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Master's thesis

Realizability with Scott's Graph Model

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Chapter 1

Introduction

In computability theory one studies computable functions. Intuitively, a function on the natural numbers is computable if it can be calculated using an algorithm. Since algorithms need not terminate on every input, it is natural (and necessary) to consider partial functions. Abstractly speaking, an algorithm is simply a finite set of data, and as such it can be encoded by a natural number. This allows us to think of applying the number m to the number n as calculating the output of the algorithm encoded by m on input n. This then blurs the distinction between the functions and their input. A fundamental result in computability theory is that there are encodings that are computable themselves. This means that there is a universal algorithm, an algorithm u such that u applied to input $2^n 3^m$ yields the output of the algorithm with code n on input m. We call u universal, because it can simulate any other algorithm for us (provided that we know its encoding).

One can generalize this as follows: consider a structure $\mathcal{A}$ consisting of a non-empty set $\mathbb{A}$ of 'programs' and a binary partial map (the application map). We impose some properties on $\mathcal{A}$ to ensure that the application map is represented (in some sense) by a program in $\mathbb{A}$. Such a structure is known as a *partial combinatory algebra* (*pca*) and can be thought of as a model of computation.

A partial combinatory algebra gives rise to an interesting and rich category: the category of *assemblies*. Intuitively, assemblies are sets with some computational content. Arrows in this category are simply functions that are 'computable' in the partial combinatory algebra. Although there is much structure on the category of assemblies, it falls short of being a topos. This, for example, means that we cannot interpret higher-order logic in it.

Fortunately, this situation can be remedied: it is possible to construct a topos over a pca $\mathcal{A}$, known as the *realizability topos of* $\mathcal{A}$ and denoted by $\mathsf{RT}(\mathcal{A})$, such that it contains the category of assemblies as a full subcategory. One may view $\mathsf{RT}(\mathcal{A})$ as a mathematical universe with some built-in notion of computation.

The name "realizability topos" warrants some explanation. *Realizability* was invented by Kleene [Kle45] to study constructive mathematics. It is a technique used to endow constructive proofs with computational content. For example, a proof of $\forall x \exists y \varphi(x, y)$

should be some computable function f that, given an x, produces a witness y and a proof that that witness is correct, viz. $f(x)$ should be some pair $(f_0(x), f_1(x))$ such that $f_1(x)$ proves (or realizes) $\varphi(x, f_0(x))$.

The first and most well-known example of a realizability topos is Hyland's [Hyl82] *Effective Topos* (denoted here by Eff). One of the remarkable aspects of Eff is that its internal logic is governed by Kleene-realizability. This means that for a statement φ about the natural numbers, φ is true in Eff if and only if φ is Kleene-realizable, i.e. there is some computable function that proves it as described above. Hyland's construction thus connected topos theory and realizability and was generalized via *tripos theory* [HJP80].

The Effective Topos arises as the realizability topos of *Kleene's first model*: the pca with the natural numbers as its underlying set and partial recursive application. Another example of a pca is *Scott's graph model* [Sco76], denoted here by $\mathcal{S}$. The realizability topos $\mathsf{RT}(\mathcal{S})$ of this pca is the object of study in this master's thesis.

1.1 Overview

In this section I give a brief description of each chapter.

Chapter 2 develops the theory of partial combinatory algebras and provides examples of pcas. We study Scott's graph model in particular.

We continue in Chapter 3 by studying assemblies and in Chapter 4 we develop some tripos theory and are finally able to construct and describe realizability toposes.

Chapter 5 identifies a particular subcategory of the realizability topos $\mathsf{RT}(\mathcal{S})$ of Scott's graph model: the subcategory of *order-discrete* objects. These objects will reappear in Chapter 6 where we investigate first and second order arithmetic in realizability toposes and $\mathsf{RT}(\mathcal{S})$ in particular.

In Chapter 7 we take our first steps in synthetic domain theory and we examine a particular *dominance* on $\mathsf{RT}(\mathcal{S})$. We also look at *Lambek algebras* for the *lift functor*. The dominance gives rise to a *model structure* on $\mathsf{RT}(\mathcal{S})$, which we describe in Chapter 8, together with the basic theory of model structures.

Finally, Chapter 9 lists some questions for future research.

1.2 Preliminaries

Familiarity with partial combinatory algebras, assemblies or realizability (toposes) is not required, as I treat all of this in detail in the coming three chapters. This thesis should be readable by anyone with knowledge of category theory: in particular, adjunctions, cartesian closed categories, elementary toposes and categorical logic. Familiarity with basic computability theory is useful, but not strictly necessary (for the bulk of this thesis). For example, we mention partial recursive functions in Example 2.3.2 and primitive recursive functions in the chapter on arithmetic. Moreover, one would certainly benefit from having studied intuitionistic logic. Lastly, I use the axiom of choice freely and (often) without mention.

Chapter 2

Partial Combinatory Algebras

This chapter introduces partial combinatory algebras. These structures give rise to interesting categories which we will study later: categories of assemblies and realizability toposes. Although we will mostly be interested in one particular partial combinatory algebra, I have decided to treat most of the material in full generality. The reason is twofold. Firstly, it allows us to compare our topos to other realizability toposes. Secondly, many results can be obtained for the general setting without much additional effort.

This chapter contains no original results. We follow Chapter 1 of Van Oosten's comprehensive book *Realizability* [Oos08]. I have also consulted [Zoe18, Chapter 1] for comparison.

2.1 Basic definitions

Definition 2.1.1. A *partial applicative structure* (*pas*) $\mathcal{A}$ is a non-empty set $\mathbb{A}$ with a partial map $\mathbb{A} \times \mathbb{A} \rightharpoonup \mathbb{A}$ called *application*. It is denoted by juxtaposition: $(a, b) \mapsto ab$. Our convention is that application associates to the left, i.e. we write abc for $(ab)c$.

In this section $\mathcal{A}$ will always denote a pas with $\mathbb{A}$ as its underlying set.

Definition 2.1.2. Fix a countably infinite set of *variables* V. The set of *terms over* $\mathcal{A}$ is the least set $T(\mathcal{A})$ such that:

(i) every variable is a term over $\mathcal{A}$;

(ii) for each $a \in \mathbb{A}$, we have a constant (also denoted by a) that is also a term over $\mathcal{A}$;

(iii) if s and t are terms over $\mathcal{A}$, then so is (st).

A term with no variables is called *closed*. We adopt the same convention concerning parentheses as above. Furthermore, when the context is clear, we will simply speak of terms.

If we read the juxtaposition of terms as application, then we might view a term t with variables $x_1, \ldots, x_n$ as a partial function $\mathbb{A}^n \rightharpoonup \mathbb{A}$. Accordingly, closed terms should be viewed as elements of $\mathbb{A}$. A partial combinatory algebra will be a pas that is a 'model of computation'. Thus, we would like it to able to represent these partial functions inside our pas itself. Providing substance to this idea is the motivation for the following definitions and results.

Definition 2.1.3. We define the relation $t{\downarrow}a$ (read as *t denotes a*) between closed terms and elements of $\mathbb{A}$ as the least relation satisfying:

(i) $a{\downarrow}a$ for any $a \in \mathbb{A}$;

(ii) $(st){\downarrow}a$ if and only if there are $b, c \in \mathbb{A}$ with $s{\downarrow}b$, $t{\downarrow}c$, bc is defined and $bc = a$.

We will write $t{\downarrow}$ (read as *t denotes*) if there is an $a \in A$ such that $t{\downarrow}a$.

One easily shows that if t denotes both a and b, then a must be equal to b. Thus, notationally, we will not distinguish between a closed term that denotes and the element that it denotes. For example, $ab{\downarrow}$ if and only if (a, b) is in the domain of the application map. Finally, observe that if a term denotes, then all of its subterms must also denote.

Next, we define substitution and equality on closed terms.

Definition 2.1.4. For two closed terms s and t, we write

(i) $s = t$ if and only if t and s both denote the same element of $\mathbb{A}$;

(ii) $s \simeq t$ for the *Kleene equality*, viz. either s and t do not denote, or s and t both denote the same element of $\mathbb{A}$.

Definition 2.1.5. For a term s and a term t with variable x, we will write $t[s/x]$ for the result of *substituting* s for x in t. Moreover, if we display all variables of a term $t(x_1, \ldots, x_n)$ then we will write $t(a_1, \ldots, a_n)$ for the result of substituting each a_i for x_i, where $a_i \in \mathbb{A}$.

We are now in position to define partial combinatory algebras.

Definition 2.1.6. We say that $\mathcal{A}$ is *combinatorially complete* if, for any integer $n \in \mathbb{N}$ and term $t(x_1, \ldots, x_{n+1})$, there exists an element $a \in \mathbb{A}$ such that for all $a_1, \ldots, a_{n+1} \in \mathbb{A}$, we have:

(i) $aa_1 \cdots a_n{\downarrow}$;

(ii) $aa_1 \cdots a_{n+1} \simeq t(a_1, \ldots, a_{n+1})$.

A *partial combinatory algebra* (*pca*) is a combinatorially complete pas.

Thus, we may view a pca as a pas which for each term has an element that 'computes' this term. An inconvenience of this definition is that using it to check that a pas is actually a pca can be quite difficult. Fortunately, there is an easier characterization, which is due to Feferman.

Theorem 2.1.7 (Feferman). *Let $\mathcal{A}$ be a pas. Then $\mathcal{A}$ is a pca if and only if there exist elements $\mathsf{k}, \mathsf{s} \in \mathbb{A}$ such that for any $a, b, c \in \mathbb{A}$:*

(i) $\mathsf{k}ab = a$;

(ii) $\mathsf{s}ab\downarrow$;

(iii) $\mathsf{s}abc \simeq ac(bc)$.

Proof. Suppose first that $\mathcal{A}$ is a pca. Consider the term $t(x, y) = x$. Combinatorial completeness for this term immediately provides an element of $\mathbb{A}$ that satisfies (i). For (ii) and (iii), apply combinatorial completeness to the term $t(x, y, z) = xz(yz)$.

To prove the converse, assume we have elements k, s satisfying (i) – (iii). We first develop some convenient notation. For any variable x and term t, define a term $\lambda^*x.t$ by recursion on t:

$\lambda^*x.x$ is the term skk;

$\lambda^*x.t$ is the term $\mathsf{k}t$ if t is a constant from $\mathbb{A}$ or any variable different from x;

$\lambda^*x.(t_1 t_2)$ is the term $\mathsf{s}(\lambda^*x.t_1)(\lambda^*x.t_2)$.

It may be proven by induction on terms that for any term $t(x, x_1, \ldots, x_n)$ and any $a, a_1, \ldots, a_n \in \mathbb{A}$, the following hold:

the variables of $\lambda^*x.t$ are exactly those of t minus x;

$(\lambda^*x.t)(a_1, \ldots, a_n)\downarrow$;

$(\lambda^*x.t)(a_1, \ldots, a_n)a \simeq t(a, a_1, \ldots, a_n)$.

For example, $\lambda^*x.x$ denotes, because skk always denotes by (ii) in the definition of combinatorial completeness. Furthermore, $(\lambda^*x.x)a \simeq \mathsf{skk}a \simeq \mathsf{k}a(\mathsf{k}a) = a$.

We are now ready to prove combinatorial completeness of $\mathcal{A}$. Let $t(x_1, \ldots, x_{n+1})$ be any term and let us write $\lambda^*x_1, \ldots x_{n+1}.t$ for $\lambda^*x_1.(\lambda^*x_2.(\ldots(\lambda^*x_{n+1}.t)\ldots))$. Then $\lambda^*x_1 \cdots x_{n+1}.t$ denotes and it functions as the required element in the definition of combinatorial completeness. For, if $a_1, \ldots, a_{n+1}$ are elements of $\mathbb{A}$, then

$$(\lambda^*x_1 \cdots x_{n+1}.t)a_1 \cdots a_n \simeq (\lambda^*x_{n+1}.t)(a_1, \ldots, a_n),$$

which denotes by above and

$$(\lambda^*x_1 \cdots x_{n+1}.t)a_1 \cdots a_{n+1} \simeq (\lambda^*x_{n+1}.t(a_1, \ldots, a_n))a_{n+1} \simeq t(a_1, \ldots, a_{n+1}),$$

as desired. ∎

Remark 2.1.8. The elements k and s are called *combinators*[1]. It may be the case that multiple elements of $\mathbb{A}$ satisfy the requirements of the k and s combinators. From now on, we will assume that we have made a choice for k and s for any pca. This allows us to freely employ the notation $\lambda^* x_1 \cdots x_{n+1}.t$ when working with pcas.

Remark 2.1.9. The notation λ^* is suggestive, as we have something that resembles λ-abstraction. However, some care is required. For example, one might expect $(\lambda^* x.t_1)t_2 \simeq t_1[t_2/x]$ to hold, but in general it does not. If we take t_1 to be a constant $b \in \mathbb{A}$, then $(\lambda^* x.t_1)t_2 \simeq \mathsf{k}bt_2$, while $t_1[t_2/x] = b$ and $\mathsf{k}bt_2 \simeq b$ does not hold if t_2 does not denote.

2.2 Basic combinators

To back up our claim that pcas serve as models of computation, we show in this section that we can define basic programming constructions in pcas.

We have already seen the combinators k and s. We also saw that $\mathsf{skk}a = a$ for any $a \in \mathbb{A}$, so have an *identity combinator* $\mathsf{i} = \mathsf{skk}$. Now define $\overline{\mathsf{k}}$ as ki. Observe that for any $a, b \in \mathbb{A}$, we have $\overline{\mathsf{k}}ab \simeq \mathsf{ki}ab \simeq \mathsf{i}b = b$, so $\overline{\mathsf{k}}$ works like k, but it outputs the second element instead of the first.

If we interpret k as 'true' and $\overline{\mathsf{k}}$ as 'false', then the i combinator functions as an if-else-operator ('if true, then a, else b'):

$$\mathsf{ik}ab = \mathsf{k}ab = a, \quad \mathsf{i}\overline{\mathsf{k}}ab = \overline{\mathsf{k}}ab = b.$$

One may extend this to closed terms: given closed terms s and t, define the closed term

$$r = \lambda^* x.x(\lambda^* y.s)(\lambda^* y.t)\mathsf{k}$$

and observe that

$$r\mathsf{k} \simeq \mathsf{k}(\lambda^* y.s)(\lambda^* y.t)\mathsf{k} \simeq (\lambda^* y.s)\mathsf{k} \simeq s \quad \text{and} \quad r\overline{\mathsf{k}} \simeq \overline{\mathsf{k}}(\lambda^* y.s)(\lambda^* y.t)\mathsf{k} \simeq (\lambda^* y.t)\mathsf{k} \simeq t.$$

Remark 2.2.1. A note on a subtlety of this case distinction operator is in order. It seems more natural to take the simpler $\lambda^* x.xst$. However, this does not work. For suppose s denotes, but t does not. Then, $(\lambda^* x.xst)\mathsf{k} \simeq \mathsf{k}st$ does not denote while s does. Consequently, $(\lambda^* x.xst)\mathsf{k} \not\simeq s$. But a term of the form $\lambda^* y.t$ always denotes by construction, explaining the need for its appearance above.

We also have pairing in our pca. Define the closed term $\mathsf{p} = \lambda^* xyz.zxy$ and observe that $\mathsf{p}ab \simeq \lambda^* z.zab$ always denotes. Let $\mathsf{p}_0 = \lambda^* w.w\mathsf{k}$ and $\mathsf{p}_1 = \lambda^* w.w\overline{\mathsf{k}}$ and note that:

$$\mathsf{p}_0(\mathsf{p}ab) \simeq \mathsf{p}ab\mathsf{k} \simeq (\lambda^* z.zab)\mathsf{k} \simeq \mathsf{k}ab = a \text{ and similarly, } \mathsf{p}_1(\mathsf{p}ab) = b.$$

Thus, we think of $\mathsf{p}ab$ as the (coded) pair (a, b) and p_0 and p_1 as the projections. We call p the *pairing combinator* and p_0 and p_1 the *projection combinators*.

[1] The letters "k" and "s" come from Moses Schönfinkel's combinatory logic. They respectively come from the German words "Konstanzfunktion" (constant function) and "Verschmelzungfunktion" (merge function). Of course, "Verschmelzungsfunktion" starts with a "v", but Schönfinkel had to avoid confusion as there was also a swap-arguments combinator called the "Vertauschungsfunktion" [Sch24].

Remark 2.2.2. Before we go on, we would like to exclude some trivialities. So, from now on, $\mathcal{A}$ will be a non-trivial pca, that is, its underlying set $\mathbb{A}$ should have more than one element. It is not hard to show that this is equivalent to demanding that k and s do not coincide (see Proposition 1.3.1 in [Oos08]).

We continue by showing that we have a copy of $\mathbb{N}$ inside an pca and that we can perform recursion inside our pca.

Definition 2.2.3. The *Curry numerals* are defined inductively as follows:

$$\overline{0} = \mathsf{i};$$
$$\overline{n+1} = \mathsf{p}\overline{\mathsf{k}}\overline{n}.$$

One may show that our assumption that $\mathcal{A}$ is non-trivial guarantees that all Curry numerals are distinct.

The next proposition shows us that the Curry numerals really behave as natural numbers.

Proposition 2.2.4. *There are successor, predecessor and zero-test combinators in $\mathcal{A}$, denoted by S, P and Z, viz. for all $n \in \mathbb{N}$ the following hold:*

$$\mathsf{S}\overline{n} = \overline{n+1}; \quad \mathsf{P}\overline{0} = \overline{0}; \quad \mathsf{P}\overline{n+1} = \overline{n}; \quad \mathsf{Z}\overline{0} = \mathsf{k}; \quad \mathsf{Z}\overline{n+1} = \overline{\mathsf{k}}.$$

Proof. Define $\mathsf{Z} = \mathsf{p}_0$. This works since i was defined as skk. The successor is also easily found: put $\mathsf{S} = \mathsf{p}\overline{\mathsf{k}}$. For the predecessor, $\mathsf{P} = \lambda^* x.\mathsf{p}_0 x\overline{0}(\mathsf{p}_1 x)$ does the job:

$$\mathsf{P}\overline{0} = \mathsf{p}_0\overline{0}\,\overline{0}(\mathsf{p}_1\overline{0}) = \mathsf{k}\overline{0}(\mathsf{p}_1\overline{0}) = \overline{0} \quad \text{(recall that } \mathsf{p}_0\overline{0} = \mathsf{p}_0(\mathsf{skk}) = \mathsf{skkk} = \mathsf{ik} = \mathsf{k});$$
$$\mathsf{P}\overline{n+1} = \mathsf{p}_0\overline{n+1}\,\overline{0}(\mathsf{p}_1\overline{n+1}) = \overline{\mathsf{k}}\,\overline{0}\,\overline{n} = \overline{n}.$$
∎

Proposition 2.2.5. *There are* fixed point combinators y, z *in $\mathcal{A}$ such that for all $f, a \in \mathbb{A}$ the following hold:*

(i) $\mathsf{y}f \simeq f(\mathsf{y}f)$;

(ii) $\mathsf{z}f\downarrow$ *and* $\mathsf{z}fa \simeq f(\mathsf{z}f)a$.

Proof. Write $w = \lambda^* xy.y(xxy)$ and put $\mathsf{y} = ww$. Note that ww indeed denotes, because w denotes by construction, so that $ww \simeq (\lambda^* y.y(xxy))[w/x]$, which denotes (see the proof of Theorem 2.1.7). Observe that:

$$\mathsf{y}f \simeq wwf \simeq f(wwf) \simeq f(\mathsf{y}f),$$

as desired.

For z we can do something similar, but with an extra variable: put $u = \lambda^* xyz.y(xxy)z$ and let $\mathsf{z} = uu$. Then,

$$\mathsf{z}f \simeq uuf \simeq \lambda^* z.f(uuf)z,$$

which denotes and moreover,

$$\mathsf{z}fa \simeq f(uuf)a \simeq f(\mathsf{z}f)a,$$

as we wished. ∎

The fixed point combinator z allows to perform primitive recursion in our pca.

Proposition 2.2.6. *There is a* primitive recursion combinator R *in* $\mathcal{A}$ *such that for all* $f, a \in \mathbb{A}$ *and* $n \in \mathbb{N}$*, we have:*

$$\begin{aligned} \mathsf{R}af\overline{0} &= a; \\ \mathsf{R}af\overline{n+1} &\simeq f\overline{n}(\mathsf{R}af\overline{n}). \end{aligned}$$

Proof. The existence of R seems plausible, since, in principle, all we need is a zero test, a predecessor and repeated application. These are provided by Z, P and z. Now, one may define

$$R = \lambda^* rxgm.\mathsf{Z}m(\mathsf{k}x)(\lambda^* y.g(\mathsf{P}m)(rxg(\mathsf{P}m)\mathsf{i}))$$

and

$$\mathsf{R} = \lambda^* xgm.\mathsf{z}Rxgm\mathsf{i}.$$

Observe:

$$\begin{aligned} \mathsf{R}af\overline{0} &\simeq \mathsf{z}Raf\overline{0}\mathsf{i} \\ &\simeq R(\mathsf{z}R)af\overline{0}\mathsf{i} \\ &\simeq \mathsf{Z}\overline{0}(\mathsf{k}a)(\lambda^* y.f(\mathsf{P}\overline{0})(\mathsf{z}Raf(\mathsf{P}\overline{0})\mathsf{i}))\mathsf{i} \\ &\simeq \mathsf{k}(\mathsf{k}a)(\lambda^* y.f(\mathsf{P}\overline{0})(\mathsf{z}Raf(\mathsf{P}\overline{0})\mathsf{i}))\mathsf{i} \\ &\simeq \mathsf{k}a\mathsf{i} = a; \end{aligned}$$

and

$$\begin{aligned} \mathsf{R}af\overline{n+1} &\simeq \mathsf{z}Raf\overline{n+1}\mathsf{i} \\ &\simeq R(\mathsf{z}R)af\overline{n+1}\mathsf{i} \\ &\simeq \mathsf{Z}\overline{n+1}(\mathsf{k}a)(\lambda^* y.f(\mathsf{P}\overline{n+1})(\mathsf{z}Raf(\mathsf{P}\overline{n+1})\mathsf{i}))\mathsf{i} \\ &\simeq \overline{\mathsf{k}}(\mathsf{k}a)(\lambda^* y.f\overline{n}(\mathsf{z}Raf\overline{n}\mathsf{i}))\mathsf{i} \\ &\simeq (\lambda^* y.f\overline{n}(\mathsf{z}Raf\overline{n}\mathsf{i}))\mathsf{i} \\ &\simeq f\overline{n}(\mathsf{z}Raf\overline{n}\mathsf{i}) \\ &\simeq f\overline{n}(\mathsf{R}af\overline{n}); \end{aligned}$$

as desired. ∎

Given the above, it will come as no surprise that, using the Curry numerals, one can code finite sequences of elements of $\mathbb{A}$ inside $\mathcal{A}$. Moreover, this can be done such that all elementary operations on these sequences (e.g. concatenation and computing lengths) are represented in $\mathcal{A}$. This coding is straightforward, but tedious. Details may be found in Section 3.5 of [Oos08].

2.3 Examples of pcas

In this section we present four examples of pcas. We skip the details in most cases, except for one. *Scott's graph model* will be studied further in this thesis, so we work out all the details there. The other examples are mostly provided for comparison.

Example 2.3.1. Our first example is a degenerate pca and is only included for completeness. It is the trivial pca: the underlying set is a singleton $\{*\}$ and the application is given by $(*, *) \mapsto *$.

Example 2.3.2. Our second example is the prime example of a pca, it is known as *Kleene's first model* $\mathcal{K}_1$. Its underlying set is $\mathbb{N}$ and the application is given by $(n, m) \mapsto \varphi_n(m)$ where φ is a partial recursive enumeration function.

Example 2.3.3. Another example is *Kleene's second model* $\mathcal{K}_2$. Its underlying set is $\mathbb{N}^{\mathbb{N}}$. The application is somewhat involved. First of all, we fix a bijection $\langle - \rangle$ from the set of finite sequences of $\mathbb{N}$ to $\mathbb{N}$. Furthermore, we write $\overline{\alpha}n = \langle \alpha(0), \ldots, \alpha(n-1) \rangle$ and $\overline{\alpha}0$ for the empty sequence $\langle\rangle$. Now each $\alpha \in \mathbb{N}^{\mathbb{N}}$ determines a partial map $F_\alpha \colon \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}$ as follows:

$$F_\alpha(\beta) = \begin{cases} \alpha(\overline{\beta}n) - 1 & \text{if } n \in \mathbb{N} \text{ is the least } k \in \mathbb{N} \text{ such that } \alpha(\overline{\beta}k) > 0 \\ \text{undefined} & \text{if no such integer exists.} \end{cases}$$

For $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^{\mathbb{N}}$, write $\langle n \rangle * \alpha$ for the function $\beta \colon \mathbb{N} \to \mathbb{N}$ defined by $\beta(0) = n$ and $\beta(k+1) = \alpha(k)$. Finally, we can define an application on $\mathbb{N}^{\mathbb{N}}$ by:

$$\alpha\beta = \begin{cases} \text{the function } n \mapsto F_\alpha(\langle n \rangle * \beta) & \text{if } F_\alpha(\langle m \rangle * \beta) \text{ is defined for each } m \in \mathbb{N} \\ \text{undefined} & \text{else.} \end{cases}$$

One may endow $\mathbb{N}^{\mathbb{N}}$ with a natural topology, the Baire topology, which is obtained by giving $\mathbb{N}$ the discrete topology and $\mathbb{N}^{\mathbb{N}}$ the product topology. Interestingly, there is a connection between continuous functions for this topology and the application we just defined. If we write G_α for the partial endofunction on $\mathbb{N}^{\mathbb{N}}$ given by $\beta \mapsto \alpha\beta$, then one may check that $G_\alpha \colon \mathrm{dom}(G_\alpha) \to \mathbb{N}^{\mathbb{N}}$ is continuous (where $\mathrm{dom}(G_\alpha) \subseteq \mathbb{N}^{\mathbb{N}}$ has the subspace topology). Conversely, any partial endofunction on $\mathbb{N}^{\mathbb{N}}$ that is continuous on its domain may be extended to a function of the form G_α for some $\alpha \in \mathbb{N}^{\mathbb{N}}$.

Example 2.3.4. Our last example is *Scott's graph model* $\mathcal{S}$. Its underlying set is the powerset of $\mathbb{N}$, for which we shall write $\mathbb{S}$. Since we want to model computations, it seems natural that the elements of our pca act only on a finite amount of data. Therefore, we first define a bijection from the set of finite subsets of $\mathbb{N}$ to $\mathbb{N}$. Such a bijection is given by mapping a finite subset of $\mathbb{N}$ to its characteristic string: $\chi \colon \mathcal{P}_{\text{fin}}(\mathbb{N}) \to \mathbb{N}, p \mapsto \Sigma_{i \in p} 2^i$. Let us write $e_{(-)}$ for the inverse of χ, so $e_0 = \emptyset, e_1 = \{0\}, e_2 = \{1\}, e_3 = \{0, 1\}$, etc. Next, we fix a bijective pairing $\langle -, - \rangle \colon \mathbb{N}^2 \to \mathbb{N}$. Finally, we define the application as follows:

$$UV = \{m \in \mathbb{N} \mid \exists n (e_n \subseteq V, \langle n, m \rangle \in U)\}.$$

We think of U as the graph of some function acting on finite subsets of V.

As with Kleene's second model, there is an interesting connection with continuous functions. First, identify $\mathbb{S}$ with $\{0,1\}^{\mathbb{N}}$ and equip $\{0,1\}$ with the *Sierpiński topology* (the open sets are $\emptyset, \{0,1\}$ and $\{1\}$). We then topologize $\mathbb{S}$ by giving it the product topology (with $\mathbb{N}$ as index set). This topology is known as the *Scott topology*. Since $\{1\}$ is the only non-trivial open of the Sierpiński topology, the basic open sets of $\mathbb{S}$ are of the form $\uparrow p = \{U \in \mathbb{S} \mid U \supseteq p\}$ for some finite subset p of $\mathbb{N}$. Therefore, it is not surprising that a function $F\colon \mathbb{S} \to \mathbb{S}$ is continuous if and only if F is completely determined by its values on $\mathcal{P}_{\text{fin}}(\mathbb{N})$, viz. $F(U) = \bigcup\{F(p) \mid p \subseteq U \text{ finite}\}$ for any $U \in \mathbb{S}$. Indeed, if F is continuous, then $F^{-1}(\uparrow\{m\})$ is a union of basic opens for any $m \in \mathbb{N}$. Hence, $m \in F(U)$ if and only if there is some finite $q \subseteq U$ with $m \in F(q)$. Conversely, given such an F, we have:

$$
\begin{aligned}
F^{-1}(\uparrow p) &= \{V \in \mathbb{S} \mid F(V) \supseteq p\} \\
&= \{V \in \mathbb{S} \mid \bigcup\{F(q) \mid q \subseteq V \text{ finite}\} \supseteq p\} \\
&= \bigcup\{\uparrow q \mid F(q) \supseteq p\},
\end{aligned}
$$

so $F^{-1}(\uparrow p)$ is open and F is continuous.

From the definition of the application it easy to see it is continuous. Hence, for any positive integer k, the map $\mathbb{S}^k \to \mathbb{S}$ given by $(U_1, \ldots, U_k) \mapsto U_1 \cdots U_k$ is also continuous. Conversely, given any continuous function $F\colon \mathbb{S}^k \to \mathbb{S}$, we can define a set $U \in \mathbb{S}$ such that $F(U_1, \ldots, U_k) = UU_1 \cdots U_k$ for any $(U_1, \ldots, U_k) \in \mathbb{S}^k$. Indeed, if we define the *graph of F* as

$$
\operatorname{graph}(F) = \{\langle n_1, \langle n_2, \ldots, \langle n_k, m\rangle \ldots\rangle\rangle \mid n_1, \ldots, n_k \in \mathbb{N}, m \in F(e_{n_1}, \ldots, e_{n_k})\},
$$

then $\operatorname{graph}(F)$ has the desired property. For if $m \in F(U_1, \ldots, U_k)$, then, by continuity, we find $n_1, \ldots, n_k \in \mathbb{N}$ such that $m \in F(e_{n_1}, \ldots, e_{n_k})$ and $e_{n_1} \subseteq U_1, \ldots, e_{n_k} \subseteq U_k$. Hence, $\langle n_1, \langle n_2, \ldots, \langle n_k, m\rangle \ldots\rangle\rangle \in \operatorname{graph}(F)$ and thus, $m \in \operatorname{graph}(F)U_1 \cdots U_k$. Similarly, one proves that $\operatorname{graph}(F)U_1 \cdots U_k \subseteq F(U_1, \ldots, U_k)$.

It is now easy to prove that $\mathcal{S}$ is in fact a pca. A k combinator exists, because the function from $\mathbb{S}^2$ to $\mathbb{S}$ defined as $(U, V) \mapsto U$ is continuous. Further, an s combinator exists, because the assignment $(U, V, W) \mapsto UW(UV)$ is continuous.

Thus, $\mathcal{S}$ is a pca and we may think of its elements as (graphs of) continuous functions (w.r.t. the Scott topology).

Remark 2.3.5. When working with $\mathcal{S}$, it will be convenient to fix a particular pairing and some notation for it. The pair of two subsets U and V of $\mathbb{N}$ is given by the set

$$
[U, V] = \{2n \mid n \in U\} \cup \{2m + 1 \mid m \in V\}.
$$

It is easily verified that the map $(U, V) \mapsto [U, V]$ is a continuous bijection and that there are continuous projections $[U, V] \mapsto U$ and $[U, V] \mapsto V$.

