Chiral anomalies and the Atiyah-Singer index theorem

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Abstract

Symmetries and their corresponding conservation laws are useful tools in particle physics. However some laws are violated when we they are extended in quantum field theory. In this thesis we review the chiral symmetry and we show its anomalous behaviour in perturbation theory and in the path integral formalism. The last method is related to the heat kernel proof of the Atiyah-Singer index theorem for Dirac operators. By extensively reviewing this proof we calculate the anomalous behaviour of the chiral symmetry in curved spacetime.

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Introduction

Symmetries play an important role in the study of classical and quantum mechanical systems. Namely, using Noethers theorem we can deduce a conservation law for each symmetry there is in a physical system. There is a quantum mechanical analog of Noethers theorem, but it is possible that the classical conservation law is violated when we extend it to the quantum realm. When this happens we call a symmetry anomalous. Anomalies are problematic in particle physics. This is because well defined quantum theories are required to be renormalizable and anomalies can break renormalization. Secondly, particle physics is heavily based on gauge theory: Every fundamental particle is characterized by the group under which it is invariant. Therefore, detecting and canceling anomalies is a substantial part of particle physics.

In this thesis we mainly focus on the chiral anomaly. Namely, each Dirac particle has a left-handed and a right-handed component. We assume that the two components can rotate independently without changing the theory. This is the chiral symmetry. Calculating the quantum mechanical consequences of this symmetry is not straight forward. Most calculations contain diverging integrals or are expressed as divergent power series. Regularization techniques are needed to extract the physical data. We show in this thesis the anomalous behavior of the chiral symmetry using perturbation theory and using the path integral formalism. The first method is due to Bardeen [1969] and the second is due to Fujikawa [1980]

Seemingly unrelated there is a family of theorems in mathematics that relate analytical and geometrical data to the topology of a manifold. For example, Gauss-Bonnet theorem relates the curvature of a surface with the Euler characteristic and the Riemann-Roch theorem relates the properties of meromorphic functions with the genus. All these theorems are specific cases of the Atiyah-Singer index theorem. It states that that a specific class of operators have a finite dimensional kernel and cokernel and that the difference between the dimensions of the kernel and the cokernel is topological of nature. This difference is called the index of an operator and Atiyah and Singer explicitly calculated this index. For Dirac operators the Atiyah-Singer index theorem is formulated as follows:

Theorem 0.1. *Let $S \to (M, g)$ be a Clifford bundle on a compact oriented n-dimensional Riemannian manifold M. Let D be the Dirac operator. If S is canonically graded, then the index of D is the integral over the n-form part of*

$\hat{A}(TM) \wedge \mathrm{ch}^{rel}(S)$. *That is,*

$$\mathrm{Index}(D) = \int_M \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)\exp\left(F^S\right)$$

where R is the Riemann curvature and F^S is the twisting curvature.

We review the proof of Theorem 0.1 using a method proposed by Atiyah et al. [1973] and Getzler [1983]. We compare this proof with the method Fujikawa [1980] used to calculate the chiral anomaly. We show that not only the results are equal, but we also show that they used the same methodology. In this thesis we model the proof of the Atiyah-Singer index theorem using the Fujikawa method and we show how the index theorem generalizes the Fujikawa method for curved spacetime.

This thesis is organized as follows: In chapter 1 we review the theory of Clifford algebras, we define the Dirac operator and we show the Weitzenbock formula. In chapter 2 we recall Noethers theorem and we show that the chiral current is anomalous using perturbation theory. After this chapter we study anomalies in the path integral formalism and we explain the Fujikawa method. From chapter 4 and onwards we generalize our study to compact curved spacetime and we start formally proving the index theorem. For this proof we need heat kernels and in chapter 4 we show that heat kernels exists and are unique. In chapter 5 we consider traces on infinitely dimensional vector spaces and we show that the heat kernel has a well-defined trace. We analyze the eigenvalues of the heat kernel and using this we show that the index of a Dirac operator is well-defined. In chapter 6 there is an intermezzo where we shortly revisit the theory of characteristic classes and we introduce the topological notions the Atiyah-Singer index theory refers to. Finally in chapter 7 we prove the index theory.

This thesis is aimed for people who have a basic understanding in particle physics and know the basics of geometry. For physicists who are not used to the mathematical notation of vector bundles there is a glossary at the end which relates mathematical notation in terms of physics.

Finally, in this thesis we will use Einstein notation. That is, if we use the same symbol as an upper index and a lower index we silently assume that we sum over that index.

1 Clifford Algebras

In the study of spin-$\frac{1}{2}$ particles, physicists use gamma matrices. They are defined as

$$\gamma_0 = \begin{pmatrix} -i & 0 & & \\ 0 & -i & & \\ & & i & 0 \\ & & 0 & i \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} & & 0 & -i \\ & & -i & 0 \\ 0 & i & & \\ i & 0 & & \end{pmatrix}$$
$$\gamma_2 = \begin{pmatrix} & & 0 & -1 \\ & & 1 & 0 \\ 0 & 1 & & \\ -1 & 0 & & \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} & & -i & 0 \\ & & 0 & i \\ i & 0 & & \\ 0 & -i & & \end{pmatrix} \tag{1.1}$$

and they satisfy the commutation relation

$$\{\gamma_\mu, \gamma_\nu\} := \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = \begin{cases} 2 & \text{if} \quad \mu = \nu = 0 \\ -2 & \text{if} \quad \mu = \nu \text{ and } \mu, \nu \in \{1, 2, 3\} \\ 0 & \text{else} \end{cases} .$$

The algebra these matrices is one of the first examples of an object now called Clifford algebra. In this chapter we give a rigorous definition of this algebra and we derive its basic properties. Using this algebra we define a differential operator called the Dirac operator. We also introduce the physics notation for dealing with Clifford algebras.

In the end, we prove the Weitzenbock formula. This equation relates the square of a Dirac operator with the Laplacian and we show that the difference can be given in terms of the curvature and the gamma matrices. For this we need an explicit expression for the adjoint of the Dirac operator and the connection. In the third paragraph we perform these calculations and we show that the Dirac operator is self-adjoint.

The material covered in this section is standard. For more information see Roe [1998]

1.1 Definitions

Definition 1.1. *Let V be a vector space with a symmetric 2-form g. A* **Clifford algebra** *for V is an unital algebra A such that the following holds:*

1. *There exists a linear map $\phi\colon V \to A$ such that $\phi(v)^2 = -g(v,v)\,\mathrm{Id}$ for all $v \in V$.*

2. *A satisfies the universal property. That is, if there exists another map $\phi'\colon V \to A'$ that have the same property of ϕ, then there exists an unique algebra homomorphism $A \to A'$ such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[dr, "\phi'"'] & A \arrow[d] \\ & A' \end{tikzcd}$$

Given a vector space V and a bilinear symmetric map g we can construct a Clifford algebra. Indeed, consider the tensor algebra

$$T(V) = \bigoplus_{k=0}^{\infty} V \otimes V \otimes \ldots \otimes V$$

and take the quotient by the ideal that is generated by $v \otimes v + g(v,v)\,\mathrm{Id}$. Denote this quotient as $\mathrm{Cl}(V)$ or $\mathrm{Cl}(V,g)$. By construction it satisfies the first part of Definition 1.1. The universal property follows from the universal property of tensor algebras. It states that for all linear maps $\phi\colon V \to A$ there exists a unique algebra homomorphism $\phi'\colon T(V) \to A$ such that

$$\begin{tikzcd} V \arrow[r] \arrow[dr, "\phi"'] & T(V) \arrow[d, "\phi'"] \\ & A \end{tikzcd}$$

commutes. Now assume that $\phi(v)^2 = -g(v,v)\,\mathrm{Id}$ for all $v \in V$. This relation extends on ϕ' to

$$\phi'(v \otimes v + g(v,v)\,\mathrm{Id}) = \phi'(v)^2 + g(v,v)\phi'(\mathrm{Id}) = 0.$$

So ϕ' factors over the ideal generated by $v \otimes v + g(v,v)\,\mathrm{Id}$ and so ϕ' is a unique algebra homomorphism between $\mathrm{Cl}(V)$ and A. Hence,

Proposition 1.2. *For any vector space V with a bilinear symmetric map, there exists a Clifford algebra and it is unique.*

The requirement in Definition 1.1 that $\phi(v)^2 = -g(v,v)\cdot\mathrm{Id}$ for all $v \in V$ is equivalent to the requirement that $\phi(u)\phi(v) + \phi(v)\phi(u) = -2g(u,v)\cdot\mathrm{Id}$ for all $u,v \in V$.

This can be easily seen if we consider $\phi(u+v)^2$. By linearity this equals $\phi(u)^2 + \phi(u)\phi(v) + \phi(v)\phi(u) + \phi(v)^2$ and so the anti-commutator between $\phi(u)$ and $\phi(v)$ is $\phi(u+v)^2 - \phi(u)^2 - \phi(v)^2$. However, by the commutation relation it follows that

$$\begin{aligned}\phi(u)\phi(v) + \phi(v)\phi(u) &= -\left(g(u+v, u+v) - g(u,u) - g(v,v)\right) \cdot \mathrm{Id} \\ &= -\,2g(u,v) \cdot \mathrm{Id}\,.\end{aligned}$$

Hence these two conditions are equivalent.

Example 1.3 (Gamma matrices)**.** Consider the vector space $V = \mathbb{R}^4$ and equip it with the Minkowski metric η. In the basis (t, x, y, z) the Minkowski metric is given by, $\eta_{tt} = -1$, $\eta_{xx} = \eta_{yy} = \eta_{zz} = 1$ and $\eta_{\mu\nu} = 0$ else. Hence, the gamma matrices satisfy the relation $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2\eta_{\mu\nu}$. and so $\gamma_\mu^2 = -\eta_{\mu\mu} \cdot \mathrm{Id}_{\mathbb{C}^4}$. The unital matrix subalgebra A generated by the gamma matrices is a Clifford algebra. Indeed, the map $\mathrm{Cl}(V, \eta) \to A$ that is defined by $t^i \cdot x^j \cdot y^k \cdot z^l \mapsto \gamma_t^i \circ \gamma_x^j \circ \gamma_y^k \circ \gamma_z^l$ for all $i, j, k, l, \in \mathbb{N}$ is an isomorphism between two algebras. It is a well-defined linear map and by the definition of A it is surjective. It is a homomorphism, because both sides satisfy the same commutation relations. We only need to show that it is injective. For this we need to work out all matrix multiplications and this is left for the reader.

In field theory the gamma matrices act on spin-$\frac{1}{2}$ fermions which are represented by fields. Mathematically particle fields are sections of a certain vector bundle. So to formalize spin-$\frac{1}{2}$ fermions we need to let the Clifford action act on section

Definition 1.4. *Let (M, g) be a (pseudo)-Riemannian manifold. The vector bundle $S \to M$ is a* **bundle of Clifford modules** *if there exists a smooth bundle map $\mathrm{Cl}(TM, g) \otimes \mathbb{C} \times S \to S$ which makes each fiber S_p a left-module over $\mathrm{Cl}(T_pM, g) \otimes \mathbb{C}$.*

Example 1.5 (Spin-$\frac{1}{2}$ particles)**.** A spin-$\frac{1}{2}$ particle is represented by a section over the trivial bundle $\mathbb{R}^4 \times \mathbb{C}^4 \to \mathbb{R}^4$. Clearly, the gamma matrices forms a left-module on $\mathbb{C}^4$. So $\mathbb{R}^4 \otimes \mathbb{C}^4$ is a bundle of Clifford modules.

For a bundle of Clifford modules $S \to M$, we define $\gamma\colon \Gamma(TM) \to \Gamma(\mathrm{End}(S))$ as the composition of the left-module action and the map ϕ from Definition 1.1. In local coordinates $e_\mu \in \Gamma(TM)$, this map is given by $\gamma(e_\mu)s = \phi(e_\mu) \cdot s$ for all sections $s \in \Gamma(S)$. If we use the shorthand $\gamma_\mu = \gamma(e_\mu)$, we see that γ_μ extends the gamma matrices from Equation 1.1 to any bundle of Clifford modules. We call $\gamma\colon \Gamma(TM) \to \Gamma(\mathrm{End}(S))$ the **Clifford action**.

Given a bundle of Clifford modules $S \to M$, a vector bundle $E \to M$ and a section $A \in \Gamma(TM \otimes S \otimes E)$ we formally define the **Feynman slash $\not{A}$ of A** as follows: It is the element $\not{A} \in \Gamma(S \otimes E)$ that is given by the composition of the following maps:

$$\begin{array}{ccccc} \Gamma(TM \otimes S \otimes E) & \xrightarrow{\phi} & \Gamma(\mathrm{Cl}(TM) \otimes S \otimes E) & \longrightarrow & \Gamma(S \otimes E) \\ A = \sum_\mu A^\mu \otimes e_\mu & \longmapsto & \sum_\mu A^\mu \otimes \gamma_\mu & \longmapsto & \not{A} = \gamma_\mu A^\mu. \end{array}$$

The second map denotes the multiplication defined by the left-module. We used $\{e_\mu\}$ as a local basis on TM. We only need it for comparison with the definition physicists use and we see that they indeed coincide.

In physics there is an operator called the Dirac operator. For the generalization of this operator we need to consider the interplay between a connection and the Clifford action. In the next definition we give explicit requirements:

Definition 1.6. *A* **Clifford bundle** *is a bundle of Clifford modules S over a (pseudo)-Riemannian manifold (M, g) equipped with a Hermitian metric and compatible connection such that*

1. *The Clifford action is skew adjoint. That is, for all $p \in M$, $v \in T_pM$ and $s_1, s_2 \in S_p$, we have*

$$\langle \gamma(v) \cdot s_1, s_2 \rangle + \langle s_1, \gamma(v)\, s_2 \rangle = 0,$$

2. *The connection on S is compatible with the Levi-Civita connection on M. So for all $u, v \in \Gamma(TM)$ and $s \in \Gamma(S)$, we have*

$$\nabla_u \gamma(v)\, s = \gamma(\nabla_u v)\, s + \gamma(v)\, \nabla_u s.$$

Example 1.7 (Spin-$\frac{1}{2}$ particles)**.** Note that the gamma matrices from Equation 1.1 are skew adjoint under the Euclidean metric. Also the compatibility condition reduces to the Leibniz rule for a flat connection. Hence, if we equip $\mathbb{R}^4 \times \mathbb{C}^4 \to \mathbb{R}^4$ with the Euclidean metric of $\mathbb{C}^4$ and the flat connection, we get that $\mathbb{R}^4 \times \mathbb{C}^4$ is a Clifford bundle.

Definition 1.8. *Let $S \to (M, g)$ be a Clifford bundle with compatible connection ∇ and the Clifford action γ. Interpret g as the isomorphism $g\colon \Gamma(TM) \to \Gamma(T^*M)$ which is given by $v \mapsto g(v, \cdot)$. The differential operator D is a* **Dirac operator** *if it is the composition of the following maps:*

$$\Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{g^{-1}} \Gamma(TM \otimes S) \xrightarrow{\gamma} \Gamma(S)$$

Following the steps above we can compute the Dirac operator in local coordinates. Denote $\{e_\mu\}$ as a local basis of TM and s as a section of S. Let $e_\mu^\flat$ be the dual of e^μ, i.e. $e_\mu^\flat(e^\nu) = \delta_{\mu\nu}$. By orthonormality it follows that $\sum_\mu e_\mu e_\mu^\flat = \mathrm{Id}$ and

$$\nabla s = \sum_\mu \nabla_\mu s \otimes e_\mu^\flat.$$

Under g^{-1} this maps to $\sum_\mu \nabla_\mu s \otimes e_\mu$ and so the Dirac operator is locally given by

$$\gamma^\mu \nabla_\mu s. \tag{1.2}$$

In the example of the spin-$\frac{1}{2}$ particles we use the flat connection $\nabla_\mu = \partial_\mu$. Then the Dirac operator is given by $\partial_\mu \gamma^\mu$ which is the expression normally used in quantum field theory.

1.2 Graded Clifford bundles

For spin-$\frac{1}{2}$ Dirac fermions, we distinguish left-handed and right-handed particles. This chirality is related to the ± 1 eigenvalues of the operator

$$\gamma_5 := -\gamma_0\gamma_1\gamma_2\gamma_3,$$

That is, if $\gamma_5\psi = \psi$, then we call ψ right handed and if $\gamma_5\psi = -\psi$ we call ψ left-handed. We see that γ_5 defines a grading on $\mathbb{R}^4 \times \mathbb{C}^4$. In general we define the grading as follows.

Definition 1.9. *Let S be a Clifford bundle. We say that S is a* **graded Clifford bundle** *if S can be decomposed into $S^+ \oplus S^-$ such that*

1. *The Clifford action γ maps $\Gamma(TM \otimes S^\pm)$ to $\Gamma(S^\mp)$.*
2. *The metric and the connection of S respects the grading.*

In the example of $\mathbb{R}^4 \times \mathbb{C}^4$ the operator γ_5 is fully determined by how it acts on left- resp. right-handed particles. Its acts by $+1$ on right-handed fields and by -1 on left-handed field. This can be generalized to any graded Clifford bundle.

Definition 1.10. *Let S be a graded Clifford bundle. We define the* **grading operator** $\gamma_5 \colon \Gamma(S) \to \Gamma(S)$ *by*

$$\begin{aligned}\gamma_5|_{S^+} &= \mathrm{Id} \\ \gamma_5|_{S^-} &= -\mathrm{Id}\,.\end{aligned}$$

The grading operator anti-commutes with the Clifford action. Indeed if $S + S^+ \oplus S^-$ is a graded Clifford bundle and $s \in \Gamma(S^\pm)$. then for all $v \in \Gamma(TM)$ the section $\gamma(v)s$ is an element of $\Gamma(S^\mp)$ and hence

$$\gamma_5\gamma(v)s = \mp\gamma(v)s = -\gamma(v)\gamma_5 s.$$

Even more, the Dirac operator anti-commutes with the grading operator. This can be easily seen if we use local coordinates and write $D = \sum_\mu \gamma_\mu \nabla_\mu$.

Gradings on Clifford bundles are not unique. However, there is a canonical method to induce a grading.

Definition 1.11. *Let $S \to (M, g)$ be a Clifford bundle on a $2n$ dimensional manifold and let $\{e_\mu\}$ be a local positively oriented orthonormal basis of TM. The element*

$$\omega = i^n \gamma_1 \cdot \ldots \cdot \gamma_{2n} \in \Gamma(\mathrm{End}(S))$$

is called the **canonical grading operator**.

Lemma 1.12 (Roe [1998], Remark 4.4)**.** *The canonical grading operator ω from Definition 1.11 does not depend on the choice of local orthonormal frame and hence it is globally defined.*

Proof. Let $\{e^\mu\}$ and $\{\tilde{e}^\mu\}$ be two orthonormal local frames of TM with identical orientation. Denote γ_μ and $\tilde{\gamma}_\mu$ as the Clifford action on e_μ resp. $\tilde{e}_\mu$. We expand $\tilde{e}^\mu$

as $A^\mu_\nu e^\nu$. In local coordinates the canonical grading operator $\tilde{\omega}$ w.r.t. the basis $\{\tilde{e}^\mu\}$ equals

$$\tilde{\omega} := i^n \tilde{\gamma}_1 \cdot \ldots \cdot \tilde{\gamma}_{2n} = i^n A^{\mu_1}_1 \ldots A^{\mu_{2n}}_{2n} \, \gamma_{\mu_1} \ldots \gamma_{\mu_{2n}}. \tag{1.3}$$

We view A an an orthogonal matrix on $\{\tilde{e}^\mu\}$. In the case that $\mu_1 = \mu_2$ it follows that

$$\sum_{\mu_1} A^{\mu_1}_1 A^{\mu_2}_2 = (AA^T)_{12} = \mathrm{Id}_{12} = 0.$$

This shows that all μ_i's in Equation 1.3 must be unique. If S_{2n} is the set of all permutations of $2n$-elements, then $\tilde{\omega}$ equals

$$\begin{aligned}
\tilde{\omega} =& i^n \sum_{\sigma \in S_{2n}} A^{\mu_{\sigma(1)}}_{\sigma(1)} \ldots A^{\mu_{\sigma(2n)}}_{\sigma(2n)} \, \gamma_{\mu_{\sigma(1)}} \ldots \gamma_{\mu_{\sigma(2n)}} \\
=& i^n \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) A^{\mu_{\sigma(1)}}_{\sigma(1)} \ldots A^{\mu_{\sigma(2n)}}_{\sigma(2n)} \, \gamma_1 \ldots \gamma_{2n} \\
=& \det(A) \cdot \omega.
\end{aligned}$$

Because $A \in SO(2n)$ the determinant of A equals one and $\tilde{\omega}$ equals the canonical grading operator w.r.t. the basis $\{e^\mu\}$. □

In the next proposition we show that the canonical grading operator ω indeed defines a graded Clifford bundle and so we see ω as the generalization of γ_5.

Proposition 1.13 (Roe [1998], Page 142). *Let $S \to (M, g)$ be a Clifford bundle on a $2n$ dimensional manifold and let ω be the canonical grading operator. Then $\omega^2 = \mathrm{Id}$ and the ± 1 eigenspaces of ω, $S^\pm$, are subbundles of S, therefore decomposing it as graded Clifford bundle $S = S^+ \oplus S^-$.*

Proof. The square of ω can be easily calculated and is indeed equal to one. Therefore we can split S into $S^+ \oplus S^-$ where $S^\pm$ are the ± 1 eigenspaces of ω. Also, the commutation identity

$$\omega\gamma + \gamma\omega = 0$$

can be easily calculated in a suitable basis. This proves the first part of Definition 1.9.

To show that g respects the grading we assume that $v_+ \in \Gamma(S^+)$ and $v_- \in \Gamma(S^-)$. By the skew-adjointness of the Clifford action and $\omega v_\pm = \pm v_\pm$ we van calculate $g(v_+, v_-)$.

$$\begin{aligned} g(v_+, v_-) =& g(\omega v_+, v_-) = i^n g(\gamma^1 \dots \gamma^{2n} v_+, v_-) \\ =&(-i)^n g(v_+, \gamma^2 n \dots \gamma^1 v_-) = (i)^n g(v_+, \gamma^1 \dots \gamma^{2n} v_-) \\ =& g(v_+, \omega v_-) = -g(v_+, v_-). \end{aligned}$$

This shows $g(S^+, S^-) = 0$ and hence g respects the grading.

To show that the connection on S respects the grading we need to pick a suitable coordinate system on M. Fix $p \in M$ and pick the local coordinate frame $\{\frac{\partial}{\partial x^\mu}\}$ that follows from the exponential map $\exp_p\colon T_pM \to M$. This frame is called the **Riemannian normal coordinate system** and it has the property that at p

$$\nabla \frac{\partial}{\partial x^\mu} = 0$$

where ∇ denote the Levi-Civita connection and $\frac{\partial}{\partial x^\mu}$ is a basis element in $\Gamma(TM)$. Let $s \in \Gamma(S)$, $v \in \Gamma(TM)$ and use ∇ also for the connection on S. Using the compatibility property of the connection we get

$$\begin{aligned} \nabla_v(\omega s) =& i^n \nabla(\gamma^1 \dots \gamma^{2n} s) \\ =& i^n \gamma(\nabla_v(\partial/\partial x^1)) \cdot \gamma^2 \dots \gamma^{2n} s + i^n \gamma^1 \nabla_v(\gamma^2 \dots \gamma^{2n} s) \\ =& i^n \gamma^1 \nabla_v(\gamma^2 \dots \gamma^{2n} s) \\ =& \dots = \omega \nabla_\nu s. \end{aligned}$$

This shows that ∇ also respects the grading. $\square$

Remark 1.14. Unless not stated otherwise we assume that a Clifford bundle is canonically graded.

Remark 1.15. If γ is the Clifford action and $\{e^\mu\}$ a basis of TM we simplify $\gamma(e^\mu)$ into γ_μ. However, if we explicitly[1] write γ_5, then it does not mean $\gamma(e_5)$, but it means the grading operator. This might be confusing, but it is standard notation in physics.

Remark 1.16. In higher dimensions physicists mostly use the notation γ_{n+1} where n is the dimension of the spacetime. However, when we consider higher dimensions in physics, we will use the Veltman-'t Hooft regularization. There it is custom to use γ_5 instead of γ_{n+1}. To prevent confusion we never use γ_{n+1}, but use γ_5.

[1] In $\gamma_0 \dots \gamma_k$ the operator γ_5 is not explicitly written we so we don't refer here to the grading operator.

1.3 The adjoint of a Dirac operator

In this paragraph we will compute the formal adjoint of a Dirac operator D. Since D is the composition of the Clifford action γ and the covariant derivative ∇. Therefore we compute the adjoint of γ and ∇ separately. These will be calculated in the next two lemmas. We conclude this paragraph by showing that D is self-adjoint.

Lemma 1.17. *Let $E \to M$ be a vector bundle with Hermitian metric g and compatible connection ∇ on an oriented Riemannian manifold M. Let $v \in \Gamma(TM)$ such that the Lie derivative along v of the Riemannian volume form $\mathrm{Vol}(M)$ on M is zero. Then, for all $s \in \Gamma(E)$ with compact support $\nabla^*(v^\flat \otimes s) = -\nabla_v(s)$.*

Proof. Let $s, t \in \Gamma_c(E)$ be two sections on E with compact support. We use $\langle \cdot, \cdot \rangle = \int_M g(\cdot, \cdot)\,\mathrm{Vol}(M)$ as the inner product on $\Gamma_c(E)$. Using Cartans magic formula, $\mathcal{L}_v = \mathrm{d}\,\iota_v + \iota_v\,\mathrm{d}$, and Stokes theorem we have

$$\int_M \mathcal{L}_v(g(s,t) \wedge \mathrm{Vol}(M)) = \int_M (\mathrm{d}\,\iota_v + \iota_v\,\mathrm{d})(g(s,t) \wedge \mathrm{Vol}(M)) = 0.$$

We expand $\mathcal{L}_v(g(s,t) \wedge \mathrm{Vol}(M))$. It follows from $\mathcal{L}_v \mathrm{Vol}(M) = 0$ that

$$\mathcal{L}_v(g(s,t) \wedge \mathrm{Vol}(M)) = \mathcal{L}_v\, g(s,t) \wedge \mathrm{Vol}(M).$$

Because ∇ is a compatible connection, we can write $\mathcal{L}_v(s,t)$ in terms of ∇_v:

$$\mathcal{L}_v(g(s,t) \wedge \mathrm{Vol}(M)) = g(\nabla_v s, t) \wedge \mathrm{Vol}(M) + g(s, \nabla_v t) \wedge \mathrm{Vol}(M)$$

This concludes

$$\langle \nabla_v s, t \rangle = -\langle s, \nabla_v t \rangle. \tag{1.4}$$

Finally, denote $(\cdot, \cdot)$ as the inner product on $\Gamma_c(T^*M \otimes E)$ and let $\{e_\mu\}$ be an orthonormal frame on TM. Using the identity $\sum_\mu e_\mu e_\mu^\flat = \mathrm{Id}$, we calculate

$$\begin{aligned} \langle s, \nabla^*(v^\flat \otimes t) \rangle =& (\nabla s, v^\flat \otimes t) \\ =& \sum_{\mu,\nu} v(e_\nu) \cdot (e_\mu^\flat \otimes \nabla e_\mu s, e_\nu^\flat \otimes t) \\ =& \sum_\mu \langle \nabla_v s, t \rangle. \end{aligned} \tag{1.5}$$

Combining Equation 1.4 and 1.5 we conclude

$$\langle s, \nabla^*(v^\flat \otimes t) \rangle = -\langle \nabla_v s, t \rangle.$$

$\square$

Recall that for a vector bundle $E \to M$ with a metric and compatible connection the **Laplacian** is defined as

$$\nabla^*\nabla \colon \Gamma(E) \to \Gamma(E).$$

Using Lemma 1.17 we can calculate the Laplacian in local coordinates.

Corollary 1.18. *Let $E \to M$ be a vector bundle with Hermitian metric g and compatible connection ∇ on an oriented Riemannian manifold M. Let $x \in M$ and let $\{\frac{\partial}{\partial x^\mu}\}$ be the Riemannian normal coordinate frame centered at y. Then at the origin of the chart the Laplacian $\nabla^*\nabla$ satisfies*

$$\nabla^*\nabla = -\sum_\mu \nabla_\mu \nabla_\mu$$

where $\nabla_\mu = \nabla_{\frac{\partial}{\partial x^\mu}}$.

Proof. Let n be the dimension of the manifold M. By Cartans magic formula we can easily check that $\mathcal{L}_{\frac{\partial}{\partial x^\mu}} \mathrm{Vol}(M) = 0$. Indeed, it equals

$$\begin{aligned}\mathcal{L}_{\frac{\partial}{\partial x^\mu}} \mathrm{Vol}(M) =& \,\mathrm{d}\,\iota_{\frac{\partial}{\partial x^\mu}} \,\mathrm{d}\,x^1 \wedge \ldots \wedge \mathrm{d}\,x^n \\ =&(-1)^\mu \,\mathrm{d}(\mathrm{d}\,x^1 \wedge \ldots \wedge \mathrm{d}\,x^{\mu-1} \wedge \mathrm{d}\,x^{\mu+1} \wedge \ldots \wedge \mathrm{d}\,x^n) \\ =&0.\end{aligned}$$

Thus we can use Lemma 1.17 and by the identity $\sum_\mu \frac{\partial}{\partial x^\mu} \,\mathrm{d}\,x^\mu = \mathrm{Id}$ we have for all $s \in \Gamma_c(E)$

$$\nabla^*\nabla s = \sum_\mu \nabla^*(\mathrm{d}\,x^\mu \otimes \nabla_\mu s) = -\sum_\mu \nabla_\mu \nabla_\mu s.$$

□

Lemma 1.19. *Let $S \to (M, g)$ be a Clifford bundle and let γ be the Clifford action. In the local orthonormal basis $\{e^\mu\}$ of TM the formal adjoint of $\gamma \colon \Gamma(TM \otimes S) \to S$ is given by*

$$\gamma^*(\cdot) = -e^\mu \otimes \gamma_\mu(\cdot).$$

Proof. Let $s, t \in \Gamma(S)$ and let $v \in \Gamma(TM)$. Denote $\langle \cdot, \cdot \rangle$ as the metric on $\Gamma(TM \otimes S)$ and $(\cdot, \cdot)$ as the metric on S. We write down the definition of the formal adjoint and use the skew symmetry of the Clifford action:

$$\langle v \otimes s, \gamma^* t \rangle = g(\gamma(v)s, t) = -g(s, \gamma(v)\, t) \tag{1.6}$$

We compare this to $\langle v \otimes s, -e_\mu \otimes (\gamma^\mu\, t)\rangle$. Expand v as $v = v_\nu e^\nu$. Then,

$$\langle v \otimes s, -e^\mu \otimes (\gamma_\mu\, t)\rangle = \langle v_\nu e^\nu \otimes s, -e^\mu \otimes (\gamma_\mu\, t)\rangle = v_\nu \cdot g(e^\nu, e^\mu) \cdot (s, \gamma_\nu t).$$

By orthonormality $v_\nu \cdot g(e^\nu, e^\mu)$ simplifies to v_μ and hence

$$\langle v \otimes s, -e^\mu \otimes (\gamma_\mu\, t)\rangle = v_\mu \cdot \langle s, \gamma_\nu t\rangle = -(s, \gamma(v)\, t). \tag{1.7}$$

Comparing Equation 1.6 and 1.7 we conclude the result. ☐

Proposition 1.20 (Roe [1998],Proposition 3.11)**.** *Let $S \to M$ be a Clifford bundle and let D be a Dirac operator. Then D is formally self-adjoint.*

Proof. Recall that D is defined as the composition of the following maps:

$$\Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{g^{-1}} \Gamma(TM \otimes S) \xrightarrow{\gamma} \Gamma(S).$$

So the formal adjoint of D is given by

$$\Gamma(S) \xrightarrow{\gamma^*} \Gamma(TM \otimes S) \xrightarrow{g} \Gamma(T^*M \otimes S) \xrightarrow{\nabla^*} \Gamma(S).$$

Let $x \in M$ and consider the Riemannian normal coordinates centered at y. At $x \in M$ we get for any $s \in \Gamma(S)$ that

$$s \mapsto -\frac{\partial}{\partial x^\mu} \otimes \gamma^\mu s \mapsto -\sum_\mu \mathrm{d}\, x^\mu \otimes \gamma_\mu s \mapsto \nabla_\mu \gamma_\mu s.$$

At the origin of our chart, we have $\nabla_\mu e^\mu = 0$. So by the compatibility condition we get

$$D^* s = \nabla_\mu \gamma^\mu s = \gamma^\mu \nabla_\mu s = Ds.$$

Therefore, D is formally self-adjoint. ☐

1.4 The Weitzenbock formula

In this paragraph we compare the difference between the square of an Dirac operator and the Laplacian. The result will be the so called Weitzenbock formula, which gives the difference in terms of the Clifford action and the curvature. Before we do this, we need to generalize the Feynman-slash operator for 2-forms.

Definition 1.21. *Let $S \to (M, g)$ be a Clifford bundle with the Clifford action γ and let $K \in \Omega^2(M, \mathrm{End}(S))$ be a 2-form with values in $\mathrm{End}(S)$. Define the map $\Omega^2(M) \to \Gamma(T^*M \otimes T^*M)$ as $\alpha \wedge \beta \mapsto \frac{1}{2}(\alpha \otimes \beta - \beta \otimes \alpha) \in \Gamma(T^*M \otimes T^*M)$. Interpret g as the isomorphism $g \colon \Gamma(TM) \to \Gamma(T^*M)$ which is given by $v \mapsto g(v, \cdot)$. The* **Clifford contraction of** *K is the composition of the maps*

$$\Gamma(S) \xrightarrow{K} \Omega^2(M, S) \hookrightarrow \Gamma(T^*M \otimes T^*M \otimes S) \xrightarrow{g^{-1}} \Gamma(TM \otimes TM \otimes S) \xrightarrow{\gamma} \Gamma(S).$$

The Clifford contraction of K is denoted as K.

In a local orthonormal frame $\{e^\mu\}$ on TM, K can be written as $K = \sum_{\mu<\nu} e^{\mu\flat} \wedge e^{\nu\flat} \otimes K(e_\mu, e_\nu)$. Therefore, the Clifford contraction of K equals It can be shown that the Clifford contraction of K equals

$$\mathrm{K} = \sum_{\mu<\nu} \gamma_\mu \gamma_\nu K(e^\mu, e^\nu). \tag{1.8}$$

We are now able to show the Weitzenbock formula.

Theorem 1.22 (Weitzenbock)**.** *Let $S \to M$ be a Clifford bundle with compatible connection ∇. Let D be the Dirac operator and K be the curvature w.r.t. ∇. Denote K as the Clifford contraction of the curvature. Then*

$$D^2 = \nabla^* \nabla + \mathrm{K}.$$

Proof. Let $x \in M$ and let $\{\frac{\partial}{\partial x^\mu}\}$ be the Riemannian normal coordinate frame centered at y. Let $\{\gamma^\mu, \gamma^\nu\}$ and $[\gamma^\mu, \gamma^\nu]$ be the (anti)-commutator. That is,

$$\begin{aligned} \{\gamma^\mu, \gamma^\nu\} &= \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \\ [\gamma^\mu, \gamma^\nu] &= \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu \qquad \forall \mu, \nu. \end{aligned}$$

Using Equation 1.2 we calculate the square of the Dirac operator

$$\begin{aligned} D^2 &= \sum_{\mu\nu} \gamma^\mu \nabla_\mu \gamma^\nu \nabla_\nu = \sum_{\mu\nu} \gamma^\mu \gamma(\nabla_\mu e^\nu)\nabla_\nu + \gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu \\ &= \sum_{\mu\nu} \gamma^\mu\gamma^\nu\nabla_\mu\nabla_\nu. \end{aligned}$$

This can be written in terms of the commutator and the anti-commutator and so

$$\begin{aligned}
D^2 =& \frac{1}{2}\sum_{\mu\nu}\{\gamma_\mu,\gamma_\nu\}\nabla_\mu\nabla_\nu + [\gamma_\mu,\gamma_\nu]\nabla_\mu\nabla_\nu \\
=& \sum_{\mu\nu} -g\left(\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial x^\nu}\right)\nabla_\mu\nabla_\nu + \frac{1}{2}\gamma_\mu\gamma_\nu[\nabla_\mu,\nabla_\nu] \\
=& -\sum_{\mu}\nabla_\mu\nabla_\mu + \sum_{\mu<\nu}\gamma^\mu\gamma^\nu K\left(\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial x^\nu}\right).
\end{aligned}$$

From Corollary 1.18 and Equation 1.8 we conclude that $D^2 = \nabla^*\nabla + \mathrm{K}$. □

Next, we will prove a refined version of the Weitzenbock formula, namely the Lichnerowicz formula. We show this by considering the commutation relations between the the curvature K and the Clifford action. Then K will split into the scalar curvature and a the Clifford contraction of a 2-form called the Riemann endomorphism.

Definition 1.23. *Let $S \to (M,g)$ be a Clifford bundle, let γ the Clifford action and let R be the Riemann curvature tensor. The* **Riemann endomorphism** R^S **of** S *is the following composition of maps applied to the Riemann curvature tensor R:*

$$\begin{tikzcd}
\Gamma(\Omega^2(M)\otimes TM\otimes T^*M) \arrow[r, "g^{-1}"] \arrow[dd, dashed, bend right, "R\mapsto R^S"'] & \Gamma(\Omega^2(M)\otimes TM\otimes TM) \arrow[dl, "\gamma"'] \\
\Gamma(\Omega^2(M)\otimes \mathrm{End}(S)\otimes \mathrm{End}(S)) \arrow[r, "\circ"] & \Gamma(\Omega^2(M)\otimes \mathrm{End}(S)) \arrow[dl, "1/4"'] \\
\Gamma(\Omega^2(M)\otimes \mathrm{End}(S)) &
\end{tikzcd}$$

Here $\circ$ denotes the point-wise composition of endomorphisms and $\frac{1}{4}$ denotes devision by four.

Given a local orthonormal frame $\{e^\mu\}$ we calculate the Riemann endomorphism by following the diagram:

$$\begin{tikzcd}
R = \sum_{\mu\nu} \langle Re^\mu, e^\nu \rangle \otimes e^\mu \otimes e^\flat_\nu \arrow[r, "g^{-1}"] & \sum_{\mu\nu} \langle Re^\mu, e^\nu \rangle \otimes e^\mu \otimes e^\nu \arrow[dl, "\gamma"'] \\
\sum_{\mu\nu} \langle Re^\mu, e^\nu \rangle \otimes \gamma_\mu \otimes \gamma_\nu \arrow[r, "\circ"] & \sum_{\mu\nu} \langle Re^\mu, e^\nu \rangle \gamma_\mu \gamma_\nu \arrow[dl, "1/4"'] \\
\frac{1}{4} \sum_{\mu\nu} \langle Re^\mu, e^\nu \rangle \gamma_\mu \gamma_\nu &
\end{tikzcd}$$

Hence, in a local orthonormal frame $\{e^\mu\}$ on TM, the Riemann endomorphism is given by

$$R^S(X, Y) = \frac{1}{4} \gamma_\mu \gamma_\nu \langle R(X, Y) e^\mu, e^\nu \rangle \qquad \forall X, Y \in \Gamma(TM).$$

Lemma 1.24 (Roe [1998], Lemma 3.13 and 3.15)**.** *Let $S \to M$ be a Clifford bundle, let K be the curvature on S and R be the Riemann curvature on M. Use γ for the Clifford action and let R^S be the Riemann endomorphism. Then for all $u, v, w \in \Gamma(TM)$*

$$[K(u, v), \gamma(w)] = [R^S(u, v), \gamma(w)] \tag{1.9}$$

where $[\cdot, \cdot]$ is the commutator.

Proof. By the definition of the curvature, the left hand side of Equation 1.9 equals

$$[K(u, v), \gamma(w)] = [\nabla_u \nabla_v - \nabla_v \nabla_u, \gamma(w)].$$

According to the Jacobi identity this is

$$\begin{aligned}
[K(u, v), \gamma(w)] = & \left[\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}, \gamma(w)\right] \\
= & \nabla_u [\nabla_v, \gamma(w)] + [\nabla_u, \gamma(w)] \nabla_v - \\
& - \nabla_v [\nabla_u, \gamma(w)] - [\nabla_v, \gamma(w)] \nabla_u - \left[\nabla_{[u,v]}, \gamma(w)\right] \\
= & [\nabla_u, [\nabla_v, \gamma(w)]] - [\nabla_v, [\nabla_u, \gamma(w)]] - \left[\nabla_{[u,v]}, \gamma(w)\right].
\end{aligned}$$

From Definition 1.6 follows that the commutator $[\nabla_v, \gamma(w)] = \gamma(\nabla_u w)$ and hence

$$\begin{aligned}
[K(u, v), \gamma(w)] = & [\nabla_u, \gamma(\nabla_v w)] - [\nabla_v, \gamma(\nabla_u w)] - \gamma(\nabla_{[u,v]} w) \\
= & \gamma(\nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w) \\
= & \gamma(R(u, v) w).
\end{aligned} \tag{1.10}$$

This gives an explicit result for the left hand side. To calculate the right hand side of Equation 1.9 we need a local orthonormal frame $\{e^\mu\}$ of TM. Using the identity

$$[\gamma(u)\gamma(v), \gamma(w)] = 2g(u, w)\gamma(v) - 2g(v, w)\gamma(u)$$

we get that the right hand side equals

$$\begin{aligned}\left[R^S(u, v), \gamma(w)\right] =& \frac{1}{4}\langle R(u, v)e^\mu, e^\nu\rangle \cdot [\gamma_\mu\gamma_\nu, \gamma(w)] \\ =& \frac{1}{2}\langle R(u, v)w, e^\nu\rangle\gamma_\nu - \frac{1}{2}\langle R(u, v)e^\mu, w\rangle\gamma_\mu.\end{aligned}$$

By the anti-symmetry property $\langle R(u, v)e^\mu, w\rangle = -\langle R(u, v)w, e^\mu\rangle$ it follows that

$$\left[R^S(u, v), \gamma(w)\right] = \langle R(u, v)w, e^\nu\rangle\gamma_\nu = \gamma(R(u, v)w). \tag{1.11}$$

Comparing Equation 1.10 and 1.11 we conclude the result. □

Definition 1.25. *Let $S \to (M, g)$ be a Clifford bundle over a Riemannian manifold and let $E \to M$ be a vector bundle. A* **Clifford endomorphism** *is a section F of $\mathrm{End}_{\mathbb{C}}(S) \otimes E$ such that for all $v \in TM$, $F \circ \gamma(v)$ equals $\gamma(v) \circ F$.*

By Lemma 1.24 we conclude that $K - R^S \in \Omega^2(\mathrm{End}(S))$ is a Clifford endomorphism. This difference will play an important role in the index theorem and so we give it a name.

Definition 1.26. *Let $S \to M$ be a Clifford bundle and let K be its curvature. The* **twisting curvature** *F^S is the Clifford endomorphism $K - R^S$.*

Using the twisting curvature and the Weitzenbock formula we can write

$$D^2 = \nabla^*\nabla + \mathrm{F}^S + \mathrm{R}^S.$$

However, this can be further reduced.

