Master's Thesis

The Gluing Construction for Path Categories

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June 18, 2018

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Acknowledgements

I would like to give special thanks to some people who have helped me throughout this thesis. First of all, I am grateful to dr. Benno van den Berg for his general supervision as well as his insights at times where I got stuck. I would also like to express my appreciation for dr. Jaap van Oosten and prof. dr. Ieke Moerdijk for being my UU supervisor and second reader, respectively. Furthermore, I thank my fellow student Tom de Jong for the many discussions we had and the time he spend proofreading this thesis. I would also like to thank the many other students working in the mathematics library at Utrecht University, and the Logic LAN Party in particular, for an excellent atmosphere to write this thesis in. Finally, I give a thanks to my father, Ton de Boer, for his suggestions on spelling mistakes and grammatical errors.

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Chapter 1

Introduction

In [LS88], the *Freyd Cover* was used to proof canonicity results for higher categorical logic. Fore example, it was shown that every morphism $1 \to N$ in the free topos, where N is a natural numbers object, is of the form $S^k 0 \colon 1 \to N$ for some $k \in \mathbb{N}$. Hence, one can derive that each term of the natural number type is equal to a numeral. The Freyd Cover is a special case of a more general construction: the *gluing construction*. It turns out that this construction is a powerful tool in showing canonicity results for deductive systems.

In this thesis we will show that this construction will also be possible for a specific type of categories called *path categories*. These categories were first introduced in [BM18], where also many basic properties were established. We are partly motivated by the successes in [Shu15], where a similar construction was shown for type theoretic fibration categories and some canonicity results were proven as an application of this. The deductive system that these categories model is that of *homotopy type theory*.

This form of type theory, which is a flavour of Per Martin Löf's *intensional type theory*, is characterized by the inclusion of the so called *identity type* $\mathrm{id}_A(-,-)$ for a type A. If $a, b \colon A$, then a term of the type $\mathrm{id}_A(a,b)$ can be interpreted as a proof that a and b are equal. If the type $\mathrm{id}_A(a,b)$ is inhabited we say that a and b are *propositional equal*.

An identity type $\mathrm{id}_A(-,-)$ is inductively defined by *reflexivity terms* $r(a) : \mathrm{id}_A(a,a)$ for $a \colon A$. At first, one might expect that $\mathrm{id}_A(a,b)$ is inhabited precisely when $a = b : A$ is derivable, i.e. when a and b are *judgementally equal*, and that any term will be equal to a reflexivity terms (or at least propositionally). However, it was shown in [HS98] that this is in fact not the case.

A difference with other categorical models of homotopy type theory is that a path category does not come equipped with a weak factorisation system. Instead, if we have a commuting square with an equivalence on the left and fibration on the right, the diagonal filler makes the upper triangle commute only up to fibrewise homotopy. A consequence of this is that these categories model a homotopy type theory in which the elimination rule of the identity types holds propositionally instead of judgementally. We will call such identity types *propositional identity types*. It was shown in [Ber16] that the syntactical category of a homotopy type theory with propositional identity types is a path category.

It is no coincidence that the author of [Ber16] is interested in categorical models of homotopy type theory with propositional identity types. In fact, as mentioned in the introduction of [Ber18], he wishes to replace all judgemental equality in elimination rules by propositional ones, not just for the identity types. Because of this, the usual constructions, such as a natural numbers object and exponentials, will have a homotopy counterpart in a path category where all uniqueness and commutativity hold only up to (fibrewise) homotopy. We will refer to these kind of objects as *homotopy universal constructions*. The ones that will be discussed in this thesis were already introduced in [BM18].

Let us give an overview of the content of this thesis. In Chapter 2 we start by giving the definition of a path category and some of its basic properties. An important concept of homotopy type theory (and thus of path categories) is that of transport, which will be introduced in Chapter 3. The homotopy universal constructions will be treated in Chapter 4. And finally, in Chapter 5, we will introduce the gluing construction for path categories and show when such a construction preserves the homotopy universal constructions. After this we will discuss some options for future research in Chapter 6.

Throughout this thesis, we assume the reader is familiar with basic notions of category theory. Although a good part of the motivation is based on type theory, we do not presuppose any knowledge of this subject. It can however help to appreciate some of the results as well as give some intuition behind the different concepts.

Chapter 2

Path Categories

In this chapter we give a general description of path categories and in particular the notion of homotopy in such a category. All results are already discussed in [Ber16], [BM18] and/or [Ber18].

2.1 Definition of a path category

Let us start by giving the definition of a path category.

Definition 2.1.1 ([Ber18])**.** A *path category* is a category $\mathcal{C}$ together with two classes of maps called *fibrations* and *equivalences*. In diagrams these will be represented respectively by $\twoheadrightarrow$ and $\rightsquigarrow$. Maps that are both an equivalence and a fibration are called *trivial fibrations*. Given an object X and a factorisation of the diagonal $X \rightsquigarrow PX \twoheadrightarrow X \times X$ as an equivalence followed by a fibration, we call PX a *path object* of X. We will denote these maps by r_X and $\langle s_X, t_X \rangle$, respectively and drop the subscript if no confusion can arise. Furthermore, the following axioms should hold:

1. Isomorphisms are fibrations and fibrations are closed under composition.
2. $\mathcal{C}$ has a terminal object 1 and $X \to 1$ is always a fibration.
3. The pullback of a fibration along any other map exists and is again a fibration.
4. Isomorphisms are equivalences and equivalences satisfy 2-out-of-6. That is, if f, g, h are maps such that hg and gf are equivalences, then f, g, h and hgf are equivalences as well.
5. Every object X has at least one path object.
6. Trivial fibrations are stable under pullback along arbitrary maps.
7. Trivial fibrations have sections.

Remark 2.1.2. We will start with a couple of observations about these axioms:

- Axioms 2 and 3 imply that $\mathcal{C}$ has finite products and that all projections from products are fibrations.

- As $\mathcal{C}$ has finite products it makes sense to ask whether a given object X has a path object. This justifies the inclusions of axiom 5, without explicitly demanding that $\mathcal{C}$ has binary products.
- If a map $\langle f, g\rangle : Y \to X \times X$ is a fibration, both f and g will also be fibrations.
- As equivalences satisfy 2-out-of-6, they will also satisfy the weaker statement 2-out-of-3, i.e. if f and g are maps and two of f, g and gf are equivalences then the other will also be an equivalence. Throughout this thesis we will only explicitly need 2-out-of-3.
- Given a path object PX of X with fibration $\langle s, t\rangle : PX \twoheadrightarrow X \times X$, both s and t are trivial fibrations.
- As we will rely heavily on results from [BM18], it might not become clear from this thesis why a path category should satisfy these specific axioms. If the reader wishes to acquire a better understanding of some of the choices made for this we highly recommend reading [BM18] beforehand.

Lemma 2.1.3 ([BM18])**.** *In a path category any map $f : Y \to X$ factors as $f = p_f w_f$ where p_f is a fibration and w_f is a section of a trivial fibration (and hence an equivalence).*

Proof. Because parts of this proof will play a role throughout this thesis, we will give the details. Following the remark above, we can take the following pullback

$$\begin{tikzcd} P_f \arrow[r, "\pi_{PX}"] \arrow[d, squiggly, "\pi_Y"] & PX \arrow[d, squiggly, "s"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

as s is a trivial fibration. We put $w_f := \langle 1, rf\rangle : Y \to P_f$ and $p_f := t\pi_{PX} : P_f \to X$. Then $p_f w_f = f$ and w_f is a section of π_Y, and hence an equivalence by 2-out-of-3. One can show that the following diagram

$$\begin{tikzcd} P_f \arrow[r, "\pi_{PX}"] \arrow[d, "\langle \pi_Y, p_f\rangle"'] & PX \arrow[d, "\langle s,t\rangle"] \\ Y \times X \arrow[r, "f\times 1"] & X \times X \end{tikzcd}$$

is a pullback as well. Thus we conclude that $\langle \pi_Y, p_f\rangle$ is a fibration, from which it follows that p_f is also a fibration by Remark 2.1.2. $\square$

Corollary 2.1.4 ([BM18])**.** *Any equivalence $f : Y \to X$ factors as $f = p_f w_f$ where p_f is a trivial fibration and w_f is a section of a trivial fibration.*

An important fact about path categories is that they are stable under slicing.

Definition 2.1.5 ([BM18]). Let $\mathcal{C}$ be a path category and X an object of $\mathcal{C}$. We define $\mathcal{C}(X)$ as the full subcategory of $\mathcal{C}/X$ whose objects are fibrations with codomain X. This is again a path category with the equivalences and fibrations as in $\mathcal{C}$. Note that path objects in $\mathcal{C}(X)$ can be constructed by a factorisation of $\Delta_Y : Y \to Y \times_X Y$ in $\mathcal{C}$. We write $P_X Y$ for the path object of $Y \twoheadrightarrow X$ in $\mathcal{C}(X)$. We call such an object a *fibred path object* of Y over X.

Remark 2.1.6. Suppose we have a path category $\mathcal{C}$ and a fibration

$$f\colon (q\colon X \twoheadrightarrow I) \twoheadrightarrow (p\colon I' \twoheadrightarrow I) \text{ in } \mathcal{C}(I).$$

Then f is an object in $(\mathcal{C}(I))(p)$, i.e. the full subcategory of $\mathcal{C}(I)/p$ whose objects are fibrations with codomain p. It is not difficult to see that $P_{I'}X \twoheadrightarrow I'$ defines a fibred path object of $P_p f$.

It is also the case that a base change functor preserves the structure of a path category.

Lemma 2.1.7 ([Ber18]). *Let $\mathcal{C}$ be a path category. For any morphism $f\colon B \to A$ the functor $f^*\colon \mathcal{C}(A) \to \mathcal{C}(B)$ obtained by pulling back fibrations along f, preserves all the structure of a path category. That is, it preserves fibrations, equivalences, the terminal object and pullbacks of fibrations along arbitrary maps*

From this, the following result can be shown.

Lemma 2.1.8 ([BM18]). *The pullback of an equivalence along a fibration is again an equivalence.*

Remark 2.1.9. Because of Lemma 2.1.7 we will introduce the following convention. Whenever we have a pullback

$$\begin{tikzcd} A \arrow[r, "\pi_B"] \arrow[d, two heads, "\pi_C"'] & B \arrow[d, two heads, "g"] \\ C \arrow[r, "f"] & D \end{tikzcd}$$

where g is a fibration, we choose $f^*(P_D B)$ as a fibred path object $P_C A$. Indeed, as the base change preserves both equivalences and fibrations, this object will be a path object of A in $\mathcal{C}(C)$. Informally this means, that paths in A in the fibre of a point c in C are just paths in B in the fibre of $f(c)$. As the fibration $A \twoheadrightarrow C$ can arise from multiple pullback squares, it should always be clear for which pullback square one uses this convention.

2.2 Homotopy in a path category

In a path category $\mathcal{C}$ one can define a homotopy relation on parallel arrows as follows:

Definition 2.2.1 ([BM18]). Two parallel arrows $f, g\colon Y \to X$ are *homotopic*, if there is a path object PX for X and a map $H\colon Y \to PX$ (the *homotopy*) such that $sH = f$ and $tH = g$. We write $f \simeq g$ or $H\colon f \simeq g$ if we want to stress the homotopy.

Remark 2.2.2. It was established in [BM18] that homotopy defines a congruence on the hom-sets of $\mathcal{C}$. Hence, we can define the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ where we identify all homotopic maps in $\mathcal{C}$.

It is now natural to define a notion of homotopy equivalence.

Definition 2.2.3. A map $f\colon X \to Y$ is a *homotopy equivalence* if there exists a map $g\colon Y \to X$ (a *homotopy inverse*) such that the compositions fg and gf are homotopic to the identities on Y and X, respectively. If such a homotopy equivalence between X and Y exists, then we say that X and Y are *homotopy equivalent*.

There is also a notion of fibrewise homotopy as follows

Definition 2.2.4. Suppose $p\colon X \twoheadrightarrow I$ is a fibration and $f, g\colon Y \to X$ are parallel arrows such that $pf = pg$. If there is a map $H\colon Y \to P_I X$ such that $sH = f$ and $tH = g$, then we say that f and g are *fibrewise homotopic* and we write $f \simeq_I g$ or $H\colon f \simeq_I g$ if we want to stress the homotopy.

Remark 2.2.5. Note that if the composite $pf = pg$ is also a fibration, then fibrewise homotopy reduces to the usual homotopy in $\mathcal{C}(I)$. Also, if $I = 1$, fibrewise homotopy reduces to the usual homotopy in $\mathcal{C}$.

Remark 2.2.6. Recall from Remark 2.1.9 that if we have a pullback

$$\begin{tikzcd} A \arrow[r, "\pi_B"] \arrow[d, "\pi_C"'] & B \arrow[d, two heads, "g"] \\ C \arrow[r, "f"] & D \end{tikzcd}$$

we choose $f^*(P_D B)$ as the path object of A in $\mathcal{C}(C)$. Suppose we have two maps $h_1, h_2\colon Z \to A$. In order to show that $h_1 \simeq_C h_2$ it is sufficient to show that $\pi_C h_1 = \pi_C h_2$ and $\pi_B h_1 \simeq_D \pi_B h_2$.

As mentioned in the introduction, a path category does not possess a weak factorisation system. However, we do have a weaker form of a diagonal filler, that we will call a *lower filler*.

Theorem 2.2.7 ([BM18])**.** *If*

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, squiggly, "f"] & C \arrow[d, two heads, "p"] \\ B \arrow[r, "n"] & D \end{tikzcd}$$

is a commutative square with f an equivalence and p a fibration, then there is a $l\colon B \to C$ such that $n = pl$ and $lf \simeq_D m$. We call such l a lower filler*. Moreover, such a lower filler is unique up to fibrewise homotopy over D.*

The existence of lower fillers will prove to be very useful in showing the next few properties of path categories. We start with a lemma that can help to show when two maps are (fibrewise) homotopic.

Lemma 2.2.8 ([Ber16]). *Suppose we have $f\colon X \twoheadrightarrow I$ and $g, h\colon Y \to X$ such that $fg = fh$. If $H\colon gw \simeq_I hw$ with $w\colon Z \rightsquigarrow Y$ an equivalence, then $g \simeq_I h$.*

Proof. The following diagram

$$\begin{tikzcd} Z \arrow[r, "H"] \arrow[d, rightsquigarrow, "w"'] & P_I X \arrow[d, twoheadrightarrow, "\langle s,t\rangle"] \\ Y \arrow[r, "\langle g,h\rangle"] & X \times_I X \end{tikzcd}$$

has a lower filler as w is an equivalence and $\langle s, t\rangle$ is a fibration. This is precisely a homotopy showing $g \simeq_I h$. □

Remark 2.2.9. A type-theorist might consider the previous lemma an generalisation of path induction, since all reflexivity maps are equivalences. Indeed, a lot of proofs using this lemma come down to precomposing with a reflexivity map in one way or another.

Another consequence of lower fillers is the existence of a groupoid structures.

Lemma 2.2.10 ([Ber18]). *Every object in a path category carries a groupoid structure up to homotopy. More precisely, if A is an object in a path category, PA is a path object for A and $PA \times_A PA$ is the pullback*

$$\begin{tikzcd} PA \times_A PA \arrow[r, rightsquigarrow, "\pi_1"] \arrow[d, rightsquigarrow, "\pi_0"'] & PA \arrow[d, rightsquigarrow, "t"] \\ PA \arrow[r, rightsquigarrow, "s"] & A \end{tikzcd}$$

then there are maps $\mu\colon PA \times_A PA \to PA$ and $\sigma\colon PA \to PA$ with $\langle s,t\rangle\mu = \langle s\pi_0, t\pi_1\rangle$ and $\langle s,t\rangle\sigma = \langle t,s\rangle$ such that:

1. $\mu\langle \pi_0, \mu\langle \pi_1, \pi_0\rangle\rangle \simeq_{A\times A} \mu\langle \mu\langle \pi_0, \pi_1\rangle, \pi_2\rangle\colon PA \times_A PA \times_A PA \to PA,$

2. $\mu\langle 1, rs\rangle \simeq_{A\times A} 1\colon PA \to PA,$

3. $\mu\langle rt, 1\rangle \simeq_{A\times A} 1\colon PA \to PA,$

4. $\mu\langle 1, \sigma\rangle \simeq_{A\times A} rs\colon PA \to PA,$

5. $\mu\langle \sigma, 1\rangle \simeq_{A\times A} rt\colon PA \to PA.$

Moreover, such μ and σ will be unique up to homotopy over $A \times A$.