Chapter 3

Assemblies

Partial combinatory algebras give rise to an interesting category called the category of assemblies. It has many enjoyable properties, e.g. it is regular, cartesian closed, finitely cocomplete and it has a natural numbers object.

Again, we follow [Oos08, Section 1.5] and [Zoe18, Section 2.5], and $\mathcal{A}$ will denote an arbitrary pca with $\mathbb{A}$ as its underlying set throughout.

3.1 Assemblies and their morphisms

Definition 3.1.1. An *assembly* X *(over* $\mathcal{A}$*)* is a pair $(|X|, E_X)$ with $|X|$ a set and $E_X \colon |X| \to \mathcal{P}^*(\mathbb{A})$ a function from $|X|$ to the set of non-empty subsets of $\mathbb{A}$.

We sometimes say that $a \in E_X(x)$ *realizes* x or that it is a *realizer* of x. We also refer to $E_X(x)$ as the *realizing set of* x.

We think of $E_X(x)$ as computational data accompanying x. Later, we will see it as a set of computational witnesses of the existence of x. Accordingly, a morphism of assemblies should take the computational data into account.

Definition 3.1.2. Let X and Y be assemblies. A *morphism of assemblies* $X \to Y$ is a function $f \colon |X| \to |Y|$ such that there exists $a \in \mathbb{A}$ satisfying: for any $x \in X$ and $b \in E_X(x)$, we have $ab{\downarrow}$ and $ab \in E_Y(f(x))$.

We say that f is *tracked by* a and call a a *tracker of* f. In other words, given a realizer of x, the tracker computes a realizer of $f(x)$.

It is important to note that trackers are not necessarily unique and are *not* part of the morphism.

Proposition 3.1.3. *The assemblies and their morphisms form a category, called the* category of assemblies (over $\mathcal{A}$) *and denoted by* $\mathsf{Asm}(\mathcal{A})$.

Proof. It suffices to show that the identity function is tracked and that if $f \colon X \to Y$ and $g \colon Y \to Z$ are morphisms of assemblies, then their function composition gf is tracked. Observe that the identity function is tracked by the i combinator. Further, if a tracks f

and b tracks g, then $\lambda^* u.b(au)$ tracks gf. Indeed, if $x \in X$ and $c \in E_X(x)$, then $ac\downarrow$ and $ac \in E_Y(f(x))$, so $b(ac)\downarrow$ and $b(ac) \in E_Z(gf(x))$. ∎

3.2 Properties of the category of assemblies

Proposition 3.2.1. *The category* $\mathsf{Asm}(\mathcal{A})$ *has finite (co)limits.*

Proof. First of all, observe that $(\{*\}, * \mapsto \mathbb{A})$ is a terminal object, as for any assembly X, the unique function $|X| \to \{*\}$ is tracked by i.

The product $X \times Y$ of two assemblies X and Y is the assembly $(|X| \times |Y|, E_{X\times Y})$ where

$$E_{X\times Y}(x,y) = \{\mathsf{p}ab \mid a \in E_X(x), b \in E_Y(y)\}.$$

Let us verify the desired universal property. It suffices to show that the usual maps in Set are tracked. The projections $\pi_0 \colon X \times Y \to X$ and $\pi_1 \colon X \times Y \to Y$ are clearly tracked by p_0 and p_1, respectively. Suppose we have arrows $f \colon Z \to X$ and $g \colon Z \to Y$, tracked by a_f and a_g, respectively. Then the map $\langle f, g\rangle \colon Z \to X \times Y$ is tracked by $\lambda^* x.\mathsf{p}(a_f x)(a_g x)$.

To establish that $\mathsf{Asm}(\mathcal{A})$ has finite limits, we show $\mathsf{Asm}(\mathcal{A})$ also has equalizers. Suppose we have parallel morphisms $f, g \colon X \to Y$ of assemblies. Let X' be the assembly $(\{x \in X \mid f(x) = g(x)\}, E_{X'})$, where $E_{X'}$ is the restriction of E_X to $|X'|$. The inclusion $i \colon |X'| \to |X|$ is obviously tracked by i and we have $fi = gi$. We must show that any morphism of assemblies $h \colon Z \to X$ satisfying $fh = gh$, factors through i. From considerations in Set, it follows that there is a unique function $k \colon |Z| \to |X'|$ with $ik = h$. It is defined as $z \mapsto h(z)$. It remains to show that k is tracked, but it is, since any tracker of h also tracks k. Thus, X' is an equalizer and $\mathsf{Asm}(\mathcal{A})$ has finite limits.

For finite cocompleteness, observe that $(\emptyset, \emptyset)$ is an initial object and that the coproduct $X + Y$ of two assemblies X and Y is given by $(|X| + |Y|, E_{X+Y})$ where

$$E_{X+Y}(0,x) = \{\mathsf{pk}a \mid a \in E_X(x)\} \text{ and } E_{X+Y}(1,y) = \{\mathsf{p}\bar{\mathsf{k}}b \mid b \in E_Y(y)\}.$$

To verify the universal property, it again suffices to show that the appropriate maps in Set are tracked. Observe that the inclusions $X \to X+Y$ and $Y \to X+Y$ are respectively tracked by pk and $\mathsf{p}\bar{\mathsf{k}}$. Suppose $f \colon X \to Z$ and $g \colon Y \to Z$ are morphisms of assemblies, tracked by a_f and a_g, respectively. The map $[f, g] \colon X + Y \to Z$ is tracked by a variation of the case distinction operator (c.f. Remark 2.2.1):

$$[a_f, a_g] = \lambda^* u.\mathsf{p}_0 u(\lambda^* v.a_f(\mathsf{p}_1 u))(\lambda^* w.a_g(\mathsf{p}_1 u))\mathsf{k}.$$

Indeed, for $b \in E_X(x)$, we have:

$$\begin{aligned}[a_f, a_g]\mathsf{pk}b &= \mathsf{k}(\lambda^* v.a_f(\mathsf{p}_1 u))[\mathsf{pk}b/u](\lambda^* w.a_g(\mathsf{p}_1 u))[\mathsf{pk}b/u]\mathsf{k}\\ &= (\lambda^* v.a_f(\mathsf{p}_1 u))[\mathsf{pk}b/u]\mathsf{k}\\ &= a_f(\mathsf{p}_1 u)[\mathsf{pk}b/u]\\ &= a_f b \in E_Z(f(x)),\end{aligned}$$

as desired. (Note our careful handling of the substitution in light of Remark 2.1.9.) Similarly, $[a_f, a_g]\mathsf{p}\overline{\mathsf{k}}b \in E_Z(g(y))$ for $b \in E_Y(y)$.

Next, we construct coequalizers. Suppose we have parallel morphisms $f, g\colon X \to Y$. Let $q\colon |Y| \to |Y|/{\sim}$ be their coequalizer in Set. Define the assembly $Y' = (|Y|/{\sim}, E_{Y'})$ with $E_{Y'}([y]) = \bigcup_{y' \in [y]} E_Y(y')$. Observe that q is a morphism from Y to Y' as it is tracked by i. Now suppose $r\colon Y \to W$ is a morphism with $rf = rg$. We must show that it factors uniquely through q. Since $|Y'|$ is the coequalizer of f and g in Set, there is a unique $k\colon |Y'| \to |W|$ such that $kq = r$. Moreover, it is tracked, because any tracker of r also tracks k (since q is tracked by i). We conclude that $\mathsf{Asm}(\mathcal{A})$ has finite colimits, as desired. ■

It will be convenient to have a characterization of regular epimorphisms in $\mathsf{Asm}(\mathcal{A})$.

Lemma 3.2.2. *A morphism $e\colon X \to Y$ is a regular epimorphism if and only if e is surjective and the surjectivity is witnessed in $\mathcal{A}$, that is: there is some $a \in \mathbb{A}$ such that for any $y \in |Y|$ and $b \in E_Y(y)$, $ab{\downarrow}$ and $ab \in E_X(x)$ for some $x \in |X|$ with $e(x) = y$.*

Proof. First of all, observe that the requirement that the surjectivity is witnessed is equivalent to requiring that $\mathrm{id}_{|Y|}$ is tracked as a morphism from Y to $Y' = (|Y|, E_{Y'})$ with $E_{Y'}(y) = \bigcup_{e(x)=y} E_X(x)$.

Suppose first that $e\colon X \to Y$ is a regular epimorphism. From our description of coequalizers above, we see that e must be surjective. Furthermore, the function e is tracked (by i) as a morphism $X \to Y'$ and therefore, it factors through $e\colon X \to Y$, so we see that $\mathrm{id}_{|Y|}$ is tracked as a morphism from Y to Y'.

Conversely, suppose $e\colon X \to Y$ is a morphism satisfying both properties. Since $\mathrm{id}_{|Y|}$ is always tracked as a morphism $Y' \to Y$, we see that Y is isomorphic to Y'. By our description of coequalizers, it is clear that

$$(\{(x, x') \mid e(x) = e(x')\}, E) \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} X \xrightarrow{\;e\;} Y',$$

where E is the appropriate restriction of $E_{X \times X}$, is a coequalizer diagram in $\mathsf{Asm}(\mathcal{A})$. Hence, e is regular epic. ■

Proposition 3.2.3. *The category $\mathsf{Asm}(\mathcal{A})$ is regular.*

Proof. It remains to show that regular epis are stable under pullback. We apply the previous lemma. Suppose $e\colon X \twoheadrightarrow Y$ is a regular epi. Let

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & X \\ \big\downarrow & \lrcorner & \big\downarrow e \\ Z & \xrightarrow{\;f\;} & Y \end{array}$$

be a pullback. By our description of finite limits, we have

$$X \times_Y Z = (\{(x, z) \in |X| \times |Z| \mid e(x) = f(z)\}, E_{X \times_Y Z}),$$

where $E_{X\times_Y Z}$ is the appropriate restriction of $E_{X\times Z}$.

The map $X \times_Y Z \to Z$ is obviously surjective. We must find an element $a \in \mathbb{A}$ witnessing it. Since e is assumed to regular epic, take $b \in \mathbb{A}$ witnessing its surjectivity and let c track f. We claim that $a = \lambda^* u.\mathsf{p}(b(cu))u$ does the job. Indeed, if $z \in |Z|$ and $d \in E_Z(z)$, then $cd\downarrow$ and $cd \in E_Y(f(z))$, so $b(cd)\downarrow$ and $b(cd) \in E_X(x)$ for some $x \in |X|$ with $e(x) = f(z)$. Hence, $\mathsf{p}(b(cd))d \in E_{X\times_Y Z}(x, z)$, as desired. ∎

Proposition 3.2.4. *The category* $\mathsf{Asm}(\mathcal{A})$ *is cartesian closed.*

Proof. Let X and Y be two assemblies. Define the assembly Y^X as

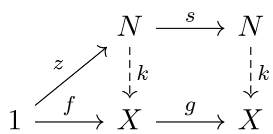
$$Y^X = (\{f\colon |X| \to |Y| \mid f \text{ is tracked}\}, E_{Y^X})$$

with $E_{Y^X}(f) = \{a \in \mathbb{A} \mid a \text{ tracks } f\}$. We have an evaluation morphism $\mathrm{ev}\colon Y^X \times X \to Y$ given by $(f, x) \mapsto f(x)$ and tracked by $\lambda^* u.\mathsf{p}_0 u(\mathsf{p}_1 u)$. Suppose we have an assembly Z and a map $Z \times X \xrightarrow{g} Y$. We must show that there is a unique $Z \xrightarrow{\tilde{g}} Y^X$ such that $\mathrm{ev}(\tilde{g} \times \mathrm{id}_X) = g$. Since Set is cartesian closed, it suffices to prove that $\tilde{g}$ defined as $\tilde{g}(z) = (x \mapsto g(z, x))$ is well-defined and tracked as a morphism from Z to Y^X. If a tracks g, then $\lambda^* uv.a(\mathsf{p}uv)$ tracks $\tilde{g}$. Moreover, $E_Z(z)$ is non-empty for any $z \in |Z|$, so $\tilde{g}(z)$ has a tracker for any $z \in |Z|$. Thus, $\tilde{g}$ is a well-defined morphism from Z to Y^X, as we wished to show. ∎

Proposition 3.2.5. *The category* $\mathsf{Asm}(\mathcal{A})$ *has a natural numbers object.*

Proof. Let N be the assembly $(\mathbb{N}, E_N)$ with $E_N(n) = \{\overline{n}\}$ (where $\overline{n}$ is the Curry numeral from Definition 2.2.3). Note that we have maps $1 \xrightarrow{z} N \xrightarrow{s} N$ given by $z(*) = 0$ and $s(n) = n + 1$ and tracked by Z and S (recall Proposition 2.2.4), respectively.

Suppose we have $1 \xrightarrow{f} X \xrightarrow{g} N$. We must show that it factors through $1 \xrightarrow{z} N \xrightarrow{s} N$, i.e. there must be a unique $k\colon N \to X$ such that



commutes. By inspection, we have no choice but to define k recursively as:

$$k(0) = f(*) \text{ and } k(n+1) = g(k(n)).$$

We must show that this is tracked. For this, let a be any element of $E_X(f(*))$, let t track g and put $t' = \lambda^* uv.t(\overline{\mathsf{k}}uv)$. We show by induction that k is tracked by $\mathsf{R}at'$. Since we defined k recursively, it comes as no surprise that we need the primitive recursion combinator R from Proposition 2.2.6. Remember that $\mathsf{R}at'\overline{0} = a \in E_X(f(*)) = E_X(k(0))$. Further, if we assume that $\mathsf{R}at'\overline{n} \in E_X(k(n))$, then $\mathsf{R}at'\overline{n+1} \simeq t'\overline{n}(\mathsf{R}at'\overline{n}) \simeq t(\mathsf{R}at'\overline{n})$. Now, the latter is an element of $E_X(g(k(n))) = E_X(k(n+1))$, because t tracks g and $\mathsf{R}at'\overline{n} \in E_X(k(n))$ by assumption. This concludes our proof. ∎

Definition 3.2.6. An assembly X is called *discrete* if the realizing sets are disjoint, viz. $E_X(x) \cap E_X(y) = \emptyset$ for all distinct $x, y \in |X|$.

We will also refer to a discrete assembly as a *modest set*. The full subcategory of $\mathsf{Asm}(\mathcal{A})$ on modest sets will be denoted by $\mathsf{Mod}(\mathcal{A})$.

Example 3.2.7. The natural numbers object N is an example of a modest set (provided that the pca $\mathcal{A}$ is non-trivial).

Lemma 3.2.8. *Let X be an assembly and Y a modest set. If there is an injective morphism from X to Y, then X is a modest set as well. In particular, the notion of modest set is stable under isomorphism.*

Proof. Let $f\colon X \to Y$ be an injective morphism tracked by $a \in \mathbb{A}$. Assume that we have an element $b \in E_X(x) \cap E_X(x')$ for certain $x, x' \in |X|$. We prove that $x = x'$. Observe that $ab\downarrow$ and $ab \in E_Y(f(x)) \cap E_Y(f(x'))$, so that $f(x) = f(x')$ (as Y is modest). By injectivity of f, we get that x and x' are equal, as desired. ∎

Proposition 3.2.9. *The category $\mathsf{Mod}(\mathcal{A})$ is an exponential ideal in $\mathsf{Asm}(\mathcal{A})$.*

Proof. Suppose Y is a modest and X is an assembly. Consider the exponential Y^X. Suppose f and g are two different elements of $|Y^X|$, i.e. different morphisms from X to Y. Assume for a contradiction that we have some element $a \in \mathbb{A}$ that tracks both f and g. Since f and g are distinct, we may find $x \in |X|$ with $f(x) \neq g(x)$. Let $b \in E_X(x)$. Then $ab\downarrow$ and $ab \in E_Y(f(x)) \cap E_Y(g(x))$. But this is impossible, because Y is a modest set and $f(x) \neq g(x)$. We conclude that Y^X is a modest set. ∎

Proposition 3.2.10. *The category $\mathsf{Mod}(\mathcal{A})$ is regular. Moreover, the inclusion functor $\mathsf{Mod}(\mathcal{A}) \to \mathsf{Asm}(\mathcal{A})$ is regular.*

Proof. It is straightforward to verify this using the description of finite limits and regular epimorphisms given above. ∎

Chapter 4

Realizability Toposes

In the previous chapter we introduced the category of assemblies, which had quite a bit of structure. It falls short of being a topos, however. In this chapter we construct a topos over a pca, known as the realizability topos of the pca. We also investigate its structure and logic and give characterizations of some categorical properties. Finally, we show that the realizability topos may be seen as a generalization of the category of assemblies.

We follow the expositions in [Oos08, Chapter 2] and [Zoe18, Chapter 3]. Again, let $\mathcal{A}$ denote an arbitrary, but fixed pca with $\mathbb{A}$ as its underlying set. We start by studying $\mathcal{P}(\mathbb{A})$-valued predicates and tripos theory. Both are paramount in describing the internal logic of realizability toposes.

4.1 $\mathcal{P}(\mathbb{A})$-valued predicates

Definition 4.1.1. Given any set X, a $\mathcal{P}(\mathbb{A})$*-valued predicate on* X is a function φ from X to the powerset $\mathcal{P}(\mathbb{A})$ of $\mathbb{A}$. For $x \in X$ and $a \in \varphi(x)$, we will say that *a realizes* $\varphi(x)$ or that a is a *realizer* for $\varphi(x)$.

Given a $\mathcal{P}(\mathbb{A})$-valued predicate φ and $x \in X$, we think of $a \in \varphi(x)$ as a proof or witness that the "predicate" φ holds for x. The partial combinatory structure of $\mathcal{A}$ allows us to turn the set of $\mathcal{P}(\mathbb{A})$-valued predicates into a Heyting prealgebra, which we define now.

Definition 4.1.2. Let $(P, \leq)$ be a preorder.

(i) The *poset reflection* of the preorder $(P, \leq)$ is the poset obtained by identifying $p, q \in P$ for which $p \leq q$ and $q \leq p$.

(ii) A *Heyting prealgebra* is a preorder whose poset reflection is a Heyting algebra.

Definition 4.1.3. For a fixed set X, we define a relation $\leq$ on the set of $\mathcal{P}(\mathbb{A})$-valued predicates $\mathcal{P}(\mathbb{A})^X$ by putting $\varphi \leq \psi$ if we can uniformly obtain ψ from φ, viz. there is an element $a \in \mathbb{A}$ such that for any $x \in X$ and $b \in \varphi(x)$, we have $ab\downarrow$ and $ab \in \psi(x)$. We also say that the element *a realizes* $\varphi \leq \psi$.

Proposition 4.1.4. *For any set X, the pair $\big(\mathcal{P}(\mathbb{A})^X, \leq\big)$ is a Heyting prealgebra.*

Proof. We commence by showing that $\leq$ is indeed a preorder. Reflexivity holds by existence of the i combinator. For transitivity, suppose that $\varphi \leq \psi$ and $\psi \leq \chi$ are realized by a and b, respectively. We claim that $\varphi \leq \chi$ is then realized by $\lambda^* u.b(au)$. Indeed, if $x \in X$ and $c \in \varphi(x)$, then $ac \in \psi(x)$, so $b(ac) \in \chi(x)$, as desired.

We have top $\top$ and bottom $\bot$ elements given by $\top(x) = \mathbb{A}$ and $\bot(x) = \emptyset$ for any $x \in X$. Indeed, i realizes the inequalities $\bot \leq \varphi \leq \top$ for any $\varphi\colon X \to \mathcal{P}(\mathbb{A})$.

We proceed by defining meet ($\wedge$) and join ($\vee$) operations. For $x \in X$, put:

$$(\varphi \wedge \psi)(x) = \{\mathsf{p}ab \mid a \in \varphi(x), b \in \psi(x)\};$$
$$(\varphi \vee \psi)(x) = \{\mathsf{pk}a \mid a \in \varphi(x)\} \cup \{\mathsf{p}\overline{\mathsf{k}}b \mid b \in \psi(x)\}.$$

Proving that these operations satisfy the desired universal properties is similar to proving that the category of assemblies has (co)products, as we have seen in the previous section. Therefore, we omit the details here.

Finally, we define the Heyting implication by:

$$(\varphi \to \psi)(x) = \{a \in \mathbb{A} \mid \forall b \in \varphi(x), ab{\downarrow} \text{ and } ab \in \psi(x)\} \quad \text{for any } x \in X.$$

For suppose $\chi \wedge \varphi \leq \psi$ is realized by a. Then $\chi \leq \varphi \to \psi$ is realized by $\lambda^* uv.a(\mathsf{p}uv)$. Conversely, if b realizes $\chi \leq \varphi \to \psi$, then $\chi \wedge \varphi \leq \psi$ is realized by $\lambda^* u.b(\mathsf{p_0}u)(\mathsf{p_1}u)$. ∎

At the beginning of this section, we mentioned the importance of $\mathcal{P}(\mathbb{A})$-valued predicates in categorical logic. The proposition above yields quite a bit of logical structure, but we should also consider quantifiers. This is what we do next.

Lemma 4.1.5. *Suppose $(P, \leq)$ and $(Q, \leq)$ are preorders. Let $f\colon Q \to P$ and $g\colon P \to Q$ be functions such that $f(q) \leq p$ if and only if $q \leq g(p)$. Then f is left adjoint to g when seen as functors between preorder categories.*

Proof. It suffices to show that f and g are functors, i.e. order preserving. Suppose $p \leq p' \in P$. Since $g(p) \leq g(p)$ holds, we have $f(g(p)) \leq p \leq p'$ and therefore, $g(p) \leq g(p')$. Thus, g is order preserving. Similarly, f is. ∎

Definition 4.1.6. For $f\colon X \to Y$, let us write f^* for the function $f^*\colon \mathcal{P}(\mathbb{A})^Y \to \mathcal{P}(\mathbb{A})^X$ defined by $f^*(\varphi)(x) = \varphi(f(x))$ for any $\varphi\colon Y \to \mathcal{P}(\mathbb{A})$ and $x \in X$.

Proposition 4.1.7. *For any $f\colon X \to$, the function f^* is a morphism of Heyting prealgebras. Moreover, f^* has left and right adjoints, for which we will write $\exists_f$ and $\forall_f$, respectively.*

Proof. The proof of the first claim is straightforward, we omit it here.

For the second claim, we apply Lemma 4.1.5. Define

$$\exists_f(\varphi)(y) = \bigcup_{f(x)=y} \varphi(x);$$
$$\forall_f(\varphi)(y) = \{a \in \mathbb{A} \mid \forall b \in \mathbb{A}\, \forall x \in X\, (\text{if } f(x) = y,\ \text{then } ab{\downarrow} \text{ and } ab \in \varphi(x))\},$$

for any $\varphi\colon X \to \mathcal{P}(\mathbb{A})$ and $y \in Y$.

It is immediate from the definitions that a realizes $\exists_f(\varphi) \le \psi$ if and only if a realizes $\varphi \le f^*(\psi)$.

Suppose a realizes $\psi \le \forall_f(\varphi)$. If $b \in \psi(f(x))$, then $ab\downarrow$ and $ab \in \forall_f(\varphi)(f(x))$, so that $ab\mathsf{k} \in \psi(x)$. Hence, $\lambda^* u.au\mathsf{k}$ is a realizer of $f^*(\psi) \le \varphi$.

Conversely, assume b realizes $f^*(\psi) \le \varphi$. We claim that $\lambda^* uv.bu$ realizes $\psi \le \forall_f(\varphi)$. Indeed, for $c \in \psi(y)$, we have $(\lambda^* uv.bu)c \simeq (\lambda^* v.bu)[u/c]$, so if $f(x) = y$ and $d \in \mathbb{A}$, then $(\lambda^* v.bu)[u/c]d = bc \in \varphi(x)$. ■

One notices a certain asymmetry in the definitions of the adjoints: $\forall_f$ is a bit more complicated than $\exists_f$. This situation is prettier if f is surjective (which it always will be in later use).

Lemma 4.1.8. *If $f\colon X \to Y$ is surjective, then a right adjoint of f^* is given by*

$$\forall_f(\varphi)(y) = \bigcap_{f(x)=y} \varphi(x),$$

where $\varphi\colon X \to \mathcal{P}(\mathbb{A})$ and $x \in X$.

Proof. It is not hard to verify that a realizes $\psi \le \forall_f(\varphi)$ if and only if a realizes $f^*(\psi) \le \varphi$. Surjectivity of f is used in the if-direction, as follows. Suppose a realizes $f^*(\psi) \le \varphi$. If $b \in \psi(y)$, then for any $x \in X$ with $f(x) = y$, we have $ab\downarrow$ and $ab \in \varphi(x)$. Since f is surjective, there is at least one such x. Hence, ab always denotes. Thus, a realizes $\psi \le \forall_f(\varphi)$, as desired. ■

Example 4.1.9. Let φ and ψ be two $\mathcal{P}(\mathbb{A})$-valued predicates on a set X. Consider the unique function $X \to \{*\}$, which is obviously surjective (or empty if $X = \emptyset$). Then we see that $\varphi \le \psi$ if and only if $\bigcap_{x \in X}(\varphi(x) \to \psi(x))$ is non-empty.

4.2 Realizability triposes

In this section we show that the $\mathcal{P}(\mathbb{A})$-predicates and the maps $f^*, \forall_f, \exists_f$ allow us to interpret many-sorted predicate logic without equality.

Definition 4.2.1. We will write $\mathsf{Heytpre}$ for the category of Heyting prealgebras: its objects are Heyting prealgebras and morphisms are Heyting prealgebra morphisms.

The *realizability tripos of $\mathcal{A}$* is the functor $\mathsf{P}\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{Heytpre}$ defined on objects by $\mathsf{P}(X) = (\mathcal{P}(\mathbb{A})^X, \le)$ and on morphisms by $\mathsf{P}(f\colon X \to Y) = f^*\colon \mathsf{P}(Y) \to \mathsf{P}(X)$.

Definition 4.2.2. A Set-*typed language* $\mathcal{L}$ is a set of relation symbols assigning to each relation symbol a finite sequence sequence of sets, called its *type*.

Next, we define $\mathcal{L}$-*terms* recursively:

(i) for each set X, we assume to have variables $x_1^X, x_2^X, \ldots$ of type X and these are all $\mathcal{L}$-terms;

(ii) if $t_1, \dots, t_n$ are $\mathcal{L}$-terms of types $X_1, \dots, X_n$ respectively and f is a function from $X_1 \times \cdots \times X_n \to Y$, then we have a function symbol (also denoted by f) and $f(t_1, \dots, t_n)$ is a term of type Y.

Finally, we recursively define $\mathcal{L}$-*formulas*:

(i) $\bot$ and $\top$ are $\mathcal{L}$-formulas;

(ii) if R is a relation symbol of type $(X_1, \dots, X_n)$ and $t_1, \dots, t_n$ are $\mathcal{L}$-terms of types $X_1, \dots, X_n$ respectively, then $R(t_1, \dots, t_n)$ is an $\mathcal{L}$-formula;

(iii) if φ and ψ are $\mathcal{L}$-formulas, then so are $\varphi \vee \psi, \varphi \wedge \psi, \varphi \to \psi$ and $\neg\varphi$;

(iv) if φ is a $\mathcal{L}$-formula and x is a variable, then $\forall x \varphi$ and $\exists x \varphi$ are $\mathcal{L}$-formulas.

In the remainder we will often simply speak of terms and formulas, suppressing reference to the language when it is clear from the context.

Remark 4.2.3. We will assume that $\to$ has the lowest precedence (w.r.t. $\neg, \vee$ and $\wedge$), e.g. we write $\varphi \wedge \psi \to \chi$ for $(\varphi \wedge \psi) \to \chi$.

Next, we turn to interpreting $\mathcal{L}$ in a realizability tripos P.

Definition 4.2.4. Let P be a realizability tripos and let $\mathcal{L}$ be a Set-typed language. An *interpretation of* $\mathcal{L}$ *in* P assigns to every relation symbol R in $\mathcal{L}$ of type $(X_1, \dots, X_n)$ an element $[R]$ of $\mathsf{P}(X_1 \times \cdots \times X_n)$.

It will be convenient to introduce the following notation: if a term t has free variables $x_1^{X_1}, \dots, x_n^{X_n}$, then we write $[\mathrm{fv}(t)]$ for $X_1 \times \cdots \times X_n$. If a term has no free variables, then $[\mathrm{fv}(t)] = 1$, the terminal object in Set. We adopt a similar notation for formulas.

Further, for every term t of type X we define a function $[t] \colon [\mathrm{fv}(t)] \to X$ by recursion:

(i) $[x^X]$ is the identity map on X;

(ii) $[f(t_1, \dots, t_n)]$ is defined as the composition

$$[\mathrm{fv}(f(t_1, \dots, t_n))] \to [\mathrm{fv}(t_1)] \times \cdots \times [\mathrm{fv}(t_n)] \xrightarrow{[t_1] \times \cdots \times [t_n]} X_1 \times \cdots \times X_n \xrightarrow{f} X.$$

Now we are ready to define an interpretation $[\varphi]$ of an $\mathcal{L}$-formula φ in P. A formula φ will be interpreted as an element $[\varphi] \in \mathsf{P}([\mathrm{fv}(\varphi)])$. We do so recursively of course:

(i) $[\bot]$ and $[\top]$ are the bottom and top elements of $\mathsf{P}(1)$ (where 1 is a terminal object of Set);

(ii) if R is a relation symbol of type $(X_1, \dots, X_n)$ and $t_1, \dots, t_n$ are terms of types $X_1, \dots, X_n$ respectively, then we define $[R(t_1, \dots, t_n)]$ as $\langle [t_1]\pi_1, \dots, [t_n]\pi_n \rangle^*[R]$, where π_i is the projection $[\mathrm{fv}(R(t_1, \dots, t_n))] \to [\mathrm{fv}(t_i)]$;

(iii) $[\varphi \wedge \psi]$ is defined as the meet $\pi_0^*[\varphi] \wedge \pi_1^*[\psi]$ in $\mathsf{P}([\mathrm{fv}(\varphi \wedge \psi)])$ where π_0 and π_1 are the projections from $[\mathrm{fv}(\varphi \wedge \psi)]$ to $[\mathrm{fv}(\varphi)]$ and $[\mathrm{fv}(\psi)]$ respectively; similarly, one defines $[\varphi \vee \psi], [\varphi \to \psi]$ using the appropriate Heyting prealgebra operations;

(iv) $[\neg\varphi]$ is defined as $[\varphi \to \bot]$;

(v) $[\forall x\varphi]$ and $[\exists x\varphi]$ are defined as $\forall_\pi[\varphi]$ and $\exists_\pi[\varphi]$, respectively, where π is the projection from $[\mathrm{fv}(\varphi)]$ to $[\mathrm{fv}(\forall x\varphi)] = [\mathrm{fv}(\exists x\varphi)]$.

Finally, we say that a sentence φ ***is true in*** P (written as $\mathsf{P} \models \varphi$) if $[\varphi]$ is the top element of $\mathsf{P}(1)$. That is, $[\varphi](*)$ has a realizer, viz. $[\varphi] \subseteq \mathbb{A}$ is non-empty.

Remark 4.2.5. For convenience, we employ a lower-case, upper-case correspondence between variables and types, viz. we write $x, x', x'', \ldots$ for variables of type X. Furthermore, given a predicate $\varphi \in \mathsf{P}(X_1, \ldots, X_n)$, we will also write φ for the relation symbol of type $(X_1, \ldots, X_n)$ that is interpreted by this predicate.

