1)^\infty$. Since V_3/V_1 is good, we also have that $(V_3/V_1)^\infty = \overline{V_3}/\overline{V_1}$, so by Lemma 42 we get that $\overline{V_2}/\overline{V_1} = (V_2/V_1)^\infty_{\max}$, which means the minimal and maximal globalizations coincide, i.e. V_2/V_1 is good. $\square$

3.2 Matrix coefficients

We have seen in the previous part that a Harish-Chandra module V is good if and only if all G-continuous norms on V are Sobolev-equivalent. In this subsection, we will formulate related theorems in terms of bounds on matrix coefficients.

Given a Harish-Chandra module V, we fix a finite-dimensional $\mathcal{Z}(\mathfrak{g})$-invariant space of generators $\Xi \subset \widetilde{V}$ of the dual of V. Let $\xi_1, \dots, \xi_k$ be a basis of Ξ. We define balls in G by setting

$$B_r := \{g \in G| \, \|g\| < r\},$$

for any $r > 0$, with $\|g\|$ the maximal scale structure, see Definition 14. We set

$$r_0 := \min\{\|g\| \, |g \in G\} \geq 1. \tag{1}$$

Now, we can formulate the following theorem:

Theorem 44 (Theorem 7.1)**.** *Let V be a Harish-Chandra module, and fix a choice of $\Xi \subset \widetilde{V}$. Then V is good if and only if for all G-continuous norms q on V^∞ there exist constants $c_1, c_2, c_3, C > 0$ such that*

$$\left(\sum_{j=1}^{k} \int_{B_r} |\xi_j(\pi(g)v)|^2 dg\right)^{\frac{1}{2}} \geq \frac{c_2}{(1 + \|\tau\|)^{c_3}} \cdot q(v) \tag{2}$$

for all $\tau \in \hat{K}, v \in V[\tau]$ and $r > \max\{r_0, C(1 + \|\tau\|)^{c_1}\}$ with r_0 given by (1).

In view of the local Sobolev Lemma this is equivalent to a pointwise version:

Theorem 45 (Theorem 7.2)**.** *Let V be a Harish-Chandra module, and fix a choice of $\Xi \subset \widetilde{V}$. Then V is good if and only if for all G-continuous norms q on V^∞ there exist constants $c_1, c_2, c_3, C > 0$ such that for all $\tau \in \hat{K}$ and $v \in V[\tau]$ there exists a $g_\tau \in G$ such that $\|g_\tau\| \leq C(1 + \|\tau\|)^{c_1}$ and*

$$\max_{1 \leq j \leq k} |\xi_j(\pi(g_\tau)v)| \geq \frac{c_2}{(1 + \|\tau\|)^{c_3}} \cdot q(v). \tag{3}$$

We will not prove this in full detail, but we will outline the method of proof. We first assume that the lower bound in (2) holds for some G-continuous norm q, and define a Hermitian norm by

$$p(v)^2 := \sum_{j=1}^{k} \int_G |\xi_j(\pi(g)v)|^2 \frac{\mathrm{d}g}{\|g\|^N},$$

for $N > 0$ large enough so that the integral converges. Then by the lower bound in (2), it follows that $q \preccurlyeq p$. On the other hand, we can take N large enough so that $p \preccurlyeq q$, and it follows that every G-continuous norm is equivalent to p, which means that V is good. This proves one implication.

To prove the other implication, we assume that V is good. We have estimates

$$|\xi_j(\pi(g)v)| \leq C \cdot \|g\|^n q(v),$$

for some $n \in \mathbb{N}$ and $C > 0$. Using these estimates, we can define norms

$$p_N(v) := \max_{1 \leq j \leq k} \sup_{g \in G} \frac{|\xi_j(\pi(g)v)|}{\|g\|^N} \quad (v \in V)$$

for any $N \geq n$. Using these norms, we can look at the Banach completions of V^∞ with respect to two $N, N' \geq n$. Since V is good, these two globalizations are isomorphic for any $N, N' \geq n$. We take some $N' > N \geq n$ and fix $\tau \in \hat{K}, v \in V[\tau]$, and define $g_\tau \in G$ such that

$$g \mapsto \max_{1 \leq j \leq k} \frac{|\xi_j(\pi(g)v)|}{\|g\|^{N'}}$$

becomes maximal at g_τ. Applying theory on K-Sobolev norms (see Proposition 3.9 in [4]), we then find estimates

$$\|g_\tau\| \leq C(1+\|\tau\|)^{c_1},$$
$$\max_{1\leq j\leq k} |\xi_j(\pi(g_\tau)v)| \geq \frac{c_2\|g_\tau\|^{N'}}{(1+\|\tau\|)^{c_3}} q(v).$$

Since $\|g_\tau\| \geq 1$, the lower bound in 3 follows. For full details, see [4], p.78-79.

3.3 Minimal principal series representations

Now that we have established that a module being good can be determined by looking at its matrix coefficients, we look at a specific case: the minimal principal series representation. The following theorem gives us a description of the structure of the representation:

Theorem 46 (Theorem 8.1(i))**.** *Let $V = I(W)$ be a minimal principal series representation of G. Let $\xi_1, \ldots, \xi_k$ be a set of generators of V. We take the Hilbert globalization $\mathcal{H} = L^2(W \times_M K)$. Then there exist constants $c_1, c_2, C_1, C_2 > 0$ such that for all $\tau \in \hat{K}$ and $v_\tau \in V[\tau]$ there exist functions $f_{\tau,1}, \ldots, f_{\tau,k} \in C_c^\infty(G)$ with the following properties:*

(a) $\sum_{j=1}^k \Pi(f_{\tau,j})\xi_j = v_\tau$.

(b) $\mathrm{supp}(f_{\tau,j}) \subset \{g \in G | \; \|g\| < C_1(1+\|\tau\|)^{c_1}\}$ *for all* $1 \leq j \leq k$.

(c) $\sum_{j=1}^k \|f_{\tau,j}\|_1 \leq C_2\|v_\tau\|(1+\|\tau\|)^{c_2}$, *where* $\|\cdot\|_1$ *refers to the* $L^1(G)$*-norm.*

The proof of this theorem will be given in the next two sections. Using this theorem, we can now prove an important result, on which the proof of the main theorem will be based:

Theorem 47 (Theorem 8.1(iii))**.** *Let $V = I(W)$ be a minimal principal series representation of G. Then V is good.*

Proof. We will establish the estimate in (3), so we fix a space of generators Ξ and a basis $\xi_1, \ldots, \xi_k$ in $\widetilde{V}$, we fix a G-continuous norm q and take $v \in V[\tau]$ for some τ. Instead of working with the norm q explicitly, we will work with a linear functional ξ. Recall from Theorem 29 that q can be taken to be a Hilbert norm. In particular, this means that for any $w \in V[\tau']$ for some $\tau' \neq \tau$, and for all $\lambda \in \mathbb{C}$, we have

$$q(\lambda v + w) \geq q(\lambda v) + q(w). \tag{4}$$

Now, we define

$$\xi' : \bigoplus_{\substack{\tau' \in \hat{K} \\ \tau' \neq \tau}} V[\tau'] \oplus \mathbb{C}v \to \mathbb{C}, \quad w + \lambda v \mapsto \lambda q(v).$$

By the estimate in (4), we have that

$$|\xi'(w+\lambda v)| = |\lambda| q(v) = q(\lambda v) \leq q(\lambda v) + q(w) \leq q(\lambda v + w).$$

Therefore, by Hahn-Banach, we can extend ξ' to a linear functional ξ on all of V, such that $|\xi(v')| \leq q(v')$ for all $v' \in V$, such that $\xi(v) = q(v)$ and such that $\xi(w) = 0$ for $w \in V[\tau']$ with $\tau' \neq \tau$. Now, if we denote by $\xi_{\tau'}$ the τ'-isotypical part of ξ, we have that $\xi_{\tau'} = \dim(\tau')\Pi^*(\overline{\chi}_{\tau'})\xi$, where $\chi_{\tau'}$ denotes the character of the K-representation τ', and π^* is the contragredient representation. Applying this to some $w \in V$ yields

$$\begin{aligned}
\xi_{\tau'}(w) &= \int_K \dim(\tau')\overline{\chi_{\tau'}(k)}\xi(\pi(k^{-1})w)\mathrm{d}k \\
&= \int_K \dim((\tau')^*)\overline{\chi_{(\tau')^*}(k')}\xi(\pi(k')w)\mathrm{d}k' \\
&= \xi\left(\int_K \dim((\tau')^*)\overline{\chi_{(\tau')^*}(k')}\pi(k')w\mathrm{d}k'\right) \\
&= \xi(w_{(\tau')^*}),
\end{aligned}$$

where $w_{(\tau')^*}$ denotes the $(\tau')^*$-isotypical component of w. Here in the second line we switched to $k' = k^{-1}$, and we use that $\chi_{\tau'}(k) = \chi_{(\tau')^*}(k^{-1})$. So, by definition of our ξ, which is only non-zero on the τ-isotypical component, we have that $\xi = \xi_{\tau^*}$. But then, using Theorem 46 and the fact that $\widetilde{I(W)} = I(W^*)$, we can decompose ξ in terms of our generators $\xi_1, \dots, \xi_k$ with functions $f_1, \dots, f_k$ satisfying the required estimates from Theorem 46. Note that the norms of τ and τ^* are the same, so that we get appropriate bounds. Writing this out, we get that

$$\begin{aligned}
q(v) = \xi(v) &\leq |\sum_{j=1}^k \Pi^*(f_j)\xi_j(v)| \\
&\leq \sum_{j=1}^k |\int_G f_j(g)\xi_j(\pi(g^{-1})v)\mathrm{d}g|.
\end{aligned}$$

If we now pick out the index j for which the summand is maximal, we get some j such that

$$\frac{q(v)}{k} \leq |\int_G f_j(g)\xi_j(\pi(g^{-1})v)\mathrm{d}g|.$$

Now, we can estimate the integral from above by $\|f_j\|_1\|\xi(\pi(\cdot^{-1})v)\|_\infty$, where the supremum-norm of the second function is taken over the support of f_j. Writing this out yields

$$\begin{aligned}
\frac{q(v)}{k} &\leq \|f_j\|_1\|\xi_j(\pi(\cdot^{-1})v)\|_\infty \\
&\leq C_2\|\xi\|(1+\|\tau\|)^{c_2}\|\xi_j(\pi(\cdot^{-1})v)\|_\infty,
\end{aligned}$$

for some constants C_2, c_2 independent of τ. Here we used part (c) of Theorem 46, and $\|\xi\|$ is the operator norm of ξ. If we move all factors but the supremum norm to the left-hand side, and combine the constants together, we get that

$$\frac{C}{(1+\|\tau\|)^{c_2}} q(v) \leq \|\xi_j(\pi(\cdot^{-1})v)\|_\infty.$$

Here we used that $|\xi(v')| \leq q(v')$ for all $v' \in V$, and $\xi(v) = q(v)$, so that $\|\xi\| = 1$. Now, the support of f_j was bounded by $C_1(1+\|\tau^*\|)^{c_1} = C_1(1+\|\tau\|)^{c_1}$ for some constants C_1, c_1 independent of τ, and we have that $\|g\| = \|g^{-1}\|$ by Lemma 15, so there is some g_τ with $\|g_\tau\| \leq C_1(1+\|\tau\|)^{c_1}$ such that

$$\frac{C}{(1+\|\tau\|)^{c_2}} q(v) \leq |\xi_j(\pi(g_\tau)v)|.$$

Up to relabeling of the constants, this is exactly what we wanted to show, and it follows from Theorem 45 that V is good. □

3.4 Reduction steps

Now that we have proven that the main theorem holds for a specific case of Harish-Chandra modules, we will show that we can transfer this result to other modules. In doing so, we will eventually prove the theorem for the most general case.

We begin with extensions:

Lemma 48 (Lemma 9.1)**.** *Let*

$$0 \to U \to L \to V \to 0$$

be an exact sequence of Harish-Chandra modules. If U and V are good, then L is good.