Proof. We will start by proving the existence of μ and σ. They arise as lower fillers of the following diagrams

$$\begin{tikzcd}
PA \arrow[r, rightsquigarrow, "1"] \arrow[d, rightsquigarrow, "\langle 1, rt\rangle"] & PA \arrow[d, "\langle s,t\rangle"] \\
PA \times_A PA \arrow[r, "\langle s\pi_0, t\pi_1\rangle"] & A \times A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, rightsquigarrow, "r"] \arrow[d, rightsquigarrow, "r"] & PA \arrow[d, "\langle s,t\rangle"] \\
PA \arrow[r, "\langle t,s\rangle"] & A \times A
\end{tikzcd}$$

where $\langle 1, rt\rangle$ is an equivalence as it is the map w_t in the application of Lemma 2.1.3 to $t\colon PA \to A$. All other groupoid structure follows by some application of Lemma 2.2.8. Now suppose μ' and σ' define a different groupoid structure. Then μ' will be a lower filler of the first diagram and σ' will be a lower filler of the second diagram, since

$$r = rsr \simeq_{A\times A} \mu'\langle 1, \sigma'\rangle r = \mu'\langle r, \sigma' r\rangle = \mu'\langle rsr, \sigma r\rangle = \mu'\langle rt, 1\rangle \sigma' r \simeq_{A\times A} \sigma' r.$$

Thus, the final statement follows as lower fillers are unique up to homotopy over $A \times A$ by Theorem 2.2.7. □

We also have the possibility of lifting morphism to paths.

Lemma 2.2.11 ([BM18]). *Let $f\colon Y \to X$ be any morphism. Then there is a morphism $Pf\colon PY \to PX$ which commutes with the s and t maps and for which we have the homotopy $(Pf)r_Y \simeq_{X\times X} r_X f$. In particular, homotopy is independent of path object.*

Proof. The map Pf arises as a lower filler of

$$\begin{tikzcd}
Y \arrow[r, "r_X f"] \arrow[d, rightsquigarrow, "r_Y"] & PX \arrow[d, "\langle s_X, t_X\rangle"] \\
PY \arrow[r, "\langle f s_Y, f t_Y\rangle"] & X \times X
\end{tikzcd}$$

which exists, as r_Y is a weak equivalence and $\langle s_X, t_X\rangle$ is a fibration. □

It was established in [BM18] that the homotopy equivalences and the equivalences coincide. To show this, one uses the following property, which we state for future reference.

Lemma 2.2.12 ([BM18]). *Suppose $f\colon Y \to X$ is a section of an equivalence $g\colon X \rightsquigarrow Y$. Then f is a homotopy inverse of g and the homotopy $H\colon fg \simeq 1$ can be chosen in such a way that $Hf \simeq_{X\times X} r_X f$.*

Proof. From $gf = 1$ and 2-out-of-3 we conclude that f is an equivalence. Hence we can find H as a lower filler of

$$\begin{tikzcd}
Y \arrow[r, "r_X f"] \arrow[d, rightsquigarrow, "f"] & PX \arrow[d, "\langle s_X, t_X\rangle"] \\
X \arrow[r, "\langle fg, 1\rangle"] & X \times X.
\end{tikzcd}$$

□

Corollary 2.2.13 ([BM18])**.** *A map $f\colon Y \to X$ is a homotopy equivalence if and only if it is an equivalence.*

Finally, we end this chapter with a uniqueness result about the factorisation in a path category. This is also a direct consequence of Theorem 2.2.7.

Corollary 2.2.14 ([BM18])**.** *If a map $k\colon Y \to X$ can be written as $k = pa = qb$ where $a\colon Y \rightsquigarrow A$ and $b\colon Y \rightsquigarrow B$ are equivalences and $p\colon A \twoheadrightarrow X$ and $q\colon B \twoheadrightarrow X$ are fibrations, then A and B are homotopy equivalent. Moreover, the homotopy equivalence $f\colon A \to B$ and homotopy inverse $g\colon B \to A$ can be chosen such that $qf = p$, $pg = q$, $fa \simeq_X b$, $gb \simeq_X a$, $gf \simeq_X 1$ and $fg \simeq_X 1$.*

In particular, any two path objects of an object X are homotopic and the homotopy can be chosen in such a way that it behaves nicely with the r, s and t maps. This also justifies that in some situations we can choose a preferred (fibred) path object, like in Remark 2.1.9. There will be more instances of this later on.

Chapter 3

Transport

An key concept in homotopy type theory is that of transport. In a path category, this is modelled by the notion of a *transport structure* on fibrations. In this chapter we will discuss some properties that transport structures possess. A lot of these are needed later on. Most results have already been discussed in [Ber16], [BM18] and/or [Ber18], but some new properties will be shown as well.

3.1 Transport structure

Before we introduce the concept of a transport structure, let us first revisit the definition of P_f in the proof of Lemma 2.1.3 with the extra assumption that $f\colon Y \to X$ is a fibration. Being a pullback of f and s, we can think of an elements of P_f as a pair (y, p) where p is a path in X from $f(y)$ to some x'. The idea behind transport is that given such a pair, we want to transport y along p to a point y' in the fibre above x'. In addition, when p is a reflexivity path, we want the resulting y' to fibrewise homotopic to y. This is formulated in the following definition.

Definition 3.1.1 ([BM18])**.** Let $f\colon Y \twoheadrightarrow X$ be a fibration. A *transport structure* on f is a morphism $\Gamma_f\colon P_f \to Y$ such that $f\Gamma_f = p_f$ and $\Gamma_f w_f \simeq_X 1_Y$. We will sometimes omit the subscript f, if it can be inferred from context.

The fact that transport structures exists is yet another consequence of the existence of lower fillers.

Lemma 3.1.2 ([BM18])**.** *Every fibration $f\colon Y \twoheadrightarrow X$ carries a transport structure. Moreover, transport structures are unique up to fibrewise homotopy over X.*

Proof. A lower filler of the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "1"] \arrow[d, rightsquigarrow, "w_f"'] & Y \arrow[d, twoheadrightarrow, "f"] \\ P_f \arrow[r, "p_f"'] & X \end{tikzcd}$$

gives the desired result. □

A related notion is that of a connection structure. The idea here is to find a path $q\colon y \to y'$ such that, in a sense, $f(q) = p : f(y) \to x'$.

Definition 3.1.3 ([BM18])**.** Let $f\colon Y \twoheadrightarrow X$ be a fibration and PX a path object for X. A *connection structure* on f consists of a path object PY for Y, a fibration $Pf\colon PY \twoheadrightarrow PX$ commuting with the r, s and t maps of PX and PY, together with a morphism $\nabla\colon P_f \to PY$ such that the following diagrams

$$\begin{tikzcd} P_f \arrow[r, "\nabla"] \arrow[dr, "\pi_{PX}"'] & PY \arrow[d, "Pf"] \\ & PX \end{tikzcd} \qquad \begin{tikzcd} P_f \arrow[r, "\nabla"] \arrow[dr, "\pi_X"'] & PY \arrow[d, "s"] \\ & Y \end{tikzcd}$$

commute.

The following theorem was proved in [BM18] and assures the existence of a connection structure. Moreover, it also interacts nicely with transport.

Theorem 3.1.4 ([BM18])**.** *Let $f\colon Y \twoheadrightarrow X$ be a fibration in a path category $\mathcal{C}$ and assume that PX is a path object for X and $\Gamma\colon P_f \to Y$ is a transport structure on f. Then we can construct a path object PY on Y and a fibration $Pf\colon PY \twoheadrightarrow PX$ with the following properties:*

1. *Pf commutes with the r, s and t maps on PX and PY.*
2. *There exists a connection structure $\nabla\colon P_f \to PY$ with $t\nabla = \Gamma$.*

In particular, every fibration $f\colon Y \twoheadrightarrow X$ carries a connection structure.

Proof. We will only present the construction, as details are worked out in [BM18]. Let $H\colon \Gamma w_f \simeq_X 1$ and define PY as the pullback

$$\begin{tikzcd} PY \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] & P_X Y \arrow[d, squiggly, "s"] \\ P_f \arrow[r, "\Gamma"] & Y \end{tikzcd}$$

with $r_Y := \langle w_f, H\rangle\colon Y \to PY$ and $\langle s_Y, t_Y\rangle := \langle \pi_Y\pi_0, t\pi_1\rangle\colon PY \to Y \times Y$. Finally, we put $Pf := \pi_{PX}\pi_0$ and $\nabla := \langle 1, r\Gamma\rangle\colon P_f \to PY$. Informally, one can think of an element of PY as a point y of Y, a path $p\colon f(y) \to x'$ and a path $q\colon \Gamma_f\langle y, p\rangle \to y'$ in the fibre of x'. This then defines a path $y \to y'$ in Y by concatenating $\nabla\langle y, p\rangle$ with q. □

Remark 3.1.5. We introduce the following convention. Whenever we are presented with a fibration $Y \to X$ and have chosen some path object PX together with a transport structure on f, we will assume that PY is constructed as in the previous theorem. This means that we will also have a fibration Pf and a connection structure ∇ at our disposal.

An consequence of Theorem 3.1.4 is the next result.

Lemma 3.1.6 ([BM18]). *If a triangle*

$$\begin{tikzcd} & Y \arrow[d, two heads, "p"] \\ Z \arrow[ur, "f"] \arrow[r, "g"'] & X \end{tikzcd}$$

with a fibration p on the right commutes up to a homotopy $H\colon pf \simeq g$, then we can find a map $f'\colon Z \to Y$, homotopic to f, such that for f' the triangle commutes strictly, that is, $pf' = g$.

Proof. See [BM18] for details. A good thing to note, and something that will be used throughout this thesis, is that $f' := \Gamma_p\langle f, H\rangle$. □

3.2 Properties of transport

Transport structures have numerous nice properties. These results are often a consequence of Lemma 2.2.8. The following properties of transport are already known.

Lemma 3.2.1 ([Ber18]). *Let $f\colon Y \twoheadrightarrow X$ be a fibration and Γ a transport structure on f. Then,*

1. *Γ preserves fibrewise homotopy. More precisely, the maps*

$$\Gamma\langle s\pi_0, \pi_1\rangle, \Gamma\langle t\pi_0, \pi_1\rangle\colon P_X Y \times_X PX \to Y$$

   are fibrewise homotopic over X.

2. *If two paths in X share the same endpoints and are homotopic relative those endpoints, then the transports of a point in Y along both paths are fibrewise homotopic. More precisely, the maps*

$$\Gamma\langle \pi_0, s\pi_1\rangle, \Gamma\langle \pi_0, t\pi_1\rangle\colon Y \times_X P_{X\times X}(PX) \to Y$$

   are fibrewise homotopic over X.

3. *The transport of a point in Y along the concatenation of two paths is fibrewise homotopy to first transporting it along the first and then along the second path. More precisely, the maps*

$$\Gamma\langle \pi_0, \mu\langle \pi_1, \pi_2\rangle\rangle, \Gamma\langle \Gamma\langle \pi_0, \pi_1\rangle, \pi_2\rangle\colon Y \times_X PX \times_X PX \to Y$$

   are fibrewise homotopic.

Remark 3.2.2. These properties will validate the following abuse of notation. Suppose we are given a sequence of paths $p_1, \ldots, p_n \colon Z \to PX$ such that the target of p_i is equal to the source of p_{i+1} together with a point $y \colon Z \to Y$ above the source of p_1. We will write $\Gamma\langle y, p_1, \ldots, p_n\rangle \colon Z \to Y \times_X PX \times_X \cdots \times_X PX \to Y$ for the result of transporting y along all of the p_i. The previous lemma, combined with Lemma 2.2.10, asserts that it does not matter, up to fibrewise homotopy over X, whether we first concatenate some of the paths and/or transport in multiple steps, and thus we will usually omit this information.

Corollary 3.2.3. *Let $f \colon Y \twoheadrightarrow X$ be a fibration with a transport structure Γ. The result of transporting a point y in Y back and forth along a path in X stays fibrewise homotopic to y. More precisely, the maps*

$$\Gamma\langle \pi_Y, \pi_{PX}, \sigma\pi_{PX}\rangle, \pi_0 \colon P_f \to Y$$

are fibrewise homotopic over X.

Proof. From Lemma 2.2.10 we conclude

$$\Gamma\langle \pi_Y, \pi_{PX}, \sigma\pi_{PX}\rangle \simeq_X \Gamma\langle \pi_Y, \mu\langle 1, \sigma\rangle\pi_{PX}\rangle \simeq_X \Gamma\langle \pi_Y, rs\pi_{PX}\rangle = \Gamma\langle 1, rf\rangle\pi_Y \simeq_X \pi_Y.$$

Note that in the expression $\Gamma\langle \pi_Y, \pi_{PX}, \sigma\pi_{PX}\rangle$, it does not matter whether this represents $\Gamma\langle\Gamma\langle \pi_Y, \pi_{PX}\rangle, \sigma\pi_{PX}\rangle$ or $\Gamma\langle \pi_Y, \mu\langle \pi_{PX}, \sigma\pi_{PX}\rangle\rangle$ as both are fibrewise homotopic over X anyway. □

Remark 3.2.4. A similar result holds if the point y lies above the end point of the path instead, i.e $\Gamma\langle \pi_Y, \sigma\pi_{PX}, \pi_{PX}\rangle \simeq_X \pi_0 \colon Y \times_X PX \to Y$, but here $Y \times_X PX$ is the pullback

$$\begin{tikzcd}
Y \times_X PX \arrow[r] \arrow[d, squiggly] & PX \arrow[d, squiggly, "t"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

instead of P_f. To prevent confusion we did not include it explicitly in Corollary 3.2.3, but the proof is essentially the same.

We end this section with two properties, that to our knowledge have not yet been presented elsewhere The first resolves around the interplay between sections and transport. It is also a good example of an application of Lemma 2.2.8.

Lemma 3.2.5. *Suppose $f \colon Y \to X$ is a fibration with transport structure Γ. Suppose that $g \colon X \to Y$ is a section of f. Then $\Gamma\langle gs, 1\rangle \simeq_X gt \colon PY \to X$.*

Proof. We will apply Lemma 2.2.8 with the equivalence r. As g is a section of f we have

$$f\Gamma\langle gs, 1\rangle = t\pi_{PX}\langle gs, 1\rangle = t = fgt \text{ and } \Gamma\langle gs, 1\rangle r = \Gamma\langle g, r\rangle = \Gamma\langle 1, rf\rangle g \simeq_X g = gtr.$$

We conclude that $\Gamma\langle gs, 1\rangle \simeq_X gt$. □

The second result states that one can pull back a transport structure in the following sense.

Lemma 3.2.6. *Suppose the following square*

$$\begin{tikzcd} A \arrow[r, "v"] \arrow[d, "g"] & B \arrow[d, "f"] \\ C \arrow[r, "w"] & D \end{tikzcd}$$

is a pullback square with f a fibration and we have chosen a transport structure Γ_f on f. Then $\langle t\pi_{PC}, \Gamma_f(v \times Pw)\rangle \colon A \times_C PC \to A$ is a transport structure on g.

Proof. Consider the two pullbacks

$$\begin{tikzcd} P_g \arrow[r, squiggly, "\pi_A"] \arrow[d, "\pi_{PC}"] & A \arrow[r, "v"] \arrow[d, "g"] & B \arrow[d, "f"] \\ PC \arrow[r, squiggly, "s_C"] & C \arrow[r, "w"] & D. \end{tikzcd}$$

First, recall from Lemma 2.2.11 that there is a morphism $Pw \colon PC \to PD$ that commutes with the s and t maps and is such that $(Pw)r_C \simeq_{C\times C} r_D w$. Therefore, we see that

$$s_D(Pw)\pi_{PC} = ws_C\pi_{PC} = fv\pi_A \colon P_g \to D,$$

and thus there is $v \times Pw \colon P_g \to P_f$. Secondly, we have

$$f\Gamma_f(v \times P(w)) = t_D\pi_{PD}(v \times Pw) = t_D(Pw)\pi_{PC} = wt_C\pi_{PC} \colon P_g \to D,$$

resulting in a map $\langle t_C\pi_{PC}, \Gamma_f(v \times Pw)\rangle \colon P_g \to A$. This map has the property that $g\langle t_C\pi_{PC}, \Gamma_f(v \times Pw)\rangle = t_C\pi_{PC}$. We choose $w^*(P_D B)$ as the fibred path object of A over C in accordance with Remark 2.1.9 and by applying some of the properties of a transport structure we see that

$$\begin{aligned} \langle t_C\pi_{PC}, \Gamma_f(v \times Pw)\rangle\langle 1, r_C g\rangle &= \langle g, \Gamma_f\langle v, (Pw)r_C g\rangle\rangle \\ &\simeq_C \langle g, \Gamma_f\langle v, r_D wg\rangle\rangle \\ &= \langle g, \Gamma_f\langle v, r_D fv\rangle\rangle \\ &= \langle g, \Gamma_f\langle 1, r_D f\rangle v\rangle \\ &\simeq_C \langle g, v\rangle = 1_A. \end{aligned}$$

We conclude that $\langle t_C\pi_{PC}, \Gamma_f(v \times Pw)\rangle$ defines a transport structure on g. $\square$

Chapter 4

Homotopy Universal Constructions

It is common to consider categories with some special types of objects, e.g. a natural numbers object and exponentials. Path categories are no exception. However, as mentioned in the introduction, we are interested in modelling homotopy type theory with propositional elimination rules. The objects of our interest will therefore be a bit different than usual: the universal properties will hold only up to (fibrewise) homotopy. These objects were first introduced in [BM18] and some basic properties were established. In this chapter we will also prove some new properties that these objects possess.