Example 4.2.6. It may be instructive to write out the quantifier cases above. Let φ be a formula with one free variable x and assume $X \neq \emptyset$. Let $\pi\colon [\mathrm{fv}(\varphi)] \to 1$ be the unique map. By definition of $\exists_\pi$, we find $\mathsf{P} \models \exists x\varphi(x)$ if and only if $\bigcup_{a\in X}[\varphi](a) \neq \emptyset$. In light of Lemma 4.1.8 and the fact that projections are surjective, we see that similarly: $\mathsf{P} \models \forall x\varphi(x)$ if and only if $\bigcap_{a\in X}[\varphi](a) \neq \emptyset$.

Moreover, by Example 4.1.9, $[\varphi(x)] \leq [\psi(x)]$ if and only if $\mathsf{P} \models \forall x(\varphi(x) \to \psi(x))$.

Example 4.2.7. Let φ be a formula. Since $[\bot]$ maps any element to the empty set, we see that $[\neg\varphi] = [\varphi \to \bot]$ has a realizer if and only if $[\varphi] = [\bot]$. In fact, since any realizer will work if $[\varphi] = [\bot]$, we have:

$$[\neg\varphi](x) = \begin{cases} \mathbb{A} & \text{if } [\varphi](x) = \emptyset; \\ \emptyset & \text{else}; \end{cases}$$

for any $x \in [\mathrm{fv}(\varphi)]$.

Remark 4.2.8. We employ the following (potentially confusing) notation. If φ is a formula with free variables $x_1^{X_1}, \ldots, x_n^{X_n}$ and $a_1 \in X_i, \ldots, a_n \in X_n$, then we write $[\varphi(a_1, \ldots, a_n)]$ for $[\varphi](a_1, \ldots, a_n)$.

Theorem 4.2.9 (Soundness Theorem)**.** *Let φ be a sentence in a* Set*-typed relational language $\mathcal{L}$. If φ is provable in intuitionistic predicate logic without equality, then* $\mathsf{P} \models \varphi$ *for every interpretation of $\mathcal{L}$ in* P*.*

Proof. This can be done by induction on φ and the proof tree for φ. Further, it relies on a substitution lemma and on the Beck-Chevalley condition. We do not go into these matters here, but one may consult [Oos08, Theorem 2.1.6]. ■

4.3 Objects and morphisms of realizability toposes

Throughout this section, let P be the realizability tripos of the pca $\mathcal{A}$.

Definition 4.3.1. For a set X, a *partial equivalence relation* (*over* P) *on* X is an element $\sim$ of $\mathsf{P}(X \times X)$ such that:

$$\mathsf{P} \models \forall x x' (x \sim x' \to x' \sim x) \qquad (\sim \text{ is } \textit{symmetric});$$
$$\mathsf{P} \models \forall x x' x'' (x \sim x' \wedge x' \sim x'' \to x \sim x'') \qquad (\sim \text{ is } \textit{transitive}).$$

(Recall our convention from Remark 4.2.5.)

Explicitly, this means that there are elements $s, t \in \mathbb{A}$ witnessing the symmetry and transitivity, respectively, viz. for any $x, x', x'' \in X$, if $a \in [x \sim x']$, then $sa\downarrow$ and $sa \in [x' \sim x]$ and if $a \in [x \sim x'], b \in [x' \sim x'']$, then $t(\mathsf{p}ab)\downarrow$ and $t(\mathsf{p}ab) \in [x \sim x'']$.

Observe that $\sim$ is really partial, since we do not require $\mathsf{P} \models \forall x(x \sim x)$. We think of elements of $[x \sim x]$ as realizers witnessing the existence of x.

The reason for stating the definition using the tripos, without explicit mention of the realizers is twofold. For one, the language of the tripos is convenient. Moreover, the Soundness Theorem makes it easy to derive further properties. For example, it yields $\mathsf{P} \models \forall x x'(x \sim x' \to x \sim x \wedge x' \sim x')$, so if elements are related, then they both exist.

The objects of a realizability topos will be pairs of sets with partial equivalence relations. Next, we turn to defining morphisms.

Definition 4.3.2. Let $(X, \sim_X)$ and $(Y, \sim_Y)$ be two sets with partial equivalence relations. A *functional relation* (*over* P) *from* $(X, \sim_X)$ *to* $(Y, \sim_Y)$ is an element F of $\mathsf{P}(X \times Y)$ such that:

$$\mathsf{P} \models \forall xy(F(x,y) \to x \sim_X x \wedge y \sim_Y y) \qquad (F \text{ is } \textit{strict});$$
$$\mathsf{P} \models \forall x x' y y'(F(x,y) \wedge x \sim_X x' \wedge y \sim_Y y' \to F(x',y')) \qquad (F \text{ is } \textit{relational});$$
$$\mathsf{P} \models \forall x y y'(F(x,y) \wedge F(x,y') \to y \sim_Y y') \qquad (F \text{ is } \textit{single-valued});$$
$$\mathsf{P} \models \forall x(x \sim_X x \to \exists y F(x,y)) \qquad (F \text{ is } \textit{total}).$$

We are now in position to formulate the definition of a realizability topos.

Definition 4.3.3. The *realizability topos* $\mathsf{RT}(\mathcal{A})$ *of* $\mathcal{A}$ is the category defined as follows. An object X is a pair $(|X|, \sim_X)$ with $|X|$ a set and $\sim_X$ a partial equivalence relation on $|X|$. A *morphism* in $\mathsf{RT}(\mathcal{A})$ from X to Y is an isomorphism class in $P(|X| \times |Y|)$ of functional relations from X to Y. If $f \colon X \to Y$ is such a morphism and F is an element of the isomorphism class f, then we say that the functional relation F *represents* f.

Remark 4.3.4. We extend our convention from Remark 4.2.5 to variables $x, x', x'', \ldots$ and objects $X = (|X|, \sim_X)$.

We should of course verify that this is a valid definition, i.e. that this really defines a category. We will do so shortly. First, we introduce a useful lemma.

Lemma 4.3.5. *Let X and Y be two objects of $\mathsf{RT}(\mathcal{A})$ and let F and G be two functional relations from X to Y. If $F \leq G$, then F and G are isomorphic. Hence, they define the same morphism in $\mathsf{RT}(\mathcal{A})$.*

Proof. This is most easily proved using the Soundness Theorem. Thus, we argue informally in intuitionistic predicate logic. Suppose $F \leq G$, viz. $\mathsf{P} \models \forall xy(F(x,y) \to G(x,y))$. We wish to show that $G \leq F$ holds. To this end, suppose $G(x,y)$ is the case. By strictness, we find $x \sim_X x$. By totality of F, we find $F(x,y')$ for some y'. Hence, $G(x,y')$, because $F \leq G$. As G is single-valued, we get $y \sim_Y y'$. Finally, F is relational, so $F(x,y)$, as desired. ∎

Proposition 4.3.6. $\mathsf{RT}(\mathcal{A})$ *is a category.*

Proof. For every object X, define $\mathrm{id}_X \colon X \to X$ as the isomorphism class of $\sim_X$. Symmetry and transitivity of $\sim_X$ ensure that $\sim_X$ is indeed a functional relation from X to X. Given morphisms $X \to Y$ and $Y \to Z$ represented by F and G, respectively, define their composition as the isomorphism class of $[\exists y(F(x,y) \wedge G(y,z))] \in \mathsf{P}(|X| \times |Z|)$. It is now not hard to use the Soundness Theorem to verify that composition is well-defined, associative and that composition with id_X does nothing. ∎

Example 4.3.7. The realizability topos $\mathsf{RT}(\mathcal{K}_1)$ over Kleene's first model is the first and most well known example of a realizability topos. It is commonly known as the *Effective Topos* and denoted here by Eff.

Example 4.3.8. In the upcoming chapters we will mainly be interested in $\mathsf{RT}(\mathcal{S})$: the realizability topos over Scott's graph model.

4.4 Properties of realizability toposes

It will be convenient to characterize the isomorphisms in $\mathsf{RT}(\mathcal{A})$. It is also a nice exercise in dealing with functional relations.

Lemma 4.4.1. *Let $f \colon X \to Y$ be a morphism of $\mathsf{RT}(\mathcal{A})$ represented by F. Then f is an isomorphism if and only if*

$$\mathsf{P} \models \forall x'x'y(F(x,y) \wedge F(x',y) \to x \sim_X x'); \tag{4.4.1}$$

$$\mathsf{P} \models \forall y(y \sim_Y y \to \exists x F(x,y)). \tag{4.4.2}$$

Proof. Assume first that f is an isomorphism with inverse g represented by G. Then,

$$\mathsf{P} \models \forall xx'(\exists y(F(x,y) \wedge G(y,x')) \leftrightarrow x \sim_X x'); \tag{$*$}$$

$$\mathsf{P} \models \forall yy'(\exists x(G(y,x) \wedge F(x,y')) \leftrightarrow y \sim_Y y'). \tag{$**$}$$

We use the Soundness Theorem. Equation (4.4.2) is immediately obtained from $(**)$. Now reason informally in intuitionistic logic. Suppose we have $F(x,y) \wedge F(x',y)$. By strictness of F, we get $y \sim_Y y$, so by $(**)$ we obtain $G(y,x'')$ for some x''. Thus, we have $G(y,x'') \wedge F(x,y) \wedge F(x',y)$, so that $(*)$ yields $x \sim_X x'' \wedge x' \sim_X x''$. So by transitivity and symmetry, $x \sim_X x'$, as desired.

Conversely, assume F satisfies (4.4.1) and (4.4.2). One easily checks that $G(y,x) = F(x,y)$ is a functional relation from Y to X. Using (4.4.1) and (4.4.2) and the fact that

F is a functional relation, it is straightforward to verify that $(*)$ and $(**)$ hold for this G. Hence, F represents an isomorphism. ■

Remark 4.4.2. Later (c.f. Lemma 4.4.7 and Example 4.5.1), we shall see that f is a monomorphism/epimorphism if and only if 4.4.1/4.4.2 holds. Of course, once we know that $\mathsf{RT}(\mathcal{A})$ is a topos, the lemma above follows from this. We prove it here already as it is convenient for showing that $\mathsf{RT}(\mathcal{A})$ is indeed a topos.

Given the fact that the objects of $\mathsf{RT}(\mathcal{A})$ are *sets* with a partial equivalence relation, it seems natural to ask ourselves how we may relate a morphism $X \to Y$ of $\mathsf{RT}(\mathcal{A})$ to functions from $|X|$ to $|Y|$. The following lemmas shed some light on these matters.

Lemma 4.4.3. *Let X and Y be objects of $\mathsf{RT}(\mathcal{A})$ and let $f\colon |X| \to |Y|$ be any function. If f satisfies*

$$\mathsf{P} \models \forall x x'(x \sim_X x' \to f(x) \sim_Y f(x')), \tag{4.4.3}$$

then f induces a morphism from X to Y represented by

$$F_f = [x \sim_X x \wedge f(x) \sim_Y y] \in \mathsf{P}(|X| \times |Y|).$$

Proof. Observe that $[x \sim_X x \wedge f(x) \sim_Y y]$ is always strict and single-valued. By the additional requirement, it is also easily seen to be total and relational. ■

Moreover, in some cases, any morphism is induced by a function on sets.

Lemma 4.4.4. *Let X and Y be objects of $\mathsf{RT}(\mathcal{A})$ and write $S(|X|, |Y|)$ for the set of functions from $|X|$ to $|Y|$ that satisfy Equation* (4.4.3)*. Suppose $y \sim_Y y' = \emptyset$ for any distinct $y, y' \in |Y|$. Moreover, assume that $x \sim_X x \neq \emptyset$ for any $x \in |X|$. Then, there is a bijective correspondence between $S(|X|, |Y|)$ and $\mathsf{RT}(\mathcal{A})(X, Y)$ given by $f \mapsto [F_f]$, where $[F_f]$ denotes the isomorphism class of F_f.*

Proof. The previous lemma showed us that the above map is well-defined. It remains to show that it is bijective.

For surjectivity, suppose F represents any morphism $X \to Y$. We wish to define a function $f\colon |X| \to |Y|$ such that F_f and F are isomorphic in $\mathsf{P}(|X| \times |Y|)$. To this end, let $x \in |X|$ be arbitrary. Since $x \sim_X x$ is assumed to be non-empty, there is, by totality, a $y \in |Y|$ such that $F(x, y) \neq \emptyset$. Now if $F(x, y)$ and $F(x, y')$ are both non-empty, then by single-valuedness of Y, we must have that $y \sim_Y y' \neq \emptyset$, which implies that y are y' are equal. Thus, we may define a function $f\colon |X| \to |Y|$ by sending x to the unique $y \in Y$ with $F(x, y) \neq \emptyset$. To show that F and F_f represent the same morphism, we apply Lemma 4.3.5 and prove that $F \leq F_f$. But this is immediate by strictness of F and the our choice of f.

For injectivity, assume F_f and F_g represent the same morphism. That is, $\mathsf{P} \models \forall x(x \sim_X x \to f(x) \sim_Y g(x))$. Since $x \sim_X x$ is non-empty for any $x \in |X|$, we find that $f(x) \sim_Y g(x)$ is non-empty for any $x \in |X|$. Hence, by our assumption on Y, we have $f(x) = g(x)$ for any $x \in |X|$. We conclude that $f = g$, as desired. ■

In the lemma above we had to assume that $x \sim_X x \neq \emptyset$ for any $x \in |X|$. The next lemma shows that we may generally do so.

Lemma 4.4.5. *Any object X of $\mathsf{RT}(\mathcal{A})$ is isomorphic to an object Y of $\mathsf{RT}(\mathcal{A})$ such that $y \sim_Y y \neq \emptyset$ for any $y \in |Y|$.*

Proof. Let X be an any object of $\mathsf{RT}(\mathcal{A})$. Define the set $X' = \{x \in |X| \mid x \sim_X x \neq \emptyset\}$, let $\sim$ be the restriction of $\sim_X$ to $X' \times X'$ and write $Y = (X', \sim)$. Observe that the inclusion $X' \to |X|$ satisfies Equation (4.4.3), so we have a morphism $i \colon Y \to X$ represented by $[x \sim x \wedge x \sim_X y] \cong [x \sim_X y]$. We use Lemma 4.4.1 to show that i is an isomorphism. By symmetry and transitivity we have (4.4.1). Moreover, (4.4.2) requires us to find an element of $\bigcap_{x\in|X|}(x \sim_X x \to \bigcup_{y\in X'} x \sim y)$. But one may take i; for $x \sim_X x$ is non-empty if and only if $x \in X'$. We conclude that X and Y are isomorphic. ∎

Proposition 4.4.6. *The category $\mathsf{RT}(\mathcal{A})$ has finite limits.*

Proof. For the terminal object, define $1 = (\{*\}, \sim)$ with $* \sim * = \mathbb{A}$. Let X be an arbitrary object of $\mathsf{RT}(\mathcal{A})$. We must show that there is a unique morphism from X to 1. By Lemma 4.4.5, we may assume that $x \sim_X x \neq \emptyset$ for any $x \in |X|$. So we can apply Lemma 4.4.4, and since there is exactly one function from $\{*\}$ to $|X|$, there is a unique morphism from X to 1.

If X and Y are two objects, then we define their product as $X \times Y = (|X| \times |Y|, \sim_{X\times Y})$ where $(x, y) \sim_{X\times Y} (x', y')$ is defined as $[(x \sim_X x') \wedge (y \sim_Y y')]$. One easily shows that $\sim_{X\times Y}$ is symmetric and transitive. The projections $\pi_0 \colon X \times Y \to X$ and $\pi_1 \colon X \times Y \to Y$ are represented by $[x \sim_X x' \wedge y \sim_Y y] \in \mathsf{P}(|X| \times |Y| \times |X|)$ and $[y \sim_Y y' \wedge x \sim_X x] \in \mathsf{P}(|X| \times |Y| \times |Y|)$, respectively. Given morphisms $f \colon Z \to X$ and $g \colon Z \to Y$ represented by F and G, we have an arrow $\langle f, g\rangle \colon Z \to X \times Y$ represented by $[F(z, x) \wedge G(z, y)]$. Now, $\pi_0\langle f, g\rangle$ is represented by $[\exists x' y(F(z, x') \wedge G(z, y) \wedge x' \sim_X x \wedge y \sim_Y y]$ and this is smaller than $[F(z, x)]$ because F is relational. So by Lemma 4.3.5, we see that $\pi_0\langle f, g\rangle = f$. Similarly, $\pi_1\langle f, g\rangle = g$. It is not much harder to check that $\langle f, g\rangle$ is unique with this property.

Next, we show that $\mathsf{RT}(\mathcal{A})$ has equalizers. Let $f, g \colon X \to Y$ be two arrows respectively represented by F and G. Construct the object $E = (|X|, \sim_E)$ with partial equivalence relation $[x \sim_E x'] = [x \sim_X x' \wedge \exists y(F(x, y) \wedge G(x, y))]$. The identity on $|X|$ induces a morphism $i \colon E \to X$ represented by $\sim_E$. The composite fi is represented by

$$\begin{aligned}
[\exists x'(x \sim_E x' \wedge F(x', y))] &\cong [\exists x'(x \sim_X x' \wedge F(x, y) \wedge \exists y'(F(x, y') \wedge G(x, y')))] \\
&\cong [\exists x' y'(x \sim_X x' \wedge F(x, y) \wedge F(x, y') \wedge G(x, y'))] \\
&\cong [\exists x' y'(x \sim_X x' \wedge G(x, y) \wedge F(x, y') \wedge G(x, y'))] \\
&\cong [\exists x'(x \sim_E x' \wedge G(x', y))],
\end{aligned}$$

where the penultimate equivalence holds because F and G are relational and single-valued. So we see see that $fi = gi$. Now suppose $h \colon Z \to X$ is an arrow such that $fh = gh$. We must show that it factors uniquely through i. If H represents h, then we have

$$\mathsf{P} \models \forall yz(\exists x(H(z, x) \wedge F(x, y)) \leftrightarrow \exists x(H(z, x) \wedge G(x, y))).$$

Using this, one verifies that H is also a functional relation from Z to E and that it yields the required unique factorisation of h through i. We conclude that $\mathsf{RT}(\mathcal{A})$ has finite limits. ■

Lemma 4.4.7. *A functional relation F from X to Y represents a monomorphism if and only if F satisfies Equation* (4.4.1).

Proof. Suppose F represents an arrow f and consider the kernel pair of f. By our description of finite limits above, the pullback is given by the object $E = (|X| \times |X|, \approx)$ where $(x_0, x_1) \approx (x_0', x_1')$ is defined as $[x_0 \sim_X x_0' \wedge x_1 \sim_X x_1' \wedge \exists y(F(x_0, y) \wedge F(x_1, y))]$ for $x_0, x_1, x_0', x_1' \in |X|$.

Now, f is monic if and only if the map $X \to E$ represented by $[x \sim_X x_0 \wedge x \sim_X x_1]$ (for $x, x_0, x_1 \in |X|$) is an isomorphism. Using Lemma 4.4.1, it is then not hard to show (using the Soundness Theorem) that the latter is equivalent to F satisfying Equation (4.4.1). ■

Now that we have characterized monomorphisms, let us turn to subobjects.

Definition 4.4.8. A *strict relation φ on an object* X is an element $\varphi \in \mathsf{P}(|X|)$ such that φ is strict and relational, i.e.

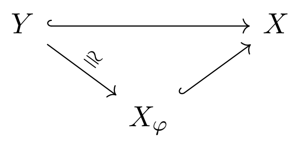
$$\mathsf{P} \models \forall x(\varphi(x) \to x \sim_X x);$$
$$\mathsf{P} \models \forall x x'(\varphi(x) \wedge x \sim_X x' \to \varphi(x')).$$

For a strict relation φ on X, let us write X_φ for the object $(|X|, \sim_\varphi)$ where $x \sim_\varphi x' = [x \sim_X x' \wedge \varphi(x)]$.

Lemma 4.4.9. *Let φ be a strict relation on X. The map $X_\varphi \to X$ represented by $\sim_\varphi$ is a mono. Moreover, if $f\colon Y \to X$ is a mono, represented by F, then Y is isomorphic as a subobject of X to X_ψ where $\psi = [\exists y(F(y, x))]$.*

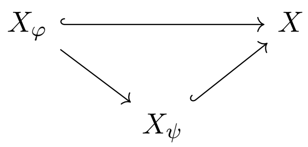
Further, given two strict relations φ and ψ on X, we have $X_\varphi \leq X_\psi$ as subobjects if and only if $\varphi \leq \psi$ as $\mathcal{P}(\mathbb{A})$-valued predicates on $|X|$.

Proof. The first claim is routine to check. For the second one, suppose F represents a monomorphism from Y to X. Note that $\varphi = [\exists y F(y, x)]$ is a strict relation on X. One readily verifies that F is a functional relation from Y to X_φ. Of course, it satisfies Equation (4.4.1) and by our choice of φ Equation (4.4.2) also holds. Therefore, the map $Y \to X_\varphi$ is an iso. Lastly, the triangle



clearly commutes. Hence, the subobject Y is isomorphic to X_φ.

Finally, observe that we have a commutative triangle



if and only if $\mathsf{P} \models \forall x(\varphi(x) \to \psi(x))$ which is equivalent to $\varphi \leq \psi$ (c.f. Example 4.1.9). ■

It will be convenient to describe pullbacks of subobjects.

Lemma 4.4.10. *Let φ be a strict relation on X and let $f\colon Y \to X$ be any morphism. The pullback of $X_\varphi \hookrightarrow X$ along f is given by the strict relation $[\exists x(F(y,x) \wedge \varphi(x))]$ on Y, where F is a representative of f.*

Proof. By our description of finite limits, the pullback $X_\varphi \times_X Y$ is given by the object $(|X| \times |Y|, \approx)$ with

$$(x,y) \approx (x',y') = [x \sim_X x' \wedge y \sim_Y y' \wedge \exists x''(x'' \sim_X x \wedge \varphi(x'') \wedge F(y,x''))].$$

The map $X_\varphi \times_X Y \hookrightarrow Y$ is represented by

$$(x,y,y') \mapsto [(x,y) \approx (x,y) \wedge y \sim_Y y'].$$

A routine calculation shows that

$$\mathsf{P} \models \forall xy((x,y) \approx (x,y) \leftrightarrow F(y,x) \wedge \varphi(x)),$$

so that $X_\varphi \times_X Y \hookrightarrow Y$ is also represented by

$$(x,y,y') \mapsto [F(y,x) \wedge \varphi(x) \wedge y \sim_Y y'].$$

By Lemma 4.4.9, the subobject $X_\varphi \times_X Y$ of Y is isomorphic to Y_ψ with

$$\psi(y') = [\exists xy(F(y,x) \wedge \varphi(x) \wedge y \sim_Y y'] \cong [\exists x(F(y',x) \wedge \varphi(x))],$$

as desired. ■

Having a good understanding of subobjects and their pullbacks allows us to prove the following.

Proposition 4.4.11. *The category $\mathsf{RT}(\mathcal{A})$ has power objects.*

Proof. Intuitively, a power object of X should be all the subobjects of X. Hence, for an object X we define its power object $\mathcal{P}X$ as the object $\big(\mathcal{P}(\mathbb{A})^{|X|}, \sim_{\mathcal{P}X}\big)$ with

$$\varphi \sim_{\mathcal{P}X} \varphi' = [\forall x(\varphi(x) \to x \sim_X x) \wedge \forall xx'(x \sim_X x' \wedge \varphi(x) \to \varphi(x')) \wedge \forall x(\varphi(x) \leftrightarrow \varphi'(x))],$$

where φ, φ' are variable of type $\mathcal{P}(\mathbb{A})^{|X|}$. Observe that for $\mathcal{P}(\mathbb{A})$-valued predicates φ and ψ on $|X|$, the relation $\varphi \sim_{\mathcal{P}X} \psi$ expresses that φ is a strict relation on X and that

$\varphi \cong \psi'$. It is easy to show that $\sim_{\mathcal{P}X}$ is symmetric and transitive (check that φ' is also a strict relation in the internal logic of P).

Next, we must define a subobject $\in_X$ of $X \times \mathcal{P}X$. We may do so by giving a strict relation on $|X| \times \mathcal{P}(\mathbb{A})^{|X|}$: define $\in_X$ as $x \in_X \varphi = [\varphi \sim_{\mathcal{P}X} \varphi \wedge \varphi(x)]$ and observe that it is indeed a strict relation.

Finally, suppose we are given a subobject $(X \times Y)_\psi \hookrightarrow X \times Y$. We are tasked with showing that there is a unique arrow $f \colon Y \to \mathcal{P}X$ such that

$$\begin{array}{ccc} (X \times Y)_\psi & \longrightarrow & \in_X \\ \big\downarrow & \lrcorner & \big\downarrow \\ X \times Y & \xrightarrow{\mathrm{id}_X \times f} & X \times \mathcal{P}X \end{array}$$

is a pullback.

Suppose first that F represents such an arrow. By the previous lemma and the second part of Lemma 4.4.9, this means precisely that

$$\mathsf{P} \models \forall xy(\psi(x,y) \leftrightarrow \exists \varphi(F(y,\varphi) \wedge x \in_X \varphi)).$$

Combining this with the fact that F is a functional relation allows us to deduce:

$$\mathsf{P} \models \forall y\varphi(F(y,\varphi) \to y \sim_Y y \wedge \varphi \sim_{\mathcal{P}X} \varphi \wedge \forall x(x \in_X \varphi \leftrightarrow \psi(x,y))). \qquad (*)$$

Now one may show that $[y \sim_Y y \wedge \varphi \sim_{\mathcal{P}X} \varphi \wedge \forall x(x \in_X \varphi \leftrightarrow \psi(x,y))]$ is itself a functional relation from Y to $\mathcal{P}X$ (use ψ for verifying totality). Moreover, by Lemma 4.3.5 and $(*)$ it is unique up to isomorphism (and so the arrow it represents is unique). We conclude that $\mathsf{RT}(\mathcal{A})$ has power objects. ■

Combining the above, we obtain the following result.

Theorem 4.4.12. *The category* $\mathsf{RT}(\mathcal{A})$ *is a topos.*

For future reference, we describe the subobject classifier of $\mathsf{RT}(\mathcal{A})$.

Lemma 4.4.13. *The subobject classifier* Ω *of* $\mathsf{RT}(\mathcal{A})$ *is given by* $(\mathcal{P}(\mathbb{A}), \leftrightarrow)$*, where* $\leftrightarrow$ *is the Heyting bi-implication in* $\mathcal{P}(\mathbb{A})^{\{*\}} \cong \mathcal{P}(\mathbb{A})$*. We will write* p, q, r *for variables of type* $|\Omega| = \mathcal{P}(\mathbb{A})$*. The true map* $t \colon 1 \hookrightarrow \Omega$ *is represented by* $[p \leftrightarrow \top]$*. Given a subobject* X_φ *of an object* X*, the unique map* $\chi_\varphi \colon X \to \Omega$ *such that*

$$\begin{array}{ccc} X_\varphi & \longrightarrow & 1 \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle t} \\ X & \xrightarrow{\chi_\varphi} & \Omega \end{array}$$

is a pullback is given by $[p \leftrightarrow \varphi(x) \wedge x \sim_X x]$

Proof. This follows from our construction of power objects and the fact that $\mathcal{P}1$ is a subobject classifier. ■

4.5 Logic in realizability toposes

In this section we show that the internal logic of realizability toposes is governed by the internal logic of the tripos and the partial equivalence relations.

Suppose $\mathcal{L}$ is a many-sorted first-order language with relation symbols, function symbols and equality. An interpretation of $\mathcal{L}$ in $\mathsf{RT}(\mathcal{A})$ consists of the following:

(i) for every sort σ of $\mathcal{L}$, an object $[\![\sigma]\!] = X$ of $\mathsf{RT}(\mathcal{A})$;

(ii) for every relation symbol R of $\mathcal{L}$ of type $(\sigma_1, \ldots, \sigma_n)$, a subobject $[\![R]\!]$ of $[\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!]$ in $\mathsf{RT}(\mathcal{A})$;

(iii) for every function symbol f of $\mathcal{L}$ of type $(\sigma_1, \ldots, \sigma_n \to \tau)$, a morphism $[\![f]\!] \colon [\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!] \to [\![\tau]\!]$ in $\mathsf{RT}(\mathcal{A})$.

Given such an interpretation, terms and formulas of $\mathcal{L}$ are interpreted in $\mathsf{RT}(\mathcal{A})$ in the standard way using inductive clauses (c.f. Section 4.2). A term t of $\mathcal{L}$ of type $(\sigma_1, \ldots, \sigma_n \to \tau)$ is interpreted as a morphism $[\![t]\!] \colon [\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!] \to [\![\tau]\!]$. A formula φ with free variables of sorts $\sigma_1, \ldots, \sigma_n$ is interpreted as a subobject $[\![\varphi]\!]$ of $[\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!]$. Recall that subobjects of $\mathsf{RT}(\mathcal{A})$ are essentially strict relations.

In particular, if t and s are terms of the same sort τ, then $[\![t = s]\!]$ is given by the equalizer of

$$\mathrm{dom}([\![t]\!]) \times \mathrm{dom}([\![s]\!]) \underset{[\![s]\!]\pi_s}{\overset{[\![t]\!]\pi_t}{\rightrightarrows}} [\![\tau]\!],$$

where π_t and π_s are the obvious projections. If $[\![t]\!]$ and $[\![s]\!]$ are represented by functional relations F and G, respectively, then by our description of finite limits in $\mathsf{RT}(\mathcal{A})$, we see that $[\![t = s]\!]$ is represented by the strict relation $[\exists y(F(x,y) \wedge G(x,y))]$.

We can show that the logical structure on the lattice of subobjects is essentially given by the logical structure of the tripos P. To this end, let X be an arbitrary object of $\mathsf{RT}(\mathcal{A})$. The greatest subobject of X is represented by the strict relation $[x \sim_X x] \in \mathsf{P}(|X|)$ on X. The least subobject is given by the strict relation $\bot \in \mathsf{P}(|X|)$. Given two strict relations φ and ψ on X, one easily verifies that the meet and join of these subobjects are respectively given by strict relations $[\varphi(x) \wedge \psi(x)]$ and $[\varphi(x) \vee \psi(x)]$. With implication one has to be a bit careful, because $[\varphi(x) \to \psi(x)]$ is not generally strict. One can check that $[x \sim_X x \wedge (\varphi(x) \to \psi(x))]$ does the job, however.

Further, suppose $[\![\varphi(x,y)]\!]$ is a subobject of $X \times Y$, represented by the strict relation $R_\varphi(x,y) \in \mathsf{P}(|X| \times |Y|)$. Then $[\![\exists y \varphi(x,y)]\!]$ is represented by $[\exists y R_\varphi(x,y)] \in \mathsf{P}(|X|)$. Moreover, $[\![\forall y \varphi(x,y)]\!]$ is the strict relation $[x \sim_X x \wedge \forall y(y \sim_Y y \to R_\varphi(x,y))] \in \mathsf{P}(|X|)$.

Finally, we can interpret higher-order logic in $\mathsf{RT}(\mathcal{A})$ as well, because we can interpret a higher-order language with quantifiers $\exists X, \forall X$ intending to range over subsets of sort σ, by letting them range over the power object $\mathcal{P}([\![\sigma]\!])$ (recall Proposition 4.4.11).