Proof. Let $(\pi, \overline{L})$ be any smooth Fréchet globalization of L. We can then take the closure of U in $\overline{L}$ to get a smooth Fréchet globalization $(\pi_U, \overline{U})$ of U, and we can take the quotient $\overline{V} := \overline{L}/\overline{U}$ to get a smooth Fréchet globalization $(\pi_V, \overline{V})$ of $V = L/U$. Now, since U and V are good, these globalizations coincide with the minimal ones, which means that $\overline{U} = \Pi_U(\mathcal{S}(G))U$ and $\overline{V} = \Pi_V(\mathcal{S}(G))V$. Since $0 \to \overline{U} \to \overline{L} \to \overline{V} \to 0$ is exact as well, this implies that $\Pi(\mathcal{S}(G))L = \overline{L}$. Indeed, we can look at the following commuting diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{S}(G)\otimes U & \longrightarrow & \mathcal{S}(G)\otimes L & \longrightarrow & \mathcal{S}(G)\otimes V & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{U} & \longrightarrow & \overline{L} & \longrightarrow & \overline{V} & \longrightarrow & 0
\end{array}$$

Here, both the upper and lower sequence are exact, and the vertical maps are given by applying the functions from $\mathcal{S}(G)$ to the space it is tensored with. The outer two vertical maps are surjective, so by standard diagram chasing we see that the middle vertical map is surjective as well. Therefore, any globalization of L coincides with the minimal globalization, i.e. L is good. □

Harish-Chandra modules admit finite composition series, which yields:

Theorem 49 (Corollary 9.2)**.** *To show that all Harish-Chandra modules are good, it is sufficient to show that all irreducible Harish-Chandra modules are good.*

Now, we look at tensor products with finite-dimensional representations. The techniques here will be used later as well. Let V be a Harish-Chandra module, with minimal globalization V^∞. Let (σ, W) denote a finite-dimensional representation of G, and set $\mathbf{V} = V \otimes W$. We will show that if V is good, then $\mathbf{V}$ is good, and that in general the minimal globalization of $\mathbf{V}$ is $V^\infty \otimes W$. We will not actually need the results for the main proof, but the techniques here will be used later.

We fix a θ-covariant inner product $\langle \cdot, \cdot \rangle$ on W, and let $w_1, \dots, w_k$ be a corresponding orthonormal basis of W. We define the $C^\infty(G)$-valued $k \times k$ matrix of the representation:

$$\mathfrak{S} := (\langle \sigma(g) w_i, w_j \rangle)_{1 \le i,j \le k}.$$

Then the matrix is invertible, and all coefficients of $\mathfrak{S}$ and $\mathfrak{S}^{-1}$ are of moderate growth, which yields:

Lemma 50 (Lemma 9.3)**.** *Applying the matrix $\mathfrak{S}$ yields a linear isomorphism in the following two cases:*

(i) $\mathcal{S}(G)^k \to \mathcal{S}(G)^k$,

(ii) $[C_c^\infty(G)]^k \to [C_c^\infty(G)]^k$.

Now, using the first of these two, we can prove the following:

Lemma 51 (Lemma 9.4)**.** *Let V be a Harish-Chandra module and (σ, W) be a finite-dimensional representation of G. If we denote $\mathbf{V} = V \otimes W$, then*

$$\mathbf{V}^\infty = V^\infty \otimes W.$$

Proof. We denote by π_1 the representation $\pi \otimes \sigma$ on $V^\infty \otimes W$. We want to show that $v \otimes w_j$ lies in $\Pi_1(\mathcal{S}(G))\mathbf{V}$ for all $v \in V^\infty$ and $1 \le j \le k$. By linearity, the claim then follows. We fix a $v \in V^\infty$, and without loss of generality we set $j = 1$. Since $v \in V^\infty$, we find some $\xi \in V$ and $f \in \mathcal{S}(G)$ such that $\Pi(f)\xi = v$. By the above lemma, we can now find some $\mathbf{f} = (f_1, \dots, f_k) \in \mathcal{S}(G)^k$ such that $\mathfrak{S}^t(\mathbf{f}) = (f, 0, \dots, 0)$. Now, we claim that

$$\sum_{j=1}^k \Pi_1(f_j)(\xi \otimes w_j) = v \otimes w_1.$$

Indeed, taking the inner product with w_i on the left-hand side yields

$$\begin{aligned}(\mathrm{id} \otimes w_i^*) \left(\sum_{j=1}^k \Pi_1(f_j)(\xi \otimes w_j) \right) &= \sum_{j=1}^k \int_G f_j(g) \langle \sigma(g) w_j, w_i \rangle \pi(g) \xi \mathrm{d}g \\ &= \delta_{1i} \int_G f(g) \pi(g) \xi \mathrm{d}g \\ &= \delta_{1i} v.\end{aligned}$$

This proves the claim, and the result follows. □

The second mentioned result is straightforward to prove:

Theorem 52 (Proposition 9.5)**.** *Let V be a good Harish-Chandra module, and (σ, W) be a finite-dimensional representation of G. Then $\mathbf{V} = V \otimes W$ is good.*

Proof. We take any norm q on W. We claim now that any G-continuous norm on $\mathbf{V}$ is equivalent to some $p \otimes q$ with p a G-continuous norm. Then, since all G-continuous norms on V are Sobolev-equivalent, the result follows.

To prove the claim, we note that $V \otimes W \cong \mathrm{Hom}(W^*, V)$. Any norm p on V then gives a norm on $V \otimes W$, which under this isomorphism becomes

$$(p \otimes q)(T) = \sup_{\substack{w^* \in W^* \\ q^*(w^*) \leq 1}} p(T(w^*)).$$

Conversely, any norm p' on $V \otimes W$ gives a norm p'_V on V by setting

$$p'_V(v) = \sup_{\substack{w \in W \\ q(w) \leq 1}} p'(v \otimes w).$$

Now, combining these operations, we see that

$$\begin{aligned}(p'_V \otimes q)(T) &= \sup_{\substack{w^* \in W^* \\ q^*(w^*) \leq 1}} p'_V(T(w^*)) \\ &= \sup_{\substack{w^* \in W^* \\ q^*(w^*) \leq 1}} \sup_{\substack{w \in W \\ q(w) \leq 1}} p'(T(w^*) \otimes w).\end{aligned}$$

This is equivalent to p', so the result follows. □

Note that this theorem also implies Lemma 51, but the method of proof we used for the lemma will be used again later, so we introduced it here.

As a next step, we look at induction. We will state the following theorem without proof, as the proof relies on techniques and theories that are not needed elsewhere in this work.

Theorem 53 (Proposition 9.6)**.** *Let $P \supseteq P_{min}$ be a parabolic subgroup of G with Langlands decomposition $P = N_P A_P M_P$. Let V_σ be an irreducible good Harish-Chandra module for M_P. Then for all $\lambda \in (\mathfrak{a}_P)^*_{\mathbb{C}}$ the induced Harish-Chandra module $V_{\sigma,\lambda}$ is good. In particular, $V^\infty_{\sigma,\lambda} = E_{\sigma,\lambda}$.*

Proof. See [4], p.84-85. □

In this proof, we need that minimal principal series representations are good, so this could not have been used to replace Theorem 47.

Another result, which we will not prove in detail, is the following:

Theorem 54 (Proposition 10.4)**.** *Suppose that $I : U \to W$ is an intertwiner of good Harish-Chandra modules which allows holomorphic deformations $\mathcal{I} : \mathcal{U} \to \mathcal{W}$. Then im I is good.*

Here $\mathcal{U} = \mathcal{O}(D, U)$ denotes the space of maps f from the open unit disk D to U such that for all $\xi \in \widetilde{U}$ the contraction $\xi \circ f$ is holomorphic. We take a holomorphic family of Harish-Chandra modules, i.e. a family of Harish-Chandra modules $(U_s)_{s\in D}$ such that they are all isomorphic to U as K-modules, and such that for all $X \in \mathfrak{g}, v \in U, \xi \in \widetilde{U}$ the map $s \mapsto \xi(X_s \cdot v)$ is holomorphic. Here $X_s \cdot v$ denotes the action of X in U_s. Then we can put a $(\mathfrak{g}, K)$-structure on $\mathcal{U}$ by setting $(X \cdot f)(s) = X_s \cdot f(s)$. We denote by $\mathbf{U}_k$ the Harish-Chandra module $\mathcal{U}/s^k\mathcal{U}$.

A morphism $\mathcal{I} : \mathcal{U} \to \mathcal{W}$ is then defined as a family of $(\mathfrak{g}, K)$-maps $I_s : U_s \to W_s$ such that for all $U \in U$ and $\xi \in \widetilde{W}$ the assignments $s \mapsto \xi(I_s(u))$ are holomorphic. We denote by $H = \ker(I_0)$. We now define the following:

Definition 55. We say that $I : U \to W$ is *holomorphically deformable* or *allows holomorphic deformations* $\mathcal{I} : \mathcal{U} \to \mathcal{W}$, if we have a morphism $\mathcal{I} : \mathcal{U} \to \mathcal{W}$ of holomorphic families of Harish-Chandra modules, satisfying the following two properties:

- I_s is invertible for all $s \neq 0$.
- There exists a $k \in \mathbb{N}_0$ such that $J(s) := s^k I_s^{-1}$ is holomorphic on D.

Under these additional requirements, one can show ([4], p.86-88) that there is an inclusion chain

$$V_1 \subset V_2 \subset V_3$$

with $V_1 = s^k\mathbf{H}_k$, $V_2 = s^k\mathcal{U}/s^{2k}\mathcal{U}$ and $V_3 = \mathbf{H}_k + s^k\mathcal{U}/s^{2k}\mathcal{U}$. So then we have

$$V_2/V_1 \simeq \mathbf{U}_k/\mathbf{H}_k, \qquad V_2 \simeq \mathbf{U}_k \quad \text{and} \quad V_3/V_1 \simeq \mathbf{W}_k.$$

Since U and W are good, it can be shown that $\mathbf{U}_k$ and $\mathbf{W}_k$ are as well, so by Lemma 43 it follows that $\mathbf{U}_k/\mathbf{H}_k$ is good. Now, using a short exact sequence

$$0 \to U/H \simeq s^{k-1}U/s^{k-1}H \to \mathbf{U}_k/\mathbf{H}_k \to U/H_{k,1} \to 0,$$

with $H_{k,1}$ the projection of $\mathbf{H}_k$ to $\mathbf{U}_1 \simeq U$, we can show that U/H is good. This was what we wanted to show, as H was the kernel of $I_0 = I$, so the image im I is isomorphic to U/H.

As the final piece of the proof, we now look at Harish-Chandra modules belonging to the discrete series. We denote by $Z < G$ the center of G, and we assume that V is a unitarizable irreducible Harish-Chandra module, i.e. there exists a unitary irreducible globalization $(\pi, \mathcal{H})$ of V.

Definition 56 (p.88-89)**.** We say that V is *square integrable* or that V *belongs to the discrete series* if for all $v \in V$ and $\xi \in \widetilde{V}$, one has

$$\int_{G/Z} |m_{\xi,v}(g)|^2 \mathrm{d}(gZ) < \infty.$$

Here $m_{\xi,v}$ denotes a matrix coefficient in the globalization $\mathcal{H}$, i.e.

$$m_{\xi,v}(g) = \xi(\pi(g)v).$$

Then we have:

Theorem 57 (Proposition 10.5)**.** *Let V be a Harish-Chandra module of the discrete series. Then V is good.*

We will not prove this in full detail. The idea of the proof is to embed V into a minimal principal series representation, and use this embedding to put an inner product on V. Using this inner product, we can show that any unitary norm is both maximal and minimal, from which it follows that V is good. The full details are in [4], p.89. Note that for this proof we need that minimal principal series representations are good.