Proposition 1.27 (Lichnerowicz [1963])**.** *Let $S \to (M, g)$ be a Clifford bundle. Let D be the Dirac operator and F^S be the Clifford contraction of the twisting curvature. Denote κ as the scalar curvature. Then,*

$$D^2 = \nabla^*\nabla + \mathrm{F}^S + \frac{1}{4}\kappa.$$

Proof. It is sufficient to show that $\mathrm{R}^S = \frac{1}{4}\kappa$. In a local orthonormal frame $\{e_\mu\}$ of TM, the Clifford contraction of the Riemann endomorphism is given by

$$\mathrm{R}^S = \frac{1}{8}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\langle R(e_\mu, e_\nu)e_\rho, e_\sigma\rangle.$$

It follows from the Bianchi identity that

$$\mathrm{R}^S = -\frac{1}{8}\langle R(e_\mu, e_\nu)e_\rho, e_\sigma\rangle(\gamma^\rho\gamma^\mu\gamma^\nu\gamma^\sigma + \gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma).$$

Using the commutation relation of the Clifford action, we can reorder the gammas back into $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ and so R^S equals

$$\mathrm{R}^S = -\frac{1}{8}\langle R(e_\mu, e_\nu)e_\rho, e_\sigma\rangle(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma - 4\delta^{\mu\rho}\gamma^\nu\gamma^\sigma + 2\delta^{\nu\rho}\gamma^\mu\gamma^\sigma).$$

By the antisymmetry of 2-forms, this simplifies into

$$\begin{aligned}\mathrm{R}^S &= -\frac{1}{8}\langle R(e_\mu, e_\nu)e_\rho, e_\sigma\rangle(2\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - 6\delta^{\mu\rho}\gamma^\nu\gamma^\sigma)\\ &= -2\mathrm{R}^S + \frac{3}{4}\langle R(e_\mu, e_\nu)e_\mu, e_\sigma\rangle\gamma^\nu\gamma^\sigma.\end{aligned}$$

Because $\langle R(e_\mu, e_\nu)e_\rho, e_\sigma\rangle = \langle R(e_\rho, e_\sigma)e_\mu, e_\rho\rangle$, the Clifford contraction of the Riemann endomorphism becomes

$$\begin{aligned}\mathrm{R}^S =& \frac{1}{4}\langle R(e_\mu, e_\nu)e_\mu, e_\sigma\rangle\gamma^\nu\gamma^\sigma\\ =& \frac{1}{8}\langle R(e_\mu, e_\nu)e_\mu, e_\sigma\rangle(\gamma^\nu\gamma^\sigma + \gamma^\sigma\gamma^\nu)\\ =& -\frac{1}{4}\langle R(e_\mu, e_\nu)e_\mu, e_\nu\rangle.\end{aligned}$$

We conclude that $\mathrm{R}^S = \frac{1}{4}\kappa$. □

2 Calculating anomalies using Feynman diagrams

In this chapter we introduce the notion of anomalies. For this we revisit Noethers theorem and we investigate if there are obstructions when we generalize it to quantum field theory. These obstructions are called anomalies:

> **Definition 2.1.** *We call classical conservation law that is not satisfied in quantum field theory an* **anomaly**.

In this chapter we use the perturbative approach to quantum field theory. That is, we calculate the amplitude of Feynman diagrams that relate to classical conservation currents. We work out examples where the classical currents are not conserved.

Most calculations are done by van Nieuwenhuizen [1989], but without much detail.

2.1 Revisit of Noethers theorem

Informally, Noethers theorem can be stated as follows:

> If a classical system has a continuous symmetry, then there are corresponding quantities whose values are conserved.

To elaborate this, we work out three examples. In each example we define a physical system by an action and we assume that the system is invariant under a symmetry. We then calculate what happens to the action when we apply this symmetry and we find that there must exist conserved quatities.

The first example is a massless complex scalar field $\phi \in \Gamma(\mathbb{R}^4 \otimes \mathbb{C})$. We describe the physics by the action

$$S = \int_{\mathbb{R}^4} \mathrm{d}\, x \partial_\mu \phi^*(x) \partial^\mu \phi(x).$$

Assume that this system is invariant under the variation

$$\begin{aligned} \phi \mapsto & \tilde{\phi} = e^{i\alpha}\phi \\ \phi^* \mapsto & \tilde{\phi}^* = e^{-i\alpha}\phi. \end{aligned}$$

where α is a smooth but infinitely small real valued function. This means that the action must not change under this transformation. We calculate $\tilde{S}$ explicitly:

$$\begin{aligned}
\tilde{S} :=& \int_{\mathbb{R}^4} \mathrm{d}\, x \partial_\mu \tilde{\phi}^* \partial^\mu \tilde{\phi} \\
=& \int_{\mathbb{R}^4} \mathrm{d}\, x \partial_\mu \left(e^{-i\alpha(x)} \phi^*\right) \partial^\mu \left(e^{i\alpha(x)} \phi\right) \\
=& S - i \int_{\mathbb{R}^4} \mathrm{d}\, x\, \partial_\mu \left(\alpha(x)\right) \phi^* \partial^\mu \phi + i \partial^\mu (\alpha(x)) \partial_\mu \left(\phi^*\right) \phi + \mathcal{O}(\alpha^2) \\
=& S + i \int_{\mathbb{R}^4} \mathrm{d}\, x\, \alpha(x) \cdot \left(\partial_\mu \left(\phi^* \cdot \partial^\mu \phi - \partial^\mu \phi^* \cdot \phi\right)\right) + \mathcal{O}(\alpha^2).
\end{aligned}$$

Invariance implies that $\int_{\mathbb{R}^4} \mathrm{d}\, x\ \alpha(x) \cdot \left(\partial_\mu \left(\phi^* \cdot \partial^\mu \phi - \partial^\mu \phi^* \cdot \phi\right)\right) = 0$. This is only possible when

$$\partial_\mu \left(\phi^* \cdot \partial^\mu \phi - \partial^\mu \phi^* \cdot \phi\right) = 0.$$

This equation implies that $j^\mu := \phi^* \cdot \partial^\mu \phi - \partial^\mu \phi^* \cdot \phi$ is conserved. This result is predicted by Noethers theorem. It states that for all symmetries, there exists a **current** j^μ such that $\partial_\mu j^\mu = 0$.

In this thesis we mainly look at the chiral symmetry. We study the fermions $\psi \in \Gamma(\mathbb{R}^4 \oplus \mathbb{C}^4)$ with mass m which are coupled to external vector field $V^\mu \in \Gamma(T\mathbb{R}^4)$ and axial-vector field $A^\mu \in \Gamma(T\mathbb{R}^4)$. The external fields are not necessary Abelian. So let $\mathfrak{g}$ be a lie algebra and let $\{\lambda_a\}$ be a set of generators of $\mathfrak{g}$. We write $V^\mu = V_a^\mu \lambda^a \in \Gamma(T\mathbb{R}^4 \otimes \mathfrak{g})$ and $A^\mu = A_a^\mu \lambda^a \in \Gamma(T\mathbb{R}^4 \otimes \mathfrak{g})$. The physics is described by the action

$$S = \int_{\mathbb{R}^4} -\bar{\psi}(\partial\!\!\!/ + m)\psi + i\bar{\psi}(V\!\!\!/ + A\!\!\!/\gamma_5)\psi\ \mathrm{d}\, x. \tag{2.1}$$

The **Abelian chiral symmetry** is given by

$$\begin{aligned}
\psi \mapsto& e^{i\alpha(x)\gamma_5} \psi \\
\bar{\psi} \mapsto& \bar{\psi} e^{i\alpha(x)\gamma_5}.
\end{aligned}$$

where α is an infinitely small real valued function. Under this symmetry the action transforms into

$$\begin{aligned}
S \mapsto& \int_{\mathbb{R}^4} \mathrm{d}\, x\ - \bar{\psi} e^{i\alpha(x)\gamma_5} (\partial\!\!\!/ + m) e^{i\alpha(x)\gamma_5} \psi + i \bar{\psi} e^{i\alpha(x)\gamma_5} (V\!\!\!/ + A\!\!\!/\gamma_5) e^{i\alpha(x)\gamma_5} \psi \\
S \mapsto& S + \int_{\mathbb{R}^4} \mathrm{d}\, x\ - \bar{\psi}\gamma^\mu \gamma_5 \psi \cdot \partial_\mu \alpha(x) - m \bar{\psi}(e^{2i\alpha(x)\gamma_5} - 1)\psi \\
S \mapsto& S + \int_{\mathbb{R}^4} \mathrm{d}\, x\, \alpha(x) \left(\partial_\mu \left(\bar{\psi}\gamma^\mu \gamma_5 \psi\right) - 2im\, \bar{\psi}\gamma_5 \psi\right) + \mathcal{O}(\alpha^2).
\end{aligned} \tag{2.2}$$

The invariance of the action implies that

$$\partial_\mu \left(\bar{\psi}\gamma^\mu\gamma_5\psi\right) = 2im\,\bar{\psi}\gamma_5\psi.$$

If $m = 0$, we notice that $j^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$ is conserved. This is Noethers theorem for the Abelian chiral symmetry.

There is also the **non-Abelian chiral symmetry**. Again we consider the action given in Equation 2.1 but now we assume that S is invariant under

$$\begin{aligned}\psi \mapsto& e^{i\alpha(x)\lambda_a\gamma_5}\psi\\ \bar{\psi} \mapsto& \bar{\psi}e^{i\alpha(x)\lambda_a\gamma_5}.\end{aligned}$$

In the same manner we can calculate the variation of S. Note that λ_a does not commute with A and V. Thefore, the variation of S is not equal to Equation 2.2, but is actually equal to

$$S + \int_{\mathbb{R}^4} \mathrm{d}\,x\,\alpha(x)\left(\partial_\mu\left(\bar{\psi}\gamma^\mu\gamma_5\lambda_a\psi\right) + \bar{\psi}\gamma^\mu\gamma_5\left(iV^b_\mu + iA^b_\mu\gamma_5\right)[\lambda_a,\lambda_b]\psi - 2im\,\bar{\psi}\gamma_5\lambda_a\psi\right).$$

By the invariance of the action we conclude

$$\partial_\mu\left(\bar{\psi}\gamma^\mu\gamma_5\lambda_a\psi\right) = -\bar{\psi}\gamma^\mu\gamma_5\left(iV^b_\mu + iA^b_\mu\gamma_5\right)[\lambda_a,\lambda_b]\psi + 2im\,\bar{\psi}\gamma_5\lambda_a\psi.$$

To simplify the notation, let $T \in \{\mathrm{Id}\} \cup \mathfrak{g}$. We define the **chiral current** as

$$j^\mu_T = \bar{\psi}\gamma^\mu\gamma_5 T\psi. \tag{2.3}$$

If $T = \mathrm{Id}$, then j^μ_T is the current for the Abelian chiral symmetry. If $T = \lambda_a$, then j^μ_a is the current for the non-Abelian chiral symmetry. In both cases, the chiral current satisfies

$$\partial_\mu\left(\bar{\psi}\gamma^\mu\gamma_5 T\psi\right) = -\bar{\psi}\gamma^\mu\gamma_5\left(iV^b_\mu + iA^b_\mu\gamma_5\right)[T,\lambda_b]\psi + 2im\,\bar{\psi}\gamma_5 T\psi. \tag{2.4}$$

2.2 Chiral currents in triangle diagrams

In the previous section, we applied Noethers theorem to chiral currents. The result we found was based on classical mechanics. Now we calculate the same chiral currents in quantum field theory. In this chapter we use one-loop approximations to calculate the amplitudes of scattering processes that are linearly dependent on the chiral current.

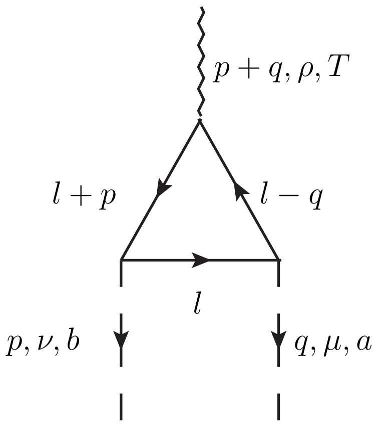


Figure 1: *Example of an anomalous scattering process. On top a particle of an axial-vector field couples to a massive fermion/anti-fermion which then decays into two vector fields. The labels p, q and l are used to denote the momentum, μ, ν, ρ are used to denote the momentum index and a and b are used to denote the gauge index. In case of the Abelian anomaly we set $T = \mathrm{Id}$, else T is an element of the gauge group.*

We compare these amplitudes with the classically expected results and we will notice that they do not coincide.

Consider the AVV diagram depicted in Figure 1. Using the following Feynman rules, we can calculate the scattering amplitude $M^{\mu\nu\rho}_{abT}$:

1. For each loop, add an integral $\int \frac{\mathrm{d}^4 l}{(2\pi)^4}$.
2. For each vertex between ψ, $\bar{\psi}$ and V^μ_a, add the term $-\gamma^\mu \lambda_a$.
3. For each vertex between ψ, $\bar{\psi}$ and A^μ_a, add the term $-\gamma^\mu \gamma_5 \lambda_a$.
4. For each internal ψ-propagator with momentum k add the term $\frac{-i\not{k}+m}{k^2+m^2}$
5. Take the trace over the gauge group and the Clifford algebra.
6. The above terms appear in the same order in the trace as they appear in the loop. However, if the current flows clockwise, then we add the terms counterclockwise and vise versa.

According to these rules, the scattering amplitude becomes

$$M_{abT}^{\mu\nu\rho} = \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \operatorname{tr}\left[\gamma_5 \gamma^\rho T \frac{-i(\not{l} - \not{q}) + m}{(l-q)^2 + m^2} \gamma_\mu \lambda_a \frac{-i\not{l} + m}{l^2 + m^2} \gamma_\nu \lambda_b \frac{-i(\not{l} + \not{p}) + m}{(l+p)^2 + m^2}\right]. \tag{2.5}$$

Comparing Equation 2.5 with Equation 2.3, we see that the term $\gamma^\rho \gamma_5 T$ is related to the chiral current. To mimic Equation 2.4, we want to calculate $\partial_\rho M_{abs}^{\mu\nu\rho}$. Recall however that Feynman diagrams apply a Fourier transformation on the fields. Hence, we consider contraction $(p+q)_\rho M_{abT}^{\mu\nu\rho}$.

By power counting we see that $M_{abT}^{\mu\nu\rho}$ is proportional to $\int \frac{\mathrm{d}^4 l}{l^3}$ and so is linearly divergent. To overcome this problem, we have to regularize the amplitude. That is, we modify the theory such that the amplitude is finite. The physical situation is when the alteration is negligible small. We work out two different regularization methods.

2.2.1 Dimensional regularization

One method is called dimensional regularization. The idea is to generalize equation 2.5 to n dimensions. Also the Clifford algebra is generalized. However, the grading operator γ_5 is still

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3.$$

The result is that γ_5 anti-commutes with γ^0, γ^1, γ^2 and γ^3, but commutes with the other γ_μ. Furthermore, we still assume that the external fields are four dimensional. This method was introduced by 't Hooft and Veltman [1972] and is also called 't Hooft-Veltman regularization.

For dimensional regularization we need to decompose l into $k + k_\perp$, with $k \in \mathbb{R}^4$ and $k_\perp \in \mathbb{R}^{n-4}$. The regularized $(p+q)_\rho M_{abT}^{\mu\nu\rho}$ becomes

$$\begin{aligned}(p+q)_\rho M_{abT}^{\mu\nu\rho} = \int \frac{\mathrm{d}^n l}{(2\pi)^n} \operatorname{tr}\Bigg[\gamma_5(\not{p} + \not{q}) & \frac{-i(\not{k} + \not{k}_\perp - \not{q}) + m}{(k + k_\perp - q)^2 + m^2} \gamma_\mu \frac{-i(\not{k} + \not{k}_\perp) + m}{(k+k_\perp)^2 + m^2} \gamma_\nu \times \\ & \frac{-i(\not{k} + \not{k}_\perp + \not{p}) + m}{(k + k_\perp + p)^2 + m^2}\Bigg] \operatorname{tr}\left[T\lambda_a\lambda_b\right].\end{aligned} \tag{2.6}$$

A little calculation shows that we can write

$$\begin{aligned}\gamma_5(\not{q} + \not{p}) =& \gamma_5(\not{q} + \not{p}) - \gamma_5\not{l} - \not{l}\gamma_5 + 2\gamma_5\not{k}_\perp \\ =& -\gamma_5(\not{l} - \not{q} - im) - (\not{l} + \not{p} - im)\gamma_5 - 2im\gamma_5 + 2\gamma_5\not{k}_\perp.\end{aligned} \tag{2.7}$$

The terms $-\gamma_5(\not{l}-\not{q}-im)-(\not{l}+\not{p}-im)\gamma_5$ in equation 2.7 are related to $\bar{\psi}\gamma^\mu\gamma_5(iV^b_\mu+iA^b_\mu\gamma_5)[T,\lambda_b]\psi$ from equation 2.4. Indeed, the term $-\gamma_5(\not{l}-\not{q}-im)$ simplifies the trace in equation 2.6 into

$$\mathrm{tr}\left[\gamma_5\gamma_\mu\frac{-i(\not{k}+\not{k}_\perp)+m}{(k+k_\perp)^2+m^2}\gamma_\nu\frac{-i(\not{k}+\not{k}_\perp+\not{p})+m}{(k+k_\perp+p)^2+m^2}\right]\mathrm{tr}\left[T\lambda_a\lambda_b\right]. \tag{2.8}$$

This can be interpreted as a one-loop diagram with two external fields where one vertex is connected to an external (axial-) vector field with gauge index $T\lambda_a$. This is up to the factor $-i$ in one to one correspondence with

$$\bar{\psi}\gamma_5\gamma^\mu(iV^a_\mu+iA^a_\mu\gamma_5)T\lambda_a\psi.$$

Even more, this part of the scattering amplitude vanishes in dimensional regularization. The numerator in equation 2.8 is equal to $-4i\epsilon^{\mu\nu\sigma\tau}k_\sigma p_\tau\cdot\mathrm{tr}\left[T\lambda_a\lambda_b\right]$. After a shift $k\to k-p/2$, the amplitude of this terms becomes

$$-4i\epsilon^{\mu\nu\sigma\tau}p_\tau\int\frac{\mathrm{d}^n\, l}{(2\pi)^n}\frac{k_\sigma}{((l-p/2)^2+m^2)((l+p/2)^2+m^2)}.$$

This term vanishes, because it is anti-symmetric in k.

The only part in equation 2.7 that is not related to the classical conservation law, is the term $2\gamma_5\not{k}_\perp$. In the rest of this section we show that this term does not vanish by explicitly calculating

$$\begin{aligned}M^{\mu\nu}_{abT,an}:=2\int\frac{\mathrm{d}^n\, l}{(2\pi)^n}\,\mathrm{tr}\left[\gamma_5\not{k}_\perp\frac{-i(\not{k}+\not{k}_\perp-\not{q})+m}{(k+k_\perp-q)^2+m^2}\gamma_\mu\frac{-i(\not{k}+\not{k}_\perp)+m}{(k+k_\perp)^2+m^2}\gamma_\nu\times\right.\\ \left.\frac{-i(\not{k}+\not{k}_\perp+\not{p})+m}{(k+k_\perp+p)^2+m^2}\right]\mathrm{tr}\left[T\lambda_a\lambda_b\right].\end{aligned} \tag{2.9}$$

The denominator is symmetric in $k_\perp$. Therefore all odd orders of $k_\perp$ in the numerator vanish by antisymmetry. Also the quartic order of $k_\perp$ disappear, because these terms are proportional to

$$\mathrm{tr}(\gamma_5 k_\perp^4)=\|k_\perp\|^4\cdot\mathrm{tr}(\gamma_5)=0$$

The 2$^{\text{nd}}$ order terms of $k_\perp^2$ are

$$\begin{aligned}&\mathrm{Tr}\left[\gamma_5\not{k}_\perp\not{k}_\perp\gamma_\mu(-i\not{k}+m)\gamma_\nu(-i(\not{k}+\not{p})+m)\right]\\ &+\mathrm{Tr}\left[\gamma_5\not{k}_\perp(-i(\not{k}-\not{q})+m)\gamma_\mu\not{k}_\perp\gamma_\nu(-i(\not{k}+\not{p})+m)\right]\\ &+\mathrm{Tr}\left[\gamma_5\not{k}_\perp(-i(\not{k}-\not{q})+m)\gamma_\mu(-i\not{k}+m)\gamma_\nu\not{k}_\perp\right].\end{aligned}$$

This trace can be calculated using the commutation rules and the trace identities[2]. It is equal to

$$4i\|k_\perp\|^2\epsilon^{\mu\nu\rho\sigma}q_\rho p_\sigma. \tag{2.10}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol[3]. Combining equation 2.10 and 2.9 we get that $M^{\mu\nu}_{abT,an}$ equals

$$8\epsilon^{\mu\nu\rho\sigma}q_\rho p_\sigma\cdot\int\frac{\mathrm{d}^n\, l}{(2\pi)^n}\frac{k_\perp^2}{((k-q)^2+k_\perp^2+m^2)(k^2+k_\perp^2+m^2)((k+p)^2+k_\perp^2+m^2)}. \tag{2.11}$$

This integral cannot vanish, because the integrand is positive. It can happen that it is linearly dependent on the dimensions of $k_\perp$. Then it vanishes when $n\to 4$. This is not the case and we will spend the rest of this paragraph to show this.

First we simplify equation 2.11 by using the Feynman trick

$$\frac{1}{ABC}=2\int_0^1\mathrm{d}\,x\int_0^{1-x}\mathrm{d}\,y\frac{1}{(xA+yB+(1-x-y)C)^3} \tag{2.12}$$

where we choose the following values for A, B and C:

$$A=(k+p)^2+k_\perp^2+m^2\quad B=(k-q)^2+k_\perp^2+m^2\quad C=k^2+k_\perp^2+m^2. \tag{2.13}$$

The denominator in equation 2.12 becomes

$$xA+yB+(1-x-y)C=(k+xp-yq)^2+k_\perp^2+m^2+xp^2+yq^2-(xp-yq)^2.$$

If we shift k to $k-xp+yq$ and define

$$a=m^2+xp^2+yq^2-(xp-yq)^2,$$

then the denominator equals $(l^2+a^2)^3$. The anomalous term $M^{\mu\nu}_{abT,an}$ simplifies into

$$16\epsilon^{\mu\nu\rho\sigma}q_\rho p_\sigma\cdot\int_0^1\mathrm{d}\,x\int_0^{1-x}\mathrm{d}\,y\int\frac{\mathrm{d}^n\, l}{(2\pi)^n}\frac{k_\perp^2}{(l^2+a^2)^3}\,\mathrm{tr}[T\lambda_a\lambda_b]. \tag{2.14}$$

[2]These are $\mathrm{tr}(\gamma_5)=0$, $\mathrm{tr}(\gamma_5\gamma_{\mu_1}\dots\gamma_{\mu_{2k+1}})=0$, $\gamma(\gamma_5\gamma_\mu\gamma_n u)=0$ and $\mathrm{tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)=4i\epsilon^{\mu\nu\rho\sigma}$.

[3]$\epsilon^{\mu\nu\rho\sigma}=\pm1$ depending on the permutation of the indices. If two indices are equal, the Levi-Civita symbol is vanishes.

From symmetry arguments we can replace $\frac{k_\perp^2}{(l^2+a^2)^3}$ with $\frac{n-4}{n} \cdot \frac{l^2}{(l^2+a^2)^3}$. Indeed, notice that

$$\begin{aligned}\frac{\mathrm{d}^n l}{(2\pi)^n}\frac{k_\perp^2}{(l^2+a^2)^3} =&(n-4)\frac{\mathrm{d}^n l}{(2\pi)^n}\frac{k_n^2}{(l^2+a^2)^3}\\ =&\frac{n-4}{n}\frac{\mathrm{d}^n l}{(2\pi)^n}\frac{k_1^2+\ldots+k_n^2}{(l^2+a^2)^3}\\ =&\frac{n-4}{n}\frac{\mathrm{d}^n l}{(2\pi)^n}\frac{l^2}{(l^2+a^2)^3}.\end{aligned}$$

Hence, the integrand from equation 2.14 equals

$$\frac{n-4}{n}\frac{l^2}{(l^2+a^2)^3}\operatorname{tr}[T\lambda_a\lambda_b] =\frac{n-4}{n}\left(\frac{1}{(l^2+a^2)^2}-\frac{a^2}{(l^2+a^2)^3}\right)\operatorname{tr}[T\lambda_a\lambda_b].$$

The integral $\int \frac{\mathrm{d}^n l}{(l^2+a^2)^\alpha}$ is called an Feynman integral [1972] and its result is set to

$$i\pi^{n/2}\frac{\Gamma(\alpha-\frac{1}{2}n)}{\Gamma(\alpha)}(a^2)^{\frac{1}{2}n-\alpha}.$$

The term $\frac{n-4}{n}\frac{a^2}{(l^2+a^2)^3}$ will be a multiple of $n-4$ after integration. Therefore, this term disappears in the limit $n \to 4$. The other term, $\frac{1}{(l^2+a^2)^2}$, induces a factor $\Gamma(2-\frac{n}{2})$ which approximates $-\frac{2}{n-4}$ in the limit $n \to 4$. Therefore, there is a non-zero constant c,independent of x and y, such that

$$\lim_{n\to 4}\int\frac{\mathrm{d}^n l}{2\pi^n}\frac{k_\perp^2}{(l^2+a^2)^3} = c.$$

The anomalous part $M_{abT,an}^{\mu\nu}$ is equal to

$$\begin{aligned}M_{ab,an}^{\mu\nu} =&16c\epsilon^{\mu\nu\rho\sigma}q_\rho p_\sigma\cdot\int_0^1\mathrm{d}\,x\int_0^{1-x}\mathrm{d}\,y\operatorname{tr}[T\lambda_a\lambda_b]\\ =&8c\cdot\epsilon^{\mu\nu\rho\sigma}q_\rho p_\sigma\cdot\operatorname{tr}[T\lambda_a\lambda_b].\end{aligned}$$

In the Abelian case, this does not vanish. For the non-Abelian case, $M_{ab,an}^{\mu\nu}$ vanishes if and only if $\operatorname{tr}[T\lambda_a,\lambda_b]=0$. We summarize this result in a theorem:

Theorem 2.2. *The classical conservation laws for the chiral current does not hold in quantum field theory. Therefore, the chiral symmetry is called anomalous.*

2.2.2 Regularization using Pauli-Villars method

Another regularization method is the Pauli-Villars method. The idea is to subtract the same amplitude, but we replace the mass m of the Dirac fermion with some mass M (van Nieuwenhuizen [1989]). This is depicted in figure 2. The physical limit is when M tends to infinity, because in this limit the regulating particle vanishes.

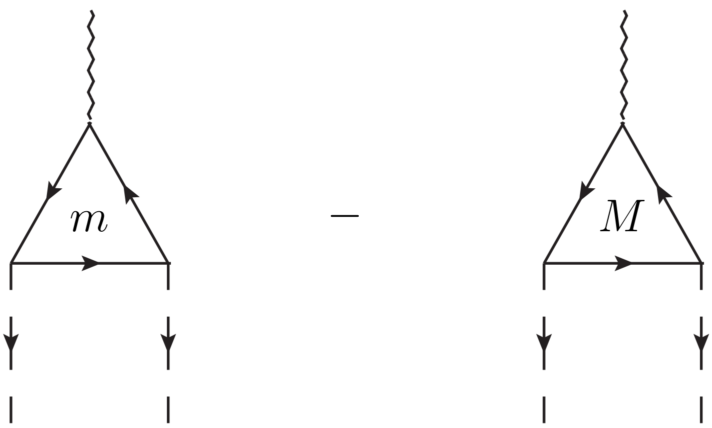


Figure 2: *Pauli-Villars regularization. We subtract the same diagram from the original, but we replace the mass m of the Dirac fermion with some mass M. The physical limit is when M tends to infinity.*

In the example of the AVV-diagram, we show that this method indeed regulates the divergences. The denominator in equation 2.5 is of order l^6. Thus the current $(p+q)_\rho M^{\mu\nu\rho}_{abT}$ diverges if the nominator is of degree l^2 or larger. We compare the orders of l w.r.t. the orders of m and the number of gamma matrices. It is summarized in table 1. From this table we conclude that the only the mass independent part of the numerator need regularization. In this case the numerator of $(p+q)_\rho M^{\mu\nu\rho}_{abT}$ equals

$$\mathrm{tr}\left[\gamma_5(\not{p}+\not{q})(\not{l}-\not{q})\gamma^\mu \not{l}\gamma^\nu(\not{l}+\not{p})\right]\cdot \mathrm{tr}(T\lambda_a\lambda_b).$$

A simple calculation shows that

Cl	m	l	Vanishes?	Convergent?
3	m^3	l^0	Yes, by trace identities	-
4	m^2	l^0 - l^1	No	Yes, by power counting
5	m^1	l^0 - l^2	Yes, by trace identities	-
6	m^0	l^1 - l^3	Only after regularization	Only after regularization

Table 1: *Order analysis of the numerator in equation 2.5. For a given number of gamma matrices (Cl), we give the possible orders of the mass m and the loop momenta l. We also state if the given Clifford order vanishes and if it is convergent.*

$$\begin{aligned}(p+q)_\rho M^{\mu\nu\rho}_{abT}(m)|_{m \text{ indep. num.}} - (p+q)_\rho M^{\mu\nu\rho}_{abT}(m)|_{M \text{ indep. num.}} =& \\ =i\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\operatorname{tr}\left[\gamma_5(\not{p}+\not{q})(\not{l}-\not{q})\gamma_\mu\not{l}\gamma_\nu(\not{l}+\not{p})\right]\cdot \operatorname{Tr}[T\lambda_a\lambda_b]\times& \\ \times\frac{1}{((l-q)^2+m^2)(l^2+m^2)((l+p)^2+m^2)}& \\ \times\frac{1}{((l-q)^2+M^2)(l^2+M^2)((l+p)^2+M^2)}& \\ \times\left(3l^4*(M^2-3m^2)+l^4\cdot\mathcal{O}(l^{-1})\right).&\end{aligned} \tag{2.15}$$

By power counting, we see that the integral in equation 2.15 converges. Hence, the Pauli-Villars method is a valid regularization method.

If one sets $k_\perp = 0$ equation 2.7 is also valid for the Pauli-Villars regularization. Using the same argument as in dimensional regularization, we conclude that the first two terms in equation 2.7 vanishes in the integral. Therefore the physical current is proportional to $\lim_{M\to\infty}\langle -2im\gamma_5 + 2iM\gamma_5\rangle$ and the chiral current is anomalous when $\lim_{M\to\infty}\langle 2iM\gamma_5\rangle$ doesn't vanish. The amplitude $\langle 2iM\gamma_5\rangle$ is given by

$$\langle 2iM\gamma_5\rangle = \int\frac{\mathrm{d}^n l}{(2\pi)^n}\operatorname{tr}\left[2iMT\gamma_5\frac{-i(\not{l}-\not{q})+M}{(l-q)^2+M^2}\gamma_\mu\lambda_a\frac{-i\not{l}+M}{l^2+M^2}\gamma_\nu\lambda_b\frac{-i(\not{l}+\not{p})+M}{(l+p)^2+M^2}\right]. \tag{2.16}$$

From the trace identities[4] and the commutation rules we calculate the trace. Equation

[4]These are $\operatorname{tr}(\gamma_5)=0$, $\operatorname{tr}(\gamma_5\gamma_{\mu_1}\dots\gamma_{\mu_{2k+1}})=0$, $\gamma(\gamma_5\gamma_\mu\gamma_n u)=0$ and $\operatorname{tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)=4i\epsilon^{\mu\nu\rho\sigma}$.

2.16 equals

$$\langle 2iM\gamma_5\rangle = \int \frac{\mathrm{d}^n\, l}{(2\pi)^n} \frac{8M^2\epsilon^{\mu\nu\rho\sigma p_\rho q_\sigma} \cdot \mathrm{tr}\left[T\lambda_a\lambda_b\right]}{((l-q)^2+M^2)(l^2+M^2)((l+p)^2+M^2)}. \tag{2.17}$$

We use the same Feynman tick and Feynman integral we used in dimensional regularization. The integral over the denominator in equation 2.17 equals

$$\int \mathrm{d}^4\, k \frac{1}{(k^2+\mu^2)^3} = I(4,3) = i\pi^2\frac{\Gamma(3-2)}{\Gamma(3)}(\mu^2)^{2-3}$$

and this is proportional to $\frac{1}{M^2+\ldots}$. So in the limit $M \to \infty$, the term $\langle 2iM\gamma_5\rangle$ is not zero, but is a multiple of $\epsilon^{\mu\nu\rho\sigma} q_\rho p_\sigma \cdot \mathrm{Tr}[T\lambda_a\lambda_b]$. This matches the result found in dimensional regularization.

2.3 Other triangle diagrams

Not only the AVV-diagram contains an anomaly. There is another one-loop diagram that shows that the chiral current is anomalous. It is the AAA diagram and it uses the same diagram shown in figure 1, but all external fields are axial-vector fields. We calculate the anomalous part of the part of the scattering using dimensional regularization. That is, we need to find

$$\begin{aligned} M^{\mu\nu,AAA}_{abT,an} :=& 2\int \frac{\mathrm{d}^n\, l}{(2\pi)^n}\, \mathrm{tr}\left[\gamma_5 \not{k}_\perp \frac{-i(\not{k}+\not{k}_\perp-\not{q})+m}{(k+k_\perp-q)^2+m^2}\gamma_\mu\gamma^5 \frac{-i(\not{k}+\not{k}_\perp)+m}{(k+k_\perp)^2+m^2}\gamma_\nu\gamma^5 \times \right. \\ & \left. \frac{-i(\not{k}+\not{k}_\perp+\not{p})+m}{(k+k_\perp+p)^2+m^2}\right] \mathrm{tr}\left[T\lambda_a\lambda_b\right]. \end{aligned} \tag{2.18}$$

The difference between equation 2.9 and 2.18 is that we have replaced γ^μ and γ^ν with $\gamma^\mu\gamma_5$ resp. $\gamma^\nu\gamma_5$. This correspond with the replacement of the external vector fields with axial-vector fields. Using the trace identities we calculate the trace of the numerator and it equals

$$4ik_\perp^2\epsilon^{\mu\nu\rho\sigma}\left(2k_\rho(p+q)_\sigma - q_\rho p_\sigma\right)\cdot \mathrm{Tr}(T\lambda_a\lambda_b).$$

Using the same Feynman trick as in equation 2.12 and 2.13, we conclude that $M^{\nu\nu,AAA}_{Tab,an}$ is

$$16\epsilon^{\mu\nu\rho\sigma}\int_0^1 \mathrm{d}\, x \int_0^{1-x} \mathrm{d}\, y \int \frac{\mathrm{d}^n\, l}{(2\pi)^n} \frac{k_\perp^2\cdot(2(k-xp+yq)_\rho(p+q)_\sigma - q_\rho q_\sigma)}{(l^2+a^2)^3}\, \mathrm{tr}[T\lambda_a\lambda_b]. \tag{2.19}$$

(Note that in the Feynman trick we shifted k and therefore we have the term $k-xp+yq$ in equation 2.19.) This can be simplified if we use the antisymmetry of ϵ and the fact that the linear order of k disappears. Hence equation 2.19 can be written as

$$16\epsilon^{\mu\nu\rho\sigma}q_\rho q_\sigma \int \frac{\mathrm{d}^n l}{(2\pi)^n}\frac{k_\perp^2}{(l^2+a^2)^3}\,\mathrm{tr}[T\lambda_a\lambda_b]\cdot\int_0^1 \mathrm{d}\,x\int_0^{1-x}\mathrm{d}\,y(2(x+y)-1). \quad (2.20)$$

Comparing equation 2.20 with 2.14, we see that the anomalous amplitude for the AVV- and the AAA-diagram differs by the constant

$$\frac{\int_0^1 \mathrm{d}\,x\int_0^{1-x}\mathrm{d}\,y(2(x+y)-1)}{\int_0^1 \mathrm{d}\,x\int_0^{1-x}\mathrm{d}\,y}$$

and by integration over x and y we conclude that this constant equals $1/3$.

2.4 Anomalies in box- and pentagon-diagrams

Not only triangle diagrams, but also box and pentagon diagrams have anomalous behavior. In this paragraph we calculate the box and pentagon anomaly using dimensional regularization. The calculation is similar to the triangle diagram. However, we first perform order analysis such that the equations doesn't become page filling. We follow the following steps:

1. Get an expression for the scattering amplitude using Feynman rules and contract it with the momenta of the A-field. This expression is proportional to the chiral current.

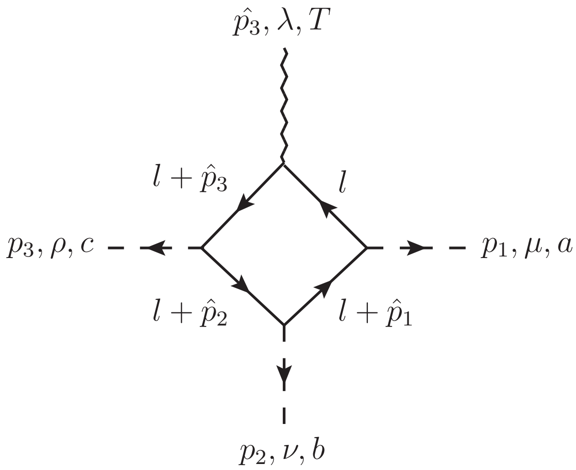
2. Generalize $\gamma_5(\not{p}+\not{q}) = -\gamma_5(\not{l}-\not{q}-im)-(\not{l}+\not{p}-im)\gamma_5-2im\gamma_5+2\gamma_5\not{k}_\perp$ for the box- and pentagon diagram so that we determine the anomalous part.

3. Apply the Feynman trick and make the denominator in the integrand symmetric.

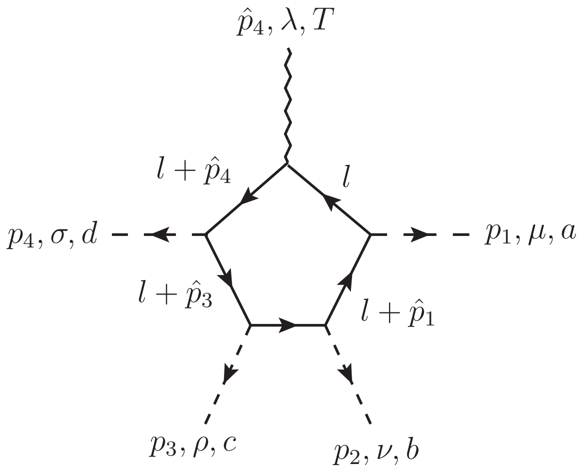
4. Determine which orders of $k_\perp$ and k do vanish and which do not.

5. Calculate the non-vanishing terms.

Consider the box- and pentagon diagrams shown in figure 3. We use the shorthand notation $\hat{p}_i = p_1+\ldots+p_i$ and $\hat{p}_0 = 0$. From the Feynman rules we deduce an



(a) *Box diagram*



(b) *Pentagon diagram*

Figure 3: *Example of an anomalous scattering processes with four resp. five external fields. In these examples we assume that the top external field is an axial-vector field and the other external fields are vector fields.*

expression for the scattering amplitude. These are

$$
\begin{aligned}
M^{\mu\nu\rho\lambda,4}_{abcT} &= \int \frac{\mathrm{d}\, l^n}{(2\pi)^n} \operatorname{Tr}\left[\frac{\gamma_5\gamma^\lambda T(\not{l} + im)\gamma^\mu \lambda_a (\not{l} + \hat{\not{p}}_1 + im)\gamma^\nu \lambda_b}{(l^2+m^2)((l+\hat{p}_1)^2+m^2)} \times \right. \\
&\qquad\qquad \left. \times \frac{(\not{l} + \hat{\not{p}}_2 + im)\gamma^\rho \lambda_c (\not{l} + \hat{\not{p}}_3 + im)}{((l+\hat{p}_2)^2+m^2)((l+\hat{p}_3)^2+m^2)} \right] \\
M^{\mu\nu\rho\sigma\lambda,5}_{abcdT} &= \int \frac{\mathrm{d}\, l^n}{(2\pi)^n} \operatorname{Tr}\left[\frac{\gamma_5\gamma^\lambda T(\not{l} + im)\gamma^\mu \lambda_a (\not{l} + \hat{\not{p}}_1 + im)\gamma^\nu \lambda_b}{(l^2+m^2)((l+\hat{p}_1)^2+m^2)} \times \right. \\
&\qquad \left. \times \frac{(\not{l} + \hat{\not{p}}_2 + im)\gamma^\rho \lambda_c (\not{l} + \hat{\not{p}}_3 + im)\gamma^\sigma \lambda_d (\not{l} + \hat{\not{p}}_4 + im)}{((l+\hat{p}_2)^2+m^2)((l+\hat{p}_3)^2+m^2)((l+\hat{p}_4)^2+m^2)} \right].
\end{aligned}
$$

The vertex $\gamma_5\gamma^\lambda$ is proportional to the chiral current. To show that these diagrams are anomalous, we need to contract these expressions with $\hat{p}_{3,\lambda}$ resp. $\hat{p}_{4,\lambda}$. Then we can compare the results with classical mechanics.

We continue with the second step. Recall that in dimensional regularization, γ_5 anti-commutes with $\gamma_0, ..., \gamma_3$, but commutes with the other gamma matrices. If expand l

into $k + k_\perp$ with $k \in \mathbb{R}^4$ and $k_\perp \in \mathbb{R}^{n-4}$, we get the identity

$$\begin{aligned}\gamma_5 \hat{\not{p}}_3 &= -\gamma_5(\not{l} - im) - (\not{l} + \hat{\not{p}}_3 - im)\gamma_5 - 2im\gamma_5 + 2\gamma_5 \not{k}_\perp \\ \gamma_5 \hat{\not{p}}_4 &= -\gamma_5(\not{l} - im) - (\not{l} + \hat{\not{p}}_4 - im)\gamma_5 - 2im\gamma_5 + 2\gamma_5 \not{k}_\perp .\end{aligned} \tag{2.21}$$

The first three terms in equation 2.21 are in one-to-one correspondence with the classical current conservation law. Therefore the box and pentagon diagrams are anomalous if

$$\begin{aligned} M^{\mu\nu\rho,4}_{abcT,an} &= \int \frac{\mathrm{d}\, l^n}{(2\pi)^n} \mathrm{Tr}\left[\frac{2\gamma_5 \not{k}_\perp T(\not{l} + im)\gamma^\mu \lambda_a (\not{l} + \hat{\not{p}}_1 + im)\gamma^\nu \lambda_b}{(l^2+m^2)((l+\hat{p}_1)^2+m^2)} \times \right. \\ &\qquad \left. \times \frac{(\not{l} + \hat{\not{p}}_2 + im)\gamma^\rho \lambda_c (\not{l} + \hat{\not{p}}_3 + im)}{((l+\hat{p}_2)^2+m^2)((l+\hat{p}_3)^2+m^2)}\right] \\ \text{and } M^{\mu\nu\rho\sigma,5}_{abcdT,an} &= \int \frac{\mathrm{d}\, l^n}{(2\pi)^n} \mathrm{Tr}\left[\frac{2\gamma_5 \not{k}_\perp T(\not{l} + im)\gamma^\mu \lambda_a (\not{l} + \hat{\not{p}}_1 + im)\gamma^\nu \lambda_b}{(l^2+m^2)((l+\hat{p}_1)^2+m^2)} \times \right. \\ &\left. \times \frac{(\not{l} + \hat{\not{p}}_2 + im)\gamma^\rho \lambda_c (\not{l} + \hat{\not{p}}_3 + im)\gamma^\sigma \lambda_d (\not{l} + \hat{\not{p}}_4 + im)}{((l+\hat{p}_2)^2+m^2)((l+\hat{p}_3)^2+m^2)((l+\hat{p}_4)^2+m^2)}\right] \end{aligned} \tag{2.22}$$

does not vanish.