Example 4.5.1. In any topos, a map $f \colon X \to Y$ is a (regular) epimorphism if and only if $[\![\exists x(f(x) = y]\!]$ is the greatest subobject of Y. From the above paragraphs, we can see that a morphism $f \colon X \to Y$ of $\mathsf{RT}(\mathcal{A})$, represented by F, is a (regular) epimorphism if and only if Equation (4.4.2) holds.

Example 4.5.2. Given a strict relation φ on an object X, we see that its double negation is interpreted as:

$$\begin{aligned}[\![\neg\neg\varphi(x)]\!] &= [\![\neg\varphi(x) \to \bot]\!] \\ &= [x \sim_X x \wedge (\neg\varphi(x) \to \bot)] \\ &= [x \sim_X x \wedge \neg\neg\varphi(x)].\end{aligned}$$

4.6 Assemblies in realizability toposes

The category $\mathsf{RT}(\mathcal{A})$ can be seen as a generalization of the category of assemblies.

Proposition 4.6.1. *The category* $\mathsf{Asm}(\mathcal{A})$ *is equivalent to a full subcategory of* $\mathsf{RT}(\mathcal{A})$.

Proof. Let us write $\mathcal{D}$ for the full subcategory of $\mathsf{RT}(\mathcal{A})$ on those objects X such that $[x \sim_X x] \neq \emptyset$ for any $x \in |X|$ and $[x \sim_X x'] = \emptyset$ for any distinct $x, x' \in |X|$.

For an assembly $X = (|X|, E_X)$, write $X' = (|X|, \sim_X)$ for the object of $\mathsf{RT}(\mathcal{A})$ with

$$[x \sim_X x'] = \begin{cases} E_X(x) & \text{if } x = x'; \\ \emptyset & \text{else.} \end{cases}$$

Note that X' is an object of $\mathcal{D}$.

Define a functor $I\colon \mathsf{Asm}(\mathcal{A}) \to \mathcal{D}$ on objects by $X \mapsto X'$ and on arrows by $f \mapsto [F_f]$, where F_f is as in Lemma 4.4.3. Note that if $f \in \mathsf{Asm}(\mathcal{A})(X, Y)$, then f satisfies Equation (4.4.3) because f is tracked, so by Lemma 4.4.3 F_f indeed represents a morphism from X' to Y'.

It is easy to see that I preserves identities. Moreover, recall that if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of assemblies, then $I(g)I(f)$ is represented by $[\exists y(F_f(x, y) \wedge F_g(y, z))] \cong [E_X(x) \wedge E_Y(f(x)) \wedge E_Z(g(f(x))]$. But this is isomorphic (since f is tracked) to $[E_x(x) \wedge E_Z(g(f(x)))]$, which in turn represents $I(gf)$. Thus I is a indeed a functor.

Moreover, Lemma 4.4.4 tells us that I is fully faithful. Lastly, given an object X of $\mathcal{D}$, the pair $(|X|, E_X)$ with $E_X(x) = [x \sim_X x]$ is easily seen to be an assembly that gets mapped to X by I. Thus, the functor I is also (essentially) surjective and we may conclude that it is an equivalence. ■

We will often identify the full subcategory above with $\mathsf{Asm}(\mathcal{A})$ and simply write $X = (|X|, E_X)$ for an object of the full subcategory.

We can also characterize the assemblies in $\mathsf{RT}(\mathcal{A})$ logically. Recall that on any topos, we have the double negation ($\neg\neg$) Lawvere-Tierney topology.

Proposition 4.6.2. *An object X of* $\mathsf{RT}(\mathcal{A})$ *is $\neg\neg$-separated if and only if X is isomorphic to an assembly.*

Proof. Firstly, recall that X is $\neg\neg$-separated if and only if the diagonal $\Delta\colon X \to X \times X$ is $\neg\neg$-closed [Joh02a, Lemma A4.3.6(a)]. The diagonal, as a subobject, is represented by the strict relation $[x \sim_X x']$ on $X \times X$. Its $\neg\neg$-closure is, c.f. Example 4.5.2, given by the strict relation $[x \sim_X x \wedge x' \sim_X x' \wedge \neg\neg(x \sim_X x')]$. Hence, by the third part of Lemma 4.4.9, an object X is $\neg\neg$-separated if and only if

$$\mathsf{P} \models \forall x x'(x \sim_X x \wedge x' \sim_X x' \wedge \neg\neg(x \sim_X x') \to x \sim_X x'). \tag{$*$}$$

Suppose first that $X = (|X|, E_X)$ is an assembly. Then (recall Example 4.2.7)

$$[x \sim_X x'] = \begin{cases} E_X(x) & \text{if } x = x'; \\ \emptyset & \text{else;} \end{cases} \quad \text{and} \quad [\neg\neg(x \sim_X x')] = \begin{cases} \mathbb{A} & \text{if } x = x'; \\ \emptyset & \text{else.} \end{cases} \tag{$**$}$$

Therefore, it is easy to see that $[\forall x x'(x \sim_X x \wedge x' \sim_X x' \wedge \neg\neg(x \sim_X x') \to x \sim_X x')]$ has a realizer.

Now suppose X is an object satisfying $(*)$. We construct an assembly Y such that X and Y are isomorphic. First of all, assume without loss of generalization that $x \sim_X x \neq \emptyset$ for any $x \in |X|$. Then the relation $\approx$ on $|X| \times |X|$ defined as

$$x \approx x' \Leftrightarrow x \sim_X x' \neq \emptyset$$

is an equivalence relation. Write q for the quotient map $|X| \to |X|/\approx$. Define the assembly $Y = (|X|/\approx, E)$ with $E(y) = \bigcup_{x \in y} x \sim_X x$. Then by Lemma 4.4.3 the map q induces a functional relation F_q from X to Y. It remains to show that F_q represents an isomorphism. The fact that F_q satisfies Equation (4.4.2) is easily checked by our choice of E. To show that F_q also validates Equation (4.4.1), consider the following argument. By construction of Y and $(**)$ we see that

$$[\forall x x' y(x \sim_X x \wedge q(x) \sim_Y y \wedge x' \sim_X x' \wedge q(x') \sim_Y y \to \neg\neg(x \sim_X x))]$$

is realized by the i combinator. As $(*)$ also holds, we see that F_q indeed satisfies Equation (4.4.1). This completes the proof. ∎

In later sections we will often calculate products or exponentials in $\mathsf{RT}(\mathcal{A})$ of assemblies. Therefore, the following propositions will be quite useful.

Proposition 4.6.3. *The category* $\mathsf{Asm}(\mathcal{A})$ *is an exponential ideal in* $\mathsf{RT}(\mathcal{A})$.

Proof. The $\neg\neg$-separated objects of any topos form an exponential ideal, see Lemma A4.4.3(ii) in [Joh02a]. ∎

Proposition 4.6.4. *The functor I has a left adjoint. Consequently, I preserves all finite limits.*

Proof. Let $(X, \sim)$ be an arbitrary object of $\mathsf{RT}(\mathcal{A})$. Construct the object $(X, \approx)$ with $[x \approx x'] = [x \sim x \wedge x' \sim x' \wedge \neg\neg(x \sim x')]$. Using $(*)$ from Proposition 4.6.2, it is easily seen that $(X, \approx)$ is an assembly. Using Lemma 4.4.3, one quickly verifies that the identity on X induces a morphism $\eta_X \colon (X, \sim) \to (X, \approx)$.

Now suppose that $f \colon (X, \sim) \to Y$ is a morphism in $\mathsf{RT}(\mathcal{A})$ with Y an assembly. We are to prove that there is a unique morphism $\tilde{f} \colon (X, \approx) \to Y$ such that $\tilde{f}\eta_X = f$.

By Lemma 4.4.5 we may assume that $x \sim x \neq \emptyset$ for any $x \in X$. Moreover, since Y is an assembly, we can use Lemma 4.4.4. Hence, f is induced by a function f from X to $|Y|$. Similarly, any morphism from $(X, \approx)$ to Y will also come from a function. Thus, by Lemma 4.4.3, we see that we are done if

$$\mathsf{P} \models \forall x x'(x \approx x' \to f(x) \sim_Y f(x')). \qquad (**)$$

Now we know that

$$\mathsf{P} \models \forall x x'(x \sim x' \to f(x) \sim_Y f(x')).$$

By definition of $\approx$ and the Soundness Theorem, we thus obtain

$$\mathsf{P} \models \forall x x'(x \approx x' \to f(x) \sim_Y f(x) \wedge f(x') \sim_Y f(x') \wedge \neg\neg(f(x) \sim_Y f(x'))).$$

But Y is $\neg\neg$-separated, so this implies $(**)$, as we wished. ■

Proposition 4.6.5. *The functor I preserves regular epimorphisms.*

Proof. This follows from Example 4.5.1 and Lemma 3.2.2 as writing out Equation (4.4.2) yields the criterion described in Lemma 3.2.2. ■

Proposition 4.6.6. *The functor I preserves the natural numbers object.*

Proof. By Corollary A2.5.11 in [Joh02a], any natural numbers object in a topos is $\neg\neg$-separated. Thus, by Proposition 3.2.5, it suffices to show that $\mathsf{RT}(\mathcal{A})$ has a natural numbers object. By [Fre72, Theorem 5.44] (or [Joh77, Corollary 6.15]), it suffices to exhibit an object X of $\mathsf{RT}(\mathcal{A})$ such that $1 + X$ and X are isomorphic. However, in $\mathsf{Asm}(\mathcal{A})$, the natural numbers object is such an object (one may verify this directly or consult [Fre72, Theorem 5.43] or [Joh02a, Lemma A2.5.5] and [Joh02b, Theorem 5.43]). Since any functor preserves isomorphisms, we are done. ■

Chapter 5

Order-discrete Objects

In this chapter we introduce the order-discrete objects of $\mathsf{RT}(\mathcal{S})$. These objects reappear in subsequent chapters; we meet them again when investigating choice axioms and when examining homotopy in $\mathsf{RT}(\mathcal{S})$.

The notion of order-discreteness is (as far as I am aware) original (although order-discrete *modest sets* already make an appearance in [Lie99]). The treatment is similar to that of the discrete objects in Section 3.2.6 of [Oos08]. In particular, Proposition 5.3.1 is very similar to [Oos08, Proposition 3.2.19(iii)]. Moreover, the exposition has benefited from comments by my supervisor Jaap van Oosten; the (formulation of the) first two definitions are due to him, for example.

Throughout this chapter, let us write $\overline{0} = \{0\}$ and $\overline{1} = \{1\}$.

5.1 The Sierpiński assembly and order-discrete objects

Definition 5.1.1. The *Sierpiński assembly* Σ is the assembly (over $\mathcal{S}$) with $|\Sigma| = \{0, 1\}$ and $E_\Sigma(0) = \{\emptyset\}$ and $E_\Sigma(1) = \{\overline{1}\}$.

In later chapters Σ will play an important role as a dominance. For now, we use it to define the order-discrete objects.

Definition 5.1.2. An object X of $\mathsf{RT}(\mathcal{S})$ is called *order-discrete* if the diagonal $X \xrightarrow{\delta} X^\Sigma$ is an isomorphism.

Lemma 5.1.3. *Let $\mathcal{C}$ be any cartesian closed category and let X be any object of $\mathcal{C}$. If $A \twoheadrightarrow B$ is an epi in $\mathcal{C}$, then $X^B \hookrightarrow X^A$ is a mono.*

This proof was communicated to me by my supervisor Jaap van Oosten.

Proof. Write G for the functor $X^{(-)}\colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ and $\overline{G}$ for the functor $X^{(-)}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$. Observe that we have the following chain of natural isomorphisms:

$$\mathcal{C}^{\mathrm{op}}(GA, B) = \mathcal{C}(B, GA) \cong \mathcal{C}(B \times A, X) \cong \mathcal{C}(A, X^B) \cong \mathcal{C}(A, \overline{G}B).$$

Hence, $G \vdash \overline{G}$. Now, if $A \to B$ is epic in $\mathcal{C}$, then it is monic in $\mathcal{C}^{\mathrm{op}}$, so $\overline{G}(A \to B) = X^B \to X^A$ is a mono in $\mathcal{C}$, as right adjoints preserves finite limits. ∎

Corollary 5.1.4. *An object X is order-discrete if and only if δ is epic.*

Proof. It suffices to show that the diagonal δ is always monic, but this follows by the lemma as the unique map $\Sigma \to 1$ is easily seen to be epic. ■

Definition 5.1.5. An assembly over a pca $\mathcal{A}$ is called *partitioned* if it is isomorphic to an assembly X such that each realizing set $E_X(x)$ is a singleton.

Lemma 5.1.6. *Let $\mathcal{A}$ be any pca. If P is a partitioned assembly and X is any object of $\mathsf{RT}(\mathcal{A})$, then the exponential X^P is isomorphic to the object $(|X|^{|P|}, \approx)$ where*

$$f \approx g = [\forall p(E_P(p) \to f(p) \sim_X g(p))].$$

Proof. See [Oos08, pp. 136–137]. ■

We can characterize the order-discrete objects in terms of their realizers.

Proposition 5.1.7. *An object X is order-discrete if and only if there is $A \in \mathbb{S}$ such that for any $x, x' \in |X|$: if $U \in [x \sim_X x]$ and $V \in [x' \sim_X x']$ with $U \subseteq V$, then $AUV \in [x \sim_X x']$.*

Proof. Suppose first that X is order-discrete. We construct the desired element $A \in \mathbb{S}$. Assume we have $x, x' \in |X|$ and $U \in [x \sim_X x], V \in [x' \sim_X x']$ with $U \subseteq V$. By Lemma 5.1.6 we have $X^\Sigma \cong (|X|^{\{0,1\}}, \approx)$. Define $f\colon \{0,1\} \to |X|$ by $f(0) = x$ and $f(1) = x'$. The map $H\colon \mathbb{S}^2 \to \mathbb{S}$ defined as

$$(W, W') \mapsto \{\langle 0, n\rangle \mid n \in W\} \cup \{\langle 2, n\rangle \mid n \in W'\}$$

is easily seen to be continuous. Write $G = \mathrm{graph}(H)$. We claim that $GUV \in [f \approx f]$. Indeed, $GUV = \{\langle 0, n\rangle \mid n \in U\} \cup \{\langle 2, n\rangle \mid n \in V\}$, so that $GUV\emptyset = U \in [f(0) \sim_X f(0)]$. Moreover, $GUV\overline{1} = \{n \in \mathbb{N} \mid \langle 0, n\rangle \in U \text{ or } \langle 2, n\rangle \in V\} = U \cup V = V \in [f(1) \sim_X f(1)]$, since $e_0 = \emptyset$ and $e_2 = \overline{1}$ (recall Example 2.3.4) and $U \subseteq V$. Thus, $GUV \in [f \approx f]$. Now let $R \in \mathbb{S}$ be a realizer of the fact that δ is epic. Then $R(GUV)$ is an element of $[\forall p(E_\Sigma(p) \to x_0 \sim_X f(p))]$ for some $x_0 \in |X|$. Thus, $R(GUV)\emptyset \in [x_0 \sim_X x]$ and $R(GUV)\overline{1} \in [x_0 \sim_X x']$. Finally, let $t, s \in \mathbb{S}$ respectively realize transitivity and symmetry of $\sim_X$. Then, we see that

$$\lambda^* uv.t(\mathsf{p}(s(R(Guv)\emptyset))(R(Guv)\overline{1}))$$

is the desired element A.

Conversely, suppose we have an $A \in \mathbb{S}$ as in the proposition. Let $f\colon \{0,1\} \to |X|$ be arbitrary. By Corollary 5.1.4, it suffices to show that from an element of $[f \approx f]$, we can continuously find an $x \in |X|$ and an element of $[\forall p(E_\Sigma(p) \to x \sim_X f(p))]$. Let $F \in [f \approx f]$. Then $F\emptyset \in [f(0) \sim_X f(0)]$, $F\overline{1} \in [f(1) \sim_X f(1)]$ and $F\emptyset \subseteq F\overline{1}$, so $A(F\emptyset)(F\overline{1}) \in [f(0) \sim_X f(1)]$ and $A(F\emptyset)(F\emptyset) \in [f(0) \sim_X f(0)]$. Hence, if we set $x = f(0)$, then the graph of the continuous function

$$\emptyset \mapsto A(F\emptyset)(F\emptyset), \quad W \neq \emptyset \mapsto A(F\emptyset)(F\overline{1})$$

is the desired element. ■

Corollary 5.1.8. *An assembly X is order-discrete if and only if the existence of realizers $U \in E_X(x)$ and $V \in E_X(y)$ with $U \subseteq V$ implies that x and y are equal.*

Proof. Immediate. ■

Definition 5.1.9. Let $\mathcal{A}$ be any pca. Write $\nabla(2)$ for the assembly $(\{0,1\}, E)$ where $E(x) = \mathbb{A}$ for any $x \in \{0,1\}$. An object X of a realizability topos $\mathsf{RT}(\mathcal{A})$ is called *discrete* if the diagonal $X \to X^{\nabla(2)}$ is an isomorphism.

By [Oos08, Proposition 3.2.18], this definition is in line with the definition of a discrete assembly from Definition 3.2.6.

We can use our characterization of order-discrete objects to see that any order-discrete object is discrete (as one might expect from the terminology).

Proposition 5.1.10. *Any order-discrete object is discrete.*

Proof. This follows from Proposition 5.1.7 and [Oos08, Corollary 3.2.20]. The statement in [Oos08] carries over to arbitrary realizability toposes. ■

5.2 Closure properties of order-discrete objects

Proposition 5.2.1. *The class of order-discrete objects is closed under finite products and forms an exponential ideal in $\mathsf{RT}(\mathcal{S})$.*

Proof. Let X and Y be order-discrete objects of $\mathsf{RT}(\mathcal{S})$. Then we have natural isomorphisms:

$$(X \times Y)^\Sigma \cong X^\Sigma \times Y^\Sigma \cong X \times Y,$$

so $X \times Y$ is again order-discrete.

Now if Z is any object, then we have natural isomorphisms:

$$\left(X^Z\right)^\Sigma \cong \left(X^\Sigma\right)^Z \cong X^Z,$$

so X^Z is order-discrete. ■

The notion of internally projective objects will play a role in the following proposition. We define it here.

Definition 5.2.2. An object P in a topos $\mathcal{E}$ is called *internally projective* if the endofunctor $(-)^P \colon \mathcal{E} \to \mathcal{E}$ preserves epimorphisms.

Lemma 5.2.3. *An object is internally projective if and only if it is isomorphic to a partitioned assembly.*

Proof. This follows from [Oos08, pp. 135–137]. The proofs generalize to an arbitrary realizability topos. ■

Proposition 5.2.4. *The order-discrete objects are closed under subobjects and quotients in $\mathsf{RT}(\mathcal{S})$.*

Proof. By the lemma, Σ is internally projective. Furthermore, it is easily checked that the unique map $\Sigma \to 1$ is epic. The claim now follows from [HRR90, Lemma 2.3 and Lemma 2.8]. ■

The object Σ is strongly indecomposable, in the following sense.

Proposition 5.2.5. *For any two objects X and Y of $\mathsf{RT}(\mathcal{S})$, we have $(X+Y)^\Sigma \cong X^\Sigma + Y^\Sigma$.*

Proof. Let X and Y be arbitrary objects. Construct their coproduct as $(|X|+|Y|, \sim_{X+Y})$ where

$$[(i,x) \sim_{X+Y} (j,y)] = \begin{cases} \{\overline{0}\} \wedge [x \sim_X y] & \text{if } i=j=0; \\ \{\overline{1}\} \wedge [x \sim_Y y] & \text{if } i=j=1; \\ \emptyset & \text{else.} \end{cases}$$

The coprojections yield maps $X^\Sigma \to (X+Y)^\Sigma$ and $Y^\Sigma \to (X+Y)^\Sigma$, so that we get a morphism $i\colon X^\Sigma + Y^\Sigma \to (X+Y)^\Sigma$. Since we are working in a topos (in particular, a distributive category), i is a mono. Therefore, it suffices to prove that it is epic.

Render $(X+Y)^\Sigma$ as $((|X|+|Y|)^{\{0,1\}}, \approx)$ using Lemma 5.1.6. We claim that if $f\colon \{0,1\} \to |X|+|Y|$ is such that $[f \approx f]$ is non-empty, then f factors through $|X|$ or $|Y|$. To this end, suppose $R \in [f \approx f]$ and assume without loss of generalization that $f(0) \in |X|$. Then $\overline{0} = \mathsf{p}_0(R\emptyset) \subseteq \mathsf{p}_0(R\overline{1}) \in \{\overline{0}, \overline{1}\}$. Hence, $\mathsf{p}_0(R\overline{1}) = \overline{0}$, since $\overline{0} \not\subseteq \overline{1}$, so $f(1) \in |X|$ as well. This proves our claim. From our claim, it is straightforward to prove that i is also an epi, and thus an isomorphism. ■

Corollary 5.2.6. *The class of order-discrete objects is closed finite coproducts in $\mathsf{RT}(\mathcal{S})$.*

Proof. It is immediate from the previous proposition that the diagonal $X+Y \to (X+Y)^\Sigma$ is an isomorphism if X and Y are order-discrete. ■

5.3 Order-discrete reflection

This section shows that the order-discrete objects give rise to an adjunction.

Proposition 5.3.1. *The full subcategory of $\mathsf{RT}(\mathcal{S})$ on order-discrete objects is reflective, viz. the inclusion functor has a left adjoint.*

Proof. We are tasked with the following: for every object X, we must construct an order-discrete object X_{od} and an map $\eta_X\colon X \to X_{od}$ such that for every map $f\colon X \to Y$ with Y order-discrete, f factors uniquely through η_X.

Let X be an arbitrary object of $\mathsf{RT}(\mathcal{S})$. In the construction of X_{od} we use the coding of finite sequences in $\mathcal{S}$ (recall Section 2.2). We will write $[U_1, \ldots, U_n]$ for the code of the sequence $(U_1, \ldots, U_n)$ of elements of $\mathbb{S}$ and write $*$ for concatenation.

We construct X_{od} as follows: $X_{od} = (|X|, \approx_X)$ where $x \approx_X x'$ is the set of codes of sequences

$$[I_0, U_0, V_0, A_1, I_1, U_1, V_1, \ldots, A_n, I_n, U_n, V_n]$$

with $n \geq 0$ such that for every $0 \leq i \leq n$:

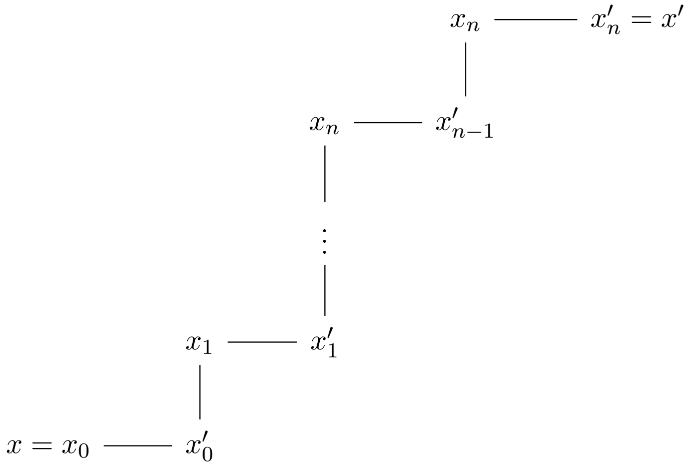
(i) $I_i \in \{\overline{0}, \overline{1}\}$ and $I_i = \overline{0} \Leftrightarrow U_i \subseteq V_i$ and $I_i = \overline{1} \Leftrightarrow V_i \subsetneq U_i$;

(ii) there are $x_0, \ldots, x_n, x'_0, \ldots, x'_n \in |X|$ such that for all $i + 1 \leq n$, it holds that $U_i \in [x_i \sim_X x_i], V_i \in [x'_i \sim_X x'_i]$ and $A_{i+1} \in [x'_i \sim_X x_{i+1}]$.

(iii) $x_0 = x$ and $x'_n = x'$;

We will say that the sequence above has length $n + 1$.

We think of a realizer of $x \approx_X x'$ as a path through realizers with source x and target x'. One might picture such a realizer as:



The horizontal steps are inclusions or reserve inclusion (as indicated by the I_i), while the vertical steps are realized by the A_i.

We should check that X_{od} is indeed an object of $\mathsf{RT}(\mathcal{S})$, i.e. that $\approx_X$ is a partial equivalence relation in the tripos P. Transitivity holds because if $\sigma \in [x \approx_X x']$ and $\tau \in [x' \approx_X x'']$, then from σ we effectively obtain $U \in [x' \sim_X x']$ and $\sigma * [U] * \tau \in [x \approx_X x'']$. Pictorially, we stack two paths on top of each other. Symmetry holds, because we can simply reverse the sequence and flip each I_i.

We proceed by showing that X_{od} is indeed order-discrete. We use Proposition 5.1.7. Note that the map $\mathbb{S}^2 \to \mathbb{S}$ given by

$$(U, V) \mapsto [\overline{0}, U, V]$$

is continuous. Further, if we write A for its graph, then $AUV \in [x \approx_X x']$ whenever $U \in [x \approx x]$ and $V \in [x' \approx_X x']$ with $U \subseteq V$. Thus, X_{od} is order-discrete, as desired.

Next, we define a map $\eta_X \colon X \to X_{od}$. Observe that

$$\mathsf{P} \models \forall x x' (x \sim_X x' \to x \approx_X x'), \qquad (*)$$

because from an element $A \in [x \sim_X x']$, we can effectively obtain realizers $U \in [x \sim_X x]$ and $V \in [x' \sim_X x']$, and from these we get $[\overline{0}, U, U, A, \overline{0}, V, V]$ as an element of $x \approx_X x'$.

Pictorially,

$$\begin{array}{ccc} & x' & \text{———} \; x' \\ & | & \\ x \; \text{———} & x & \end{array}$$

Thus, by Lemma 4.4.3, the identity on $|X|$ induces a morphism $\eta_X : X \to X_{od}$. It is easily established that η_X satisfies Equation (4.4.2), so η_X is epic.

Finally, let $f : X \to Y$ be any with Y order-discrete. We must show that f factors through η_X. That the factorization is unique follows from the fact that η_X is epic. Let F be a functional relation representing f. We show that F is also a functional relation from X_{od} to Y. Since F is a functional relation from X to Y, it remains to show that F is strict, total and relational with respect to $\approx_X$. The first two are easily checked, because $(*)$ (see above) and $\mathsf{P} \models \forall x(x \approx_X x \to x \sim_X)$ hold and because F is strict and total w.r.t. $\sim_X$.

For the proof that F is relational w.r.t $\approx_X$, we describe how to recursively define the required algorithm. Since Y is order-discrete, let A be as in Proposition 5.1.7. Assume $F(x, y)$ holds. Given $\sigma \in [x \approx_X x']$, inspect its length $n + 1$.

If $n = 0$, then we have $\sigma = [I, U, V]$ with $U \in [x \sim_X x], V \in [x' \sim_X x']$ and $I \in \{\overline{0}, \overline{1}\}$. Using totality and strictness of F w.r.t $\sim_X$ and $\sim_Y$, we obtain $y', y'' \in |Y|$ such that $F(x, y')$ and $F(x', y'')$ and realizers $U' \in [y' \sim_Y y']$ and $V' \in [y'' \sim_Y y'']$. By single-valuedness of F, we get $y \sim_Y y'$. Since application is monotone and $U \subseteq V$ or $V \subseteq U$, we have $U' \subseteq V'$ or $V' \subseteq U'$, respectively. If $0 \in I$, then $AU'V' \in [y' \sim_Y y'']$ and if $1 \in I$, then $AV'U' \in [y'' \sim_Y y']$. Since F is relational w.r.t $\sim_Y$, we get $F(x', y')$. As we also had $y \sim_Y y'$, we get $F(x', y)$, as desired.

If $n > 0$, then $\sigma = \tau * [A_n, I_n, U_n, V_n]$ with $\tau \in [x \approx_X x'_{n-1}]$, $U_n \in [x_n \sim_X x_n]$, $V_n \in [x'_n \sim_X x'_n]$ and $A_n \in [x'_{n-1} \sim_X x_n]$. By induction, we get $F(x'_{n-1}, y)$ from τ. Using A_n we obtain $F(x_n, y)$. Finally, using a similar argument as above, we use I_n, U_n and V_n to get $F(x'_n, y) = F(x', y)$, as we wished.

Thus, F is a functional relation from X_{od} to Y, completing our proof. ■

Chapter 6

Arithmetic in $\mathsf{RT}(\mathcal{S})$

In this chapter we examine some of the logical properties of the realizability topos $\mathsf{RT}(\mathcal{S})$ over Scott's graph model. We look at first and second order arithmetic. It will turn out that first order arithmetic is simply (classically) true arithmetic, but second order arithmetic will prove to be more interesting.

6.1 First order arithmetic

This section is based on [Oos08, Section 3.1], but suitably adapted from Eff to $\mathsf{RT}(\mathcal{S})$. The final theorem in the section is my own.

We start out with an easy but useful lemma.

Lemma 6.1.1. *Let $f\colon \mathbb{N}^k \to \mathbb{N}$ be any function with $k \geq 1$. Then we have $F \in \mathbb{S}$ such that $F\{n_1\}\dots\{n_k\} = \{f(n_1,\dots,n_k)\}$ for every $(n_1,\dots,n_k) \in \mathbb{N}^k$. Similarly, we can represent any function $\mathbb{N}^k \to \mathbb{S}$ in the pca $\mathcal{S}$.*

Proof. Given $f\colon \mathbb{N}^k \to \mathbb{N}$, define $F'\colon \mathbb{S}^k \to \mathbb{S}$ by mapping

$$(U_1,\dots,U_k) \mapsto \bigcup_{n_1 \in U_1,\dots,n_k \in U_k} \{f(n_1,\dots,n_k)\}.$$

Then F' is continuous by design, so we have $F \in \mathbb{S}$ with $FV_1\dots V_k = F'(V_1,\dots,V_k)$ for any $V_1,\dots,V_k \in \mathbb{S}$. Clearly then, $F\{n_1\}\dots\{n_k\} = \{f(n_1,\dots,n_k)\}$ for any $n \in \mathbb{N}$. The second claim is proved similarly. ■

Proposition 6.1.2. *The natural numbers object N in $\mathsf{RT}(\mathcal{S})$ is the assembly $(\mathbb{N}, \{\{-\}\})$.*

Proof. We prove this by showing that $(\mathbb{N}, \{\{-\}\})$ is isomorphic to the standard natural numbers object $N = (\mathbb{N}, E_N)$ with $E_N(n) = \{\overline{n}\}$ (here $\overline{n}$ is the nth Curry numeral in the pca $\mathcal{S}$; recall Proposition 3.2.5 and Proposition 4.6.6). Since both objects are assemblies, we may work in $\mathsf{Asm}(\mathcal{S})$.

Observe that the function $\mathbb{N} \to \mathbb{S}$ given by $n \mapsto \overline{n}$ is tracked by the previous lemma. Therefore, the function $\mathrm{id}_{\mathbb{N}}$ is tracked as a morphism from $(\mathbb{N}, \{\{-\}\})$ to N. It remains to show that it is tracked as a morphism $N \to (\mathbb{N}, \{\{-\}\})$. To this end, let R be a primitive recursive combinator in our pca $\mathcal{S}$ (recall Proposition 2.2.6) and let $S \in \mathbb{S}$ be such that $S\{n\} = \{n+1\}$ for any $n \in \mathbb{N}$ (possible by the lemma above). Then $\mathrm{id}_{\mathbb{N}} \colon N \to (\mathbb{N}, \{\{-\}\})$ is tracked by $\mathsf{R}\{0\}\lambda^* xy.Sy$, as one easily verifies. ∎

Remark 6.1.3. In light of the above proposition, we will henceforth write $\overline{n}$ for the singleton $\{n\}$, with $n \in \mathbb{N}$, when working with the pca $\mathcal{S}$.