Now, we can prove the final result:

Theorem 58 (Theorem 10.6)**.** *All Harish-Chandra modules are good.*

Proof. We saw in Theorem 49 that we can assume that our module V is irreducible. Using the Langlands classification ([9], Ch. VIII, Theorem 8.54) and our previous theorems on deformation and induction, Theorem 54 and Theorem 53, we can reduce to the case that V is tempered. Now, using Proposition 5.2.5 from [12], this means we can assume that V is of the discrete series. This case has already been proved in Theorem 57, which concludes the proof. □

3.5 Comparing with the article

In this section we treated most of Sections 6 through 10 of the article by Krötz and Bernstein. There are several side results that we have skipped, such as part (ii) of Theorem 8.1 and the discussion in Section 10.4, and of course some proofs were not treated in full detail.

We have also added several explanations, and rephrased certain results as lemmas that got glossed over in the article. Most importantly of all, the proof of our Theorem 47, their 8.1(iii), has been added, which in the original article was treated in a single line (the remark that $\widetilde{I(W)} = I(W^*)$).

We have made some changes to the results claimed in Theorems 7.1 and 7.2 in the article, particularly the addition of the constants C in the statements of our Theorems 44 and 45. Without these constants, the proofs in the article no longer line up with the statements, and the way the theorems are used would also be incorrect. We assume this was merely a typographical error, and have changed the result accordingly.

Overall, we found the article generally very clear in these sections, apart from the proof of Theorem 8.1(iii), and the additions made here were predominantly to emphasize certain parts that will return later.

4 Reduction steps for minimal principal series representations

In this section, we reduce the proof of Theorem 46 to the case where $W = \mathbb{C}_\chi$ for some character χ on A. We repeat here the theorem we want to prove:

Theorem 46 (Theorem 8.1(i))**.** *Let $V = I(W)$ be a minimal principal series representation of G. Let $\xi_1, \dots, \xi_k$ be a set of generators of V. We take the Hilbert globalization $\mathcal{H} = L^2(W \times_M K)$. Then there exist constants $c_1, c_2, C_1, C_2 > 0$ such that for all $\tau \in \hat{K}$ and $v_\tau \in V[\tau]$ there exist functions $f_{\tau,1}, \dots, f_{\tau,k} \in C_c^\infty(G)$ with the following properties:*

(a) $\sum_{j=1}^k \Pi(f_{\tau,j})\xi_j = v_\tau$.

(b) $\operatorname{supp}(f_{\tau,j}) \subset \{g \in G | \, \|g\| < C_1(1 + \|\tau\|)^{c_1}\}$ *for all* $1 \leq j \leq k$.

(c) $\sum_{j=1}^k \|f_{\tau,j}\|_1 \leq C_2 \|v_\tau\|(1 + \|\tau\|)^{c_2}$, *where* $\| \cdot \|_1$ *refers to the* $L^1(G)$*-norm.*

We first note that if the theorem holds for any set of generators, it holds for all sets of generators. Indeed, if we take the decomposition of a vector v_τ in terms of some set of generators ξ_j, then for another set ξ_j', we can write the ξ_j in terms of the ξ_j' and plug that into the decomposition to obtain another decomposition. By changing the values of the constants, we then see that the theorem still holds. Because of this, we introduce the following definition:

Definition 59. Let V be a Harish-Chandra module, with some Hilbert globalization $\mathcal{H}$. We say that V is of $\mathcal{D}$-type if for any set of generators $\xi_1, \dots, \xi_k$ of V, there exist constants $c_1, c_2, C_1, C_2 > 0$ such that for all $\tau \in \hat{K}$ and $v_\tau \in V[\tau]$ there exist functions $f_{\tau,1}, \dots, f_{\tau,k} \in C_c^\infty(G)$ with the following properties:

(a) $\sum_{j=1}^k \Pi(f_{\tau,j})\xi_j = v_\tau$.

(b) $\operatorname{supp}(f_{\tau,j}) \subset \{g \in G | \, \|g\| < C_1(1 + \|\tau\|)^{c_1}\}$ for all $1 \leq j \leq k$.

(c) $\sum_{j=1}^k \|f_{\tau,j}\|_1 \leq C_2 \|v_\tau\|(1 + \|\tau\|)^{c_2}$, where $\| \cdot \|_1$ refers to the $L^1(G)$-norm.

Here the $\mathcal{D}$ in the notation "of $\mathcal{D}$-type" refers to the notation $C_c^\infty(G) = \mathcal{D}(G)$, commonly used in the study of distributions. To prove that a certain representation is of $\mathcal{D}$-type, we only have to check one specific set of generators of our choosing, by the discussion above.

Now, in this notation we can rephrase Theorem 46 as follows:

Theorem 60. *Let $V = I(W)$ be a minimal principal series representation of G. Then V is of $\mathcal{D}$-type.*

Recall that we need this theorem to prove that minimal principal series representations are good. This is then used both in proving that being good is preserved under induction, and that Harish-Chandra modules of the discrete series are good, which then results in the general statement of the Casselman-Wallach globalization theorem.

4.1 Reduction steps

To begin the reduction, we first construct a Jordan-Hölder series of W, i.e. a sequence $0 \subset W_0 \subset W_1 \subset \cdots \subset W_n = W$ of submodules, such that for each inclusion the quotient W_{i+1}/W_i is irreducible. But then, the exact sequence $0 \to W_i \to W_{i+1} \to W_{i+1}/W_i \to 0$ induces an exact sequence $0 \to I(W_i) \to I(W_{i+1}) \to I(W_{i+1}/W_i) \to 0$. Suppose now that we have for irreducible W' that $I(W')$ is of $\mathcal{D}$-type. With the following lemma, it then follows by induction that $I(W)$ is of $\mathcal{D}$-type.

Lemma 61. *Let $0 \to U \xrightarrow{g} V \xrightarrow{h} W \to 0$ be a short exact sequence of Harish-Chandra modules, and let U and W be of $\mathcal{D}$-type. Then V is of $\mathcal{D}$-type.*

Proof. First, we pick a set of generators for the module V. We take any set of generators $\xi'_1, \ldots, \xi'_k$ for U, and map it under g to $\xi_i = g(\xi'_i)$. Then, we extend this to a set of generators by picking $\xi_{k+1}, \ldots, \xi_l$ such that applying h to these yields a set of generators of W. (We can do this, since h is surjective.) The result is indeed a set of generators of V, since for any $v \in V$ we can first write $h(v)$ in terms of the $h(\xi_j)$, which gives us a way to write v in terms of the ξ_j, modulo the image of U, but any element in this image can be written in terms of the $g(\xi'_i) = \xi_i$, so they form a set of generators.

Now, we take an isotypical component $\tau \in \hat{K}$, and a vector $v_\tau \in V[\tau]$. Then $h(v_\tau)$ is a (possibly zero) vector in $W[\tau]$, so since W is of $\mathcal{D}$-type, we can decompose it in any set of generators. Using the $h(\xi_j)$, we find $f_{\tau,j}$ that satisfy the estimates from Definition 59, and such that $\sum_{j=k+1}^{l} \Pi(f_{\tau,j})h(\xi_j) = h(v_\tau)$. Now, we can take the h outside, and get that $h(\sum_{j=k+1}^{l} \Pi(f_{\tau,j})\xi_j) = h(v_\tau)$, which shows that $\sum_{j=k+1}^{l} \Pi(f_{\tau,j})\xi_j - v_\tau$ is in the kernel of h. We then project this difference onto the τ-isotypical component, by applying $\dim(\tau)\Pi(\bar{\chi}_\tau)$. Note that χ_τ is a function on K, so here we only look at the K-action. Then, v_τ remains unchanged, and in the sum we get terms $\dim(\tau)\Pi(\bar{\chi}_\tau * f_{\tau,j})\xi_j$, where we take the convolution of a function on K and a function on G to obtain a function on G. The resulting function will still be bounded in the L^1-norm, and the support will still be polynomially bounded, since K is bounded and the support of $f_{\tau,j}$ was bounded. After this projection, the result $\sum_{j=k+1}^{l} \dim(\tau)\Pi(\bar{\chi}_\tau * f_{\tau,j})\xi_j - v_\tau$ is still in the kernel of h, which is the image of g, and it is in $V[\tau]$. But then there is a $v \in U[\tau]$ such that $g(v) = \sum_{j=k+1}^{l} \dim(\tau)\Pi(\bar{\chi}_\tau * f_{\tau,j})\xi_j - v_\tau$. This v can now be decomposed, since U is of $\mathcal{D}$-type, so we can write $v = \sum_{j=1}^{k} \Pi(f_{\tau,j})\xi'_j$. But then, applying g, we get that $g(v) = \sum_{j=1}^{k} \Pi(f_{\tau,j})\xi_j$, which we can rewrite as $v_\tau = \sum_{j=1}^{k} \Pi(f_{\tau,j})\xi_j + \sum_{j=k+1}^{l} \dim(\tau)\Pi(\bar{\chi}_\tau * f_{\tau,j})\xi_j$. Since all the resulting functions satisfy the needed bounds, the result now follows. $\square$

It follows that we only have to prove Theorem 60 in the case that W is irreducible. But then we can simplify things even further: since W is irreducible, N has to act trivially on W. Indeed, if we look at the infinitesimal action of $\mathfrak{n}$, a nilpotent Lie algebra, by Engel's Theorem we can pick a basis for W so that all elements of $\mathfrak{n}$ have a strictly upper triangular form. But this implies that there is a vector w that gets mapped to 0 by all of $\mathfrak{n}$, or equivalently, that N fixes that vector. But, since $M \times A$ normalizes N, and M

and A commute, this means that P fixes the subspace MAw, because $(MAN)MAw = (MA)MANw = MAw$. So, MAw is a non-trivial subrepresentation, which means that it has to be all of W. But then, any element of W can be written as MAw. If we apply an element $n \in N$ to maw, we get $n(maw) = (ma)n'w = maw$ for some $n' \in N$, using again that MA normalizes N. So, N acts trivially on W.

This means that we can factor the P_{min}-representation W into a $P_{min}/N \simeq M \times A$-representation. Since M and A commute, this means that we can write the representation as $\sigma \times \chi$ for representations σ and χ of M and A respectively. As A is abelian, we apply Schur's Lemma, and the representation χ of A is scalar. Since W is irreducible as a P-representation, the action of N is trivial, and the action of A is scalar, the representation σ of M has to be irreducible. Now, we use two results:

Lemma 62. *If W is a P-module, and F a finite-dimensional representation of G, then $I(W \otimes F|_P) \simeq I(W) \otimes F$ in a natural way.*

Here $W \otimes F|_P$ is a P-module, and $I(W) \otimes F$ is a G-module.

Proof. We define the map $Q : I(W) \otimes F \to I(W \otimes F|_P)$ by setting

$$Q(f \otimes v) : g \mapsto f(g) \otimes \pi(g^{-1})v,$$

and extending linearly to all of $I(W) \otimes F$. Here π denotes the representation on F. This lands in the right space: for any $h \in P$, we have that $Q(f \otimes v)(gh) = f(gh) \otimes \pi(h^{-1}g^{-1})v = \xi(h^{-1})f(g) \otimes \pi(h^{-1})\pi(g^{-1})v = h^{-1} \cdot (f(g) \otimes \pi(g^{-1})v) = h^{-1} \cdot Q(f \otimes v)(g)$, where ξ denotes the representation on W. It is easily checked that the map is G-equivariant, and it is injective: suppose that some $\sum_i f_i \otimes v_i$ gets mapped to 0. If we pick a basis $e_1, \ldots, e_n$ of F, we can expand all of the v_i in terms of this basis, and then group the resulting terms, to write $\sum_i f_i \otimes v_i = \sum_j g_j \otimes e_j$. Note that any element of $I(W) \otimes_G F$ can be written in one unique such way. Then, applying Q, we get $0 = \sum_j Q(g_j \otimes e_j)$, so applying this to any element x of the group, we get $0 = g_j(x) \otimes \pi(x^{-1})e_j$. Now, since the $\pi(x^{-1})e_j$ still form a basis of F, we conclude that $g_j(x) = 0$ for all x, so all the g_j were zero. This means that $\sum_i f_i \otimes v_i = 0$, which is what we wanted to prove.