In the next step we symmetrize the denominator by applying the Feynman trick. In the general case, the Feynman trick is

$$\frac{1}{A_1 \cdot \ldots \cdot A_m} = (m-1)! \int_0^1 \mathrm{d}\, z_1 \ldots \mathrm{d}\, z_m \, \frac{\delta(1 - \sum_i z_i)}{(\sum_i z_i A_i)^m}. \tag{2.23}$$

Comparing equation 2.22 and 2.23 we choose $A_i = (k + \hat{p}_i)^2 + k_\perp^2 + m^2$. The denominators in equation 2.22 can be written as

$$\begin{aligned}\left(\sum_i A_i z_i\right)^m &= \left(\sum_i z_i \left(k^2 + k_\perp^2 + m^2 + 2k \cdot \hat{p}_i + \hat{p}_i^2\right)\right)^m \\ &= \left(\left(k + \sum_i z_i \hat{p}_i\right)^2 + k_\perp^2 + m^2 + \sum_i z_i \hat{p}_i^2 - \left(\sum_i z_i \hat{p}_i\right)^2\right)^m .\end{aligned}$$

Let $\mu^2 = m^2 + \sum_i z_i \hat{p}_i^2 - \left(\sum_i z_i \hat{p}_i\right)^2$ and shift k to $k - \sum_i z_i \hat{p}_i$. If we set $B_i =$

$\hat{p}_i - \sum_j z_j \hat{p}_j$, equation 2.22 simplifies to

$$\begin{aligned} M^{\mu\nu\rho,4}_{abcT,an} &= \int \frac{\mathrm{d}\, l^n}{(2\pi)^n} \operatorname{Tr}\left[\frac{2\gamma_5 \not{k}_\perp T(\not{l} + \not{B}_0 + im)\gamma^\mu \lambda_a (\not{l} + \not{B}_1 + im)\gamma^\nu \lambda_b}{(l^2+\mu^2)^4} \times \right. \\ &\qquad \left. \times\, (\not{l} + \not{B}_2 + im)\gamma^\rho \lambda_c (\not{l} + \not{B}_3 + im)\right] \\ M^{\mu\nu\rho\sigma,5}_{abcdT,an} &= \int \frac{\mathrm{d}\, l^n}{(2\pi)^n} \operatorname{Tr}\left[\frac{2\gamma_5 \not{k}_\perp T(\not{l} + \not{B}_0 + im)\gamma^\mu \lambda_a (\not{l} + \not{B}_1 + im)\gamma^\nu \lambda_b}{(l^2+\mu^2)^5} \times \right. \\ &\qquad \left. \times\, (\not{l} + \not{B}_2 + im)\gamma^\rho \lambda_c (\not{l} + \not{B}_3 + im)\gamma^\sigma \lambda_d (\not{l} + \not{B}_4 + im)\right]. \end{aligned} \tag{2.24}$$

Note that the denominator is symmetric in k and $k_\perp$. The numerator is symmetric if the terms are a multiple of $k_\perp^2 (l^2)^m$ for some $m \in \mathbb{N}$. For these terms we can use the following symmetry argument:

$$\int \frac{\mathrm{d}^n\, l}{(2\pi)^n} \frac{k_\perp^2 l^{2m}}{(l^2+\mu^2)^d} = \frac{n-4}{n} \int \frac{\mathrm{d}^n\, l}{(2\pi)^n} \frac{l^{2m+2}}{(l^2+\mu^2)^d}.$$

We expand the numerator of equation 2.24 in a polynomial in l. In table 2 and 3 we determine which orders will vanish. There we notice that all terms are a multiple of $\frac{(\mu^2)^\alpha}{(l^2+\mu^2)^\beta}$. In dimensional regularization this integrates to $i\pi^{n/2} \frac{\Gamma(\beta - n/2)}{\Gamma(\beta)} (\mu^2)^{n/2+\alpha-\beta}$. By the following proposition, we show that for some values of β it vanishes.

Proposition 2.3. *Let $\beta \in \mathbb{N}$ and f a smooth map. The limit*

$$\lim_{n\to 4} \frac{n-4}{n} \cdot i\pi^{\frac{n}{2}} \frac{\Gamma(\beta - \frac{n}{2})}{\Gamma(\beta)} \int_0^1 \mathrm{d}\, z_1 \ldots \mathrm{d}\, z_m \delta\left(1 - \sum_i z_i\right) f(z_i) \cdot (\mu^2)^{\frac{n}{2}+\alpha-\beta}$$

is zero if $\beta > 2$ and converges if $\beta = 0$.

We postpone the proof of this proposition to the end of this chapter. From table 2 and 3 we conclude that the only non-vanishing terms of equation 2.24 are

$$\begin{aligned} M^{\mu\nu\rho,4}_{abcT,an} &= \int \frac{\mathrm{d}\, l^n}{(2\pi)^n} \frac{\operatorname{Tr}[T\lambda_a\lambda_b\lambda_c]}{(l^2+\mu^2)^4} \times \\ &\times \left(\operatorname{Tr}\left[2\gamma_5 \not{k}_\perp (\not{B}_0 + im)\gamma^\mu \not{l}\gamma^\nu \not{l}\gamma^\rho \not{l}\right] + \operatorname{Tr}\left[2\gamma_5 \not{k}_\perp \not{l}\gamma^\mu (\not{B}_1 + im)\gamma^\nu \not{l}\gamma^\rho \not{l}\right] + \right. \\ &\left. \operatorname{Tr}\left[2\gamma_5 \not{k}_\perp \not{l}\gamma^\mu \not{l}\gamma^\nu (\not{B}_2 + im)\gamma^\rho \not{l}\right] + \operatorname{Tr}\left[2\gamma_5 \not{k}_\perp \not{l}\gamma^\mu \not{l}\gamma^\nu \not{l}\gamma^\rho (\not{B}_3 + im)\right]\right) \end{aligned} \tag{2.25}$$

$$M^{\mu\nu\rho\sigma,5}_{abcdT,an} = \int \frac{\mathrm{d}\, l^n}{(2\pi)^n} \frac{\operatorname{Tr}[T\lambda_a\lambda_b\lambda_c\lambda_d]}{(l^2+\mu^2)^5} \operatorname{Tr}\left[2\gamma_5 \not{k}_\perp \not{l}\gamma^\mu \not{l}\gamma^\nu \not{l}\gamma^\rho \not{l}\gamma^\sigma \not{l}\right].$$

Order(s) of l	**Vanishes?**
0,2,4	Yes, Then the numerator of equation 2.24 is antisymmetric in k or $k_\perp$.
1	Yes, the first order of equation 2.24 is proportional to $$\begin{aligned}\frac{k_\perp^2}{(l^2+\mu^2)^4} &= \frac{n-4}{n}\frac{l^2}{(l^2+\mu^2)^4} \\ &= \frac{n-4}{n}\left(\frac{1}{(l^2+\mu^2)^3} - \frac{\mu^2}{(l^2+\mu^2)^4}\right).\end{aligned}$$ By Proposition 2.3 this vanishes.
3	No

Table 2: *Order analysis of equation 2.24 for the box diagram. We expand this equation in l and we determine which orders vanishes.*

Using the Clifford algebra, we work out the spin traces. We conclude that for the box diagram this trace is a multiple of $\epsilon^{\mu\nu\rho\tau}p_{(i),\tau}$. For the pentagon diagram equation 2.25 is a multiple of $\epsilon^{\mu\nu\rho\sigma}$.

We finish this paragraph with the proof of Proposition 2.3. We simplify our notation with $\int_0^1 \mathrm{d}\, z_1 \dots \mathrm{d}\, z_m \delta\left(1 - \sum_i z_i\right) = \int_\Delta \mathrm{d}\, z_i$. Observe that

$$\begin{aligned}&\lim_{n\to4}\frac{n-4}{n}\cdot i\pi^{\frac{n}{2}}\frac{\Gamma(\beta-\frac{n}{2})}{\Gamma(\beta)}\int_\Delta \mathrm{d}\, z_i f(z_i)\cdot(\mu^2)^{\frac{n}{2}-\beta} \\ &= \left(\lim_{n\to4}\frac{-2i\pi^{\frac{n}{2}}}{n\Gamma(\beta)}\right)\left(\lim_{n\to4}(2-n/2)\,\Gamma\left(\beta-n/2\right)\right)\left(\lim_{n\to4}\int_\Delta \mathrm{d}\, z_i f(z_i)\cdot(\mu^2)^{\frac{n}{2}-\beta}\right).\end{aligned} \tag{2.26}$$

The first part of equation 2.26 can be easily calculated and is equal to $\frac{-i\pi^2}{2\Gamma(\beta)}$. The second term vanishes if $\beta > 2$. If $\beta = 2$, the second term does not vanish, but is equal to

$$\lim_{n\to4}(2-n/2)\,\Gamma\left(2-n/2\right) = \lim_{n\to4}\Gamma\left(3-n/2\right) = 1.$$

We finally focus on the last term of equation 2.26. We need to show that this integral

Order(s) of l	**Vanishes?**
0,2,4	Yes, Then the numerator of equation 2.24 is antisymmetric in k or $k_\perp$.
1	Yes, the first order of equation 2.24 is proportional to $$\begin{aligned}\frac{k_\perp^2}{(l^2+\mu^2)^5} =& \frac{n-4}{n}\frac{l^2}{(l^2+\mu^2)^5} \\ =& \frac{n-4}{n}\left(\frac{1}{(l^2+\mu^2)^4} - \frac{\mu^2}{(l^2+\mu^2)^5}\right).\end{aligned}$$ By Proposition 2.3 this vanishes.
3	Yes, the third order of equation 2.24 is proportional to $$\begin{aligned}\frac{k_\perp^2 l^2}{(l^2+\mu^2)^5} =& \frac{n-4}{n}\frac{l^4}{(l^2+\mu^2)^5} \\ =& \frac{n-4}{n}\left(\frac{1}{(l^2+\mu^2)^3} - \frac{2\mu^2}{(l^2+\mu^2)^4} + \frac{\mu^4}{(l^2+\mu^2)^5}\right).\end{aligned}$$ By Proposition 2.3 this vanishes.
5	No

Table 3: *Order analysis of equation 2.24 for the pentagon diagram. We expand this equation in l and we determine which orders vanishes.*

converges. Recall that

$$\mu^2 - m^2 = \sum_i z_i \hat{p}_i^2 - \left(\sum_i z_i \hat{p}_i\right)^2 \geq 0.$$

Hence the map $f \cdot (\mu^2)^{n/2-\beta}$ does not have $\frac{1}{0}$ behavior on Δ. This concludes that $f \cdot (\mu^2)^{n/2-\beta}$ is integrable for each value of $n \in \mathbb{R}$. This proves the proposition. $\square$

2.5 Non-Abelian Anomalies in Feynman diagrams

Till now, we ignored the traces like $\mathrm{Tr}(T\lambda_a \ldots \lambda_c)$. These terms are determined by the properties of the gauge group. It turns out that in some cases these terms vanishes.

We give two examples of this phenomena. In both cases we assume Bose symmetry. Recall that this is the symmetry when one interchanges external lines.

> **Example 2.4.** If $T = 1$, then the pentagon anomaly vanishes due to Bose symmetry.

Indeed, the pentagon anomaly is proportional to

$$\epsilon^{\mu\nu\rho\sigma} \cdot \mathrm{tr}(T\lambda_a\lambda_b\lambda_c\lambda_d). \tag{2.27}$$

Bose symmetry requires that the above expression is symmetric under the permutation of the pairs (a,μ), (b,ν), (c,ρ) and (d,σ). By adding all the permuted versions of equation 2.27 we get an expression with 24 terms, which can be written as

$$\epsilon^{\mu\nu\rho\sigma} \cdot \mathrm{tr}\left(T\{[\lambda_a,\lambda_b],[\lambda_c,\lambda_d]\} + T\{[\lambda_a,\lambda_c],[\lambda_d,\lambda_b]\} + T\{[\lambda_a,\lambda_d],[\lambda_b,\lambda_c]\}\right). \tag{2.28}$$

We now assume that $T = 1$. Note that due to the cyclic property of the trace, the anti-commutator in equation 2.28 simplifies to

$$2\epsilon^{\mu\nu\rho\sigma} \cdot \mathrm{tr}\left([\lambda_a,\lambda_b][\lambda_c,\lambda_d] + [\lambda_a,\lambda_c][\lambda_d,\lambda_b] + [\lambda_a,\lambda_d][\lambda_b,\lambda_c]\right)$$

and this can be written as

$$2\epsilon^{\mu\nu\rho\sigma} \cdot \mathrm{tr}\left(\lambda_a([\lambda_b,[\lambda_c,\lambda_d]] + [\lambda_c,[\lambda_d,\lambda_b]] + [\lambda_d,[\lambda_b,\lambda_c]])\right).$$

From the Jacobi identity we conclude that the Abelian pentagon anomaly vanishes.

> **Example 2.5.** If all triangle anomalies vanishes, then the box and pentagon anomalies vanishes due to Bose symmetry.

Under Bose symmetry the triangle anomaly is proportional to

$$\epsilon^{\mu\nu\rho\sigma} q_\rho p_\sigma \cdot \mathrm{tr}(T\lambda_a\lambda_b) + \epsilon^{\nu\mu\rho\sigma} p_\rho q_\sigma \cdot \mathrm{tr}(T\lambda_b\lambda_a).$$

which equals $\epsilon^{\mu\mu\rho\sigma} q_\rho p_\sigma \cdot \mathrm{tr}(T\{\lambda_a,\lambda_b\})$. Hence all triangle anomalies vanishes if and only if $\mathrm{tr}(\{\lambda_a,\lambda_b\}\lambda_c) = 0$ for all generators of the lie algebra $\lambda_a,\lambda_b,\lambda_c$. We call $\mathrm{tr}(\{\lambda_a,\lambda_b\}\lambda_c)$ to be the d-**symbol** and we denote it as d_{abc}. We show that the box and pentagon anomalies can be written as linear combinations of d-symbols.

Assume that $T = \lambda_e$. We can write equation 2.28 in terms of d-symbols using the structure constants[5] f^c_{ab}:

$$\epsilon^{\mu\nu\rho\sigma} \cdot \left(f^f_{ab} f^g_{cd} + f^f_{ac} f^g_{db} + f^f_{ad} f^g_{bc}\right) d_{efg}$$

Clearly this vanishes if the d-symbol vanishes. We repeat the calculation for the box diagram. Under Bose symmetry, the anomaly is proportional to

$$\begin{aligned} &\epsilon^{\mu\nu\rho\tau} p_{(i),\tau} \cdot \mathrm{tr}(\lambda_e \lambda_a \lambda_b \lambda_c) + \epsilon^{\mu\nu\rho\tau} p_{(i),\tau} \cdot \mathrm{tr}(\lambda_e \lambda_a \lambda_b \lambda_c) \\ +&\epsilon^{\nu\rho\mu\tau} p_{(i),\tau} \cdot \mathrm{tr}(\lambda_e \lambda_b \lambda_c \lambda_a) + \epsilon^{\nu\mu\rho\tau} p_{(i),\tau} \cdot \mathrm{tr}(\lambda_e \lambda_b \lambda_a \lambda_c) \\ +&\epsilon^{\rho\mu\nu\tau} p_{(i),\tau} \cdot \mathrm{tr}(\lambda_e \lambda_c \lambda_a \lambda_b) + \epsilon^{\rho\nu\mu\tau} p_{(i),\tau} \cdot \mathrm{tr}(\lambda_e \lambda_c \lambda_b \lambda_a). \end{aligned}$$

We write this in terms of (anti)-commutators

$$\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} p_{(i),\sigma} \,\mathrm{tr}(\lambda_d(\{\lambda_a, [\lambda_b, \lambda_c]\}) + \{\lambda_b, [\lambda_c, \lambda_a]\} + \{\lambda_c, [\lambda_a, \lambda_b]\}).$$

This is a linear combination of d-symbols, because it equals

$$\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} p_{(i),\sigma} \left(f^e_{bc} d_{aed} + f^e_{ca} d_{bed} + f^e_{ab} d_{ced}\right).$$

So when the d-symbol vanishes, the box and pentagon anomaly also vanishes. Georgi and Glashow calculated the d-symbol for many matrix groups and showed that in most cases the d-symbol vanishes. For more information, see Georgi and Glashow [1972].

Now we calculated the chiral anomaly in the one-loop approximation we might ask if there is any anomalous behavior in higher order loop approximations. Adler and Bardeen [1969] calculated higher order loops and they came to the conclusion

> **Theorem 2.6** (Adler and Bardeen [1969])**.** *The chiral anomaly can be fully determined in the one loop approximation.*

In the next chapter we consider another method to calculate anomalies and there we prove this theorem.

[5]Recall: f^c_{ab} are uniquely defined by $[\lambda_a, \lambda_b] = f^c_{ab} \lambda_c$

3 Calculating anomalies using path integrals

In the previous chapter, we have seen that anomalies are related to the braking of classical conservation laws when we consider its results in one-loop diagrams. In this chapter we examine another point of view, namely we show that anomalies arise due to the fact that the path integral measure is not invariant under the symmetry.

To show this, we first consider the simple case of massless Dirac fermions in the path integral formalism. Using the method introduced by Fujikawa [1980], we again show that the chiral symmetry is anomalous. After this, we look at other examples and we relate them to the results found when using perturbation theory.

Finally we analyze Fujikawas method from a more mathematical perspective. We show that the anomaly only depends on the topology of the gauge bundle and is a specific application of the Atiyah-Singer index theorem. In the next chapters we prove the index theorem using Fujikawas method.

3.1 The Fujikawa method

In classical mechanics we investigated how the action D changed under symmetry transformations of the fields ψ. From this we deduced conserved quantities we called Noethers currents. In quantum field theory the dynamics of a field are not determined by the action, but by the generating functional

$$\mathcal{Z} = \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi \exp\left(iS\right).$$

Here $\int \mathcal{D}\psi$ is the path integral over all fields ψ. With Fujikawas method(1980, 2004) we mimic Noethers theorem for generating functionals. We get a result that differs from the classical theory and this difference is the anomaly. As an example we consider a Dirac fermion ψ of mass m in quantum electrodynamics. Let A_μ be the electromagnetic gauge potential and let $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ be the field strength. The action for this fermion is given by

$$S = \int_{\mathbb{R}^4} \mathrm{d}^4 x\, \bar{\psi} i\gamma^\mu(\partial_\mu - iA_\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{3.1}$$

As usual we work with a flat four dimensional spacetime, but instead of using the Minkowski metric, we use the Euclidean metric. For this we need to replace the i with -1 in equation 3.1 We use the following properties of path integrals:

1. The path integral is different for fermionic than for bosonic particles. For fermionic particles physicists use Berezin integration for which the Jacobian is given by the inverse of regular Jacobian.
2. The Dirac field ψ is actually a bispinor. That is, $\mathbb{C}^4$ has 2 irreducible spin-representations and thus describes 2 spin particles. Therefore we treat $\bar{\psi}$ and ψ as separate particles and so we integrate them seperately.

We assume that the Dirac fermion is invariant under the transformation

$$\begin{aligned}\psi \mapsto& e^{i\alpha(x)\gamma_5}\psi\\ \bar{\psi} \mapsto& \bar{\psi}e^{i\alpha(x)\gamma_5}.\end{aligned}$$

where α is an arbitrary but infinitely small real-valued map on $\mathbb{R}^4$. Under this symmetry the action S changes into

$$\tilde{S} = S + \int_{\Re^4} \mathrm{d}^4\, x\, \alpha(x) \left[\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) + 2im\bar{\psi}\gamma_5\psi\right].$$

If we denote the Jacobian of the path integral as J_ψ, then the generating functional $\mathcal{Z}$ transforms into

$$\tilde{\mathcal{Z}} = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\; J_{\bar{\psi}} J_\psi \exp\left(-S - \int_{\mathbb{R}^4} \mathrm{d}^4\, x\, \alpha(x) \left[\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) + 2im\bar{\psi}\gamma_5\psi\right]\right). \tag{3.2}$$

Clearly, we get the classical chiral current conservation if and only if the Jacobians are equal to one. To show that the chiral symmetry is anomalous we need to calculate the Jacobians. For this we must assume that α is small enough such that it behaves as a constant. Formally, the Jacobian is given by

$$J_\psi \equiv \det{}^{-1} \exp(i\alpha\,\gamma_5) = \exp(-i\,\mathrm{Tr}(\alpha\,\gamma_5)) + \mathcal{O}(\dot{\alpha}, \alpha^2)$$

where Tr is the trace over spacetime and all spin indices. The expression is exactly the same for $J_{\bar{\psi}}$ and therefore is the combined Jacobian

$$J = \exp(-2i\,\mathrm{Tr}(\alpha\,\gamma_5)) + \mathcal{O}(\dot{\alpha}, \alpha^2). \tag{3.3}$$

To evaluate this explicitly, we consider an orthonormal basis of eigenvectors ϕ_n for the operator $D = \gamma^\mu(\partial_\mu - iA_\mu)$. We denote the corresponding eigenvalues with λ_n. Formally the trace equals $\mathrm{Tr}(\cdot) = \sum_n \int \mathrm{d}^4\, x\langle\phi_n(x)|\cdot|\phi_n(x)\rangle$ and so

$$J = \exp\left[-2i\sum_n \int \mathrm{d}^4\, x\langle\phi_n(x)|\alpha(x)\gamma_5|\phi_n(x)\rangle\right] + \mathcal{O}(\dot{\alpha}, \alpha^2).$$

This is a formal calculation and may not converge. Indeed, there may be infinitely many eigenspaces and each eigenspace may be infinite dimensional. It is also possible that the eigenvectors are not normalizable. That is, they diverge when we integrate over them. We tackle this problem by modifying the Jacobian slightly. For this observe the following: Recall that the sum over an infinite sequence $\{c_n\}$ is defined as the limit of the partial sums $\sum_{n=0}^{M} c_n$. Consider the step function $\theta\colon \mathbb{R} \to \{0,1\}$ with $\theta(x) = 1$ if $x \leq 1$ and $\theta(x) = 0$ if $x > 1$. The partial sum equals

$$\sum_{n=0}^{M} c_n = \sum_{n=0}^{M} c_n \theta\left(\frac{n}{M}\right) = \sum_{n=0}^{\infty} c_n \theta\left(\frac{n}{M}\right).$$

We modify the Jacobian by replacing the step function with another smooth map $f(x)$ that rapidly approaches zero when x is large and $f(0) = 1$. See figure 4. For a well-chosen map f the Jacobian converges absolutely. An example is e^{-x}. We study this regulator more thoroughly in chapter 4. In this chapter we investigate the second method and we assume that the combined Jacobian equals

$$J = \exp\left[-2i \lim_{t\to 0} \sum_n \int \mathrm{d}^4\, x\, \alpha(x) \langle \phi_n(x)| \gamma_5\, f\left(t \cdot \lambda_n^2\right) |\phi_n(x)\rangle\right] + \mathcal{O}(\dot{a}, a^2). \quad (3.4)$$

Using functional calculus we rewrite equation 3.4 as

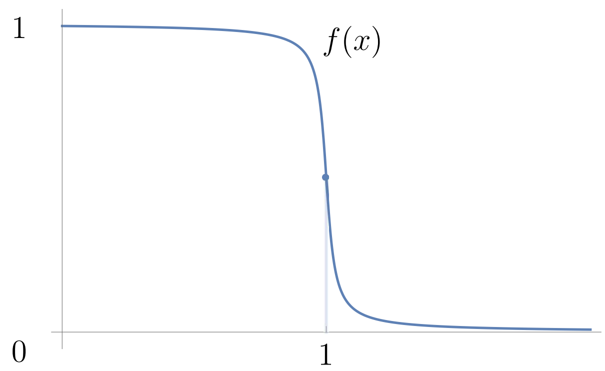


Figure 4: *The regulator $f(x)$,used in the Fujikawa method, mimics the behavior of the step function. It must value 1 at $x = 0$ and it must rapidly decrease at infinity.*

$$\begin{aligned}\log J =& -2i \lim_{t\to 0} \sum_n \int \mathrm{d}^4\, x\, \alpha(x) \langle \phi_n(x)| \gamma_5\, f\left(t \cdot D^2\right) |\phi_n(x)\rangle + \mathcal{O}(\dot{a}, a^2)\\ =& -2i \lim_{t\to 0} \mathrm{Tr}\left[\gamma_5\, \alpha \cdot f\left(t \cdot D^2\right)\right] + \mathcal{O}(\dot{a}, a^2). \quad (3.5)\end{aligned}$$

Next we perform a change of basis. Namely, we expand the Jacobian in terms of the plane-waves e^{ikx}. Because plane-waves form only a basis over $\mathbb{R}^4$, we still have to take the trace over the spin indices. We denote the trace over the spin indices as tr. In the plane-wave basis equation 3.5 becomes

$$\begin{aligned}\log J =& -2i \lim_{t\to 0} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \langle k| \,\mathrm{tr}\left[\gamma_5\, \alpha \cdot f\left(t \cdot D^2\right)\right] |k\rangle \\ =& -2i \lim_{t\to 0} \int \mathrm{d}^4 x\, \alpha(x) \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik\cdot x}\,\mathrm{tr}\left[\gamma_5\, f\left(t\cdot D^2\right)\right] e^{ik\cdot x}.\end{aligned}$$

Note that the operator D is the Dirac operator with respect to the covariant derivative $\nabla_\mu = \partial_\mu - iA_\mu$. Theorem 1.22 states that $D^2 = -\nabla^\mu\nabla_\mu - \frac{i}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}$ and so

$$\log J = -2i \lim_{t\to 0} \int \mathrm{d}^4 x\, \alpha(x) \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-ik\cdot x}\,\mathrm{tr}\left[\gamma_5\, f\left(-t\nabla^\mu\nabla_\mu - \frac{it}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right)\right] e^{ik\cdot x}. \tag{3.6}$$

We simplify the equation 3.6 by pulling $e^{ik\cdot x}$ to the left. For this we use the Leibniz rule $[\nabla_\mu, e^{ik\cdot x}] = ik_\mu$ and hence

$$\log J = -2i \lim_{t\to 0} \int \mathrm{d}^4 x\, \alpha(x) \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{tr}\left[\gamma_5\, f\left(-t(\nabla^\mu + ik_\mu)^2 - \frac{it}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right)\right]. \tag{3.7}$$

We calculate the non vanishing part of this integral. First we rescale k_μ by $t^{-1/2}k_\mu$ and the integrand of $\int \mathrm{d}^4 x$ in equation 3.7 equals

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} t^{-2}\,\mathrm{tr}\left[\gamma_5\, f\left(-(t^{1/2}\nabla_\mu + ik_\mu)^2 - \frac{it}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right)\right]. \tag{3.8}$$

We consider the Taylor series of f in $t^{\frac{1}{2}}$ at $t = 0$. We only have to consider the first four orders, because all higher orders are linear to $t^{1/2}$ and vanishes when we perform the limit $t \to 0$. In table 4 we analyze these orders and we see that only the t-constant term doesn't vanish. Hence, when t tends to zero, then equation 3.8 equals

$$-\frac{1}{32} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} f''\left(k^2\right) \mathrm{tr}\left[\gamma_5\, [\gamma^\mu,\gamma^\nu][\gamma^\rho,\sigma^\sigma]\right] F_{\mu\nu}F_{\rho\sigma}.$$

t	Clifford order	Vanishes?
t^{-2}	$\mathrm{tr}(\gamma_5)$	Yes
$t^{-3/2}$	$\mathrm{tr}(\gamma_5)$	Yes
t^{-1}	$\mathrm{tr}(\gamma_5)$	Yes
	$\mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu)$	Yes
$t^{-1/2}$	$\mathrm{tr}(\gamma_5)$	Yes
	$\mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu)$	Yes
t^0	$\mathrm{tr}(\gamma_5)$	Yes
	$\mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu)$	Yes
	$\mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma)$	No

Table 4: *Order analysis of the Taylor approximation of equation 3.8 in $t^{1/2}$ at $t = 0$. For a given order of $t^{1/2}$, we give the possible traces over the spin indices. Using the trace identities we conclude if a given order vanishes*

From the trace identity $\gamma(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\epsilon^{\mu\nu\rho\sigma}$ where ϵ is the Levi-Civita symbol, the regularized Jacobian becomes

$$\begin{aligned}\log J =& \frac{i}{4}e^{\mu\nu\rho\sigma}\int \mathrm{d}^4\, x\, \alpha(x)\int \frac{\mathrm{d}^4\, k}{(2\pi)^4}f''(k^2)F_{\mu\nu}F_{\rho\sigma}\\ =& \frac{i}{16}e^{\mu\nu\rho\sigma}\int \mathrm{d}^4\, x\, \alpha(x)F_{\mu\nu}F_{\rho\sigma}.\end{aligned} \tag{3.9}$$

In the last step we used the that $f(0) = 1$ and $f(\infty) = 0$. We use equation 3.9 in the transformed generating functional and so equation 3.2 equals

$$\tilde{\mathcal{Z}} = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, \exp\left(-S[\psi] - \int_{\Re^4} \mathrm{d}^4\, x\, \alpha(x)\left[\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) + 2im\bar{\psi}\gamma_5\psi - \frac{i}{16}e^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right]\right).$$

Because $\mathcal{Z}$ is conserved under the chiral transformation we get the conservation law

$$\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) = -2im\bar{\psi}\gamma_5\psi + \frac{i}{16}e^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}.$$

This shows that the chiral current is anomalous.

3.2 The covariant anomaly

Previously, we have only considered Fujikawas method for abelian anomalies. It is natural to ask whether this method also works for non-abelian gauge fields. Most steps we can copy directly, but there are some subtleties we have to be aware of. For reference see Bertlmann [1996], which give a detailed overview of different non-Abelian anomalies.

In the rest of this paragraph we consider the chiral symmetry for a fermion ψ with mass m we studied in chapter 2.1. The action S is given by

$$S = \int_{\mathbb{R}^4} -\bar{\psi}(\partial\!\!\!/ + m)\psi + i\bar{\psi}(V\!\!\!/ + A\!\!\!/\gamma_5)\psi \; \mathrm{d}\,x$$

where V_μ and A_μ are gauge fields. In the case of the chiral symmetry we assume that the physical system is invariant under the transformation

$$\psi \mapsto e^{i\alpha(x)T\gamma_5}\psi \qquad \bar{\psi} \mapsto \bar{\psi}e^{i\alpha(x)T\gamma_5}. \tag{3.10}$$

Here α is a small real valued function. When we consider the Abelian chiral symmetry we assume that T equals the identity. Otherwise we assume that T is a generator of the Lie algebra. We rewrite the action and the symmetry in terms of the operators $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$. These operators project the fields into the ± 1 eigenspace of γ_5. Because they are projection operators they obey the properties

$$P_\pm^2 = P_\pm, \quad P_+P_- = P_-P_+ = 0, \quad P_+ + P_- = 1, \quad \text{and} \quad \gamma_5 P_\pm = \pm P_\pm.$$

We write $\psi_\pm$ for the projection of ψ to the ± 1 eigenspace of γ_5. In terms of these new fields the action can be written as

$$S = \int_{\mathbb{R}^4} -\bar{\psi}_+(\partial\!\!\!/ + m - iV\!\!\!/ - iA\!\!\!/)\psi_+ + -\bar{\psi}_-(\partial\!\!\!/ + m - iV\!\!\!/ + iA\!\!\!/)\psi_- \; \mathrm{d}\,x.$$

If we denote $A^\pm = V \pm A$ and define $D_\pm = \partial\!\!\!/ - iA\!\!\!/^\pm$ the action splits into the action of two non-interacting Dirac particles. That is, the action becomes

$$S = \int_{\mathbb{R}^4} -\bar{\psi}_+(D_+ + m)\psi_+ + -\bar{\psi}_-(D_- + m)\psi_- \; \mathrm{d}\,x. \tag{3.11}$$

Using Fujikawas method we calculate how the generating functional changes under the chiral symmetry. The generating functional $\mathcal{Z} = \int \mathcal{D}\bar{\psi}_+\mathcal{D}\psi_+\mathcal{D}\bar{\psi}_-\mathcal{D}\psi_- \exp[-S]$ transforms under equation 3.10 into

$$\tilde{\mathcal{Z}} = \int \mathcal{D}\bar{\psi}_+\mathcal{D}\psi_+\mathcal{D}\bar{\psi}_-\mathcal{D}\psi_- \; J_{\bar{\psi}_+}J_{\psi_+}J_{\bar{\psi}_-}J_{\psi_-} \exp[-\tilde{S}] \tag{3.12}$$

where $\tilde{S}$ equals

$$\tilde{S} = S + \int_{\mathbb{R}^4} \mathrm{d}^4\, x\, \alpha(x) \left[\partial_\mu \left(\bar{\psi}\gamma^\mu\gamma_5 T\psi\right) + \bar{\psi}\gamma^\mu\gamma_5 \left(iV^b_\mu + iA^b_\mu\gamma_5\right)[T, \lambda_b]\psi - 2im\, \bar{\psi}\gamma_5 T\psi\right].$$

The transformed action $\tilde{S}$ was found using Noethers theorem and was calculated in equation 2.4. We note that $P_\pm\gamma_5 = \gamma_5 P_\pm$ and hence the chiral symmetry equals

$$\psi_\pm \mapsto e^{i\alpha(x)T\gamma_5}\psi_\pm \qquad \bar{\psi}_\pm \mapsto \bar{\psi}_\pm e^{i\alpha(x)T\gamma_5}.$$

By definition of the fermionic path integral the Jacobian equals

$$J_{\bar{\psi}_\pm} = J_{\psi_\pm} = \det{}^{-1} \exp(i\alpha\, S\gamma_5).$$

If we assume that α and $\dot{\alpha}$ are infinitely small, we get by the Baker–Campbell–Hausdorff formula

$$J_{\bar{\psi}_\pm} = J_{\psi_\pm} = \exp \mathrm{Tr}(-i\alpha\, T\gamma_5) + \mathcal{O}(\alpha^2, \dot{\alpha}).$$

Again, this might diverge and we need to regulate this. We regulate the same way as we regulated the Jacobian in the previous paragraph. For this we need to pick a real valued maps $f_+, f_- \colon \mathbb{R} \to \mathbb{R}$ that vanishes at infinity and $f_\pm(0) = 1$. We regulate $J_{\bar{\psi}_\pm}$ and $J_{\psi_\pm}$ with $D_\pm$. The combined regularized Jacobian equals

$$J = \lim_{t_\pm \to 0} \exp \mathrm{Tr}\left[-2i\, \alpha\, T\gamma_5 \cdot (f_+(t_+\, D_+^2) + f_-(t_-\, D_-^2)\right]. \tag{3.13}$$

Here Tr denotes the trace over the fields, spin-indices and the Lie algebra of the gauge group. We split this trace into the trace $\mathrm{tr}_\mathfrak{g}$ over the Lie algebra and the trace Tr over the fields and spin indices. Equation 3.13 then equals

$$\log J = \mathrm{tr}_\mathfrak{g}\left[T \cdot \left(\lim_{t_+ \to 0} \mathrm{Tr}\left[-2i\alpha\ \cdot \gamma_5\, f_+(t_+ D_+^2)\right] + \lim_{t_- \to 0} \mathrm{Tr}\left[-2i\alpha\ \cdot \gamma_5\, f_-(t_- D_-^2)\right]\right)\right].$$

We already calculated $\lim_{t_\pm \to 0} \mathrm{Tr}\left[-2i\alpha\ \cdot \gamma_5\, f_\pm(t_\pm D_\pm^2)\right]$ in the previous paragraph. By equation 3.9 we conclude

$$\log J = \frac{i}{16} e^{\mu\nu\rho\sigma} \int \mathrm{d}^4\, x\, \alpha(x)\, \mathrm{tr}_\mathfrak{g}\left[T \cdot \left(F^+_{\mu\nu}F^+_{\rho\sigma} + F^-_{\mu\nu}F^-_{\rho\sigma}\right)\right]$$

where $F^{\pm}_{\mu\nu}$ is the field strength with respect to $A^{\pm}_{\mu}$. Hence equation 3.12 equals

$$\tilde{\mathcal{Z}} = \int \mathcal{D}\bar{\psi}_+ \mathcal{D}\psi_+ \mathcal{D}\bar{\psi}_- \mathcal{D}\psi_- \ \exp\left(-S\right) \times$$
$$\times \left(-\int_{\Re^4} \mathrm{d}^4 x\, \alpha(x) \left[\partial_\mu(\bar{\psi}\gamma^\mu\gamma_5\psi) - \bar{\psi}\gamma^\mu\gamma_5 \left(iV^b_\mu + iA^b_\mu\gamma_5\right)[T,\lambda_b]\psi - 2im\bar{\psi}\gamma_5\psi\right] \right) \times$$
$$\times \left(+ \int_{\Re^4} \mathrm{d}^4 x\, \alpha(x)\, \mathrm{tr}_{\mathfrak{g}} \left[T \cdot \left(F^+_{\mu\nu}F^+_{\rho\sigma} + F^-_{\mu\nu}F^-_{\rho\sigma}\right)\right] \right).$$

Because $\mathcal{Z}$ is conserved under the chiral transformation we get a conservation law. However it does not coincide with the classical conservation equation. This shows that the Abelian chiral current is anomalous. We call this anomaly the **covariant anomaly**.

3.3 The Bardeen anomaly

In Fujikawas method, we have to regulate in order to get a well-defined Jacobian. However, the result depends on the method of regularization. In this paragraph we consider another regulator for the chiral current. Again the result is anomalous, but this time it coincides with the one-loop calculations.
We consider the chiral symmetry given in equation 3.10:

$$\psi \mapsto e^{i\alpha(x)T\gamma_5}\psi \qquad \bar{\psi} \mapsto \bar{\psi}e^{i\alpha(x)T\gamma_5} \tag{3.10}$$

The generating functional $\mathcal{Z}$ transforms under this transformation and the Jacobian of the path integral equals

$$J_\psi = J_{\bar{\psi}} = \det{}^{-1}\exp(i\alpha\gamma_5) = \exp(-i\,\mathrm{Tr}[\alpha\cdot\gamma_5]).$$

We need to regulate the Jacobian in order to get a well-defined result. For the covariant anomaly we used $D_\pm$ as the regulator. We now try to regulate the Jacobian with the differential operator $\partial\!\!\!/ - iV\!\!\!/ - iA\!\!\!/\gamma_5$. This operator is not Hermitian. Even more, it doesn't commute with its adjoint. Hence it is not suitable to use in functional calculus. To solve this, we take the analytic continuation of A_μ and we transform A_μ to iA_μ. We regulate the Jacobian with the differential operator $D = \partial\!\!\!/ - iV\!\!\!/ + A\!\!\!/\gamma_5$. At the end of our calculation we undo the transformation. Note that $\mathrm{Tr}[\alpha\,\gamma_5\exp(-tD^2)]$ diverges in the limit $t \to 0$. This is due to the factor γ_5 in the differential operator D. There are two approaches to solve this problem:

1. We can renormalize the theory such that the divergent terms are canceled.[6]

[6] For the mathematicians: Renormalization is the process of modifying the action with t dependent terms such that the final result is finite.

2. According to Hu et al. [1984], the t-divergent terms cancels if we choose a different regulator for $\mathrm{d}\,\bar{\psi}$ and $\mathrm{d}\,\psi$: They regulate the Jacobian for $\mathrm{d}\,\psi$ with $e^{-t\,D^2}$ but they use $e^{-t\,\bar{D}^2}$ (with $\bar{D}_\mu := \partial_\mu - iV_\mu - A_\mu$) for $\mathrm{d}\,\bar{\psi}$.

We use the second method, but to prevent extra terms from the Baker-Campbell-Hausdorff formula we regulate both $\mathrm{d}\,\psi$ and $\mathrm{d}\,\bar{\psi}$ with $\frac{1}{2}(e^{-t\,D^2}+e^{-t\,\bar{D}^2})$. The regulated Jacobian equals

$$J = \lim_{t\to 0} \exp\left[-i\,\mathrm{Tr}(\gamma_5\,\alpha\,(e^{-t\,D^2}+e^{-t\,\bar{D}^2}))\right]$$

where Tr is the trace over spinor fields and over all spin indices. We expand this trace using the plane wave basis $e^{ik\cdot x}$. This yields an expression for the trace over space-time, but we still need a trace over the spin-indices. If we denote the trace over the spin indices as tr, we get

$$\log J = -\,i\lim_{t\to 0}\int \mathrm{d}^4\,x\,\alpha(x)\cdot\int\frac{\mathrm{d}^4\,k}{(2\pi)^4}\,\mathrm{tr}\left(\gamma_5 e^{-ik\cdot x}\,(e^{-t\,D^2}+e^{-t\,\bar{D}^2})e^{ik\cdot x}\right).$$

Using the Leibniz rule $[D, e^{ik\cdot x}] = i\not{k}e^{ik\cdot x}$, we pull $e^{ik\cdot x}$ to the left

$$\log J = -\,i\lim_{t\to 0}\int \mathrm{d}^4\,x\,\alpha(x)\cdot\int\frac{\mathrm{d}^4\,k}{(2\pi)^4}\,\mathrm{tr}\left(\gamma_5\left(e^{-t\,(D+i\not{k})^2}+e^{-t\,(\bar{D}+i\not{k})^2}\right)\right)$$

and as before we rescale k_μ with $t^{-1/2}k_\mu$:

$$\log J = -\,i\lim_{t\to 0}\int \mathrm{d}^4\,x\,\alpha(x)\cdot\int\frac{\mathrm{d}^4\,k}{(2\pi)^4}t^{-2}\,\mathrm{tr}\left(\gamma_5\left(e^{-(t^{1/2}D+i\not{k})^2}+e^{-(t^{1/2}\bar{D}+i\not{k})^2}\right)\right). \tag{3.14}$$

We consider the Taylor series in $t^{1/2}$ at $t=0$. In this approximation the integrand of equation 3.14 equals

$$\begin{aligned} e^{-k^2}\,\mathrm{tr}[\gamma_5(2t^{-2}+ &\\ & -it^{-3/2}\left(\{D,\not{k}\}+\{\bar{D},\not{k}\}\right)\\ & -t^{-1}\left(D^2+\bar{D}^2+\frac{1}{2}\left(\{D,\not{k}\}\right)^2+\frac{1}{2}\left(\{\bar{D},\not{k}\}\right)^2\right)\\ & \frac{i}{2}t^{-1/2}\left(D^2\{D,\not{k}\}+\bar{D}^2\{\bar{D},\not{k}\}\right))]+\ldots \end{aligned}$$

The order t^{-2} vanishes because it is proportional to $\mathrm{tr}(\gamma_5) = 0$. We notice that the orders $t^{-3/2}$ and $t^{-1/2}$ are odd in k. Hence these orders also vanishes when we integrate over k. To calculate the t^{-1} order let $D_0 = \not{\partial} + i\not{V}$ and notice that

$$D^2 + \bar{D}^2 + \frac{1}{2}\left(\{D, \not{k}\}\right)^2 + \frac{1}{2}\left(\{\bar{D}, \not{k}\}\right)^2 = 2D_0^2 + 2A^2 + \left(\{D_0, \not{k}\}\right)^2 + \left([\not{k}, \not{A}]\right)^2. \tag{3.15}$$

All notions of γ_5 disappears. We apply the trace identities by counting the Clifford degree. The terms D_0^2 and A^2 vanishes in the trace. The other terms in equation 3.15 are proportional to $\epsilon^{\mu\nu\rho\sigma} k_\mu k_\nu$. However, the integral $\int \mathrm{d}^4\, k e^{-k^2} k_\mu k_\nu$ equals $\frac{\pi^2}{2}\eta_{\mu\nu}$ and so the t^{-1} order vanishes completely. We conclude that the Jacobian does not diverge when $t \to 0$.

The calculation of the t-independent part of equation 3.14 is tedious and hence we use computer algebra to solve this problem. We calculate the Jacobian using FORM(Vermaseren [2000]) and the source code is given in the appendix. The basic structure of the calculation is as follows:

1. We expand the exponent in equation 3.14 using the series $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. We only consider the taylor series with five orders. All higher order terms vanishes because they are a multiple of $t^{1/2}$.
2. We work out the trace using the gamma matrix identities

$$\mathrm{tr}(\gamma_5) = 0 \qquad \mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu) = 0 \qquad \mathrm{tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\epsilon^{\mu\nu\rho\sigma}.$$

3. We integrate over k using the identities:

$$\int d^4k e^{-k^2} = \pi^2 \qquad \int d^4k e^{-k^2} k_\mu k_\nu = \frac{\pi^2}{2}\eta_{\mu\nu}$$
$$\int d^4k e^{-k^2} k_\mu k_\nu k_\rho k_\sigma = \frac{\pi^2}{4}(\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}).$$

4. We simplify the result and we recall that we transformed iA_μ to A_μ.

After the calculation we conclude that the Jacobian equals

$$J = \exp\left[\int \frac{\mathrm{d}^4\, k}{16\pi^2}\alpha(x)\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left(F_{\mu\nu}F_{\rho\sigma} + \frac{1}{3}G_{\mu\nu}G_{\rho\sigma} + \frac{32}{3}A_\mu A_\nu A_\rho A_\sigma - \right.\right.$$
$$\left.\left. -\frac{8}{3}\left(A_\mu A_\nu F_{\rho\sigma} + A_\mu F_{\nu\rho} A_\sigma + F_{\mu\nu} A_\rho A_\sigma\right)\right)\right]$$

where

$$\begin{aligned} F_{\mu\nu} =& \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu] + [A_\mu, A_\nu] \\ G_{\mu\nu} =& \partial_\mu A_\nu - \partial_\nu A_\mu + [V_\mu, A_\nu] + [A_\mu, V_\nu]. \end{aligned}$$

This result coincide with the results in Bertlmann [1996] and van Nieuwenhuizen [1989] and was first found by Bardeen [1969]. Therefore, $F_{\mu\nu}$ and $G_{\mu\nu}$ are called Bardeen curvatures.