4.1 Homotopy natural numbers object

A homotopy natural numbers object can be seen as a collection of disjoint contractible spaces $(C_i)_{i\in\omega}$ together with a point z in C_0 and a function $S\colon \bigcup_{i\in\omega} C_i \to \bigcup_{i\in\omega} C_i$ such that for a point x in C_i we have $S(x_i)$ in C_{i+1}.

Definition 4.1.1. An object $\mathbb{N}$ together with maps $z\colon 1 \to \mathbb{N}$ and $S\colon \mathbb{N} \to \mathbb{N}$ is a *homotopy natural numbers object* (*hnno*) if for any commutative diagram of the form

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "p"] & X \arrow[d, "p"] \\ 1 \arrow[ur, "x"] \arrow[r, "z"'] & \mathbb{N} \arrow[r, "S"'] & \mathbb{N} \end{tikzcd}$$

where p is a fibration, there is a section $a\colon \mathbb{N} \to X$ of p such that $az \simeq_{\mathbb{N}} x$ and $aS \simeq_{\mathbb{N}} fa$.

Remark 4.1.2. Notice the slight difference with the definition of a hnno in [BM18]. The definition presented there is equivalent to this one except the homotopies are not fibrewise (see Proposition 4.13 therein). After some discussion with the author of [BM18] we feel that this definition is a better reflection of type theory. We do however still have the weaker property:

Lemma 4.1.3. *Suppose* $(\mathbb{N}, z, S)$ *is a hnno. If there are maps* $y\colon 1 \to Y$ *and* $g\colon Y \to Y$*, then there is a map* $h\colon \mathbb{N} \to Y$*, unique up to homotopy, such that* $hz \simeq y$ *and* $hS \simeq gh$*.*

Proof. As our definition of a hnno is a strengthening of the right hand side of the biconditional statement of [BM18, Proposition 4.13], this follows by that same proposition. □

Remark 4.1.4. From the previous lemma it follows that a hnno becomes a natural numbers object in the homotopy category $\mathrm{Ho}(\mathcal{C})$.

We end this section with some properties of a homotopy natural numbers object. As far as we know these properties have not yet been stated elsewhere. The first one shows that, as one would expect, the only relevant information about z and $S(x_i)$ to define a hnno is their path component.

Lemma 4.1.5. *Suppose* $(\mathbb{N}, z, S)$ *is a hnno and* $H_z\colon z' \simeq z$ *and* $H_S\colon S' \simeq S$*, then* $(\mathbb{N}, z', S')$ *is a hnno.*

Proof. Suppose we are given a commutative diagram

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "p"] & X \arrow[d, "p"] \\ 1 \arrow[ur, "x"] \arrow[r, "z'"'] & \mathbb{N} \arrow[r, "S'"] & \mathbb{N} \end{tikzcd}$$

with p a fibration. Note that this also gives rise to the following diagram

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "p"] & X \arrow[d, "p"] \\ 1 \arrow[ur, "x"] \arrow[r, "z"'] & \mathbb{N} \arrow[r, "S"] & \mathbb{N} \end{tikzcd}$$

which commutes up to homotopy by the homotopies H_z and $H_S p$, respectively. As p is a fibration, we can use Lemma 3.1.6 to find $x' = \Gamma\langle x, H_z\rangle$ and $f' = \Gamma\langle f, H_S p\rangle$ making this diagram commute strictly. Hence, we find a section $a\colon \mathbb{N} \to X$ of p such that $az \simeq_{\mathbb{N}} x'$ and $aS \simeq_{\mathbb{N}} f'a$. But then, by the interplay between sections and transport structure presented in Lemma 3.2.5, it holds that $az' = at\sigma H_z \simeq_{\mathbb{N}} \Gamma\langle as, 1\rangle\sigma H_z = \Gamma\langle az, \sigma H_z\rangle$. Using Corollary 3.2.3, we have the homotopy

$$\Gamma\langle az, \sigma H_z\rangle \simeq_{\mathbb{N}} \Gamma\langle x', \sigma H_z\rangle = \Gamma\langle x, H_z, \sigma H_z\rangle \simeq_{\mathbb{N}} x.$$

Similarly, we have $aS' = at\sigma H_S \simeq_{\mathbb{N}} \Gamma\langle as, 1\rangle\sigma H_S = \Gamma\langle aS, \sigma H_S\rangle$ and

$$\Gamma\langle aS, \sigma H_S\rangle \simeq_{\mathbb{N}} \Gamma\langle f'a, \sigma H_S\rangle = \Gamma\langle fa, H_S pa, \sigma H_S\rangle = \Gamma\langle fa, H_S, \sigma H_S\rangle \simeq_{\mathbb{N}} fa.$$

Hence, $(\mathbb{N}, z', S')$ is also a hnno. □

Another property of a hnno that one would expect is that a hnno is stable under homotopic equivalence, i.e. if $(\mathbb{N}, z, S)$ is a hnno and $f\colon \mathbb{N} \rightsquigarrow X$ is an equivalence then X can be made into a hnno. However, the question whether this is true remains open. We did manage to proof this property when f is a section of a trivial fibration. There is a clear connection between this and the factorisation in $\mathcal{C}$ as stated in Lemma 2.1.3, which will become more apparent in the following chapter.

Theorem 4.1.6. *Suppose* $(\mathbb{N}, z, S)$ *is a hnno and* $f\colon \mathbb{N} \to X$, *is a section of a trivial fibration* $g\colon X \rightsquigarrow \mathbb{N}$. *Then* (X, fz, fSg) *is a hnno.*

Proof. Note first that by Lemma 2.2.12 we can conclude that f is a homotopy inverse of g and that we can choose $H\colon fg \simeq 1$ such that $Hf \simeq_{X\times X} rf$. Suppose we have the following commutative diagram

$$\begin{tikzcd} & Y \arrow[r, "h"] \arrow[d, "p"] & Y \arrow[d, "p"] \\ 1 \arrow[ur, "y"] \arrow[r, "fz"] & X \arrow[r, "fSg"] & X \end{tikzcd}$$

where p is a fibration. Construct the pullback

$$\begin{tikzcd} f^*Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_{\mathbb{N}}"] & Y \arrow[d, "p"] \\ \mathbb{N} \arrow[r, "f"] & X \end{tikzcd}$$

and choose $f^*(P_Y X)$ to be the fibred path object of f^*Y like in Remark 2.1.9. Notice that $fz = py\colon 1 \to X$ and

$$ph\pi_Y = fSgp\pi_Y = fSgf\pi_{\mathbb{N}} = fS\pi_{\mathbb{N}}\colon f^*Y \to X$$

so there are maps $\langle z, y\rangle\colon 1 \to f^*Y$ and $S\times h\colon f^*Y \to f^*Y$. This results into the following commutative diagram

$$\begin{tikzcd} & f^*Y \arrow[r, "S\times f"] \arrow[d, "\pi_{\mathbb{N}}"] & f^*Y \arrow[d, "\pi_{\mathbb{N}}"] \\ 1 \arrow[ur, "\langle z,y\rangle"] \arrow[r, "z"] & \mathbb{N} \arrow[r, "S"] & \mathbb{N} \end{tikzcd}$$

from which we find a section $a\colon \mathbb{N} \to f^*Y$ of $\pi_{\mathbb{N}}$ such that $az \simeq_{\mathbb{N}} \langle z, y\rangle$ and $aS \simeq_{\mathbb{N}} (S\times h)a$. By our choice of fibre path object of f^*Y, this implies that $\pi_Y az \simeq_X y$ and $\pi_Y aS \simeq_X h\pi_Y a$. We claim that $\Gamma\langle \pi_Y ag, H\rangle\colon X \to Y$ is the desired section of p, where Γ is transport strucure on p. Indeed, it is a section as $p\Gamma\langle\pi_Y ag, H\rangle = tH = 1$, so it remains to show the necessary homotopies. For the first one, we calculate

$$\Gamma\langle\pi_Y ag, H\rangle fz = \Gamma\langle\pi_Y agfz, Hfz\rangle = \Gamma\langle\pi_Y az, Hfz\rangle \simeq_X \Gamma\langle y, rfz\rangle = \Gamma\langle 1, rp\rangle y \simeq_X y,$$

where we have used some of the properties of transport, the fact that $fz = py$ and $Hf \simeq_{X\times X} rf$.

For the second one, we have to show that $\Gamma_{\langle}\pi_Y ag, H\rangle fSg \simeq_X h\Gamma\langle\pi_Y ag, H\rangle$. Recall that f is an equivalence, so we can apply Lemma 2.2.8. Postcomposing both sides with p results in fSg so we need to check that they are fibrewise homotopic after precomposing with f. On one hand we have

$$\Gamma\langle\pi_Y ag, H\rangle fSgf = \Gamma\langle\pi_Y aS, HfS\rangle \simeq_X \Gamma\langle\pi_Y aS, rfS\rangle = \Gamma\langle 1, rp\rangle\pi_Y aS \simeq_X \pi_Y aS,$$

where the last equality follows from $f = f\pi_{\mathbb{N}}a = p\pi_Y a$. On the other hand we have

$$h\Gamma\langle\pi_Y ag, H\rangle f = h\Gamma\langle\pi_Y a, Hf\rangle \simeq_X h\Gamma\langle\pi_Y a, rf\rangle \simeq_X h\Gamma\langle 1, rp\rangle\pi_Y a \simeq_X h\pi_Y a \simeq_X \pi_Y aS$$

and thus the result indeed follows from Lemma 2.2.8. □

4.2 Homotopy function spaces

Just as there is a notion of a homotopy natural numbers object, there is a notion of homotopy function spaces. In particular, we will consider (weak) homotopy exponentials and (weak) homotopy $\Pi-$types. Let us start by giving the definitions.

Definition 4.2.1. For objects X and Y in $\mathcal{C}$ a *weak homotopy exponential* is an object X^Y together with a map $\mathrm{ev}_{X,Y}\colon X^Y \times Y \to X$ such that for any map $h\colon A \times Y \to X$ there is a map $\bar{h}\colon A \to X^Y$ (called the *transpose* of h) such that

$$\begin{tikzcd} X^Y \times Y \arrow[r, "\mathrm{ev}_{X,Y}"] & X \\ A \times Y \arrow[u, "\bar{h}\times 1"] \arrow[ur, "h"'] & \end{tikzcd}$$

commutes up to homotopy. If such a transpose $\bar{h}$ is unique up to homotopy, then X^Y is a *homotopy exponential*. We will often omit the second subscript in the evaluation map.

Remark 4.2.2. It is not difficult to see that a homotopy exponential in $\mathcal{C}$ becomes an exponential in the homotopy category $\mathrm{Ho}(\mathcal{C})$.

Definition 4.2.3. Let $f\colon X \twoheadrightarrow J$ and $\alpha\colon J \twoheadrightarrow I$ be two fibrations. A *weak homotopy* Π-*type* for these fibrations is an object $\Pi_\alpha f\colon \Pi_\alpha X \twoheadrightarrow I$ in $\mathcal{C}(I)$ together with an evaluation map $\mathrm{ev}_{f,\alpha}\colon J \times_I \Pi_\alpha X \to X$ over J. Is has the following weak universal property: if there are maps $g\colon Y \to I$ and $m\colon J \times_I Y \to X$ with $fm = \pi_J$, then there exists a map $n\colon Y \to \Pi_\alpha X$ such that $(\Pi_\alpha f)n = g$ and $m\colon J \times_I Y \to X$ and $\mathrm{ev}_{f,\alpha}(1 \times n)\colon J \times_I Y \to X$ are fibrewise homotopic over J. If the map n is unique with this property up to fibrewise homotopy over I, then we call $\Pi_\alpha f$ and $\mathrm{ev}_{f,\alpha}$ a *homotopy* Π-*type*. We will omit one or both subscripts from the evaluation map, if no confusion can arise.

Remark 4.2.4. One can think of an element h of $\Pi_\alpha X$ above a point i in I as a section of f defined only on the fibre of i in J, i.e. when we have a point j in J above i then $f(\mathrm{ev}(h, j)) = j$.

Remark 4.2.5. It is possible to formulate the definition of Π-types as a kind of right adjoint to the base change functor α^*, but one has to be a bit careful (see for instance [Ber18]). For this reason we prefer the definition presented here, but it is a good idea to keep this connection in mind.

Some useful properties about these objects were established in [BM18]. As was the case there, we will only give the constructions.

Lemma 4.2.6 ([BM18])**.** *If $\mathcal{C}$ has (weak) homotopy Π-types, then each $\mathcal{C}(I)$ has (weak) homotopy exponentials.*

Proof. Given two objects Y and Z of $\mathcal{C}(I)$ one defines Z^Y in $\mathcal{C}(I)$ as $\Pi_\alpha(\pi_Y)$ where $\alpha\colon Y \twoheadrightarrow I$ and $\pi_Y\colon Z \times_I Y \twoheadrightarrow Y$. □

Lemma 4.2.7 ([BM18])**.** *Let $\mathcal{C}$ be a path category with (weak) homotopy Π-types. Given a fibration $p\colon Z \twoheadrightarrow Y$ and a (weak) homotopy exponential (Y^X, ev_Y), there is a (weak) homotopy exponential (Z^X, ev_Z) and a fibration $p^X\colon Z^X \twoheadrightarrow Y^X$ such that*

1. *The diagram*

$$\begin{tikzcd} Z^X \times X \arrow[r, "\mathrm{ev}_Z"] \arrow[d, two heads, "p^X \times 1"'] & Z \arrow[d, two heads, "p"] \\ Y^X \times X \arrow[r, "\mathrm{ev}_Y"] & Y \end{tikzcd}$$

commutes.

2. *For each T the diagram*

$$\begin{tikzcd} \mathrm{Ho}(\mathcal{C})(T, Z^X) \arrow[r] \arrow[d] & \mathrm{Ho}(\mathcal{C})(T \times X, Z) \arrow[d] \\ \mathrm{Ho}(\mathcal{C})(T, Y^X) \arrow[r] & \mathrm{Ho}(\mathcal{C})(T \times X, Y) \end{tikzcd}$$

in **Sets** *is a (quasi-)pullback.*

Proof. Given (Y^X, ev_Y) let q be the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, two heads, "q"] & Z \arrow[d, two heads, "p"] \\ Y^X \times X \arrow[r, "\mathrm{ev}_Y"] & Y \end{tikzcd}$$

and let Z^X be $\Pi_{\pi_0}(q)$, where $\pi_0\colon Y^X \times X \to Y^X$. Define $\mathrm{ev}_Z \coloneqq \pi_Z \, \mathrm{ev}_{q,\pi_0}\colon Z^X \times X \to Z$. In particular, one can construct a map $n\colon T \to Z^X$ by giving a map $g\colon T \to Y^X$ and $m\colon T \times X \to Z$ such that $\mathrm{ev}_Y(g \times 1) = pm$. We choose $\mathrm{ev}_Y^* P_Y Z$ as fibred path object of P over $Y^X \times X$ to conclude that n is such that $p^X n = g$ and $\mathrm{ev}_Z(n \times 1) \simeq_X m$. Note that in this construction we use that $(Y^X \times X) \times_{Y^X} (-)^X \cong (-)^X \times X$, resulting in the $(-) \times 1$ instead of $1 \times (-)$. □

A consequence of the preceding lemma is the following.

Lemma 4.2.8 ([BM18]). *Suppose $\mathcal{C}$ is a path category with weak homotopy Π-types, and let X^Y be a weak homotopy exponential in $\mathcal{C}$. We may choose $X^Y \times X^Y$ as a suitable weak homotopy exponential $(X \times X)^Y$ with $\mathrm{ev}_{X\times X} = \langle \mathrm{ev}_X(\pi_0 \times 1), \mathrm{ev}_X(\pi_1 \times 1)\rangle$ and choose $(PX)^Y$ as in Theorem 3.1.4, so that $\langle s^Y, t^Y\rangle \colon (PX)^Y \twoheadrightarrow X^Y \times X^Y$ is a fibration. Then the following are equivalent:*

1. *X^Y is a homotopy exponential.*
2. *There is a morphism $e\colon (PX)^Y \to P(X^Y)$ such that $\langle s, t\rangle e = \langle s^Y, t^Y\rangle$.*

Also, if both X^Y and $(PX)^Y$ are homotopy exponentials, then the canonical morphism $P(X^Y) \to (PX)^Y$ is an equivalence.