Now, to prove surjectivity, we take any element ψ of $I(W \otimes_P F|_P)$. At every point g, we can decompose the outcome of ψ as $\psi(g) = \sum_i \psi_i(g) \otimes e_i$, similar to above. However, it turns out to be more convenient to use the basis consisting of $\pi(g^{-1})e_i$, and we write

$$\psi(g) = \sum_i \phi_i(g) \otimes \pi(g^{-1})e_i.$$

Since ψ was an element of $I(W \otimes F|_P)$, we know for any $g_0 \in P$ that $\psi(gg_0) = g_0^{-1} \cdot \psi(g)$, which means that

$$\sum_i \phi_i(gg_0) \otimes \pi(g_0^{-1})\pi(g^{-1})e_i = \xi(g_0^{-1})\phi_i(g) \otimes \pi(g_0^{-1})\pi(g^{-1})e_i.$$

Since the $\pi(g_0^{-1})\pi(g^{-1})e_i$ form a linear basis, we conclude that $\phi_i(gg_0) = \xi(g_0^{-1})\phi_i(g)$, which means that the ϕ_i are elements of $I(W)$. But then, we can look at the element $\sum_i \phi_i \otimes e_i$

in $I(W) \otimes F$, and apply Q to it, to yield the function $g \mapsto \sum_i \phi_i(g) \otimes \pi(g^{-1})e_i = \psi(g)$. This shows surjectivity, so the two spaces are indeed isomorphic. Since the map Q was natural, they are isomorphic in a natural way. $\square$

Theorem 63. *If W is an irreducible $M \times A$-module as above, there exists a finite-dimensional representation F of G, with space of N-invariants F^N such that W embeds in $F^N \otimes \mathbb{C}_\psi$ for some character ψ of A.*

We postpone the proof of this embedding theorem to the next subsection.

Using these two, we can embed W into $F \otimes \mathbb{C}_\psi$, which means we can embed $I(W)$ into $I(F \otimes \mathbb{C}_\psi) \simeq I(\mathbb{C}_\psi) \otimes F$. Now, using two more lemmas, we conclude that we only have to show that $I(\mathbb{C}_\psi)$ is of $\mathcal{D}$-type for any character ψ to conclude that any $I(W)$ is of $\mathcal{D}$-type:

Lemma 64. *If V is of $\mathcal{D}$-type, and $U \subset V$ is a submodule, then U is of $\mathcal{D}$-type.*

Proof. Unfortunately, we have not been able to prove this yet. We will nevertheless proceed as if this or a similar result holds, and refer the reader to the Discussion in Section 6 for further details. $\square$

Lemma 65. *If V is of $\mathcal{D}$-type, and F is a finite-dimensional G-representation, then $V \otimes F$ is of $\mathcal{D}$-type.*

Proof. We denote the representation on V by π, the representation on G by σ, and the representation on $V \otimes F$ by π_1.

We take a θ-covariant inner product on F, and fix an orthonormal basis $f_1, \ldots, f_k$ of F with respect to this inner product. We now define the matrix $\mathfrak{S} := (\langle \sigma(g) f_i, f_j \rangle)_{1 \leq i,j \leq k}$ as before, and recall from Lemma 50 that the matrix is invertible, so that it defines a linear isomorphism from $C_c^\infty(G)^k \to C_c^\infty(G)^k$.

Now, to prove that $V \otimes F$ is of $\mathcal{D}$-type, we will use the following set of generators: take $\{\xi_i\}$ a set of generators of V, then the set of $\xi_i \otimes f_j$ are a set of generators of $V \otimes F$. Suppose now that we have some $v = \sum_i v_i \otimes f_i \in (V \otimes F)[\tau]$. We can then decompose every v_i into isotypical components, to get

$$v = \sum_i (\sum_j (v_{i,j}) \otimes f_i),$$

where each of the $v_{i,j}$ is in an isotypical component. Since weights add up under tensor products, and since F only has finitely many weights, we see that if $v_{i,j} \in V[\tau']$, then $\|\tau'\| \leq \|\tau\| + C$ for some constant C. In particular, we can find constants C_1', C_2' such that $C_1(1 + \|\tau'\|)^{c_1} < C_1'(1 + \|\tau\|)^{c_1}$ and $C_2(1 + \|\tau'\|)^{c_2} < C_2'(1 + \|\tau\|)^{c_2}$, where the unprimed constants come from the definition of V being of $\mathcal{D}$-type. Finally we note that since the weights lie on a lattice, we can find a polynomial bound on the number of weights with $\|\tau'\| \leq \|\tau\| + C$ in terms of $\|\tau\|$.

Now, since V was of $\mathcal{D}$-type, we can decompose every $v_{i,j}$ as $\sum_l(\Pi(f_{i,j,l})\xi_l)$, for functions $f_{i,j,l}$ that satisfy

$$\begin{aligned}
&\operatorname{supp}(f_{i,j,l}) \subset \{g \in G \mid \|g\| < C_1(1+\|\tau'\|)^{c_1} < C_1'(1+\|\tau\|)^{c_1}\} \\
&\sum_{l=1}^{n} \|f_{i,j,l}\|_1 \leq C_2\|v_{i,j}\|(1+\|\tau'\|)^{c_2} < C_2'\|v\|(1+\|\tau\|)^{c_2},
\end{aligned}$$

for the appropriate τ'. Now, analogous to the proof of Lemma 51, we find functions $f_{i,j,l,m} \in C_c^\infty(G)$ such that $\mathfrak{S}^t((f_{i,j,l,m})_m) = (0,\dots,f_{i,j,l},\dots,0)$, with the non-zero term being on the i-th position. It then follows that $\sum_{m=1}^k \Pi_1(f_{i,j,l,m})(\xi_l \otimes f_m) = (\Pi(f_{i,j,l})\xi_l)\otimes f_i$. The functions $f_{i,j,l,m}$ now also have their supports in $\{g \in G \mid \|g\| < C_1'(1+\|\tau\|)^{c_1}\}$ since they are linear combinations of the previous $f_{i,j,l}$, and since the functions we used as coefficients are bounded on this support, the L_1-norm of the result still satisfies $\|f_{i,j,l,m}\| < C_2''\|v\|(1+\|\tau\|)^{c_2}$ for some constant C_2''. Now, if we define functions $f_{l,m}$ by $\sum_{i,j} f_{i,j,l,m}$, we sum up some number of terms that is polynomially bounded in $\|\tau\|$, and we have

$$\begin{aligned}
\sum_{l,m}\Pi_1(f_{l,m})(\xi_l\otimes f_m) &= \sum_{i,j,l,m}\Pi_1(f_{i,j,l,m})(\xi_l\otimes f_m) \\
&= \sum_{i,j,l}(\Pi(f_{i,j,l})\xi_l)\otimes f_i \\
&= \sum_i(\sum_j v_{i,j}\otimes f_i) \\
&= v.
\end{aligned}$$

So, these functions can be used to decompose v in terms of the chosen generators. Their supports satisfy the needed bound, since each of the $f_{i,j,l,m}$ did, and their L_1-norms satisfy a polynomial bound, since we summed polynomially many functions that each satisfied it, so by the triangle inequality the required result follows. □

4.2 Proof of Theorem 63

To show the reasoning, we will prove Theorem 63 in the case of G being semisimple. For a reductive Lie group the result still holds, but the arguments are slightly different.

We first prove a result about extending characters to a torus. Suppose we have a compact finite-dimensional torus T, i.e. a compact connected abelian finite-dimensional Lie group. Assume that we have some involution σ on this torus. We denote by $\hat{T}$ the set of all characters of T, that is the set of all Lie group homomorphisms $\xi : T \to \mathbb{C}^*$. Since T is abelian, the exponential map is a group homomorphism from the Lie algebra $\mathfrak{t}$ to T. The kernel of the exponential map is therefore a lattice Γ, since it is a discrete subgroup, and it spans $\mathfrak{t}$ over $\mathbb{R}$. We define the dual lattice $\Gamma^\vee$ as the set of all $\lambda \in \mathfrak{t}^*$ such that $\lambda(X) \in \mathbb{Z}$ for all $X \in \Gamma$. With this notation, the following result is well-known:

Lemma 66. *The map $\hat{T} \to \mathfrak{t}_{\mathbb{C}}^*, \xi \mapsto \xi_* = T_e\xi$ is a bijection from $\hat{T}$ to $2\pi i\Gamma^\vee$.*

The involution σ induces a linear involution on $\mathfrak{t}$, which we also denote by σ. Then, we can decompose $\mathfrak{t}$ into the two eigenspaces of σ, and write $\mathfrak{t} = \mathfrak{t}_+ \oplus \mathfrak{t}_-$. If we look at the set T^σ of fixed points of σ in T, we see that it is a closed subgroup of T, with as its Lie algebra the fixed points $\mathfrak{t}^\sigma = \mathfrak{t}_+$. Similarly, we can look at the subgroup $S = \{t \in T | \sigma(t) = t^{-1}\}$, which is a subgroup since T is abelian. Its Lie algebra is given by $\mathfrak{t}_-$, and its identity component is given by $T_- = \exp(\mathfrak{t}_-)$. This is a submanifold, hence a closed subgroup of T again. The intersection

$$F := T^\sigma \cap T_-$$

is then also a closed subgroup, but its Lie algebra is trivial, which means that it is discrete, hence finite. Now, using this notation, we can prove a sequence of results:

Lemma 67. *The multiplication map $m : T^\sigma \times T_- \to T$ is a surjective Lie group homomorphism, which is a local diffeomorphism everywhere. Its kernel is given by pairs (t, t^{-1}), with $t \in F$.*

Proof. At (e, e), the tangent map is bijective by the decomposition of $\mathfrak{t}$ we gave before. By homogeneity, this makes m a local diffeomorphism everywhere. The image of m is therefore an open subgroup of M, and since T is connected, this makes it surjective. Any element of the kernel has to be of the form (t, t^{-1}), and t has to be in both T^σ and T_-, hence in F. $\square$

Lemma 68. $F = \{t \in T_- | t^2 = e\}$.

Proof. An element in F clearly has to be in T_-, and it has to have $t = \sigma(t) = t^{-1}$, so $t^2 = e$. On the other hand, if $t \in T_-$, then $t^{-1} = \sigma(t)$, but if $t^2 = e$, then $t = t^{-1}$, so $t \in T^\sigma$ and $t \in F$. $\square$

We denote now by $\Gamma_\pm$ the kernels of exp restricted to $\mathfrak{t}_\pm$, that is $\Gamma_\pm = \Gamma \cap \mathfrak{t}_\pm$. Then our previous lemma becomes $F = \exp(\frac{1}{2}\Gamma_-)$. Now we can formulate our first substantial result:

Lemma 69. *Restriction to T^σ induces a surjective map $\hat{T} \to \hat{T^\sigma}$.*

Proof. Let ξ be a character of T^σ. We claim that there exists a character η on T_- which agrees with ξ on the intersection F. From this claim and Lemma 67, the result then follows.

To prove the claim, we fix a $\mathbb{Z}$-basis for $\frac{1}{2}\Gamma_-$, and denote it by $\gamma_1, \ldots, \gamma_r$. If we denote $\epsilon_j = \xi(\exp \gamma_j)$, we have $\epsilon_j^2 = 1$, so $\epsilon_j = \pm 1$. This means we can select $k_j \in \{0, 1\}$ such that $e^{k_j \pi i} = \epsilon_j$. We then define a functional $\mu \in i\mathfrak{t}_-^*$ by setting

$$\mu(\gamma_j) = k_j \pi i.$$

Then we have $\mu(2\gamma_j) \in 2\pi i\mathbb{Z}$ for all j, which means that $\mu \in 2\pi i \Gamma_-^\vee$. So, by Lemma 66, we have that $\mu = \eta_*$ for a unique $\eta \in \hat{T_-}$. Now, we have for any j that

$$\eta(\exp \gamma_j) = e^{\eta_*(\gamma_j)} = e^{\mu(\gamma_j)} = \epsilon_j = \xi(\exp \gamma_j).$$

The elements $\exp \gamma_j$ generate F, so we have that $\eta = \xi$ on all of F, which proves the claim and therefore the lemma. $\square$

Now, we will recall some results about weights and representations. Let G be a compact semisimple Lie group, with Lie algebra $\mathfrak{g}$. If we take $\mathfrak{t}$ a maximal torus in $\mathfrak{g}$, and $T = \exp(\mathfrak{t})$, then T is an open subgroup of $Z_G(\mathfrak{t})$, so it is closed and hence compact in G. The exponential map of G, restricted to $\mathfrak{t}$, coincides with the exponential map of T, so we can describe the characters of T via $\Gamma_T = \ker \exp$ as before. We see that we have a bijection $\xi \mapsto \xi_*$ from $\hat{T}$ onto $\Lambda_T := 2\pi i \Gamma_T^\vee$.