We compare the difference between the Bardeen anomaly and the covariant anomaly. We notice that the Bardeen anomaly is not covariant under gauge transformation. From this we immediately see that the Covariant anomaly and the Bardeen anomaly must differ. There are two possible factors where the difference can come from:

1. We used different regulators for the different anomalies.
2. We changed A_μ to iA_μ when we calculated the Bardeen anomaly.

When we set A_μ to zero, the Dirac operators D, $\bar{D}$ and $D^\pm$ become equal and hence the regulators are equal. The act of making making A_μ imaginary is irrelevant, when $A_\mu = 0$. Hence the results must be equal when $A_\mu = 0$. This is indeed true, because the Bardeen anomaly simplifies to

$$\epsilon^{\mu\nu\rho\sigma} \left(\partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu]\right) \left(\partial_\rho V_\sigma - \partial_\sigma V_\rho + [V_\rho, V_\sigma]\right).$$

This is equal to the covariant anomaly.

Earlier we calculated the chiral anomaly using Feynman diagrams. Recall that the rules for Feynman diagrams comes from the perturbation theory of the path integral formalism. In table 5 we compare our earlier calculated results with the results found by Bardeen. For example, the term $\epsilon^{\mu\nu\rho\sigma} A_\mu A_\nu A_\rho A_\sigma$ corresponds to an interaction with four external A-fields. Up to first loop approximation, this is calculated in a pentagon diagram[7]. In this diagram we denote

$$\begin{aligned} F^{lin}_{\mu\nu} =& \partial_\mu V_\nu - \partial_\nu V_\mu \\ G^{lin}_{\mu\nu} =& \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned}$$

See that the factor $\frac{1}{3}$ difference in the AVV and AAA diagram corresponds to the factor $\frac{1}{3}$ between $F^{lin}_{\mu\nu} F^{lin}_{\rho\sigma}$ and $G^{lin}_{\mu\nu} G^{lin}_{\rho\sigma}$. We see that the one loop approximation fully

[7]Although a pentagon diagram has 5 external fields, one is contracted to calculate the anomaly. Hence, it correspond to $\epsilon^{\mu\nu\rho\sigma} A_\mu A_\nu A_\rho A_\sigma$

captures the anomaly. This is first found by Adler and Bardeen [1969] and is called the Adler-Bardeen theorem:

Theorem 3.1 (Adler and Bardeen [1969])**.** *The chiral anomaly can be fully determined in the one loop approximation.*

Bardeen anomaly	**Factor**	**Feynman diagram**	**Amplitude**
$F^{lin}_{\mu\nu}F^{lin}_{\rho\sigma}$	1	AVV	$\epsilon^{\mu\nu\rho\sigma}q_\rho p_\sigma$
$G^{lin}_{\mu\nu}G^{lin}_{\rho\sigma}$	$\frac{1}{3}$	AAA	$\frac{1}{3}\epsilon^{\mu\nu\rho\sigma}q_\rho p_\sigma$
$F^{lin}_{\mu\nu}V_\rho V_\sigma + V_\mu V_\nu F^{lin}_{\rho\sigma}$	2	AVVV	$\epsilon^{\mu\nu\rho\sigma}p^{(i)}_\sigma$
$G^{lin}_{\mu\nu}V_\rho A_\sigma + G^{lin}_{\mu\nu}A_\rho V_\sigma +$ $V_\mu A_\nu G^{lin}_{\rho\sigma} + A_\mu V_\nu G^{lin}_{\rho\sigma} -$ $-F^{lin}_{\mu\nu}A_\rho A_\sigma - 4A_\mu F^{lin}_{\nu\rho}A_\sigma - A_\mu A_\nu F^{lin}_{\rho\sigma}$	$\frac{2}{3}$	AAAV	-
$V_\mu V_\nu V_\rho V_\sigma$	4	AVVVV	$\epsilon^{\mu\nu\rho\sigma}$
$V_\mu A_\nu V_\rho A_\sigma - V_\mu V_\nu A_\rho A_\sigma +$ $A_\mu V_\nu A_\rho V_\sigma - A_\mu A_\nu V_\rho V_\sigma +$ $V_\mu A_\nu A_\rho V_\sigma - 4A_\mu V_\nu V_\rho A_\sigma$	$\frac{4}{3}$	AAAVV	-
$A_\mu A_\nu A_\rho A_\sigma$	$-\frac{4}{3}$	AAAAA	-

Table 5: *The Bardeen anomaly has an one loop approximation using Feynman diagrams. In this table every term in the Bardeen anomaly is related to a triangle, box or pentagon diagram. The prefactor of each term in the Bardeen anomaly are given relative to $F^{lin}_{\mu\nu}F^{lin}_{\rho\sigma}$. Also, the results from chapter 2 are again stated.*

3.4 The consistent anomaly

When we calculated the covariant anomaly, we simplified the calculation by considering the γ_5 eigenbasis. We concluded that the action given in equation 3.11 describes two non-interacting particles. We now ask what happens to the chiral anomaly if we only consider massless right-handed particles ψ_+. That is, what is the chiral anomaly if we consider the action

$$S = \int_{\mathbb{R}^4} -\bar{\psi}_+(\partial\!\!\!/ - iA\!\!\!/^+)\psi_+ \,\mathrm{d}\,x?$$

However, this does not define a well-defined quantum theory. Using $\psi_+ = P_+\psi = \frac{1}{2}(1+\gamma_5)$ and $D_+ = \partial\!\!\!/ - iA\!\!\!/^+$ we rewrite this action as

$$S = \int_{\mathbb{R}^4} -\bar{\psi}(D_+)P_+\psi \,\mathrm{d}\,x.$$

The operator D_+P_+ maps right-handed chirality spinors into left-handed chirality spinors. Therefore, this operator has no eigenvalues and the generating functional, which is the formal determinant of iD_+P_+, cannot be defined. Alvarez-Gaumé and Ginsparg [1984] [1985] noticed that if we consider the action

$$S = \int_{\mathbb{R}^4} -\bar{\psi}(D_+P_+ + \partial\!\!\!/ P_-)\psi \,\mathrm{d}\,x$$

we get a well-defined quantum theory. Also the gauge couples only to the positive chirality spinors and the non-zero eigenmodes are all right-handed. Hence the gauge theories coincide. We assume that ψ_+ is invariant under the symmetry $\psi_+ \mapsto e^{i\alpha(x)\gamma_5}\psi_+$. Here we assume that α is an infinitely small Lie algebra valued smooth map. The action transforms into

$$\tilde{S} = S + \int_{\mathbb{R}^4} -i\bar{\psi}_+\gamma^\mu(\tilde{D}^+_\mu\alpha(x))\psi_+ \,\mathrm{d}\,x. + \mathcal{O}(\alpha^2) \tag{3.16}$$

where $\tilde{D}^+_\mu = \delta_\mu - i[A^+_\mu, \cdot]$. The generating functional $\mathcal{Z}[A^+_\mu]$ transforms into

$$\tilde{\mathcal{Z}}[A^+_\mu] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi J \cdot \exp\left(-\tilde{S}\right) \tag{3.17}$$

where J is the Jacobian of the path integral. We can use Fujikawas method to calculate the Jacobian. However, this is a special case (Andrianov and Bonora [1984]) of Bardeens anomaly if we substitute

$$V_\mu \to \frac{1}{2}A^+_\mu \qquad A_\mu \to \frac{1}{2}A^+_\mu.$$

The Jacobian J equals $\int \mathrm{d}^4\,x \,\mathrm{tr}(\alpha(x)\cdot G[A^+])$ where $G[A^+]$ is

$$G[A^+] = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\left(\frac{1}{3}A^+_\mu A^+_\nu\ (\partial_\rho A^+_\sigma) - \frac{1}{3}A^+_\mu(\partial_\nu A^+_\rho)A^+_\sigma \right.$$
$$\left. + \frac{1}{3}(\partial_\mu A^+_\nu)A^+_\rho A^+_\sigma + \frac{2}{3}(\partial_\mu A^+_\nu)(\partial_\rho A^+_\sigma)\right)$$

We simplify $G[A^+]$ into

$$G[A^+] = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(A^+_\nu(\partial_\rho A^+_\sigma) + \frac{1}{2}A^+_\nu A^+_\rho A^+_\sigma\right).$$

This result is called the **consistent anomaly**. It is called this way, because $G[A^+]$ satisfies the Wess-Zumino consistency condition [1971].

We now derive this condition. By assumption the generating functional is invariant under the chiral symmetry. So from Equations 3.16 and 3.17 it follows that

$$\begin{aligned}\mathcal{Z}[A^+_\mu + \tilde{D}^+_\mu\alpha(x)] =& \tilde{\mathcal{Z}}[A^+_\mu + \tilde{D}^+_\mu\alpha(x)]\\ =& \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(-S + \int \alpha(x)G[A^+]\ \mathrm{d}^4 x\right)\end{aligned}$$

and we conclude

$$\lim_{t\to 0}\frac{1}{t}\left(\log\mathcal{Z}[A^+_\mu + t\tilde{D}^+_\mu\alpha(x)] - \log\mathcal{Z}[A^+_\mu]\right) = \int \alpha(x)G[A^+](x)\ \mathrm{d}^4 x. \tag{3.18}$$

In terms of the variational derivative we yield the **anomalous Ward identity** $\frac{\delta}{\delta\alpha}\log\mathcal{Z} = G[A^+]$. The Ward identity can also be written in terms of differential forms[8]. For this we view $\mathcal{Z}$ as a smooth real-valued map on the space of all fields A^+_μ. That is, $\mathcal{Z}$ is a 0-form on a infinite dimensional manifold. The left hand side of equation 3.18 can be interpreted as the Lie derivative of $\log(Z)$ in the direction $\tilde{D}^+_\mu\alpha$. The right hand side is the L^2 inner product on the space of Lie algebra valued smooth maps. Hence, the anomalous Ward identity equals

$$\mathcal{L}_{\tilde{D}^+\alpha}\log\mathcal{Z} = \langle\alpha, G[A^+]\rangle_{L^2}. \tag{3.19}$$

Let $\tilde{G}$ be the exterior derivative $\mathrm{d}\log(\mathcal{Z})$. It is the unique 1-form such that $\tilde{G}(\delta A_\mu) = \mathcal{L}_{\delta A_\mu}\log(\mathcal{Z})$. By equation 3.19 we conclude $\tilde{G}(\tilde{D}^+\alpha) = \langle\alpha, G[A^+]\rangle_{L^2}$. Using Cartans magic formula we calculate the exterior derivative of $\tilde{G}$ which is

$$\begin{aligned}\mathrm{d}\,\tilde{G}(\tilde{D}^+\alpha, \tilde{D}^+\beta) =& \iota_{\tilde{D}^+\beta}\circ\iota_{\tilde{D}^+\alpha}\circ\mathrm{d}\,\tilde{G}\\ =& -\iota_{\tilde{D}^+\beta}\circ\mathrm{d}\circ\iota_{\tilde{D}^+\alpha}\tilde{G} + \iota_{\tilde{D}^+\beta}\circ\mathcal{L}_{\tilde{D}^+\alpha}\tilde{G}\\ =& \mathrm{d}\circ\iota_{\tilde{D}^+\beta}\circ\iota_{\tilde{D}^+\alpha}\tilde{G} - \mathcal{L}_{\tilde{D}^+\beta}\circ\iota_{\tilde{D}^+\alpha}\tilde{G} + \iota_{\tilde{D}^+\beta}\circ\mathcal{L}_{\tilde{D}^+\alpha}\tilde{G}.\end{aligned}$$

[8] See the appendix for a short introduction.

We notice that $\iota_{\tilde{D}^+\beta} \circ \iota_{\tilde{D}^+\alpha}\tilde{G}$ is a (-1)-form. These forms does not exists and hence $\mathrm{d} \circ \iota_{\tilde{D}^+\beta} \circ \iota_{\tilde{D}^+\alpha}\tilde{G}$ is zero. We also recall that the commutator between a Lie derivative and the interior product is the interior product of a Lie bracket. That is, $[L_x, \iota_Y] = \iota_{[X,Y]}$ and hence

$$\begin{aligned}\mathrm{d}\,\tilde{G}(\tilde{D}^+\alpha, \tilde{D}^+\beta) =& -[\mathcal{L}_{\tilde{D}^+\alpha}, \iota_{\tilde{D}^+\beta}]\tilde{G} + \mathcal{L}_{\tilde{D}^+\alpha} \circ \iota_{\tilde{D}^+\beta}\tilde{G} - \mathcal{L}_{\tilde{D}^+\beta} \circ \iota_{\tilde{D}^+\alpha}\tilde{G} \\ =& -\iota_{[\tilde{D}^+\alpha, \tilde{D}^+\beta]}\tilde{G} + \mathcal{L}_{\tilde{D}^+\alpha} \circ \iota_{\tilde{D}^+\beta}\tilde{G} - \mathcal{L}_{\tilde{D}^+\beta} \circ \iota_{\tilde{D}^+\alpha}\tilde{G}. \end{aligned} \tag{3.20}$$

By the definition of the exterior derivative $\mathrm{d}^2 = 0$, and so equation 3.20 equals zero. Using the Ward identity we simplify equation 3.20 into

$$\mathcal{L}_{\tilde{D}^+\alpha}\langle \tilde{D}^+\beta, G[A^+]\rangle_{L_2} - \mathcal{L}_{\tilde{D}^+\beta}\langle \tilde{D}^+\alpha, G[A^+]\rangle_{L_2} - \langle [\tilde{D}^+\alpha, \tilde{D}^+\beta], G[A^+]\rangle_{L_2} = 0.$$

This is the Wess-Zumino consistency condition.

3.5 Mathematical interpretation of the Fujikawa method

At last we give a mathematical interpretation to the Fujikawa method. In section 3.1 we calculated the chiral anomaly for a Dirac particle in quantum electrodynamics. The anomaly was due to the Jacobian of the path integral. By equation 3.9 the Jacobian equals

$$\log J = \frac{i}{16} e^{\mu\nu\rho\sigma} \int_{\mathbb{R}^4} \mathrm{d}\,x\, \alpha(x) F_{\mu\nu} F_{\rho\sigma}. \tag{3.9}$$

For simplicity we assume that $\alpha(x) = 1$. We rewrite J in terms of differential forms[9]. Recall that $F^{\mu\nu}$ are the components of the curvature tensor F. For quantum electrodynamics F is a complex valued 2-form. By definition the wedge product between F and itself equals $(F \wedge F)^{0123} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$. This shows that

$$\log J = \frac{i}{8} \int_{\mathbb{R}^4} F \wedge F.$$

By stokes theorem we conclude that the Jacobian is determined by the cohomology class of $F \wedge F$. In Chapter 6 we will show that this cohomology class only depends on *topology* of the gauge bundle. At the same time, the regulated Jacobian equals

$$\log J = -2i \lim_{t \to 0} \mathrm{Tr}\left[\gamma_5\, f\left(t \cdot D^2\right)\right].$$

[9] See the appendix for a basic introduction.

In the orthonormal eigenbasis ϕ_n of D where λ_n is the corresponding eigenvalue this trace is given by

$$\begin{aligned}\log J =& -2i \lim_{t\to 0} \sum_n \langle \phi_n | \gamma_5\, f\left(t \cdot D^2\right) | \phi_n \rangle \\ =& -2i \lim_{t\to 0} \sum_n f(t\cdot \lambda_n^2) \langle \phi_n | \gamma_5 | \phi_n \rangle.\end{aligned}$$

Recall that γ_5 anti-commutes with the Dirac operator D and hence $\gamma_5\phi_n$ is an eigenvector of D with eigenvalue $-\lambda_n$. By the orthonormality of the eigenspaces we conclude that $\langle \phi_n|\gamma_5|\phi_n\rangle = 0$ if $\lambda_n \neq 0$. If $n^\pm$ denotes the number of independent left- resp. right-handed zero modes of D, then

$$\begin{aligned}\log J =& -2i \sum_{\substack{\text{left-handed}\\ \text{zero modes}}} \langle \phi_n|\gamma_5|\phi_n\rangle - 2i \sum_{\substack{\text{right-handed}\\ \text{zero modes}}} \langle \phi_n|\gamma_5|\phi_n\rangle \\ =& 2i \sum_{\substack{\text{left-handed}\\ \text{zero modes}}} \langle \phi_n|\phi_n\rangle - 2i \sum_{\substack{\text{right-handed}\\ \text{zero modes}}} \langle \phi_n|\phi_n\rangle \\ =& -2i(n^+ - n^-).\end{aligned}$$

This result is found by McKean and Singer [1967]. In mathematics the quantity $n^+ - n^-$ is called the *index of* D. Using the Fujikawa method we see that the index is determined by the topology of a vector bundle. This relation is first found by Atiyah and Singer [1968] and is called the Atiyah-Singer index theorem.

In the next chapters we prove this theorem for Dirac operators on compact spaces[10] by using Fujikawas method. In chapter 4 we define the operator $\exp(-tD^2)$. We will not define it using the eigenvalues of D, but as the unique operator that satisfies $(\frac{\partial}{\partial t} + D^2)e^{-tD^2} = 0$. In chapter 5 we show that the trace over $\gamma_5 e^{-tD^2}$ is finite and does not depend on the choice of basis. Hence, we show that e^{-tD^2} is a well-chosen regulator for J.

In the next step Fujikawa considered the trace in the plane wave basis $e^{ik\cdot x}$ and showed that the index of D equals the integral over a trace. In chapter 4 we show that e^{-tD^2} has a kernel. That is, we show that there exists an operator k_t such that e^{-tD^2} is the integral over this operator. Later in chapter 5 we show that the trace over $\gamma_5 e^{-tD^2}$

[10] That is, we assume that the space-time is of finite size and we assume that it does not contain any singularities.

reduces to the integral $\int \mathrm{tr}(\gamma_5\, k_t)$.

In the last step Fujikawa calculated a Taylor series. In chapter 7 we formalize this in the theory of graded and filtered algebras. We show that the number of gamma matrices induce a grading. Also, Taylor series in k_μ forms a grading. However the grading Fujikawa used is a combination of both and is due to Getzler [1983]. Till now we explicitly calculated each term of the Taylor series. This is tedious and is not useful when we consider the general case. Hence we investigate how the Getzler grading behaves under the differential equation $\frac{\partial}{\partial t}+D^2$. This yields another differential equation that we can explicitly solve.

4 Smoothing operators and Heat kernels

In chapter 3 we saw how the Abelian anomaly was related to the index of a Dirac operator. In the following chapters we analyze this method and work out the technical details. In this chapter we question whether a Dirac operator can be exponentiated. This is indeed possible and we also show that we can write it as a integral.

We follow the approach given by Berline et al. [2004]. We construct the exponential by solving a differential equation. This is still a formal solution, because we need to show that this formal solution converges. We prove this in two steps. First we modify the formal solution into an approximate solution that does converge. Secondly we increase the accuracy of the approximate solution. So the basic steps in this chapter are

$$\text{Formal solution} \implies \text{Approximate solution} \implies \text{Existence.}$$

4.1 Definitions

We want to study e^{-tD^2}. From the Lichnerowicz formula, we know that

$$D^2 = \nabla^*\nabla + \mathrm{F}^S + \frac{1}{4}\kappa$$

where $\mathrm{F}^S + \frac{1}{4}\kappa$ is a section of $\mathrm{End}(S)$. In this chapter we don't need the Clifford structure and hence we study generalized Laplacians:

> **Definition 4.1.** *Let (M,g) be a Riemannian manifold and let E be a vector bundle over M with a positive definite inner product. A self-adjoint map $H\colon \Gamma(E) \to \Gamma(E)$ is a* **generalized Laplacian** *if there exists a compatible connection ∇ such that $H - \nabla^*\nabla$ is a section of $\mathrm{End}(E)$.*

We want to write e^{-tD^2} as an integral. Although vector-valued integration is well-defined, vector bundle valued integration is not. To solve this, we construct an operator which maps all fibers into a single one. Then vector bundle integration reduces to vector space integration which is well defined. Such an operator is called a kernel. Informally, for two vector bundles E^1 and E^2 and $x, y \in M$ a kernel is a "smooth" linear map $p(x,y)$ from E^1_y to E^2_x. To define smoothness we recall that $\mathrm{Hom}(E^1_y, E^2_x)$

is isomorphic to $(E_y^1)^* \otimes E_x^2$. Using the canonical projections pr_1 and pr_2 from $M \times M$ to M we can write this as

$$\mathrm{pr}_1^* E^2 \otimes \mathrm{pr}_2^* E^1{}_{(x,y)} \simeq (E^2)_x \otimes (E_y^1)^* \simeq \mathrm{Hom}(E_y^1, E_x^2).$$

To simplify notation we write the vector bundle $\mathrm{pr}_1^* E^2 \otimes \mathrm{pr}_2^* E^1 \to M \times M$ as $E^1 \boxtimes E^2 \to M \times M$. We see p is a section of $E^1 \boxtimes E^2$. Smoothness of a kernel follows from the fact that sections are smooth.

Definition 4.2. *Let (M,g) be a compact Riemannian manifold and let E^1 and E^2 be vector bundles over M with positive definite inner products. A* **kernel** *is a section $p \in \Gamma(E^1 \boxtimes E^2)$ (which is a linear map $p(x,y)\colon E_y^1 \to E_x^2$). A linear operator $P\colon \Gamma(M,E_1) \to \Gamma(M,E_2)$ is a* **smoothing operator** *if there exists a kernel $p \in \Gamma(E^1 \boxtimes E^2)$ such that*

$$(Ps)(x) = \int_{y\in M} p(x,y)s(y)\,\mathrm{Vol}(g).$$

The are two methods to define e^{-tD^2}. One method is to consider the eigenvalues λ_i of D^2 and construct an eigenbasis $\{v_i\}$ of D^2. Then we define e^{-tD^2} by $e^{-tD^2}v_i = e^{-t\lambda_i}v_i$. For this method we need to estimate the eigenvalues of D^2 before we can show that the exponent converges. Also, if we want to show that this is a smoothing operator, then we need to prove this separately. We follow a more direct approach: We consider all smoothing operators that satisfy $\frac{\partial}{\partial t}e^{-tD^2} = -D^2 e^{-tD^2}$. We then show that there is a unique operator satisfying this equation which has therefore the properties we would formally expect from e^{-tD^2}.

Definition 4.3. *Let (M,g) be a compact Riemannian manifold, E be a vector bundle over M with a positive definite inner product and H be a generalized Laplacian. A* **heat kernel** *for H is a section k^H of the vector bundle $E \boxtimes E \to \mathbb{R}_+ \times M \times M$ which has the following properties:*

1. *The kernel k^H is at least once continuous differentiable in the first component*
2. *The kernel k^H is at least twice continuous differentiable in the second component*
3. *The kernel k satisfies the heat equation:*

$$\left(\frac{\partial}{\partial t} + H_x\right) k^H(t,x,y) = 0 \qquad \forall x,y \in M,\ t \in \mathbb{R}_+$$

4. *The kernel k^H satisfies the boundary condition. That is, in the supremum norm*
$$\lim_{t\to 0}\int_{y\in M} k^H(t,x,y)\, s(y)\, \mathrm{Vol}(g) = s(x)$$
for all $x \in M$ and $s \in \Gamma(E)$.

We denote $k^H(t,\cdot,\cdot)$ as $k^H_t(\cdot,\cdot)$.

4.2 Examples of heat kernels

In this paragraph we give some useful examples of heat kernels for Laplacians on the real line. Although $\mathbb{R}$ is not compact we can define a heat kernel if we restrict the boundary condition to only square integrable sections of $\mathbb{R}$. Note that on the real line p_t reduces to a smooth map $\mathbb{R}\times\mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$.

The simplest generalized Laplacian is the standard Laplacian $H = -\frac{\partial}{\partial x^2}$. The heat equation $\frac{\partial}{\partial t} - \frac{\partial}{\partial x^2} = 0$ suggests that the heat kernel is a Gaussian function. By trial and error we can find that the map $k^H_t(x,y) = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}$ satisfies
$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) k^H_t(x,y) = 0.$$

This suggests that k^H_t is a heat kernel. This is indeed true.

Lemma 4.4 (Berline et al. [2004], Lemma 2.12)**.** *The map $k^H_t(x,y) = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}$ is a heat kernel for the generalized Laplacian $H = -\frac{\partial^2}{\partial x^2}$ on $\mathbb{R}$.*

Proof. We are left to show that $\int_{\mathbb{R}} k^H_t(x,y)s(y)\,\mathrm{d}\,y = s(x)$ for all $s \in L^2(\mathbb{R})$. Recall that compactly supported smooth maps are dense in $L^2(R)$ and on this compact support we can approximate these maps with polynomials. Hence it is sufficient to that this is true if $s(x) = x^k$. Consider the map
$$A(s,t) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-(x-y)^2/4t+sy}\,\mathrm{d}\,y. \tag{4.1}$$

This map converges, because if we substitute y with $y = \sqrt{4t}\tilde{y} + 2st + x$ we get

$$\begin{aligned} A(s,t) =& \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-(x-(\sqrt{4t}\tilde{y}+2st+x))^2/4t+s(\sqrt{4t}\tilde{y}+2st+x)} \,\mathrm{d}\, y \\ =& \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\tilde{y}^2+s^2t+sx} \,\mathrm{d}\, \tilde{y} \\ =& \exp(s^2t + sx). \end{aligned} \tag{4.2}$$

We calculate the Taylor series of $A(s,t)$ in s. Comparing equation 4.1 and 4.2 we see that

$$\sum_{k=0}^{\infty} \frac{s^k}{k!} \left(\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-(x-y)^2/4t} y^k \,\mathrm{d}\, y \right) = \sum_{k=0}^{\infty} \frac{s^k}{k!} \cdot (st+x)^k.$$

In the limit $t \to 0$ this simplifies to

$$\sum_{k=0}^{\infty} \frac{s^k}{k!} \left(\lim_{t\to 0} \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-(x-y)^2/4t} y^k \,\mathrm{d}\, y \right) = \sum_{k=0}^{\infty} \frac{s^k}{k!} \cdot x^k.$$

This proves $\int_{\mathbb{R}} k_t^H(x,y) s(y) \,\mathrm{d}\, y = s(x)$ for all $s \in L^2(\mathbb{R})$. ☐

Another example is the quantum mechanical harmonic oscillator in one dimension . Up to constants the Schrödinger equation for this system is

$$\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + \omega^2 x^2 = 0$$

where $\omega \in \mathbb{R}$ is the angular frequency. Notice that this is the heat equation for the generalized Laplacian

$$H = -\frac{\partial^2}{\partial x^2} + \omega^2 x^2.$$

The heat kernel is found by Mehler [1866] and in the next lemma we give the formula.

Lemma 4.5 (Mehler [1866])**.** *Let H be the generalized Laplacian on $\mathbb{R}$ that is given by $H = -\frac{\partial^2}{\partial x^2} + \omega^2 x^2$. Then the heat kernel w.r.t. H exists and equals*

$$k_t^H(x,y) = \sqrt{\frac{\omega}{2\pi \sinh(2\omega t)}} \exp\left[\frac{-\omega(x^2+y^2)\coth(2\omega t) + 2\omega \operatorname{cosech}(2\omega t) xy}{2} \right].$$

Proof. To solve the heat equation, we use the ansatz:

$$k_t^H(x,y) = \exp\left(\frac{a(t)}{2}(x^2+y^2) + b(t)xy + c(t)\right)$$

The heat equation we need to solve is then

$$\begin{aligned} 0 =& \frac{\dot\alpha}{2}x^2 + \dot\beta xy + \frac{\dot\alpha}{2}y^2 + \dot\gamma - (\alpha x + \beta y)^2 - \alpha + \omega^2 x^2 \\ =& x^2\left(\frac{\dot\alpha}{2} - \alpha^2 + \omega^2\right) + y^2\left(\frac{\dot\alpha}{2} - \beta^2\right) + xy\left(\dot\beta - 2\alpha\beta\right) + (\dot\gamma - \alpha)\,. \end{aligned}$$

This equation must be valid for any $x, y \in \mathbb{R}$. Hence, we have the following system of differential equations

$$\frac{\dot\alpha(t)}{2} = \alpha^2(t) - \omega^2 = \beta^2(t) \qquad \dot\beta(t) = 2\alpha(t)\beta(t) \qquad \dot\gamma(t) = \alpha(t).$$

The differential equation for α can be written as $\int_{\mathbb{R}} \frac{1}{\alpha^2-\omega^2}\,\mathrm{d}\,a = 2t + c_1$ where $c_1 \in \mathbb{R}$ is an integration constant. Using change of variables this simplifies to $\int_{\mathbb{R}} \frac{\dot\alpha(t)}{\alpha^2(t)-\omega^2}\,\mathrm{d}\,t = 2t + c_1$. The primitive is a cotangent hyperbolic[11] and α equals

$$\alpha(t) = -\,\omega\coth\left(2\omega t + c_1\right).$$

By integration and differentiation we find expressions for β and γ and hence we have

$$\begin{aligned} \alpha(t) &= -\,\omega\coth\left(2\omega t + c_1\right) \\ \beta(t) &= +\,\omega\,\mathrm{cosech}\left(2\omega t + c_1\right) \\ \gamma(t) &= -\,\frac{1}{2}\log\left(\sinh\left(2\omega t + c_1\right) * c_2\right). \end{aligned}$$

Here c_1 and c_2 are integration constants and we find them by using the boundary condition. Already we notice that k_t^H is a bounded map and so it is a well defined operator on $L^2(\mathbb{R})$.

When we apply the smoothing operator on 1 and x^2, we conclude that c_1 must vanish. We approximate the resulting kernel using a taylor series in t and we get that

$$k_t^H(x,y) = \frac{1}{\sqrt{4\pi t}}e^{-(x-y)^2/4t}\cdot\left(\sqrt{\frac{2\pi}{\omega c_2}} + \mathcal{O}(t)\right).$$

Using Lemma 4.4, we conclude that k_t^H satisfies the boundary condition if $c_2 = \frac{\omega}{2\pi}$. $\square$

[11] Recall that $\coth = \frac{\cosh}{\sinh}$ and $\mathrm{cosech} = \frac{1}{\sinh}$

In our last example we generalize Mehlers kernel for $\mathbb{R}^n$. This extension will be important in the proof of Atiyahs index theorem.

Lemma 4.6 (Roe [1998], Proposition 12.25)**.** *Let $R \in M_{n\times n}(\mathbb{R})$ be a skew-symmetric matrix and let $F \in \mathbb{R}$. Let H be the generalized Laplacian on $\mathbb{R}^n$ given by $H = -\sum_i \left(\frac{\partial}{\partial x_i} + \frac{1}{4}\sum_j R_{ij}x_j\right)^2 + F$. Then the heat kernel k_t^H w.r.t. H is given by*

$$k_t^H(x,0) = \frac{1}{(4\pi t)^{n/2}} \det{}^{1/2}\left(\frac{tR/2}{\sinh(tR/2)}\right)\exp\left[-\frac{1}{4t}\left\langle \frac{tR}{2}\coth\left(\frac{tR}{2}\right)x, x\right\rangle - tF\right].$$

Proof. Assume that the heat kernel k_t^H can be written as

$$k_t^H(x,y) = w_t(x,y)e^{-tF}.$$

The heat equation $(\frac{\partial}{\partial t} + H_x)K_t^H(x,y) = 0$ simplifies to

$$e^{-tF}\left[\frac{\partial w_t}{\partial t} - \sum_i\left(\frac{\partial}{\partial x_i} + \frac{1}{4}\sum_j R_{ij}x^j\right)^2 w_t\right] = 0 \tag{4.3}$$

and this gives us a differential equation for w_t. Next we consider the eigenvalue decomposition of R. Extend R to $R_{\mathbb{C}}\colon \mathbb{C}^n \to \mathbb{C}^n$ and let $\{z_i\}$ be the eigenbasis of $R_{\mathbb{C}}$. The eigenvalues λ_i w.r.t. z_i must be imaginary, because R is skew symmetric. So there exists $\theta_i \in \mathbb{R}$ such that $\lambda_i = i\theta_i$. We split z_i in its real and complex components $z_i = x_i + iy_i$. We decompose the eigenvalue equation $R_{\mathbb{C}}z_i = \lambda_i z_i$ into its real and complex components. It follows that

$$Rx_i = -\theta_i y_i \qquad Ry_i = \theta_i x_i.$$

Because $\bar{z}_i$ is an eigenvector of $\mathbb{R}_{\mathbb{C}}$ and x_i and y_i forms a linear combination of z_i and $\bar{z}_i$, we get that the set $\{x_i, y_i\}$ forms a basis of $\mathbb{R}^n$. Even more, this basis can be chosen orthonormal. This follows from the relations

$$\begin{aligned}\langle x_i, Rx_i\rangle &= -\langle Rx_i, x_i\rangle = \theta_i\langle y_i, x_i\rangle = 0\\ \langle x_i, Ry_i\rangle &= -\langle Rx_i, y_i\rangle = \theta_i\|x_i\|^2 = \theta_i\|y_i\|^2.s\end{aligned}$$

In the orthonormal basis $\{x_i, y_i\}$, equation 4.3 becomes

$$\frac{\partial w_t}{\partial t} - \sum_i\left(\frac{\partial}{\partial x_i} + \frac{1}{4}\theta_i y_i\right)^2 w_t - \sum_i\left(\frac{\partial}{\partial y_i} - \frac{1}{4}\theta_i x_i\right)^2 w_t = 0.$$

We also assume that $w_t = \prod_i u_i \cdot v_i$, where u_i is a smooth map in x_i and t and v_i is a map in y_i and t. From this ansatz the heat equation reduces a differential equation for all $i \leq n$:

$$\begin{aligned} v_i \left(\frac{\partial u_i}{\partial t} - \frac{\partial^2 u_i}{\partial x_i^2} - \frac{\theta_i}{16} x^2 u_i \right) + u_i \left(\frac{\partial v_i}{\partial t} - \frac{\partial^2 v_i}{\partial y_i^2} - \frac{\theta_i}{16} y^2 v_i \right) = \\ - \frac{\theta_i}{4} \left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right) u_i \cdot v_i. \end{aligned} \tag{4.4}$$

We further postulate that the left hand side and right hand side of equation 4.4 are independent differential equations and are both equal to zero. From the left hand side we notice that u_i and v_i are the heat kernels for the harmonic oscillator. That is

$$\begin{aligned} u_i(x_i, 0, t) =& \frac{1}{\sqrt{4\pi t}} \sqrt{\frac{it\theta_i/2}{\sinh(it\theta_i/2)}} \exp\left[-\frac{1}{8} i\theta_i x_i^2 \coth(it\theta_i/2) \right] \\ v_i(x_i, 0, t) =& \frac{1}{\sqrt{4\pi t}} \sqrt{\frac{it\theta_i/2}{\sinh(it\theta_i/2)}} \exp\left[-\frac{1}{8} i\theta_i y_i^2 \coth(it\theta_i/2) \right]. \end{aligned}$$

Notice that the product $u_i \cdot v_i$ is rotation invariant. Therefore is $\left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right) u_i \cdot v_i = 0$. Thus the right hand side of equation 4.4 also vanishes. We finally have a solution of the heat equation which is given by

$$k_t^H(x, 0) = e^{-tF} \prod_i \frac{1}{4\pi t} \frac{it\theta/2}{\sinh(it\theta/2)} \exp\left[-\frac{1}{8} i\theta (x_i^2 + y_i^2) \coth(it\theta/2) \right].$$

This equals the heat kernel proposed in the lemma. At last we need to check the boundary condition. Note that

$$\frac{tR/2}{\sinh(tR/2)} = \mathrm{Id} + \mathcal{O}(t^2) \quad \text{and} \quad \frac{tR}{2} \coth \frac{rR}{2} = \mathrm{Id} + \mathcal{O}(t^2).$$

Hence in the limit $t \to 0$, $\int_{\mathbb{R}^n} k_t^H(x, 0) s(x) = s(0)$ for all $s \in L^2(\mathbb{R}^n)$. ☐

4.3 Uniqueness of the heat kernel

Proposition 4.7 (Berline et al. [2004], Proposition 2.17). *Let (M, g) be a compact Riemannian manifold and let E be a vector bundle on M with a positive definite* *inner product. Let H be a generalized Laplacian and suppose that H admits a heat kernel $k_t^H \in \Gamma(E \boxtimes E)$. Then k_t is unique and is self adjoint in te sense that*

$$k_t^H(x,y)^* = k_t^H(y,x)$$

for all $x, y \in M$ and $t \in \mathbb{R}_+$.

Proof. Let $\tilde{k}_t^H \in \Gamma(E \boxtimes E)$ be another heat kernel of H and let K_t^H and $\tilde{K}_t^H$ be the corresponding smoothing operators. Pick two sections $u, v \in \Gamma(E)$ and denote $\langle \cdot, \cdot \rangle$ as the inner product on E. Consider the smooth map

$$f(\theta) = \langle K_\theta^H u, \tilde{K}_{t-\theta}^H v \rangle \qquad 0 < \theta < t.$$

From the heat equation it follows that

$$\frac{\partial f}{\partial \theta} = \langle -HK_\theta^H u, \tilde{K}_{t-\theta}^H v \rangle + \langle K_\theta^H u, H\tilde{K}_{t-\theta}^H v \rangle.$$

This equals zero for all $0 < \theta < t$, because H is self-adjoint and hence f is constant. Taking the limits $\theta \to 0$ and $\theta \to t$ we conclude that

$$\langle K_t^H u, v \rangle = \langle u, \tilde{K}_t^H v \rangle.$$

If we multiply u and v with bump functions, then for all $x, y \in M$ and $t \in \mathbb{R}_+$ we get $(k_t^H)^*(x,y) = \tilde{k}_t^H(y,x)$. If we choose $\tilde{k}_t^H = k_t^H$, we conclude that k_t^H is self adjoint. In the general case we get for all $x, y \in M$ and $t \in \mathbb{R}_+$

$$k_t^H(x,y) = (\tilde{k}_t^H)^*(y,x) = \tilde{k}_t^H(x,y).$$

Therefore the heat kernel is unique. □

With a slight modification of this proof we show that smoothing operators of heat kernels form a semi group. That is, $K_{s+t}^H = K_s^H \circ K_t^H$. This property will be useful later, when we show that the trace of e^{-tD^2} is well-defined.

Lemma 4.8 (Berline et al. [2004], Proposition 2.17(3)). *Let (M, g) be a compact Riemannian manifold, let E be a vector bundle on M with a positive definite inner product and let H be a generalized Laplacian. If H admits a heat kernel $k_t^H \in \Gamma(E \boxtimes E)$, then the corresponding smoothing operator K_t^H satisfies*

$$K_{s+t}^H = K_s^H \circ K_t^H$$

for all $s, t \in \mathbb{R}_+$.

Proof. Let $u, v \in \Gamma(E)$ be two sections on E. Consider the differentiable map

$$f(\theta) = \langle K^H_{\theta+t} u, K^H_{s-\theta} v \rangle \qquad -t < \theta < s.$$

From the heat equation it follows that

$$\frac{\partial f}{\partial \theta} = \langle -HK^H_{\theta+t} u, K^H_{s-\theta} v \rangle + \langle K^H_{\theta+t} u, HK^H_{s-\theta} v \rangle.$$

Again, this equals zero for all $-t < \theta < s$, because H is self-adjoint. Hence, f is constant. Comparing f at different values of θ we conclude

$$\langle K^H_t u, K^H_s v \rangle = \langle K^H_{s+t} u, v \rangle$$

and so $K^H_s \circ K^H_t = K^H_{t+s}$. □

4.4 The formal solution of the heat kernel

In section 4.2 we found that $\frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$ was the heat kernel for the Laplacian on $\mathbb{R}$. For an n dimensional Riemannian manifold this kernel can be generalized into $\frac{1}{(4\pi t)^{n/2}} \exp(-r^2/4t)$ where r is the geodesic distance. In order to construct the heat kernel k^H_t we consider the map s_t which is defined by $s_t = (4\pi t)^{n/2} \exp(r^2/4t) \cdot k^H_t$. This map measures the difference between the 'Euclidian' heat kernel and the heat kernel we are interested in. We show that if s_t is a formal power series $\sum_i t^i \Phi_i$, then s_t has a unique solution.

In this section we work with the following setup: Let (M, g) be an n dimensional Riemannian manifold and let $y \in M$. Consider a neighborhood U_y such that the map $\exp_y^{-1} \colon U_y \to T_y M$ is a chart of M. Let $E \to U_y$ be a vector bundle with a positive definite inner product and let H be a generalized Laplacian on E. The "Euclidian" heat kernel we denote by

$$q_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp(-d(x, y)^2/4t) \qquad \forall x \in U_y \tag{4.5}$$

where $d(x, y)$ is the geodesic distance between x and y.

Definition 4.9. *A formal power series $k^H_t \in \Gamma(E \boxtimes E)$ of the form*

$$k^H_t(x, y) = q_t(x, y) \sum_{i=0}^{\infty} t^i \Phi_i(x, y)$$

is a **formal solution of the heat equation** *if*

$$\left(\frac{\partial}{\partial t} + H_x\right) k_t^H = 0$$

If we let $k_t^H = q_t \cdot s_t$ be a candidate for the heat kernel, then s_t must satisfy a differential equation. In the next lemmas we make this explicit.

Lemma 4.10 (Roe [1998], Lemma 7.12 and 7.13)**.** *Let $s_t \in \Gamma(E \to U \times R_+) \otimes E_y^*$ be a time-dependent section on E and let $q_t(\cdot, y)\colon U \times \mathbb{R}_+ \to \mathbb{R}$ be the map defined in equation 4.5. Then*

$$H(q_t \cdot s_t) - q_t H s_t = (\Delta_x q_t) s_t - 2\nabla_{\nabla q_t} s_t.$$

Proof. By definition of the generalized Laplacian, the only non-commuting part of H is the Laplacian. In a Riemannian normal coordinate system $\{x_\mu\}$ it is given by $-\sum_\mu \nabla_\mu \nabla_\mu$. In this coordinate system the commutator between H and q_t is given by

$$[H, q_t] s_t = -\sum_\mu [\nabla_\mu \nabla_\mu, q_t] s_t.$$

Using the properties of the commutator this can be written as

$$\begin{aligned} [H, q_t] s_t &= -\sum_\mu \nabla_\mu [\nabla_\mu, q_t] s_t + [\nabla_\mu, q_t] \nabla_\mu s_t \\ &= -\sum_\mu [\nabla_\mu, [\nabla_\mu, q_t]] s_t + 2[\nabla_\mu, q_t] \nabla_\mu s_t. \end{aligned}$$

According to the Leibniz rule, the commutator between a connection and a smooth function is the Lie derivative and hence

$$[H, q_t] s_t = -\sum_\mu (\mathcal{L}_\mu \mathcal{L}_\mu q_t) s_t + 2(\mathcal{L}_\mu q_t) \nabla_\mu s_t.$$

Recall that $-\sum_\mu (\mathcal{L}_\mu \mathcal{L}_\mu q_t)$ is the Laplacian of q_t. We only need to show that $\sum_\mu \mathcal{L}_\mu q_t \frac{\partial}{\partial x^\mu}$ is the gradient of q_t. Indeed, the gradient is the dual of the exterior derivative. The dual of $\sum_\mu \mathcal{L}_\mu q_t \frac{\partial}{\partial x^\mu}$ equals $\sum_\mu \mathcal{L}_\mu q_t \, \mathrm{d}\, x^\mu$. This is the exterior derivative of q_t in local coordinates. ☐

We need to calculate the Laplacian and the gradient of q_t in a suitable coordinate system. Namely, take the Riemannian normal coordinate frame and pick the polar coordinates $\{r, \phi_1, \ldots, \phi_{n-1}\}$ on T_yM. In this coordinate frame the map q_t reduces to

$$q_t = \frac{1}{(4\pi t)^{n/2}} \exp(-r^2/4t).$$

For this calculation we also need the Hodge dual[12], because the Laplacian can be written as $-*\mathrm{d}*\mathrm{d}$. The next lemma calculates $*\mathrm{d}\,r$ and after this we calculate Δq_t and ∇q_t.