As was mentioned in [BM18], this lemma shows that homotopy exponentials are those weak homotopy exponentials that satisfy what type-theorists call *function extensionality*: two function are equal precisely when they are equal on any input. So, if a path category has homotopy exponentials as well as weak homotopy Π-types, we will say that function extensionality holds in $\mathcal{C}$ and one can choose $(PX)^Y$ as the path object of X^Y for any two objects X, Y of $\mathcal{C}$.

Corollary 4.2.9. *Let $\mathcal{C}$ be a path category with homotopy exponentials and weak homotopy Π-types. Then $(PX)^Y$, as constructed in Lemma 4.2.8, is a suitable path object for X^Y and $r^Y\colon X^Y \rightsquigarrow (PX)^Y$ can be chosen such that $\mathrm{ev}_{PX}(r^Y \times 1) \simeq_{X\times X} r\,\mathrm{ev}_X$.*

Proof. We have a map $r\,\mathrm{ev}_X\colon X^Y \times Y \to PX$ and a map $\Delta_{X^Y}\colon X^Y \to X^Y \times X^Y$. From

$$\langle s, t\rangle r\,\mathrm{ev}_X = \Delta_X\,\mathrm{ev}_X = \langle \mathrm{ev}_X(\pi_0 \times 1), \mathrm{ev}_X(\pi_1 \times 1)\rangle(\Delta_{X^Y} \times 1) = \mathrm{ev}_{X\times X}(\Delta_{X^Y} \times 1)$$

and the construction of $(PX)^Y$ we find a map $r^Y\colon X^Y \to (PX)^Y$ such that $(s^Y, t^Y)r^Y = \Delta_{X^Y}$ and $\mathrm{ev}_{PX}(r^Y \times 1) \simeq_{X\times X} r\,\mathrm{ev}_X$. The fact that this is an equivalence follows from the observation that homotopy exponentials become exponentials in the homotopy category (see the discussion at the end of the proof of [BM18, Proposition 5.7]). □

We end this section with a new result resolving around an interplay between transport structures and the construction of Lemma 4.2.7.

Lemma 4.2.10. *Let $\mathcal{C}$ be a path category with homotopy Π-types. Suppose $f\colon Y \twoheadrightarrow X$ is a fibration and X^Z a homotopy exponential. Let Y^Z and $(PX)^Z$ be constructed as in Lemma 4.2.7, i.e. they come with fibrations $f^Z\colon Y^Z \to X^Z$ and $\langle s^Z, t^Z\rangle\colon (PX)^Z \to X^Z \times X^Z$. Note that we can choose $(PX)^Z$ as path object of X^Z by Corollary 4.2.9. Suppose that Γ is a transport structure on f. Then one can construct a transport structure $\Gamma^Z\colon Y^Z \times_{X^Z} (PX)^Z \to Y^Z$ on f^Z in such a way that*

$$\mathrm{ev}_Y(\Gamma^Z \times 1) \simeq_X \Gamma\langle \mathrm{ev}_Y(\pi_0 \times 1), \mathrm{ev}_{PX}(\pi_1 \times 1)\rangle\colon (Y^Z \times_{X^Z} (PX)^Z) \times Z \to Y.$$

Proof. Note that

$$f\,\mathrm{ev}_Y(\pi_0\times 1) = \mathrm{ev}_X(f^Z\pi_0\times 1) = \mathrm{ev}_X(s^Z\pi_1\times 1) = s\,\mathrm{ev}_{PX}(\pi_1\times 1)\colon Y^Z\times_{X^Z}(PX)^Z\to X,$$

so there is a map $\langle \mathrm{ev}_Y(\pi_0\times 1), \mathrm{ev}_{PX}(\pi_1\times 1)\rangle\colon (Y^Z\times_{X^Z}(PX)^Z)\times Z\to Y\times_X PX$. Thus we have two maps $\Gamma\langle \mathrm{ev}_Y(\pi_0\times 1), \mathrm{ev}_{PX}(\pi_1\times 1)\rangle\colon (Y^Z\times_{X^Z}(PX)^Z)\times Z\to Y$ and $t^Z\pi_1\colon Y^Z\times_{X^Z}(PX)^Z\to X^Z$ such that

$$\begin{aligned} f\Gamma\langle \mathrm{ev}_Y(\pi_0\times 1), \mathrm{ev}_{PX}(\pi_1\times 1)\rangle &= t\pi_{PX}\langle \mathrm{ev}_Y(\pi_0\times 1), \mathrm{ev}_{PX}(\pi_1\times 1)\rangle \\ &= t\,\mathrm{ev}_{PX}(\pi_1\times 1) \\ &= \mathrm{ev}_X(t^Z\pi_1\times 1). \end{aligned}$$

So, since Y^Z is constructed by Lemma 4.2.7, we find a map $\Gamma^Z\colon Y^Z\times_{X^Z}(PX)^Z\to Y^Z$ with the property that

$$f^Z\Gamma^Z = t^Z\pi_1 \text{ and } \mathrm{ev}_Y(\Gamma^Z\times 1)\simeq_X \Gamma\langle \mathrm{ev}_Y(\pi_0\times 1), \mathrm{ev}_{PX}(\pi_1\times 1)\rangle.$$

It remains to show that $\Gamma^Z\langle 1, r^Zf^Z\rangle\simeq_{X^Z} 1_{Y^Z}$. We will use the fact that $\mathcal{C}$ has homotopy Π-types for this. We have $f^Z\Gamma^Z\langle 1, r^Zf^Z\rangle = t^Z\pi_1\langle 1, r^Zf^Z\rangle = t^Zr^Zf^Z = f^Z$ and by Corollary 4.2.9 and Lemma 4.2.8, we have the homotopy

$$\begin{aligned} \mathrm{ev}_Y(\Gamma^Z\langle 1, r^Zf^Z\rangle\times 1) &\simeq_X \Gamma\langle \mathrm{ev}_Y(\pi_0\times 1), \mathrm{ev}_{PX}(\pi_1\times 1)\rangle(\langle 1, r^Zf^Z\rangle\times 1) \\ &= \Gamma\langle \mathrm{ev}_Y, \mathrm{ev}_{PX}(r^Zf^Z\times 1)\rangle \\ &\simeq_X \Gamma\langle \mathrm{ev}_Y, r\,\mathrm{ev}_X(f^Z\times 1)\rangle \\ &= \Gamma\langle \mathrm{ev}_Y, rf\,\mathrm{ev}_Y\rangle \\ &= \Gamma\langle 1, rf\rangle\,\mathrm{ev}_Y \\ &\simeq_X \mathrm{ev}_Y\,. \end{aligned}$$

But as $f^Z1_{Y^Z} = f^Z$ and $\mathrm{ev}_Y(1_{Y^Z}\times 1) = \mathrm{ev}_Y$ we conclude that $\Gamma^Z\langle 1, r^Zf^Z\rangle\simeq_{X^Z} 1_{Y^Z}$ by the uniqueness property of homotopy Π-types. ☐

Chapter 5

The Gluing Construction for Path Categories

In this chapter we will present a gluing construction for path categories. As was pointed out in the introduction, this construction can aid in proving canonicity results fo our deductive system: homotopy type theory with proposition identity types. We will show that the resulting category is again a path category and give sufficient criteria for when it possesses certain homotopy universal constructions. To our knowledge all results in this chapter are new, but whenever they draw inspiration from previous work this will be mentioned accordingly.

5.1 The gluing of two path categories

Definition 5.1.1. We call a functor $F\colon \mathcal{C} \to \mathcal{D}$ between two path categories *exact*, if it preserves the structure of a path category, i.e. the terminal object, equivalences, fibrations and the pullbacks along a fibration.

By Lemma 2.1.7 we have already seen that a base change functor is an example of an exact functor. An exact functor will have some additional properties.

Lemma 5.1.2. *An exact functor F preserves (fibred) path objects and (fibrewise) homotopies. Moreover, if μ and σ define a groupoid structure on B in the sense of Lemma 2.2.10, then $F(\mu)$ and $F(\sigma)$ define a groupoid structure on FB for the path object $F(PB)$ of FB.*

Proof. As F preserves both equivalences as well as fibrations we immediately see that F preserves (fibred) path objects. Moreover, if $H\colon f \simeq_I g$ then $F(H)\colon F(f) \simeq_{FI} F(g)$ (where usual homotopy is the special case when $I = 1$, by Remark 2.2.5). From this it follows that F preserves a groupoid structure as well. □

Remark 5.1.3. In light of the above, given an object $B \twoheadrightarrow I$ of $\mathcal{C}(I)$ together with a fibred path object $P_I B$ and groupoid structure μ and σ, we will choose $F(P_I B)$ as the fibred path object of FB with $r_{FB} := F(r)$ and $\langle s_{FB}, t_{FB}\rangle := \langle F(s), F(t)\rangle$ and choose $F(\mu)$ and $F(\sigma)$ as its groupoid structure. Some care should be taken whenever we have two object A and B of $\mathcal{C}$ such that $FA = FB$. Namely, it should always be clear with regard to which object of $\mathcal{C}$ the choice is made.

From now one, we will assume the existence of an exact functor $F\colon \mathcal{C} \to \mathcal{D}$ between two path categories $\mathcal{C}$ and $\mathcal{D}$. We continue by given the definition of the gluing category of F.

Definition 5.1.4. The *gluing category* of F is the category $\mathrm{GL}(F)$ defined by:

- Objects: (X, A, α) where $X \in \mathcal{D}$, $A \in \mathcal{C}$ and $\alpha\colon X \twoheadrightarrow FA$ a fibration in $\mathcal{D}$.
- Morphisms: $f := (f_0, f_1)\colon (X, A, \alpha) \to (Y, B, \beta)$, where $f_0\colon X \to Y$ and $f_1\colon A \to B$ such that the following square

$$\begin{tikzcd} X \arrow[r, two heads, "\alpha"] \arrow[d, "f_0"] & FA \arrow[d, "F(f_1)"] \\ Y \arrow[r, two heads, "\beta"] & FB \end{tikzcd}$$

commutes. Composition of morphisms is defined component-wise.

The first thing to observe is that $\mathrm{GL}(F)$ is a path category.

Lemma 5.1.5. *The category* $\mathrm{GL}(F)$ *carries the structure of a path category with a morphism* $f\colon (X, A, \alpha) \to (Y, B, \beta)$ *being:*

- *an equivalence, if both f_0 and f_1 are equivalences.*
- *a fibration, if f_1 is a fibration and the induced arrow*

$$\langle f_0, \alpha\rangle\colon X \twoheadrightarrow \beta^*(FA) = Y \times_{FB} FA$$

is a fibration. In particular f_0 will also be a fibration as it is the composition of $X \twoheadrightarrow Y \times_{FB} FA \twoheadrightarrow Y$, where the second map is a fibration as it is the pullback of $F(f_1)$.

Proof. We check the axioms. We draw inspiration from constructions in [Shu15, Theorem 8.8].

1. *Isomorphisms are fibrations and fibrations are closed under composition.*
Let $f\colon (X, A, \alpha) \to (Y, B, \beta)$ be an isomorphism. Then f_0 and f_1 will be isomorphisms and thus fibrations. The pullback $Y \times_{FB} FA$ will be isomorphic to Y and so the induced arrow will simply be isomorphic to f_1 and we conclude that f is a fibration.

   Suppose now that we have a fibration $f\colon (X, A, \alpha) \twoheadrightarrow (Y, B, \beta)$ and a fibration $g\colon (Y, B, \beta) \twoheadrightarrow (Z, C, \gamma)$. Then $g_1 f_1$ will be a fibration. Moreover, the induced arrow $\langle g_0 f_0, \alpha\rangle\colon X \to Z \times_{FC} FA$ is the composition of the upper part of the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\langle f_0,\alpha\rangle"] & Y \times_{FB} FA \arrow[r] \arrow[d] & (Z \times_{FC} FB) \times_{FB} FA \cong Z \times_{FC} FA \arrow[d] \\
& Y \arrow[r, "\langle g_0,\beta\rangle"] & Z \times_{FC} FB
\end{tikzcd}$$

where the fibrations are induced by f and g, respectively. Since the square is a pullback, the top arrow is a fibration and thus the composition of the upper part is a fibration. We conclude that gf is a fibration.

2. $\mathrm{GL}(F)$ *has a terminal object and* $(X, A, \alpha) \to 1$ *is always a fibration.*
We claim that $(1_{\mathcal{D}}, 1_{\mathcal{C}}, 1)$ is the terminal object in $\mathrm{GL}(F)$. As F preserves the terminal object it is indeed an object of $\mathrm{GL}(F)$. Let (X, A, α) be any object of $\mathrm{GL}(F)$. Then we have a unique morphism $(!_{\mathcal{D}}, !_{\mathcal{C}})\colon (X, A, \alpha) \to (1_{\mathcal{D}}, 1_{\mathcal{C}}, 1)$ where $!_{\mathcal{C}}$ is a fibration and since $1_{\mathcal{D}} \times_{1_{\mathcal{D}}} FA \cong FA$ the induced map is isomorphic to α, which is a fibration. Hence $(!_{\mathcal{D}}, !_{\mathcal{C}})$ is a fibration.

3. *The pullback of a fibration along any other map exists and is again a fibration.*
Let $g\colon (Z, C, \gamma) \twoheadrightarrow (Y, B, \beta)$ be a fibration and $f\colon (X, A, \alpha) \to (Y, B, \beta)$ any map. We claim that the pullback of g along f in is given by $(X \times_Y Z, A \times_B C, \alpha \times \gamma)$ with the canonical projections. This object exists, as its components are the pullbacks of fibrations in $\mathcal{D}$ and $\mathcal{C}$ respectively and as F preserves such pullbacks the arrow $\alpha \times \gamma : X \times_Z Y \to FA \times_{FB} FC = F(A \times_B C)$ has the correct codomain. We do have to check that $\alpha \times \gamma$ is a fibration. For this, note that it is the composition of the upper part of the following commutative diagram:

$$\begin{tikzcd}
X \times_Y Z \arrow[r] \arrow[d] \arrow[rr, bend left, "1\times\gamma"] & X \times_Y (Y \times_{FB} FC) \cong X \times_{FB} FC \arrow[d] \arrow[r, "\alpha\times 1"] & FA \times_{FB} FC \arrow[d] \\
Z \arrow[r, "\langle g_0,\gamma\rangle"] & Y \times_{FB} FC \quad X \arrow[r, "\alpha"] & FA
\end{tikzcd}$$

where the bottom left arrow is a fibration as it is induced by g. Since both squares are pullbacks we see that all the upper maps are fibrations and hence $\alpha \times \gamma$ is a fibration. It is not difficult to see that this construction satisfies the universal properties of a pullback.

It remains to show that $(\pi_X, \pi_A)\colon (X \times_Y Z, A \times_B C, \alpha \times \gamma) \to (X, A, \alpha)$ is again a fibration. One can see that π_A is a fibration as it is the pullback of the fibration $F(g_1)$. To show that the induced map is a fibration, consider the following commutative diagram:

$$\begin{tikzcd}
X \times_Y Z \arrow[rr, "\pi_Z"] \arrow[d, "\langle \pi_X, \alpha\times\gamma\rangle"'] & & Z \arrow[d, "\langle g_0, \gamma\rangle"'] \arrow[dr, "\gamma"] & \\
X \times_{FA} (FA \times_{FB} FC) \arrow[rr, "f_0 \times \pi_{FC}"] \arrow[d, "\pi_X"'] & & Y \times_{FB} FC \arrow[r] \arrow[d] & FC \arrow[d, "F(g_1)"] \\
X \arrow[rr, "f_0"] & & Y \arrow[r, "\beta"] & FB
\end{tikzcd}$$

where the map $\langle g_0, \gamma\rangle \colon Z \twoheadrightarrow Y \times_{FB} FC$ is a fibration as it is induced by g. As the lower right square and the entire bottom rectangle are pullbacks, it follows that the lower left square is a pullback. Combined with the fact that the entire left rectangle is a pullback it follows that the top left square is a pullback. From this we see that the map $\langle \pi_X, \alpha \times \gamma\rangle \colon X \times_Y Z \to Z \times_{FC} (FA \times_{FB} FC)$ is a fibration and this is precisely the map induced by (π_X, π_A).

4. *Isomorphisms are equivalences and equivalences satisfy 2-out-of-6.*
   This follows immediately from the fact that equivalences in $\mathrm{GL}(F)$ are defined component-wise and that $\mathcal{C}$ and $\mathcal{D}$ are path categories.