The language of arithmetic is the first-order language with function symbols for each primitive recursive function. In every cartesian closed category with a natural numbers object N, there is a standard interpretation of the primitive recursive functions: a primitive recursive function $\mathbb{N}^k \to \mathbb{N}$ is interpreted as a morphism $N^k \to N$.

Using Lemma 6.1.1, one can show that N^k is isomorphic to the assembly $(\mathbb{N}^k, E)$ where $E((n_1, \ldots, n_k)) = \overline{\langle n_1, \ldots, n_k \rangle}$, with $\langle - \rangle$ a bijection from $\mathbb{N}^k$ to $\mathbb{N}$.

In $\mathsf{Asm}(\mathcal{S})$, morphisms from N^k to N are determined by functions $\mathbb{N}^k \to \mathbb{N}$. Thus, in $\mathsf{Asm}(\mathcal{S})$ the primitive recursive functions from N^k to N are simply the primitive recursive functions from $\mathbb{N}^k$ to $\mathbb{N}$. In $\mathsf{RT}(\mathcal{S})$, a primitive recursive function $f \colon \mathbb{N}^k \to \mathbb{N}$ is represented by the functional relation $(\vec{n}, m) \mapsto \{\mathsf{p}\overline{\langle \vec{n} \rangle}\overline{m} \mid F(\vec{n}) = m\}$ from N^k to N (remember Lemma 4.4.3).

By the results of Section 4.5, we can inductively interpret a formula $\varphi(x_1, \ldots, x_k)$ of arithmetic as a strict relation on N^k. For $\vec{n} \in \mathbb{N}^k$, we have:

$$\begin{aligned}
[\![\bot]\!](\vec{n}) &= \emptyset; \\
[\![t = s]\!](\vec{n}) &= \{\overline{\langle \vec{n} \rangle} \mid (t = s)(\vec{n}) \text{ is true}\}; \\
[\![\varphi \wedge \psi]\!](\vec{n}) &= \{\mathsf{p}UV \mid U \in [\![\varphi]\!](\vec{n}), V \in [\![\psi]\!](\vec{n})\}; \\
[\![\varphi \vee \psi]\!](\vec{n}) &= \{\mathsf{pk}U \mid U \in [\![\varphi]\!](\vec{n})\} \cup \{\mathsf{p}\bar{\mathsf{k}}V \mid V \in [\![\psi]\!](\vec{n})\}; \\
[\![\varphi \to \psi]\!](\vec{n}) &= \{\mathsf{p}\overline{\langle \vec{n} \rangle}U \mid \text{for all } V \in [\![\varphi]\!](\vec{n}),\ UV \in [\![\psi]\!](\vec{n})\} \\
[\![\exists y \varphi]\!](\vec{n}) &= \bigcup_{m \in \mathbb{N}} [\![\varphi]\!](\vec{n}, m); \\
[\![\forall y \varphi]\!](\vec{n}) &= \{\mathsf{p}\overline{\langle \vec{n} \rangle}U \mid \text{for all } m,\ U\overline{m} \in [\![\varphi]\!](\vec{n}, m)\}.
\end{aligned}$$

Theorem 6.1.4. *For any formula $\varphi(x_1, \ldots, x_n)$ and $n_1, \ldots, n_k \in \mathbb{N}$, there exists an element of $[\![\varphi]\!](n_1, \ldots, n_k)$ if and only if $\varphi(n_1, \ldots, n_k)$ is (classically) true. Thus, the first order arithmetic of $\mathsf{RT}(\mathcal{S})$ is (classically) true arithmetic.*

Proof. We use induction on the logical complexity of $\varphi(x_1, \ldots, x_n)$. We will only treat implication and universal quantification; the other cases are fairly easy.

Suppose $(\varphi \to \psi)(\vec{n})$ is true. Then, either $\varphi(\vec{n})$ is false or $\varphi(\vec{n})$ and $\psi(\vec{n})$ are both true. In the first case, we have $[\![\varphi]\!](\vec{n}) = \emptyset$ by induction hypothesis, so any element of $\mathbb{S}$ is an element of $[\![\varphi \to \psi]\!](\vec{n})$. In the second case, we have an element $W \in [\![\psi]\!](\vec{n})$ by induction hypothesis, so $\lambda^* x.W$ is in $[\![\varphi \to \psi]\!](\vec{n})$. Conversely, assume we have $U \in [\![\varphi \to \psi]\!](\vec{n})$. If $[\![\varphi]\!](\vec{n}) = \emptyset$, then by induction hypothesis, $\varphi(\vec{n})$ is false, so $(\varphi \to \psi)(\vec{n})$ is true. If we have $V \in [\![\varphi]\!](\vec{n})$, then $UV \in [\![\psi]\!](\vec{n})$, so by induction hypothesis, $\psi(\vec{n})$ is true and thus, $(\varphi \to \psi)(\vec{n})$ is true.

Suppose $(\forall y\varphi)(\vec{n})$ is true. Then by induction hypothesis, we have for each $m \in \mathbb{N}$ some $V_m \in [\![\varphi]\!](\vec{n}, m)$. Thus, using the axiom of choice, we can construct a function $v\colon \mathbb{N} \to \mathbb{S}$ such that $v(m) \in [\![\varphi]\!](\vec{n}, m)$. By Lemma 6.1.1, we have $V \in \mathbb{S}$ such that $V\overline{m} = v(m)$ for each $m \in \mathbb{N}$. Hence, $V \in [\![\forall y\varphi]\!](\vec{n})$. For the converse, suppose that $U \in [\![\forall y\varphi]\!](\vec{n})$. Then, we have an element $U\overline{m}$ of $[\![\varphi(\vec{n}, m)]\!]$ for any $m \in \mathbb{N}$. By induction hypothesis, $\varphi(\vec{n}, m)$ is true for any $m \in \mathbb{N}$. Hence, $(\forall y\varphi)(\vec{n})$ is true, as desired. ■

6.2 Axiom of choice and modest sets

The material in this section is due to [Lie99] (c.f. [Lie04, Proposition 2.3.4]). This section will be devoted to the following principle:

$$(\mathrm{AC}_{X,Y})\ \forall x{:}X\exists y{:}Y\,\varphi(x,y) \to \exists f{:}Y^X\forall x{:}X\,\varphi(x, f(x))$$

(*Axiom of Choice for X with respect to Y*).

We will only concern ourselves with objects of $\mathsf{RT}(\mathcal{S})$ that are modest sets. We will see that $\mathrm{AC}_{X,Y}$ holds for a surprisingly large class of objects. In particular, it holds for all objects of finite type (i.e. N, N^N, $N \times N$, $N^{N\times N}$, $N^{(N^N)}$, etc.).

Definition 6.2.1. We say that a modest set (in $\mathsf{RT}(\mathcal{S})$) has the *join-property* if it is isomorphic to a modest set X whose realizing sets are closed under (binary) joins, viz. if $U, V \in E_X(x)$, then $U \cup V \in E_X(x)$.

Example 6.2.2. The natural numbers object N is an example of a modest set with the join-property.

Proposition 6.2.3. *Let X and Y be modest sets. If X has the join-property and Y is order-discrete, then $\mathrm{AC}_{X,Y}$ holds in $\mathsf{RT}(\mathcal{S})$.*

Proof. First of all, recall from Corollary 5.1.8 what it means for an assembly to be order-discrete. We first prove the following claim.

Claim: Let $U \in \mathbb{S}$ be an element mapping realizers of X to realizers of Y, i.e. if $V \in E_X(x)$ for some $x \in |X|$, then $UV \in E_Y(y)$ for some $y \in |Y|$. Then U is extensional, in the sense that: if $V, W \in E_X(x)$ for some $x \in |X|$, then $UV, UW \in E_Y(y)$ for some $y \in |Y|$.

Proof of claim: Suppose $V, W \in E_X(x)$. Since X has the join-property, we find $V \cup W \in E_X(x)$. Hence, $U(V \cup W) \in E_Y(y)$ for some $y \in |Y|$. Moreover, there are $y, y' \in |Y|$ such that $UV \in E_Y(y')$ and $UW \in E_Y(y'')$. Note that $UV, UW \subseteq U(V \cup W)$ by monotonicity of the application. So by order-discreteness of Y, we have $y = y' = y''$, as desired. □

For ease of notation, we will assume that φ has only x and y as free variables. We may render $[\![\forall x{:}X\exists y{:}Y\,\varphi(x,y)]\!]$ as follows:

$$\{U \in \mathbb{S} \mid \text{for all } x \in |X|, V \in E_X(x), \text{ there is some } y \in |Y| \text{ with } \mathsf{p}_0(UV) \in E_Y(y) \text{ and } \mathsf{p}_1(UV) \in [\![\varphi(x,y)]\!]\}. \qquad (*)$$

Moreover $[\![\exists f{:}Y^X \forall x{:}X \varphi(x, f(x))]\!]$ is given by

$$\{W \in \mathbb{S} \mid \mathsf{p}_0 W \text{ tracks some } f\colon X \to Y \text{ and for all } x \in |X|, V \in E_X(x) \text{ we have } \mathsf{p}_1 WV \in [\![\varphi(x, f(x))]\!]\}. \qquad (**)$$

We are to find an element R of $\mathbb{S}$ such that for every U in $(*)$, the element RU is in $(**)$. The idea is that any element of $(*)$ yields a tracker of some map $X \to Y$. We claim that $R = \lambda^* u.\mathsf{p}(\lambda^* v.\mathsf{p}_0(uv))(\lambda^* v.\mathsf{p}_1(uv))$ is a suitable choice.

To verify our choice of R, suppose U is an element of $(*)$. By the claim and the fact that Y is modest, we see that for every $x \in |X|$, there is a *unique* $y \in |Y|$ such that for every $V \in E_X(x)$, we have $\mathsf{p}_0(UV) \in E_Y(y)$ and $\mathsf{p}_1(UV) \in [\![\varphi(x, y)]\!]$. Thus, this induces a function $f\colon |X| \to |Y|$, by sending $x \in |X|$ to this unique $y \in |Y|$.

We must show that $\mathsf{p}_0(RU)$ tracks f and that $\mathsf{p}_1(RU)V \in [\![\varphi(x, f(x))]\!]$ for any element $V \in E_X(x)$. But this follows easily by construction of f and the fact that $\mathsf{p}_0(RU)V = \mathsf{p}_0(UV)$ and $\mathsf{p}_1(RU)V = \mathsf{p}_1(UV)$. This completes our proof. ■

Proposition 6.2.4. *If X and Y are modest sets with the join-property, then their product $X \times Y$ and coproduct $X + Y$ have the join-property as well.*

Proof. Recall our choice of pairing combinators from Remark 2.3.5. The proof of the proposition boils down to the fact that $[U \cup U', V \cup V'] = [U, V] \cup [U', V']$, which is easily checked. ■

Proposition 6.2.5. *The class of modest sets with the join-property is an exponential ideal in* $\mathsf{Mod}(\mathcal{S})$.

Proof. Suppose Y is a modest set with the join-property. Suppose $f\colon X \to Y$ and let U, U' track f. We must show that $U \cup U'$ also tracks f. Let $V \in E_X(x)$. Then one checks that $(U \cup U')V = UV \cup U'V$. Since $UV, U'V \in E_X(f(x))$, we also have that $UV \cup U'V \in E_X(f(x))$. Thus, $U \cup U'$ tracks f, as we wished. ■

Corollary 6.2.6. *The scheme $\mathrm{AC}_{X,Y}$ holds in $\mathsf{RT}(\mathcal{S})$ for all finite types X and Y.*

Proof. This follows from the fact N has the join-property and is order-discrete and the fact that the order-discrete modest sets with the join property are closed under binary products and form an exponential ideal in $\mathsf{RT}(\mathcal{S})$. ■

6.3 Some logical principles involving finite types

The principles we consider here are the same as in [Oos08, Section 3.1]. The final proposition appears in [Lie99] and [Lie04, Proposition 2.3.4]. In this section we will be concerned with the following principles:

(CT) $\forall f{:}N^N \exists e{:}N \forall x{:}N \exists z{:}N (T(e,x,z) \wedge U(z) = f(x))$
(*Church's Thesis*);

(WCN) $\forall f{:}N^N \exists x{:}N \varphi(f,x) \to \forall f{:}N^N \exists xy{:}N \forall g{:}N^N (\overline{f}y = \overline{g}y \to \varphi(g,x))$,
where $\overline{f}y = \overline{g}y$ is short for $\forall z{:}N(z < y \to f(z) = g(z))$
(*Weak Continuity for Numbers*);

(BP) $\forall F{:}N^{(N^N)} \forall f{:}N^N \exists x{:}N \forall g{:}N^N (\overline{f}x = \overline{g}x \to F(f) = F(g))$
(*Brouwer's Principle*).

Using Kleene's primitive recursive predicates T and U, Church's Thesis asserts that any function on the natural numbers is given by a partial recursive function. Brouwer's Principle says that any function from N^N to N is continuous (where N^N has the Baire topology and N the discrete topology). Finally, WCN is both a continuity principle and a choice principle: it states that any total relation from N^N to N is determined by some initial values of the input function.

In Eff, both CT and BP are true [Oos08, Proposition 3.1.6] (the latter by the Kreisel-Lacombe-Shoenfield theorem), while WCN is not. In $\mathsf{RT}(\mathcal{K}_2)$ both WCN and BP are valid [Oos08, Proposition 4.3.4]. Given the topological nature of the pca $\mathcal{S}$, one might expect BP to hold in $\mathsf{RT}(\mathcal{S})$ as well. Consider the following "proof":

> The object $N^{(N^N)}$ is given by the assembly $(\{F \mid F\colon (\mathbb{N}^\mathbb{N}, E) \to N\}, E')$ where $E'(F)$ is the set of $U \in \mathbb{S}$ tracking F. For such U, we have that $UV = \overline{F(f)}$ for any $f\colon \mathbb{N} \to \mathbb{N}$ and $V \in E(f)$.
>
> Let F be any element of $\left|N^{(N^N)}\right|$ and suppose f is any function from $\mathbb{N}$ to $\mathbb{N}$. Let $U, V \in \mathbb{S}$ track F and f, respectively. By continuity of the application, find a finite subset $p \subseteq V$ such that $F(f) \in Up$. Now find $m \in \mathbb{N}$ such that if $g \in \mathbb{N}^\mathbb{N}$ and $\overline{f}m = \overline{g}m$, then $p \subseteq W$ for any $W \in \mathbb{S}$ tracking g. Now let $g \in \mathbb{N}^\mathbb{N}$ be arbitrary such that $\overline{f}m = \overline{g}m$ and let $W \in \mathbb{S}$ track g. Then, $F(f) \in Up \subseteq UW = \{F(g)\}$. Hence, $F(f) = F(g)$.

The problem with this argument is that it does not show that such m and p can be found continuously. In fact, we have the following result.

Proposition 6.3.1. *The principles* CT*,* BP *and* WCN *are all invalid in* $\mathsf{RT}(\mathcal{S})$*.*

Proof. First of all, observe that N^N is given by the assembly $(\{f \mid f\colon N \to N\}, E)$ where $E(f)$ is the set of trackers of f. In light of Lemma 6.1.1 this object is actually $(\mathbb{N}^\mathbb{N}, E)$. From this description of N^N, we see that CT indeed does not hold (take f to be any non-recursive function).

We proceed by showing that WCN and BP are equivalent in the presence of choice. Then, we prove that BP does not hold, finishing the proof.

Suppose WCN holds. Let F be of sort $N^{(N^N)}$ and define $\varphi(f,x)$ as $F(f)=x$. Then $\forall f{:}N^N \exists x{:}N \varphi(f,x)$. By WCN we obtain: $\forall f{:}N^N \exists y{:}N \forall g{:}N^N(\overline{f}y = \overline{g}y \to F(g) = F(f))$. Hence, WCN implies BP.

For the converse, we need choice. Assume BP and suppose $\forall f{:}N^N \exists x{:}N \varphi(f,x)$. By $AC_{N^N,N}$, we find $F{:}N^{(N^N)}$ such that $\forall f{:}N^N \varphi(f, F(f))$. Let $f{:}N^N$ be arbitrary. By BP, there is $y{:}N$ such that $\forall g{:}N^N(\overline{f}y = \overline{g}y \to F(f) = F(g))$. Hence, we obtain WCN, as $\forall g{:}N^N(\overline{f}y = \overline{g}y \to \varphi(g,x))$ with $x = F(f)$ holds.

Now assume for a contradiction that BP holds. Let us write z for the zero map from $\mathbb{N}$ to $\mathbb{N}$. By BP, we have:

$$\forall F{:}N^{(N^N)} \exists x{:}N \forall g{:}N^N(\overline{g}x = \overline{z}x \to F(z) = F(g)).$$

Using $\mathrm{AC}_{N^{(N^N)},N}$ on this, we find $\Phi\colon N^{\left(N^{(N^N)}\right)}$ such that

$$\forall F{:}N^{(N^N)} \forall g{:}N^N(\overline{g}\Phi(F) = \overline{z}\Phi(F) \to F(z) = F(g)). \qquad (*)$$

In others words, Φ is a modulus of continuity functional.

Now, any cartesian closed category with a natural numbers object is a model of HA^ω. Further, one easily verifies that $\mathsf{Mod}(\mathcal{S})$ (and therefore $\mathsf{RT}(\mathcal{S})$) is a model of extensional HA^ω. By [TD88, Corollary 6.11, Chapter 9], $(*)$ is inconsistent with extensional HA^ω; completing our proof. ■

6.4 Second order arithmetic

In this section we turn to second order arithmetic in realizability toposes. We first start by working in a general realizability topos over a non-trivial pca. In the second subsection we specialize to $\mathsf{RT}(\mathcal{S})$. The first section, up to Proposition 6.4.4, is a generalization of [Oos08, Section 3.1.1]; Proposition 6.4.4 and further are my own results.

6.4.1 In general realizability toposes

Throughout this section, $\mathcal{A}$ will denote a non-trivial pca with underlying set $\mathbb{A}$. For second order arithmetic, we need to consider the power object $\mathcal{P}N$ of N. By Proposition 4.4.11 and the fact that N is an assembly, we see that $\mathcal{P}N$ is given by the object $\left(\mathcal{P}(\mathbb{A})^{\mathbb{N}}, \sim_{\mathcal{P}N}\right)$ with

$$\varphi \sim_{\mathcal{P}N} \psi = [\forall n(\varphi(n) \to \{\overline{n}\}) \wedge \forall n(\varphi(n) \leftrightarrow \psi(n))].$$

We wish to give a more convenient representation of $\mathcal{P}N$. Therefore, we prove the following simple lemma.

Lemma 6.4.1. *Suppose $(X, \sim)$ is any object of $\mathsf{RT}(\mathcal{A})$ and let X' be any subset of X. Write $\sim'$ for the restriction of $\sim$ to X'. If the map $X \times X' \to \mathcal{P}(\mathbb{A})$ given by $(x, x') \mapsto [x \sim x']$ is a total relation (in the sense of the tripos), then $(X, \sim)$ is isomorphic to $(X', \sim')$.*

Proof. One quickly verifies that $\sim$ represents an arrow $(X', \sim') \to (X, \sim)$. Also observe that $\sim$ is a strict, relational and single-valued function from $X \times X'$ to $\mathcal{P}(\mathbb{A})$. Thus, if it is also total, then $\sim$ represents a morphism $(X, \sim) \to (X', \sim')$. Checking that these arrows are inverses means proving that $[x \sim y] \cong \bigcup_{z \in X'} [x \sim z] \wedge [z \sim y]$ and $[x' \sim' y'] \cong \bigcup_{z \in X} [x' \sim z] \wedge [z \sim y']$ hold for any $x, y \in X$ and $x', y' \in X'$. By Lemma 4.3.5, it suffices to show that in both cases $\geq$ holds. But this is immediate, because $\sim$ is transitive. ■

Lemma 6.4.2. *The object $\mathcal{P}N$ is isomorphic to the object $(P, \approx)$ where*

$$P = \{\varphi \colon \mathbb{N} \to \mathcal{P}(\mathbb{A}) \mid \textit{for all } n \in \mathbb{N},\ \textit{if } a \in \varphi(n),\ \textit{then } \mathsf{p_1} a = \overline{n}\}.$$

and

$$\varphi \approx \psi = [\forall n(\varphi(n) \leftrightarrow \psi(n))].$$

Proof. We use the previous lemma. Let φ be any function $\mathbb{N} \to \mathcal{P}(\mathbb{A})$. Define the function $\widetilde{\varphi} \colon \mathbb{N} \to \mathcal{P}(\mathbb{A})$ by

$$n \mapsto \{\mathsf{p} a \overline{n} \in \mathbb{A} \mid a \in \varphi(n)\}$$

and note that $\widetilde{\varphi} \in P$. Observe that $\widetilde{\varphi}(n) \to \varphi(n)$ is realized by $\mathsf{p_0}$ for every $n \in \mathbb{N}$. Further, assuming we have a realizer of $\varphi \sim_{\mathcal{P}N} \varphi$, we have an element $s \in \mathbb{A}$ witnessing the strictness of φ. Hence, from a realizer of $\varphi \sim_{\mathcal{P}N} \varphi$ we effectively obtain a realizer of $\forall n(\varphi(n) \to \widetilde{\varphi}(n))$, namely $\lambda^* x.\mathsf{p} x(sx)$.

By the previous lemma, $\mathcal{P}N$ is isomorphic to $(P, \sim_{\mathcal{P}N})$. Lastly, by the condition on the elements of P, we see that $\mathcal{P}N$ is in fact isomorphic to $(P, \approx)$. ■

For the remainder of this section, let us use upper case variables X, Y, Z for second order variables and x, y, z for first order variables ranging over natural numbers. Further, we will write α, β, γ for elements of P.

In the proof above we already noted that $\mathsf{pii} \in \bigcap_{\alpha \in P} [\alpha \approx \alpha]$, so when quantifying over second order variables, we have that $[\forall X(X \approx X \to \varphi)]$ and $[\exists X(X \approx X \wedge \varphi)]$ are isomorphic to $[\forall X \varphi]$ and $[\exists X \varphi]$, respectively. The object $\mathcal{P}N$ is said to be *uniform*. As a consequence, we have the following proposition.

Proposition 6.4.3. *The following schemes*

(UP)	$\forall X \exists x \varphi(X, x) \to \exists x \forall X \varphi(X, x)$
	(Uniformity Principle)
(IP_X)	$(\neg\varphi \to \exists X \psi) \to \exists X(\neg\varphi \to \psi)$ *(with X not free in φ)*
	(Independence of Premiss for X)
($\mathrm{DNS}_{\exists X}$)	$\neg\neg\exists X \varphi \to \exists X \neg\neg\varphi$
	(Double Negation Shift for $\exists X$)

hold in $\mathsf{RT}(\mathcal{A})$.

Proof. For the Uniformity Principle, observe that by our remark,

$$[\![\forall X \exists x \varphi(X,x)]\!] = \bigcap_{\alpha \in P} \{\mathsf{p}\overline{n}a \in \mathbb{A} \mid a \in [\![\varphi(\alpha,n)]\!], n \in \mathbb{N}\}$$

and

$$[\![\exists x \forall X \varphi(X,x)]\!] = \left\{\mathsf{p}\overline{n}a \in \mathbb{A} \mid a \in \bigcap_{\alpha \in P} [\![\varphi(\alpha,n)]\!], n \in \mathbb{N}\right\},$$

from which it is clear that UP is realized by the i combinator.

Using Example 4.2.7 and the fact that $\mathcal{P}N$ is uniform, we see that IP is realized by $\lambda^* u.(\lambda^* v.u\mathsf{i})$. Finally, $\mathrm{DNS}_\exists$ is proved similarly. ■

We shall revisit the Independence of Premiss schemes in Proposition 6.4.9 and Corollary 6.4.10 and Lemma 7.1.5. The first two show that the scheme does not hold in second order arithmetic when quantifying over first order variables. The second is specific to total pcas and the realizer object.

The Uniformity Principle is the most remarkable of these three (it is classically absurd, for instance). It states that any total relation on $\mathcal{P}N \times N$ contains a number related to *any* subset. The principle has some interesting ramifications, that we list here.

Proposition 6.4.4. *The schemes*

$\neg\varphi \vee \neg\neg\varphi$	(Weak Law of Excluded Middle)
$\neg\neg(\varphi \vee \psi) \to \neg\neg\varphi \vee \neg\neg\psi$	(Double Negation Shift for $\vee$)
$\neg\neg\exists x\varphi \to \exists x\neg\neg\varphi$	(Double Negation Shift for $\exists x$)

are all invalid in $\mathsf{RT}(\mathcal{A})$.

Proof. We will use that, in first order arithmetic, $\varphi \vee \psi$ is equivalent to the formula $\exists x((x = 0 \to \varphi) \wedge (\neg x = 0 \to \psi))$.

Suppose for the sake of a contradiction that the Weak Law of Excluded Middle were true. Then, so would $\forall X(\neg 0 \in X \vee \neg\neg 0 \in X)$. Hence,

$$\begin{aligned}\forall X(\neg 0 \in X \vee \neg\neg 0 \in X) &\leftrightarrow \forall X \exists x((x = 0 \to \neg 0 \in X) \wedge (\neg x = 0 \to \neg\neg 0 \in X)) \\ &\to \exists x \forall X((x = 0 \to \neg 0 \in X) \wedge (\neg x = 0 \to \neg\neg 0 \in X)) \text{ (by UP)} \\ &\to \exists x((x = 0 \to \forall X \neg 0 \in X) \wedge (\neg x = 0 \to \forall X \neg\neg 0 \in X)) \\ &\leftrightarrow (\forall X \neg 0 \in X) \vee (\forall X \neg\neg 0 \in X).\end{aligned}$$

But the final formula is obviously absurd.

For the second scheme, take $\psi = \neg\varphi$ and note that $\neg\neg(\varphi \vee \neg\varphi)$ is intuitionistically true. Hence, Double Negation Shift for $\vee$ would imply the Weak Law of Excluded Middle.

Finally, we show that the third scheme implies the second, since:

$$\begin{aligned}\neg\neg(\varphi \vee \psi) &\leftrightarrow \neg\neg\exists x((x = 0 \to \varphi) \wedge (\neg x = 0 \to \psi)) \\ &\to \exists x(\neg\neg((x = 0 \to \varphi) \wedge (\neg x = 0 \to \psi))) \quad \text{(assuming the third scheme)} \\ &\to \exists x((x = 0 \to \neg\neg\varphi) \wedge (\neg x = 0 \to \neg\neg\psi)) \\ &\to \neg\neg\varphi \vee \neg\neg\psi.\end{aligned}$$

■

We now turn our attention to $\neg\neg$-stable subsets of N, viz. subsets X such that $\forall x(\neg\neg(x \in X) \to x \in X)$ is true in $\mathsf{RT}(\mathcal{A})$. We will abbreviate this formula by $\mathrm{Stab}(X)$.

We have following proposition, which should be compared with Proposition 6.4.8 in the case of $\mathcal{A} = \mathcal{S}$.

Proposition 6.4.5. *Not every subset is $\neg\neg$-stable, i.e. the sentence $\forall X \mathrm{Stab}(X)$ is not valid in $\mathsf{RT}(\mathcal{A})$.*

Proof. Recall the set P from Lemma 6.4.2. Define the functions $\alpha, \beta \in P$ by:

$$\alpha(0) = \{\mathsf{p}\overline{0}\,\overline{0}\}, \alpha(n+1) = \emptyset \quad \text{and} \quad \beta(0) = \{\mathsf{p}\overline{1}\,\overline{0}\}, \beta(n+1) = \emptyset$$

for any $n \in \mathbb{N}$. Suppose for a contradiction that we have an element $R \in [\![\forall X \mathrm{Stab}(X)]\!]$. Since $[0 \in \alpha] = \alpha(0)$ and $[0 \in \beta] = \beta(0)$ are both non-empty, $R\overline{0}\emptyset$ must be an element of $[0 \in \alpha] \cap [0 \in \beta] = \alpha(0) \cap \beta(0)$. But this impossible, because α and β are disjoint. ■

In the following proposition let us write $\langle -,- \rangle$ for a primitive recursive coding of $\mathbb{N}^2$ to $\mathbb{N}$. The sentence $\forall X \exists Y (\mathrm{Stab}(Y) \wedge \forall x (x \in X \leftrightarrow \exists y \langle y, x \rangle \in Y))$ is known as *Shanin's Principle* (*SHP*). It holds in the Effective Topos. Internally, it says that every subset of N is covered by a stable subset of N.

Proposition 6.4.6. *If $|\mathbb{A}| > |\mathbb{N}|$, then Shanin's Principle does not hold in $\mathsf{RT}(\mathcal{A})$.*

Proof. Assume for the sake of contradiction that it does. Then we have a realizer

$$R \in \bigcap_{\alpha \in P} \bigcup_{\beta \in P} [\![\mathrm{Stab}(\beta) \wedge \forall x (x \in \alpha \leftrightarrow \exists y \langle y, x \rangle \in \beta)]\!].$$

Let us write $R_0 = \mathsf{p}_0 R$ and $R_1 = \mathsf{p}_1 R$. For each $a \in \mathbb{A}$, define the element $\alpha_a \in P$ by:

$$\alpha_a(0) = \{\mathsf{p}a\overline{0}\} \quad \text{and} \quad \alpha_a(n+1) = \emptyset \quad \text{for any } n \in \mathbb{N}.$$

For each $a \in \mathbb{A}$, pick some $\beta_a \in P$ such that $R \in [\![\mathrm{Stab}(\beta_a) \wedge \forall x (x \in \alpha_a \leftrightarrow \exists y \langle y, x \rangle \in \beta_a)]\!]$.

From R_1, we effectively obtain $R' \in \mathbb{A}$ such that for every $a \in \mathbb{A}$, we have:

$$R'(\mathsf{p}a\overline{0}) \in [\langle m, 0 \rangle \in \beta_a]$$

for some $m \in \mathbb{N}$. As $|\mathbb{A}| > |\mathbb{N}|$, there must be two different $a, a' \in \mathbb{A}$ such that

$$R'(\mathsf{p}a\overline{0}) \in [\langle m, 0 \rangle \in \beta_a] \text{ and } R'(\mathsf{p}a'\overline{0}) \in [\langle m, 0 \rangle \in \beta_{a'}] \qquad (*)$$

for the same $m \in \mathbb{N}$.

From R_0, we effectively obtain $s \in \mathbb{A}$ witnessing the stability of β_a and $\beta_{a'}$. Thus, by $(*)$, we can use s to get a common realizer:

$$s\overline{\langle m, 0 \rangle}\mathsf{i} \in [\langle m, 0 \rangle \in \beta_a] \cap [\langle m, 0 \rangle \in \beta_{a'}].$$

Finally, using $s\overline{\langle m, 0 \rangle}\mathsf{i}$ and R_1, we find a realizer in the intersection $[0 \in \alpha_a] \cap [0 \in \alpha_{a'}] = \alpha_a \cap \alpha_{a'}$. But this is impossible, because a and a' are different, so that α_a and $\alpha_{a'}$ are disjoint. ■

Corollary 6.4.7. *Shanin's Principle does not hold in $\mathsf{RT}(\mathcal{S})$ and $\mathsf{RT}(\mathcal{K}_2)$.*

Proof. Immediate, as $|\mathbb{S}|, |\mathbb{N}^{\mathbb{N}}| > |\mathbb{N}|$. ■

6.4.2 In $\mathsf{RT}(\mathcal{S})$

This section focusses on some of the particular features of $\mathsf{RT}(\mathcal{S})$ with respect to second order arithmetic. Although $\forall X \mathrm{Stab}(X)$ does not hold in $\mathsf{RT}(\mathcal{S})$ (as we have seen), we do have the following result. Again, let P be as in Lemma 6.4.2, but now with $\mathbb{A} = \mathbb{S}$.