Lemma 4.11. *Let $\{r, \phi_1, \ldots \phi_{n-1}\}$ be the polar coordinates on T_yM and use the exponential map as a coordinate frame around y. Then,*

$$*\mathrm{d}\,r = r^{n-1}\sqrt{g}\,\mathrm{d}\,\phi_1 \wedge \ldots \wedge \mathrm{d}\,\phi_{n-1}$$

where g is the determinant of the metric.

Proof. First we show that $*\mathrm{d}\,r$ is a multiple of $\mathrm{d}\,\phi_1 \wedge \ldots \mathrm{d}\,\phi_{n-1}$. For simplicity we write $\mathrm{d}\,\phi_1 \wedge \ldots \mathrm{d}\,\phi_{n-1} = \mathrm{d}\,\Omega$. Indeed, $\mathrm{d}\,r$ can be expanded into

$$*\mathrm{d}\,r = c_r\,\mathrm{d}\,\Omega + \sum_i c_i\,\mathrm{d}\,r \wedge \mathrm{d}\,\phi_1 \wedge \ldots \mathrm{d}\,\phi_{i-1} \wedge \mathrm{d}\,\phi_{i+1} \wedge \ldots \mathrm{d}\,\phi_{n-1}$$

where $c_r, c_i \in \mathbb{R}$ are the components of the vector field. We calculate $\mathrm{d}\,\phi_k \wedge *\mathrm{d}\,r$ in this local basis and using the Hodge dual. Comparing them gives

$$\mathrm{d}\,\phi_k \wedge *\mathrm{d}\,r = (-1)^{k+1} c_k\,\mathrm{d}\,r \wedge \mathrm{d}\,\Omega = \langle \mathrm{d}\,\phi_k, \mathrm{d}\,r\rangle\,\mathrm{Vol}(g).$$

By Gauss lemma it follows that $\langle \mathrm{d}\,\phi_k, \mathrm{d}\,r\rangle = 0$ in a local neighborhood around the origin and so $c_k = 0$ for all k. This shows that $*\mathrm{d}\,r = c_r\,\mathrm{d}\,\Omega$.

We explicitly calculate c_r by computing $\mathrm{d}\,r \wedge *\mathrm{d}\,r$. Because $\left\|\frac{\partial}{\partial r}\right\|_g = 1$ it follows that

$$\mathrm{d}\,r \wedge *\mathrm{d}\,r = c_r\,\mathrm{d}\,r \wedge \mathrm{d}\,\Omega = \langle \mathrm{d}\,r, \mathrm{d}\,r\rangle\,\mathrm{Vol}(g) = \mathrm{Vol}(g).$$

For polar coordinates the volume form is given by $r^{n-1}\sqrt{q}\,\mathrm{d}\,r \wedge \mathrm{d}\,\Omega$. Hence we conclude that $*\mathrm{d}\,r = r^{n-1}\sqrt{q}\,\mathrm{d}\,\Omega$. ☐

[12] Recall that the Hodge dual $*$ is the unique operator which is defined by the property $\alpha \wedge *\beta = \langle \alpha, \beta\rangle\,\mathrm{Vol}(g)$ for all $\alpha, \beta \in \Omega^\bullet(M)$.

Lemma 4.12. *In the coordinates defined in Lemma 4.11 the gradient of q_t is*

$$\nabla q_t = -q_t \frac{r}{2t}\frac{\partial}{\partial r}$$

and the Laplacian of q_t is

$$\Delta q_t = q_t \left(-\frac{r^2}{4t^2} + \frac{n}{2t} + \frac{r}{4gt}\frac{\partial g}{\partial r} \right)$$

where g is the determinant of the metric.

Proof. The gradient is the dual of the exterior derivative and so

$$\nabla q_t = (\mathrm{d}\, q_t)^\flat = \left(-\frac{q_t}{2t} r \,\mathrm{d}\, r \right)^\flat = -q_t \frac{r}{2t}\frac{\partial}{\partial r}.$$

For the Laplacian we need to calculate $- * d * d\, q_t$. From Lemma 4.11, we know that

$$* \,\mathrm{d}\, q_t = -\frac{q_t}{2t} r * \mathrm{d}\, r = -q_t \frac{r^n}{2t}\sqrt{g}\,\mathrm{d}\,\phi_1 \wedge \ldots \wedge \mathrm{d}\,\phi_{n-1}.$$

The Laplacian can now be easily calculated

$$\Delta q_t = - * q_t \left(\frac{r^2}{4t^2} - \frac{n}{2t} - \frac{r}{4gt}\frac{\partial g}{\partial r} \right) r^{n-1}\sqrt{g}\,\mathrm{d}\, r \wedge \mathrm{d}\,\phi_1 \wedge \ldots \wedge \mathrm{d}\,\phi_{n-1}$$

and this proves the result. □

Theorem 4.13 (Roe [1998], Theorem 7.15)**.** *For any generalized Laplacian H, there exists a unique formal solution of the heat equation of the form*

$$k_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-d(x,y)^2/4t} \sum_{i=0}^{\infty} t^i \Phi_i(x,y)$$

such that $\Phi_0(y,y)$ is the identity map on E_y.

Proof. Denote $s_t = \sum_i t^i \Phi_i$. The kernel $k_t^H = q_t s_t$ must satisfy the heat equation. By Lemmas 4.10 - 4.12 the heat equation applied on k_t^H equals

$$\begin{aligned}\left(\frac{\partial}{\partial t} + H\right) k_t^H =& \frac{\partial}{\partial t}(q_t \cdot s_t) + q_t H k_t + (\Delta q_t) s_t - 2\nabla_{\nabla q_t} s_t = 0 \\ =& \frac{\partial q_t}{\partial t} s_t + q_t \left(\frac{\partial}{\partial t} + H - \frac{r^2}{4t^2} + \frac{n}{2t} + \frac{r}{4gt}\frac{\partial g}{\partial r} - 2\frac{1}{t}\nabla_{r\partial/\partial r}\right) s_t.\end{aligned}$$

The t-derivative of q_t can be easily calculated and it equals

$$\frac{\partial}{\partial t} q_t = q_t \left(-\frac{n}{2t} + \frac{r^2}{4t^2}\right)$$

and so the heat equation simplifies to

$$\left(\frac{\partial}{\partial t} + H\right) k_t^H = q_t \left(\frac{\partial}{\partial t} + H + \frac{r}{4gt}\frac{\partial g}{\partial r} - 2\frac{1}{t}\nabla_{r\partial/\partial r}\right) s_t = 0.$$

This induces a differential equation for s_t. By expanding s_t into $\sum_i \Phi_i$ we get a differential equation for each factor of t. We get the system of equations

$$\left(\frac{r}{4g}\frac{\partial g}{\partial r} + \nabla_{r\partial/\partial r}\right) \Phi_0 = 0 \tag{4.6}$$

$$\left(i + \frac{r}{4g}\frac{\partial g}{\partial r} + \nabla_{r\partial/\partial r}\right) \Phi_i = - H\Phi_{i-1} \qquad \forall i > 0. \tag{4.7}$$

This can be simplified into

$$\nabla_{\partial/\partial r}\left(g^{1/4}\Phi_0\right) = 0 \tag{4.8}$$

$$\nabla_{\partial/\partial r}\left(r^i g^{1/4}\Phi_i\right) = - r^{i-1} g^{1/4} H \Phi_{i-1} \qquad \forall i > 0. \tag{4.9}$$

Equation 4.8 and 4.9 are first order differential equations. Both are uniquely determined by its initial value at the origin. We set the integration constant for Φ_0 such that $\Phi_0(y, y)$ is the identity map on E_y. Note that $r^i g^{1/4}\Phi_i$ will be of order r^i if and only if the integration constant is set to zero. So the requirement that Φ_i is smooth determines Φ_i uniquely. □

4.5 The existence of the heat kernel

We have seen that for any generalized Laplacian has a unique formal solution of the heat equation. However we do not know if this formal solution is globally defined and we do not know if the power series converges. In this section we construct the heat kernel by considering globally defined approximations to the formal solution. These approximations are not necessary smooth, but we require them to be l times continuous differentiable for some $l \in \mathbb{N}$. In this paragraph we denote the space of l-times differentiable sections on a vector bundle $E \to M$ as $\Gamma^l(E)$. We often use the norm

$$\|s\|_l(x) = \sup_{k \leq l} \sup_{v \in T_x M} \|(\mathcal{L}_v)^k s\|.$$

The first approximation to $k_t^H = q_t \sum_i t^i \Phi_i$ we consider is the partial sum multiplied by a bump function. The next proposition states some properties this approximation has.

In this section we work with the following setup: Let $E \to (M, g)$ be a vector bundle with a positive definite inner product over an oriented n dimensional *compact* Riemannian manifold, let H be a generalized Laplacian and let $k_t^H = \sum_i t^i \Phi_i$ be the formal solution of the heat equation. Let $y \in M$ and consider a neighborhood U_y such that the map $\exp_y^{-1} \colon U_y \to T_y M$ is a chart of M.

Proposition 4.14 (Berline et al. [2004], Theorem 2.20)**.** *For a small enough $\epsilon > 0$ pick a smooth map $\psi \colon \mathbb{R} \to \mathbb{R}$ such that*

$$\psi(x) = \begin{cases} 1 & \text{if} \quad x < \epsilon^2/4 \\ 0 & \text{if} \quad x > \epsilon^2 \end{cases}.$$

Then $k_t^{H,N}(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-d(x,y)^2/4t} \cdot \psi(d^2(x,y)) \sum_{i=0}^N t^i \Phi_i(x,y)$ is a smooth family of sections of $E \boxtimes E \to M \times M$ for which the following holds:

1. *The smoothing operators $K_t^{H,N}$ which has $k_t^{H,N}$ as their kernel, form a uniform bounded family of operators on $\Gamma^l(E)$ for all $0 \leq t \leq T$ for all T.*
2. *For all $l \in \mathbb{N}$ and $s \in \Gamma^l(E)$ the norm $\|K_t^{H,N} s - s\|_l$ tends to zero when t tends to zero.*
3. *For all $l \in \mathbb{N}$, there exists a constant $C \in \mathbb{R}$ such that the kernel satisfies the estimate*

$$\left\| \left(\frac{\partial}{\partial t} + H \right) k_t^{H,N}(x,y) \right\| \leq C t^{N - \frac{n+l}{2}}.$$

Proof. Pick $\epsilon > 0$ such that the ball of radius ϵ centered at y lies inside U_y. For this ϵ the map $k_t^{H,N}$ can be globally extended on $M \times M$ by zero. Hence $k_t^{H,N}$ is a smooth family of sections of $E \boxtimes E$. The corresponding smoothing operator equals

$$\left(K_t^{H,N} s\right)(x) = \frac{1}{(4\pi t)^{n/2}} \int_{y \in M} e^{-\frac{d(x,y)^2}{4t}} \psi(d(x,y)^2) \sum_{i=0}^{N} t^i \Phi_i(x,y) s(y) \operatorname{Vol}(g).$$

The integrand is almost everywhere zero, except in the neighborhood of $x \in M$. For this neighborhood we pick the local coordinates $y = \exp_x \mathbf{y}$ and we have

$$\left(K_t^{H,N} s\right)(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbf{y} \in T_x M} e^{-\frac{\|\mathbf{y}\|^2}{4t}} \sum_{i=0}^{N} t^i \Psi_i(x, \vec{y}) s(\exp_x(\mathbf{y})) \, \mathrm{d}^n \mathbf{y}$$

for some compactly supported $\Psi_i \in \Gamma(E \boxtimes E)$. It follows from applying the vector space transformation $\mathbf{y} = t^{1/2} v$ that

$$\left(K_t^{H,N} s\right)(x) = (4\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\|v\|^2/4} \sum_{i=0}^{N} t^i \Psi_i(x, t^{1/2} v) s(\exp_x(t^{1/2} v)) \, \mathrm{d}^n v. \quad (4.10)$$

This is bounded for all t. Because $[0, T]$ is compact for all $T > 0$, we conclude that K_t^N is uniformly bounded on $[0, T]$. This proves the first part.

In the limit $t \to 0$ equation 4.10 equals

$$\begin{aligned} \left(K_0^{H,N} s\right)(x) =& (4\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\|v\|^2/4} \Psi_0(x, 0) s(\exp_x(0)) \, \mathrm{d}^n v \\ =& (4\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\|v\|^2/4} \psi(d(x,x)^2) \Psi_0(x,x) s(x) \, \mathrm{d}^n v. \end{aligned} \quad (4.11)$$

Clearly, the distance between x and itself is zero and hence $\psi(d(x,x)^2) = 1$. Recall that $\Psi_0(x,x)$ is the identity operator on E_x. So equation 4.11 simplifies to

$$\left(K_t^{H,N} s\right)(x) = s(x) \cdot (4\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\|v\|^2/4} \, \mathrm{d}^n v.$$

By comparing this result to the Euclidean heat kernel we conclude that $\lim_{t \to 0} K_t^{H,N} s = s$. This shows the second part.

Finally we estimate $\left\|\left(\frac{\partial}{\partial t}+H\right)k_t^{H,N}(x,y)\right\|$. For simplicity we write $r_t^N=\left(\frac{\partial}{\partial t}+H\right)k_t^{H,N}(x,y)$. From lemmas Lemmas 4.10 - 4.12, we know that

$$\begin{aligned}
r_t^N(x,y)=&q_t\left(\frac{\partial}{\partial t}+H+\frac{1}{t}\nabla_{r\partial/\partial r}+\frac{r}{4gt}\frac{\partial g}{\partial r}\right)\psi(d(x,y)^2)\sum_{i=0}^{N}t^i\Phi_i(x,y)\\
=&q_t\psi(d(x,y)^2)\left(\frac{\partial}{\partial t}+H+\frac{1}{t}\nabla_{r\partial/\partial r}+\frac{r}{4gt}\frac{\partial g}{\partial r}\right)\sum_{i=0}^{N}t^i\Phi_i(x,y)+\\
&+q_t\left(\Delta\psi(r^2)-2\nabla_{\nabla\psi(r^2)}+\frac{r}{t}\frac{\partial\psi(r^2)}{\partial r}\right)\sum_{i=0}^{N}t^i\Phi_i(x,y).
\end{aligned}$$

The kernels Φ_i will cancel most terms. However, we are left with

$$\begin{aligned}
r_t^N(x,y)=&q_t\psi(d(x,y)^2)t^NH\Phi_N(x,y)+\\
&+q_t\left(\Delta\psi(r^2)-2\nabla_{\nabla\psi(r^2)}+\frac{r}{t}\frac{\partial\psi(r^2)}{\partial r}\right)\sum_{i=0}^{N}t^i\Phi_i(x,y).
\end{aligned}$$

The first terms is of order $t^{N-n/2}$. We show that this is true for

$$q_t\left(\Delta\psi(r^2)-2\nabla_{\nabla\psi(r^2)}+\frac{r}{t}\frac{\partial\psi(r^2)}{\partial r}\right)\sum_{i=0}^{N}t^i\Phi_i(x,y). \tag{4.12}$$

This vanishes if $d(x,y)<\epsilon$, because ψ is constant in this area. If this term is zero, then it is of order $t^{N-n/2}$. So we only need to show that equation 4.12 is of the right degree when $d(x,y)>\epsilon$.

If $d(x,y)>\epsilon$, then $\|t^{-N+\frac{n}{2}}q_t(x)\|_0$ is bounded by $\|t^{-N+\frac{n}{2}}q_t(\epsilon)\|_0$. This follows from the fact that q_t is a decreasing map. For $t>1$ the norm $\|t^{-N+\frac{n}{2}}q_t(x)\|_0$ is bounded by one. Because $t^{-N+\frac{n}{2}}q_t(\epsilon)$ is continuous in t and $[0,1]$ is compact follows that $t^{-N+\frac{n}{2}}q_t(\epsilon)$ has an upper bound. So r_t^N is bounded in the supremum norm by $t^{N-\frac{n}{2}}$.

To consider higher order derivatives of $r_t^N(x,y)$ in x, we note that we can only lower the degree of t by differentiating over q_t. Because $\frac{\partial}{\partial x}q_t=\mathcal{O}(t^{-1/2})$, we conclude that $\|r_t^N\|_l<Ct^{N-\frac{n+l}{2}}$. □

New kernels can be constructed using old kernels. Indeed let $p,q\in\Gamma(E\boxtimes E)$ be two kernels. The composition $\int_{z\in M}p(x,z)q(z,y)\operatorname{Vol}(g)$ is a map from E_y to E_x.

Therefore it is a section on $E \boxtimes E$. For smooth families of kernels $p_t, q_t \in \Gamma(E \boxtimes E \to M \times M \times \mathbb{R}+)$ we consider the composition

$$(x, y, t) \mapsto \int_0^t \mathrm{d}\, s \int_{z \in M} p_{t-s}(x, z) q_s(z, y) \operatorname{Vol}(g).$$

This is also a section of $\Gamma(E \boxtimes E \to M \times M \times \mathbb{R}+)$. As our second approximation attempt we consider compositions of $k_t^{H,N}$ and $\left(\frac{\partial}{\partial t} + H_x\right) k_t^{H,N}$. To simplify our notation we inductively define

$$r_t^{H,N,1}(x, y) = \left(\frac{\partial}{\partial t} + H_x\right) k_t^{H,N} \tag{4.13}$$

$$r_t^{H,N,m+1}(x, y) = \int_0^t \mathrm{d}\, s \int_{z \in M} r_{t-s}^{H,N,1}(x, z) r_s^{H,N,m}(z, y) \operatorname{Vol}(g) \tag{4.14}$$

$$k_t^{H,N,0}(x, y) = k_t^{H,N}(x, y) \tag{4.15}$$

$$k_t^{H,N,m}(x, y) = \int_0^t \mathrm{d}\, s \int_{z \in M} k_{t-s}^{H,N}(x, z) r_s^{H,N,m}(z, y) \operatorname{Vol}(g). \tag{4.16}$$

In the following lemma we give an estimation of $k_t^{H,N,m}$ and $r_t^{H,N,m}$.

Lemma 4.15 (Berline et al. [2004], Lemma 2.21 and 2.22(1))**.** *Let $k_t^{H,N}$ be the family of kernels defined in Proposition 4.14. Then for all $l, m \in \mathbb{N}$ and $N > \frac{n+l}{2}$ the kernel $r_t^{H,N,m+1}(x, y)$, which is defined in equation 4.13 and 4.14, is l-times continuous differentiable with respect to x and y and satisfies the estimate*

$$\|r_t^{H,N,m+1}\|_l \leq C^{m+1} t^{(m+1)(N - \frac{n+l}{2})} \operatorname{Vol}(M)^m \frac{t^m}{m!} \tag{4.17}$$

for some $C \in \mathbb{R}$. The kernel $k_t^{H,N,m}$, which is defined in equation 4.15 and 4.16, is also l-times continuous differentiable with respect to x and y and satisfies the estimate

$$\|k_t^{H,N,m+1}\|_l \leq \tilde{C} C^{m+1} t^{(m+1)(N - \frac{n+l}{2})} \operatorname{Vol}(M)^m \frac{t^m}{m!}.$$

for some $\tilde{C} \in \mathbb{R}$.

Proof. From Theorem 4.14 follows that $r_t^{H,N,1}$ is bounded by t. Hence we extend $r_t^{H,N,1}$ (and its derivatives) continuously to $t = 0$ and so $r_t^{H,N,1}$ has a well defined

l-norm. This is equivalent to the fact that it is l-times continuously differentiable.

We show the first estimate using induction. From Definition 4.13 and Theorem 4.14 it follows that

$$\|r_t^{H,N,0+1}\|_l \leq Ct^{N-\frac{n+l}{2}}.$$

This coincides with equation 4.17 for $m = 0$. Now assume that equation 4.17 holds for some $m \in \mathbb{N}$. Then $r_t^{H,N,m+1}$ is estimated by

$$\left\|r_t^{H,N,m+1}(x,y)\right\|_l \leq \int_0^t \mathrm{d}\, s \int_{z\in M} \left\|r_{t-s}^{H,N,1}(x,z)\right\|_l \cdot \left\|r_s^{H,N,m}(x,z)\right\|_l.$$

From the induction hypothesis it follows that

$$\left\|r_t^{H,N,m+1}(x,y)\right\|_l \leq \int_0^t \mathrm{d}\, s \int_{z\in M} C(t-s)^{N-\frac{n+l}{2}} \cdot C^m s^{m(N-\frac{n+l}{2})} \operatorname{Vol}(M)^{m-1} \frac{s^{m-1}}{m-1!}$$

and this simplifies to

$$\begin{aligned}\left\|r_t^{H,N,m+1}(x,y)\right\|_l \leq & C^{m+1} \operatorname{Vol}(M)^m \int_0^t \mathrm{d}\, s(t-s)^{N-\frac{n+l}{2}} s^{m(N-\frac{n+l}{2})} \frac{s^{m-1}}{m-1!} \\ \leq & C^{m+1} t^{(m+1)(N-\frac{n+l}{2})} \operatorname{Vol}(M)^m \int_0^t \mathrm{d}\, s \frac{s^{m-1}}{m-1!}.\end{aligned}$$

Integrating $\int_0^t \mathrm{d}\, s \frac{s^{m-1}}{m-1!}$ yields a factor $\frac{t^m}{m!}$ and this proves that equation 4.17 is satisfied for $m+1$. By induction we conclude that it is satisfied for all $m \in \mathbb{N}$.

From Theorem 4.14 it follows that the smoothing operators w.r.t. k_t^N are uniform bounded on $0 \leq t \leq T$ for all T. Hence for all $s \in [0,t]$ and $x, y \in M$

$$\left\|\int_{z\in M} k_{t-s}^{H,N}(x,z) r_s^{H,N,m}(z,y) \operatorname{Vol}(g)\right\|_l \leq \tilde{C} \|r_s^{H,N,m}(x,y)\|_l$$

for some $\tilde{C} \in \mathbb{R}$. Notice that the left hand side is a norm estimate of $k_t^{H,N,m}$. From this and equation 4.17 we conclude the result. $\square$

The kernels $r_t^{H,N,m}$ and $k_t^{H,N,m}$ are bounded by $\frac{(C \operatorname{Vol}(M) t^{N-\frac{n+l}{2}})^m}{m!}$. Hence we can use them for constructing convergent power series, because we can bound these series by $\exp(C \operatorname{Vol}(M) t^{N-\frac{n+l}{2}})$. To find which series is the heat kernel, we need to investigate how the heat equation behaves for $k_t^{H,N,m}$:

Lemma 4.16 (Berline et al. [2004], Lemma 2.22(2)). *For all $l, m \in \mathbb{N}$ and $N > \frac{n+l}{2}$ the kernel $k_t^{H,N,m}(x,y)$, which is defined in equation 4.15 and 4.16, satisfies*

$$\left(\frac{\partial}{\partial t} + H_x\right) k_t^{H,N,m} = r_t^{H,N,m+1} + r_t^{H,N,m}.$$

Proof. Consider the map $b(t,s,x,y) = \int_{z\in M} k_{t-s}^{H,N}(x,z) r_s^{H,N,m}(z,y)\,\mathrm{Vol}(g)$. The integral $\int_0^t b(t,s,x,y)\,\mathrm{d}\,s$ equals the map $k_t^{H,N,m}$. The map $b(t,s,x,y)$ is continuous in $s \in [0,t]$, because the smoothing operator K_t^N is uniform bounded. So the heat equation applied on $k_t^{H,N,m}$ equals

$$\begin{aligned}\left(\frac{\partial}{\partial t} + H_x\right) k_t^{H,N,m} =& \left(\frac{\partial}{\partial t} + H_x\right) \int_0^t b(t,s,x,y)\,\mathrm{d}\,s \\ =& b(t,t,x,y) - b(t,0,x,y) + \int_0^t \left(\frac{\partial}{\partial t} + H_x\right) b(t,s,x,y)\,\mathrm{d}\,s.\end{aligned} \tag{4.18}$$

From Theorem 4.14 it follows that $b(t,0,x,y) = \int_{z\in M} k_t^{H,N}(x,z) r_0^{H,N,m}(z,y)\,\mathrm{Vol}(g) = 0$. Also, $b(t,t,x,y) = \int_{z\in M} k_0^{H,N}(x,z) r_t^{H,N,m}(z,y)\,\mathrm{Vol}(g) = r_t^{H,N,m}(x,y)$. Equation 4.18 therefore becomes

$$\left(\frac{\partial}{\partial t} + H_x\right) k_t^{H,N,m} = r_t^{H,N,m}(x,y) + \int_0^t \left(\frac{\partial}{\partial t} + H_x\right) b(t,s,x,y)\,\mathrm{d}\,s.$$

We calculate the heat equation acted on $b(t,s,x,y)$. By definition it equals:

$$\left(\frac{\partial}{\partial t} + H_x\right) b(t,s,x,y) = \left(\frac{\partial}{\partial t} + H_x\right) \int_{z\in M} k_{t-s}^{H,N}(x,z) r_s^{H,N,m}(z,y)\,\mathrm{Vol}(g)$$

The heat equation acts only on terms which depend on t and x and so

$$\begin{aligned}\left(\frac{\partial}{\partial t} + H_x\right) b(t,s,x,y) &= \int_{z\in M} \left(\left(\frac{\partial}{\partial t} + H_x\right) k_{t-s}^{H,N}(x,z)\right) r_s^{H,N,m}(z,y)\,\mathrm{Vol}(g) \\ &= \int_{z\in M} r_{t-s}^{H,N,1}(x,z) r_s^{H,N,m}(z,y)\,\mathrm{Vol}(g).\end{aligned}$$

By equation 4.14 it follows that

$$\left(\frac{\partial}{\partial t} + H_x\right) k_t^{H,N,m} = r_t^{H,N,m}(x,y) + r_t^{H,N,m+1}(x,y)$$

and we finish the proof. ∎

Assume that the heat kernel is of the form $\sum_{m=0}^{\infty} c_m k_t^{H,N,m}$. We now search for which values of $c_m \in \mathbb{R}$ the heat equation is satisfied. From the previous lemma it follows that

$$\begin{aligned}\left(\frac{\partial}{\partial t} + H\right)\sum_{m=0}^{\infty} c_m k_t^{H,N,m} &= \sum_m c_m(r_t^{H,N,m} + r_t^{H,N,m+1}) \\ &= \sum_m r_t^{H,N,m}(c_k + c_{m-1}).\end{aligned}$$

The alternating sum of $k_t^{H,N,m}$ will satisfy the heat equation. We show that it is indeed the heat kernel.

Theorem 4.17 (Berline et al. [2004], Theorem 2.23)**.** *Let (M, g) be a compact Riemannian manifold of dimension n and let $E \to M$ be a vector bundle with a positive definite inner product. Let H be a generalized Laplacian and let k_t be the formal solution to the heat kernel. Then the following is true:*

1. *For any $N \in \mathbb{N}$ and $l \in \mathbb{N}$ such that $N > \frac{n+l+1}{2}$, the series*

$$p_t^{H,N}(x, y) = \sum_{m=0}^{\infty}(-1)^m k_t^{H,N,m}(x, y)$$

converges in the $\|\cdot\|_{l+1}$-norm over $M \times M$ and it is continuous differentiable in t. It satisfies the heat equation.

2. *The kernel $k_t^{H,N} \in \Gamma(E \boxtimes E)$ approximates p_t in the sense that*

$$\left\|\frac{\partial^m}{\partial t^m}(p_t^{H,N} - k_t^{H,N})\right\|_l = \mathcal{O}(t^{N-\frac{n+l}{2}-m+1})$$

for all $m \in \mathbb{N}$ and $N > \frac{n+l+1}{2}$ when t approaches zero.

3. *The kernel $p_t^H = p_t^{H,n+1}$ is a heat kernel for the operator H.*

Proof. In the above discussion we showed that p_t converges because it can be estimated using the exponential series. Therefore the kernel p_t converges in the $l+1$-norm. From Lemma 4.16 it follows that

$$\frac{\partial}{\partial t}k_t^{H,N,m} = r_t^{H,N,m+1} + r_t^{H,N,m} - H_x k_t^{H,N,m}$$

and we estimate

$$\begin{aligned}\left\|\frac{\partial}{\partial t}p_t\right\|_l &= \lim_{M\to\infty}\left\|\sum_{m=0}^{M}(-1)^m(r_t^{H,N,m+1}+r_t^{H,N,m}-H_x m_t^{H,N,m})\right\|_l\\ &= \lim_{M\to\infty}\left\|(-1)^M r_t^{H,N,M+1}-H_x p_t\right\|_l.\end{aligned}$$

This converges for all t and so p_t is continuous differentiable in the t component. It also satisfies the heat equation. This prove the first part.

To show the second part, recall that $q_t^{H,N,0}=k_t^{H,N}$. From Lemma 4.15 we estimate

$$\|p_t-k_t^N\|_l=\left\|\sum_{j=1}^{\infty}(-1)^j q_t^{j,N}\right\|_l=\mathcal{O}(t^{N-\frac{n+l}{2}+1}).$$

The second part of the theorem follows after m-times differentiation.

Before we show that p_t^H is a heat kernel, we prove that $p_t^{H,N}$ is a C^l-heat kernel if $N>\frac{n+l+1}{2}$. We only need to show the boundary condition

$$\lim_{t\to 0}\int_{y\in M}p_t^{H,N}(x,y)s(y)=s(x)\qquad \text{for all } s\in\Gamma^l(E).$$

From the second part of this theorem we know that $\lim_{t\to0}\|\int_{y\in M}p_t^{H,N}(x,y)s(y)-s(x)\|_l=\lim_{t\to0}\|\int_{y\in M}k_t^{H,N}(x,y)s(y)-s(x)\|$. By Theorem 4.14 is equals zero. Hence, $p_t^{H,N}$ is a C^l-heat kernel w.r.t. H.

Finally notice that $\Gamma^l(E)\subseteq\Gamma^0(E)$. Hence, $p_t^{H,N}$ and $p_t^{H,n+1}$ are both C^0-heat kernels. By unicity of the heat kernel it follows that they are equal. Hence, $p_t^{H,n+1}$ is the smooth heat kernel w.r.t. H. □

5 Traces and the Index of a Dirac operator

In chapter 3 we saw how the Fujikawa method relates the chiral anomaly to the index of a Dirac operator. Although we calculated the trace over $\gamma_5 e^{-tD^2}$ we did not check whether it converges. We also never checked if the trace was independent of basis. Now we rigorously introduce a class of operators for which the trace is well defined and show that the heat kernel is in this class.

Because the trace is the sum of the eigenvalues, we retrieve some properties of the eigenvalues of the heat kernel. Using the heat equation we get that a generalized Laplacian has countably many eigenvalues. Using this spectrum analysis we can prove McKean-Singer formula. It relates the index of a Dirac operator with the trace of the heat kernel.

Next we investigate how the trace interacts with the Clifford action. We prove the trace identities which we used in chapter 2 and we generalize them to higher dimensions.

At last we study the topological properties of traces. We study the trace over differential forms and we show that they characterize vector bundles. We introduce the characteristic classes needed for the Atiyah-Singer index theorem.

The theory of traceclass operators is standard. For more information see Murphy [1990].

5.1 Traceclass operators

Definition 5.1. *A linear operator A on a Hilbert space $\mathcal{H}$ is called* **Hilbert-Schmidt** *if $\sum_i \|Ae_i\|^2$ is finite for all orthonormal basis $\{e_i\}$.*

Hilbert-Schmidt operators are bounded operators. Indeed let $\mathcal{H}$ be a Hilbert space, A a Hilbert-Schmidt operator and let $v \in \mathcal{H}$. In an orthonormal basis $\{e^i\}$ of H we can write v as $\sum_i v_i e^i$. Let v_{max} be the largest component of v. That is, $v_{max} = \max_i |v_i|$. We can estimate

$$\frac{\|Av\|^2}{\|v\|^2} \leq \sum_i \frac{|v_i|^2}{\|v\|^2} \cdot \|Ae^i\|^2 \leq \frac{v_{max}^2}{\|v\|^2} \sum_i \|Ae^i\|^2 \leq \sum_i \|Ae^i\|^2$$

and so the operator norm is bounded by

$$\|A\|_{op}^2 = \sup_{v\in\mathcal{H}} \frac{\|Av\|^2}{\|v\|^2} \leq \sup_{v\in\mathcal{H}} \sum_i \|Ae^i\|^2 = \sum_i \|Ae^i\|^2. \tag{5.1}$$

By definition $\sum_i \|Ae^i\|^2$ is finite and hence A is bounded.

Linear combinations of Hilbert-Schmidt operators are also Hilbert-Schmidt. To see this let A and B be Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$, let $\lambda \in \mathbb{C}$ and let $\{e^i\}$ be an orthonormal basis of $\mathcal{H}$. For all $N \in N$ we estimate

$$\sum_{i=0}^{N} \|(A+\lambda B)e^i\|^2 \leq \sum_{i=0}^{N} \|Ae^i\|^2 + |\lambda|^2 \cdot \sum_{i=0}^{N} \|Be^i\|^2$$

and in the limit $N \to \infty$ we conclude that $\sum_{i=0}^{N} \|(A+\lambda B)e^i\|^2$ is finite.

The product between a Bounded operator B and a Hilbert-Schmidt operator A is also Hilbert-Schmidt. This follows from the estimate

$$\sum_i \|BAe^i\|^2 \leq \|B\|_{op}^2 \cdot \sum_i \|Ae^i\|^2.$$

This shows that the space of Hilbert-Schmidt operators is an ideal in the space of Bounded operators.

Lemma 5.2. *Let $\mathcal{H}$ be a Hilbert space. Then for all orthonormal frames $\{e^i\}$ of $\mathcal{H}$ the map $A \mapsto \sqrt{\sum_i \|Ae^i\|^2}$ is a norm on the space of Hilbert-Schmidt operators.*

Proof. Let A be a Hilbert-Schmidt operator. Clearly, $\sum_i \|Ae^i\|^2 \geq 0$ and so this norm is positive. It is also definite. In the case that $\sum_i \|Ae^i\|^2 = 0$, then by Equation 5.1 the operator norm of A is zero. Hence the norm is positive definite.

This map is also absolutely scalable: For all $N \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ the finite sum $\sum_{i=0}^{N} \|\lambda\, Ae^I\|^2$ equals $|\lambda|^2 \sum_{i=0}^{N} \|Ae^i\|^2$. Absolutely scalability follows if we take the limit $N \to \infty$.

Finally we show that it satisfies the triangle inequality. Let A and B be Hilbert-Schmidt operators. Then for all $N \in \mathbb{N}$ these operators satisfy

$$\sum_{i=0}^{N} \|(A+B)e^i\|^2 \leq \sum_{i=0}^{N} \|Ae^i\|^2 + \sum_{i=0}^{N} \|Be^i\|^2 \leq \sum_{i=0}^{\infty} \|Ae^i\|^2 + \sum_{i=0}^{\infty} \|Be^i\|^2.$$

The triangle inequality follows if we take $N \to \infty$. □

Lemma 5.3. *If A is a Hilbert-Schmidt operator on a Hilbert space $\mathcal{H}$ then the following is true:*

1. *The adjoint A^* is a Hilbert-Schmidt operator.*
2. *The norm $\sqrt{\sum_i \|A^* e^i\|^2}$ equals $\sqrt{\sum_i \|Ae^i\|^2}$.*
3. *The norm $\sqrt{\sum_i \|Ae^i\|^2}$ does not depend on the choice of orthonormal basis $\{e^i\}$.*

Proof. Let $\{e^i\}$ and $\{\tilde{e}^i\}$ be two orthonormal bases of $\mathcal{H}$. By Parsevals identity it follows that for all $N \in \mathbb{N}$

$$\sum_{i=0}^{N} \|A^* e^i\|^2 = \sum_{i=0}^{N} \sum_{j=0}^{\infty} |\langle A^* e^i, \tilde{e}^j \rangle|^2 = \sum_{j=0}^{\infty} \sum_{i=0}^{N} |\langle e^i, A\tilde{e}^j \rangle|^2 \leq \sum_{j=0}^{\infty} \|A\tilde{e}^j\|^2.$$

Because $\sum_{j=0}^{\infty} \|A\tilde{e}^j\|^2$ is finite, we conclude that A^* is Hilbert-Schmidt. Using a similar method we can show that $\sum_{i=0}^{\infty} \|Ae^i\|^2 \leq \sum_{i=0}^{\infty} \|A^* \tilde{e}^i\|^2$. Hence A has the same norm as A^* and

$$\sqrt{\sum_i \|Ae_i\|^2} = \sqrt{\sum_i \|A^* \tilde{e}_i\|^2} = \sqrt{\sum_i \|Ae_i\|^2}.$$

So the norm is independent of the choice of basis. □

Definition 5.4. *Let $\mathcal{H}$ be a Hilbert space and let $\{e^i\}$ be an orthonormal basis of $\mathcal{H}$. The* **Hilbert-Schmidt norm** *is the norm on the space of all Hilbert-Schmidt operators on $\mathcal{H}$ defined by*

$$\|A\|_{HS}^2 = \sum_i \|Ae^i\|^2.$$

Given a norm, we can ask whether it is induced from an inner product. Recall that any Hermitian inner product $\langle \cdot, \cdot \rangle$ on Hilbert space $\mathcal{H}$ satisfies the polarization identity

$$\langle u, v \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \|u + i^k v\|^2 \qquad \forall u, v \in \mathcal{H}.$$

Hence we investigate if the pairing $(A, B) \mapsto \frac{1}{4}\sum_{k=0}^{3} i^k \|A + i^k B\|_{HS}^2$ is an inner product on the space of Hilbert-Schmidt operators.

Lemma 5.5. *Let $\mathcal{H}$ be a Hilbert space. The function that maps two Hilbert-Schmidt operators A and B to*

$$\frac{1}{4}\sum_{k=0}^{3} i^k \|A + i^k B\|_{HS}^2$$

is an inner product on the space of Hilbert-Schmidt operators. Given an orthonormal basis $\{e^i\}$ this inner product is given by

$$(A, B) \mapsto \sum_i \langle Ae^i, Be^i\rangle. \tag{5.2}$$

Proof. This pairing is clearly positive definite, because the norm is positive definite. We only need to check for linearity and conjugate-symmetry. Let $\{e^i\}$ be an orthonormal basis of $\mathcal{H}$. Then the pairing of A and B equals

$$\frac{1}{4}\sum_{k=0}^{3}\sum_j i^k \|Ae^j + i^k Be^j\|_{HS}^2. \tag{5.3}$$

Applying the polarization identity on $\mathcal{H}$ Equation 5.3 simplifies to

$$\frac{1}{4}\sum_j\sum_{k=0}^{3} i^k \|Ae^j + i^k Be^j\|_{HS}^2 = \sum_j \langle Ae^i, Be^i\rangle.$$

This concludes Equation 5.2. From this Equation we directly conclude that the pairing is linear and skew-symmetric. ☐

Definition 5.6. *Let $\mathcal{H}$ be a Hilbert space, $\{e^i\}$ be an orthonormal basis of $\mathcal{H}$ and let A and B be two Hilbert-Schmidt operators. The* **Hilbert-Schmidt inner product** *is the inner product that maps A and B to*

$$\sum_i \langle Ae^i, Be^i\rangle$$

and is denoted by $\langle\cdot,\cdot\rangle_{HS}$.

For a finite dimensional Hilbert space $\mathcal{H}$, all linear operators are Hilbert Schmidt. Note that for any linear operator A, B on $\mathcal{H}$,

$$\langle A, B\rangle_{HS} = \sum_i \langle Ae_i, Be_i\rangle = \sum_i \langle B^* Ae_i, e_i\rangle = \mathrm{Tr}(B^* A). \tag{5.4}$$

Equation 5.4 suggests that we can define a trace for products of Hilbert Schmidt operators.

Definition 5.7. *A linear operator T on a Hilbert space $\mathcal{H}$ is* **traceclass** *if it is the product of two Hilbert-Schmidt operators.*

Let T be a traceclass operator. Hence, there exists two Hilbert-Schmidt operators A and B such that $T = AB$. Now suppose that there is another decomposition of T. That is, there exists also some Hilbert-Schmidt operators $\tilde{A}$ and $\tilde{B}$ such that $T = AB = \tilde{A}\tilde{B}$. In both cases we have

$$\begin{aligned}
\langle B, A^*\rangle_{HS} =& \sum_i \langle Be^i, A^* e^i\rangle = \sum_i \langle ABe^i, A^* e^i\rangle = \sum_i \langle Te^i, e^i\rangle \\
\langle \tilde{B}, \tilde{A}^*\rangle_{HS} =& \sum_i \langle \tilde{B}e^i, \tilde{A}^* e^i\rangle = \sum_i \langle \tilde{A}\tilde{B}e^i, A^* e^i\rangle = \sum_i \langle Te^i, e^i\rangle.
\end{aligned}$$

and we conclude that $\langle B, A^*\rangle_{HS}$ does not depend on the decomposition of T. Using Equation 5.4 we extend the definition of the trace to traceclass operators.

Definition 5.8. *Let T be a traceclass operator and let A and B Hilbert-Schmidt operators such that $T = AB$. The* **trace** *of T is the Hilbert-Schmidt inner product* $\mathrm{Tr}(T) = \langle B, A^*\rangle_{HS}$.

In the previous chapter we studied generalized Laplacians. We showed that for each generalized Laplacian H there exists a smoothinh operator e^{-tH} that satisfies the heat equation. We now show that e^{-tH} is traceclass. Observe that it is enough to show that e^{-tH} is Hilbert-Schmidt, because of the semi-group property. Indeed, if we show that e^{-tH} is Hilbert-Schmidt for all $t \in R_+$ then $e^{-\frac{t}{2}H} \cdot e^{-\frac{t}{2}H}$ is traceclass. By the semi-group property $e^{-\frac{t}{2}H} \cdot e^{-\frac{t}{2}H}$ simplifies to e^{-tH}.