5. *Every object (X, A, α) has at least one path object.*
   Recall by Remark 5.1.3 we choose $F(PA)$ as a path object of FA with $r_{FA} = F(r_A)$ and $\langle s_{FA}, t_{FA}\rangle = \langle F(s_A), F(t_A)\rangle$. Now let PX be constructed from the fibration $\alpha \colon X \twoheadrightarrow FA$ according to Theorem 3.1.4, i.e. $PX = P_\alpha \times_X P_{FA}X$ and it comes equipped with a fibration $P\alpha = \pi_{F(PA)}\pi_0 \colon PX \twoheadrightarrow F(PA)$ that commutes with the r, s and t maps on $F(PA)$ and PX. We claim that $(PX, PA, P\alpha)$ is a path object of (X, A, α) with $r_{(X,A,\alpha)} \coloneqq (r_X, r_A)$ and $\langle s_{(X,A,\alpha)}, t_{(X,A,\alpha)}\rangle \coloneqq (\langle s_X, t_X\rangle, \langle s_A, t_A\rangle)$. These maps are morphisms in $\mathrm{GL}(F)$ as

$$(P\alpha) r_X = \alpha F(r_A) \text{ and } (\alpha \times \alpha)\langle s_X, t_X\rangle = \langle F(s_A), F(t_A)\rangle (P\alpha).$$

Because both r_A and r_X are equivalences, so is (r_X, r_A). As $\langle s_A, t_A\rangle$ is a fibration, it remains to verify that the induced map

$$\langle \langle s_X, t_X\rangle, P\alpha\rangle \colon PX \to (X \times X) \times_{FA \times FA} F(PA)$$

is a fibration. One can show that $(X \times X) \times_{FA\times FA} F(PA) \cong P_\alpha \times_{FA} X$, which arises as the pullback

$$\begin{tikzcd}
P_\alpha \times_{FA} X \arrow[r, "\pi_X"] \arrow[d, "\pi_{P_\alpha}"] & X \arrow[d, "\alpha"] \\
P_\alpha \arrow[r, "p_\alpha"] & FA
\end{tikzcd}$$

and since $PX = P_\alpha \times_X P_{FA}X$ by construction, the induced map is isomorphic to $1 \times t \colon PX \to P_a \times_{FA} X$. Hence, it fits in the following diagram

$$\begin{tikzcd}
PX \arrow[d] & = & P_\alpha \times_X P_{FA}X \arrow[r, "\pi_1"] \arrow[d, "1\times t"] & P_{FA}X \arrow[d, "\langle s,t\rangle"] \\
(X\times X)\times_{FA\times FA} F(PA) & \cong & P_\alpha \times_{FA} X \arrow[r, "\Gamma\times 1"] \arrow[d, "\pi_{P_\alpha}"] & X\times_{FA} X \arrow[d, "\pi_0"] \\
 & & P_\alpha \arrow[r, "\Gamma"] & X
\end{tikzcd}$$

which commutes, because $(\Gamma\times 1)(1\times t) = \langle \Gamma\pi_0, t\pi_1\rangle = \langle s\pi_1, t\pi_1\rangle = \langle s,t\rangle\pi_1$. As both the lower square and the entire rectangle are pullbacks, the top square is a pullback. Hence, the induced map is indeed a fibration and we are done.

6. *Trivial fibrations are stable under pullback along arbitrary maps.*
If the situation is as in (3) and g is also an trivial, then it follows that π_X and π_A will both be equivalences, as they are pullbacks of trivial fibrations. We conclude that (π_X, π_A) is a trivial fibration.

7. *Trivial fibrations have sections.*
Suppose that $f\colon (X,A,\alpha)\to(Y,B,\beta)$ is a trivial fibration. Then f_1 is a trivial fibration and we find a section s_1. So, $F(f_1)$ has a section $F(s_1)$ and therefore we get a section $\langle \mathrm{id}_Y, F(s_1)\beta\rangle\colon Y\to Y\times_{FB} FA$ of π_Y. Notice that π_Y is a trivial fibration as it is the pullback of $F(f_1)$. As f_0 itself is a weak equivalence it follows by 2-out-of-3 that the induced morphism $\langle f_0,\alpha\rangle\colon X\to Y\times_{FB} FA$ is a trivial fibration and thus has a section s_0. We claim that $(s_0\langle \mathrm{id}_Y, F(s_1)\beta\rangle, s_1)\colon (Y,B,\beta)\to(X,A,\alpha)$ is the desired section of f. First we must check that it is even a morphism in $\mathrm{GL}(F)$. But this is true from the equality

$$\alpha s_0\langle \mathrm{id}_Y, F(s_1)\beta\rangle = \pi_{FA}\langle f_0,\alpha\rangle s_0\langle \mathrm{id}_Y, F(s_1)\beta\rangle = \pi_{FA}\langle \mathrm{id}_Y, F(s_1)\beta\rangle = F(s_1)\beta.$$

That it is a section of f follows from

$$f_0 s_0\langle \mathrm{id}_Y, F(s_1)\beta\rangle = \pi_Y\langle f_0,\alpha\rangle s_0\langle \mathrm{id}_Y, F(s_1)\beta\rangle = \pi_Y\langle \mathrm{id}_Y, F(s_1)\beta\rangle = \mathrm{id}_Y$$

and the fact that s_1 is a section of f_1. □

Remark 5.1.6. The choice for the class of fibrations might strike one as odd, but this is an example of a general approach known as *reedy fibrations.* See for instance [Shu15, Definition 8.1].

Now that we have established that $\mathrm{GL}(F)$ is indeed a path category, it will be important to characterize its homotopy structure. We will do this first for ordinary homotopy.

Lemma 5.1.7. *Let $f,g\colon (X,A,\alpha)\to(Y,B,\beta)$ be two parallel arrows in $\mathrm{GL}(F)$. Then $f\simeq g$ if and only if there is $H_1\colon f_1\simeq g_1$ and $\Gamma_\beta\langle f_0, F(H_1)\alpha\rangle \simeq_{FB} g_0$.*

Proof. Let $(PY, PB, P\beta)$ be the path object of (Y, B, β) in $\mathrm{GL}(F)$ as constructed in the previous lemma. Recall that in the construction of PY we defined $s_Y = \pi_Y \pi_0$ and $t_Y = t\pi_1$.

($\Rightarrow$) Suppose we have a morphism $(H_0, H_1)\colon (X, A, \alpha) \to (PY, PB, P\beta)$ such that $\langle s, t\rangle(H_0, H_1) = (f, g)$. Note first that this implies that $H_0\colon f_0 \simeq g_0$ and $H_1\colon f_1 \simeq g_1$. Recall that $PY = P_\beta \times_Y P_{FB}Y$ and comes equipped with a projection $\pi_1\colon PY \to P_{FB}Y$. We claim that $\pi_1 H_0\colon \Gamma_\beta\langle f_0, F(H_1)\alpha\rangle \simeq_{FB} g_0$. First note that by construction of PY we have $\pi_Y\pi_0 H_0 = s_Y H_0 = f_0$ and $\pi_{F(PB)}\pi_0 H_0 = (P\beta)H_0 = F(H_1)\alpha$ from the fact that (H_0, H_1) is a morphism in $\mathrm{GL}(F)$. Thus we conclude $s\pi_1 H_0 = \Gamma_\beta \pi_0 H_0 = \Gamma_\beta\langle f_0, F(H_1)\alpha\rangle$ and together with $t\pi_1 H_0 = t_Y H_0 = g_0$, we can indeed conclude that $\pi_1 H_0\colon \Gamma_\beta\langle f_0, F(H_1)\alpha\rangle \simeq_{FB} g_0$.

($\Leftarrow$) Suppose that $H_1\colon f_1 \simeq g_1$ and that $H_0\colon \Gamma_\beta\langle f_0, F(H_1)\alpha\rangle \simeq_{FB} g_0$. Since we have chosen $PY = P_\beta \times_Y P_{FB}Y$ we conclude that there is a map $\langle\langle f_0, F(H_1)\alpha\rangle, H_0\rangle\colon X \to PY$. The equality

$$P\beta\langle\langle f_0, F(H_1)\alpha\rangle, H_0\rangle = \pi_{F(PB)}\pi_0((f_0, F(H_1)\alpha), H_0) = F(H_1)\alpha$$

tells us that that $(\langle\langle f_0, F(H_1)\alpha\rangle, H_0\rangle, H_1)\colon (X, A, \alpha) \to (PY, PB, P\beta)$ is a morphism in $\mathrm{GL}(F)$. Using this we see that

$$\begin{aligned}\langle s, t\rangle(\langle\langle f_0, F(H_1)\alpha\rangle, H_0\rangle, H_1) &= (\langle s_Y, t_Y\rangle\langle\langle f_0, F(H_1)\alpha\rangle, H_0\rangle, \langle s_B, t_B\rangle H_1)\\ &= (\langle f_0, g_0\rangle, \langle f_1, g_1\rangle) = \langle f, g\rangle.\end{aligned}$$

Hence $f \simeq g$. $\square$

Before we characterize fibrewise homotopy we first try to get a better understanding of the category $\mathrm{GL}(F)((Z, C, \gamma))$ for an object (Z, C, γ) of $\mathrm{GL}(F)$. Suppose we have a fibration $f\colon (Y, B, \beta) \twoheadrightarrow (Z, C, \gamma)$ in $\mathrm{GL}(F)$. Then this tells us that f_1 and the induced morphism $\langle f_0, \beta\rangle\colon Y \to \gamma^*(FB)$ are fibrations. Observe that this data can be represented in the following way:

$$\left(\begin{tikzcd} Y \arrow[d, two heads, "f_0"] \\ Z, \end{tikzcd} \quad \begin{tikzcd} B \arrow[d, two heads, "f_1"] \\ C, \end{tikzcd} \quad \begin{tikzcd} Y \arrow[rr, two heads, "\langle f_0,\beta\rangle"] \arrow[dr, two heads, "f_0"'] & & \gamma^*(FB) \arrow[dl, "\gamma^*(F(f_1))"] \\ & Z & \end{tikzcd}\right)$$

This motivates the next lemma.

Lemma 5.1.8. *Let (Z, C, γ) be an object of $\mathrm{GL}(F)$. Then the functor $\gamma^* F_C\colon \mathcal{C}(C) \to \mathcal{D}(Z)$ is exact and we have a functor $G\colon \mathrm{GL}(\gamma^* F_C) \to \mathrm{GL}(F)((Z, C, \gamma))$ which preserves equivalences and fibrations. Moreover, this functor is an isomorphism.*

Proof. Since F is an exact functor we see that F_C must also preserve equivalences, fibrations and pullbacks of fibrations along arbitrary maps, as these are calculated in $\mathcal{C}$ and $\mathcal{D}$, respectively. It also clearly preserve the terminal object. We already know that the base change functor is exact by Lemma 2.1.7. We conclude that the composition $\alpha^* F_A$ is exact.

Let the functor $G\colon \mathrm{GL}(\gamma^* F_C) \to \mathrm{GL}(F)((Z, C, \alpha))$ be given on objects as

$$(h_0\colon Y \twoheadrightarrow Z, h_1\colon B \twoheadrightarrow C, \langle h_0, \beta\rangle\colon Y \twoheadrightarrow \gamma^*(FB)) \mapsto (Y, B, \beta\colon Y \twoheadrightarrow FB) \overset{(h_0,h_1)}{\twoheadrightarrow} (Z, C, \gamma)$$

and the identity on morphisms. Suppose

$$f\colon (X \twoheadrightarrow Z, A \twoheadrightarrow C, \langle j_0, \alpha\rangle\colon X \twoheadrightarrow \gamma^*(FA)) \to (Y \twoheadrightarrow Z, B \twoheadrightarrow C, \langle h_0, \beta\rangle\colon Y \twoheadrightarrow \gamma^*(FB))$$

is a morphism in $\mathrm{GL}(\gamma^* F_A)$. If f is an equivalence, then f_0 and f_1 are both equivalences in $\mathcal{C}(A)$ and $\mathcal{D}(X)$ and thus also in $\mathcal{C}$ and $\mathcal{D}$. We conclude that $G(f)$ is an equivalence. Suppose now that f is a fibration. Then we have that f_1 is a fibration in $\mathcal{C}(A)$ and therefore also in $\mathcal{C}$. Looking at the following diagram

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right, "f_0"'] & & & \\
& P \arrow[r] \arrow[d] & \gamma^* FA \arrow[r] \arrow[d, "\gamma^*(F(f_1))"] & FA \arrow[d, "F(f_1)"] \\
& Y \arrow[r] \arrow[dr] & \gamma^* FB \arrow[r] \arrow[d] & FB \arrow[d] \\
& & Z \arrow[r, "\gamma"] & FC
\end{tikzcd}$$

we see that f and $G(f)$ induce the same arrow in $\mathcal{C}$, so it follows that $G(f)$ is also a fibration. The inverse of H is given on objects by

$$(Y, B, \beta) \overset{f}{\twoheadrightarrow} (Z, C, \gamma) \mapsto (Y \overset{f_0}{\twoheadrightarrow} X, B \overset{f_1}{\twoheadrightarrow} A, \langle f_0, \beta\rangle\colon Y \twoheadrightarrow \alpha^* FB)$$

and again the identity on morphisms. We leave verifications to the reader as we shall not need that G is an isomorphism. $\square$

Corollary 5.1.9. *Let $p\colon (Y, B, \beta) \twoheadrightarrow (Z, C, \gamma)$ be a fibration in $\mathrm{GL}(F)$. Choose $F(P_C B)$ as the fibred path object of FB over FC and $\gamma^*(F(P_C B))$ as the fibred path object of $\gamma^*(FB)$ over Z and some transport structure $\Gamma_{\langle p_0,\beta\rangle}$ in $\mathcal{C}(Z)$ on the induced fibration $\langle p_0, \beta\rangle\colon Y \twoheadrightarrow \gamma^*(FB)$. Finally, choose $P_{\gamma^*(FB)} Y$ as the fibred path object $P_{\gamma^*(F_C B)} p_0$ in $(\mathcal{C}(Z))(\gamma^*(FB))$ in accordance with Remark 2.1.6.*

Now, construct the path object of Y over Z using the induced fibration $\langle p_0, \beta\rangle\colon Y \twoheadrightarrow \gamma^*(FB)$ and Theorem 3.1.4 in $\mathcal{D}(Z)$, i.e. $P_Z Y = P_{\langle p_0,\beta\rangle} \times_Y P_{\gamma^*(FB)} Y$ where $P_{\langle p_0,\beta\rangle}$ is the pullback

$$\begin{tikzcd}
P_{\langle p_0,\beta\rangle} \arrow[r, "\pi_{\gamma^*(F(P_C B))}"] \arrow[d, squiggly, "\pi_Y"] & \gamma^*(F(P_C B)) \arrow[d, squiggly, "\gamma^*(F(s_B))"] \\
Y \arrow[r, "\langle p_0,\beta\rangle"] & \gamma^*(FB).
\end{tikzcd}$$

Recall that it comes equipped with a fibration $P_Z\langle p_0, \beta\rangle \colon P_Z Y \twoheadrightarrow \gamma^*(F(P_C B))$ *in* $\mathcal{D}(Z)$. *Then* $(P_Z Y, P_C B, \pi_1 P_Z\langle p_0, \beta\rangle \colon P_Z Y \twoheadrightarrow F(P_C B))$ *is a fibred path object of* (Y, B, β) *over* (Z, C, γ).

Proof. Recall that Lemma 5.1.5 allows us to describe a path object in $\mathrm{GL}(\gamma^* F_C)$ as follows: We can choose $(P_Z Y \twoheadrightarrow Z, P_C B \twoheadrightarrow C, P_Z\langle p_0, \beta\rangle \colon P_Z Y \twoheadrightarrow \gamma^*(F(P_C B)))$ as a path object of $(Y \overset{p_0}{\twoheadrightarrow} Z, B \overset{p_1}{\twoheadrightarrow} C, \langle p_0, \beta\rangle \colon Y \to \gamma^*(FB))$. As the functor G from the previous lemma preserves equivalence and fibrations the result follows by applying G. $\square$

Now that we have a description of the fibred path objects of $\mathrm{GL}(F)$, we can characterize fibrewise homotopy.

Lemma 5.1.10. *Let* $f, g \colon (X, A, \alpha) \to (Y, B, \beta)$ *and* $p \colon (Y, B, \beta) \twoheadrightarrow (Z, C, \gamma)$ *with* $pf = pg$. *Then* $f \simeq_{(Z,C,\gamma)} g$ *if and only if*

$$H_1 \colon f_1 \simeq_C g_1 \text{ and } \Gamma_{\langle p_0, \beta\rangle}\langle f_0, \langle p_0 f_0, F(H_1)\alpha\rangle\rangle \simeq_{\gamma^*(FB)} g_0,$$

where the transport structure $\Gamma_{\langle p_0,\beta\rangle}$ *is constructed in* $\mathcal{D}(Z)$.