Proposition 6.4.8. *For any $\alpha \in P$, we have a realizer of* $\mathrm{Stab}(\alpha)$. *Hence, the sentence* $\forall X \neg\neg \mathrm{Stab}(X)$ *is true in* $\mathsf{RT}(\mathcal{S})$.

Proof. Let $\alpha \in P$. Note that there is a function $f_\alpha \colon \mathbb{N} \to \mathbb{S}$ such that

$$f_\alpha(n) \in \begin{cases} [n \in \alpha] & \text{if } [n \in \alpha] \neq \emptyset; \\ \{\emptyset\} & \text{else.} \end{cases}$$

By Lemma 6.1.1 we have $F_\alpha \in \mathbb{S}$ such that $F_\alpha \overline{n} = f_\alpha(n)$. Now note that $\lambda^* xy.F_\alpha x$ realizes $\mathrm{Stab}(\alpha)$. ■

We finish the chapter by turning our attention to some Independence of Premiss schemes, as alluded to after Proposition 6.4.3. The following is a propositional version of the Independence of Premiss scheme and is known as the *Kreisel-Putnam* scheme.

Proposition 6.4.9. *The scheme* $(\neg\varphi \to \psi \vee \chi) \to (\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)$ *does not hold in* $\mathsf{RT}(\mathcal{S})$.

Proof. Suppose the scheme were true. Then we would have

$$R \in [\![\forall XYZ((\neg 0 \in X \to 0 \in Y \vee 0 \in Z) \to (\neg 0 \in X \to 0 \in Y) \vee (\neg 0 \in X \to 0 \in Z))]\!].$$

Take $\alpha \in P$ such that $\neg 0 \in \alpha$ is false in $\mathsf{RT}(\mathcal{S})$. Then $[\![\neg 0 \in \alpha]\!] = \emptyset$, so W is an element of $[\![\neg 0 \in \alpha \to 0 \in \beta \vee 0 \in \gamma]\!]$ for any $W \in \mathbb{S}$ and $\beta, \gamma \in P$. Thus, $\mathsf{p_0}(RW) \in \{\mathsf{k}, \overline{\mathsf{k}}\}$ for any $W \in \mathbb{S}$. In particular, $\mathsf{p_0}(R\emptyset) \in \{\mathsf{k}, \overline{\mathsf{k}}\}$. Since $\mathsf{k} \not\subseteq \overline{\mathsf{k}}$ and $\overline{\mathsf{k}} \not\subseteq \mathsf{k}$, this implies that $\mathsf{p_0}(RW) = \mathsf{k}$ for any $W \in \mathbb{S}$ or $\mathsf{p_0}(RW) = \overline{\mathsf{k}}$ for any $W \in \mathbb{S}$.

Assume without loss generalization that that $\mathsf{p_0}(RW) = \mathsf{k}$ for any $W \in \mathbb{S}$. Now take $\alpha, \beta, \gamma \in P$ such that $\neg 0 \in \alpha, 0 \in \gamma$ are true and $0 \in \beta$ is false in $\mathsf{RT}(\mathcal{S})$. Fix $W' \in [\![0 \in \gamma]\!]$. Then $V = \lambda^* x.\mathsf{p}\overline{\mathsf{k}}W'$ is an element of $[\![\neg 0 \in \alpha \to 0 \in \beta \vee 0 \in \gamma]\!]$. Thus, RV is an element of $[\![(\neg 0 \in \alpha \to 0 \in \beta) \vee (\neg 0 \in \alpha \to 0 \in \gamma)]\!]$. By assumption, $\mathsf{p_0}(RV) = \mathsf{k}$, so $\mathsf{p_1}(RV) \in [\![\neg 0 \in \alpha \to 0 \in \beta]\!]$. But this is impossible, since $[\![0 \in \beta]\!] = \emptyset$, while $[\![\neg 0 \in \alpha]\!]$ is not. ■

Corollary 6.4.10. *The scheme* $(\neg\varphi \to \exists x \psi) \to \exists x(\neg\varphi \to \psi)$ *is not valid in* $\mathsf{RT}(\mathcal{S})$.

Proof. Again, we define $\psi \vee \chi$ by $\exists x((x = 0 \to \psi) \wedge (\neg x = 0 \to \chi))$. Let us write ρ for $(x = 0 \to \psi) \wedge (\neg x = 0 \to \chi)$. Then, $(\neg\varphi \to \exists x \rho) \leftrightarrow (\neg\varphi \leftrightarrow \psi \vee \chi)$, while

$$\begin{aligned} \exists x(\neg\varphi \to \rho) &\equiv \exists x((\neg\varphi \to (x = 0 \to \psi)) \wedge (\neg\varphi \to (\neg x = 0 \to \chi))) \\ &\leftrightarrow \exists x((x = 0 \to (\neg\varphi \to \psi)) \wedge (\neg x = 0 \to (\neg\varphi \to \chi))) \\ &\leftrightarrow (\neg\varphi \to \psi) \vee (\neg\varphi \to \chi). \end{aligned}$$

Thus, the validity of the scheme in the corollary contradicts the proposition above. ■

Chapter 7

A Dominance in $\mathsf{RT}(\mathcal{S})$

Domain theory was pioneered by Dana Scott [Sco76]. It came about as solution to the problem of finding (denotational) semantics for theories like the untyped lambda calculus. In synthetic domain theory ([Hyl91], [OS00]), one develops domain theory not by constructing particular sets with desirable properties, but by imposing these properties directly using axioms. In search of semantics for synthetic domain theory, one may turn to toposes. The cornerstone of this idea is the notion of a dominance in a topos. From there, one considers the associated lift functor and algebras for this functor.

In this section we define and investigate some properties of a particular dominance in $\mathsf{RT}(\mathcal{S})$. We will write $S = (\mathbb{S}, \{-\})$ for the object of realizers. The material is an adaptation of [Oos08, Sections 3.6.3 and 3.6.4] and also based on [OS00]. In particular, Propositions 7.2.1 and 7.2.2 are similar to [Oos08, Propositions 3.2.27 and 3.2.28]. The calculations in Sections 7.3–7.5 are my own, but were inspired by the examples in [OS00].

7.1 Basic definitions

The following definition is formulated in the internal language of a topos.

Definition 7.1.1. In a topos $\mathcal{E}$ a *dominance* is a subobject D of the subobject classifier Ω satisfying:

(i) $\top \in D$;

(ii) $\forall p, q{:}\Omega(p \in D \wedge (p \to (q \in D)) \to (p \wedge q) \in D)$.

We will show that the Sierpiński assembly Σ (recall Definition 5.1.1) is a dominance in $\mathsf{RT}(\mathcal{S})$.

The following lemma is due to my supervisor Jaap van Oosten and originated from my (incorrect) conjecture that the object of realizers S is isomorphic to the exponential $(1+1)^N$.

Lemma 7.1.2. *The object of realizers $S = (\mathbb{S}, \{-\})$ is isomorphic to the exponential Σ^N with N the natural numbers object.*

Proof. We first prove that the underlying set of Σ^N is $\{0,1\}^{\mathbb{N}}$. Since any morphism from N to Σ is in particular a function from $\mathbb{N}$ to $\{0,1\}$, one inclusion is clear. Conversely, if $f\colon \mathbb{N} \to \{0,1\}$, then f is tracked by graph(F), where $F\colon \mathbb{S} \to \mathbb{S}$ is the continuous function defined by:

$$\overline{n} \mapsto \begin{cases} \overline{1} & \text{if } f(n) = 1; \\ \emptyset & \text{if } f(n) = 0. \end{cases}$$

Thus, Σ^N is the assembly $(\{0,1\}^{\mathbb{N}}, E)$ where $E(f) \subseteq \mathbb{S}$ is the (non-empty) set of trackers of f.

We have a canonical bijection

$$\chi\colon \mathbb{S} \to \{0,1\}^{\mathbb{N}}, \quad U \mapsto \chi_U$$

(where χ_U is the characteristic function of U) with inverse

$$\chi^{-1}\colon \{0,1\}^{\mathbb{N}} \to \mathbb{S}, \quad f \mapsto f^{-1}(\{1\}).$$

It remains to prove that these functions are tracked. For χ, consider the continuous function $F\colon \mathbb{S} \to \mathbb{S}$ given by $U \mapsto \{\langle 2^n, 1\rangle \mid n \in U\}$. Then χ is tracked by graph(F). Indeed, if $U \in \mathbb{S}$ and $n \in \mathbb{N}$, then graph$(F)U\overline{n} = F(U)\overline{n}$ and this is $\overline{1}$ if $n \in U$ and $\emptyset$ otherwise.

For the inverse of χ, we define $G\colon \mathbb{S} \to \mathbb{S}$ continuous by $U \mapsto \{n \in \mathbb{N} \mid U\overline{n} = \overline{1}\}$. (This is continuous, because the application of $\mathbb{S}$ is continuous.) We claim that χ^{-1} is tracked by graph(G). Indeed, for $f \in \{0,1\}^{\mathbb{N}}$ and $U \in E(f)$, we have $n \in G(U)$ if and only if $f(n) = 1$, since U tracks f. ∎

Definition 7.1.3. Define a relation $\in$ between N and S by taking the following pullback

$$\begin{array}{ccc} \in & \xrightarrow{\qquad\qquad\qquad} & 1 \\ \big\downarrow & \lrcorner & \big\downarrow t \\ N \times S \cong N \times \Sigma^N & \xrightarrow{\text{ev}} \Sigma \longrightarrow & \Omega \end{array}$$

where $\Sigma \to \Omega$ is the morphism induced by the function $0 \mapsto \emptyset$ and $1 \mapsto \mathbb{S}$.

Remark 7.1.4. Observe that $\in$ is given by

$$[n \in U] = \begin{cases} \mathsf{p}\overline{n}U & \text{if } n \text{ is an element of } U; \\ \emptyset & \text{else.} \end{cases}$$

In particular, $\in$ is $\neg\neg$-stable.

Lemma 7.1.5. *Let $\mathcal{A}$ be a arbitrary* total *pca and let* $\mathsf{RT}(\mathcal{A})$ *be its realizability topos. Let $A = (\mathbb{A}, \{-\})$ be the object of realizers in the topos. The scheme*

$$(\mathrm{IP}_A) \quad (\neg\varphi \to \exists x{:}A\psi) \to \exists x{:}A(\neg\varphi \to \psi)$$

with x not free in φ is valid in $\mathsf{RT}(\mathcal{A})$.

Proof. It is not hard to verify that $\lambda^* u.\mathsf{p}(\mathsf{p_0}(u\mathsf{k}))(\lambda^* v.\mathsf{p_1}(u\mathsf{k}))$ realizes the scheme. ∎

Proposition 7.1.6. *The subobject Ω' of Ω given by*

$$\Omega' = [\![\exists X{:}S(p \leftrightarrow 1 \in X)]\!]$$

with p ranging over Ω, is a dominance in $\mathsf{RT}(\mathcal{S})$. Moreover, it is $\neg\neg$-separated, i.e. $\forall p{:}\Omega'(\neg\neg p \to p)$. Furthermore, Ω' is closed under finite joins in Ω, viz. $\bot \in \Omega'$ and $\forall p, q{:}\Omega(p, q \in \Omega' \to p \vee q \in \Omega')$.

Proof. Double negation separation is immediate by Remark 7.1.4. Further, it is clear that $\top \in \Omega'$ (take $X = \overline{1}$).

Suppose $p \in \Omega'$ and $p \to (q \in \Omega')$. Take $U \in \mathbb{S}$ such that $p \leftrightarrow 1 \in U$. Then, $1 \in U \to \exists X{:}S(q \leftrightarrow 1 \in X)$, so by Lemma 7.1.5 and Remark 7.1.4, we get that $\exists X{:}S(1 \in U \to (q \leftrightarrow 1 \in X))$. Take such $V \in \mathbb{S}$. Then, $p \wedge q \leftrightarrow 1 \in U \cap V$. Hence, $p \wedge q \in \Omega'$ and Ω' is a dominance.

The final claim is also easily proven, since $\emptyset \in \mathbb{S}$ and because we can take unions. ∎

Definition 7.1.7. Let D be a dominance in a topos $\mathcal{E}$. A subobject $m\colon A \hookrightarrow B$ is called a *D-subobject* if the classifying map χ_m of m factors through D. Equivalently (by pullback pasting), if the square

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle m}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle t} \\ B & \longrightarrow & D \end{array}$$

is a pullback. We write $A \subseteq_D B$ in this case. We also say that m is a *D-map*.

7.2 Assemblies and their Σ-subobjects

We have already remarked that the object Ω' from Proposition 7.1.6 is $\neg\neg$-separated. Indeed, it is isomorphic to an assembly.

Proposition 7.2.1. *The object Ω' is isomorphic to Σ.*

Proof. First of all, observe that Ω' is isomorphic to the object $(\mathcal{P}(\mathbb{S}), \sim)$ where

$$\begin{aligned} &\mathcal{U} \sim \mathcal{V} = \mathcal{U} \leftrightarrow \mathcal{V} \wedge E(\mathcal{U}), \text{with} \\ &\quad E(\mathcal{U}) = \{[W, U] \in \mathbb{S} \mid \text{if } 1 \in W, \text{ then } U \in \mathcal{U} \text{ and if } \mathcal{U} \neq \emptyset, \text{ then } 1 \in W\}. \end{aligned}$$

Define a continuous function $F\colon \mathbb{S} \to \mathbb{S}$ by

$$F(\emptyset) = \emptyset \text{ and } F(V) = \overline{1} \text{ for any non-empty } V.$$

Next, define $\Phi\colon \mathcal{P}(\mathbb{S}) \times \{0, 1\} \to \mathcal{P}(\mathbb{S})$ by

$$(\mathcal{U}, i) \mapsto \{[W, U, C] \mid [W, U] \in E(\mathcal{U}), C \in E_\Sigma(i) \text{ and } i = 1 \Leftrightarrow 1 \in W\}.$$

We show that Φ is a functional relation from $(\mathcal{P}(\mathbb{S}), \sim)$ to Σ. Strictness is immediate. For single-valuedness, suppose we have $[W, U, C] \in \Phi(\mathcal{U}, i)$ and $[W', U', C'] \in \Phi(\mathcal{U}, j)$. We show that $i = j$. By definition, we have

$$i = 1 \Leftrightarrow 1 \in W \Leftrightarrow U \in \mathcal{U} \Rightarrow 1 \in W' \Leftrightarrow j = 1$$

and similarly, $i = 0 \Rightarrow j = 0$. Thus, $i = j$, as desired. Suppose $[W, U, C] \in \Phi(\mathcal{U}, i)$ and $B \in [\mathcal{U} \leftrightarrow \mathcal{V}]$. We must effectively obtain an element of $\Phi(\mathcal{V}, i)$. But if B_0 realizes $\mathcal{U} \to \mathcal{V}$, then one easily sees that $[W, B_0U, C]$ is an element of $\Phi(\mathcal{V}, i)$. So, Φ is relational. For totality, suppose $[W, U] \in E(\mathcal{U})$, then $[W, U, F(W)] \in \Phi(\mathcal{U}, i)$ for some $i \in \{0, 1\}$, by construction of E and F. We conclude that Φ is a functional relation.

Moreover, (the arrow represented by) Φ is easily seen to be epic. For, if $C \in E_\Sigma(i)$, then $[F(C), F(C), C] \in \Phi(\mathcal{U}_i, i)$ with $\mathcal{U}_1 = \{\overline{1}\}$ and $\mathcal{U}_0 = \emptyset$ by construction of F and definition of E_Σ.

Finally, we prove that Φ is monic and hence that Φ represents an isomorphism, as desired. Suppose we have $[W, U, C] \in \Phi(\mathcal{U}, i)$ and $[W', U', C] \in \Phi(\mathcal{V}, i)$. It suffices to effectively provide an element of $\mathcal{U} \leftrightarrow \mathcal{V}$, since $[W, U]$ is an element of $E(\mathcal{U})$ already. But $[\lambda^* x.U', \lambda^* x.U]$ is easily seen to do the job. ■

Observe that a Σ-subobject of an assembly is again an assembly as $\mathsf{Asm}(\mathcal{S})$ is closed under finite limits in $\mathsf{RT}(\mathcal{S})$. The following proposition characterizes these Σ-subobjects and justifies the name Sierpiński assembly (as the Sierpiński space is the classifying space for the Scott topology).

Proposition 7.2.2. *Let X be an assembly. There is a bijective correspondence between morphisms $X \to \Sigma$ and subsets $X' \subseteq |X|$ for which there is an open $\mathcal{U} \subseteq \mathbb{S}$ with the following properties:*

$$x \in X' \Rightarrow E_X(x) \subseteq \mathcal{U};$$
$$x \notin X' \Rightarrow E_X(x) \cap \mathcal{U} = \emptyset.$$

Moreover, an assembly Y is a Σ-subobject of X if and only if Y is isomorphic to some assembly (X', E) where $X' \subseteq |X|$ is as above and E is the restriction of E_X to X'.

Proof. Let f be a morphism from X to Σ that is tracked by $U \in \mathbb{S}$. Set

$$X' = \{x \in X \mid f(x) = 1\} \quad \text{and} \quad \mathcal{U} = \{V \in \mathbb{S} \mid UV = \overline{1}\}.$$

We show that $\mathcal{U}$ is open. Let $\mathcal{Q} := \{p \subseteq \mathbb{S} \mid p \text{ is finite and } Up = \overline{1}\}$. Recall the notation $\uparrow p = \{V \in \mathbb{S} \mid p \subseteq V\}$. By continuity of the application, one can show that $\mathcal{U} = \bigcup_{p \in \mathcal{Q}} \uparrow p$. Thus, $\mathcal{U}$ is an open of $\mathbb{S}$.

From the definition of $\mathcal{U}$ and the fact that U tracks f, it is immediate that $\mathcal{U}$ has the desired properties.

For the converse, assume we are given an open $\mathcal{U} \subseteq \mathbb{S}$ and a subset $X' \subseteq |X|$ with the properties stated. Define $f \colon X \to \Sigma$ by $f(x) = 1$ if $x \in X'$ and $f(x) = 0$ if $x \notin X'$. We claim that it is tracked by $\mathrm{graph}(F)$ where $F(U) = \{1 \mid U \in \mathcal{U}\}$. That this F is continuous follows from the assumption that $\mathcal{U}$ is open. Now if $x \in X$ and $U \in E_X(x)$, then either $f(x) = 1$, in which case $E_X(x) \subseteq \mathcal{U}$, so that $F(U) = \overline{1} \in E_\Sigma(f(x))$; or $f(x) = 0$, in which case $E(x) \cap \mathcal{U} = \emptyset$, so that $F(U) = \emptyset \in E_\Sigma(f(x))$. So f is tracked, as desired.

That the operations above are each other's inverse is readily verified. The final claim follows immediately from the construction above and the description of pullbacks in the category of assemblies. ■

7.3 The lift functor on assemblies

Definition 7.3.1. Let D be a subobject of Ω in an arbitrary topos $\mathcal{E}$. A *D-partial map classifier* for an object Y is an arrow $Y \xrightarrow{\eta_Y} \tilde{Y}$ such that for every $U \subseteq_D X$ and $f\colon U \to Y$ (we regard this as a partial map from X to Y with domain U) there is a unique $\tilde{f}\colon X \to \tilde{Y}$ such that

$$\begin{array}{ccc} U & \xrightarrow{f} & Y \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \eta_Y} \\ X & \xrightarrow{\tilde{f}} & \tilde{Y} \end{array}$$

is a pullback.

Given a dominance D in a topos $\mathcal{E}$, there is an endofunctor L (called the *lift functor*) on $\mathcal{E}$ and a natural transformation $\eta\colon \mathrm{id}_\mathcal{E} \Rightarrow L$ such that $Y \xrightarrow{\eta_Y} L(Y)$ is a D-partial map classifier for Y (see [Oos08, pp. 221–222] and [OS00, pp. 237–238]). The following proposition describes this lift functor when restricted to $\mathsf{Asm}(\mathcal{S})$, for the dominance Σ.

Proposition 7.3.2. *The lift functor L on $\mathsf{Asm}(\mathcal{S})$ is given by on objects by:*

$$L(X) = \left(|X| \cup \{\bot_X\}, E_{L(X)}\right),$$

where $\bot_X$ is some element not in $|X|$ and

$$E_{L(X)}(\bot_X) = \{\emptyset\} \quad \textit{and} \quad E_{L(X)}(x) = \{[U, \overline{1}] \mid U \in E_X(x)\} \textit{ for } x \in |X|.$$

Given an arrow $f\colon X \to Y$, we define $L(f)$ as the unique extension of f satisfying $\bot_X \mapsto \bot_Y$. The natural transformation $\eta\colon \mathrm{id}_{\mathsf{Asm}(\mathcal{S})} \to L$ is defined as $\eta_X(x) = x$.

Proof. Given a morphism $f\colon X \to Y$ of $\mathsf{Asm}(\mathcal{S})$ tracked by $U_f \in \mathbb{S}$, note that $L(f)$ is tracked, as

$$[V_0, V_1] \mapsto \begin{cases} [U_f V_0, \overline{1}] & \text{if } 1 \in V_1; \\ \emptyset & \text{else;} \end{cases}$$

is a continuous map $\mathbb{S} \to \mathbb{S}$. Verifying that L is indeed a functor is routine. Also, note that η_X is tracked by $\lambda^* u.[u, \overline{1}]$. That η is natural is easily checked.

Finally, suppose we have a morphism $f\colon U \to Y$ and $U \subseteq_\Sigma X$. By Proposition 7.2.2, we may assume that we have $|U| \subseteq |X|$ and an open $\mathcal{U}$ such that for $x \in |X|$:

$$\text{if } x \in |U|, \text{then } E_X(x) \subseteq \mathcal{U} \quad \text{and} \quad \text{if } x \notin |U|, \text{then } E_X(x) \cap \mathcal{U} = \emptyset.$$

Define $\tilde{f}\colon X \to L(Y)$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in |U|; \\ \bot_Y & \text{else.} \end{cases}$$

Note that $\tilde{f}$ is tracked, for if f is tracked by U_f, then

$$U \mapsto \begin{cases} U_f U & \text{if } U \in \mathcal{U}; \\ \emptyset & \text{else} \end{cases}$$

is a continuous map $\mathbb{S} \to \mathbb{S}$, because $\mathcal{U}$ is open.

If we have morphisms $g\colon Z \to X$ and $h\colon Z \to Y$ such that $\tilde{f}g = \eta_Y h$, then we must have that $g(z) \in |U|$ for any $z \in |Z|$. Hence, g factors uniquely through (U, E_U). This proves that $\tilde{f}$ makes the square into a pullback.

It remains to show that it is unique with this property. To this end, suppose we have $f'\colon X \to L(Y)$, such that the square is a pullback. From the commutativity of the square, it follows that for $x \in |U|$ we must have $f'(x) = f(x) = \tilde{f}(x)$. Now suppose for a contradiction that we have $x_0 \in |X| \setminus |U|$ and $f'(x_0) \in |Y|$. The universal property of the pullback then yields a map $U \cup \{x_0\} \to U$ such that the inclusion $|U| \cup \{x_0\} \to |X|$ factors through $|U|$, but this is impossible, as $x_0 \notin |U|$. Hence, no such x_0 exists and therefore, f' and $\tilde{f}$ coincide. ■

Remark 7.3.3. Observe that $L(1) \cong \Sigma$ and that $\eta_1 = 1 \xrightarrow{t} \Sigma$.

Lemma 7.3.4. *For any X, we have $X \subseteq_\Sigma L(X)$ via η_X.*

Proof. It is straightforward to verify that

$$\begin{array}{ccc} X & \longrightarrow & 1 \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle t} \\ L(X) & \xrightarrow{\chi_X} & \Sigma \end{array}$$

with $\chi_X(x) = 1$ and $\chi_X(\bot_X) = 0$ is a pullback. (Note the map $F(V) = \overline{1}$ if $1 \in \mathsf{p}_1 V$ and $\emptyset$ otherwise is continuous and its graph tracks χ_X.) ■

It is easy to check that the lift functor L is actually a monad on $\mathsf{Asm}(\mathcal{S})$. The multiplication $\mu\colon L^2(X) \to L(X)$ is given by the map $x \mapsto x, \bot_X \mapsto \bot_X, \bot_{L(X)} \mapsto \bot_X$.

Definition 7.3.5. An *object with* $\bot$ is an algebra for the monad L. A *strict map* between objects with $\bot$ is an L-algebra homomorphism.

7.4 Lift functor for slices

Since one can define the lift functor using the internal logic of the topos (as done in [Oos08, pp. 221–222] and [OS00, pp. 237–238]), it follows that one can also generalize the lift functor to an endofunctor on a slice.

Let us first look at the dominance in the slice. Given a topos $\mathcal{E}$ and an object Y of $\mathcal{E}$, the functor $\mathcal{E} \to \mathcal{E}/Y$ given by $X \mapsto (X \times Y \xrightarrow{\pi_1} Y)$ is known to be logical. That is, it preserves the logical structure of $\mathcal{E}$. In particular, if D is a dominance in $\mathcal{E}$, then $D \times Y \xrightarrow{\pi_1} Y$ is a dominance in $\mathcal{E}/Y$. One may also verify this directly. So let us look at $\Sigma \times Y$ subobjects of $X \xrightarrow{f} Y$ in $\mathsf{RT}(\mathcal{S})/Y$. These subobjects are given as pullbacks of

$$\begin{array}{ccc} & & Y \\ & & \big\downarrow{\scriptstyle \langle t!_Y, \mathrm{id}_Y\rangle} \\ X & \xrightarrow{\langle \chi, f\rangle} & \Sigma \times Y \end{array}$$

(where $!_Y$ is the unique map $Y \to 1$). If $f\colon X \to Y$ is a morphism of assemblies, then using Proposition 7.2.2 and the description of pullbacks in $\mathsf{Asm}(\mathcal{S})$, we see that these pullbacks all are given by assemblies (X', E) as in Proposition 7.2.2. The map from $(X', E) \to Y$ is simply the restriction of f.

In this section we describe, given an assembly Y, how to generalize the lift functor on $\mathsf{Asm}(\mathcal{S})$ to a *lift functor* L_Y *over* Y on the slice $\mathsf{Asm}(\mathcal{S})/Y$. Of course, there is again a natural transformation $\eta_Y\colon \mathrm{id}_{\mathsf{Asm}(\mathcal{S})/Y} \Rightarrow L_Y$ and this structure classifies Σ-partial maps.

By the above analysis, given

$$\begin{array}{ccccc} X & \hookleftarrow & (X', E) & \longrightarrow & Z \\ & \searrow & \big\downarrow & \swarrow{\scriptstyle h} & \\ & & Y & & \end{array}$$

in $\mathsf{Asm}(\mathcal{S})/Y$ with (X', E) as in Proposition 7.2.2, there is a unique morphism from X to $\mathrm{dom}(L_Y(a))$ of $\mathsf{Asm}(\mathcal{S})/Y$ such that

$$\begin{array}{ccc} (X', E) & \hookrightarrow & X \\ \big\downarrow & \lrcorner & \big\downarrow \\ Z & \xrightarrow{\eta_{Y h}} & \mathrm{dom}(L_Y(h)) \end{array}$$

is a pullback in $\mathsf{Asm}(\mathcal{S})/Y$.

Proposition 7.4.1. *Let Y be any assembly. The lift functor L_Y on $\mathsf{Asm}(\mathcal{S})/Y$ is given on objects by*

$$L_Y(f\colon X \to Y) = Y \sqcup_f X \to Y,$$

where $Y \sqcup_f X$ is the assembly $(Y + X, E_f)$ with

$$\begin{aligned} E_f(0, y) &= \{[\emptyset, V] \mid V \in E_Y(y)\} \\ E_f(1, x) &= \{[\,[U, \overline{1}], V] \mid U \in E_X(x), V \in E_Y(f(x))\}; \end{aligned}$$

and the arrow $Y \sqcup_f X \to Y$ is given by $[\mathrm{id}_Y, f]\colon (0, y) \to y, (1, x) \mapsto f(x)$.

On arrows, L_Y is defined as:

$$L_Y\left(\begin{array}{ccc} X & \xrightarrow{h} & Z \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & Y & \end{array}\right) = \begin{array}{ccc} Y \sqcup_f X & \xrightarrow{L_Y(h)} & Y \sqcup_g Z \\ & {\scriptstyle [\mathrm{id}_Y, f]}\searrow \quad \swarrow{\scriptstyle [\mathrm{id}_Y, g]} & \\ & Y & \end{array}$$

where $L_Y(h)\colon (0,y) \mapsto (0,y), (1,x) \mapsto (1,h(x))$.

Finally, the natural transformation $\eta_Y\colon \mathrm{id}_{\mathsf{Asm}(\mathcal{S})/Y} \to L_Y$ is given by $\eta_{Y\,f}(x) = (1,x)$. We will henceforth simply write η_f for this map.

Proof. Firstly, observe that $[\mathrm{id}_Y, f]$ is tracked by p_1. Moreover, the map $L_Y(h)$ is tracked by the graph of the continuous function

$$[V_0, V_1] \mapsto \begin{cases} [U_h(\mathsf{p}_0 V_0), V_1] & \text{if } V_0 \neq \emptyset; \\ [\emptyset, V_1] & \text{else;} \end{cases}$$

where U_h is a tracker of h. Obviously, L_Y is a functor.

Further, $\eta_f\colon X \to Y \sqcup_f X$ is tracked by $\lambda^* v.[\,[v, \bar{1}], U_f v]$ where U_f tracks f. This clearly defines a natural transformation $\eta\colon \mathrm{id} \Rightarrow L_Y$.

It remains to verify the desired pullback property. Assume we are given two commutative triangles in $\mathsf{Asm}(\mathcal{S})$

$$\begin{array}{ccccc} X & \hookleftarrow & (X', E) & \xrightarrow{g} & Z \\ & {\scriptstyle f}\searrow & \downarrow & \swarrow{\scriptstyle h} & \\ & & Y & & \end{array}$$

where again (X', E) is as in Proposition 7.2.2. Define $\tilde{g}\colon X \to Y \sqcup_h Z$ by

$$\tilde{g}(x) = \begin{cases} \eta_h g(x) & \text{if } x \in X'; \\ (0, f(x)) & \text{else.} \end{cases}$$

Let $\mathcal{U}$ be an open as in Proposition 7.2.2 and let U_g and U_f be trackers of g and f, respectively. Then $\tilde{g}$ is tracked, because the assignment

$$V \mapsto \begin{cases} [\,[U_g V, \bar{1}], U_f V] & \text{if } V \in \mathcal{U}; \\ [\emptyset, U_f V] & \text{else;} \end{cases}$$

is continuous as $\mathcal{U}$ is open.

Note that for $x \in X'$, we have $[\mathrm{id}_Y, h]\tilde{g}(x) = hg(x) = f(x)$ and for $x \in |X| \setminus X'$, we see that $[\mathrm{id}_Y, h]\tilde{g}(x)) = f(x)$. Hence, $\tilde{g}$ is in fact an arrow of the slice $\mathsf{Asm}(\mathcal{S})/Y$.