We denote by R the root system of $\mathfrak{t}$ in $\mathfrak{g}$, by Λ the lattice of integral weights (which is a discrete additive subgroup of $i\mathfrak{t}^*$), and by Q the root lattice of $(\mathfrak{t}, \mathfrak{g})$, i.e. the additive subgroup of Λ generated by R. We then recall the following result:

Lemma 70. *Let G be compact connected semisimple. We have the following inclusion of lattices:*

$$Q \subset \Lambda_T \subset \Lambda.$$

Furthermore, we have:

- *G is adjoint if and only if $Q = \Lambda_T$*
- *G is simply connected if and only if $\Lambda_T = \Lambda$.*

With G still a compact connected semisimple Lie group, it is a well known result that the inclusion $T \to G$ induces an epimorphism of fundamental groups $\Pi_1(T) \to \Pi_1(G)$ based at e. We also have that $\exp : \mathfrak{t} \to T$ is a universal covering, so we have $\Pi_1(T) \simeq \Gamma_T$. This yields a natural epimorphism, which has:

Lemma 71. *The natural epimorphism $\Gamma_T \to \Pi_1(G)$ has kernel $2\pi i \Lambda^\vee$.*

From this it follows that $\Pi_1(G) \simeq \Lambda_T^\vee / \Lambda^\vee$. It can also be shown that there is a natural one-to-one correspondence from the lattices between Q and Λ, to the isomorphism classes of connected Lie groups with Lie algebra $\mathfrak{g}$. This works as follows:
The lattice Q gives rise to the class represented by $\mathrm{Int}(\mathfrak{g})$, and Λ gives rise to the class of the universal covering $\widetilde{G}$ of $\mathrm{Int}(\mathfrak{g})$. If Λ_0 is a lattice in between, then $C = \exp(2\pi i \Lambda_0^\vee)$ is a finite group in the center of $\widetilde{G}$, and $G := \widetilde{G}/C$ represents the isomorphism class associated to Λ_0.

Now, we apply this to representation theory, to describe which irreducible representations of $\mathfrak{g}$ lift to irreducible representations of G. We choose positive roots R^+ and let Λ^+ be the associated set of dominant integral weights. For $\lambda \in \Lambda^+$ we denote by π_λ the associated irreducible representation of $\mathfrak{g}$ with highest weight λ. Then, $\lambda \mapsto \pi_\lambda$ induces a bijection from Λ^+ onto the equivalence classes of finite dimensional irreducible representations of $\mathfrak{g}$. We now have the following result, which we state without proof:

Lemma 72. *Assume that G is connected compact semisimple, and let $\lambda \in \Lambda$. Then π_λ lifts to a representation of G, which is automatically irreducible, if and only if $\lambda \in \Lambda_T$.*

Now finally, we can prove the result we need. As we assume that G is a linear group, we can embed G into a connected complex semisimple Lie group $G_{\mathbb{C}}$. Therefore, we can assume that $\mathfrak{g}$ is a real semisimple Lie algebra, $\mathfrak{g}_{\mathbb{C}}$ its complexification, $G_{\mathbb{C}}$ a connected semisimple Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, and G the analytic subgroup with Lie algebra $\mathfrak{g}$. Then, the center of G is contained in the center of $G_{\mathbb{C}}$, which is finite since the group is semisimple. That means that G has a finite center, which implies that G has some maximal compact subgroup K. Let θ be the associated Cartan involution, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition. Pick a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$.

Now, $\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{p}$ is a real Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$, which is actually a real form of $\mathfrak{g}_{\mathbb{C}}$. The analytic subgroup U generated by $\mathfrak{u}$ is a maximal compact subgroup of $G_{\mathbb{C}}$, and we have that $K = G \cap U$. The complex linear extension of θ to $\mathfrak{g}_{\mathbb{C}}$ lifts to a complex analytic involution of $G_{\mathbb{C}}$, which we also denote by θ. It leaves U invariant, and we have that $K = U^{\theta}$.

If we denote by M the centralizer of $\mathfrak{a}$ in K, then its Lie algebra $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. The centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ is then $\mathfrak{m} \oplus \mathfrak{a}$. We now fix a maximal abelian subspace $\mathfrak{t}_+$ of $\mathfrak{m}$. Then $\mathfrak{t} = \mathfrak{t}_+ \oplus i\mathfrak{a}$ is a maximal torus in $\mathfrak{u}$, that is invariant under θ. If we denote by σ the restriction of θ to $\mathfrak{t}$, then in the same notation as before, we have $i\mathfrak{a} = \mathfrak{t}_-$. If T denotes the maximal torus $\exp(\mathfrak{t})$ in U, then T is invariant under the Cartan involution, and $\sigma = \theta|_T$ is the lift of the infinitesimal σ to T. Again in the notations from before, we define T^{σ} and T_-, so that we have $T = T^{\sigma}T_-$, and $F = T^{\sigma} \cap T_-$. Then, we have:

Lemma 73. *We have $F = K \cap \exp(i\mathfrak{a})$ and $M = M_e F$.*

Proof. Since $T^{\sigma} \subset U^{\theta} = K$, we have $F \subset K \cap \exp i\mathfrak{a}$. Conversely, if $X \in i\mathfrak{a}$, and $\exp X \in K$, by applying the Cartan involution we see $(\exp X)^2 = e$, so that $\exp X \in F$ by Lemma 68.

To prove the second equality, we recall that the centralizer $Z_U(S)$ of any subset S of $\mathfrak{t}$ is a connected closed subgroup of U. Applying this to $S = i\mathfrak{a}$, we get that $Z_U(i\mathfrak{a})$ is connected. The Lie algebra of this centralizer is $\mathfrak{m} \oplus i\mathfrak{a}$, so the multiplication map $M_e \times \exp(i\mathfrak{a}) \to U$ is a group homomorphism that is a local diffeomorphism, so its image is an open subset of $Z_U(i\mathfrak{a})$. This centralizer is connected, so we see that the multiplication is surjective, so $M_e \exp(i\mathfrak{a}) = Z_U(i\mathfrak{a})$. In particular, we have $M \subset M_e \exp(i\mathfrak{a})$. So, for any $m \in M$, there exists $m_0 \in M_e, a \in \exp(i\mathfrak{a})$ such that $m = m_0 a$. Then $a \in M \cap \exp(i\mathfrak{a}) \subset F$, so $m \in M_e F$, which shows that $M \subset M_e F$, and the reverse inclusion is obvious, so the result follows. □

Lemma 74. *$T^{\sigma} = Z_M(\mathfrak{t}_+)$.*

Proof. The right-hand side is easily seen to be included in the left-hand side. For the other inclusion, note that $T = Z_U(\mathfrak{t})$ so that $T^{\sigma} \subset Z_{U^{\theta}}(\mathfrak{t}) = Z_K(\mathfrak{t}) = Z_M(\mathfrak{t}) = Z_M(\mathfrak{t}_+)$. □

Now, we can finally prove a result that directly implies our theorem:

Lemma 75. *Let (ξ, V_{ξ}) be an irreducible representation of M. Then there exists an irreducible finite-dimensional representation (π, V) of G such that V_{ξ} can be realized as a submodule of the M-module V^N.*

Proof. The representation ξ induces an infinitesimal representation ξ_* of $\mathfrak{m}$, which is irreducible since F is central in M, and $M = M_e F$. By compactness, we have that $\mathfrak{m}$ is reductive, so ξ_* restricts to an irreducible representation of the semisimple part of $\mathfrak{m}$, i.e. of $\mathfrak{m}_s := [\mathfrak{m}, \mathfrak{m}]$. If we denote by R_M the root system of $\mathfrak{t}_+$ in $\mathfrak{m}$, then via extension by zero we can view them as the roots in $R = R(\mathfrak{t}, \mathfrak{u})$ that vanish on $i\mathfrak{a}$. We can now identify the roots in R with the roots of $\mathfrak{t}_\mathbb{C}$ in $\mathfrak{g}_\mathbb{C}$, and under this identification R_M becomes the set of roots in R that vanish on $\mathfrak{a}$. Note that roots from $R \backslash R_M$ restrict to roots of Σ, the root system of $\mathfrak{a}$ in $\mathfrak{g}$, and that all roots of Σ can be obtained in this way.

We now fix a choice R^+ of positive roots for R, such that all $\alpha \in R^+$ restrict to $\alpha|_\mathfrak{a} \in \Sigma^+$. Then $R_M^+ = R_M \cap R^+$ is a positive system for R_M too. Let now $v \in V_\xi \backslash \{0\}$ be a highest weight vector for $\xi_*|_{\mathfrak{m}_s}$. Then $T^\sigma = Z_M(\mathfrak{t}_+)$ acts by some character ψ on v. So, by Lemma 69 we can extend it to a character ψ' on T. The derivative of this character is then a weight $\lambda \in \Lambda_T$. We denote by (π, V) the irreducible representation of U of highest weight λ, which exists by Lemma 72, and we extend it to a finite dimensional complex analytic representation of $G_\mathbb{C}$. This latter representation restricts to a representation of G in V, which we also denote by π.

Now, we select a highest weight vector $v_\lambda \in V$. Then, it is a well known result that $V = U(\mathfrak{g}_\mathbb{C})v_\lambda$. It is easily seen that v_λ is annihilated by $\mathfrak{n}$, so that $v_\lambda \in V^N$. Since $\mathfrak{m}$ normalizes $\mathfrak{n}$, it follows that $W := U(\mathfrak{m})v_\lambda \subset V^N$. We now claim that this W can be seen as an embedded copy of (ξ, V_ξ).

The center of $\mathfrak{m}$ is contained in $\mathfrak{t}$, hence it preserves $\mathbb{C}v_\lambda$. Therefore, W is a cyclic $\mathfrak{m}_s$ module, which is finite dimensional, so it is irreducible. Furthermore, there is a unique $\mathfrak{m}_s$-intertwining operator $A : V_\xi \to W$ that maps v to v_λ. If we denote by M_s the analytic subgroup of M generated by $\mathfrak{m}_s$, we see that A also intertwines the M_s-module structures on V_ξ and W, since both already had these structures. If we look at how T^σ acts on V_ξ and on W, we see that it acts by the character ψ on v, and by $\psi'|_{T^\sigma} = \xi$ on v_λ, so A also intertwines the T^σ-actions. Finally, since $M = M_eF = M_sT^\sigma$, we see that A intertwines the M-module structures of V_ξ and W, showing that it embeds V_ξ, as W, into V^N. $\square$

Now, to finish up the proof: we had W an $M \times A$ representation $\sigma \times \chi$. This means that we can also view it as $W \otimes \mathbb{C}_\chi$, where now only M acts on W. We can then embed this M-module W in some finite-dimensional representation F of G by the above lemma. However, A also acts on F, by some character χ', so to correct for this, we look at $F \otimes \mathbb{C}_{\chi - \chi'}$. Then since both the M- and A-actions match, our desired result follows with $\psi = \chi - \chi'$.