Proof. We will use the fibred path object of (Y, B, γ) over (Z, C, γ) as constructed in the previous lemma.

($\Rightarrow$) Suppose that $(H_0, H_1) \colon (X, A, \alpha) \to P_{(Z,C,\gamma)}(Y, B, \beta)$ is a morphism such that $\langle s, t\rangle(H_0, H_1) = \langle f, g\rangle$. Observe that $H_0 \colon f_0 \simeq_Z g_0$ and $H_1 \colon f_1 \simeq_C g_1$. By construction $P_Z Y = P_{\langle p_0, \beta\rangle} \times_Y P_{\gamma^*(FB)} Y$, so there is a projection $\pi_1 \colon P_Z Y \to P_{\gamma^*(FB)} Y$. We claim that $\pi_1 H_0 \colon \Gamma_{\langle p_0,\beta\rangle}\langle f_0, \langle p_0 f_0, F(H_1)\alpha\rangle\rangle \simeq_{\gamma^*(FB)} g_0$. By construction we have

$$t\pi_1 H_0 = t_Y H_0 = g_0 \text{ and } \pi_Y \pi_0 H_0 = s_Y H_0 = f_0.$$

We wish to also show that

$$\pi_{\gamma^*(F(P_C B))}\pi_0 H_0 = \langle p_0 f_0, F(H_1)\alpha\rangle \colon X \to \gamma^*(F(P_C B)).$$

Recall that in constructing $P_Z Y$ in Theorem 3.1.4 we define

$$P_Z\langle p_0, \beta\rangle = \pi_{\gamma^*(F(P_C B))}\pi_0 \colon P_Z Y \to \gamma^*(F(P_C B)).$$

. Since (H_0, H_1) is a morphism in $\mathrm{GL}(F)$ we conclude that

$$\pi_1 \pi_{\gamma^*(F(P_C B))}\pi_0 H_0 = \pi_1(P_Z\langle p_0, \beta\rangle) H_0 = F(H_1)\alpha.$$

On the other hand the following diagram

$$\begin{tikzcd}
\gamma^*(F(P_C B)) \arrow[r, "\pi_1"] \arrow[d, "\gamma^*(F(s_B))"] \arrow[dd, bend right=60, "\pi_Z"'] & F(P_C B) \arrow[d, "F(s_B)"] \\
\gamma^*(FB) \arrow[r, two heads] \arrow[d, two heads, "\gamma^*(F(p_1))"] & FB \arrow[d, two heads, "F(p_1)"] \\
Z \arrow[r, two heads, "\gamma"] & FC
\end{tikzcd}$$

tells us that $\pi_Z = \gamma^*(F(p_1))\gamma^*(F(s_B))$. Since we have the equality

$$\begin{aligned}\gamma^*(F(p_1))\gamma^*(F(s_B))\pi_{\gamma^*(F(P_C B))}\pi_0 H_0 &= \gamma^*(F(p_1))\langle p_0, \beta\rangle \pi_Y \pi_0 H_0 \\ &= p_0 \pi_Y \pi_0 H_0 \\ &= p_0 s_Y H_0 \\ &= p_0 f_0,\end{aligned}$$

We conclude that $s\pi_1 H_0 = \Gamma_{\langle p_0,\beta\rangle}\pi_0 H_0 = \Gamma_{\langle p_0,\beta\rangle}\langle f_0, \langle p_0 f_0, F(H_1)\alpha\rangle\rangle$.

($\Leftarrow$) Suppose that $H_1\colon f_1 \simeq_C g_1$ and $H_0\colon \Gamma_{\langle p_0,\beta\rangle}\langle f_0, \langle p_0 f_0, F(H_1)\alpha\rangle\rangle \simeq_{\gamma^*(FB)} g_0$. This gives us a map $\langle\langle f_0, \langle p_0 f_0, F(H_1)\alpha\rangle\rangle, H_0\rangle\colon X \to P_Z Y$ such that

$$\pi_1(P_Z\langle p_0, \beta\rangle)\langle\langle f_0, \langle p_0 f_0, F(H_1)\alpha\rangle\rangle, H_0\rangle = F(H_1)\alpha,$$

so there is $(\langle\langle f_0, \langle p_0 f_0, F(H_1)\alpha\rangle\rangle, H_0\rangle, H_1)\colon (X, A, \alpha) \to P_{(Z,C,\gamma)}(Y, B, \beta)$ in $\mathrm{GL}(F)$. Moreover,

$$\begin{aligned}\langle s,t\rangle(\langle\langle f_0, \langle p_0 f_0, F(H_1)\alpha\rangle\rangle, H_0\rangle, H_1) &= (\langle s_Y, t_Y\rangle\langle\langle f_0, \langle p_0 f_0, F(H_1)\alpha\rangle\rangle, H_0\rangle, \langle s_B, t_B\rangle H_1) \\ &= (\langle f_0, g_0\rangle, \langle f_1, g_1\rangle) = \langle f, g\rangle\end{aligned}$$

showing that $f \simeq_{(Z,C,\gamma)} g$. □

5.2 Homotopy natural numbers object in a gluing category

Now that we have an understanding of the (fibrewise) homotopy in $\mathrm{GL}(F)$ it is time to tackle the homotopy universal construction. Our first result will bet that $\mathrm{GL}(F)$ has a hnno whenever both $\mathcal{C}$ and $\mathcal{D}$ have one.

Theorem 5.2.1. *Suppose that $\mathcal{C}$ and $\mathcal{D}$ have hnnos $(\mathbb{N}_1, z_1, S_1)$ and $(\mathbb{N}_0, z_0, S_0)$, respectively. Then $\mathrm{GL}(F)$ has a hnno.*

Proof. As F is exact, it preserves the terminal object. This results in a diagram

$$1 \xrightarrow{F(z_1)} F\mathbb{N}_1 \xrightarrow{F(S_1)} F\mathbb{N}_1$$

in $\mathcal{D}$. By Lemma 4.1.3 we find $h\colon \mathbb{N}_0 \to F\mathbb{N}_1$ such that $hz_0 \simeq F(z_1)$ and $hS_0 \simeq F(S_1)h$. Factor h as an equivalence followed by a fibration $\mathbb{N}_0 \overset{f}{\rightsquigarrow} X \overset{q}{\twoheadrightarrow} F\mathbb{N}_1$. Recall that from Lemma 2.1.3 this equivalence f is actually a section of an acyclic fibration $g\colon X \to \mathbb{N}_0$. Hence, by Theorem 4.1.6 we see that $(X, fz_0, fS_0 g)$ is a hnno in $\mathcal{D}$. We also have the following diagram

$$\begin{tikzcd} 1 \arrow[r, "fz_0"] \arrow[d, equal] & X \arrow[r, "fS_0 g"] \arrow[d, two heads, "q"] & X \arrow[d, two heads, "q"] \\ 1 \arrow[r, "F(z_1)"] & F\mathbb{N}_1 \arrow[r, "F(S_1)"] & F\mathbb{N}_1 \end{tikzcd}$$

which commutes up to homotopy, since $qfz_0 = hz_0 \simeq F(z_1)$ and

$$qfS_0g = hS_0g \simeq F(S_1)hg = F(S_1)qfg \simeq F(S_1)q.$$

Because q is a fibration we can use Lemma 3.1.6 to find morphisms $z_X : 1 \to X$ and $S_X : X \to X$ such that the above diagram commutes strictly with the property that $z_X \simeq fz_0$ and $S_X \simeq fS_0g$. By Lemma 4.1.5 we conclude that (X, z_X, S_X) is a hnno in $\mathcal{D}$ and that we have an object $(X, \mathbb{N}_1, q)$ in $\mathrm{GL}(F)$. The commutativity of the above diagrams now shows that $(z_X, z_1) : 1 \to (X, \mathbb{N}_1, q)$ and $(S_X, S_1) : (X, \mathbb{N}_1, q) \to (X, \mathbb{N}_1, q)$ are both morphisms in $\mathrm{GL}(F)$. We claim that $((X, \mathbb{N}_1, q), (z_X, z_1), (S_X, S_1))$ is a hnno in $\mathrm{GL}(F)$.

Suppose we are given a commutative diagram of the form

$$\begin{tikzcd}
 & (Y,B,\beta) \arrow[r, "f"] \arrow[d, "p"] & (Y,B,\beta) \arrow[d, "p"] \\
1 \arrow[ur, "{(y,b)}"] \arrow[r, "{(z_X,z_1)}"'] & (X,\mathbb{N}_1,q) \arrow[r, "{(S_X,S_1)}"'] & (X,\mathbb{N}_1,q)
\end{tikzcd}$$

with p is a fibration. Then this induces a diagram in $\mathcal{C}$ which gives us a section $a_1 : \mathbb{N}_1 \to B$ of p_1 such that $H_1 : a_1 z_1 \simeq_{\mathbb{N}_1} b$ and $H_1' : a_1 S_1 \simeq_{\mathbb{N}_1} f_1 a_1$. To find an appropriate section of p_0, consider the following diagram

$$\begin{tikzcd}
P \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \arrow[d, "{\langle p_0,\beta\rangle}"'] \arrow[dr, "\beta"] & \\
X \arrow[r, "{\langle 1,F(a_1)q\rangle}"] \arrow[dr, equal] & q^*(FB) \arrow[r] \arrow[d] & FB \arrow[d, "F(p_1)"] \\
 & X \arrow[r, "q"] & F\mathbb{N}_1
\end{tikzcd}$$

where P is constructed as a pullback. This is possible as the induced arrow $\langle p_0, \beta\rangle : Y \to q^*(FB)$ is a fibration. It follows from the diagram that $\pi_X = p_0\pi_Y : P \to X$. We apply the convention of Remark 2.1.9 to the fibred path objects of P over X and $q^*(FB)$ over X. We have $z_X : 1 \to X$ and $y : 1 \to Y$. Because $z_X = p_0 y : 1 \to X$ and

$$F(a_1)qz_X = F(a_1)F(z_1) \simeq_{F\mathbb{N}_1} F(b) = \beta y : 1 \to FB$$

we conclude that the following diagram

$$\begin{tikzcd}
1 \arrow[r, "y"] \arrow[d, "z_X"] & Y \arrow[d, "{\langle p_0,\beta\rangle}"] \\
X \arrow[r] & q^*(FB)
\end{tikzcd}$$

commutes up to homotopy over X where the homotopy is given by

$$\langle p_0 y, F(H_1)\rangle : 1 \to P_X(q^*(FB)) = q^*(F(P_{\mathbb{N}_1}B)).$$

By Lemma 3.1.6 applied in the path category $\mathcal{D}(X)$, we find $y' := \Gamma_{\langle p_0,\beta\rangle}\langle y, \langle p_0 y, F(H_1)\rangle\rangle$ that makes the diagram commute strictly. Note that here $\Gamma_{\langle p_0,\beta\rangle}$ is constructed in $\mathcal{C}(X)$.

Similarly, we have two maps $S_X\pi_X \colon P \to X$ and $f_0\pi_Y \colon P \to Y$. Because of $S_X\pi_X = S_X p_o \pi_Y = p_0 f_0 \pi_Y$ and

$$F(a_1)qS_X\pi_X = F(a_1)F(S_1)q\pi_X \simeq_{F\mathbb{N}_1} F(f_1)F(a_1)q\pi_X = F(f_1)\beta\pi_Y = \beta f_0\pi_Y,$$

the following diagram

$$\begin{tikzcd}
P \arrow[r, "f_0\pi_Y"] \arrow[d, "S_X\pi_X"'] & Y \arrow[d, "\langle p_0,\beta\rangle"] \\
X \arrow[r, "F(a_1)q"] & q^*(FB)
\end{tikzcd}$$

commutes up to homotopy over X by the homotopy

$$\langle p_0 f_0 \pi_Y, F(H_1')q\pi_X\rangle \colon P \to P_X(q^*(FB)) = q^*(F(P_{\mathbb{N}_1}B)).$$

Therefore, we find $h := \Gamma_{\langle p_0,\beta\rangle}\langle f_0\pi_Y, \langle p_0 f_0\pi_Y, F(H_1')q\pi_X\rangle\rangle \colon P \to Y$ that makes the diagram commute strictly.

In summary, we have the following commutative diagram

$$\begin{tikzcd}
 & P \arrow[r, "\langle S_X\pi_X, h\rangle"] \arrow[d, "\pi_X"] & P \arrow[d, "\pi_X"] \\
1 \arrow[ur, "\langle z_X, y'\rangle"] \arrow[r, "z_X"] & X \arrow[r, "S_X"] & X
\end{tikzcd}$$

and there is a section $a_X \colon X \to P$ of π_X such that

$$H_0 \colon a_X z_X \simeq_X \langle z_X, y'\rangle \text{ and } H_0' \colon a_X S_X \simeq_X \langle s_X\pi_X, h\rangle a_X.$$

Now put $a := (\pi_Y a_X, a_1) \colon (X, \mathbb{N}_1, q) \to (Y, B, \beta)$, which is a morphism in $\mathrm{GL}(F)$ because $\beta\pi_Y a = F(a_1)q\pi_X a = F(a_1)q$. It is also a section of p since $p_0\pi_Y a_X = \pi_X a_X = 1$ and $p_1 a_1 = 1$. It remains to show the desired homotopies in $\mathrm{GL}(F)$.

For the first one, it holds that $H_1 \colon a_1 z_1 \simeq_{\mathbb{N}_1} b$ and, by choice of fibred path object of P over X, $\pi_1 H_0 \colon \pi_Y a_X z_X \simeq_{q^*(FB)} y' = \Gamma_{\langle p_0,\beta\rangle}\langle y, \langle p_0 y, F(H_1)\rangle\rangle$. By Lemma 5.1.7 this implies that $(\pi_Y a_X, a_1)(z_X, z_1) \simeq_{(X,\mathbb{N}_1,q)} (y, b)$.

Similarly we have $H_1' \colon a_1 S_1 \simeq_{\mathbb{N}_1} f_1 a_1$ and

$$\begin{aligned}
\pi_1 H_0' \colon \pi_Y a_X S_X \simeq_{q^*(FB)} h a_X &= \Gamma_{\langle p_0,\beta\rangle}\langle f_0\pi_Y, \langle p_0 f_0\pi_Y, F(H_1')q\pi_X\rangle\rangle a_X \\
&= \Gamma_{\langle p_0,\beta\rangle}\langle f_0\pi_Y a_X, \langle p_0 f_0 \pi_Y a_X, F(H_1')q\rangle\rangle.
\end{aligned}$$

Again, Lemma 5.1.7 implies that $(\pi_Y a_X, a_1)(S_X, S_1) \simeq_{(X,\mathbb{N}_1,q)} (f_0, f_1)(\pi_Y a_X, a_1)$. □

5.3 Homotopy exponentials in a gluing category

In this section we will give the sufficient criteria for $\mathrm{GL}(F)$ to have (weak) homotopy exponentials. We start with weak homotopy exponentials. Let (X, A, α) and (Y, B, β) be objects of $\mathrm{GL}(F)$. What would one expect $(Y, B, \beta)^{(X,A,\alpha)}$ to look like? A good start might be to draw inspiration from the construction in toposes. Looking at [Joh02], we would construct the pullback

$$\begin{tikzcd} E \arrow[r, "k"] \arrow[d, "h"] & Y^X \arrow[d, "\beta^X"] \\ F(B^A) \arrow[r, "\overline{F(\mathrm{ev})(1\times\alpha)}"] & FB^X \end{tikzcd}$$

where $\overline{F(\mathrm{ev})(1 \times \alpha)}$ is the transpose of $F(B^A) \times X \to F(B^A) \times FA \to FB$ with $H\colon \mathrm{ev}(\overline{F(\mathrm{ev})(1 \times \alpha)} \times 1) \simeq F(\mathrm{ev})(1 \times \alpha)$. One way to ensure that this pullback actually exists, is by demanding that $\beta^X\colon Y^X \to FB^X$ is a fibration. From Lemma 4.2.7 it follows that this can be done by requiring that $\mathcal{D}$ has weak homotopy exponentials. This seems like a reasonable requirement as we are not just interested in arrows $X \to Y$ in $\mathcal{D}$ but arrows that map inside a certain fibre. Indeed this approach will work, however the homotopy H will turn out to give an extra (unnecessary) level of complexity to the proof. This is due to the fact that one can think of an element of E as a morphism $A \to B$ and a morphism $X \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d] & FA \arrow[d] \\ Y \arrow[r, "\beta"] & FB \end{tikzcd}$$

commutes up to the homotopy H. As $\beta\colon Y \twoheadrightarrow FB$ is a fibration, this is enough to define a morphism in $\mathrm{GL}(F)$ by Lemma 3.1.6, but one would have to constantly transport back and forth along H when evaluating.