It remains to show that $\tilde{g}$ is the unique arrow in $\mathsf{Asm}(\mathcal{S})/Y$ making the square in the proposition into a pullback. But this follows from essentially the same arguments as in Proposition 7.3.2. ■

Observe that the L_Y really is a generalization of L, because the functor L_1 is isomorphic to L. We can also generalize Lemma 7.3.4.

Lemma 7.4.2. *For any $f\colon X \to Y$, we have $X \subseteq_\Sigma Y \sqcup_f X$ via η_f.*

Proof. Similarly to the proof of Lemma 7.3.4, one can show that

$$\begin{array}{ccc} X & \longrightarrow & 1 \\ \big\downarrow{\scriptstyle \eta_f} & & \big\downarrow{\scriptstyle t} \\ Y \sqcup_f X & \xrightarrow{\chi_X} & \Sigma \end{array}$$

with $\chi_X(0, y) = 0$ and $\chi_X(1, x) = 1$ is a pullback square. ∎

Finally, one may also show that L_Y is actually a monad, the multiplication μ_f for some $f\colon X \to Y$ is given by the map $Y \sqcup_{[\mathrm{id}_Y, f]} (Y \sqcup_f X) \to Y \sqcup_f X$ defined as $(0, y) \mapsto (0, y), (1, (0, y)) \mapsto (0, y)$ and $(1, (1, x)) \mapsto (1, x)$.

7.5 Lambek algebras for the lift functor

Definition 7.5.1. Let $\mathcal{C}$ be an arbitrary category and let F be an endofunctor on $\mathcal{C}$. A *Lambek algebra* for F is a morphism $\alpha\colon F(X) \to X$. A *morphism between two Lambek algebras* $F(X) \xrightarrow{\alpha} X$ and $F(Y) \xrightarrow{\beta} Y$ is an arrow $f\colon X \to Y$ compatible with the algebra structure, i.e. such that

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \beta} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

Definition 7.5.2. A *Lambek coalgebra* is the dual of a Lambek algebra, viz. a morphism $\beta\colon X \to F(X)$. Similarly, we have a notion of *morphisms between Lambek coalgebras*.

Initial Lambek algebras are interesting for various reasons. For example, *Lambek's Lemma* states that any initial Lambek algebra is an isomorphism. This means that the functor has a fixed point. Similarly, any terminal Lambek coalgebra is an isomorphism. The proof of Lambek's Lemma is a nice exercise in working with Lambek algebras.

Lemma 7.5.3 (Lambek's Lemma [Lam68])**.** *Any initial Lambek algebra is an isomorphism.*

Proof. Suppose $F(I) \xrightarrow{\sigma} I$ is an initial Lambek algebra for an endofunctor F on a category $\mathcal{C}$. We construct an inverse $\tau\colon I \to F(I)$. Consider the Lambek algebra $F^2(I) \xrightarrow{F(\sigma)} F(I)$. As σ is initial, this yields a unique $\tau\colon I \to F(I)$ such that $\tau\sigma = F(\sigma)F(\tau) = F(\sigma\tau)$. Since F preserves the identity, it suffices to prove that $\sigma\tau = \mathrm{id}_I$. As σ is initial, we are done if we can show that $\sigma\tau$ is a morphism of Lambek algebras. That is, $\sigma F(\sigma\tau) = \sigma\tau\sigma$ should hold, but it does by our choice of τ. ∎

By [OS00, Theorem 1.4 and pp. 238–239] and [Oos08, Theorem 3.6.5], the initial Lambek algebra and terminal Lambek algebra for the lift functor both exist. We explicitly construct them below.

Proposition 7.5.4. *The initial Lambek algebra for the lift functor given by $I = (\mathbb{N}, E_I)$ where $E_I(n) = \{\{0, \dots, n\}\}$ and $\sigma\colon L(I) \to I$ defined as $\sigma(\bot_I) = 0$ and $\sigma(n) = n + 1$.*

Proof. First of all, note that I is indeed a Lambek algebra, since σ is tracked by the graph of the continuous function $V \mapsto \{0\} \cup \{x + 1 \mid x \in \mathsf{p}_0 V\}$. Now suppose that $L(X) \xrightarrow{\alpha} X$ is an arbitrary Lambek algebra for the lift functor. We must find $f\colon I \to X$ such that $f\sigma = \alpha L(f)$. This equality implies that we have no choice but to put $f(n) = \alpha^{n+1}(\bot_X)$. It remains to show that this function is tracked. Suppose U_α tracks α. Consider the set $U = \{\langle \sum_{i=0}^{n} 2^i, m\rangle \mid n \in \mathbb{N}, m \in U_\alpha^{n+1}\emptyset\}$. By the monotonicity of the application, we have $U_\alpha V \supseteq U_\alpha \emptyset$ for any $V \in \mathbb{S}$. In particular, we obtain a chain $\cdots \supseteq U_\alpha^3\emptyset \supseteq U_\alpha^2\emptyset \supseteq U_\alpha\emptyset$. Thus U tracks f, because for any $n \in \mathbb{N}$ we have

$$U\{0, \dots, n\} = \bigcup_{k=0}^{n} U\{0, \dots, k\} = \bigcup_{k=0}^{n} U_\alpha^{k+1}\emptyset = U_\alpha^{n+1}\emptyset \in E_X(\alpha^{n+1}(\bot_X)). \qquad \blacksquare$$

Proposition 7.5.5. *The terminal (also called final) Lambek coalgebra for the lift functor is given by $F = (\mathbb{N} \cup \{\infty\}, E_F)$ where $E_F(n) = \{\{0, \dots, n\}\}$ and $E_F(\infty) = \{\mathbb{N}\}$ and the coalgebra structure is given by $\tau\colon F \to L(F)$ with $\tau(0) = \bot_F, \tau(n + 1) = n$ and $\tau(\infty) = \infty$.*

Proof. First of all, define the set $T = \{\langle \sum_{i=0}^{n} 2^i, m\rangle \mid n \in \mathbb{N}, 0 \leq m < n\}$. One can check that $T\{0\} = \emptyset$ and $T\{0, \dots, n + 1\} = \{0, \dots, n\}$ for any $n \in \mathbb{N}$. Moreover, the function $G\colon \mathbb{S} \to \mathbb{S}$ defined as

$$G(V) = \begin{cases} \emptyset & \text{if } V = \emptyset; \\ [V, \overline{1}] & \text{else.} \end{cases}$$

is obviously continuous. Hence, τ is tracked by $\lambda^* u.\mathrm{graph}(G)(Tu)$, so it is a morphism of assemblies.

Let $X \xrightarrow{\beta} L(X)$ be any Lambek coalgebra. We must show that there is a unique $f\colon X \to F$ such that

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & F \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \tau} \\ L(X) & \xrightarrow{L(f)} & L(F) \end{array}$$

commutes. We first prove that the commutativity of this square completely determines f. We prove by induction that for any $m \in \mathbb{N}$ and $x \in |X|$ if m is the least such that $\beta^{m+1}(x) = \bot_X$, then $f(x) = m$. For the base case $m = 0$, consider $x \in |X|$ such that $\beta(x) = \bot_X$. Then $\bot_F = L(f)(\beta(x)) = \tau(f(x))$, so $f(x) = 0 = m$. Now suppose the statement is true for m. Suppose we have $x \in |X|$ such that $m + 1$ is the least such that $\beta^{m+2}(x) = \bot_X$. By induction hypothesis applied to $\beta(x)$, we find that $f(\beta(x)) = m$. Hence, $m = f(\beta(x)) = L(f)(\beta(x)) = \tau(f(x))$, so $f(x) = m + 1$. Thus, if $x \in |X|$ is such that $\beta^{m+1} = \bot_X$ for some $m \in \mathbb{N}$, then we must define $f(x)$ as the least such m. We prove that if no such m exists, then we must put $f(x) = \infty$. For suppose we had $x \in |X|$ such that $\beta^k(x) \neq \bot_X$ for any $k \in \mathbb{N}$ and $f(x) = m$ for some $m \in \mathbb{N}$. Then, $\bot_X = \tau^{m+1}(f(x)) = \tau^m(L(f)(\beta(x))) = \cdots = L(f)(\beta^{m+1}(x))$, so $\beta^{m+1}(x) = \bot_X$, contradicting our choice of x. We conclude that the map $f\colon X \to F$ defined by

$$x \mapsto \begin{cases} m & \text{if } m \text{ is the least such that } \beta^{m+1}(x) = \bot_X; \\ \infty & \text{if no such } m \text{ exists.} \end{cases}$$

is unique.

Finally, f is tracked, because if U_β tracks β, then $V \mapsto \{m \in \mathbb{N} \mid 1 \in \mathsf{p_1}(U_\beta^{m+1}V)\}$ is a continuous map $\mathbb{S} \to \mathbb{S}$ and its graph tracks f. Indeed, for any $x \in |X|$ and $V \in E_X(x)$, it holds that $1 \in \mathsf{p_1}(U_\beta^{m+1}V)$ if and only if $\beta^{m+1}(x) \neq \bot_X$. ∎

We will write ι for the inclusion $I \to F$. Note that we have an arrow (tracked by i) from X^F to X^I by precomposing with ι. We will write X^ι for this morphism.

Definition 7.5.6. An object X is *complete* if $X^\iota\colon X^F \to X^I$ is an isomorphism.

Proposition 7.5.7. *The object Σ is complete.*

Proof. We define an inverse of Σ^ι. Let $f\colon I \to \Sigma$ be any morphism. We prove that there is a unique morphism $f'\colon F \to \Sigma$ extending f. Define $f'\colon F \to \Sigma$ by $f'(n) = f(n)$ and

$$f(\infty) = \begin{cases} 0 & \text{if } f(n) = 0 \text{ for all } n \in \mathbb{N}; \\ 1 & \text{else.} \end{cases}$$

If f is tracked by U_f, then f' is also tracked by $U_{f'}$ by our choice of E_I and E_F. Furthermore, f' is the unique extension of f, because by continuity, $U_f\mathbb{N} = \bigcup_{V \subseteq \mathbb{N}} U_f V$ and therefore, $U_f\mathbb{N} = \overline{1}$ if and only if $f(n) = 1$ for some $n \in \mathbb{N}$. Thus, $f \mapsto f'$ (tracked by i) is the required inverse of Σ^ι. ∎

7.6 Algebraic compactness

We conclude this chapter by sketching future developments of the material above. I only provide a sketch as it is beyond the scope of this thesis to elaborate on it.

In domain theory, we can construct a category such that many functors have fixed points. In synthetic domain theory, we therefore wish to consider algebraically compact categories.

Definition 7.6.1. A category $\mathcal{C}$ is called *algebraically compact* if for every endofunctor $T\colon \mathcal{C} \to \mathcal{C}$ there exists an initial Lambek algebra $T(I) \xrightarrow{\sigma} I$ and a final Lambek coalgebra $F \xrightarrow{\tau} T(F)$, and moreover the map $\iota\colon I \to F$ such that

$$\begin{array}{ccc} T(I) & \xrightarrow{T(\iota)} & T(F) \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau^{-1}} \\ I & \xrightarrow{\iota} & F \end{array}$$

commutes, is an isomorphism. (Note that ι exists by Lambek's Lemma and the initiality of σ.)

Definition 7.6.2. Let $\mathcal{E}$ be a topos with a dominance D and associated lift monad L.

(i) A *category of predomains* is a full internal subcategory $\mathcal{C}$ of $\mathcal{E}$ which consists of complete objects and is closed under L.

(ii) The *associated category of domains* is the category of algebras over the L-monad on $\mathcal{C}$.

Given such a topos $\mathcal{E}$, one may then want to look for a category of predomains such that its associated category of domains is complete and algebraically compact.

In [LS97], a possible candidate for such a category of predomains is identified by considering well-complete objects, which we define now.

Definition 7.6.3. An object X is *well-complete* if its lift $L(X)$ is complete.

Finally, observe that in our case, if X is a modest set, then so is $L(X)$. This means that L can be viewed as an internal monad on the internal category of internal modest sets (see [Oos08, Section 3.4.1]). The article [LS97] then suggests looking at the internal category of well-complete modest sets with $\perp$.

Chapter 8

A Model Structure on a Subcategory of $\mathsf{RT}(\mathcal{S})$

Frumin and van den Berg [Fv18] have recently given an axiomatic setup for constructing a model structure on a full subcategory of an elementary topos. Model structures are interesting, because they allow us to 'do homotopy'. As such, model structures can also provide semantics for homotopy type theory.

In this section we use the setup of [Fv18] to give a model structure on a subcategory of $\mathsf{RT}(\mathcal{S})$.

The notions and structure of this chapter are due to [Fv18]. The particular model structure on $\mathsf{RT}(\mathcal{S})$ and the characterizations (of contractible objects, for example) are (as far as I am aware) original.

8.1 Basic definitions and setup

We describe the setup of [Fv18]. We assume to be working in a topos $\mathcal{E}$ with a dominance D that is closed under finite joins in Ω. We wish to have an interval object in $\mathcal{E}$, which we define now.

Definition 8.1.1. An *interval object* $\mathbb{I}$ is an object $\mathbb{I}$ with a mono $[\partial_0, \partial_1]\colon 1+1 \to \mathbb{I}$ and *connections* $\wedge, \vee\colon \mathbb{I} \times \mathbb{I} \to \mathbb{I}$ such that the following equalities hold:

$$
\begin{aligned}
&\wedge(\mathrm{id}_{\mathbb{I}} \times \partial_0) = \partial_0\pi_1\colon \mathbb{I}\times 1 \to \mathbb{I}; &\quad &\vee(\mathrm{id}_{\mathbb{I}} \times \partial_1) = \partial_1\pi_1\colon \mathbb{I}\times 1 \to \mathbb{I};\\
&\wedge(\partial_0 \times \mathrm{id}_{\mathbb{I}}) = \partial_0\pi_0\colon 1\times \mathbb{I} \to \mathbb{I}; &&\vee(\partial_1 \times \mathrm{id}_{\mathbb{I}}) = \partial_1\pi_0\colon 1\times \mathbb{I} \to \mathbb{I};\\
&\wedge(\mathrm{id}_{\mathbb{I}} \times \partial_1) = \pi_0\colon \mathbb{I}\times 1 \to \mathbb{I}; &&\vee(\mathrm{id}_{\mathbb{I}} \times \partial_0) = \pi_0\colon \mathbb{I}\times 1 \to \mathbb{I};\\
&\wedge(\partial_1 \times \mathrm{id}_{\mathbb{I}}) = \pi_1\colon 1\times \mathbb{I} \to \mathbb{I}; &&\vee(\partial_0 \times \mathrm{id}_{\mathbb{I}}) = \pi_1\colon 1\times \mathbb{I} \to \mathbb{I}.
\end{aligned}
$$

The requirements for the connections say that $\wedge$ and $\vee$ behave somewhat like the meet and join operators of a lattice.

Finally, we require that the map $[\partial_0, \partial_1]\colon 1+1 \to \mathbb{I}$ to be a D-map (recall Definition 7.1.7). We refer to the D-maps as *cofibrations*.

Definition 8.1.2. Let f, g and be two morphisms. Then we say that f has the *left lifting property (LLP) with respect to* g and g has the *right lifting property (RLP) with respect to* f and we write $f \pitchfork g$ if for any commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\ C & \longrightarrow & D \end{array}$$

there is a *diagonal filler*, i.e. an arrow $l \colon C \to B$ making the resulting two triangles commute. If $\mathcal{M}$ is some class of morphisms, then we write $\mathcal{M}^{\pitchfork}$ for the class of morphisms having the RLP with respect to every morphism in $\mathcal{M}$, and ${}^{\pitchfork}\mathcal{M}$ for the class of morphisms having the LLP with respect to every morphism in $\mathcal{M}$.

The dominance D gives rise to a weak factorization system on the topos $\mathcal{E}$.

Definition 8.1.3. A *weak factorization system* on category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms of $\mathcal{C}$ such that

(i) every morphism $f \colon X \to Y$ of $\mathcal{C}$ factors as a morphism in $\mathcal{L}$ followed by a morphism in $\mathcal{R}$;

(ii) $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\pitchfork}$.

The next proposition is [Fv18, Proposition 2.6], but the proof is sparse on details. We present a detailed proof here, due to my supervisor Jaap van Oosten. For a better overview, we first present a lemma.

Lemma 8.1.4. *Let $\mathcal{C}$ be any category and suppose $\mathcal{M}$ is a class of morphisms of $\mathcal{C}$. Then*

(i) $\mathcal{M}^{\pitchfork}$ is closed under retracts;

(ii) if $\mathcal{M}$ is a class of monomorphisms closed under pullbacks in $\mathcal{C}$, then $\mathcal{M}$ is also closed under retracts.

Proof. The first item is easily shown, so we focus on the second. Suppose $f \colon A \to B$ is in $\mathcal{M}$ and let $f' \colon A' \to B'$ be a retract of f. That is, we have a commutative diagram

$$\begin{array}{ccccc} A' & \xrightarrow{i_0} & A & \xrightarrow{r_0} & A' \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f'} \\ B' & \xrightarrow{i_1} & B & \xrightarrow{r_1} & B' \end{array} \qquad r_0 i_0 = \mathrm{id}_{A'}, \quad r_1 i_1 = \mathrm{id}_{B'}$$

Of course, we prove that $f' \in \mathcal{M}$ by showing that f' is a pullback of f. First, we establish that i_0 is the equalizer of $i_0 r_0$ and id_A. For, suppose g is such that $i_0 r_0 g = g$. Then, g

factors through i_0 via $r_0 g$. The factorization is unique, because i_0 is monic. Similarly, i_1 is the equalizer of $i_1 r_1$ and id_B. We are now ready to show that

$$\begin{array}{ccc} A' & \xrightarrow{i_0} & A \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{i_1} & B \end{array}$$

is a pullback. Suppose we have arrows $a\colon C \to A$ and $b\colon C \to B'$ satisfying $fa = i_1 b$. Then

$$f i_0 r_0 a = i_1 f' r_0 a = i_1 r_1 f a = i_1 r_1 i_1 b = i_1 b = fa,$$

so that, since f is monic, $i_0 r_0 a = a$. Because i_0 is the equalizer of $i_0 r_0$ and id_A, this implies that we can factor a uniquely as $a = i_0 c$ for some $c\colon C \to A'$. It remains to show that $f'c = b$. We have seen that $i_1 r_1 f a = fa$, so fa factors uniquely through i_1. But observe that b and $f'c$ are two such factorizations. Hence, they are equal as desired. This establishes f' as the pullback of f. As $\mathcal{M}$ is assumed to be closed under pullbacks, we are done. ∎

We are now in position to prove that the dominance D yields a weak factorization system on $\mathcal{E}$.

Proposition 8.1.5. *Let us write $\mathcal{D}$ for the class of D-maps. The pair $(\mathcal{D}, \mathcal{D}^{\pitchfork})$ is a weak factorization system on $\mathcal{E}$.*

Proof. By the previous lemma $\mathcal{D}$ and $\mathcal{D}^{\pitchfork}$ are closed under retracts. So, by the retract argument [Rie09, Lemma 11.2.3], only the factorization needs to be proved. Thus, let $h\colon B \to A$ be a morphism. We need to factor it as a map in $\mathcal{D}$ followed by a map in $\mathcal{D}^{\pitchfork}$.

In this proof let us employ set-theoretic notation for subobjects, that is, we write $\{x \in X \mid \varphi(x)\}$ for the subobject of the object X defined by the formula φ.

Define $\widetilde{B} = \{\alpha \in \mathcal{P}B \mid \forall x, y \in B(x \in \alpha \wedge y \in \alpha \to x = y\}$, the object of subsingletons of B. There is an obvious inclusion $B \hookrightarrow \widetilde{B}$, which we denote by η_B. Next, define $L(h)$ as

$$\{(a, \sigma, \alpha) \in A \times D \times \widetilde{B} \mid \alpha \subseteq h^{-1}(a), \mathrm{inhab}(\alpha) = \sigma\},$$

where $\mathrm{inhab}(\alpha) \equiv \exists x(x \in \alpha)$ (i.e. α is inhabited). We prove that h factors through $L(h)$. Note that $\langle h, t!_B, \eta_B\rangle\colon B \to A \times D \times \widetilde{B}$. One easily checks that this map equalizes χ and $A \times D \times \widetilde{B} \to 1 \xrightarrow{t} \Omega$, with χ the characteristic map of the subobject $L(h)$. Thus, we obtain a map $f\colon B \to L(h)$. We will say that we have defined f by $f(b) = (h(b), \top, \{b\})$. Similarly, we define $g\colon L(h) \to A$ by $g(a, \sigma, \alpha) = a$. Then clearly, h factorizes as gf.

It remains to prove that $f \in \mathcal{D}$ and that $g \in \mathcal{D}^{\pitchfork}$. Using the definition of $L(h)$, it is not hard to verify that B is the pullback of the projection $\pi_D\colon L(h) \to D$ along $t\colon 1 \to D$. Thus, $f \in \mathcal{D}$, as D-maps are closed under pullbacks (by pullback pasting).

To show that $g \in \mathcal{D}^{\pitchfork}$, assume we have a lifting problem

$$\begin{array}{ccc} U & \xrightarrow{l} & L(h) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{k} & A \end{array}$$

where i is a D-map. Write $l(u) = (k(u), p_1(u), p_2(u))$ with $u \in U$. Define the subobject $U' = \{u \in U \mid p_1(u)\}$, the pullback of p_1 along $t\colon 1 \to D$. Then $U' \subseteq_D U$, so $U' \subseteq_D V$ (by pullback pasting). Let $m\colon V \to L(h)$ be given by

$$m(v) = (k(v), (v \in U'), \{x \in B \mid v \in U' \wedge x \in p_2(v)\}) = (k(v), q_1(v), q_2(v)).$$

We should check that m is well-defined, i.e. $q_2(v) \subseteq h^{-1}(k(v))$ and $\mathrm{inhab}(q_2(v)) = q_1(v)$. For the former, suppose $y \in q_2(v)$. Then $v \in U'$ and $y \in p_2(v)$, so $h(y) = k(v)$, as desired. For the latter, note that:

$$\mathrm{inhab}(q_2(v)) = (v \in U' \wedge \mathrm{inhab}(p_2(v))) = v \in U' \wedge p_1(v) = v \in U' = q_1(v),$$

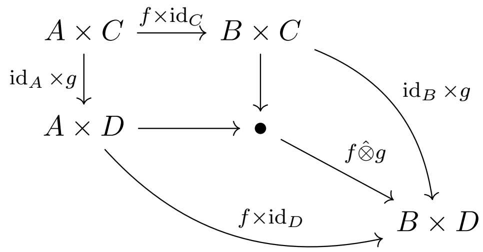
by definition of U'. Thus, m is well-defined.

Obviously, $gm = k$. Finally, if $u \in U$, then

$$\begin{aligned} m(u) &= (k(u), (u \in U'), \{x \in B \mid x \in p_2(u) \wedge u \in U'\}) \\ &= (k(u), p_1(u), \{x \in B \mid x \in p_2(u) \wedge p_1(u)\}) \\ &= (k(u), p_1(u), p_2(u)), \end{aligned}$$

so m is a solution to the lifting problem. ■

Definition 8.1.6. Suppose $f\colon A \to B$ and $g\colon C \to D$ are morphisms of $\mathcal{E}$. The *Leibniz product* (or *pushout product*) of f and g is the unique morphism $f \hat{\otimes} g$ making



commute, with the square being a pushout.

Definition 8.1.7. A morphism f is a *fibration* if it has the right lifting property with respect to all morphisms of the form $\partial_i \hat{\otimes} u$ with $i \in \{0, 1\}$ and u a cofibration. An object X is called *fibrant* if the unique map $X \to 1$ is a fibration.

Remark 8.1.8. For a topological viewpoint, observe that this definition means that an object is fibrant if and only if it enjoys the homotopy extension property (with respect to cofibrations).

Definition 8.1.9. Let $f, g\colon X \to Y$ be two parallel arrows in $\mathcal{E}$. We say that f and g are *homotopic* if there is a map $H\colon X \times \mathbb{I} \to Y$, a *homotopy*, such that $H(\mathrm{id}_X \times \partial_0) = f$ and $H(\mathrm{id}_X \times \partial_1) = g$. We will write $f \simeq g$ in this case.

Definition 8.1.10. A morphism $f\colon X \to Y$ in $\mathcal{E}$ is a *homotopy equivalence* if there is a morphism $g\colon Y \to X$ in $\mathcal{E}$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$. We call g the *homotopy inverse* of f.

Definition 8.1.11. A *model structure* on a category $\mathcal{C}$ is a triple (Cof, Fib, Weq) of classes of morphisms of $\mathcal{C}$ (called cofibrations, fibrations and weak equivalences, respectively) satisfying:

(i) Weq contains all isomorphisms and is closed under 2-out-of-3, viz. if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of $\mathcal{C}$ and two out of the tree morphisms f, g, gf are in Weq, then so is the third.

(ii) (Cof, Fib $\cap$ Weq) and (Cof $\cap$ Weq, Fib) are two weak factorization systems on $\mathcal{C}$.

Theorem 8.1.12. *The full subcategory $\mathcal{E}_f$ f of $\mathcal{E}$ on fibrant objects carries a model structure, where the fibrations and cofibrations are defined as above and the weak equivalences are the homotopy equivalences.*

Proof. See [Fv18, Theorem 4.4]. ■

Example 8.1.13. In [Fv18], the authors take $\mathcal{E} = \mathsf{Eff}$ and $D = \Omega$ (so the cofibrations are simply all monomorphisms). The interval object $\mathbb{I}$ is $\nabla(2)$ (recall Definition 5.1.9).

8.1.1 The model structure on the fibrant objects of $\mathsf{RT}(\mathcal{S})$

In $\mathsf{RT}(\mathcal{S})$, the Sierpiński assembly Σ is a dominance that is closed under finite products, as we have seen. Accordingly, the cofibrations are the Σ-maps. For the interval object, we want an object $\mathbb{I}$ such that there is a Σ-map from $1 + 1$ to $\mathbb{I}$. Observe that $1 + 1 \cong 2$, where 2 is the assembly $(\{0, 1\}, E_2)$ with $E_2(i) = \{\bar{i}\}$. In light of Lemma 7.3.4, it seems natural to take $\mathbb{I} = L(2)$, so that the required Σ-map is $\eta_2\colon 2 \to L(2)$. The connections are given as prescribed above and by putting $\bot_2 \wedge \bot_2 = \bot_2 = \bot_2 \vee \bot_2$. The map $\vee$ is tracked, because there is a continuous function $F\colon \mathbb{S}^2 \to \mathbb{S}$ such that

$$
\begin{aligned}
&F(\emptyset, \emptyset) = F(\emptyset, \bar{0}) = F(\bar{0}, \emptyset) = \emptyset;\\
&F(\emptyset, \bar{1}) = F(\bar{1}, \emptyset) = F(\bar{0}, \bar{1}) = F(\bar{1}, \bar{0}) = F(\bar{1}, \bar{1}) = \bar{1};\\
&F(\bar{0}, \bar{0}) = \bar{0};
\end{aligned}
$$

by case inspection of finite sets and the fact that this assignment is monotone. Similarly, $\wedge$ is tracked. Hence, $L(2)$ is indeed an interval object.

By Theorem 8.1.12, we have a model structure on the full subcategory $\mathsf{RT}(\mathcal{S})_f$ on fibrant objects of $\mathsf{RT}(\mathcal{S})$.

In fact, we can show that there is even a model structure on the category of fibrant assemblies.

Proposition 8.1.14. *If the dominance D is $\neg\neg$-separated, then the model structure of Theorem 8.1.12 restricts to a model structure on the full subcategory of $\mathcal{E}_f$ on fibrant $\neg\neg$-separated objects.*

In particular, since Σ is an assembly, the model structure on $\mathsf{RT}(\mathcal{S})_f$ restricts to a model structure on the category $\mathsf{Asm}(\mathcal{S})_f$ of fibrant assemblies.

Proof. It suffices to prove that the factorizations also exist in the subcategory on $\neg\neg$-separated objects.

One of the required factorizations is described in [Fv18, Proposition 4.3] by taking a pullback. Since the $\neg\neg$-separated objects of $\mathcal{E}$ are closed under finite limits in $\mathcal{E}$ [Joh02a, Lemma A4.4.3(i)], we are done here.

It remains to show that if $h\colon A \to B$ is a morphism between $\neg\neg$-separated objects, then then the object $L(h)$ from Proposition 8.1.5 is again $\neg\neg$-separated. We argue informally in the internal logic of the topos $\mathcal{E}$. The proof is a nice exercise in intuitionistic reasoning. We must show that the diagonal is $\neg\neg$-closed, viz.

$$\neg\neg((a, \sigma, \alpha) = (a', \sigma', \alpha')) \to (a, \sigma, \alpha) = (a', \sigma', \alpha')$$

should hold where a, a' are variables of sort A, σ, σ' are variables of sort D and α, α' are variables of sort $\tilde{B}$ satisfying $\sigma = \text{inhab}(\alpha), \sigma' = \text{inhab}(\alpha')$ and $\alpha \subseteq h^{-1}(a), \alpha' \subseteq h^{-1}(a')$.

Assume $\neg\neg((a, \sigma, \alpha) = (a', \sigma', \alpha'))$. Since A and D are $\neg\neg$-separated, this yields:

$$a = a' \wedge \sigma = \sigma' \wedge \neg\neg(\alpha = \alpha').$$

We are to prove that $\alpha = \alpha'$. By symmetry, it suffices to show that $\alpha \subseteq \alpha'$. To this end, suppose $b \in \alpha$. Then α is inhabited. Since $\text{inhab}(\alpha) = \sigma = \sigma' = \text{inhab}(\alpha')$, we see that α' is inhabited as well. Thus, we get some b' with $b' \in \alpha'$. We show that $b = b'$. Since B is $\neg\neg$-separated, it suffices to prove that $\neg\neg(b = b')$.

To do so, assume $\neg(b = b')$. As α' is a subsingleton, we see that $\neg(b \in \alpha')$. Hence, $\neg\forall x{:}B(x \in \alpha \to x \in \alpha')$. But, in combination with $\neg\neg(\alpha = \alpha')$, this yields a contradiction, as desired. We conclude that $\neg\neg(b = b')$, so we are done. ∎

In the next sections we examine this model structure on the assemblies. An interesting feature of our model structure is that the domain theoretic notions from the previous chapter reappear in the study of the model structure. For example, the contractible objects will be the objects with $\perp$ from Definition 7.3.5.

Remark 8.1.15. I am aware that many of the results in the coming sections can be generalized to any topos $\mathcal{E}$ with a dominance D such that the lift of $1 + 1$ is an interval object. For the sake of definiteness, however, we will only study the assemblies over $\mathcal{S}$.

8.2 Contractible assemblies and trivial fibrations

Definition 8.2.1. A morphism f of a category with a model structure is called a *trivial fibration* if it has the RLP with respect to every cofibration.

Definition 8.2.2. An object X of a category with a model structure is *contractible* if the unique morphism $X \to 1$ is a trivial fibration.

The following proposition characterizes the contractible objects, using the lift functor from the previous section.

Proposition 8.2.3. *An assembly X is contractible if and only if $X \xrightarrow{\eta_X} L(X)$ has a retraction.*

Proof. Suppose $f\colon X \to 1$ is a trivial fibration. By Lemma 7.3.4, f has the RLP with respect to η_X, so we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle \eta_X}\downarrow & \overset{r}{\nearrow} & \downarrow \\ L(X) & \longrightarrow & 1 \end{array}$$

whence the desired retraction.