Proposition 5.9. *Let E is a vector bundle with a positive definite inner product over a compact Riemannian manifold (M,g) and let H be a generalized Laplacian. Let k_t^H be the heat kernel and let e^{-tH} be the corresponding smoothing operator of k_t^H. Then e^{-tH}, which we view as a operator on the space of sections of E with L^2 norm, is traceclass.*

Proof. First we observe that it is enough to prove that e^{-tH} is Hilbert-Schmidt for all $t \in \mathbb{R}+$, since in this case $e^{-tH} = e^{-\frac{t}{2}H} \cdot e^{-\frac{t}{2}H}$ by the semi-group property. Pick an orthonormal basis $\{e^i\}$ on the space of sections of E. The Hilbert-Schmidt norm of e^{-tH} equals

$$\|e^{-tH}\|_{HS}^2 = \sum_{i=0}^{\infty} \|e^{-tH}e^i\|_{L^2} = \sum_{i=0}^{\infty} \left\langle e^{-tH}e^i, e^{-tH}e^i \right\rangle_{L^2}.$$

In Lemma 4.7 we showed that e^{-tD^2} is self-adjoint. Using this and the fact that e^{-tH} is a smoothing operator we conclude that the Hilbert-Schmidt norm of e^{-tH} equals

$$\begin{aligned}\|e^{-tH}\|_{HS}^2 &= \sum_{i=0}^{\infty} \left\langle e^{-2tH}e^i, e^i \right\rangle_{L^2}\\ &= \sum_{i=0}^{\infty} \int_{x\in M} \left\langle k_{2t}(\cdot,x)e^i(x), e^i \right\rangle_{L^2}.\end{aligned}$$

Let $\{f_j\}$ be an orthonormal frame of the fiber E_x. Using the identity $e^i(x) = \sum_{j=0}^{\dim E} \langle e^i(x), f^j\rangle f^j$ we see

$$\|e^{-tH}\|_{HS}^2 = \sum_{j=0}^{\dim E} \sum_{i=0}^{\infty} \int_{x\in M} \left\langle \left\langle k_{2t}(\cdot,x)f^j, e^i \right\rangle_{L^2} \cdot e^i(x), f^j \right\rangle.$$

Because $\{e^i\}$ is a basis of $\Gamma(E)$ we conclude that $\sum_{i=0}^{\infty} \langle k_{2t}(\cdot,x)f^j, e^i\rangle_{L^2} \cdot e^i$ equals $k_{2t}(\cdot,x)$ and hence

$$\|e^{-tH}\|_{HS}^2 = \sum_{j=0}^{\dim E} \int_{x\in M} \left\langle k_{2t}(x,x)f^j, f^j \right\rangle = \int_{x\in M} \mathrm{tr}(k_{2t}(x,x)).$$

Because the manifold is compact, we conclude that e^{-tH} is Hilbert-Schmidt. □

Proposition 5.10 (Berline et al. [2004], Proposition 2.32)**.** *Let E is a vector bundle with a positive definite inner product over a compact Riemannian manifold (M, g), let H be a generalized Laplacian, let k_t^H be its heat kernel and let e^{-tH} be the corresponding smoothing operator. Let B be a bounded operator on $\Gamma(E)$. The operator Be^{-tH} is trace-class and its trace is given by*

$$\operatorname{Tr}\left(Be^{-tH}\right) = \int_{x\in M} \operatorname{tr}(Bk_t(x,x)).$$

Proof. Because Hilbert-Schmidt operators is an ideal in the space of bounded operators it follows that $Be^{-t/2\,H}$ is Hilbert-Schmidt. Hence the product $Be^{-\frac{t}{2}H} \cdot e^{-\frac{t}{2}H} = Be^{-tH}$ is traceclass.

To show the explicit expression of the trace, let $\{e^i\}$ be an orthonormal basis of the space of all sections on E. Equip $\Gamma(E)$ with the L^2 norm. In this orthonormal basis the trace equals

$$\operatorname{Tr}(Be^{-tH}) = \sum_{i=0}^{\infty} \langle Be^{-tH} e^i, e^i \rangle_{L^2}.$$

From the definition of a smoothing operator follows that

$$\operatorname{Tr}(Be^{-tH}) = \sum_{i=0}^{\infty} \int_{x\in M} \langle Bk_t(\cdot, x) e^i(x), e^i \rangle_{L^2}.$$

Let f^j be an orthonormal basis of E_x. Using the identity $e^i(x) = \sum_{j=1}^{\dim E_x} \langle e^i(x), f^j \rangle f^j$ we see

$$\begin{aligned}
\operatorname{Tr}(Be^{-tH}) &= \sum_{j=1}^{\dim E_x} \sum_{i=0}^{\infty} \int_{x\in M} \langle e^i(x), f^j \rangle \cdot \langle Bk_t(\cdot, x) f^j, e^i \rangle_{L^2} \\
&= \sum_{j=1}^{\dim E_x} \sum_{i=0}^{\infty} \int_{x\in M} \left\langle \langle Bk_t(\cdot, x) f^j, e^i \rangle_{L^2} \cdot e^i(x), f^j \right\rangle.
\end{aligned}$$

Because $\{e^i\}$ is a basis of $\Gamma(E)$ we conclude that $\sum_{i=0}^{\infty} \langle k_t(\cdot, x) f^j, e^i \rangle_{L^2} \cdot e^i$ equals

$k_t(\cdot, x)$ and hence

$$\begin{aligned}\operatorname{Tr}(Be^{-tH}) &= \sum_{j=1}^{\dim E_x} \int_{x \in M} \left\langle Bk_t(x,x)f^j, f^j \right\rangle \\ &= \int_{x \in M} \operatorname{tr}\left(Bk_t(x,x)\right).\end{aligned}$$

□

The grading operator γ_5 is a bounded operator. On a Clifford bundle with Dirac operator D we have the generalized Laplacian D^2. We conclude that $\gamma_5 e^{-tD^2}$ is traceclass and so $\operatorname{Tr}(\gamma_5 e^{-tD^2})$ is well-defined. In the next section we will study the eigenvalues of the Dirac operator so that we can express $\operatorname{Tr}(\gamma_5 e^{-tD^2})$ in terms of these eigenvalues.

5.2 Spectral theory of generalized Laplacians

In functional analysis there is a generalized notion of eigenvalues and it is called the spectrum. The spectrum of an operator A is the collection $\lambda \in \mathbb{C}$ such that $A - \lambda \operatorname{Id}$ is not invertible. There is a class of operators for which the spectrum behaves well, namely the class of compact[13] operators. We recall the following theorems from functional analysis. For more information see Murphy [1990].

Theorem 5.11. *On a Hilbert space, the space of finite rank operators lies dense in the space of compact operators.*

Using this theorem we can create compact operator by considering the limit of a converging sequence of finite rank operators. The next two theorems, we describe the spectrum of compact operators

Theorem 5.12. *If A is a compact operator on a Banach space $\mathcal{B}$, then the spectrum of A is at most countable, each non-zero element of the spectrum is an eigenvalue and all eigenvalues are isolated.*

[13] Do not confuse it with the notion of compact spaces!

Not all elements in the spectrum are eigenvalues, but all eigenvalues are part of the spectrum. The previous theorem states that for compact operators, the spectrum and the set of eigenvalues coincide (except for zero). Isolated means that eigenvalues does not lie infinitely close to each other.

Theorem 5.13. *If A is a compact self-adjoint operator on a Hilbert space $\mathcal{H}$, then $\mathcal{H}$ can be decomposed in a countable family of finite eigenspaces.*

We show that Hilbert-Schmidt operators are compact. Especially we get that smoothing operators of heat kernels are compact. Using the eigenspace decomposition of we then calculate the spectrum of a generalized Laplacian.

Lemma 5.14. *Hilbert-Schmidt operators are compact.*

Proof. Let $K\colon \mathcal{H} \to \mathcal{H}$ be a Hilbert-Schmidt operator on a Hilbert space $\mathcal{H}$. Let $\{e^i\}$ be an orthonormal basis of $\mathcal{H}$. Denote K^N as the projection of K to the span of $\{e^1, \ldots, e^N\}$. That is $K^N = \sum_{i=1}^N K(e^i)e_i^\flat$ where $\flat$ denotes the dual. We estimate the difference of K and K^N in the operator norm:

$$\|K - K^N\|_{op} \leq \sum_{i=N+1}^{\infty} \|(Ke^i)e_i^\flat\|_{op} \leq \sum_{i=N+1}^{\infty} \|Ke^i\|.$$

This is arbitrary small when N tends to infinity, because K is Hilbert-Schmidt. Hence, K is a limit of finite rank operators and thus compact. □

Smoothing operators e^{-tH} of heat kernels are compact operators. Hence, for each $t \in \mathbb{R}_+$, we have a eigenvalue decomposition of $\Gamma_{L^2}(E)$ with respect to e^{-tH}. If we defined e^{-tH} using functional calculus, we get that λ is an eigenvalue of H if and only if $e^{-t\lambda}$ is an eigenvalue of e^{-tH}. We replicate this result using the heat equation. First we relate the different eigenvalue decompositions of e^{-tH}. That is, we show that there only one eigenvalue decomposition for e^{-tH} and the rest is related using the exponential map.

Lemma 5.15. *Let E be a vector bundle with a positive definite inner product over a compact Riemannian manifold (M, g) and let H be a generalized Laplacian. Let k_t^H be the heat kernel and let e^{-tH} be the corresponding smoothing operator. The operator e^{-tH} has only positive eigenvalues and its kernel is empty.*

Proof. Let λ be an eigenvalue of e^{-tH} and let $\mathcal{H}_\lambda$ be the eigenspace of e^{-tH} corresponding to the eigenvalue λ. Because e^{-tH} is self-adjoint, we have that λ is real. Let $z \in \mathcal{H}_\lambda$. From the semi-group property of the heat kernel follows that

$$e^{-tH}(e^{-t/2\,H}z) = e^{-\frac{3}{2}t\,H} = e^{-t/2\,H}e^{-tH}z = \lambda(e^{-t/2\,H}z).$$

This shows $e^{-t/2\,H}(\mathcal{H}_\lambda)$ is a subset of $\mathcal{H}_\lambda$. From the spectral theory of compact operators it follows that

$$\mathcal{H}_\lambda = \bigoplus_i \mathcal{H}_{\lambda,\mu_i}$$

where $\mathcal{H}_{\lambda,\mu_i}$ is the eigenspace of $e^{-t/2H}$ with eigenvalue μ_i and the eigenspace of e^{-tH} with eigenvalue λ. Also μ_i is real valued, because $e^{-t/2H}|_{\mathcal{H}_\lambda}$ is self-adjoint. So for all $z \in \mathcal{H}_{\lambda_1,\mu_i}$ we get

$$\lambda z = e^{-tH}z = (e^{-t/2\,H})^2 z = \mu^2 z.$$

We conclude that the eigenvalues of e^{-tH} are non-negative.

If e^{-tH} has a non-empty kernel, then we can construct a sequence $\{t_i\}$ converging to zero such that $e^{-t_iH}z = 0$ for all t_i. Hence the limit of $e^{-tH}z$ tends to zero when t tends to zero. This is in conflict with the boundary condition of the heat kernel. Therefore, the kernel is empty. □

Theorem 5.16. *Let E be a vector bundle with a positive definite inner product over a compact Riemannian manifold (M,g) and let H be a generalized Laplacian. Let k_t^H be the heat kernel and let e^{-tH} be the corresponding smoothing operator. The space $\Gamma_{L^2}(E)$ can be decomposed in an countable family of finite eigenspaces $\mathcal{H}_\lambda$ such that*

$$e^{-tH}|_{H_\lambda} = \lambda^t \cdot \mathrm{Id}\,.$$

Proof. Let λ be an eigenvalue of e^{-H} and let $\mathcal{H}_\lambda$ be the eigenspace of e^{-H} corresponding to the eigenvalue λ. Fix $n,m \in \mathbb{N}$. The operators e^{-H} and $e^{-\frac{n}{m}H}$ commute and so $e^{-\frac{n}{m}H}(\mathcal{H}_\lambda) \subseteq \mathcal{H}_\lambda$. From the spectral theory of compact operators it follows that

$$\mathcal{H}_\lambda = \bigoplus_i \mathcal{H}_{\lambda_1,\mu_i}$$

where $\mathcal{H}_{\lambda,\mu_i}$ is also the eigenspace of $e^{-\frac{n}{m}H}$ with eigenvalue μ_i. Let $z \in \mathcal{H}_{\lambda,\mu_i}$. By comparing eigenvalues we get

$$\mu_i^m z = (e^{-\frac{n}{m}H})^m z = e^{-nH}z = (e^{-H})^n z = \lambda^n z.$$

Because μ_i and λ are strictly larger then zero, we conclude that $\mu_i = \lambda^{n/m}$ and is proves the theorem for all $t \in \mathbb{Q}_+$. Finally recall that e^{-tH} is continuous in t and thus the theorem holds for all $t > 0$. $\square$

Finally, we study the eigenvalues of H. We use the heat equation to show that there is a one to one correspondence between the spectrum of e^{-tH} and H. The proof is straight forward, but we need that H and e^{-tH} commute. This is proven in the following lemma.

Lemma 5.17. *Let E be a vector bundle with a positive definite inner product over a compact Riemannian manifold (M, g) and let H be a generalized Laplacian. Let k_t^H be the heat kernel and let e^{-tH} be the corresponding smoothing operator. The operators H and e^{-tH} commute for all $t > 0$.*

Proof. Because H and e^{-tH} are self-adjoint, we get

$$He^{-tH} - e^{-tH}H = He^{-tH} - (He^{-tH})^*.$$

Using the heat equation, we rewrite this into

$$He^{-tH} - e^{-tH}H = -\frac{\partial}{\partial t}e^{-tH} + \left(\frac{\partial}{\partial t}e^{-tH}\right)^* = -\frac{\partial}{\partial t}\left(e^{-tH} - (e^{-tH})^*\right).$$

Because e^{-tH} is self-adjoint we conclude that H and e^{-tH} commute. $\square$

Theorem 5.18. *Let E be a vector bundle with a positive definite inner product over a compact Riemannian manifold (M, g) and let H be a generalized Laplacian. Let k_t^H be the heat kernel and let e^{-tH} be the corresponding smoothing operator. A section $s \in \Gamma_{L^2}(E)$ is an eigenvector of H with eigenvalue λ if and only if it is an eigenvector of e^{-H} with eigenvalue $e^{-\lambda}$. Therefore, H has countably many eigenvalues which are isolated.*

Proof. Let $e^{-\lambda}$ be an eigenvalue of e^{-H}. Let $s \in \Gamma_{L^2}(E)$ such that $e^{-H}s = e^{-\lambda}s$. From the heat equation and Theorem 5.16 it follows that

$$0 = \left(\frac{\partial}{\partial t} + H\right)e^{-tH}s = \left(\frac{\partial}{\partial t} + H\right)e^{-\lambda t}s = (-\lambda + H)\, e^{-\lambda t}s.$$

After the limit $t \to 0$, one concludes that $Hs = \lambda s$. Because $\Gamma_{L^2}(E)$ can be fully decomposed by the eigenvalues of e^{-H}, we know that H has no other eigenvalues. $\square$

5.3 The McKean-Singer formula

When we considered the Abelian anomaly using the Fujikawa method we showed that the anomaly was proportional to $\mathrm{Tr}(\gamma_5 e^{-tD^2})$ where γ_5 is the grading operator and D is the Dirac operator. We already found that this is a well-defined quantity for Clifford bundles on compact Riemannian manifolds. We now study this trace using the eigenvalue decomposition we found in the previous section.

Theorem 5.19 (McKean and Singer [1967]). *Let S be a Clifford bundle over a compact Riemannian manifold (M, g) and let D be the Dirac operator. Let k_t^H be the heat kernel of D^2 and let e^{-tD^2} be the corresponding smoothing operator. Let γ_5 be the grading operator and let $D_\pm$ be the restriction of D to the ± 1 eigenspace of γ_5. Then,*

$$\mathrm{Tr}(\gamma_5 e^{-tD^2}) = \dim\ker D_+ - \dim\ker D_-.$$

This result is called the **McKean-Singer formula**

Proof. By Theorem 5.18 we can decompose $\Gamma(S)$ into the eigenspaces

$$\Gamma(S) = \bigoplus_i H_{\lambda_i}$$

where H_{λ_i} is the $\lambda_i\mathbb{R}$ eigenspace of D. Denote $\{s_j^{\lambda_i}\}$ as an orthonormal basis of H_{λ_i}. The trace of $\gamma_5 e^{-tD^2}$ equals

$$\mathrm{Tr}(\gamma_5 e^{-tD^2}) = \sum_i \sum_j \langle s_j^{\lambda_i}, \gamma_5 e^{-tD^2} s_j^{\lambda_i}\rangle = \sum_i e^{-t\lambda_i^2} \sum_j \langle s_j^{\lambda_i}, \gamma_5 s_j^{\lambda_i}\rangle.$$

However, $\langle s_j^{\lambda_i}, \gamma_5 s_j^{\lambda_i}\rangle$ will vanish if $\lambda_i \neq 0$. Indeed, the grading operator and the Dirac operator anti-commute and so for all i and j and so

$$D\gamma_5 s_j^{\lambda_i} = -\gamma_5 D s_j^{\lambda_i} = -\lambda_i \gamma_5 s_j^{\lambda_i}.$$

This shows that $\gamma_5 s_j^{\lambda_i}$ is an element of $H_{-\lambda_i}$. From the orthonormality of the eigenspaces it follows that $\langle s_j^{\lambda_i}, \gamma_5 s_j^{\lambda_i}\rangle = 0$ if $\lambda_i \neq 0$. Therefore, the trace simplifies to

$$\mathrm{Tr}(\gamma_5 e^{-tD^2}) = \sum_j \langle s_j^0, \gamma_5 s_j^0\rangle.$$

The grading operator decomposes the kernel of D into two eigenspaces $\ker D_+ \oplus \ker D_-$. Let $\{s_j^\pm\}$ be an orthonormal basis of $\ker D_\pm$. We conclude that the trace of $\gamma_5 e^{-tD^2}$ equals

$$\mathrm{Tr}(\gamma_5 e^{-tD^2}) = \sum_j \langle s_j^+, s_j^+\rangle - \langle s_j^-, s_j^-\rangle = \dim\ker D_+ - \dim\ker D_-.$$

□

Notice that the adjoint of D_+ is D_-. Therefore, we can write the McKean-Singer formula as $\mathrm{Tr}(\epsilon e^{-tD^2}) = \dim\ker D_+ - \dim\operatorname{coker} D_+$. This quantity is called the index of the operator D_+. When considering Dirac operators some writers(Roe [1998], Berline et al. [2004]) call this the index of D. Although they look similar, those definitions does not coincide. Indeed, recall that D is a formally self-adjoint operator. Hence the kernel equals the cokernel and $\dim\ker D - \dim\operatorname{coker} D$ is zero. In this thesis we only calculate the index of Dirac operators and hence define the index as follows:

Definition 5.20. *Let S be a graded Clifford bundle over a compact Riemannian manifold (M, g). Let D be the Dirac operator and let e^{-tD^2} be the smoothing operator for which the kernel is the heat kernel $k_t^{D^2}$. The* **index of** *D, which we denote by $\mathrm{Ind}(D)$, is defined as*

$$\mathrm{Ind}(D) = \dim\ker D_+ - \dim\ker D_-$$

where $D_\pm$ is the restriction of D on the ± 1 eigenspace of the grading operator.

One of the main results of the Atiyah-Singer index theorem is that the index of a Dirac operator is topologically determined. That is, it does not depend on the choice of metric or connection, but only depends on the topology of the vector bundle. A similar result can be deduced from the McKean-Singer formula.

Corollary 5.21 (Roe [1998], Proposition 11.13)**.** *Let D_t be a continuous family of graded Dirac operators. That is, Let g_t, be a continuous family of Riemannian metrics on a compact manifold M, γ_t be a continuous family of Clifford actions on a Clifford bundle $E \to M$ and let ∇_t be a continuous family of compatible connections on E. D_t is the resulting Dirac operator. Then the index of D_t is independent of t.*

Proof. The map $(s,t) \mapsto \mathrm{Tr}(\gamma_{5,t}e^{-sD_t^2})$ is a continuous map. By the McKean-Singer formula this map equals $(s,t) \mapsto \mathrm{Index}(D_t)$. Notice that the index is an integer. The only continuous maps from $\mathbb{R}$ to $\mathbb{Z}$ are the constant maps and hence $\mathrm{Index}(D_t)$ is constant. $\square$

In theory the index of a Dirac operator D can be calculated using the formal solution of the heat equation. Indeed, we know that the index of D equals $\int_{x\in M} \mathrm{tr}[\gamma_5 k_t(x,x)] \,\mathrm{Vol}(g)$ where k_t is the heat kernel. We approximate this using the formal solution of the heat equation $k_t(x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-d(x,y)^2/4t} \sum_i t^i \Phi_i(x,y)$. The index is independent of t and hence

$$\mathrm{Index}(D) = \begin{cases} \frac{1}{(4\pi)^{n/2}} \int_{x\in M} \mathrm{tr}[\gamma_5 \Phi_{n/2}(x,x)] & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Although this calculation is straight forward, it is not easy to calculate $\Phi_{n/2}$ by hand. To solve this equation we need to study the behavior of the heat kernel and the Clifford action with the trace. In the next section we study the interaction between the Clifford action and the trace.

5.4 The trace and the Clifford action

When we considered chiral anomalies using perturbation theory, we often used the 'trace identities'

$$\mathrm{Tr}(\gamma_5) = \mathrm{Tr}(\gamma_5\gamma^\mu) = \mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho) = 0$$

for indices μ, ν and ρ and

$$\mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4i\epsilon^{\mu\nu\rho\sigma}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is ± 1 depending on the permutation of $\{\mu,\nu,\rho,\sigma\}$. One can find these results by calculating the trace over the gamma matrices. However these identities can also be deduced from the Clifford algebra. We show this in the following lemma.

Lemma 5.22 (Roe [1998], Lemma 11.5)**.** *Let V be a $2n$ dimensional vector space with a positive definite inner product g. Let $\mathrm{Cl}(V)$ be the Clifford algebra and let S be a left $\mathrm{Cl}(V)$-module. Let γ_5 be a grading operator, let $F \in \mathrm{End}_{\mathrm{Cl}(V)}(S)$ and let $\gamma\colon V \to \mathrm{End}(S)$ be the Clifford action. For any orthonormal basis $\{s_i\}$ of V there is*

$$\begin{aligned} \mathrm{Tr}(\gamma_5 F) &= 0 \\ \mathrm{Tr}(\gamma_5\gamma(s_0)\dots\gamma(s_k)F) &= 0 \qquad \forall k < 2n. \end{aligned}$$

If γ_5 is the canonical grading operator then,

$$\operatorname{Tr}(\gamma_5\gamma(s_0)\dots\gamma(s_{2n})F) =(-i)^n \operatorname{Tr}(F).$$

Proof. Let $v \in V$ with norm 1. By the anti-commutation property of the Clifford algebra it follows that $\gamma(v)^2 = -\operatorname{Id}_S$ and hence $\operatorname{Tr}(\gamma_5 F) = -\operatorname{Tr}(\gamma_5\gamma(v)\gamma(v)F)$. The Clifford action anti-commutes with the grading operator and so

$$\operatorname{Tr}(\gamma_5 F) = -\operatorname{Tr}(\gamma_5\gamma(v)\gamma(v)F) = +\operatorname{Tr}(\gamma(v)\gamma_5\gamma(v)F) = +\operatorname{Tr}(\gamma_5\gamma(v)F\gamma(v)).$$

Because F commutes with the Clifford action it follows that $\operatorname{Tr}(\gamma_5 F) = 0$. This trick can be repeated for $\operatorname{Tr}(\gamma_5\gamma(s_0)\dots\gamma(s_{2k})F)$ for all $k < n$. That is, we add the term $-\gamma(s_{2k+1})\gamma(s_{2k+1})$ and we cyclically permute. Hence, $\operatorname{Tr}(\gamma_5\gamma(s_0)\dots\gamma(s_{2k})F) = 0$ for all $k < n$.

To show that $\operatorname{Tr}(\gamma_5\gamma(s_0)\dots\gamma(s_{2k+1})F) = 0$ for all $2k+1 \in \mathbb{N}$ we just pull γ_5 through the other side of the trace. That is, we use that γ_5 anti-commutes with the Clifford action and commutes with F. Hence,

$$\operatorname{Tr}(\gamma_5\gamma(s_0)\dots\gamma(s_{2k+1})F) =(-1)^{2k+1} \operatorname{Tr}(\gamma(s_0)\dots\gamma(s_{2k+1})F\gamma_5).$$

By the cyclic property of the trace and $(-1)^{2k+1} = -1$ we conclude

$$\operatorname{Tr}(\gamma_5\gamma(s_0)\dots\gamma(s_{2k+1})F) = 0.$$

Finally we assume that γ_5 is the canonical grading operator and we calculate $\operatorname{Tr}(\gamma_5 \cdot \gamma(s_0)\dots\gamma(s_{2n})F)$. Recall that $\gamma_5 = i^n\gamma(s_0)\dots\gamma(s_{2n})$ and thus

$$\operatorname{Tr}(\gamma_5\gamma(s_0)\dots\gamma(s_{2n})F) = (-i^n) \operatorname{Tr}(\gamma_5^2 F).$$

From $\gamma_5^2 = 1$ follows the result. $\square$

With this lemma we characterize the trace for all elements in $\operatorname{Cl}(V) \otimes \operatorname{End}_{\operatorname{Cl}(V)}(S)$. Clearly these are endomorphisms on S. The next question is whether all endomorphisms on S can be decomposed into $\operatorname{Cl}(V) \otimes \operatorname{End}_{\operatorname{Cl}(V)}(S)$. That is, is every linear map the product of Clifford actions and a Clifford endomorphism? The following proposition shows that this is indeed true.

Proposition 5.23. *Let V be a $2n$-dimensional vector space with a positive definite inner product and let S be a left $\mathrm{Cl}(V)$-module. Assume that the Clifford action is normal with respect to the norm on S. Then the map which injects $\mathrm{Cl}(V) \otimes \mathrm{End}_{\mathrm{Cl}}(S)$ into $\mathrm{End}_{\mathbb{C}}(S)$ using the Clifford action is an isomorphism.*

Proof. Because elements of the Clifford algebra are invertible we have that $\mathrm{Cl}(V) \otimes \mathrm{End}_{\mathrm{Cl}}(S) \to \mathrm{End}_{\mathbb{C}}(S)$ is injective. We only need to show that is map is surjective. Let $F \in \mathrm{End}_{\mathbb{C}}(S)$ be a linear map on S and let $v \in V$ be a vector of length one. We first show that F can be decomposed into two terms that (anti)-commute with $\gamma(v)$. Indeed, Let $P_v^{\pm}$ be bounded linear operators on S defined by $P_v^{\pm} = \frac{1 \mp i\,\gamma(v)}{2}$. These operators have the properties:

$$\begin{aligned} P_v^{\pm} \cdot P_v^{\pm} &= P_v^{\pm} \qquad & P_v^{\pm} \cdot P_v^{\mp} &= 0 \\ P_v^{+} + P_v^{-} &= \mathrm{Id}_S \qquad & P_v^{\pm}\,\gamma(v) &= \gamma(v)\,P_v^{\pm} = \pm i P_v^{\pm}. \end{aligned}$$

Hence $P_v^{\pm}$ are the orthogonal eigenspace projectors of $\gamma(v)$. From these properties it follows that

$$P_v^{\pm} F P_v^{\pm}\,\gamma(v) = \pm\, i\, P_v^{\pm} F P_v^{\pm} = +\gamma(v)\, P_v^{\pm} F P_v^{\pm} \tag{5.5}$$
$$P_v^{\mp} F P_v^{\pm}\,\gamma(v) = \pm\, i\, P_v^{\mp} F P_v^{\pm} = -\gamma(v)\, P_v^{\mp} F P_v^{\pm}. \tag{5.6}$$

and thus we decompose F into

$$\begin{aligned} F =&(P_v^{+} + P_v^{-}) F (P_v^{+} + P_v^{-}) \\ =&(P_v^{+} F P_v^{+} + P_v^{-} F P_v^{-}) + (P_v^{+} F P_v^{-} + P_v^{-} F P_v^{+}). \end{aligned}$$

We denote $F_v^{\pm}$ for the operator that is defined by $F_v^{\pm} = P_v^{+} F P_v^{\pm} F + P_v^{-} F P^{\mp}$. By equations 5.5 and 5.6 it follows that F_v^{+} commutes and F_v^{-} anti-commutes with $\gamma(v)$. Hence, $F = F^{+} + F^{-}$ is the sum of (anti)-commuting terms w.r.t $\gamma(v)$.

Assume w.l.o.g. that F (anti)-commutes with $\gamma(v)$. That is, suppose that $F\gamma(v) = \epsilon\gamma(v)F$ for some $\epsilon \in \{0, 1\}$. Let $w \in V$ be of length one and normal to v. By the anti-commutation property of the Clifford algebra it follows that $P_w^{\pm}\gamma(v) = \gamma(v)P_w^{\mp}$. However, $F_w^{\pm}$ is defined such that $F_w^{\pm}\gamma(v) = \epsilon\gamma(v)F_w^{\pm}$. By induction we conclude that F can be decomposed into the sum of terms that (anti)-commute with a whole orthonormal basis $\{e^{\mu}\}$ of V.

Now pick an orthonormal basis $\{e_i\}$ of V and assume w.l.o.g. that F commutes with $e_1, \ldots, e_k$ and anti-commutes with $e_{k+1} \ldots e_{2n}$. Let $\tilde{F}^+, \tilde{F}^- \in \operatorname{End}(S)$ be operators defined as $\tilde{F}^+ = \gamma_1 \ldots \gamma_k F$ and $\tilde{F}^- = \gamma_{k+1} \ldots \gamma_{2n} F$ By the anti-commutation property of the Clifford algebra it follows that

$$\begin{aligned}
\gamma_\mu \tilde{F}^+ &= (-1)^{k-1} \gamma_1 \ldots \gamma_k \gamma_\mu F = (-1)^{k-1} \tilde{F}^+ \gamma_\mu && \forall \mu \leq k \\
\gamma_\mu \tilde{F}^+ &= (-1)^{k} \gamma_1 \ldots \gamma_k \gamma_\mu F = (-1)^{k+1} \tilde{F}^+ \gamma_\mu && \forall \mu > k \\
\gamma_\mu \tilde{F}^- &= (-1)^{2n-k} \gamma_{k+1} \ldots \gamma_{2n} \gamma_\mu F = (-1)^{2n-k} \tilde{F}^- \gamma_\mu && \forall \mu \leq k \\
\gamma_\mu \tilde{F}^- &= (-1)^{2n-k-1} \gamma_{k+1} \ldots \gamma_{2n} \gamma_\mu F = (-1)^{2n-k} \tilde{F}^- \gamma_\mu && \forall \mu > k.
\end{aligned}$$

This calculation shows that if k is even, then $\tilde{F}^-$ is a Clifford endomorphism and if k is odd, then $\tilde{F}^+$ is a Clifford bundle endomorphism. So in both cases, F has a representative in $\operatorname{Cl}(V) \otimes \operatorname{End}_{\operatorname{Cl}(V)}(S)$. So $\operatorname{Cl}(V) \otimes \operatorname{End}_{\operatorname{Cl}(V)}(S)$ is isomorphic to $\operatorname{End}_{\mathbb{C}} S$. $\square$

In the next chapter we decompose the heat kernel k_t into $\operatorname{Cl}(V) \otimes \operatorname{End}_{\operatorname{Cl}(V)}(S)$ and we will calculate the index of a Dirac operator.

6 Characteristic classes

In this chapter we consider the following question: "How can we distinguish two different vector bundles?" The answer to this question is studied in the theory of characteristic classes. We introduce this theory by following the Chern-Weil method.

6.1 Chern-Weil method

Definition 6.1. *Given a complex vector space V, an* **invariant polynomial** p *on V is an element of $\oplus_k \operatorname{Sym}^k \operatorname{End}^*_{\mathbb{C}} V$ that is invariant under conjugation. That is, if $M_i \in \operatorname{End}_C V$ and $S \in Gl(V)$ then*

$$p(SM_1S^{-1}, \ldots, SM_kS^{-1}) = p(M_1, \ldots, M_k).$$

To simplify the notation we denote the space of polynomials as $\operatorname{Sym}^\bullet \operatorname{End}^*_{\mathbb{C}} V$. Clearly the trace and the determinant are invariant polynomials on any vector space V. They are both invariant under cyclic permutations. This also holds for invariant polynomials and we will show this in the next lemma.

Lemma 6.2. *Let $p \in \operatorname{Sym}^k \operatorname{End}^*_C V$ be a degree k polynomial. Then p is invariant under conjugation if and only if it is invariant under cyclic permutations.*

Proof. Let $M_1, \ldots, M_k \in \operatorname{End}(V)$ and let $S \in Gl(V)$. Suppose that p is invariant under cyclic permutations. Then it satisfies

$$\begin{aligned} p(SM_1S^{-1}, \ldots SM_kS^{-1}) =& p(M_1S^{-1}S, \ldots M_kS^{-1}S) \\ =& p(M_1, \ldots M_k) \end{aligned}$$

and we conclude that it is invariant under conjugation. Now assume that p is invariant under conjugation. By definition it satisfies

$$\begin{aligned} p(M_1S, \ldots M_kS) =& p(SM_1SS^{-1}, \ldots SM_kSS^{-1}) \\ =& p(SM_1, \ldots SM_k) \end{aligned}$$

and so it is invariant under cyclic permutations of invertible elements. Because $GL(V)$ is dense in $\operatorname{End}(V)$ we conclude that p is invariant under all cyclic permutations. □

Given a polynomial $p \in \mathrm{Sym}^\bullet \mathrm{End}^*_{\mathbb{C}}(V)$ we can interpret p as a map from V to $\mathbb{C}$. Indeed, we inject $\mathrm{Sym}^\bullet \mathrm{End}^*_{\mathbb{C}}(V)$ into $\otimes^\bullet \mathrm{End}_{\mathbb{C}}(V)$ by

$$p_1 \odot \ldots \odot p_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} p_{\sigma(1)} \otimes \ldots p_{\sigma(n)}.$$

The tensor algebra satisfies the following universal property: If A is a complex algebra and $f\colon \mathrm{End}_{\mathbb{C}}(V) \to A$ is a linear map, then there exist a unique algebra homomorphism $\tilde{f}\colon \otimes^\bullet \mathrm{End}_{\mathbb{C}}(V) \to \mathbb{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{End}_{\mathbb{C}}(V) \arrow[r] \arrow[dr, "f"'] & \otimes^\bullet \mathrm{End}_{\mathbb{C}}(V) \arrow[d, "\tilde{f}"] \\
& A
\end{tikzcd}$$

Now let $A = \mathbb{C}$ and $M \in \mathrm{End}_{\mathbb{C}}(V)$ and consider the map $\iota_M\colon \mathrm{End}_{\mathbb{C}}(V) \to \mathbb{C}$ that is defined by $\iota_M(p) = p(M)$. Using this universal property there is a unique algebra homomorphism $\tilde{\iota}_M\colon \otimes^\bullet \mathrm{End}_{\mathbb{C}}(V) \to \mathbb{C}$ extending ι_M. Combining $\tilde{\iota}_M$ with the injection of $\mathrm{Sym}^\bullet \mathrm{End}_{\mathbb{C}}(V)$ we constructed a map from $\mathrm{Sym}^\bullet \mathrm{End}_{\mathbb{C}}(V) \to \mathbb{C}$. Notice that this maps $p \in \mathrm{Sym}^k \mathrm{End}_{\mathbb{C}}(V)$ to $p(M, \ldots, M)$. The map $\iota_\bullet p$ is a map from V to $\mathbb{C}$ which we also denote as p.

Example 6.3. Consider the vector space $\mathbb{C}^2$ and let M be a complex 2×2 matrix. The components of M we denote as M_{ij}. Let $p_{ij}\colon M_2(\mathbb{C}) \to \mathbb{C}$ be the projection onto the $(i,j)^{\text{th}}$ component. In this case the map $\iota_M\colon M_2(\mathbb{C})^* \to \mathbb{C}$ equals $\iota_M(p_{ij}) = M_{ij}$ for all i and j. We now consider the determinant which is formally the element $p_{00} \odot p_{11} - p_{01} \odot p_{10}$ in $\mathrm{Sym}^\bullet M_2(\mathbb{C})^*$. We inject p into $\otimes^\bullet M_2(\mathbb{C})$ to

$$\frac{1}{2}\left(p_{00} \otimes p_{11} + p_{11} \otimes p_{00} - p_{01} \otimes p_{10} - p_{10} \otimes p_{01}\right).$$

Calculating $\tilde{\iota}_M(\det)$ we get

$$\begin{aligned}
\tilde{\iota}_M(\det) =& \frac{1}{2}\tilde{\iota}_M\left(p_{00} \otimes p_{11} + p_{11} \otimes p_{00} - p_{01} \otimes p_{10} - p_{10} \otimes p_{01}\right) \\
=& \frac{1}{2}(\tilde{\iota}_M(p_{00}) \cdot \tilde{\iota}_M(p_{11}) + \tilde{\iota}_M(p_{11}) \cdot \tilde{\iota}_M(p_{00}) \\
& \quad - \tilde{\iota}_M(p_{01}) \cdot \tilde{\iota}_M(p_{10}) - \tilde{\iota}_M(p_{10}) \cdot \tilde{\iota}_M(p_{01})) \\
=& \tilde{\iota}_M(p_{00}) \cdot \tilde{\iota}_M(p_{11}) - \tilde{\iota}_M(p_{01}) \cdot \tilde{\iota}_M(p_{10}) \\
=& M_{00}M_{11} - M_{10}M_{01}.
\end{aligned}$$

Hence $\tilde{\iota}_M(\det)$ equals the determinant of M.

We can replace $\mathbb{C}$ with any commutative algebra A. Indeed, let $p \in \operatorname{Sym}^\bullet \operatorname{End}^*_{\mathbb{C}}(V)$ be a polynomial, let M be an element of $\operatorname{End}_{\mathbb{C}}(V) \otimes A$ and consider the map $\iota_M \colon \operatorname{End}_{\mathbb{C}}(V) \to A$ that is defined by $\iota_M(p) = p(M)$. By the same universal property we used before, there exists a unique algebra homomorphism $\tilde{\iota}_M \colon \otimes^\bullet \operatorname{End}^*_{\mathbb{C}}(V) \to A$ that extends ι_M. Because A is commutative, we have for any polynomial

$$\begin{aligned}
\tilde{\iota}_M(p_1 \odot \ldots \odot p_n) =& \tilde{\iota}_M \left(\frac{1}{n!} \sum_{\sigma \in S_n} p_{\sigma(1)} \otimes \ldots \otimes p(\sigma(n)) \right) \\
=& \frac{1}{n!} \sum_{\sigma \in S_n} \iota_M(p_{\sigma(1)}) \cdot \ldots \cdot \iota_M(p_{\sigma(n)}) \\
=& \iota_M(p_1) \cdot \ldots \cdot \iota_M(p_n)
\end{aligned}$$

and so we interpret polynomials as maps from V to A.

Example 6.4. We review Example 6.3. Consider $\mathbb{C}^2$ and let A be a commutative algebra. We denote the matrix multiplication with $*$. Let M be a matrix on $\mathbb{R}^2$ with values in A. We want to interpret the determinant of M as

$$\det(M) = M_{00} * M_{11} - M_{01} * M_{10}.$$

With the above construction this is indeed possible. Indeed, the map ι_M is given by $\iota_M(p_{ij}) = M_{ij} \in A$ for all i, j. We conclude that

$$\begin{aligned}
\tilde{\iota}_M(\det) =& \frac{1}{2} \tilde{\iota}_M \left(p_{00} \otimes p_{11} + p_{11} \otimes p_{00} - p_{01} \otimes p_{10} - p_{10} \otimes p_{01} \right) \\
=& \frac{1}{2} (\tilde{\iota}_M(p_{00}) * \tilde{\iota}_M(p_{11}) + \tilde{\iota}_M(p_{11}) * \tilde{\iota}_M(p_{00}) \\
& \quad - \tilde{\iota}_M(p_{01}) * \tilde{\iota}_M(p_{10}) - \tilde{\iota}_M(p_{10}) * \tilde{\iota}_M(p_{01})) \\
=& \tilde{\iota}_M(p_{00}) * \tilde{\iota}_M(p_{11}) - \tilde{\iota}_M(p_{01}) * \tilde{\iota}_M(p_{10}) \\
=& M_{00} * M_{11} - M_{10} * M_{01}.
\end{aligned}$$

Hence we define the determinant of an element $M \in M_2(A)$ as $\iota_M(\det)$.

Let $E \to M$ be a rank k complex vector bundle over an n-dimensional manifold. We view invariant polynomials on E as sections of $\operatorname{Sym}^\bullet \operatorname{End}_{\mathbb{C}}(E)$ which are invariant in each fiber. Consider the vector bundle $\operatorname{End}_{\mathbb{C}}(E) \otimes (\oplus_k \Lambda^{2k} T^*M)$ which we denote as $\operatorname{End}_{\mathbb{C}}(E) \otimes \Lambda^{ev} T^*M$. The algebra on $\Gamma(\Lambda^{ev} T^*M)$ is a commutative algebra and hence we can use the construction described above.

Let $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ be a connection on E and let p be an invariant polynomial. The curvature K is a section of $\mathrm{End}(E) \otimes \Lambda^2 T^*M$. We show that $p(K) \in \Gamma(\Lambda^{ev}T^*M)$ is a useful measure for some invariant polynomials. For this we need the following lemma.

Theorem 6.5 (Milnor and Stasheff [1974], Page 298)**.** *Let $E \to M$ be a vector bundle and let p be an invariant polynomial such that for all curvature tensors K the form $p(K)$ is closed. Then the cohomology class of $p(K)$ does not depend on the choice of connection.*

Proof. Let $\pi\colon M \times \mathbb{R} \to M$ be defined as $\pi(x,t) = x$ and let for all $t \in \mathbb{R}$, $\iota_t\colon M \to M \times \mathbb{R}$ be defined as $\iota_t(x) = (x,t)$. Let π^*E and $\iota_t^*\pi^*E$ be the pull-back bundles. Because $\pi \circ \iota$ is the identity map we have that the pull-back bundle $\iota_t^*\pi^*E \to M$ is equal to $E \to M$. Now consider the following diagram:

$$\begin{tikzcd}
E \cong \iota_t^*\pi^*E \arrow[d] & \pi^*E \arrow[d] & E \arrow[d] \\
M \arrow[r, "\iota_t"] & M \times \mathbb{R} \arrow[r, "\pi"] & M
\end{tikzcd}$$

Let ∇^0 and ∇^1 be two connections on $E \to M$. Using the pull-back[14] we get two connections $\pi^*\nabla^0$ an $\pi^*\nabla^1$ on π^*E. Also the linear combination

$$\nabla' = t\pi^*\nabla^1 + (1-t)\pi^*\nabla^0$$

is a connection on π^*E. We pull-back ∇' on $\iota_t^*\pi^*E$ and we calculate $\iota_0\pi^*\nabla'$. Let $s \in \Gamma(E)$ and note that $s = \iota_t^*\pi^*s$ for all $t \in \mathbb{R}$. Hence for all $u \in \Gamma(T^*M)$ we have

$$(\iota_0^*\nabla')_u s = (\iota_0^*\nabla')_u(\iota_0^*\pi^*s) = \iota_0^*(\nabla'_{\mathrm{d}\,\iota_t(u)}\pi^*s).$$

Note that for $\iota_0^*(\nabla'_{\mathrm{d}\,\iota_t(u)}\pi^*s)$ we only consider the fibers of $(\pi^*E)_{(\cdot,0)}$ and so $\iota_0^*(\nabla'_{\mathrm{d}\,\iota_t(u)}\pi^*s)$ equals $\iota_0^*((\pi^*\nabla^0)_{\mathrm{d}\,\iota_t(u)}\pi^*s)$. By the definition of the pull-back connection $\pi^*\nabla^0$ $(\iota_0^*\nabla')_u s$ simplifies to

$$(\iota_0^*\nabla')_u s = \iota_0^*\pi^*\nabla^0_{\mathrm{d}\,\pi\,\circ\,\mathrm{d}\,\iota_t(u)} s = \nabla^0.$$

[14] Recall that for a vector bundle $E \to M$ and a smooth map between manifolds $\phi\colon N \to M$ the pull-back connection is the unique connection on ϕ^*E such that $(\phi^*\nabla)_X(\phi^*s) = \phi^*(\nabla_{\mathrm{d}\,\phi(X)}s)$ for all $X \in T^*N$ and $s \in \Gamma(\phi^*E)$

Using a similar argument we can show that $\iota_1^*\nabla' = \nabla^1$.

Let K^0, K^1 and K' be the curvature with respect to the curvatures ∇^0, ∇^1 and ∇'. Note that ι_t is a homotopy between ι_0 and ι_1. Hence the induced maps $\iota_0^*, \iota_1^* \colon H^\bullet(M\times\mathbb{R}) \to H^\bullet(M)$ are isomorphic. Therefore, we have

$$[p(K^0)] = \iota_0^*[p(K')] = \iota_1^*[p(K')] = [p(K^1)] \in H^\bullet(M).$$

So the cohomology class of $p(K)$ does not depend on the choice of connection. □

Definition 6.6. *Let $E \to M$ be a complex vector bundle and let p be an invariant polynomial such that $p(K)$ is closed for all curvature tensors K. A* **characteristic class** *of E w.r.t. p is the cohomology class of $p(K)$ which we denote as $p(E)$.*

Notice that the characteristic class w.r.t. p only depends on the isomorphism class of a vector bundle, because p is invariant. Hence, characteristic classes are useful tools to distinguish different vector bundles using their topology. To construct characteristic classes we need to define covariantly constant polynomials. For this we need to extend the connection on a vector bundle E to a connection on $\operatorname{End}_{\mathbb{C}}(E) \otimes \Lambda^\bullet T^*M$ and $\operatorname{Sym}^\bullet \operatorname{End}_{\mathbb{C}}^*(E)$.