It turns out to be possible to construct the exponential in such a way that it represent strictly commutative squares using weak homotopy Π-types in $\mathcal{D}$. This will be the approach in the following theorem.

Theorem 5.3.1. *Suppose $\mathcal{C}$ has weak homotopy exponentials and $\mathcal{D}$ has weak homotopy Π-types. Then $\mathrm{GL}(F)$ has weak homotopy exponentials.*

Proof. Let (X, A, α) and (Y, B, β) be two objects of $\mathrm{GL}(F)$. Let B^A be a weak homotopy exponential in $\mathcal{C}$ with evaluation map $\mathrm{ev}_B\colon B^A \times A \to B$. Now consider the following diagram

$$\begin{tikzcd} Q \arrow[rr, "\pi_Y"] \arrow[d, "\pi_0"] & & Y \arrow[d, "\beta"] \\ F(B^A) \times X \arrow[r, "1\times\alpha"] & F(B^A) \times FA \arrow[r, "F(\mathrm{ev}_B)"] & FB \end{tikzcd}$$

where Q is constructed as a pullback. Let the fibred path object of Q over $F(B^A) \times X$ be chosen by pulling back $P_{FB}X$ along $F(\mathrm{ev}_B)(1 \times \alpha)$ like in Remark 2.1.9. So, now we have a pair of fibrations $Q \twoheadrightarrow F(B^A) \times X \twoheadrightarrow F(B^A)$ and we construct its weak homotopy Π-type. Denote this object by $\beta^\alpha \colon Y_\alpha^X \twoheadrightarrow F(B^A)$ and its evaluation map by $\mathrm{ev}_{\pi_0} \colon Y_\alpha^X \times X \to Q$ over $F(B^A) \times X$. Now define $\mathrm{ev}_Y := \pi_Y \, \mathrm{ev}_{\pi_0} \colon Y_\alpha^X \times X \to Y$. We claim that $(Y_\alpha^X, B^A, \beta^\alpha)$ is a weak homotopy exponential in $\mathrm{GL}(F)$ with evaluation map $\mathrm{ev}_{(X,A,\alpha),(Y,B,\beta)} := (\mathrm{ev}_Y, \mathrm{ev}_B) \colon (Y_\alpha^X \times X, B^A \times A, \beta^\alpha \times \alpha) \to (Y, B, \beta)$. The fact that $\mathrm{ev}_{(X,A,\alpha),(Y,B,\beta)}$ is a morphism in $\mathrm{GL}(F)$ follows from

$$\beta \, \mathrm{ev}_Y = F(\mathrm{ev}_B)(1 \times \alpha)(\beta^\alpha \times 1) = F(\mathrm{ev}_B)(\beta^\alpha \times \alpha).$$

Informally, an element of this object consists of a morphism $f_1 \colon A \to B$ and a morphism $f_0 \colon Y \to X$ above $F(f_1)$ with the property that $\beta(\mathrm{ev}_Y\langle f_0, x\rangle) = F(\mathrm{ev}_B)\langle F(f_1), \alpha(x)\rangle$.

Suppose we have $f \colon (Z \times X, C \times A, \gamma \times \alpha) \to (Y, B, \beta)$. In particular we have $f_1 \colon C \times A \to B$ and hence a transpose $\bar{f}_1 \colon C \to B^A$ such that $H_f \colon \mathrm{ev}_B(\bar{f}_1 \times 1) \simeq f_1$. From this we find a map $F(\bar{f}_1)\gamma \colon Z \to F(B^A)$. On the other hand we have the map $f_0 \colon Z \times X \to Y$. As f is a morphism in $\mathrm{GL}(F)$ we see that

$$\beta f_0 = F(f_1)(\gamma \times \alpha) \simeq F(\mathrm{ev}_B(\bar{f}_1 \times 1))(\gamma \times \alpha) = F(\mathrm{ev}_B)(1 \times \alpha)(F(\bar{f}_1)\gamma \times 1),$$

where the homotopy is given by $\sigma F(H_f)(\gamma \times \alpha)$. Since β is a fibration, we can apply Lemma 3.1.6 to conclude that

$$\begin{tikzcd}
Z \times X \arrow[r, "\Gamma_\beta\langle f_0, \sigma F(H_f)(\gamma\times\alpha)\rangle"] \arrow[d, "F(\bar{f}_1)\gamma\times 1"'] & Y \arrow[d, two heads, "\beta"] \\
F(B^A) \times X \arrow[r, "F(\mathrm{ev}_B)(1\times\alpha)"] & FB
\end{tikzcd}$$

commutes, resulting in a map $\langle F(\bar{f}_1)\gamma \times 1, \Gamma_\beta\langle f_0, \sigma F(H_f)(\gamma \times \alpha)\rangle\rangle \colon Z \times X \to Q$ over $F(B^A) \times X$. Thus, by the property of weak homotopy Π-types we find a map $n_0 \colon Z \to Y_\alpha^X$ such that $\beta^\alpha n_0 = F(\bar{f}_1)\gamma$ and

$$\mathrm{ev}_{\pi_0}(n_0 \times 1) \simeq_{F(B^A)\times X} \langle F(\bar{f}_1)\gamma \times 1, \Gamma_\beta\langle f_0, \sigma F(H_f)(\gamma \times \alpha)\rangle\rangle.$$

By choice of the fibred path object of Q this implies that

$$\mathrm{ev}_Y(n_0 \times 1) \simeq_{FB} \Gamma_\beta\langle f_0, \sigma F(H_f)(\gamma \times \alpha)\rangle.$$

We put $\bar{f} := (n_0, \bar{f}_1) \colon (Z, C, \gamma) \to (Y_\alpha^X, B^A, \beta^\alpha)$, which is indeed a morphism in $\mathrm{GL}(F)$ by the above. It remains to show that $\mathrm{ev}_{(X,A,\alpha),(Y,B,\beta)}(\bar{f} \times 1) \simeq f$. We have already seen that $H_f \colon \mathrm{ev}_B(\bar{f}_1 \times 1) \simeq f_1$. Moreover,

$$\Gamma_\beta\langle \mathrm{ev}_Y(n_0 \times 1), F(H_f)(\gamma \times \alpha)\rangle \simeq_{FB} \Gamma_\beta\langle f_0, \sigma F(H_f)(\gamma \times \alpha), F(H_f)(\gamma \times \alpha)\rangle \simeq_{FB} f_0,$$

by Remark 3.2.4. We conclude that $\mathrm{ev}_{(X,A,\alpha),(Y,B,\beta)}(\bar{f} \times 1) \simeq f$ by Lemma 5.1.7. Hence, $(Y_\alpha^X, B^A, \beta^\alpha)$ is a weak homotopy exponential. $\square$

Before we can move on to homotopy exponentials, we will need a lemma about a nice choice of transport structure on β^α. The proof is very similar to Lemma 4.2.10. In fact, Lemma 4.2.10 is instance of the following lemma in the case that $\mathcal{C} = \mathcal{D}, F = 1_{\mathcal{D}}$ and one puts $(X, A, \alpha) := (Z, Z, 1_Z)$ and $(Y, B, \beta) := (Y, X, f\colon Y \twoheadrightarrow X)$.

Lemma 5.3.2. *Suppose $\mathcal{C}$ has homotopy exponentials as well as weak homotopy Π-types and $\mathcal{D}$ has homotopy Π-types. Let $\beta^\alpha\colon Y_\alpha^X \to F(B^A)$ be constructed as in the previous lemma. Then we can choose a transport structure Γ_{β^α} on β^α in such a way that*

$$\mathrm{ev}_Y(\Gamma_{\beta^\alpha} \times 1) \simeq_{FB} \Gamma_\beta\langle \mathrm{ev}_Y(\pi_0 \times 1), F(\mathrm{ev}_{PB})(\pi_1 \times \alpha)\rangle\colon (Y_\alpha^X \times_{F(B^A)} F((PB)^A)) \times X \to Y.$$

Proof. Compare Lemma 4.2.10. As $\mathcal{C}$ has both homotopy exponentials as well as weak homotopy Π-types, we can choose $(PB)^A$ as a suitable path object of B^A in accordance with Corollary 4.2.9. In particular, we have $\langle s^A, t^A\rangle$ such that $\mathrm{ev}_B(s^A \times 1) = s\,\mathrm{ev}_{PB}$ and $\mathrm{ev}_B(t^A \times 1) = t\,\mathrm{ev}_{PB}$ and r^A is such that $\mathrm{ev}_{PB}(r^A \times 1) \simeq_{B\times B} r\,\mathrm{ev}_B$. The equality

$$\begin{aligned}
\beta\,\mathrm{ev}_Y(\pi_0 \times 1) &= F(\mathrm{ev}_B)(1 \times \alpha)(\beta^\alpha \times 1)(\pi_0 \times 1)\\
&= F(\mathrm{ev}_B)(\beta^\alpha\pi_0 \times \alpha)\\
&= F(\mathrm{ev}_B)(F(s^A)\pi_1 \times \alpha)\\
&= F(\mathrm{ev}_B(s^A \times 1))(\pi_1 \times \alpha)\\
&= F(s)F(\mathrm{ev}_{PB})(\pi_1 \times \alpha).
\end{aligned}$$

shows that there is a map

$$\langle \mathrm{ev}_Y(\pi_0 \times 1), F(\mathrm{ev}_{PB})(\pi_1 \times \alpha)\rangle\colon (Y_\alpha^X \times_{F(B^A)} F((PB)^A)) \times X \to Y \times_{FB} F(PB).$$

So, we have two maps $\Gamma_\beta\langle \mathrm{ev}_Y(\pi_0 \times 1), F(\mathrm{ev}_{PB})(\pi_1 \times \alpha)\rangle$ and $F(t^A)\pi_1$ with the property that

$$\begin{aligned}
\beta\Gamma_\beta\langle \mathrm{ev}_Y(\pi_0 \times 1), F(\mathrm{ev}_{PB})(\pi_1 \times \alpha)\rangle &= F(t)\pi_1\langle \mathrm{ev}_Y(\pi_0 \times 1), F(\mathrm{ev}_{PB})(\pi_1 \times \alpha)\rangle\\
&= F(t)F(\mathrm{ev}_{PB})(\pi_1 \times \alpha)\\
&= F(\mathrm{ev}_B(t^A \times 1))(\pi_1 \times \alpha)\\
&= F(\mathrm{ev}_B)(1 \times \alpha)(F(t^A)\pi_1 \times 1).
\end{aligned}$$

So, by the universal property of Y_α^X we find $\Gamma_{\beta^\alpha}\colon Y_\alpha^X \times_{F(B^A)} F((PB)^A) \to Y_\alpha^X$ such that

$$\beta^\alpha\Gamma_{\beta^\alpha} = F(t^A)\pi_1 \text{ and } \mathrm{ev}_Y(\Gamma_{\beta^\alpha} \times 1) \simeq_{FB} \Gamma_\beta\langle \mathrm{ev}_Y(\pi_0 \times 1), F(\mathrm{ev}_{PB})(\pi_1 \times \alpha)\rangle.$$

Finally, we will show that $\beta^\alpha\Gamma_{\beta^\alpha}\langle 1, F(r^A)\beta^\alpha\rangle \simeq_{F(B^A)} 1_{Y_\alpha^X}$ by making use of the fact that $\mathcal{D}$ has homotopy Π-types. We have the equality

$$\beta^\alpha\Gamma_{\beta^\alpha}\langle 1, F(r^A)\beta^\alpha\rangle = F(t^A)\pi_1\langle 1, F(r^A)\beta^\alpha\rangle = F(t^A)F(r^A)\beta^\alpha = \beta^\alpha.$$

and the homotopy

$$\begin{aligned}
\mathrm{ev}_Y(\Gamma_{\beta^\alpha}\langle 1, F(r^A)\beta^\alpha\rangle \times 1) &\simeq_{FB} \Gamma_\beta\langle \mathrm{ev}_Y, F(\mathrm{ev}_{PB})(F(r^A)\beta^\alpha \times \alpha)\rangle \\
&= \Gamma_\beta\langle \mathrm{ev}_Y, F(\mathrm{ev}_{PB}(r^A \times 1))(\beta^\alpha \times \alpha)\rangle \\
&\simeq_{FB} \Gamma_\beta\langle \mathrm{ev}_Y, F(r)F(\mathrm{ev}_B)(1 \times \alpha)(\beta^\alpha \times 1)\rangle \\
&= \Gamma_\beta\langle \mathrm{ev}_Y, F(r)\beta\,\mathrm{ev}_Y\rangle \\
&= \Gamma_\beta\langle 1, F(r)\beta\rangle\,\mathrm{ev}_y \\
&\simeq_{FB} \mathrm{ev}_Y,
\end{aligned}$$

using Corollary 4.2.9 and properties of transport. But because

$$\beta^\alpha 1_{Y^X_\alpha} = \beta^\alpha \text{ and } \mathrm{ev}_Y(1_{Y^X_\alpha} \times 1) = \mathrm{ev}_Y,$$

we conclude that $\Gamma_{\beta^\alpha}\langle 1, F(r^A)\beta^\alpha\rangle \simeq_{F(B^A)} 1_{Y^X_\alpha}$ by the uniqueness property of homotopy Π-types. □

Theorem 5.3.3. *Suppose $\mathcal{C}$ has homotopy exponentials as well as weak homotopy* Π*-types and $\mathcal{D}$ has homotopy* Π*-types. Then* $\mathrm{GL}(F)$ *has homotopy exponentials.*

Proof. Let $(Y^X_\alpha, B^A, \beta^\alpha)$ and $\bar{f}$ be constructed as in Theorem 5.3.1.

Suppose $g\colon (Z, C, \gamma) \to (Y^X_\alpha, B^A, \beta^\alpha)$ is such that $\mathrm{ev}_{(X,A,\alpha),(Y,B,\beta)}(g \times 1) \simeq f$. In particular we have

$$H_g\colon\ \mathrm{ev}_B(g_1 \times 1) \simeq f_1 \text{ and } \Gamma_\beta\langle \mathrm{ev}_Y(g_0 \times 1), F(H_g)(\gamma \times \alpha)\rangle \simeq_{FB} f_0.$$

Like the preceding lemma, choose $(PB)^A$ as the path object of B^A with evaluation map $\mathrm{ev}_{PB}\colon (PB)^A \times A \to PB$ using Corollary 4.2.9. As there are maps $\langle g_1, \bar{f}_1\rangle\colon C \to B^A \times B^A$ and $\mu\langle H_g, \sigma H_f\rangle\colon C \times A \to PB$ such that

$$\langle s, t\rangle\mu\langle H_g, \sigma H_f\rangle = \langle \mathrm{ev}_B(g_1 \times 1), \mathrm{ev}_B(\overline{f_1} \times 1)\rangle = \mathrm{ev}_{B\times B}(\langle g_1, \bar{f}_1\rangle \times 1)\colon C \times A \to B \times B$$

we conclude, by construction of $(PB)^A$ as a weak homotopy Π-type, the existence of $K\colon C \to (PB)^A$ such that $\langle s^A, t^A\rangle K = \langle g_1, \bar{f}_1\rangle$ and $\mathrm{ev}_{PB}(K \times 1) \simeq_{B\times B} \mu\langle H_g, \sigma H_f\rangle$. In particular $K\colon g_1 \simeq \bar{f}_1$. Let Γ_{β^α} be constructed as in Lemma 5.3.2. We wish to show that $\Gamma_{\beta^\alpha}\langle g_0, F(K)\gamma\rangle \simeq_{F(B^A)} n_0$. For this we use the uniqueness property of n_0. Recall that $n_0\colon Z \to Y^X_\alpha$ came as a consequence of the morphisms

$$F(\bar{f}_1)\gamma\colon Z \to F(B^A) \text{ and } \Gamma_\beta\langle f_0, \sigma F(H_f)(\gamma \times \alpha)\rangle\colon Z \times X \to Y$$

We have $\beta^\alpha\Gamma_{\beta^\alpha}\langle g_0, F(K)\gamma\rangle = tF(K)\gamma = F(\bar{f}_1)\gamma$, and thanks to Lemma 5.3.2 it holds that

$$\begin{aligned}
\mathrm{ev}_Y(\Gamma_{\beta^\alpha}\langle g_0, F(K)\gamma\rangle \times 1) &\simeq_{FB} \Gamma_\beta\langle \mathrm{ev}_Y(g_0 \times 1), F(\mathrm{ev}_{PB})(F(K)\gamma \times \alpha)\rangle \\
&\simeq_{FB} \Gamma_\beta\langle \mathrm{ev}_Y(g_0 \times 1), F(H_g)(\gamma \times \alpha), \sigma F(H_f)(\gamma \times \alpha)\rangle \\
&\simeq_{FB} \Gamma_\beta\langle f_0, \sigma F(H_f)(\gamma \times \alpha)\rangle.
\end{aligned}$$

By uniqueness of n_0 we can conclude that $\Gamma_{\beta^\alpha}\langle g_0, F(K)\gamma\rangle \simeq_{F(B^A)} n_0$, from which it follows that $g \simeq \bar{f}$. Thus, $(Y^X_\alpha, B^A, \beta^\alpha)$ is a homotopy exponential. □

Remark 5.3.4. One might have expected that it would be enough to demand that $\mathcal{C}$ has just homotopy exponentials. Sadly, this does not seem to be enough. Let us see what could be the problem. Suppose $g\colon (Z, C, \gamma) \to (Y_\alpha^X, B^A, \beta^\alpha)$ is another potential transpose of f. One would have $H_g\colon \mathrm{ev}_B(g_1 \times 1) \simeq f_1$ and, as we now have homotopy exponentials in $\mathcal{C}$, we immediately find some $K\colon g_1 \simeq \bar{f}$. But this is were the trouble begins. As $\mathcal{C}$ has only homotopy exponentials we have no idea how this morphism K looks like, only that it exists. This turns out to be a problem in showing that $\Gamma_{\beta^\alpha}\langle g_0, F(K)\gamma\rangle \simeq_{F(B^A)} n_0$. One would wish that $K\colon Z \to (PB)^A$ and that at each element a of A it evaluates to $\mu\langle H_g(a), \sigma H_f(a)\rangle$, or at least up to homotopy. So, it seems to be the case that function extensionality needs to hold in $\mathcal{C}$. Corollary 4.2.9 tells us that we can fix this by requiring that $\mathcal{C}$ has weak homotopy Π-types. Until this point it is not certain whether this extra requirements on $\mathcal{C}$ is actually necessary for $\mathrm{GL}(F)$ to have homotopy exponentials.