Conversely, suppose we have a retraction $r\colon L(X) \to X$ of η_X. Let $i\colon U \to Y$ be a cofibration and suppose we have a lifting problem

$$\begin{array}{ccc} U & \xrightarrow{g} & X \\ \downarrow{\scriptstyle i} & & \downarrow \\ Y & \longrightarrow & 1 \end{array}$$

By Proposition 7.3.2, there is a morphism $\tilde{g}$ such that

$$\begin{array}{ccc} U & \xrightarrow{g} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \eta_X} \\ Y & \xrightarrow{\tilde{g}} & L(X) \end{array}$$

commutes. We claim that $Y \xrightarrow{\tilde{g}} L(X) \xrightarrow{r} X$ is the desired filler. Indeed, $r\tilde{g}i = r\eta_X g = g$. We conclude that f is a trivial fibration. ■

Corollary 8.2.4. *The following are equivalent for an assembly X:*

(*i*) *X is contractible;*

(*ii*) *X is an object with $\bot$;*

(*iii*) *there is a continuous $F\colon \mathbb{S} \to \mathbb{S}$ such that $F(\emptyset) \in \bigcup_{x \in |X|} E_X(x)$ and $F(U) \in E_X(x)$ for any $U \in E_X(x)$.*

Proof. If X is an object with $\bot$, then there is a retraction of η_X, so X is contractible. If X is contractible, then we have a retraction r of η_X. It remains to show that the diagram

$$\begin{array}{ccc} L^2(X) & \xrightarrow{L(r)} & L(X) \\ \big\downarrow{\scriptstyle \mu_X} & & \big\downarrow{\scriptstyle r} \\ L(X) & \xrightarrow{\quad r \quad} & X \end{array}$$

commutes. But this is easily verified. Hence, (i) $\Leftrightarrow$ (ii).

The equivalence (i) $\Leftrightarrow$ (iii) follows by writing out what it means to have a retraction that is tracked. ∎

Corollary 8.2.5. *Let X be an assembly. If $\emptyset \in \bigcup_{x\in|X|} E_X(x)$, then X is contractible.*

Proof. Take the identity function in (iii) above. ∎

Example 8.2.6. The previous propositions allow us to produce some (non-)examples of contractible objects.

(i) Any assembly of the form $L(Y)$ is contractible (μ_Y is the desired retraction). In particular, the interval object $\mathbb{I} = L(2)$ and $\Sigma \cong L(1)$ are contractible. One should compare this to [Fv18, Proposition 6.3], since Σ and $\mathbb{I}$ are thus contractible, but not uniform.

(ii) Any non-empty uniform assembly is contractible. For given such an assembly X, take $x_0 \in X$ and pick $U \in \bigcap_{x\in X} E_X(x)$. A retraction $L(X) \to X$ is then given by mapping $\bot$ to x_0. The map is tracked by $\mathsf{k}U$.

(iii) The assemblies 2 and N are not contractible, because there are no non-constant morphisms $L(2) \to 2$, $L(N) \to N$.

We can generalize the previous proposition by considering the generalized lift functor for slices.

Proposition 8.2.7. *A morphism $f\colon X \to Y$ of assemblies is a trivial fibration if and only if $[\mathrm{id}_Y, f]\colon Y \sqcup_f X \to Y$ is a retract of f in $\mathsf{Asm}(\mathcal{S})/Y$. Explicitly, the latter means that we have a morphism $r\colon Y \sqcup_f X \to X$ such that*

$$\begin{array}{ccccc} & & \overset{\mathrm{id}_X}{\frown\!\frown\!\frown} & & \\ X & \xrightarrow{\eta_f} & Y \sqcup_f X & \xrightarrow{r} & X \\ & {\scriptstyle f}\searrow & \big\downarrow & \swarrow{\scriptstyle f} & \\ & & Y & & \end{array}$$

is a commutative diagram.

Proof. Let $f\colon X \to Y$ be a trivial fibration. By Lemma 7.4.2, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ \downarrow{\scriptstyle \eta_f} & \nearrow{\scriptstyle r} & \downarrow{\scriptstyle f} \\ Y \sqcup_f X & \longrightarrow & Y \end{array}$$

whence the desired retraction.

Conversely, let $r\colon Y \sqcup_f X \to X$ be a retraction of η_f in $\mathsf{Asm}(\mathcal{S})/Y$. Suppose we have a lifting problem

$$\begin{array}{ccc} U & \xrightarrow{g} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{h} & Y \end{array}$$

where i is a cofibration. By Proposition 7.4.1, there is a morphism $\tilde{g}$ in $\mathsf{Asm}(\mathcal{S})/Y$ such that

$$\begin{array}{ccccc} U & & \xrightarrow{g} & & X \\ \downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle \eta_f} \\ Z & & \xrightarrow{\tilde{g}} & & Y \sqcup_f X \\ & {\scriptstyle h}\searrow & & \swarrow{\scriptstyle [\mathrm{id}_Y, f]} & \\ & & Y & & \end{array}$$

commutes. We claim that $r\tilde{g}$ is the desired filler. Indeed, $r\tilde{g}i = r\eta_f g = g$ and $fr\tilde{g} = [\mathrm{id}_Y, f]\tilde{g} = h$, as desired. ■

Corollary 8.2.8. *The following are equivalent for a morphism of assemblies $f\colon X \to Y$:*

(*i*) *f is a trivial fibration;*

(*ii*) *f is an algebra for the L_Y-monad;*

(*iii*) *there is a continuous $F\colon \mathbb{S} \to \mathbb{S}$ such that for any $y \in |Y|$ and $V \in E_Y(y)$, we have $F(\mathsf{p}\emptyset V) \in E_X(x_0)$ for some $x_0 \in |X|$ with $f(x_0) = y$ and for any $x \in |X|$ with $f(x) = y$ and $U \in E_X(x)$, it holds that $F([\,[U, \overline{1}], V] \in E_X(x)$.*

Proof. Similar to Corollary 8.2.4. ■

8.3 Fibrant assemblies

Definition 8.3.1. For any assembly X, the morphisms ∂_0 and ∂_1 induce morphisms $X^{\partial_0}\colon X^{\mathbb{I}} \to X$ and $X^{\partial_1}\colon X^{\mathbb{I}} \to X$, which we respectively refer to as the source map s_X and target map t_X. If the context is clear, then we will drop the subscripts in the source and target maps.

Lemma 8.3.2 (Proof of Theorem 6.10 in [Fv18])**.** *An assembly X is fibrant if and only if the source map $s\colon X^{\mathbb{I}} \to X$ is a trviial fibration.*

Proof. By Proposition 3.5 in [Fv18], X is fibrant if and only if s and t are trivial fibrations. Observe that our interval object $\mathbb{I} = L(2)$ comes with a twist map $\mathrm{tw}\colon 2 \to 2$, defined by $0 \mapsto 1, 1 \mapsto 0, \bot \mapsto \bot$, which is a self-inverse and that satisfies $X^{\mathrm{tw}}s = t$ and $X^{\mathrm{tw}}t = s$. Thus, s is a trivial fibration if and only if t is. So we conclude that X is fibrant precisely when s is a trivial fibration. ∎

Definition 8.3.3 (Definition 6.9 in [Fv18])**.** Let X be an assembly and pick $x \in |X|$. A *path-connected component of* x, denoted as $[x]$ is the set of $y \in |X|$ such that there is a morphism $p\colon \mathbb{I} \to X$ (a *path in* X) with $s(p) = x$ and $t(p) = y$. We also say that x *is path connected to* y. If for any $x, y \in |X|$ we have $y \in [x]$, then we call X *path-connected.*

Proposition 8.3.4. *Let X be an assembly. If X is contractible, then X is path-connected.*

Proof. Let $x, y \in |X|$ be fixed, but arbitrary. If X is contractible, then the lifting problem

$$\begin{array}{ccc} 1+1 & \xrightarrow{[x,y]} & X \\ {\scriptstyle [\partial_0,\partial_1]}\big\downarrow & & \big\downarrow \\ \mathbb{I} & \longrightarrow & 1 \end{array}$$

has a solution. Hence, $y \in [x]$, as desired. ∎

Proposition 8.3.5. *Let X be an assembly and $x, y \in |X|$. Then, $x \in [y]$ if and only if there exist $z \in |X|$ and $U \in E_X(z), V \in E_X(x)$ and $W \in E_X(y)$ with $U \subseteq V$ and $U \subseteq W$.*

Proof. Suppose first that we have a path $p\colon \mathbb{I} \to X$ with $s(p) = y$ and $t(p) = x$. If P tracks p, then $P\emptyset \in E_X(p(\bot))$, while $P\emptyset \subseteq P\overline{0} \in E_X(y)$ and $P\emptyset \subseteq P\overline{1} \in E_X(x)$, by monotonicity of the application.

Conversely, given such $z \in |X|$ and U, V, W in $E_X(z), E_X(x)$ and $E_X(y)$, we may define $p\colon \mathbb{I} \to X$ by $p(0) = y, p(\bot) = z, p(1) = x$. Observe that p is tracked by the graph of the continuous function

$$A \mapsto \begin{cases} V \cup W & \text{if } 0, 1 \in A; \\ V & \text{if } 1 \in A, 0 \notin A; \\ W & \text{if } 0 \in A, 1 \notin A; \\ U & \text{else.} \end{cases}$$

Thus, $x \in [y]$, as we wished to prove. ∎

The following lemma gives a necessary condition for fibrancy of an assembly.

Lemma 8.3.6. *If X is a fibrant assembly, then for each $x \in |X|$, there are $w \in |X|$ and $U \in E_X(w)$ such that for each $y, z \in [x]$ there are $V \in E_X(y)$ and $W \in E_X(z)$ with $U \subseteq V, W$.*

Proof. Let X be a fibrant object. By Lemma 8.3.2 and Proposition 8.2.7, there exists a retraction $r\colon X \sqcup_s X^{\mathbb{I}} \to X^{\mathbb{I}}$. Suppose it is tracked by R. Let $x \in |X|$ be fixed, but arbitrary and let A be any element of $E_X(x)$. Set $w = tr(0,x)$ and $U = R[\emptyset, A]\overline{1}$. Now suppose $y, z \in [x]$. Find $\alpha, \beta \in X^{\mathbb{I}}$ with $s(\alpha) = s(\beta) = x$ and $t(\alpha) = y$ and $t(\beta) = z$. Further, let U_α and U_β track α and β, respectively. Set $V = R([\,[U_\alpha, \overline{1}], A])\overline{1}$ and put $W = R([\,[U_\beta, \overline{1}])A])\overline{1}$. From the fact that R tracks r, we find $U \in E_X(w), V \in E_X(y)$ and $W \in E_X(z)$. By monotonicity of the application, we furthermore see that $U \subseteq V, W$. ■

The previous lemma allows to give an example of an assembly that is not fibrant.

Example 8.3.7. The assembly $X = (\{a,b,c\}, E)$ with $E(a) = \{\{0,1\}\}, E(b) = \{\{0\}\}$ and $E(c) = \{\{1\}\}$ is *not* fibrant. Indeed, by Proposition 8.3.5, we have $b, c \in [a]$, but for every $w \in |X|$ and $U \in E(w)$ we find $U \not\subseteq \{0\}$ or $U \not\subseteq \{1\}$.

Another interesting example is the following assembly.

Example 8.3.8. Consider the assembly $X = (\mathbb{N}, E)$ with $E(0) = \{\mathbb{N}\}$ and $E(n+1) = \{\mathbb{N} \setminus \{0, \ldots, n\}\}$. By the previous lemma, it is not fibrant. Interestingly, X is an example of a path-connected assembly that is not contractible. The former is easily verified using Proposition 8.3.5. To see that it is not contractible, suppose $r\colon L(X) \to X$ is a retraction of $\eta_X\colon X \to L(X)$. Write $m = r(\bot_X)$ and let $U_r \in \mathbb{S}$ track the retraction r. By continuity, $U_r\emptyset \subseteq U_r[\mathbb{N} \setminus \{0, \ldots, m\}, \overline{1}]$, while $U_r\emptyset \in E(m)$ and $U_r[\mathbb{N} \setminus \{0, \ldots, m\}, \overline{1}] \in E(m+1)$, since r is a retraction of η_X. But there are no $V \in E(m)$ and $V' \in E(m+1)$ with $V \subseteq V'$, so X is indeed not contractible.

8.4 Order-discrete assemblies again

In [Fv18, Lemma 7.2], an assembly X is discrete if and only if every map $p\colon \mathbb{I} \to X$ factors through the terminal object. Observe that in our case, $\mathbb{I}$ is discrete, but the twist map does not factor through 1. However, we have something analogous: we can characterize the assemblies with trivial homotopy as the order-discrete assemblies.

Proposition 8.4.1. *An assembly X is order-discrete if and only if every map $p\colon \mathbb{I} \to X$ factors through the terminal object.*

Proof. If X is order-discrete, then it is immediate from Proposition 8.3.5 that any path in X factors through the terminal object.

Conversely, assume we have $x, y \in |X|$ and $U \in E_X(x), V \in E_X(y)$ with $U \subseteq V$. Define $p\colon \mathbb{I} \to X$ by $p(\bot) = x, p(0) = p(1) = y$. Note that is tracked by the graph of the continuous function

$$W \mapsto \begin{cases} V & \text{if } W \neq \emptyset; \\ U & \text{else.} \end{cases}$$

Now p factors through the terminal object, so $x = y$, as desired. ■

Corollary 8.4.2. *Every order-discrete assembly is fibrant.*

Proof. It is easy to see that if X is order-discrete, then the source map is an isomorphism. In particular, it is a trivial fibration, so by Lemma 8.3.2, we are done. ■

Example 8.4.3. The assemblies 2 and N are order-discrete and thus fibrant.

We have already remarked that 2 and N are not contractible. The following corollary shows that (non-terminal) order-discrete assemblies are never contractible. Note that we really need the strong version of discreteness, as $\mathbb{I}$ is discrete and also contractible, as we have already shown.

Corollary 8.4.4. *A non-terminal, order-discrete assembly is never path-connected. In particular, it is never contractible.*

Proof. Immediate from Proposition 8.4.1 and Proposition 8.3.4. ■

Remark 8.4.5. In Section 8.2 of [Fv18], it is remarked that in the model of [Fv18] every two paths in an assembly are equal if they have the same endpoints. In particular, the authors are unsure whether there exists a fibrant object that has non-homotopic paths with the same endpoints.

The situation is different here. Consider $X = \nabla(2)$ (recall Definition 5.1.9) and the paths $p, q\colon \mathbb{I} \to X$ given by $p(0) = p(\bot) = 0, p(1) = 1$ and $q(0) = 0, q(\bot) = q(1) = 1$. These paths have the same endpoints, but are not equal. Also note that X is fibrant (even contractible) by Example 8.2.6.

In the setting of [Fv18], it shown (Proposition 7.10) that each fibrant assembly X is homotopy equivalent to its assembly of path components. This is then used to show that the homotopy category of fibrant assemblies is equivalent to the category of modest sets. One might hope for similar results in our setting.

Proposition 8.4.6. *The full subcategory of* $\mathsf{Asm}(\mathcal{S})$ *on order-discrete assemblies of* $\mathsf{Asm}(\mathcal{S})$ *is reflective.*

Proof. Let X be an assembly. Consider the coequalizer

$$X^{\mathbb{I}} \underset{t}{\overset{s}{\rightrightarrows}} X \xrightarrow{q} X_{pc}$$

and observe that X_{pc} is the assembly $(|X|/{\sim}, E)$ where $x \sim y$ iff $x \in [y]$ and $E([x]) = \bigcup_{y\in[x]} E_X(y)$. So we may think of X_{pc} as the object of path components of X.

We will show that X_{pc} is order-discrete. To this end, one may adapt the proof of [Fv18, Proposition 7.9], but we prefer to give a direct proof here. Assume we have $[x],[y] \in X_{pc}$ and $U \in E([x]), V \in E([y])$ such that $U \subseteq V$. Find $x' \in [x]$ and $y' \in [y]$ such that $U \in E_X(x')$ and $V \in E_X(y')$. By Proposition 8.3.5, we find $x' \in [y']$, so $[x] = [y]$. Thus, X_{pc} is order-discrete.

We now prove the required universal property. Suppose $f\colon X \to Y$ is a morphism with Y order-discrete. We must prove that f factors uniquely through q. By the universal property of the coequalizer, it suffices to show that f equalizes the source and target maps of X. But this is easy, for $fs_X = s_Y f^{\mathbb{I}}$ and $ft_X = t_Y f^{\mathbb{I}}$, but $s_Y = t_Y$ by Proposition 8.4.1, so we are done. ■

Definition 8.4.7. Let $\mathcal{C}$ be a category with a model structure. The *homotopy category* of $\mathcal{C}$, denoted by $\mathrm{Ho}(\mathcal{C})$, has as objects the objects of $\mathcal{C}$ that are both fibrant and cofibrant and as morphisms the homotopy classes of morphisms of $\mathcal{C}$.

Since the order-discrete objects have no homotopy, the situation above carries over to $\mathrm{Ho}(\mathsf{Asm}_f(\mathcal{S}))$.

Proposition 8.4.8. *Let us write* $\mathsf{OrdDisAsm}_f$ *for the full subcategory of* $\mathsf{Asm}(\mathcal{S})_f$ *on fibrant order-discrete assemblies. The quotient functor* $\mathsf{OrdDisAsm}_f \to \mathrm{Ho}(\mathsf{Asm}_f(\mathcal{S}))$ *has a left adjoint.*

Proof. Given a morphism $[f]\colon X \to Y$ in $\mathrm{Ho}(\mathsf{Asm}_f(\mathcal{S}))$, we show that it factors uniquely through $[q]\colon X \to X_{pc}$. By the previous proposition, there is a unique morphism $\tilde{f}\colon X_{pc} \to Y$ such that $\tilde{f}q = f$. Clearly, then $[\tilde{f}]$ makes $[f]$ factor through $[q]$. It remains to show that it is unique. But this must be, as there is no homotopy on Y. Indeed, if $[g]$ is such that gq and f are homotopic, then $gq(x) = f(x)$ for any $x \in |X|$ as any path in Y factors through the terminal object. Hence, $g = \tilde{f}$ by the universal property in $\mathsf{Asm}_f(\mathcal{S})$. ∎

8.5 Closure properties of fibrant assemblies

So far, the only fibrant assemblies we have encountered were quite extreme from a homotopical viewpoint (either contractible or without any non-trivial paths). In this section we show that the full subcategory on fibrant assemblies has finite (co)products. We will then use this to give an example of a fibrant assembly that is not contractible and has non-trivial paths.

The following proposition follows from the dual of Lemma 11.1.4 in [Rie09]. We give a simple, direct proof here.

Proposition 8.5.1. *The class of fibrant assemblies is closed under finite products in* $\mathsf{Asm}(\mathcal{S})$.

Proof. It is easy to verify that 1 is fibrant. Now let $c\colon U \to Z$ be any cofibration and let

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & X \times Y \\ {\scriptstyle c\hat{\otimes}\partial_i}\downarrow & & \downarrow \\ Z \times \mathbb{I} & \longrightarrow & 1 \end{array}$$

be a lifting problem. If X and Y are fibrant, then the lifting problems

$$\begin{array}{ccc} \bullet & \xrightarrow{\pi_0 f} & X \\ {\scriptstyle c\hat{\otimes}\partial_i}\downarrow & & \downarrow \\ Z \times \mathbb{I} & \longrightarrow & 1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \bullet & \xrightarrow{\pi_1 f} & Y \\ {\scriptstyle c\hat{\otimes}\partial_i}\downarrow & & \downarrow \\ Z \times \mathbb{I} & \longrightarrow & 1 \end{array}$$

have solutions l_0 and l_1, respectively. A solution to the original lifting problem is then given by $\langle l_0, l_1 \rangle$. Hence, $X \times Y$ is fibrant as well. ∎

Remark 8.5.2. Observe that one may similarly show that if X, Y are contractible, then so is $X \times Y$.

Example 8.5.3. By the previous proposition, $X = \nabla(2) \times 2$ is fibrant. Observe that it is not order-discrete ($(0,0)$ and $(1,0)$ share realizers). So by Proposition 8.4.1, X has non-trivial paths. Moreover, X is not contractible. For suppose we had retraction $r\colon L(X) \to X$. Assume without loss of generalization that $\pi_1 r(\bot) = 0$. Then, $\mathsf{p}_1(R\emptyset) = \overline{0}$, where R tracks r. But this implies that $\pi_1 r(x) = 0$ for any $x \in |X|$. So r is not a retraction.

Proposition 8.5.4. *The class of fibrant assemblies forms an exponential ideal in* $\mathsf{Asm}(\mathcal{S})$.

Proof. We are to prove that if X is fibrant, then so is X^Y for any assembly Y. We will need an intermediate result: if f and g are cofibrations, then $f \times g$ is again a cofibration. It is not hard to provide the required pullback diagrams.

We use Lemma 8.3.2. Observe that $\left(X^Y\right)^{\mathbb{I}}$ is naturally isomorphic to $\left(X^{\mathbb{I}}\right)^Y$, so it suffices to show that s^Y is a trivial fibration where s is the source map $X^{\mathbb{I}} \to X$. Now suppose we are given a lifting problem

$$\begin{array}{ccc} U & \longrightarrow & \left(X^{\mathbb{I}}\right)^Y \\ \big\downarrow{\scriptstyle c} & & \big\downarrow{\scriptstyle s^Y} \\ Z & \longrightarrow & X^Y \end{array}$$

where $c\colon U \to Z$ is a cofibration. Consider the transpose lifting problem

$$\begin{array}{ccc} U \times Y & \longrightarrow & X^{\mathbb{I}} \\ {\scriptstyle c\times \mathrm{id}_Y}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ Z \times Y & \longrightarrow & X \end{array}$$

By our earlier remark, $c \times \mathrm{id}_Y$ is again a cofibration, so this has a solution as X is fibrant. The transpose of this solution then solves our original lifting problem. ■

Remark 8.5.5. By considering transpose lifting problems, one may also show that the class of contractible objects forms an exponential ideal in $\mathsf{Asm}(\mathcal{S})$.

The interval object $\mathbb{I}$ is strongly indecomposable, in the following sense.

Lemma 8.5.6. *For any two assemblies X and Y, we have* $(X+Y)^{\mathbb{I}} \cong X^{\mathbb{I}} + Y^{\mathbb{I}}$.

Proof. This is similar to Proposition 5.2.5. ■

Proposition 8.5.7. *The class of fibrant assemblies is closed under finite coproducts in* $\mathsf{Asm}(\mathcal{S})$.

Proof. One easily checks that the initial assembly is fibrant. Suppose X and Y are fibrant assemblies. By Lemma 8.3.2, the source maps $s_X : X^{\mathbb{I}} \to X$ and $s_Y : Y^{\mathbb{I}} \to Y$ are trivial fibrations and we must prove that the source map $s : (X+Y)^{\mathbb{I}} \to X+Y$ is a trivial fibration as well. By the previous lemma, it suffices to show that the map $s_X + s_Y : X^{\mathbb{I}} + Y^{\mathbb{I}} \to X + Y$ is a trivial fibration.

Let $c : U \to Z$ be any cofibration and let

$$\begin{array}{ccc} U & \xrightarrow{\;f\;} & X^{\mathbb{I}} + Y^{\mathbb{I}} \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle s_X + s_Y} \\ Z & \xrightarrow{\;g\;} & X + Y \end{array}$$

be a lifting problem.

Define an assembly Z_X by putting $|Z_X| = \{z \in |Z| \mid \pi_0 g(z) = 0\}$, the set of elements of $|Z|$ that get mapped into X and letting E_{Z_X} be the appropriate restriction of E_Z. Similarly, define Z_Y and assemblies U_X and U_Y with $|U_X| = \{u \in |U| \mid \pi_0 f(u) = 0\}$. Note that the restriction c_X of c to $|U_X|$ is a well-defined map from U_X to Z_X. It is not hard to check that c_X is again a cofibration. Since s_X is assumed to be a trivial fibration, we have a solution l_X to the lifting problem

$$\begin{array}{ccc} U_X & \xrightarrow{\;f_{|U_X}\;} & X^{\mathbb{I}} \\ {\scriptstyle c_X}\downarrow & \overset{l_X}{\nearrow} & \downarrow{\scriptstyle s_X} \\ Z_X & \xrightarrow[\;g_{|Z_X}\;]{} & X \end{array}$$

Similarly, we have a solution $l_Y : Z_Y \to Y^{\mathbb{I}}$. Since $|Z| = |Z_X| \cup |Z_Y|$, the functions l_X and l_Y can be patched together to obtain a function $l : |Z| \to |X^{\mathbb{I}}| + |Y^{\mathbb{I}}|$. If we can show that this map is tracked, then this yields a solution to our original lifting problem.

To this end, we first prove that $Z_X \subseteq_\Sigma Z$. Let G and P_0 be continuous functions whose graphs track g and π_0. Note that

$$\begin{aligned} |Z_X| &= \{z \in |Z| \mid \pi_0 g(z) = 0\} \\ &= \{z \in |Z| \mid (P_0 G)^{-1}(\overline{0}) \supseteq E_Z(z)\} \\ &= \{z \in |Z| \mid (P_0 G)^{-1}(\uparrow \overline{0}) \supseteq E_Z(z)\}. \end{aligned}$$

Put $\mathcal{U}_X := (P_0 G)^{-1}(\uparrow \overline{0})$ and note that this is open. Furthermore, $\mathcal{U}_X$ witnesses that $Z_X \subseteq_\Sigma Z$. Similarly, one obtains $\mathcal{U}_Y$ witnessing $Z_Y \subseteq_\Sigma Z$. Finally, it is the case that $\mathcal{U}_X \cap \mathcal{U}_Y = \emptyset$, so that the function from $\mathbb{S}$ to $\mathbb{S}$ given by

$$V \mapsto \begin{cases} [\overline{0}, L_X V] & \text{if } V \in \mathcal{U}_X; \\ [\overline{1}, L_Y V] & \text{if } V \in \mathcal{U}_Y; \\ \emptyset & \text{else}; \end{cases}$$

where L_X and L_Y respectively track l_X and l_Y, is well-defined. Moreover, it is continuous and its graph tracks l. ■

Chapter 9

Future Research

In this section I describe some aspects that warrant further research, but that (due to time constraints and scope limitations) I have been unable to treat.

9.1 Axiomatization of second order arithmetic

My supervisor and I have given considerable thought to the following question: is it possible to give an axiomatization (à la [Tro71] or [Oos94]) of the second order arithmetic of $\mathsf{RT}(\mathcal{S})$?

We found this question to be very difficult to answer. A particular problem is that it seems hard to capture the application in the arithmetic of $\mathsf{RT}(\mathcal{S})$. In Eff, we can capture the recursive function application using the primitive recursive function symbols in the language of arithmetic. In $\mathsf{RT}(\mathcal{K}_2)$, we can capture the application of Kleene's second model, by considering the exponential N^N (where N is the natural numbers object).

Again, write S for the object of realizers $(\mathbb{S}, \{-\})$ in $\mathsf{RT}(\mathcal{S})$ and recall the membership relation from Definition 7.1.3. The application map of $\mathcal{S}$ induces a morphism $S \times S \to S$, which we will also denote by juxtaposition. One might hope

$$\forall X, Y{:}S \forall x{:}N(x \in XY \to \exists y{:}N(\langle y, x\rangle \in X \wedge e_y \subseteq Y))$$

to be true in $\mathsf{RT}(\mathcal{S})$. Sadly, it is not. For if it were, then there is a realizer $U \in \mathbb{S}$ such that for any $A, B \in \mathbb{S}$ and $n \in AB$, we have: $UAB\overline{n} = \overline{k}$ with $\langle k, n\rangle \in A$ and $e_k \subseteq B$. Taking $A = \{\langle k, 0\rangle \mid k \in \mathbb{N}\}$ and $B \in \mathbb{S}$ arbitrary, we see that by continuity in A of U, we must have $U\overline{\langle 0, 0\rangle}B\overline{0} = \overline{0}$, while $U\overline{\langle 1, 0\rangle}B\overline{0} = \overline{1}$. Hence, $UAB\overline{0} \supseteq \overline{0} \cup \overline{1}$, which is impossible as $UAB\overline{0}$ must be a singleton.

9.2 Computing the homotopy category of fibrant assemblies

In [Fv18], the quotient map from a fibrant assembly X to its assembly of path components X_{pc} is a trivial fibration. This allows the authors to compute the homotopy category of the fibrant assemblies as the category of modest sets. It is a consequence of the fact that the quotient map is a trivial fibration that any fibrant, path-connected assembly is in fact contractible.

Is the quotient map also a trivial fibration in $\mathsf{Asm}_f(\mathcal{S})$? This would enable us to compute $\mathrm{Ho}(\mathsf{Asm}_f(\mathcal{S}))$ as the full subcategory on fibrant, order-discrete assemblies. I believe that the quotient map is not a trivial fibration in this case, but I have been unable to provide a counterexample. The existence of a fibrant, path-connected, non-contractible assembly (c.f. Example 8.3.8) would also settle this question. Constructing such an assembly has proved to be challenging, however. Obtaining a better understanding of fibrancy seems to be key.

9.3 Embedding of topological spaces

In his PhD thesis [Bau00], Andrej Bauer showed that the category $\mathsf{Mod}(\mathcal{S})$ of modest sets is equivalent to the category of *equilogical spaces*. In particular, there is an embedding of countably based, T_0 spaces into $\mathsf{Mod}(\mathcal{S})$. Can one use this embedding to give a model structure with a natural notion of homotopy on $\mathsf{Mod}(\mathcal{S})$ or on $\mathsf{RT}(\mathcal{S})$?

9.4 Relative realizability

Let $\mathcal{A}$ and $\mathcal{A}'$ be two partial combinatory algebras with underlying sets $\mathbb{A}$ and $\mathbb{A}'$. An *elementary inclusion of pcas* is an inclusion $\mathbb{A}' \subseteq \mathbb{A}$ such that the application of $\mathcal{A}'$ is the restriction of the application of $\mathcal{A}$ to $\mathbb{A}'$ and moreover, there are elements k, s in $\mathbb{A}'$ witnessing that $\mathcal{A}'$ and $\mathcal{A}$ are pcas.

Let us write $\mathbb{S}_{re}$ for the subset of $\mathbb{S}$ consisting of the recursively enumerable sets of natural numbers. One can define an application on $\mathbb{S}_{re}$ such that we obtain a pca $\mathcal{S}_{re}$ and moreover, the inclusion $\mathbb{S}_{re} \subseteq \mathbb{S}$ is an elementary inclusion of pcas.

This elementary inclusion gives rise to two realizability toposes. The first is $\mathsf{RT}(\mathcal{S}_{re})$ and the second is the *relative realizability topos* $\mathsf{RT}(\mathcal{S}_{re}, \mathcal{S})$. An object of the latter is an object of $\mathsf{RT}(\mathcal{S})$ such that the transitivity and symmetry are realized in $\mathcal{S}_{re}$. Similarly, a morphism in $\mathsf{RT}(\mathcal{S}_{re}, \mathcal{S})$ is a morphism of $\mathsf{RT}(\mathcal{S})$ such that the functional relation properties are realized in $\mathcal{S}_{re}$.

It would be interesting to investigate these toposes. For example, does Shanin's Principle hold in $\mathsf{RT}(\mathcal{S}_{re})$? For more information on relative realizability toposes, consult [Oos08, Section 4.5].

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