Lemma 6.7. *let $E \to M$ be a vector bundle and let ∇ be a connection on E. Then ∇ induces a connection over E^* by the relation*

$$\mathrm{d}\, f(v) = f(\nabla v) + (\nabla f)(v)$$

for all $f \in \Gamma(E^*)$ and $v \in \Gamma(E)$.

Proof. Clearly ∇f is an element of $\Gamma(T^*M \otimes E^*)$. It is also linear, because for all $f, g \in \Gamma(E^*)$, $v \in \Gamma(E)$ and $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}\nabla(f+\lambda g)(v) =& \mathrm{d}(f(v) + \lambda g(v)) - (f+\lambda g)(\nabla v)\\ =& \nabla(f)(v) + \lambda\nabla(g)(v).\end{aligned}$$

The Leibniz rule is satisfied, because for all $f \in \Gamma(E^*)$, $v \in \Gamma(E)$ and $\alpha \in \mathbb{C}^\infty(M)$ there is

$$\begin{aligned}\nabla(\alpha\, f)(v) =& \mathrm{d}(\alpha\, f(v)) - \alpha\, f(\nabla v)\\ =& \mathrm{d}\,\alpha \cdot f(v) + \alpha\ \mathrm{d}(f(v)) - \alpha\, f(\nabla v)\\ =& \mathrm{d}\,\alpha \cdot f(v) + \alpha\,(\nabla f)(v).\end{aligned}$$

So we conclude that ∇ induces a connection on E^*. $\square$

Lemma 6.8. *Let $E, F \to M$ be two vector bundles on M and let ∇_E be a connection on E and let ∇_F be a connection on F. Then ∇_E and ∇_F induces a connection on $E \otimes F$ by the relation*

$$\nabla(e \otimes f) = \nabla_E e \otimes f + e \otimes_F \nabla f.$$

Proof. By the universal property of the tensor algebra this map is well-defined. Clearly ∇ is a linear map from $\Gamma(E \otimes F)$ to $\Gamma(T^*M \otimes E \otimes F)$. We only need to show the Leibniz rule. Let $e \in \Gamma(E)$, $f \in \Gamma(F)$ and let $\alpha \in C^\infty(M)$. By the Leibniz rule of ∇_E and ∇_F we have

$$\begin{aligned}\nabla(\alpha\, e \otimes f) =&\nabla((\alpha e) \otimes f)\\ =&\,\mathrm{d}\,\alpha \otimes e \otimes f + \alpha \otimes \nabla_E e \otimes f + \alpha \otimes e \otimes \nabla_F f\\ =&\,\mathrm{d}\,\alpha \otimes e \otimes f + \alpha \otimes \nabla(e \otimes f).\end{aligned}$$

Notice that this proof is independent if we considered $\nabla(e \otimes (\alpha\, f))$. So we conclude the proof. $\square$

Recall that $\operatorname{End}_{\mathbb{C}} E$ is isomorphic to $E \otimes_{\mathbb{C}} E^*$. So the connection ∇ induces a connection on $\operatorname{End}_{\mathbb{C}}(E)$ and a connection on the space of invariant polynomials. For sections of $\operatorname{End}_{\mathbb{C}}(E) \otimes \Lambda^\bullet T^*M$ we consider the exterior covariant derivative d_∇.

Corollary 6.9. *Let $E \to M$ be a vector bundle equipped with a connection ∇. Let $p, q \in \Gamma(E)$ and let $f \in \Gamma(E^*)$. Then the induced connection on $\Gamma(\operatorname{End}(E))$ satisfies*

$$\nabla\left((p \otimes f)(q)\right) =(\nabla(p \otimes f))(q) + (p \otimes f)(\nabla q).$$

Proof. Notice that $(p \otimes f)(q)$ equals $p \cdot f(q)$ and hence

$$\nabla\left((p \otimes f)(q)\right) = \mathrm{d}(f(q)) \cdot p + f(q) \cdot \nabla p.$$

By the definition of the induced connection on E^* we conclude

$$\begin{aligned}\nabla\left((p \otimes f)(q)\right) =&(\nabla f)(q) \cdot p + f(\nabla q) \cdot p + f(q) \cdot \nabla p\\ =&(\nabla p \otimes f)(q) + (p \otimes f)(\nabla q).\end{aligned}$$

$\square$

Lemma 6.10. *Let $E \to M$ be a vector bundle with a connection and let $p \in \Gamma(\operatorname{Sym}^k \operatorname{End}_{\mathbb{C}}^* E)$ then for all $X_1, \dots X_k \in \Gamma(\operatorname{End}_{\mathbb{C}} E \otimes \Lambda^\bullet T^* M)$ we have*

$$\begin{aligned} \mathrm{d}(p(X_1, \dots, X_k)) =& (\nabla p)(X_1, \dots, X_k) + p(\mathrm{d}_\nabla X_1, X_2, \dots, X_k) \\ & + p(X_1, \mathrm{d}_\nabla X_2, X_3, \dots X_K) + \dots \end{aligned}$$

where ∇ is the induced connection.

Proof. We prove by induction. Pick a local coordinate basis $\{x^\mu\}$ of M. Assume that $k = 1$ and pick w.l.o.g. $X = \mathrm{d}\, x^1 \wedge \dots \mathrm{d}\, x^j \cdot X_{1,\dots j}$. Then for all $p \in \Gamma(\operatorname{End}_{\mathbb{C}}^*(E))$ there is

$$\mathrm{d}\, p(X) = \mathrm{d}\left(p(X_{1,\dots j}) \cdot \mathrm{d}\, x^1 \wedge \dots \mathrm{d}\, x^j\right).$$

By the previous lemma we conclude that

$$\begin{aligned} \mathrm{d}\, p(X) =& (\nabla p)(X_{1,\dots j}) \wedge \mathrm{d}\, x^1 \wedge \dots \mathrm{d}\, x^j + p(\nabla X_{1,\dots j}) \wedge \mathrm{d}\, x^1 \wedge \dots \mathrm{d}\, x^j \\ =& (\nabla p)(X) + p(\mathrm{d}_\nabla X). \end{aligned}$$

For the induction step let $p_1, \dots p_k \in \Gamma(\operatorname{End}(E))$ and let $X_1 \dots X_k \in \Gamma(E \otimes \Lambda^\bullet T^* M)$. By the definition of the symmetric algebra we get

$$\begin{aligned} \mathrm{d}(p_1 \odot \dots \odot p_k(X_1, \dots X_k)) =& \mathrm{d}\, \frac{1}{k!} \sum_{\sigma \in S_k} \prod_{j=1}^k p_j(X_{\sigma(j)}) \\ =& \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{i=0}^k \mathrm{d}\, p_i(X_{\sigma(i)}) \prod_{j \neq i} p_j(X_{\sigma(j)}). \end{aligned}$$

Using the induction base we conclude

$$\begin{aligned} \mathrm{d}(p_1 \odot \dots \odot p_k(X_1, \dots X_k)) =& \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{i=0}^k (\nabla p_i)(X_{\sigma(i)}) \prod_{j \neq i} p_j(X_{\sigma(j)}) \\ & + \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{i=0}^k p_i(\mathrm{d}_\nabla X_{\sigma(i)}) \prod_{j \neq i} p_j(X_{\sigma(j)}) \end{aligned}$$

and this simplifies into

$$\begin{aligned} \mathrm{d}(p_1 \odot \dots \odot p_k(X_1, \dots X_k)) =& \frac{1}{k!} \sum_{\sigma \in S_k} (\nabla(p_1 \otimes \dots \otimes p_k))(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \\ & + \frac{1}{k!} \sum_{\sigma \in S_k} (\nabla(p_1 \otimes \dots \otimes p_k))(X_{\sigma(1)}, \dots, \mathrm{d}_\nabla X_{\sigma(i)}, \dots, X_{\sigma(k)}). \end{aligned}$$

By rewriting the right hand side in terms of the symmetric algebra we conclude the proof. ☐

Definition 6.11. *Let $E \to M$ be a complex vector bundle equipped with a connection ∇ and a curvature K. A polynomial $p \in \Gamma(Sym^\bullet \operatorname{End}^*_C(E)$ is* **covariantly constant** *if $\nabla p = 0$.*

Theorem 6.12. *Let $E \to M$ be a complex vector bundle equipped with connection ∇ and curvature K. Let p be a covariantly constant polynomial on E. Then $p(K, \dots, K)$ is closed.*

Proof. By the previous lemma we know

$$\mathrm{d}\, p(K, \dots, K) = (\nabla p)(K, \dots K) + p(\mathrm{d}_\nabla K, K, \dots, K) + p(K, \mathrm{d}_\nabla K, K, \dots, K) + \dots$$

By assumption $\nabla p(K, \dots K)$ vanishes and by Bianchi identity $\mathrm{d}_\nabla K = 0$. This concludes that $p(K, \dots, K)$ is closed. ☐

The previous theorem states that covariantly constant invariant polynomials induce characteristic classes. Next we will show that the trace is covariantly constant and how we can use this to create many more characteristic classes.

Lemma 6.13. *Let $E \to M$ be a rank k vector bundle and let K be a curvature on E. The covariant derivative of the trace vanishes and so $\mathrm{d}\operatorname{tr}(K) = 0$.*

Proof. Let $\{e^i\}$ be a local frame of E and let $\{e^{i\flat}\}$ de the dual frame. In this local basis the trace of a section $F \in \Gamma(\operatorname{End}_{\mathbb{C}}(E) \otimes \Lambda^{ev} T^*M)$ equals $\sum_i e^{i\flat} F e^i$. Expanding the exterior derivative in terms of the connection of E and E^* we get that

$$\begin{aligned} \mathrm{d}\operatorname{Tr}(F) &= \sum_i \mathrm{d}\, e^{i\flat} F e^i \\ &= \sum_i (\nabla e^{i\flat})(F e^i) + e^{i\flat}(\mathrm{d}_\nabla F) e^i + e^{i\flat} F (\nabla e^i). \end{aligned}$$

Recall that $e^{i\flat} e^j$ is one if $i = j$ and zero otherwise. Hence the exterior derivative of $e^{i\flat} e^j$ vanishes and so

$$0 = \mathrm{d}(e^{i\flat} e^j) = (\nabla e^{i\flat})(e^j) + e^{i\flat}(\nabla e^j).$$

This concludes

$$\begin{aligned}
\mathrm{d}\,\mathrm{Tr}(F) &= \sum_i e^{i\flat}(\mathrm{d}_\nabla F)e^i + e^{i\flat}F(\nabla e^i) + \sum_{ij}(\nabla e^{i\flat})(e^j)\cdot e^{j\flat}(Fe^i)\\
&= \sum_i e^{i\flat}(\mathrm{d}_\nabla F)e^i + e^{i\flat}F(\nabla e^i) - \sum_{ij} e^{i\flat}(\nabla e^j)\cdot e^{j\flat}(Fe^i)\\
&= \sum_i e^{i\flat}(\mathrm{d}_\nabla F)e^i.
\end{aligned}$$

Hence, $\nabla\,\mathrm{tr} = 0$. For $F = K$ Bianchi identity states that $\mathrm{d}_\nabla F = 0$ and so $\mathrm{d}\,\mathrm{Tr}(K) = 0$. □

We create more examples of characteristic classes using formal power series on $\mathbb{C}$. If $E \to M$ is a vector bundle, K is a curvature tensor and $f(z) = \sum_{k=0}^\infty a_k z^k$ is a formal power series on $\mathbb{C}$, we extend f to an element of $\Gamma(\Lambda^{ev}T^*M)$ by setting $f(K) = \sum_{k=0}^\infty a_k K^k$. Notice that $\Gamma(\oplus_{k=1}^\infty \Lambda^{2k}T^*M)$ is a nilpotent algebra and so $\sum_{k=0}^\infty a_k K^k$ has only finitely many nonzero terms. This concludes that $f(K)$ is well-defined.

Lemma 6.14. *Let $E \to M$ be a vector bundle, let p be a covariantly constant polynomial and let f and g be formal power series on $\mathbb{C}$. If K is a curvature on E, then $g \circ p \circ f(K)$ is well-defined invariant polynomial and is closed.*

Proof. Clearly, $g\circ p\circ f$ is a formal power series on E. The algebra generated by K is nilpotent and so $g \circ p \circ f$ is a polynomial on E. Because f is equivariant[15] we know that $g \circ p \circ f$ is invariant.

To show closedness we need to calculate the exterior derivative of $g\circ p\circ f(K)$. Assume that $f(z) = \sum_i a_i z^i$ and $g(z) = \sum_j b_j z^j$. The exterior derivative of $g \circ p \circ f(K)$ equals

$$\begin{aligned}
\mathrm{d}\, g\circ p\circ f(K) &= \mathrm{d}\left(\sum_j b_j p\left(\sum_i a_i K^i\right)^j\right)\\
&= \sum_j b_j p\left(\sum_i a_i K^i\right)^{j-1} \times\\
&\qquad \times \left((\nabla p)(f(K)) + p(\mathrm{d}_\nabla f(K), f(K), \ldots, f(K)) + \ldots\right).
\end{aligned}$$

[15]That is, $f(SMS^{-1}) = f(M)$ for all $M \in \mathrm{End}(E)$ and $S \in Gl(E)$

We only need to show that $\mathrm{d}_\nabla f(K)$ vanishes. Note that for all $A, B \in \Gamma(\mathrm{End}_\mathbb{C}(E))$ and $v \in \Gamma(E)$ there is

$$\nabla(ABv) = (\nabla A)Bv + A(\nabla B)v + AB(\nabla v) = (\nabla(AB))v + AB(\nabla(v)).$$

This concludes that $\mathrm{d}_\nabla F^k = k(\mathrm{d}_\nabla F)F^{k-1}$. By the Bianchi identity this vanishes and so $\mathrm{d}_\nabla f(K) = 0$. ☐

Example 6.15. Using the previous lemma we show that the determinant defines a characteristic class. For this we use that the determinant of the exponent of a matrix is the exponent of its trace. For suitable $\lambda \in \mathbb{C}$ we have the matrix identity

$$\det(K + \lambda\mathbb{I}) = \exp(\mathrm{tr}\log(K + \lambda\mathbb{I})).$$

The logarithm might be ill-defined, but if $K \in \Gamma(\mathrm{End}_C(E) \otimes \oplus_{k=1}^{\infty}\Lambda^{2k}T^*M)$ then K is a section of a nilpotent algebra and the power series of the logarithm,

$$\log(K + \lambda\mathbb{I}) = \log(\lambda) - \sum_{i=1}^{\infty} \frac{(-1)^i}{i\lambda^i} K^i,$$

converges. This proves that for all rank r vector bundles

$$\det(K + \lambda\mathbb{I}) = \exp\left(k\log(\lambda) - \sum_{i=1}^{\infty} \frac{(-1)^i}{i\lambda^i}\mathrm{tr}(K^i)\right) = \lambda^k \prod_{i=1}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{j+i\cdot j}}{j!i^j\lambda^{i\cdot j}}\mathrm{tr}(K^i)^j.$$

The left hand side is a polynomial in λ and the right hand side is a (finite) Laurent polynomial in λ. This concludes that the determinant of K is a polynomial map over the traces of K^i and so the determinant defines a characteristic map.

6.2 Examples of characteristic classes

We now revisit some examples of characteristic classes. All are based on the construction described in Lemma 6.14.

Example 6.16. Let $E \to M$ be a complex vector bundle, let K be a curvature tensor on E and let f be a formal power series on $\mathbb{C}$. The **Chern f-genus** of E is the characteristic class with respect to the polynomial

$$\Pi_f(X) = \det\left[f\left(\frac{-X}{2\pi i}\right)\right]$$

and is denoted by $\Pi_f(E)$.

Example 6.17. Let $E \to M$ be a complex vector bundle. The **total Chern class** $c(E) \in H^{ev}(M)$ is the Chern f-genus with respect to the power series $f(z) = 1+z$. The i^{th} **Chern class** is the projection of the total Chern class into $H^{2i}(M) \otimes \mathbb{C}$.

Example 6.18. Let $E \to M$ be a complex vector bundle and let K be a curvature tensor on E. The **Chern character** $\mathrm{Ch}(E) \in H^{ev}(M) \otimes \mathbb{C}$ is the characteristic class with respect to the polynomial

$$P(X) = \mathrm{tr}\left[\exp\left(\frac{-X}{2\pi i}\right)\right].$$

For the Atiyah-Singer index theorem we mainly focus on the Chern Character and the Chern f-genus of TM where $f(z) = \sqrt{\frac{z/2}{\sinh(z/2)}}$. The latter is a cohomology class in $H^\bullet(M)$ and not in $H^\bullet(M) \otimes \mathbb{C}$. Indeed, if R is the curvature tensor on TM, then R is a skew-symmetric matrix. Hence, it has eigenvalues of the form $\pm i\lambda$ where $\lambda \in \Omega^2(M)$. Hence, the Chern f-genus is

$$\Pi_f(R) = \prod_j \sqrt{\frac{\lambda_j/2}{\sinh \lambda_j/2}} \cdot \sqrt{\frac{-\lambda_j/2}{\sinh -\lambda_j/2}} = \prod_j \frac{\lambda_j/2}{\sinh \lambda_j/2}.$$

This shows that $\Pi_f(TM \otimes \mathbb{C})$ indeed represents a real cohomology class. Also it does not depend on the choice of branch of the square root. This is not a special property of TM and so we define:

Example 6.19. Let $E \to M$ be a real vector bundle. The $\hat{A}$-**genus** $\hat{A}(E)$ is the Chern f-genus of $E \otimes \mathbb{C}$ where f is the holomorphic map $z \mapsto \sqrt{\frac{z/2}{\sinh z/2}}$.

6.3 Characteristic classes of Clifford bundles

When we consider a Clifford bundle $S \to (M, g)$ the curvature K is not the only section of $\mathrm{End}(S) \otimes \Lambda^{ev} T^*M$ we study. Namely, for a Clifford bundle we also have the twisting curvature F^S and the Riemann endomorphism R^S. We study if invariant polynomials over F^S and R^S also define characteristic classes.

Lemma 6.20. *Let A be an algebra and let V, W be two vector spaces. Let P be an element of $\mathrm{Sym}^\bullet(V \oplus W)^*$. Let $\tilde{P}$ be the projection of P onto $\mathrm{Sym}^\bullet W^*$. Them for all $w \in W \otimes A$, $v \in V \otimes A$ the polynomials P and $\tilde{P}$ satisfies*

$$P(w) = \tilde{P}(v + w).$$

Proof. Assume w.l.o.g. that P has a unique representation as $\sum_i \prod_{j=0}^{i}(f_{ij} + g_{ij})$ where $f_{ij} \in V^*$ and $g_{ij} \in W^*$ for all i and j. Then $\tilde{p}$ is the sum $\sum_i \prod_{j=0}^{i} g_{ij}$. Let $v \in V \otimes A$ and $w \in W \otimes A$. Because $f_{ij}(w) = 0$ and $g_{ij}(v) = 0$ there is

$$\begin{aligned} P(w) &= \sum_i \prod_{j=0}^{i}(f_{ij}(w) + g_{ij}(w)) = \sum_i \prod_{j=0}^{i} g_{ij}(w) \\ \tilde{P}(v + w) &= \sum_i \prod_{j=0}^{i}(g_{ij}(v) + g_{ij}(w)) = \sum_i \prod_{j=0}^{i} g_{ij}(w). \end{aligned}$$

This concludes that $P(w) = \tilde{P}(v + w)$. ☐

Theorem 6.21. *Let $S \to (M, g)$ be a Clifford bundle, let R^S be the Riemann endomorphism and let F^S be the twisting curvature. For all invariant polynomials P, the differential forms $P(R^S)$ and $P(F^S)$ are closed.*

Proof. Let $K \in \Gamma(\mathrm{End}(S) \otimes \Lambda^2 T^*M)$ be the curvature on S. Recall that $\mathrm{End}_{\mathbb{C}}(S)$ is isomorphic to $\mathrm{Cl}(TM) \otimes \mathrm{End}_{\mathrm{Cl}}(S)$ where $\mathrm{Cl}(TM)$ is the Clifford action and $\mathrm{End}_{\mathrm{Cl}}(S)$ is the bundle of Clifford endomorphisms. Because $\mathrm{Cl}(TM)$ is unital we can decompose $\mathrm{End}(S)$ into $V \oplus W$ where $W = \{\mathbb{C} \cdot \mathrm{Id}\} \otimes \mathrm{End}_{\mathrm{Cl}}(S)$ and V is the rest. Let $\tilde{P}$ be the projection of P onto W. By the previous lemma we know that $P(F^S) = \tilde{P}(F^S + R^S) = \tilde{P}(K)$. Notice that $\tilde{P}$ is also an invariant polynomial. Hence, $\mathrm{d}\,\tilde{P}(K) = \mathrm{d}\,P(F^S) = 0$. Using the same argument we can show that $\mathrm{d}\,P(R^S) = 0$. ☐

For Clifford bundles we can also use F^S and R^S to calculate characteristic classes. For the Atiyah-Singer theorem we only use the following:

Definition 6.22. *The* **relative Chern character** $\mathrm{Ch}^{rel}(S) \in H^{ev}(M) \otimes \mathbb{C}$ *of a Clifford bundle* $S \to M$ *on an* n *dimensional manifold is the cohomology class of* $P(F^S)$ *where* P *is the invariant polynomial*

$$P(X) = (-2)^{n/2} \operatorname{tr}\left[\exp\left(\frac{-X}{2\pi i}\right)\right].$$

7 Symbol Calculus

In chapter 3 we saw how the Fujikawa method relates the chiral anomaly to the index of a Dirac operator. In one of the steps we considered a Taylor approximation and we showed that only a single order contributed. We formalize this approach by introducing graded and filtered algebras. We show that the Taylor approximation used by Fujikawa is related to the Getzler filtration. Secondly we investigate how the heat equation changes the Getzler filtration. The result is related to the heat equation for Mehlers kernel and this proves the Atiyah-Singer index theorem.

7.1 Definitions

> **Definition 7.1.** *A* **graded algebra** *is an algebra G with a direct product decomposition $G = \prod_{i\in\mathbb{Z}} G_i$ such that $G_i \cdot G_j \subseteq G_{i+j}$ for all $i, j \in \mathbb{Z}$. An element $g \in G$ is of* **degree** *k if $g \in G_k$.*

Example 7.2 (Trivial grading)**.** We can turn any algebra G into a graded algebra. For this let $G_0 = G$ and $G_i = \{0\}$ if $i \neq 0$. Clearly G has the decomposition $\prod_i G_i$. Because G is an algebra $G_0 \cdot G_0 \subseteq G_0$. Also $\{0\} \cdot G = G \cdot \{0\} = \{0\}$ and hence $G_i \cdot G_j \subseteq G_{i+j}$. We call this the trivial grading.

Example 7.3 (Taylor series)**.** Let V be a vector space let $G = \mathbb{C}[\![x]\!]$ be the space of formal power series of maps on V. Homogeneous polynomials span this space and it even induces a direct product decomposition. Indeed, we say that a formal series is of degree[16] $-k$ and only if it is a homogeneous polynomial of degree k. Under change of coordinates a homogeneous polynomial stays homogeneous and the degree stays the same. Hence the homogeneous polynomials induce a direct product decomposition of G. Also the multiplication between a degree i and j homogeneous polynomial induces a degree $i + j$ homogeneous polynomial. Hence the property $G_i \cdot G_j \subseteq G_{i+j}$ is also satisfied.

Example 7.4 (Exterior algebra)**.** Let V be a vector space and let $G_k = \Lambda^k V$ be the k-times wedge product of elements in V. The exterior algebra is the direct product $G = \prod_{k\in\mathbb{N}} \Lambda^k V$. The wedge product between an i- and j-form is a $(i + j)$-form. Hence $G_i \wedge G_j \subseteq G_{i+j}$ and the exterior algebra is a graded algebra.

[16] Although the minus sign is not necessarily, it will be useful in later calculations.

Let V be a vector space with inner product g and let $\mathrm{Cl}_k(V,g)$ be the linear span of k products of pairwise orthonormal Clifford actions. The space $\mathrm{Cl}(V,g)$ has the direct product decomposition $\prod_k \mathrm{Cl}_k(V,g)$ but is not a graded algebra. Indeed, for all $v \in V$ the Clifford action $\gamma(v)$ is an element of $\mathrm{Cl}_1(V,g)$. However $\gamma(v)^2$ is an element of $\mathrm{Cl}_0(V,g)$ instead of $\mathrm{Cl}_2(V,g)$, because $\gamma(v)^2 = -g(v,v)\cdot \mathrm{Id}$. To assign degrees for elements in the Clifford algebra we use filtered algebras:

Definition 7.5. *A* **filtered algebra** *is an algebra A with a family of subspaces A_i, $i \in \mathbb{Z}$ such that $A_i \subseteq A_{i+1}$ and $A_i \cdot A_j \subseteq A_{i+j}$. An element $a \in A$ is of* **degree** *$k \in \mathbb{Z}$ if $a \in A_k$, but $a \notin A_{k-1}$.*

Example 7.6. (Clifford algebra) Consider the Clifford algebra $A = \mathrm{Cl}(V,g)$ and let $A_i = \prod_{j<i} \mathrm{Cl}_j(V,g)$. Clearly A is the union of all A_i and for all i and j the product of an element in A_i and in A_j is an element of A_{i+j}. Hence the Clifford algebra is a filtered algebra.

Example 7.7. (Differential operators) Recall that for a manifold M a differential operator is a smooth operator on $C^\infty(M)$ such that in a local coordinate frame $\{x^\mu\}$ the differential operator is given by

$$\sum c_{\mu_1,\ldots,\mu_k} \frac{\partial}{\partial x^{\mu_1}} \frac{\partial}{\partial x^{\mu_2}} \cdots \frac{\partial}{\partial x^{\mu_k}}.$$

Here $c_{\mu_1,\ldots,\mu_k}$ is a smooth maps on M. If we use multi-index notation we say that a differential must locally be of the form $\sum_I c_I \frac{\partial}{\partial x^I}$. Let $|I|$ be the degree of the multi index I and let A_k be the vector space generated by the differential operators that are locally of the form $\sum_{|I|\leq k} c_I \frac{\partial}{\partial x^I}$. Notice that A_k is independent of choice of basis and $A_i \cdot A_j \subseteq A_{i+j}$ for all $i+j$. Hence A_k induce a filtration on the space of differential operators. We denote differential operators as $\mathcal{D}(M)$.

Lemma 7.8. *Let $G = \prod_k G_k$ be a graded algebra and let $A_k = \prod_{j\leq k} G_j$. If there exists an $M \in \mathbb{Z}$ such that $G_k = \{0\}$ for all $k > M$, then $\bigcup_k A_k$ is a filtration of G*

Proof. Clearly G is the union of all A_k and by definition $A_k \subseteq A_{k+1}$ for all k. Because G is graded we conclude that $A_i \cdot A_j \subseteq A_{i+j}$. Hence A_i is a filtration of G. ☐

From the previous lemma we conclude that all our examples of graded algebras are also filtered algebras. Not every filtered algebra is a graded algebra. However for every filtered algebra we can construct a graded algebra by considering quotient spaces.

Definition 7.9. *Let A be a filtered algebra. The* **associated graded algebra** *is the graded algebra $G(A)$ that is defined by*

$$G(A) = \prod_k A_k/A_{k-1}.$$

Given a filtered algebra $A = \bigcup_i A_i$ consider the associated graded algebra $G(A)$. Notice that the quotient map $\pi_k \colon A_k \to A_k/A_{k-1}$ describes the relation between the filtration of A and the grading of $G(A)$. Also notice that for all $a \in A_i$ and $a' \in A_j$

$$\pi_i(a) \cdot \pi_j(a') = \pi_{i+j}(a \cdot a').$$

This is an example of a symbol map.

Definition 7.10. *Let $A = \bigcup_{k \in I} A_k$ be a filtered algebra and let $G = \prod_{k \in I} G_k$ be a graded algebra. A* **symbol map** *is a family of linear maps $\sigma_k \colon A_k \to G_k$ such that*

1. *for all $k \in I$ and $a \in A_{k-1}$ the map σ_k satisfies $\sigma_k(a) = 0$.*
2. *for all $i, j \in I$, $a \in A_i$ and $a' \in A_j$ the symbol map satisfies $\sigma_i(a) \cdot \sigma_j(a') = \sigma_{i+j}(a \cdot a')$.*

If G is the associated graded algebra of A, then the **associated symbol map** *is the family of projection maps $\pi_k \colon A_k \to A_k/A_{k-1}$, which is a symbol map between A and $G(A)$.*

Example 7.11. Consider the trivially filtered algebra A. That is, $A_i = A$ if $i \geq 0$ and $A_i = \{0\}$ else. The associated graded algebra $G(A)$ is isomorphic to the trivially graded algebra of A. Hence, the associated symbol map equals $\sigma_k = \mathrm{Id}_A$ if $k = 0$ and $\sigma_k = 0$ else.

Example 7.12. (Clifford algebra) Let V be a vector space with symmetric 2-form g. Consider the associated graded algebra of $\mathrm{Cl}(V, g)$. By the definition of a Clifford algebra the projection map π_i satisfies

$$\pi_1(v_1) \cdot \pi_1(v_2) + \pi_1(v_2) \cdot \pi_1(v_1) = \pi_2(v_1 \cdot v_2 + v_2 \cdot v_1) = -g(v1, v2) \cdot \pi_2(\mathrm{Id}) = 0 \quad (7.1)$$

for all $v_1, v_2 \in V$. This shows that the associated graded algebra of $\mathrm{Cl}(V, g)$ is isomorphic to the exterior algebra $\Lambda^\bullet V^*$. Equation 7.1 also shows that the associated symbol map interchanges the Clifford multiplication with the wedge product. That is, for all $v_1 \ldots v_k \in V$, the map $\pi_k(v_1 \cdot \ldots \cdot v_k)$ equals $v_1^\flat \wedge \ldots \wedge v_k^\flat$.

Example 7.13 (Endomorphisms of algebras)**.** Given an filtered algebra $A = \bigcup_i A_i$ we can create a filtered subalgebra on $\mathrm{End}(A)$. Indeed, $\mathrm{End}_n(A)$ be the subspace of $\mathrm{End}(A)$ such that for all $k \in \mathbb{Z}$ the map $f|_{A_k}$ maps into A_{k+n}. Because A_{k+n} is a subspace of A_{k+n+1} we conclude that $\mathrm{End}_n(A) \subseteq \mathrm{End}_{n+1}(A)$. Using the composition of maps we notice that $\mathrm{End}_n(A) \circ \mathrm{End}_m(A) \subseteq \mathrm{End}_{n+m}(A)$. Hence, $\bigcup_n \mathrm{End}_n(A)$ is a filtered algebra and a subalgebra of $\mathrm{End}(A)$. This property can also been shown for graded algebras G. That is, the subalgebra $\prod_n \mathrm{End}_n(G)$ of $\mathrm{End}(G)$ is also a graded algebra.

We construct a symbol map between $\bigcup_n \mathrm{End}_n(A)$ and $\prod_n \mathrm{End}_n(G(A))$. Fix $n \in \mathbb{Z}$ and let $f \in \mathrm{End}_n(A)$. For any $k \in \mathbb{Z}$ and $a \in A_{k-1}$ we have $f(a) \in A_{k+n-1}$. This shows that $\pi_{n+k} \circ f(a) = 0$. We conclude that $\pi_{n+k} \circ f$ factors over A_k/A_{k-1} and we can define $\sigma_n(f)\colon G(A) \to G(A)$ as the unique map such that for all $k \in \mathbb{Z}$ the following diagram commutes:

$$\begin{tikzcd}
A_k \arrow[r, "f"] \arrow[d, "\pi_k"] & A_{k+n} \arrow[d, "\pi_{k+n}"] \\
A_k/A_{k-1} \arrow[r, "\sigma_n(f)"] & A_{k+n}/A_{k+n-1}
\end{tikzcd}$$

We see that σ_n is a family of linear maps from $\mathrm{End}_n(A)$ to $\mathrm{End}_n(G(A))$. It is also a symbol map. We already showed that $\sigma_n(\mathrm{End}_{n-1}(A)) = 0$ for all $n \in \mathbb{Z}$. To show the homomorphism property let $n, m \in \mathbb{Z}$, $f \in \mathrm{End}_n(A)$ and let $g \in \mathrm{End}_m(A)$. For all $k \in \mathbb{Z}$ we get the commuting diagram:

$$\begin{tikzcd}
A_k \arrow[r, "f"] \arrow[d, "\pi_k"] & A_{k+n} \arrow[r, "g"] \arrow[d, "\pi_{k+n}"] & A_{k+n+m} \arrow[d, "\pi_{k+n+m}"] \\
A_k/A_{k-1} \arrow[r, "\sigma_n(f)"] \arrow[rr, bend right, "\sigma_{n+m}(g\circ f)"'] & A_{k+n}/A_{k+n-1} \arrow[r, "\sigma_m(g)"] & A_{k+n+m}/A_{k+n+m-1}
\end{tikzcd}$$

By uniqueness we conclude that $\sigma_{n+m}(g \circ f) = \sigma_m(g) \circ \sigma_n(f)$ and so σ is a symbol map.

Example 7.14 (Taylor series)**.** Consider formal power series $\mathbb{C}\,[\![x]\!]$ over $\mathbb{R}$. Note that for all $\alpha_i, \beta_j \in \mathbb{C}$ the differential operator $\sum_{i=0}^n \alpha_i \left(\frac{\partial}{\partial x}\right)^i$ maps the formal power series

$\sum_{j=m}^{\infty} \beta_j x^j$ to $\sum_{i=0}^{n} \sum_{j=m}^{\infty} \alpha_i \, \beta_j \cdot \frac{j!}{i!} x^{j-i}$. Hence, the space $\mathcal{D}_n(\mathbb{R})$ is a subalgebra of $\mathrm{End}_n(\mathbb{C}\,[\![x]\!])$ for all $n \in \mathbb{Z}$. However the filtrations of $\mathcal{D}_n(\mathbb{R})$ and $\mathrm{End}_n(\mathbb{C}\,[\![x]\!])$ differs. To see this note that the operator $p \mapsto x \cdot p$ is a differential operator of degree zero, but this operator is an element of $\mathrm{End}_{-1}(\mathbb{C}\,[\![x]\!])$.

7.2 Getzler filtration

In paragraph 3.1 we calculated the trace over $\gamma_5 e^{-tD^2}$ by considering a Taylor approximation in $t^{1/2}$ for Equation 3.8

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} t^{-2} \operatorname{tr}\left[\gamma_5 \, f\left(-(t^{1/2}\nabla_\mu + ik_\mu)^2 - \frac{it}{4}[\gamma^\mu, \gamma^\nu] F_{\mu\nu}\right)\right]. \tag{3.8}$$

Notice that we have a factor $t^{1/2}$ for every k_μ and for every γ^μ. Now we rephrase this in terms of filtrations. The element k_μ is the Fourier transform for a degree one differential operator. Also, γ^μ is a degree one Clifford action. By applying a Taylor approximation we combine the filtration of the Clifford action with the filtration induced by differential operators. In general we always can combine multiple filtrations using the tensor algebra.

Lemma 7.15. *Let $B = \bigcup_i B_i$ and $C = \bigcup C_i$ be filtered algebras and let A be the tensor algebra of B and C. Then A is a filtered algebra spanned by $A_k = \sum_{i=0}^{k} B_i \otimes C_{k-i}$.*

Proof. Let $b \in B$, $c \in C$ and consider $b \otimes c \in A$. Because B and C are filtrations there exists an $i, j \in \mathbb{Z}$ such that $b \in B_i$ and $c \in C_j$. Hence, $b \otimes c \in B_i \otimes C_j$. The space $B_i \otimes C_j$ is a subspace of $\sum_{i=0}^{i+j} B_i \otimes C_j = \sum_{i=0}^{i+j} B_i \otimes C_{(i+j)-i}$ which equals A_{i+j}. Hence, A is the union of the family of spaces A_k.

Now let $k \in \mathbb{Z}$ and consider A_k. By definition it equals $\sum_{i=0}^{k} B_i \otimes C_{k-i}$. The space A_k is a subspace of A_{k+1}, because

$$A_k = \sum_{i=0}^{k} B_i \otimes C_{k-i} \subseteq \sum_{i=0}^{k} B_i \otimes C_{(k+1)-i} \subseteq \sum_{i=0}^{k+1} B_i \otimes C_{(k+1)-i} = A_{k+1}.$$

Finally let $i, j, k, l \in \mathbb{Z}$ and let $b_i \in B_i$, $c_j \in C_j$, $b_k \in B_k$ and $c_l \in C_l$. Then $b_i \otimes c_j \in A_{i+j}$ and $b_k \otimes c_l \in A_{k+1}$. The tensor product between $b_i \otimes c_j$ and $b_k \otimes c_l$ equals $(b_i \cdot b_j) \otimes (c_k \cdot c_l)$. Because B and C are filtrations $(b_i \cdot b_j) \otimes (c_k \cdot c_l)$ is an element of $B_{i+j} \otimes C_{k+l}$. This proves that $b_i \otimes c_j \cdot b_k \otimes c_l \in A_{i+j+k_l}$ and so $A_{i+j} \cdot A_{j+k} \in A_{i+j+k+l}$. We conclude that A is a filtered algebra. $\square$

We want to create a filtration on the space of smoothing operators. For this let (M, g) be an even dimensional Riemannian manifold and let $y \in M$. Consider a neighborhood $U_y \subseteq M$ such that $\exp_y^{-1} \colon U_y \to T_yM$ is a chart of M. Let $S \to U_y$ be a Clifford bundle and let $p \in \Gamma(S \boxtimes S)$ be a kernel. We fix the second component of p at y and so we only consider $p(\cdot, y) \in \Gamma(S \otimes S_y^*)$. Using parallel transport and the exponential map we trivialize S into $T_yM \times S_y$. Then locally $p(\cdot, y)$ is a smooth map from T_yM to $S_y \otimes S_y^*$. Notice that $S_y \otimes S_y^*$ is isomorphic to $\mathrm{End}(S_y)$. By Lemma 5.23 the space $S_y \otimes S_y^*$ is isomorphic to $\mathrm{Cl}(T_yM) \otimes \mathrm{End}_{\mathrm{Cl}}(S_y)$ where $\mathrm{End}_{\mathrm{Cl}}(S_y)$ is the space of Clifford endomorphisms on S_y. We conclude that $p(\cdot, y)$ is locally a map from T_yM to $\mathrm{Cl}(T_yM) \otimes \mathrm{End}_{\mathrm{Cl}}(S_y)$. That is, $p(\cdot, y)$ is an element of $C^\infty(T_yM) \otimes \mathrm{Cl}(T_yM) \otimes \mathrm{End}_{\mathrm{Cl}}(S_y)$. Notice that this space is the tensor product of filtered algebras. Using Lemma 7.15 we can create a filtration on $\Gamma(S \otimes S_y^*)$.

Definition 7.16. *The* **Getzler filtration for kernels on** S *is the tensor filtration on $S \otimes S_y$ that is induced by the Taylor series on $C^\infty(T_yM)$, the Clifford action on $\mathrm{Cl}(T_yM)$ and the trivial filtration on $\mathrm{End}_{\mathrm{Cl}}(S_y)$. For operators on $\Gamma(S \otimes S_y^*)$ we define the* **Getzler filtration for operators on** $\Gamma(S)$ *and the* **Getzler symbol** *as the filtration and symbol described in Example 7.13.*

In the rest of this section we refer to the trivialization $T_yM \times \mathrm{End}_{\mathbb{C}}(S_y)$ of S as the trivialization induced by parallel transport. Without further introducing we also use $\{s_k\}$ for an orthonormal frame of T_yM and $\{\tilde{s}_k\}$ as the local frame of S induced from $\{s_k\}$ by parallel transport. Note that $\{\tilde{s}_k\}$ is also orthonormal, because the connection on S is metric compatible.

Our goal is to calculate the Getzler symbol of the operator D^2. For this we need to calculate the symbol of the Clifford action and the covariant derivative. We show this in the next lemmas.

Lemma 7.17. *Let $v_y \in T_yM$ and consider the vector field $v \in \Gamma(TM)$ induced by parallel transport. The Clifford action $\gamma(v)$ along v is locally given by $\mathrm{Id}_{T_yM} \otimes \gamma(v_y) \otimes \mathrm{Id}_{S_y} \in C^\infty(T_yM) \otimes \mathrm{Cl}(T_yM) \otimes \mathrm{End}_{\mathrm{Cl}}(S_y)$.*
The Clifford action $\gamma(v)$ has Getzler degree one and the Getzler symbol $\sigma_1(\gamma(v))$ equals $v_y^\flat$.

Proof. Let $x \in M$ and consider a radial path $p \colon [0, 1] \to T_yM$ from y to x. The local trivialization induced by parallel transport of $\gamma(v)$ is given as

$$\sum_{i,j} \langle \tilde{s}_i, \gamma(v)\tilde{s}_j\rangle_x \otimes s_i \cdot s_j^\flat \in C^\infty(T_yM) \otimes \mathrm{End}_{\mathbb{C}}(S_y).$$

The metric on S is compatible with the covariant derivative ∇ and so

$$\frac{\partial}{\partial t}\langle \tilde{s}_i, \gamma(v)\tilde{s}_j\rangle_{p(t)} = \langle \nabla_{\dot{p}(t)}\tilde{s}_i, \gamma(v)\tilde{s}_j\rangle_{p(t)} + \langle \tilde{s}_i, \nabla_{\dot{p}(t)}\gamma(v)\tilde{s}_j\rangle_{p(t)}. \tag{7.2}$$

Because $\tilde{s}_i$ is a vector field induced by parallel transport, we have the identity $\nabla_{\dot{p}(t)}\tilde{s}_i = 0$ for all t. So equation 7.2 simplifies to

$$\begin{aligned}\frac{\partial}{\partial t}\langle \tilde{s}_i, \gamma(v)\tilde{s}_j\rangle_{p(t)} &= \langle \tilde{s}_i, \nabla_{\dot{p}(t)}\gamma(v)\tilde{s}_j\rangle_{p(t)}\\ &= \langle \tilde{s}_i, \gamma(\nabla_{\dot{p}(t)}v)\ \tilde{s}_j\rangle_{p(t)} + \langle \tilde{s}_i, \gamma(v)\nabla_{\dot{p}(t)}\tilde{s}_j\rangle_{p(t)}\\ &= \langle \tilde{s}_i, \gamma(\nabla_{\dot{p}(t)}v)\ \tilde{s}_j\rangle_{p(t)}.\end{aligned}$$

Also the vector field v is induced from parallel transport and so $\langle \tilde{s}_i, \gamma(v)\tilde{s}_j\rangle_{p(t)}$ is constant. We calculate this constant by evaluating at y and we conclude that $\gamma(v)$ locally equals

$$\sum_{i,j} \langle s_i, \gamma(v_y)s_j\rangle \cdot \mathrm{Id}_{U_y} \otimes s_i \cdot s_j^\flat \in C^\infty(T_yM) \otimes \mathrm{End}_{\mathbb{C}}(S_y).$$

This simplifies to $\mathrm{Id}_{T_yM} \otimes \gamma(v_y) \in C^\infty(T_yM) \otimes \mathrm{End}_{\mathbb{C}}(S_y)$ and we conclude the result. □

Lemma 7.18. *Any endomorphism $F \in \Gamma(\mathrm{End}_{\mathbb{C}}(S))$ is a Clifford endomorphism if and only if there exists some $\alpha_i \in C^\infty(T_yM)$ and $f^i \in \mathrm{End}_{\mathrm{Cl}}(S_y)$ such that in the trivialization induced by parallel transport F equals*

$$\sum_i \alpha_i \otimes \mathrm{Id} \otimes f^i \in C^\infty(T_yM) \otimes \mathrm{Cl}(T_yM) \otimes \mathrm{End}_{\mathrm{Cl}}(S_y).$$

The Clifford bundle endomorphism F have Getzler degree zero and the Getzler symbol $\sigma_0(F)$ equals $F(y)$.