4.3 Comparing with the article

This section provides details on a part that in the article by Bernstein and Krötz, [4], gets treated in the introduction to Appendix A (Section 12). There, only a very brief outline of the argument is given. We expanded this into a full proof, which takes up this entire section. We introduced the definition of a Harish-Chandra module being of $\mathcal{D}$-type to simplify the phrasing of partial results, and to emphasize the important role it plays here. The proof given for Theorem 63 is standard, and written with the help of prof. dr. Van den Ban, while the rest of the proofs are original.

5 Proof for spherical principal series representations

Throughout this section, we match the notation of [4], to more easily reference results. As a consequence of this, induced representations will be using the right regular representation instead of the left regular representation we have been using so far.

5.1 Spherical principal series representations

We now focus on the remaining case that $W = \mathbb{C}_\chi$ for some character χ. In this case, the induced representation will consist of the K-finite vectors of the space of functions

$$\{f \in C^\infty(G) \mid f(pg) = p \cdot f(g) \ \ \forall p \in P_{\min}, \forall g \in G\}.$$

The action of p on $f(g)$ is given by $p \cdot f(g) = \chi(a)f(g)$ where $p = nam$ in the Iwasawa decomposition. Now, the character is nowhere zero, since if it were anywhere zero, it would be everywhere zero and the representation would be trivial. So, we can write $\chi(a) = a^\mu$ for some $\mu \in \mathfrak{a}^*_\mathbb{C}$, using that A is abelian and connected. If we then define the half sum $\rho \in \mathfrak{a}^*$ by $\rho(Y) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}_\mathfrak{n} Y)$, and write $\mu = \lambda + \rho$, we have that $I(\mathbb{C}_\chi)$ is exactly the space of K-finite vectors of the spherical principal series representation

$$\mathcal{H}^\infty_\lambda := \{f \in C^\infty(G) \mid f(namg) = a^{\rho+\lambda} f(g) \ \ \forall nam \in P_{\min}, \forall g \in G\}.$$

The action of G on this space is given by the right regular representation, so $(g_1 \cdot f)(g) = f(gg_1)$. The representation is smooth, and we denote it by π_λ. Now, since we know how the functions transform under multiplication by NA, restriction to K yields an isomorphism

$$\mathrm{Res}_K : \mathcal{H}^\infty_\lambda \to C^\infty(M\backslash K).$$

Here $M\backslash K$ denotes the right cosets of M in K. Using the Iwasawa decomposition $G = NAK$ with corresponding functions $g = \tilde{n}(g)\tilde{a}(g)\tilde{k}(g)$, the action of G on $C^\infty(M\backslash K)$ which we also denote by π_λ becomes:

$$[\pi_\lambda(g)f](Mk) = \tilde{a}(kg)^{\lambda+\rho} f(M\tilde{k}(kg)).$$

Note that the space of functions and the action of K is completely independent of λ, only the action of G depends on it. The action lifts to a continuous action on the Hilbert completion $\mathcal{H}_\lambda = L^2(M\backslash K)$. We note that the dual representation of $(\pi_\lambda, \mathcal{H}_\lambda)$ is isomorphic to $(\pi_{-\lambda}, \mathcal{H}_{-\lambda})$, using the pairing

$$(\cdot,\cdot) : \mathcal{H}_{-\lambda} \times \mathcal{H}_\lambda \to \mathbb{C}, \quad (\xi, v) := \int_{M\backslash K} \xi(Mk)v(Mk)d(Mk).$$

This pairing is G-equivariant, so it yields the claimed isomorphism between the dual representation of π_λ and $\pi_{-\lambda}$.

Now, we look in more detail at the structure of V_λ. For an equivalence class $[\tau] \in \hat{K}$, we let (τ, U_τ) denote a representative. We write $\hat{K}_M$ for the M-spherical equivalence classes, i.e. those classes such that U_τ has non-zero elements in

$$U_\tau^M := \{u \in U_\tau \mid \tau(m)u = u \forall m \in M\}.$$

If we denote by $(U_\tau^*)^M$ the elements of U_τ^* that are M-fixed and we obtain the following map for every $[\tau] \in \hat{K}_M$:

$$r_\tau : U_\tau \otimes (U_\tau^*)^M \to L^2(M\backslash K), \quad u \otimes \eta \mapsto (Mk \mapsto \eta(\tau(k)u)).$$

In particular, the resulting map is smooth. If we then put a K-invariant inner product on U_τ, it induces a K-invariant inner product on the dual U_τ^*. This yields an inner product on $U_\tau \otimes (U_\tau^*)^M$, which is independent of the original chosen inner product. With this inner product, and noting that the resulting function is essentially a matrix coefficient, Schur-orthogonality tells us that

$$\frac{1}{d(\tau)}\|u \otimes \eta\|^2 = \|r_\tau(u \otimes \eta)\|^2_{L_2(M\backslash K)},$$

where $d(\tau)$ denotes the dimension of U_τ. This tells us, amongst other things, that r_τ is a (linear) isomorphism to its image. Similarly, using Schur-orthogonality, we see that for non-equivalent τ, τ' the outcomes of r_τ and r'_τ are orthogonal.

We view $U_\tau \otimes (U_\tau^*)^M$ as a K-module, under the representation $\tau \otimes 1$. Then the map r_τ intertwines the K-actions on $U_\tau \otimes (U_\tau^*)^M$ and $L^2(M\backslash K)$. Now, using the Peter-Weyl theorem, we have the following:

Lemma 76. *We define the map*

$$r : \bigoplus_{\tau \in \hat{K}_M} U_\tau \otimes (U_\tau^*)^M \to V_\lambda$$

that equals r_τ on the summand indexed by τ and is linear. This is an isomorphism of K-modules.

We note here that the τ-isotypical component $V_\lambda[\tau]$ is exactly the image of $U_\tau \otimes (U_\tau^*)^M$ under r_τ. We will often identify elements of V_λ with their pre-image under this isomorphism.

Now, we denote by δ_{Me} the point-evaluation of $C^\infty(M\backslash K)$ at the base point Me. If we restrict it to V_λ, we can decompose it into K-types:

$$\delta_{Me} = \sum_{\tau \in \hat{K}_M} \delta_\tau.$$

Here, each δ_τ is given by $\delta_{Me}|_{U_\tau \otimes (U_\tau^*)^M}$, and zero on the other isotypical components. We will show that δ_τ corresponds to a specific function on $C^\infty(M\backslash K)$, where we view a function $F \in C^\infty(M\backslash K)$ as a distribution via

$$F(f) := \int_K F(k^{-1})f(k)\mathrm{d}k, \quad f \in C^\infty(M\backslash K).$$

We now claim that δ_τ is the distribution belonging to the function

$$F_\tau = r_\tau(d(\tau)\sum_{i=1}^{l(\tau)} u_i \otimes u_i^*.$$

Here we pick a basis $u_1, \dots, u_{l(\tau)}$ of U_τ^M, and denote by $u_1^*, \dots, u_{l(\tau)}^*$ the corresponding dual basis of $(U_\tau^*)^M$. We note that $F_\tau(k^{-1}) = \overline{F_\tau(k)}$, so that applying the distribution belonging to F_τ to some function f is the same as taking the inner product between f and F_τ. So, if we take some $\tau' \neq \tau$ and apply F_τ to any function in the τ'-isotypical component, we get zero by Schur orthogonality. On the other hand, an arbitrary element of $V_\lambda[\tau]$ can be written as $r_\tau(v_j \otimes u_j^*)$, with u_j^* the dual basis from before and $v_j \in U_\tau$ arbitrary. Applying the distribution belonging to F_τ to this yields, by Schur orthogonality:

$$\begin{aligned}
F_\tau\left(r_\tau(\sum_j v_j \otimes u_j^*)\right) &= \langle \sum_i u_i \otimes u_i^*, \sum_j v_j \otimes u_j^* \rangle \\
&= \sum_j \langle u_j, v_j \rangle \\
&= \sum_j u_j^*(v_j) \\
&= r_\tau(\sum_j v_j \otimes u_j^*)(Me) \\
&= \delta_\tau \left(r_\tau(\sum_j v_j \otimes u_j^*)\right).
\end{aligned}$$

So we see that indeed the distribution corresponding to F_τ is exactly δ_τ. Now, if we write

$$F_\tau^{i,j} := u_i \otimes u_j^*,$$

we have that $F_\tau = d(\tau)\sum_{i=1}^{l(\tau)} F_\tau^{i,i}$. Furthermore, we have the following properties:

- $\|F_\tau^{i,i}\|_\infty = F_\tau^{i,i}(Me) = 1$,
- $F_\tau * F_\tau = F_\tau$,
- $F_\tau * f = f$ for all $f \in \operatorname{im} r_\tau$.

The first one follows from the inner product on U_τ being K-invariant, the second one is a special case of the third, which follows by using the K-invariance of the inner product to write the convolution as an inner product, and using Schur-orthogonality again.

Now, we shift our attention to a different way of viewing $\mathcal{H}_\lambda^\infty$. Using the restriction Res_K we could identify $\mathcal{H}_\lambda^\infty$ with $C^\infty(M\backslash K)$, but we can also restrict it to a different subgroup: if we denote by $\overline{N}$ the opposite of N (i.e. the subgroup generated by the negative roots), then $NAM\overline{N}$ is open and dense in G, so the restriction map to $\overline{N}$ is injective since we know how our functions transform under left multiplication by $P_{\min}$. This gives us the restriction mapping:

$$\mathrm{Res}_{\overline{N}} : \mathcal{H}_\lambda^\infty \to C^\infty(\overline{N}), \quad f \mapsto f|_{\overline{N}}.$$

Note that this map is not onto, since not every function on $\overline{N}$ transforms properly. The model of $\mathcal{H}_\lambda^\infty$ as $C^\infty(M\backslash K)$ is called the *compact model*, while its realization embedded in $C^\infty(\overline{N})$ is called the *non-compact model*. The transfer from the compact to the non-compact model is given as follows:

$$\begin{aligned}\mathrm{Res}_{\overline{N}} \circ \mathrm{Res}_K^{-1} :& C^\infty(M\backslash K) \to C^\infty(\overline{N}), \\ & f \mapsto F;\ F(\overline{n}) := \tilde{a}(\overline{n})^{\lambda+\rho} f(\tilde{k}(\overline{n})).\end{aligned}$$

We can also transfer the Hilbert space structure from $L^2(M\backslash K)$ to the non-compact model, which results in the L^2-space $L^2(\overline{N}, \tilde{a}(\overline{n})^{-2\mathrm{Re}\,\lambda} \mathrm{d}\overline{n})$, with $\mathrm{d}\overline{n}$ an appropriately normalized Haar measure on $\overline{N}$ (we will return to the normalization later). We will also denote this L^2-space by $\mathcal{H}_\lambda$ when the context makes it clear which of the two spaces is meant. The action of G in the non-compact model is not too relevant for what follows, but we will use the action of A:

$$[\pi_\lambda(a) f](\overline{n}) = a^{\lambda+\rho} f(a^{-1}\overline{n}a),$$

for all $a \in A$, and $f \in L^2(\overline{N}, \tilde{a}(\overline{n})^{-2\mathrm{Re}\,\lambda} \mathrm{d}\overline{n})$.

5.2 Construction and properties of f_σ

We saw before that Res_K is an isomorphism from $\mathcal{H}_\lambda^\infty$ to $C^\infty(M\backslash K$, since $P_{\min}\backslash G \simeq M\backslash K$. Now, we can map $\overline{N}$ into $P_{\min}\backslash G = M\backslash K$ as an open dense subset. It turns out that the complement is algebraic, and we will describe it as the zero set of a K-finite function f on $M\backslash K$. In particular, this f can be chosen so that when restricted to $\overline{N}$ it has polynomial decay of arbitrary fixed order.

Let (σ, W) be a finite-dimensional faithful irreducible representation of G, that is K-spherical, i.e. there is a non-zero K-fixed vector v_K. It is a known fact that since σ is K-spherical, there is a real line $L \subset W$ that is fixed under $\overline{P}_{\min} = MA\overline{N}$. Let v_0 be a nonzero vector on this line, and define $\mu \in \mathfrak{a}^*$ by $\sigma(a)v_0 = a^\mu v_0$ for all $a \in A$. In other words, v_0 is a lowest weight vector of σ and μ is the corresponding lowest weight.