5.4 Weak homotopy Π-types in a gluing category

We will end this chapter by proving that $\mathrm{GL}(F)$ has weak homotopy Π-types when both $\mathcal{C}$ and $\mathcal{D}$ do. Let us start by giving some intuition about the construction presented below. Suppose we have pair of fibrations $f\colon (X, A, \alpha) \twoheadrightarrow (Y, B, \beta)$ and $g\colon (Y, B, \beta) \twoheadrightarrow (Z, C, \gamma)$ in $\mathrm{GL}(F)$. Then we can visualize this data as the following diagram

$$\begin{tikzcd}
X \arrow[dr] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "f_0"'] & & & \\
& X \times_{FB} Y \arrow[r] \arrow[d] & FA \arrow[ddr, "F(f_1)"] & \\
& Y \arrow[dr] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "g_0"'] & & \\
& & Z \times_{FC} FB \arrow[r] \arrow[d] & FB \arrow[d, "F(g_1)"] \\
& & Z \arrow[r, "\gamma"] & FC.
\end{tikzcd}$$

Suppose we have a point c and a point z of Z above $F(c)$. Then first of all, we are looking for a section h_1 of f_1 defined on the fibre of c in B and a section h_0 of f_0 defined on the fibre of z in Y and that, in a sense, lies above $F(h_1)$. Moreover, if we have a y above z, we should have the equality $\alpha(\mathrm{ev}(h_0, y)) = F(\mathrm{ev})(h_1, \beta(y))$.

Theorem 5.4.1. *Suppose $\mathcal{C}$ and $\mathcal{D}$ have weak homotopy Π-types. Then $\mathrm{GL}(F)$ has weak homotopy Π-types.*

Proof. For this proof we draw inspiration from the proof of [Shu15, Theorem 8.8].

Suppose that we have a pair of fibrations $(X, A, \alpha) \overset{f}{\twoheadrightarrow} (Y, B, \beta) \overset{g}{\twoheadrightarrow} (Z, C, \gamma)$. Then we start by constructing the weak homotopy Π-type $\Pi_{g_1} f_1\colon \Pi_{g_1} A \twoheadrightarrow C$ with evaluation map $\mathrm{ev}_{g_1} \colon B \times_C \Pi_{g_0} A \to A$ over B. Now, take the following two pullbacks

$$\begin{tikzcd}
P \arrow[r, "\pi_1"] \arrow[d, "\pi_Y"] & F(\Pi_{g_1} A) \arrow[d, "F(\Pi_{g_1} f_1)"] \\
Y \arrow[r, "F(g_1)\beta"] & FC
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, "\pi_P"] \arrow[d, "\pi_X"] & P \arrow[d, "\langle \pi_Y, F(\mathrm{ev}_{g_1})(\beta\times 1)\rangle"] \\
X \arrow[r, "\langle f_0, \alpha\rangle"] & Y \times_{FB} FA
\end{tikzcd}$$

which fit in the following larger diagram

$$\begin{tikzcd}
 & & Z \times_{FC} F(\Pi_{g_1} A) \arrow[rr, "\pi_1"] \arrow[ddd, "\pi_Z"] & & F(\Pi_{g_1} A) \arrow[ddd, "F(\Pi_{g_1} f_1)"] \\
Q \arrow[r, "\pi_P"] \arrow[d, "\pi_X"] & P \arrow[ur, "p"] \arrow[rr, "\beta\times 1"] \arrow[d] & & F(B \times_C \Pi_{g_1} A) \arrow[ur, "F(\pi_1)"] \arrow[d, "F(\mathrm{ev}_{g_1})"] & \\
X \arrow[r, "\langle f_0, \alpha\rangle"] \arrow[ddr, "f_0"] & Y \times_{FB} FA \arrow[rr] \arrow[dd, "\pi_Y"] & & FA \arrow[dd, "F(f_1)"] & \\
 & & Z \arrow[rr, "\gamma"] & & FC \\
 & Y \arrow[ur, "g_0"] \arrow[rr, "\beta"] & & FB \arrow[ur, "F(g_1)"] &
\end{tikzcd}$$

where by pullback pasting, all squares are actually pullbacks. In particular, the map $p \colon P \to Z \times_{FC} F(\Pi_{g_1} A)$ is a fibration.

So we have a pair of fibrations $Q \twoheadrightarrow P \twoheadrightarrow Z \times_{FC} F(\Pi_{g_1} A)$. This allows us to construct the weak homotopy Π-type $\Pi_p(\pi_P) \colon \Pi_p Q \to Z \times_{FC} F(\Pi_{g_1} A)$ with evaluation map $\mathrm{ev}_p \colon p^* \pi_p Q \cong Y \times_Z \Pi_p Q \to Q$ over P, i.e. $\pi_P \, \mathrm{ev}_p = 1 \times \pi_1 \Pi_p(\pi_P) \colon \Pi_p Q \times_Z Y \to P$. We claim that $(\Pi_p Q, \Pi_{g_1} A, \pi_1 \Pi_p(\pi_P) \colon \Pi_p Q \twoheadrightarrow F(\Pi_{g_1} A))$ is a weak homotopy Π-type in $\mathrm{GL}(F)$.

By construction, $(\pi_Z \Pi_p q, \Pi_{g_1} f_1) \colon (\Pi_p Q, \Pi_{g_1} A, \pi_1 \Pi_p(\pi_P) \twoheadrightarrow (Z, C, \gamma)$ is a fibration. there is a fibration To determine the evaluation map, note that

$$\begin{aligned}
\alpha \pi_X \, \mathrm{ev}_p &= F(\mathrm{ev}_{g_1})(\beta \times 1) \pi_P \, \mathrm{ev}_p \\
&= F(\mathrm{ev}_{g_1})(\beta \times 1)(1 \times \pi_1 \Pi_p(\pi_P)) \\
&= F(\mathrm{ev}_{g_1})(\beta \times \pi_1 \Pi_p(\pi_P)),
\end{aligned}$$

showing that there is a morphism

$$\mathrm{ev} \coloneqq (\pi_X \, \mathrm{ev}_p, \mathrm{ev}_{g_1}) \colon (Y \times_Z \Pi_p Q, B \times_C \Pi_{g_1} A, \beta \times \pi_1 \Pi_p(\pi_P)) \to (X, A, \alpha)$$

in $\mathrm{GL}(F)$.

Intuitively, an element of $(\Pi_p Q, \Pi_{g_1} A, \pi_1 \Pi_p(\pi_P))$ defines a section h_1 of f_1 on the fibre in B of some point c in C and a section h_0 of f_0 above $F(f_1)$ on the fibre in Y of some point z above $F(c)$. Moreover, if y lies in this fibre the equality $\alpha(\mathrm{ev}_X \langle h_0, y \rangle) = F(\mathrm{ev}_{g_1} \langle h_1, c \rangle)$ holds. This is precisely the kind of object we were looking for.

Now suppose we have a map $h\colon (W,D,\delta) \to (Z,C,\gamma)$ and a map $m\colon (Y \times_Z W, B \times_C D, \beta \times \delta) \to (X,A,\alpha)$ over (Y,B,β). This gives us a map $n_1\colon D \to \Pi_{g_1} A$ over C such that $H\colon \mathrm{ev}_{g_1}(1 \times n_1) \simeq_B m_1$. As h is a morphism in $\mathrm{GL}(F)$ we have

$$F(\Pi_{g_1} f_1)F(n_1)\delta = F(h_1)\delta = \gamma h_0\colon W \to FC$$

showing that there is $\langle h_0, F(n_1)\delta\rangle\colon W \to Z \times_{FC} F(\Pi_{g_1} A)$. The following diagram

$$\begin{tikzcd}
Y \times_Z W \arrow[r, "\pi_W"] \arrow[d, "1\times F(n_1)\delta"'] & W \arrow[d, "\langle h_0, F(n_1)\delta\rangle"] & \\
P \arrow[r, "p"] \arrow[d, "\pi_Y"] & Z \times_{FC} F(\Pi_{g_1} A) \arrow[r, "\pi_0"] \arrow[d, "\pi_Z"] & F(\Pi_{g_1} A) \arrow[d, "\Pi_{g_1} f_1"] \\
Y \arrow[r, "g_0"] & Z \arrow[r, "\gamma"] & FC
\end{tikzcd}$$

establishes that the upper left square is a pullback. Additionally, we have the map $m_0\colon Y \times_Z W \to X$. By choosing $\beta^*(F(P_B A))$ as the fibred path object of $Y \times_F BFA$ over Y, the following diagram

$$\begin{tikzcd}
Y \times_Z W \arrow[r, "m_0"] \arrow[d, "1\times F(n_1)\delta"'] & X \arrow[d, "\langle f_0, \alpha\rangle"] \\
P \arrow[r, "\langle \pi_Y, F(\mathrm{ev}_{g_1})(\beta\times 1)\rangle"] & Y \times_{FB} FA
\end{tikzcd}$$

commutes up to fibrewise homotopy over Y as

$$\pi_Y\langle \pi_Y, F(\mathrm{ev}_{g_1})(\beta \times 1)\rangle(1 \times F(n_1)\delta) = \pi_Y(1 \times F(n_1)\delta) = \pi_Y = f_0 m_0 = \pi_Y\langle f_0, \alpha\rangle m_0$$

and

$$\alpha m_0 = F(m_1)(\beta \times \delta) \simeq_{FB} F(\mathrm{ev}_{g_1}(1 \times n_1))(\beta \times \delta) = F(\mathrm{ev}_{g_1})(\beta \times 1)(1 \times F(n_1)\delta).$$

So, because $\langle f_0, \alpha\rangle$ is a fibration, we can apply Lemma 3.1.6 in the path category $\mathcal{D}(Y)$ to get $m_0' := \Gamma_{\langle f_0,\alpha\rangle}\langle m_0, \langle \pi_Y, \sigma F(H)(\beta \times \delta)\rangle\rangle$ so that the diagram commutes strictly. Note that $\Gamma_{\langle f_0,\alpha\rangle}$ is constructed in $\mathcal{C}(Y)$. This results in a map $\langle m_0', 1 \times F(n_1)\delta\rangle\colon Y \times_Z W \to Q$ over P. By the universal property of weak Π-types, this gives a map $n_0\colon W \to \Pi_p Q$ such that

$$Pi_p(\pi_P)n_0 = \langle h_0, F(n_1)\delta\rangle \text{ and } \mathrm{ev}_p(1 \times n_0) \simeq_P \langle m_0', 1 \times F(n_1)\delta\rangle.$$

By choosing the fibred path object of Q over P as the pullback of the fibred path object of X over $Y \times_{FB} FA$ along $\langle \pi_Y, F(\mathrm{ev}_{g_1})(\beta \times 1)\rangle$, this implies that

$$\pi_X \mathrm{ev}_p(1 \times n_0) \simeq_{Y \times_{FB} FA} m_0' \text{ and } \pi_P \mathrm{ev}_p(1 \times n_0) = 1 \times F(n_1)\delta.$$

Combining the above, we define $n := (n_0, n_1)\colon (W,D,\delta) \to (\Pi_p Q, \Pi_{g_1} A, \pi_1 \Pi_p(\pi_P))$ over (Z,C,γ). That this is in fact a morphism follows directly from

$$\pi_1 \Pi_p(\pi_P) n_0 = \pi_1\langle h_0, F(n_1)\delta\rangle = F(n_1)\delta.$$

It remains to show that $\mathrm{ev}(1 \times n) \simeq_{(Y,B,\beta)} m$. But this now follows readily. On one hand we had $H\colon \mathrm{ev}_{g_1}(1 \times n_1) \simeq_B m_1$ and on the other

$$
\begin{aligned}
&\Gamma_{\langle f_0,\alpha\rangle}\langle \pi_X \,\mathrm{ev}_p(1 \times n_0), \langle f_0 \pi_X \,\mathrm{ev}_p(1 \times n_0), F(H)(\beta \times \delta)\rangle\rangle \\
&= \Gamma_{\langle f_0,\alpha\rangle}\langle \pi_X \,\mathrm{ev}_p(1 \times n_0), \langle \pi_Y \pi_P \,\mathrm{ev}_p(1 \times n_0), F(H)(\beta \times \delta)\rangle\rangle \\
&\simeq_{\beta^*(FA)} \Gamma_{\langle f_0,\alpha\rangle}\langle m_0', \langle \pi_Y(1 \times F(n_1)\delta), F(H)(\beta \times \delta)\rangle\rangle \\
&= \Gamma_{\langle f_0,\alpha\rangle}\langle m_0', \langle \pi_Y, F(H)(\beta \times \delta)\rangle\rangle \\
&= \Gamma_{\langle f_0,\alpha\rangle}\langle m_0, \langle \pi_Y, \sigma F(H)(\beta \times \delta)\rangle, \langle \pi_Y, F(H)(\beta \times \delta)\rangle\rangle \\
&\simeq_{\beta^*(FA)} m_0.
\end{aligned}
$$

by Remark 3.2.4. So we can conclude that $(\Pi_p Q, \Pi_{g_1} A, \pi_1 \Pi_{\pi_P})$ is a weak homotopy Π-type. □

One would conjecture that a strengthening of the above theorem would hold where all weakness is dropped. Due to time constraints this has fallen beyond the scope of this thesis. A good place to start might be [Ber18, Proposition 2.19], where a similar result as Lemma 4.2.7 was shown, but now for homotopy Π-types. This can lead to a statement about the path objects of homotopy Π-types like Lemma 4.2.8, which would help in an analogous way as it did for the homotopy exponentials. But, as was the case with homotopy exponentials, it could also be that some extra assumptions will be needed to ensure that $\mathrm{GL}(F)$ has homotopy Π-types.

Chapter 6

Future Research

In this thesis we have seen that it is possible to construct a gluing category $\mathrm{GL}(F)$ for an exact functor $F\colon \mathcal{C} \to \mathcal{D}$ between two path categories and that the resulting category is again a path category. Moreover, we have seen some results about the existence of homotopy universal constructions in $\mathrm{GL}(F)$.

There are still some questions that could be studied. For example,

- Is it true that $\mathrm{GL}(F)$ has homotopy Π-types when both $\mathcal{C}$ and $\mathcal{D}$ do?
- Is it true that a homotopy natural number object is stable under homotopy, i.e. is X a hnno whenever $\mathbb{N} \simeq X$?
- The gluing construction in other settings has been used to show some canonicity results about the type theory it models, see for instance [Shu15]. So, can the gluing construction for path categories show canonicity results for homotopy type theory with propositional identity types?
- We have shown sufficient conditions for when $\mathrm{GL}(F)$ has homotopic universal constructions. However, are these necessary? Especially regarding the extra demand that $\mathcal{C}$ needs weak homotopy Π-types in the case of the question whether $\mathrm{GL}(F)$ has homotopy exponentials.

Index of Symbols

Index of Terms

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