We now take a θ-covariant inner product on W, which is unique up to scalar by Schur's Lemma. We fix this scalar by taking v_0 normalized, and we fix v_K such that $\langle v_0, v_K\rangle = 1$. Now, we define the function f_σ on G by

$$f_\sigma(g) := \langle \sigma(g)v_0, v_0\rangle.$$

The restriction of f_σ to K will also be denoted by f_σ. Now, if we take $\overline{n} \in \overline{N}$, and write $\overline{n} = \tilde{n}(\overline{n})\tilde{a}(\overline{n})\tilde{k}(\overline{n})$ according to the Iwasawa decomposition, we have $\tilde{k}(\overline{n}) = n'\tilde{a}(\overline{n})^{-1}\overline{n}$ for some $n' \in N$. Therefore, we have

$$f_\sigma(\tilde{k}(\overline{n})) = \tilde{a}(\overline{n})^{-\mu}.$$

One can show that f_σ exactly defines the complement $M\backslash K - \overline{N} := M\backslash K - \tilde{k}(\overline{N})$:

$$M\backslash K - \overline{N} = \{Mk \in M\backslash K \mid f_\sigma(k) = 0\}.$$

A property that is more interesting to us, is that the map $\overline{n} \mapsto f_\sigma(\overline{n})$ is the inverse of a polynomial mapping. Indeed, the map $\overline{n} \mapsto \tilde{a}(\overline{n})^\mu$ is polynomial, since it equals

$$\tilde{a}(\overline{n})^\mu = \langle \sigma(\overline{n})v_K, v_0\rangle,$$

which is polynomial by our normalizations.

It turns out that this function will play an important role, and to make some estimates later, we will introduce coordinates on $\overline{N}$. We pick a basis $X_1, \dots, X_n$ of $\overline{N}$ in such a way that every X_i is an $\mathfrak{a}$-root vector, of roots of increasing height. We write elements of $\overline{\mathfrak{n}}$ accordingly as $X = \sum_{j=1}^n x_j X_j$ for real numbers x_i. We then have the following two properties:

- The map
$$\Phi : \overline{\mathfrak{n}} \to \overline{N}, \quad X \mapsto \overline{n}(X) := \exp(x_1X_1)\cdots\exp(x_nX_n)$$
is a diffeomorphism.

- The Haar measure $\mathrm{d}\overline{n}$ on $\overline{N}$ can be normalized in such a way that $\Phi^*(\mathrm{d}\overline{n}) = \mathrm{d}x_1 \cdots \mathrm{d}x_n$.

We will assume that the measure is normalized in this way. We introduce a norm on $\overline{\mathfrak{n}}$ by

$$\|X\|^2 := \sum_{j=1}^n |x_j|^2.$$

Finally, we will also define f_σ on $\overline{\mathfrak{n}}$ via

$$f_\sigma(X) := f_\sigma(\tilde{k}(\overline{n}(X))) = \tilde{a}(\overline{n}(X))^{-\mu}.$$

Our results so far can be summarized as follows:

Lemma 77 (Lemma 12.1)**.** *Let $m > 0$. Then there exists $C > 0$ and a finite-dimensional K-spherical representation (σ, W) of G such that:*

(i) $M\backslash K - \overline{N} = \{Mk \in M\backslash K \mid f_\sigma(k) = 0\}$.

(ii) $|f_\sigma(X)| \leq C \cdot (1 + \|X\|)^{-m}$ *for all* $X \in \overline{\mathfrak{n}}$.

The first one is just the first part of our discussion, while the second part is the translation of f_σ being the inverse of a polynomial.

Instead of applying the function f_σ to elements of $\overline{N}$ directly, we can also use the transfer between the compact and the non-compact model. We then get a corresponding function $\xi = \xi_\sigma$ on $\overline{N} \simeq \overline{\mathfrak{n}}$, given by

$$\xi(X) := \tilde{a}(\overline{n}(X))^{\rho+\lambda} f_\sigma(\tilde{k}(\overline{n}(X))) = \tilde{a}(\overline{n}(X))^{\rho+\lambda-\mu}.$$

Our original function f_σ was K-finite, since W is finite-dimensional, so the corresponding function ξ is K-finite as well, in the non-compact model. It will turn out that this ξ, for sufficiently well-behaved σ, will be a cyclic vector for V_λ, and that we can prove the theorem for this generator.

We will again have a similar inequality to before,

$$|\xi(X)| \leq C \cdot (1 + \|X\|)^{-m},$$

where we can choose m as large as we wish as long as σ is sufficiently regular and large. We will take m at least large enough that ξ becomes integrable, and write $\|\xi\|_1$ for the corresponding L^1-norm.

We looked at the action of A on these functions before, and we will now look at a specific case of this: we fix an element $Y \in \mathfrak{a}$ such that $\alpha(Y) \geq 1$ for all the positive roots α, i.e. those generating $\mathfrak{n}$. For $t > 0$ we write

$$a_t := \exp((\log t)Y).$$

This means that for any $\eta \in \mathfrak{a}_\mathbb{C}^*$ we have

$$a_t^\eta = t^{\eta(Y)}.$$

Now, to explain the ideas of the rest of the proof, we assume for now that λ is real. Then ξ is a positive function, and we can look at the following functions:

$$\xi_t := \frac{a_t^{\rho-\lambda}}{\|\xi\|_1} \cdot \pi_\lambda(a_t)\xi. \quad (t > 0)$$

If we let t go to infinity, it turns out that these functions converge to the Dirac distribution. Indeed, for every $t > 0$ the functions integrate to 1, and if we look at the value at some $\overline{n}$, we get

$$\xi_t(\overline{n}) = \frac{a_t^{2\rho}}{\|\xi\|_1} \xi(a_t^{-1}\overline{n}a_t) = \frac{t^{2\rho(Y)}}{\|\xi\|_1} \xi(a_t^{-1}\overline{n}a_t),$$

which goes to zero outside of $\overline{n} = e$ because of the arbitrary polynomial decrease of ξ, and at $\overline{n} = e$ it goes to infinity.

If we transfer this back to the compact picture, we get that

$$\lim_{t\to\infty} \frac{a_t^{\rho-\lambda}}{\|\xi\|_1} \cdot \pi_\lambda(a_t) f_\sigma = \delta_{Me},$$

where the convergence is in terms of distributions. We decomposed δ_{Me} into K-types before as $\delta_{Me} = \sum \delta_\tau$, which corresponded to the functions F_τ. Now, we take the τ-isotypical part of the functions, and see how well they approximate the functions F_τ already. It turns out that we can choose our t polynomially in τ to get a reasonably well approximation, even if λ is not real, which yields the following result:

Theorem 78 (Theorem 12.2)**.** *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $N > 0$. Then there exists a choice of σ and hence of $\xi = \xi_\sigma \in V_\lambda$, and constants $c > 2, C > 0$ such that for all $\tau \in \hat{K}_M$ we have*

$$[\pi_\lambda(a_{t(\tau)})\xi]_\tau = a_{t(\tau)}^{\lambda-\rho} \cdot I_\xi \cdot D_\tau + R_\tau,$$

where D_τ is the transfer of the function F_τ that belonged to δ_τ, to the non-compact model, $t(\tau) := (1 + \|\tau\|)^c$,

$$I_\xi := \int_{\overline{N}} \xi(\overline{n}) d\overline{n} \neq 0$$

and the remainder $R_\tau \in \mathcal{H}_\lambda[\tau]$ satisfies

$$\frac{\|R(\tau)\|}{|a_{t(\tau)}^{\lambda-\rho}|} \leq \frac{C}{(1 + \|\tau\|)^N}.$$

Proof. The full proof can be found in [4], p.102 and on, we will only treat the main ideas here. Since the functions we are working with are M-fixed, we can decompose them into the $F_\tau^{i,j}$ from before: if we denote by $D_\tau^{i,j}$ their transfer to the non-compact model, we can write

$$[\pi_\lambda(a_t)\xi]_\tau = \sum_{i,j=1}^{l} b_{i,j}(t) \cdot d(\tau) \cdot D_\tau^{i,j}.$$

The coefficients $b_{i,j}(t)$ can be obtained via the inner product on $L^2(\overline{N}, \tilde{a}(\overline{n})^{-2\mathrm{Re}\,\lambda} \mathrm{d}\overline{n})$ as follows:

$$b_{i,j}(t) = \langle \pi_\lambda(a_t)\xi, D_\tau^{i,j} \rangle = \int_{\overline{\mathfrak{n}}} (\pi_\lambda(a_t)\xi)(X) \cdot \overline{D_\tau^{i,j}(X)} \tilde{a}(\overline{n}(X))^{-2\mathrm{Re}\,\lambda} \mathrm{d}X,$$

where $\mathrm{d}X = \mathrm{d}x_1 \cdots \mathrm{d}x_n$ in our coordinates from before. Now, we can split the integral into a part around 0, and a part away from 0. By estimating and calculating the integrals, we get that the integral away from 0 decays quickly enough to be put into the remainder term, and the part around 0 yields the D_τ-term, and another remainder that we can put into the R_τ. □

It follows from the more detailed proof that we can make the approximation uniformly over any compact subset Q of $\mathfrak{a}_{\mathbb{C}}^*$. Now, $a_{t(\tau)}$ is bounded from above and below by powers of $1 + \|\tau\|$, so if we switch to the compact model again, and start denoting f_σ by ξ as well to match the notation of Theorem 46, we get the following version of the theorem:

Theorem 79 (Theorem 12.3)**.** *Let $Q \subset \mathfrak{a}_{\mathbb{C}}^*$ be compact, and $N > 0$. Then there exists $\xi \in V_\lambda$ (realized as functions on $M \backslash K$) and constants $c_1, c_2 > 0$ such that for all $\tau \in \hat{K}_M, \lambda \in Q$ there exists $a_\tau \in A$, independent of λ, with $\|a_\tau\| \leq (1 + \|\tau\|)^{c_1}$, and numbers $b(\lambda, \tau) \in \mathbb{C}$ such that*

$$\|[\pi_\lambda(a_\tau)\xi]_\tau - b(\lambda,\tau)F_\tau\| \leq \frac{1}{(\|\tau\| + 1)^{N+c_2}}$$

and

$$|b(\lambda,\tau)| \geq \frac{1}{(1+\|\tau\|)^{c_2}}.$$

Here the $\|\cdot\|$ refers to the L^2-norm.

Now, with this theorem in hand, we can finally prove Theorem 46 for this specific case, which by the previous section proves it in general.

5.3 Proving the theorem

In the article, Bernstein and Krötz prove something slightly different from what we need, but if we reduce their proof to the case from Theorem 46, the result will follow. Using our previous approximations of $\pi_\lambda(a_\tau)$, we can find the following:

Lemma 80 (Lemma 12.6)**.** *Let U be an $Ad(K)$-invariant neighborhood of e in G, and $\mathcal{F}(U)$ the space of $Ad(K)$-invariant test functions supported in U. Then there exists a holomorphic map*

$$Q \to \mathcal{F}(U), \quad \lambda \mapsto h_\lambda$$

such that $\Pi_\lambda(h_\lambda)\xi = \xi$.

Proof. We will not treat the proof here, as it involves techniques we do not need elsewhere. The proof can be found on p.106 of [4]. $\square$

Now, we define $h_{\lambda,\tau}$ by

$$h_{\lambda,\tau} := \delta_\tau * \delta_{a_{t(\tau)}} * h_\lambda,$$

where $\delta_{a_{t(\tau)}}$ denotes the Dirac delta-distribution at $a_{t(\tau)}$. These functions will still be compactly supported, since h_λ was compactly supported, convolution with a delta-distribution just shifts it, and δ_τ was compactly supported as well. In particular, since $a_{t(\tau)}$ was polynomially bounded, the support is polynomially bounded. The article now goes on to use these functions to construct a function $f : Q \times C^\infty(M\backslash K) \to \mathcal{S}(G)$ such that $\Pi_\lambda(f(\lambda, v))\xi = v$, with the function being holomorphic in the first variable and linear in the second. However, this is not exactly the result we need, but in the construction of the function, isotypical components are treated separately, and we can show the following result:

Theorem 81. *Let V_λ be the spherical principal series representation of G with parameter λ. Then there exists an element $\xi \in V_\lambda$, and constants $c_1, c_2, C_1, C_2 > 0$ such that for all $\tau \in \hat{K}_M$ and $v_\tau \in V[\tau]$ there exists a function f_τ with the following properties:*

(a) $\Pi_\lambda(f_\tau)\xi = v_\tau$.

(b) $\operatorname{supp}(f_\tau) \subset \{g \in G \mid \|g\| < C_1(1 + \|\tau\|)^{c_1}\}$.

(c) $\|f_\tau\|_1 \leq C_2 \cdot \|v_\tau\| \cdot (1 + \|\tau\|)^{c_2}$.

Proof. By our earlier discussion about convolutions with the F_τ that belonged to δ_τ, it follows that to create any function f in $V[\tau]$ it suffices to first create F_τ, and by convolving with the distribution belonging to f we then get the required result. This will only modify the support by a constant, and the estimate on the L^1-norm is handled by the $\|v_\tau\|$-factor. So, we assume now that $v_\tau = F_\tau$.

We recall the number $b(\lambda, \tau)$ from Theorem 79 and set

$$f'_\tau := \frac{h_{\lambda,\tau}}{b(\lambda, \tau)}.$$

We then have that

$$\begin{aligned} \Pi_\lambda(f'_\tau)\xi &= \frac{1}{b(\lambda, \tau)} \Pi_\lambda(\delta_\tau * \delta_{a_{t(\tau)}} * h_\lambda)\xi \\ &= \frac{1}{b(\lambda, \tau)} [\pi_\lambda(a(t(\tau))\xi]_\tau. \end{aligned}$$

By Theorem 79, this is close to F_τ, and we can write it as

$$\Pi_\lambda(f'_\tau)\xi = F_\tau + R_\tau$$

for some remainder term R_τ that has norm smaller than $\frac{1}{(1+\|\tau\|)^N}$. Now, we can get rid of this remainder term by adding in an extra convolution with the Neumann series of $(\mathrm{id} + R_\tau)^{-1}$. (Here we use that the convolution of F_τ with R_τ is again R_τ.) Now for the resulting function f_τ the required bounds hold, since the support of h_λ was compact, and it only gets modified by convolving with distributions with supports that satisfy similar bounds (δ_τ and the distribution at the end have supports in K, and $\delta_{a_{t(\tau)}}$ satisfies a polynomial bound), and the L^1-norm similarly does not grow too much, by the bound on $\|R_\tau\|$, and since δ_τ has a bounded L^1-norm. □

This is exactly what we wanted for the case of a spherical principal series, and by our reduction in the previous chapter, it now follows that the main theorem holds for any $P_{\min}$-module W.

5.4 Comparing with the article

In this section, we treat Appendix A (Section 12) from the article by Krötz and Bernstein, [4], minus the introduction. For the first part of the appendix, we follow their treatment, adding more explanations at some points, and leaving out certain details that are less relevant for us. We deviate rather strongly in our Theorem 81, which is based on the proof of Lemma 12.7 and Theorem 12.8 in the article. Here they prove a more general result than is needed for the proof of Theorem 46, which is generalized even more in Section 11, but on a first reading it is not immediately clear how this result implies the needed Theorem 81. Therefore, we chose to not treat the more general result, and instead focus only on the specific part we need, and how that follows from similar arguments.

6 Discussion

Almost everything we need for the proof was either proven in the original article ([4]), or the details have been worked out in this thesis. Unfortunately, the proof of Lemma 64 is an exception to this. In the original article, the lemma was not stated in this exact way, but it or a similar result was implicitly used in the reduction steps in Appendix A. In order to present the arguments as written in the article, we therefore chose to assume this result holds, and proceed with the rest of the argument. We recall the importance of the lemma here briefly: we want to show that minimal principal series representations are of $\mathcal{D}$-type, in other to prove that they are good. From the fact that they are good, we can then prove that being good is preserved under induction, and that Harish-Chandra modules of the discrete series are good. Both of those results are needed in order to prove the general Casselman-Wallach globalization theorem. In order to prove that minimal principal series representations are of $\mathcal{D}$-type, we show that they can be embedded into $I(\mathbb{C}_\chi) \otimes F$ for some finite-dimensional G-representation. This larger module is shown to be of $\mathcal{D}$-type, and we need Lemma 64 to then conclude that minimal principal series representations are of $\mathcal{D}$-type.

Because of time restrictions, we have not been able to continue work on proving this lemma, but some first steps have been made, and some ideas have been tried, which we will explain here.

We can freely choose the generators we work with to prove that U is of $\mathcal{D}$-type, and we know that the necessary properties hold for V with respect to any set of generators. So, if we take a set of generators of U, we can always extend it to a set of generators of V, in terms of which every element of an isotypical component $V[\tau]$ can be decomposed. Since $U[\tau] \subset V[\tau]$, this gives us a decomposition of every vector in $U[\tau]$ in terms of the generators of U, and some extra in $V \backslash U$. The problem that is left to solve is to eliminate the need for these other generators.

By using the finite composition series of Harish-Chandra modules, we can reduce the general lemma to the specific case of either assuming that U is irreducible, or that the quotient V/U is irreducible, but not both at the same time. In the first case, U would only have one generator, while V has an arbitrary number, and in the second case U has an arbitrary number of generators, but adding any element of $V \backslash U$ yields a set of generators of V. So in this second case, we can decompose any vector in $U[\tau]$ in terms of the chosen set of generators of U and exactly one other arbitrary vector, but the bounds on the functions do depend on this extra generator. If we write $\{\xi_1, \dots, \xi_k\}$ for the generators of U, and ξ for the extra generator in $V \backslash U$, we have the decomposition

$$u_\tau = \sum_{j=1}^{k} \Pi(f_{\tau,j})\xi_j + \Pi(f)\xi,$$

and since $u_\tau \in U$ and all the terms in the sum are in the globalization of U (within the larger globalization of V), it follows that $\Pi(f)\xi$ also has to be in this globalization of U. Based on this, an approach to the problem that could be explored further would be to look at particular choices of ξ that reach as little as possible of this globalizaton of U, to limit the influence this last term can have on the total sum. For instance, if a finite number of ξs could be chosen so that for each isotypical component at least one of them could not reach the component, then via projection to the isotypical component, one could find a decomposition of u_τ in only the ξ_j, and since there would only be a finite amount of different ξs used, the constants could be adjusted to account for this.

A different approach could be to make the requirements of the lemma more strict. We only need the lemma for the specific case of a module being embedded into $I(\mathbb{C}_\chi)\otimes F$, so one could consider using the properties of it being a tensor product, or even the properties of $I(\mathbb{C}_\chi)$. We have not yet explored the implications of assuming these stronger requirements, but it could lead to a method of proving a weaker version of the lemma that is still sufficient for the arguments we need. However, in this case one would have to be more careful in using reduction steps like the above, since then the modules that are worked with will probably change in these steps.

One final approach that could be taken to at least prove the main theorem, would be to look at how we use that minimal principal series are of $\mathcal{D}$-type, namely we only use it in the dual representation. In the proof of Theorem 47, we decompose the constructed ξ in terms of the generators ξ_j of $\widetilde{I(W)} = I(W^*)$. If we would only be able to use that $I(W^*)$ is embedded into some module of $\mathcal{D}$-type, we would be able to decompose ξ into a larger set of generators. If one studies what this decomposition tells us about $q(v) = \xi(v)$, it might be possible to still prove the main result without having to use Lemma 64. However, in doing so some of the side results of the article would no longer necessarily hold, such as Theorem 8.1(ii) which used the fact that minimal principal series representations are of $\mathcal{D}$-type to find a continuous linear section from V^∞ to $\mathcal{S}(G)^k$. Therefore, it would be preferable to prove the lemma or a similar version of it, instead of trying to bypass it completely.

7 Conclusion

To answer the main question of how to prove the Casselman-Wallach globalization theorem, we summarize the proof. To prove the uniqueness of globalizations, we define the minimal and maximal globalizations. (Definition 36 and Definiton 39) These are connected to maximal and minimal G-continuous norms respectively, with respect to the Sobolev-ordering. Therefore, proving the theorem reduces to proving that all G-continuous norms on any Harish-Chandra module are equivalent. This can be proved by finding lower bounds on matrix coefficients of the action of G on the minimal globalization, in terms of any G-continuous norm (Theorem 44/Theorem 45). We then use the fact that minimal principal series representations are of $\mathcal{D}$-type, which we try to prove in Sections 4 and 5, and show these lower bounds, so that all minimal principal series representations are good (Theorem 47). Having proven the globalization theorem for this set of Harish-Chandra modules, we then look at results to extend the statement to other modules as well, in particular extension, induction and holomorphic deformation (Lemma 48, Theorem 53 and Theorem 54), and study tensoring with finite-dimensional G-representations as well. Using these results, the general case of any Harish-Chandra module is reduced to the case of a square-integrable module. This case is then proven by embedding it into a minimal principal series representation (Theorem 57), which proves the general theorem.

For most of the thesis, we follow the article, but there are a few specific cases where we deviate to a more significant degree than just fixing typos. In the statements of Theorem 7.1 and 7.2 from the article, we have added an extra constant that seemed to be missing. We have elaborated a lot more on the proof of Theorem 8.1(iii), that was handled in one remark in the article, and in our case is handled in Theorem 47. In proving Theorem 8.1(i), that minimal principal series representations are of $\mathcal{D}$-type, we have gone into more detail on the reduction steps. In particular, we have added the definition of being of $\mathcal{D}$-type, and recorded some results in lemmas and theorems instead of handling it in a few lines, which was done in the article. Unfortunately, one of these lemmas has not been proven yet, which leaves a small gap in the proof of the main theorem. Finally, in our treatment of proving that minimal principal series representations are of $\mathcal{D}$-type, we focus more on the exact result that is needed, instead of proving a related result that is more tailored towards a side-result from the article.

References

[1] E. van den Ban. Harmonic Analysis, Lecture Notes, Spring 2015.

[2] E. van den Ban and H. Schlichtkrull. Asymptotic expansions and boundary values of eigenfunctions on Riemannian symmetric spaces. *J. reine angew. Math*, 380:108–165, 1987.

[3] J. Bernstein. On Casselman-Wallach globalization theorem. Representation Theory and Lie Theory 2009, https://www.newton.ac.uk/seminar/20090623153016301, retrieved July 4th 2018.

[4] J. Bernstein and B. Krötz. Smooth Fréchet globalizations of Harish-Chandra modules. *Israel J. Math*, 199(1):45–111, 2014.

[5] W. Casselman. Canonical extensions of Harish-Chandra modules to representations of G. *Canad. J. Math*, 41(3):385–438, 1989.

[6] L. Gårding. Vecteurs analytiques dans les représentations des groupes de Lie. *Bull. Soc. Math. France*, 88(1):73–93, 1960.

[7] S. Helgason. Some results on eigenfunctions on symmetric spaces and eigenspace representations. *Mathematica Scandinavica*, 41(1):79–89, 1978.

[8] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima, and M. Tanaka. Eigenfunctions of invariant differential operators on a symmetric space. *Annals of Mathematics*, 107(1):1–39, 1978.

[9] A.W. Knapp. *Representation theory of semisimple groups. An overview based on examples.* Reprint of the 1986 original. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001.

[10] A.W. Knapp. *Lie groups beyond an introduction*, volume 140. Springer Science & Business Media, 2013.

[11] N. Wallach. Asymptotic expansions of generalized matrix entries of representations of real reductive groups. In *Lie group representations I*, pages 287–369. Springer, 1983.

[12] N.R. Wallach. *Real Reductive Groups I*, volume 132. Academic press, 1988.