Proof. Let $f \in \mathrm{End}_{\mathrm{Cl}}(S_y)$ and let $\alpha \in C^\infty(T_yM)$. Let F be a section of $\mathrm{End}_{\mathbb{C}}(S)$ such that F is locally given by $\alpha \otimes \mathrm{Id} \otimes f \in C^\infty(T_yM) \otimes \mathrm{Cl}(T_yM) \otimes \mathrm{End}_{\mathrm{Cl}}(S_y)$. Using the orthonormality of $\{\tilde{s_k}\}$ we see that F equals

$$F = \sum_{ij} \alpha \cdot \langle s_i, f s_j\rangle \tilde{s}_i \cdot \tilde{s}_j^\flat.$$

Let $v \in \Gamma(TM)$ be the parallel transport of $v_y \in T_yM$. By the identity $\sum_k \tilde{s}_k \tilde{s}_k^\flat = \mathrm{Id}$ we conclude

$$F\gamma(v) = \sum_{ijk} \alpha \cdot \langle s_i, f s_j\rangle \langle \tilde{s}_j, \gamma(v)\tilde{s}_k\rangle \tilde{s}_i \cdot \tilde{s}_k^\flat.$$

By Lemma 7.17 we know that $\langle \tilde{s}_j, \gamma(v)\tilde{s}_k\rangle$ is constant and hence equal to $\langle s_j, \gamma(v_y)s_k\rangle$. This simplifies $F\gamma(v)$ into

$$F\gamma(v) = \sum_{ijk} \alpha \cdot \langle s_i, f\gamma(v_y)s_k\rangle \tilde{s}_i \cdot \tilde{s}_k^\flat.$$

By definition f must commute with $\gamma(v_y)$ and so F commutes with $\gamma(v)$. Hence F is a Clifford bundle endomorphism.

At last consider the subbundle of $\mathrm{End}(S)$ spanned by the Clifford action and all Clifford endomorphisms that are locally of the form $\sum_i \alpha_i \otimes \mathrm{Id} \otimes f^i \in C^\infty(T_yM) \otimes \mathrm{Cl}(T_yM) \otimes \mathrm{End}_{\mathrm{Cl}}(S_y)$. By dimension counting we conclude it spans the whole bundle and so all Clifford bundle endomorphisms are of the form $\sum_i \alpha_i \otimes \mathrm{Id} \otimes f^i \in C^\infty(U_y) \otimes \mathrm{Cl}(T_yM) \otimes \mathrm{End}_{\mathrm{Cl}}(S_y)$. □

Lemma 7.19 (Roe [1998], Proposition 12.22)**.** *Let $\{x^\mu\}$ be an orthonormal coordinate frame of T_yM and extend this to an orthonormal coordinate frame on TM using parallel transport. Let R be the Riemann curvature tensor. Then for all μ the covariant derivative ∇_μ is a differential operator of Getzler degree one and the Getzler symbol of ∇_μ equals*

$$\sigma_1(\nabla_\mu) = \frac{\partial}{\partial x^\mu} + \frac{1}{4}\sum_\nu \left\langle \frac{\partial}{\partial x^\mu}, R(\cdot,\cdot)\frac{\partial}{\partial x^\nu}\right\rangle x^\nu.$$

Proof. Let p be a section of $S \otimes S_y^*$ and assume without loss of generality that $p = \sum_{ij} p_{ij} \cdot \tilde{s}_i \cdot s_j^\flat$ where $p_{ij} \in C^\infty(U_y)$. In the trivialization induced by the parallel transport p equals $\sum_{ij} p_{ij} \cdot s_i \cdot s_j^\flat$. We show that in this trivialization $\nabla_\mu p$ equals $\sigma_1(\nabla_\mu)\, p$ plus lower order terms. From the Leibniz rule it follows that

$$\nabla_\mu p = \sum_{ij} \nabla_\mu \left(p_{ij} \tilde{s}_i\, s_j^\flat \right) = \sum_{ij} \frac{\partial p_{ij}}{\partial x^\mu} \tilde{s}_i\, s_j^\flat + \sum_{ijk} \langle \tilde{s}_k, \nabla_\mu \tilde{s}_i \rangle \cdot p_{ij}\, \tilde{s}_k \cdot s_j^\flat.$$

Locally we can write $\nabla_\mu p$ as $(\frac{\partial}{\partial x^\mu} + \omega)\, p$ where ω is a linear endomorphism on S_y with the elements $\omega_{ki} = \langle \tilde{s}_k, \nabla_\mu \tilde{s}_i \rangle \in C^\infty(T_y M)$. Clearly $\frac{\partial}{\partial x^\mu}$ is a degree one operator and so we only need to show that ω has Getzler degree one. For this we use a trick. Let K be the curvature tensor w.r.t. the connection ∇ of S and let $\mathcal{R}$ be the radial vector field. That is, $\mathcal{R} = \sum_\nu x^\nu \frac{\partial}{\partial x^\nu}$. Note that the Lie bracket between $\frac{\partial}{\partial x^\mu}$ and $\mathcal{R}$ equals $\frac{\partial}{\partial x^\mu}$. So the matrix element ω_{ki} equals $= \langle \tilde{s}_k, \nabla_{[\frac{\partial}{\partial x^\mu}, \mathcal{R}]} \tilde{s}_i \rangle$. We write this in terms of the curvature.

$$\omega_{ki} = \langle \tilde{s}_k, \nabla_\mu \tilde{s}_i \rangle = \left\langle \tilde{s}_k, \left(-K\left(\frac{\partial}{\partial x^\mu}, \mathcal{R} \right) + \nabla_\mu \nabla_{\mathcal{R}} - \nabla_{\mathcal{R}} \nabla_\mu \right) \tilde{s}_i \right\rangle. \tag{7.3}$$

By the definition of the parallel transport it follows that $\nabla_{\mathcal{R}} \tilde{s}_i = 0$. Using that ∇ is a metric connection equation 7.3 simplifies into

$$\langle \tilde{s}_k, \nabla_\mu \tilde{s}_i \rangle = -\left\langle \tilde{s}_k, K\left(\frac{\partial}{\partial x^\mu}, \mathcal{R} \right) \tilde{s}_i \right\rangle - \mathcal{L}_{\mathcal{R}} \langle \tilde{s}_k, \nabla_\mu \tilde{s}_i \rangle.$$

Moving $\mathcal{L}_{\mathcal{R}}$ to the left hand side, we get $(1+\mathcal{L}_{\mathcal{R}})\omega = -K\left(\frac{\partial}{\partial x^\mu}, \mathcal{R}\right)$. The Weitzenbock formula states that $K = R^S + F^S$ where F^S is a Clifford endomorphism and $R^S = \frac{1}{4}\sum_{\nu\rho} \gamma_\nu \gamma_\rho \langle R(\cdot,\cdot)\frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial x^\rho}\rangle$. So the matrix ω satisfies

$$\begin{aligned}(1+\mathcal{L}_{\mathcal{R}})\omega =& -R^S\left(\frac{\partial}{\partial x^\mu}, \mathcal{R} \right) - F^S\left(\frac{\partial}{\partial x^\mu}, \mathcal{R} \right) \\ =& \frac{1}{4} \sum_{\nu\rho\sigma} \gamma_\nu \gamma_\rho \left\langle \frac{\partial}{\partial x^\mu}, R\left(\frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial x^\rho} \right) \frac{\partial}{\partial x^\sigma} \right\rangle x^\sigma - F^S\left(\frac{\partial}{\partial x^\mu}, \mathcal{R} \right). \end{aligned} \tag{7.4}$$

Now we write ω in local coordinates. Let I, J be multi-indices over $\{x^\mu\}$ with degree $|I|$ and $|J|$ and write ω as $\sum_{IJ} x^I \otimes c^J \otimes \omega_{IJ} \in C^\infty(T_y M) \otimes \mathrm{Cl}(T_y M) \otimes \mathrm{End}_{\mathrm{Cl}}(S_y)$. The left hand side of equation 7.4 equals

$$\begin{aligned}(1+\mathcal{L}_{\mathcal{R}})\omega =& \sum_{IJ} \left(1 + \sum_\tau x^\tau \frac{\partial}{\partial x^\tau} \right) x^I \otimes c^J \otimes \omega_{IJ} \\ =& \sum_{IJ} \left(1 + \sum_\tau x^\tau \frac{I(\tau)}{x^\tau} \right) x^I \otimes c^J \otimes \omega_{IJ} \\ =& \sum_{IJ} (1 + |I|)\, x^I \otimes c^J \otimes \omega_{IJ}.\end{aligned}$$

We compare this to the right hand side of equation 7.4. Because F^S is a Clifford endomorphism, we conclude that $F^S\left(\frac{\partial}{\partial x^\mu}, \mathcal{R}\right)$ is a Getzler degree zero operator. Also, $R^S\left(\frac{\partial}{\partial x^\mu}, \mathcal{R}\right)$ has Getzler degree one and so

$$\begin{aligned}&\sum_{IJ}(1+|I|)\, x^I \otimes c^J \otimes \omega_{IJ}\\ &\quad = \frac{1}{4}\sum_{\nu\rho\sigma}\gamma_\nu\gamma_\rho\left\langle \frac{\partial}{\partial x^\mu}, R\left(\frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial x^\rho}\right)\frac{\partial}{\partial x^\sigma}\right\rangle x^\sigma + \text{lower order terms.}\end{aligned}$$

We conclude for the top order that $X^I = x^\sigma$, $C^J = \gamma_\nu\gamma_\rho$ and $\omega_{IJ} = \left\langle \frac{\partial}{\partial x^\mu}, R\left(\frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial x^\rho}\right)\frac{\partial}{\partial x^\sigma}\right\rangle$. Hence ω has Getzler degree one and is given by

$$2\omega = \frac{1}{4}\sum_{\nu\rho\sigma}\gamma_\nu\gamma_\rho\left\langle \frac{\partial}{\partial x^\mu}, R\left(\frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial x^\rho}\right)\frac{\partial}{\partial x^\sigma}\right\rangle x^\sigma + \text{lower order terms.}$$

The Getzler symbol of ω equals

$$\begin{aligned}\sigma_1(\omega) &= \frac{1}{8}\sum_{\nu\rho\sigma}\left\langle \frac{\partial}{\partial x^\mu}, R\left(\frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial x^\rho}\right)\frac{\partial}{\partial x^\sigma}\right\rangle x^\sigma \cdot \mathrm{d}\, x^\nu \wedge \mathrm{d}\, x^\rho\\ &= \frac{1}{4}\sum_{\sigma}\left\langle \frac{\partial}{\partial x^\mu}, R(\cdot,\cdot)\frac{\partial}{\partial x^\sigma}\right\rangle x^\sigma\end{aligned}$$

and we finish the proof. □

Corollary 7.20 (Roe [1998], Example 12.16)**.** *The Dirac operator D has Getzler degree two and the Getzler symbol of D equals*

$$\sigma_2(D) = \sum_\mu \frac{\partial}{\partial x^\mu}\cdot \mathrm{d}\, x^\mu + \frac{1}{8}\sum_{\mu\nu\rho\sigma}\left\langle \frac{\partial}{\partial x^\mu}, R\left(\frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial x^\rho}\right)\frac{\partial}{\partial x^\sigma}\right\rangle x^\sigma \cdot \mathrm{d}\, x^\mu \wedge \mathrm{d}\, x^\rho \wedge \mathrm{d}\, x^\sigma$$

where $\{x^\mu\}$ is the Riemannian normal coordinate frame.

Proof. Recall that in the Riemannian normal coordinate frame $D = \sum_\mu \gamma_\mu \nabla_\mu$ and use Lemmas 7.17 and 7.19. □

Proposition 7.21 (Roe [1998], Proposition 12.17)**.** *Let $S \to (M, g)$ be a Clifford bundle over an even dimensional Riemannian manifold and let D be the Dirac operator. Let R be the Riemann curvature and F^s be the twisting curvature. Let $\{x^\mu\}$ be the Riemannian normal coordinate frame and denote $R_{\mu\nu} = \left\langle \frac{\partial}{\partial x_\mu}, R(\cdot,\cdot)\frac{\partial}{\partial x_\nu} \right\rangle \in \Omega^2(M)$. The operator D^2 has Getzler degree two and its symbol equals*

$$\sigma_2(D^2) = -\sum_\mu \left(\frac{\partial}{\partial x^\mu} + \frac{1}{4} \sum_\nu R_{\mu\nu} x^\nu \right)^2 + F^S.$$

Proof. Proposition 1.27 states that $D^2 = \nabla^*\nabla + \mathrm{F}^S + \frac{1}{4}\kappa$ where F^S is the Clifford contraction of the twisting curvature and κ is the scalar curvature. In Riemannian normal coordinates we write this identity as

$$D^2 = -\sum_\mu \nabla_\mu \nabla_\mu + \frac{1}{2}\sum_{\mu,\nu} \gamma_\mu \gamma_\nu F^s \left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right) + \frac{1}{4}\kappa.$$

Recall that the twisting curvature is a Clifford endomorphism. Hence by Lemmas 7.17, 7.18 and 7.19 we conclude that D^2 has Getzler degree 2 and the symbol is given by

$$\sigma_2(D^2) = -\sum_\mu \sigma_2(\nabla_\mu^2) + \frac{1}{2}\sum_{\mu,\nu} \sigma_2\left(\gamma_\mu \gamma_\nu F^s \left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right)\right) + \sigma_2\left(\frac{1}{4}\kappa\right). \tag{7.5}$$

The scalar curvature and $F^s\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$ are Clifford endomorphisms and hence it has at most degree zero. So we simplify equation 7.5 into

$$\begin{aligned}
\sigma_2(D^2) &= -\sum_\mu \sigma_1(\nabla_\mu)^2 + \frac{1}{2}\sum_{\mu,\nu} \sigma_1(\gamma_\mu)\sigma_1(\gamma_\nu)\sigma_0\left(F^s \left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right)\right) \\
&= -\sum_\mu \sigma_1(\nabla_\mu)^2 + \frac{1}{2}\sum_{\mu,\nu} F^s \left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right) \mathrm{d}\, x^\mu \wedge \mathrm{d}\, x^\nu \\
&= -\sum_\mu \sigma_1(\nabla_\mu)^2 + F^s.
\end{aligned}$$

Using Lemma 7.19 we conclude the result. ◻

7.3 The symbol of the heat kernel

In chapter 4 we showed that each Dirac operator on a compact Riemannian manifold has a heat kernel k_t. Using the heat equation we calculate the Getzler symbol of the heat kernel.

Proposition 7.22 (Roe [1998], Proposition 12.24). *Let $S \to (M, g)$ be a Clifford bundle over an n dimensional Riemannian manifold and let D be the Dirac operator. Assume that n is even. Let k_t be the heat kernel w.r.t. the generalized Laplacian D^2 and denote the formal power series of k_t as $\frac{1}{(4\pi t)^{n/2}} e^{-r^2/4t} \sum_{i=0}^{\infty} t^i \Phi_i$. Then for all i the kernel Φ_i has at most Getzler degree $2i$ and the symbol of Φ_i satisfies*

$$\begin{aligned} r\frac{\partial}{\partial r}\sigma_0(\Phi_0) =&0 \\ \left(r\frac{\partial}{\partial r}+i\right)\sigma_{2i}(\Phi_i) =&-\sigma_2(D^2)\sigma_{2i-2}(\Phi_{i-1}) \qquad \forall i \in \mathbb{N}. \end{aligned}$$

Proof. Let $y \in M$. In Theorem 4.13 we showed that k_t has a formal solution to the heat equation by proving that Φ_i is the unique solution of the differential equations

$$\left(\frac{r}{4g}\frac{\partial g}{\partial r}+\nabla_{\mathcal{R}}\right)\Phi_0(\cdot, y) =0 \tag{4.6}$$

$$\left(i+\frac{r}{4g}\frac{\partial g}{\partial r}+\nabla_{\mathcal{R}}\right)\Phi_i(\cdot, y) =-D^2\Phi_{i-1}(\cdot, y) \qquad \forall i > 0. \tag{4.7}$$

Here r is the distance from y, $\mathcal{R} = r\,\partial/\partial r$ is the radial vector field originating in y and g is the determinant of the metric. Suppose that $\Phi_i(\cdot, y)$ has Getzler degree k_i for some $k_i \in \mathbb{N}$. The symbol of Φ_i equals

$$\begin{aligned} \sigma_{k_0}\left(\left(\frac{r}{4g}\frac{\partial g}{\partial r}+\nabla_{\mathcal{R}}\right)\Phi_0(\cdot, y)\right) =&0 \\ \sigma_{k_i}\left(\left(i+\frac{r}{4g}\frac{\partial g}{\partial r}+\nabla_{\mathcal{R}}\right)\Phi_i(\cdot, y)\right) =&\sigma_{k_i}(-D^2\Phi_{i-1}(\cdot, y)) \qquad \forall i > 0. \end{aligned}$$

Multiplication by i is a degree zero operator and $\frac{r}{4g}\frac{\partial g}{\partial r}$ has degree -1. Also $\nabla_{\mathcal{R}}$ is a degree zero operator. Indeed, let $\{x^\mu\}$ be the Riemannian orthonormal coordinate and notice that $\nabla_{\mathcal{R}} = \sum_\mu x^\mu \cdot \nabla_\mu$. The symbol of $\nabla_{\mathcal{R}}$ equals

$$\sigma_0(\nabla_{\mathcal{R}}) = \sum_\mu x^\mu \frac{\partial}{\partial x^\mu} + \frac{1}{4}\sum_\nu R_{\mu\nu}x^\mu x^\nu.$$

Using the anti-symmetry in R we simplify $\sigma_0(\nabla_{\mathcal{R}})$ into $x^\mu \frac{\partial}{\partial x^\mu}$ and this equals $r\frac{\partial}{\partial r}$. Therefore the symbol of Φ_k is

$$r\frac{\partial}{\partial r}\left(\sigma_{k_0}\Phi_0(\cdot,y)\right) = 0 \tag{7.6}$$

$$\left(r\frac{\partial}{\partial r}+i\right)\sigma_{k_i}\Phi_i(\cdot,y) = -\sigma_2(D^2)\sigma_{k_i-2}(\Phi_{i-1}) \qquad \forall i>0. \tag{7.7}$$

Inductively we conclude that Φ_i has Getzler degree $2i$. □

Comparing equations 4.6 and 4.7 with 7.6 and 7.7 suggest we need to investigate the differential equation $\frac{\partial}{\partial t}+\sigma_2(D^2)=0$. Even more, this is a heat equation. Indeed, it is sufficient to show that $\sigma_2(D^2)$ is a generalized Laplacian. For this consider the trivial bundle $T_yM \times \Lambda^\bullet T_yM \otimes \mathrm{End}_{\mathrm{Cl}}(S_y) \to T_yM$. Equip $\Lambda^\bullet T_yM$ with the induced metric from T_yM and equip $\mathrm{End}_{\mathrm{Cl}}(S_y)$ with the Hilbert-Schmidt metric. The operator $\frac{\partial}{\partial x^\mu}+\frac{1}{4}\sum_\nu R_{\mu\nu}x^\nu$ is a covariant derivative for all x^μ. Denote this connection as ∇. This connection is metric compatible, because $R_{\mu\nu}$ is skew-symmetric. Hence the operator $\sigma_2(D^2)$, which equals $\nabla^*\nabla+F^S$, is a generalized Laplacian. Next we explicitly calculate the heat kernel.

Proposition 7.23 (Roe [1998], Proposition 12.24)**.** *Let $S\to(M,g)$ be a Clifford bundle over an n dimensional Riemannian manifold. Let D be the Dirac operator and let $k_t=\frac{1}{(4\pi t)^{n/2}}\exp(-r^2/4t)\sum_i t^i\Phi_i$ be the formal solution of the heat equation with respect to the generalized Laplacian D^2. Let $y\in M$ and consider the trivial vector bundle $T_yM\times\Lambda^\bullet T_yM\otimes\mathrm{End}_{\mathrm{Cl}}(S_y)\to T_yM$. Let $W_t=\frac{1}{(4\pi t)^{n/2}}\exp(-r^2/4t)\sum_i t^i\Theta_i$ be the formal solution of the heat equation with respect to the generalized Laplacian $\sigma_2(D^2)$. Then for all $x\in T_yM$ and $i\in\mathbb{N}$ the components Θ_i satisfy*

$$\Theta_i(x,y)=\sigma_{2i}(\Phi_i).$$

Proof. In Theorem 4.13 we showed that Θ_i is the solution of the differential equation

$$\left(\frac{r}{4g}\frac{\partial g}{\partial r}+\nabla_{\mathcal{R}}\right)\Phi_0(\cdot,y) = 0 \tag{4.6}$$

$$\left(i+\frac{r}{4g}\frac{\partial g}{\partial r}+\nabla_{\mathcal{R}}\right)\Phi_i(\cdot,y) = -D^2\Phi_{i-1}(\cdot,y) \qquad \forall i>0. \tag{4.7}$$

Comparing this to Equations 7.6 and 7.7 we conclude that $\sigma_{2i}\Phi_i$ satisfies the same set of differential equations. The only possible difference is in the initial conditions. This is not the case. For $i = 0$, the initial condition is that $\Phi_0(y, y)$ equals the identity map. The symbol map doesn't alter the identity map and so $\Phi_0(\cdot, y) = \sigma_0(\Phi_0)$. For $i > 0$ the initial condition is determined by the requirement that Φ_i is differentiable at the origin. Because $\sigma_{2i}(\Phi_i)$ is differentiable by construction, we conclude the result. □

If we replace $T_yM \times \Lambda^\bullet T_yM \otimes \mathrm{End}_{\mathrm{Cl}}(S_y) \to T_yM$ with the vector bundle $\mathbb{R} \times T_yM \to T_yM$ we already have a heat kernel w.r.t. $\sigma_2(D^2)$, namely Mehlers kernel. Consider the Taylor series of this kernel in R and F. If replace $R \in M_{n\times n}(\mathbb{R})$ with the Riemann curvature and F with the twisting curvature we created a kernel on $\Lambda^\bullet T_yM \otimes \mathrm{End}_{\mathrm{Cl}}$. Note that the algebra generated by the Riemann curvature and the Twisting curvature is a nilpotent commutative algebra. Hence the Taylor series converges on $\Lambda^\bullet T_yM \otimes \mathrm{End}_{\mathrm{Cl}}$ and satisfies the heat equation. So Mehlers kernels is also the heat kernel w.r.t. $\sigma_2(D^2)$ and by uniqueness we conclude

Proposition 7.24 (Roe [1998], Proposition 12.26)**.** *Let $S \to (M, g)$ be a Clifford bundle over an even dimension Riemannian manifold. Let D be the Dirac operator and let $k_t = \frac{1}{(4\pi t)^{n/2}} \exp(-r^2/4t) \sum_i t^i \Phi_i$ be the formal solution of the heat equation w.r.t. D^2. Let $y \in M$, $U_y \subseteq M$ be a neighborhood of y such that $\exp_y^{-1} \colon U_y \to T_yM$ is a chart of M, let $x \in U_y$ and let r be the distance between x and y. Then for all i the symbol $\sigma_{2i}(\Phi_i)$ satisfies*

$$\det{}^{1/2}\left(\frac{tR/2}{\sinh(tR/2)}\right) \exp\left[-\frac{1}{4t}\left\langle \frac{tR}{2}\coth\left(\frac{tR}{2}\right)x, x\right\rangle - tF\right] = \exp(-r^2/4t) \sum_i t^i \sigma_{2i}(\Phi_i).$$

7.4 Atiyah-Singer Index theorem

At last we consider the interaction between the symbol map and the trace and we prove the Atiyah-Singer index theorem. From the trace identities we know that only top Clifford degree part are non-vanishing in the trace. In the next definition and lemma we translate this in terms of Getzler symbols.

Definition 7.25. *Let $S \to (M, g)$ be a Clifford bundle on an even dimensional Riemannian manifold and let $y \in M$. The* **constant part of the Getzler symbol**

σ_k^0 is the projection of the Getzler symbol σ_k to $C_0^\infty(T_yM) \otimes \Lambda^\bullet T_yM \otimes \mathrm{End}_{\mathrm{Cl}}(S_y)$ where $C_0^\infty(T_yM)$ denotes the constant part of the Taylor grading.

Lemma 7.26. *Let $S \to (M, g)$ be a Clifford bundle on an $2n$ dimensional Riemannian manifold, let $y \in M$ and let p be a section of $S \otimes S_y$. Let γ_5 be the canonical grading operator. If S is canonically graded, then*

$$\mathrm{tr}(\gamma_5\, p(y)) \cdot \mathrm{Vol}(g) = (-i)^n\, \mathrm{tr}(\sigma_{2n}^0(p)).$$

Proof. In this proof we use the letters I and J for multi-indices and we denote their degrees with $|I|$ resp. $|J|$. Locally p is represented by an element $\sum_{IJ} x^I \otimes c^J \otimes p_{IJ}$ in $C^\infty(T_yM) \otimes \mathrm{Cl}(T_yM) \otimes \mathrm{End}_{\mathrm{Cl}}(S_y)$. The trace of $\gamma_5\, p(y)$ equals

$$\mathrm{tr}(\gamma_5\, p(y)) = \sum_{IJ} x^I \cdot \mathrm{tr}(\gamma_5\, c^J \cdot p_{IJ}).$$

Note that if $|I| \neq 0$, then $x^I = 0$ because we evaluate x at zero. Hence the trace of $p(y)$ is

$$\mathrm{tr}(\gamma_5\, p(y)) = \sum_J \cdot\, \mathrm{tr}(\gamma_5\, c^J \cdot p_{\emptyset,J}).$$

By lemma 5.22 it follows that $\mathrm{tr}(\gamma_5\, c^J \cdot p_{IJ}) = 0$ if $|J| \neq 2n$. The only non-vanishing multi-index of J is $(1, 2, \ldots, 2n)$. The same lemma states that $\mathrm{tr}(\gamma_5 c^{(1,\ldots 2n)} \cdot p_{I,(1,\ldots,2n)}) = (-i)^n\, \mathrm{tr}(p_{I,(1,\ldots,2n)})$ and we conclude

$$\mathrm{tr}(\gamma_5\, p(y)) = (-i)^n\, \mathrm{tr}(p_{\emptyset,(1,\ldots,2n)}).$$

The trace over the constant part of the Getzler symbol equals

$$\begin{aligned}\mathrm{tr}(\sigma_{2n}^0(p)) &= \sum_{\substack{I,J \\ |I|=0 \\ |I|+|J|=2n}} \mathrm{tr}(x^I \otimes \mathrm{d}\, x^J \otimes p_{IJ}) \\ &= \mathrm{tr}(p_{\emptyset,(1,2,\ldots,2n)})\, \mathrm{d}\, x^1 \wedge \mathrm{d}\, x^2 \wedge \ldots \wedge \mathrm{d}\, x^{2n}\end{aligned}$$

and so $\mathrm{tr}(\gamma_5 p(y)) = (-i)^{n/2}\, \mathrm{tr}(\sigma_{2n}^0(p))$. ☐

We finally have all the ingredients to prove the Atiyah-Singer index theorem.

Theorem 7.27 (Atiyah and Singer [1968]). *Let $S \to (M, g)$ be a Clifford bundle on a compact oriented n-dimensional Riemannian manifold M. Let D be the Dirac operator. If S is canonically graded, then the index of D is the integral over the n-form part of $\hat{A}(TM) \wedge \mathrm{ch}^{rel}(S)$. That is,*

$$\mathrm{Index}(D) = \int_M \hat{A}(TM) \wedge \mathrm{ch}^{rel}(S).$$

Proof. By Theorem 4.17 each Dirac operator has a unique heat kernel $k_t^H \in \Gamma(S \boxtimes S)$ such that the corresponding smoothing operator e^{-tD^2} satisfies $(\frac{\partial}{\partial t} + D^2)e^{-tD^2} = 0$. From Proposition 5.10 we know that the composition between the canonical grading operator γ_5 and the operator e^{-tD^2} is traceclass and its trace equals

$$\mathrm{Tr}(\gamma_5 e^{-tD^2}) = \int_{y \in M} \mathrm{tr}(\gamma_5\, k_t(y, y)) \,\mathrm{Vol}(g).$$

The McKean-Singer formula states that the trace of $\gamma_5 e^{-tD^2}$ equals the index of D and we conclude

$$\mathrm{Index}(D) = \int_{y \in M} \mathrm{tr}(\gamma_5 k_t(x, x)) \,\mathrm{Vol}(g).$$

Now Theorem 4.13 states that k_t has a formal solution $\frac{1}{(4\pi t)^{n/2}} e^{-r^2/4t} \sum_i t^i \Phi_i$. Because $\mathrm{Index}(D)$ is independent of the choice of t, all t-dependent terms in $\frac{1}{(4\pi t)^{n/2}} e^{-r^2/4t} \sum_i t^i \Phi_i$ will vanish and so

$$\mathrm{Index}(D) = \begin{cases} \frac{1}{(4\pi)^{n/2}} \int_{y \in M} \mathrm{tr}(\gamma_5 \Phi_{n/2}(x, x)) \,\mathrm{Vol}(M) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

By definition $\hat{A}(TM) \wedge \mathrm{ch}^{rel}(S)$ is an element of $H^{ev}(M) \otimes \mathbb{C}$. Hence if n is odd, then the n-form part of $\hat{A}(TM) \wedge \mathrm{ch}^{rel}(S)$ is zero and we conclude

$$\mathrm{Index}(D) = \int_M \hat{A}(TM) \wedge \mathrm{ch}^{rel}(S).$$

Now assume that n is even. By Lemma 7.26 the index of D equals

$$\mathrm{Index}(D) = \frac{1}{(4\pi i)^{n/2}} \int_M \mathrm{tr}(\sigma_n^0(\Phi_{n/2})).$$

According to Proposition 7.24 the constant part of the Getzler symbol $\sigma_n^0(\Phi_{n/2})$ equals the n-form part of

$$\det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)\exp(-F^S).$$

where R is the Riemann curvature and F^S is the twisting curvature. Up to the constant $2^{n/2} \times (2\pi i)^{n/2}$ this equals the n-form part of $\hat{A}(TM) \wedge \mathrm{ch}^{rel}(S)$ and we conclude

$$\mathrm{Index}(D) = \int_M \hat{A}(TM) \wedge \mathrm{ch}^{rel}(S)$$

for all $n \in \mathbb{N}$. □

7.5 Final remarks

In the previous section we finally concluded the proof of the Atiyah-Singer index theorem. For this we required that the manifold was compact. There is a version of the index theorem that does not require compactness. It is the local Atiyah-Singer index theorem and it states that for a Clifford bundle S on a (non)-compact oriented Riemannian manifold (M, g) the formal solution of the heat equation k_t satisfies

$$\lim_{t\to 0} \mathrm{tr}(\gamma_5 k_t(x,x)) \cdot \mathrm{Vol}(M) = \hat{A}(TM) \wedge \mathrm{ch}^{rel}(S) \tag{7.8}$$

for all $x \in M$. Compactness was needed to show the existence of the heat kernel and the existence of the trace. In the local index theorem we only take the trace over a fiber and hence the trace is well-defined. For a non-compact manifold it is uncertain if the heat kernel exists. However, there always exists a unique formal power series. The limit $\lim\limits_{t\to 0} k_t$ denotes the constant part of this formal power series and so Equation 7.8 is well-defined. Therefore, the given proof of the Atiyah-Singer index theorem also proves the local version.

Notice that the local index theorem extends the Fujikawa method for Abelian chiral anomalies to curved spacetime. Indeed, we regularize the Jacobian $J = \exp(-2i\,\mathrm{Tr}(\gamma_5\alpha))$ of Equation 3.3 as

$$\log J = -\,2i \int_{x\in M} \mathrm{Tr}(\gamma_5\, \alpha k_t(x,x)\, \mathrm{Vol}(M)).$$

By the local Atiyah-Singer index theorem this equals

$$\log J = -2i \int_{x \in M} \alpha \cdot \hat{A}(TM) \wedge \mathrm{ch}^{rel}(S).$$

By requiring that the curvatures are rapidly decreasing the Jacobian converges.

Also the non-Abelian chiral anomalies can be related to the Atiyah-Singer index theorem. Using a technique called descend equations(Stora [1985], Zumino [1983], Zumino et al. [1984], Zumino [1985]) one can relate the non-Abelian chiral anomaly in n dimensions with the Abelian chiral anomaly in $n+2$ dimensions. This technique is heavily based on Poincaré lemma and variational calculus. It can be formalized in K-theory and and the resulting theorem is called the family index theorem. For more information on descend equations see Bertlmann [1996] and for more information on the family index theorem see Berline et al. [2004].

Glossary: Differential geometry for physicists

There is a joke that "*differential geometry is the study of properties that are invariant under change of notation*" (Lee [2013]). This is especially true when we compare the notation in Riemannian geometry and gauge theory. If one is familiar with general relativity and field theory, then one has already seen many of the constructions used by mathematicians. So in this chapter we will translate the language used by mathematicians into the language used by physicists.
If one is already familiar with differential geometry, one can safely skip this chapter.

In this chapter we assume that M is a manifold, and we denote x as a point on M. In general we don't assume that M has a metric g. If we have an Euclidean metric g we call M a **Riemannian manifold**. For an Minkowski metric g we say that M is **Pseudo-Riemannian**. On a (Pseudo)-Riemannian manifold we often write the integral $\int_M f(x)\sqrt{g}\,\mathrm{d}\,x^1 \dots \mathrm{d}\,x^n$ as $\int_M f(x)\,\mathrm{Vol}(M)$. We say that $\mathrm{Vol}(M)$ is the **Riemannian volume form**.

Sections and fields

Vector fields is important in particle physics and in geometry. We see however that there physicists and mathematicians use them for different purposes and so use different notation. In physics we use fields to model the reality and we compare these models with experiments. To calculate comparable results we always need expressions in local coordinates. In geometry we study the global properties of vector fields and so we often prefer index-free notation. So if physicists denote a vector field they just write down $A^\mu(x)$ and say "$A^\mu(x)$ *is a field*". Mathematicians on the other hand write down $A \in \Gamma(TM)$ and say "A *is a section of the tangent bundle*".

An example of a vector field is given in Figure 6. Note that the vector field assigns each point of the manifold exactly one vector. Also note that each vector lies in a different vector space. Therefore we consider vector bundles: If we have for each point $x \in M$ a vector space E_x, we call the collection of all these vector spaces a **vector bundle**[17]. The collection of all tangent spaces is an example of a vector bundle. We call this bundle the tangent bundle and we denote it as TM. To emphasize the dependence of M we often write $E \to M$ for a vector bundle over M.

When we consider the tangent bundle TM of M we often consider one tangent space

[17] There are also some smoothness and compatibility requirements.

T_xM of M at x. This principle we can generalize to vector bundles. Each single vector space E_x inside of the vector bundle $E = \{E_x\}$ is called a **fiber** of E at x.

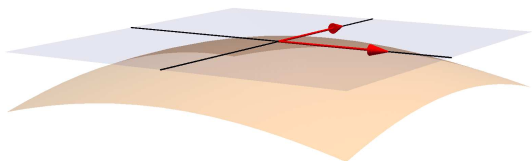


Figure 5: *An example of a vector bundle is the tangent bundle. It is the collection of all tangent spaces. A single vector space inside of a vector bundle called a fiber. So the blue tangent space depicted is an example of a fiber in a vector bundle.*

A synonym for a vector field is a section. A **section** of a vector bundle $E \to M$ is a function that assigns each point $x \in M$ a vector of E_x. This requirement is needed because we want that each base point obtains exactly one vector. We denote the space of all section of E as $\Gamma(E)$. We use this notation often in this thesis, because with a few symbols we can explain the whole structure of a vector field. For example, if you want to explain the electromagnetic four-potential A^μ in seven characters you just write "$A \in \Gamma(TM)$".

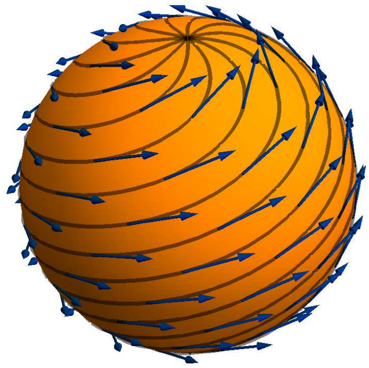


Figure 6: *An example of a vector field on a sphere. Notice that every basepoint has exactly one vector attached to it. Another word for a vector field is a section.*

We can also generalize the concept of bases of vector spaces to vector bundles. If we have a collection of vector fields which uniquely span all vector fields we call such collection a **frame**. For example the vectors $\{\frac{\partial}{\partial x^\mu}\}$ span the tangent spaces, but also defines local vector fields. Hence, $\{\frac{\partial}{\partial x^\mu}\}$ is a local frame of TM.

There is a trivial method to create a vector bundle from a vector space V. Just create a copy of V for each point $x \in M$ and bundle them in a set. Vector bundles that can be created this way are called **trivial vector bundles**. For example, A real scalar field is related to the trivial vector bundle $M \times \mathbb{R}$.

Must constructions we use to create new vector spaces from old ones we also can use to create new vector bundles. For example given a vector space we can consider the dual space, namely we can consider the space of bras instead of the space of kets. Hence if $E = \{E_x\}$ is a vector bundle then the bundle of dual spaces is also a vector bundle. We denote this vector bundle as E^*. Tensor products and direct sums are also constructions we generalize to vector bundles. In table 6 we give some examples of such constructions we show where they are used in physics.

Type	Field	Section of	Alternative notations
Real scalar field	ϕ	$M \times R$	$\Gamma(M \times \mathbb{R}) = C^\infty(M)$
Complex scalar field	ϕ	$M \times \mathbb{C}$	$\Gamma(M \times \mathbb{R}) = C^\infty(M) \otimes \mathbb{C}$
Real vector field	ϕ^μ	TM	
	ϕ_μ	T^*M	$\Gamma(T^*M) = \Omega^1(M)$
Complex vector field	ϕ^μ	$TM \otimes \mathbb{C}$	
	ϕ_μ	$T^*M \otimes \mathbb{C}$	
Real gauge Field	A^a_μ	$T^*M \otimes \mathfrak{g}$	
Tensor field	$T^{\mu\nu}$	$TM \otimes TM$	
	$T^\mu{}_\nu$	$TM \otimes T^*M$	$TM \otimes T^*M = \mathrm{End}(TM)$
	$T_{\mu\nu}$	$T^*M \otimes T^*M$	
Antisymmetric tensor field	$F^{\mu\nu}$	$TM \wedge TM$	$\Lambda^2 TM$
	$F_{\mu\nu}$	$\Lambda^2 T^*M$	$\Gamma(\Lambda^2 T^*M) = \Omega^2(M)$.
	$\epsilon_{\mu\nu\rho\sigma}$	$\Lambda^4 T^*M$	$\Gamma(\Lambda^4 T^*M) = \Omega^4(M)$
Classical Dirac field on $\mathbb{R}^4$	$\psi^{\mu,a}$	$\mathbb{R}^4 \times \mathbb{C}^4$	

Table 6: *Examples of different fields and the common used notation.*

Derivatives and curvature

In calculus we defined the derivative of a function $f(x)$ at $a \in \mathbb{R}$ as the limit $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$. For functions there is a canonical way to extend this definition to manifolds. That is, we take the flow ϕ_t in a given direction $v \in T_aM$ and we follow it for a given time t. This gives us a point $x = \phi_t(a) \in M$. The derivative $\frac{\partial f}{\partial v}$ is defined as $\lim\limits_{t\to 0} \frac{f(x)-f(a)}{t}$. However, on curved manifolds there is no canonical method to extend the directional derivative for fields. This is because there is no canonical way to compare different fibers. See figure 7. In this paragraph we study different generalizations of derivatives on fields and we study which extra structure they require.

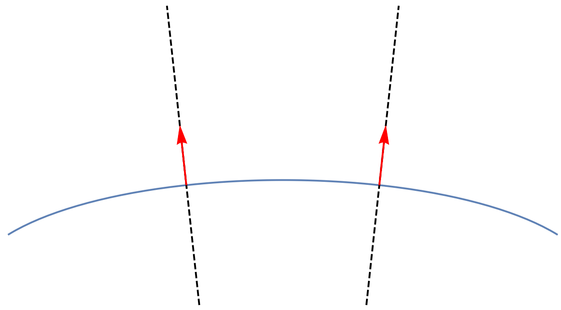


Figure 7: *In this picture we see a manifold(blue line) and different vectors(red) in a vector bundle. We cannot compare the vectors, because they are in a different fibre(dotted line). So there is no canonical method to generalize the derivative for sections on curved manifolds, because there is no canonical method to compare different fibers.*

For vector fields on the tangent bundle there is a canonical derivative and it is called the **Lie derivative**. For the tangent bundle we use the flow ϕ_t to identify different fibers. Under this identification the Lie derivative of a field $\psi \in \Gamma(TM)$ in the direction $v \in \Gamma(TM)$ at $a \in M$ is $\lim\limits_{t\to 0} \frac{\psi(\phi_t(a))-\psi(a)}{t}$. Usually we write this as $\mathcal{L}_v(\psi)$.

Note that v and ψ are the same type of field. To show this equal footing, one also writes $L_v\psi$ as $[v,\psi]$. This bracket is called the **Lie bracket**. It also behaves like a bracket, because one can show that $[v,\psi] = -[\psi,v]$. In general the flow identifies different fibers of the dual tangent bundle, but it also works for tensor bundles of TM. So for all these cases we can define an derivative and for all these cases we call it the Lie derivative.

Another method to define the derivative on fields uses the Leibniz rule. Recall that for multi-variable calculus, the Leibniz rule reads

$$\frac{\partial}{\partial v}(f\cdot\vec{g}) = \frac{\partial f}{\partial v}\cdot\vec{g} + f\cdot\frac{\partial\vec{g}}{\partial v}$$

for all $f \in \mathbb{R}^n \to \mathbb{R}$, $\vec{g} \in R^n \to \mathbb{R}^m$ and $v \in \mathbb{R}^n$. A **covariant derivative** on a vector bundle E is a map which maps a vector field of TM and a section of E to a section of E such that the Leibniz rule holds

$$\nabla_\mu(f\cdot\psi) = f\cdot\nabla_\mu\psi + \frac{\partial f}{\partial x^\mu}\psi.$$

It must also be linear in the following way:

$$\begin{aligned}\nabla_v(\psi_1+\psi_2) &= \nabla_v(\psi_1)+\nabla_v(\psi_2)\\ \nabla_{v_1+f(x)v_2}\psi &= \nabla_{v_1}\psi + f(x)\cdot\nabla_{v_2}\psi \end{aligned} \tag{7.9}$$

As you see we usually denote the vector field of TM with a subscript.

In gauge theory the covariant derivative is defined such that it acts covariant under gauge transformations. However, it also satisfies the properties listed above. Therefore, the covariant derivative in gauge theory is an example of a covariant derivative. Another example is the Levi-Civita connection. It is the unique covariant derivative on TM such that it is compatible[18] with the metric and torsion free[19]. A non-example is the Lie-derivative on vector fields. Indeed, for any smooth map $f \in \mathbb{C}^\infty(M)$ and vector fields $u, v \in \Gamma(M)$, we have

$$\begin{aligned}\mathcal{L}_{fu}v &= -\mathcal{L}_v(f\cdot u)\\ &= -\partial_v f\cdot u - f\cdot\mathcal{L}_v u\\ &= -\partial_v f\cdot u + f\cdot\mathcal{L}_u v.\end{aligned}$$

This does not satisfy condition 7.9. Another word for the covariant derivative is a **connection**. Both words are frequently used in this thesis.

For a given connection $\nabla\colon \Gamma(E)\to\Gamma(E\otimes T^*M)$ we define the **curvature tensor** $K\in\Gamma(T^*M\otimes T^*M\otimes \mathrm{End}(E))$ as

$$K(u,v) = \nabla_u\nabla_v - \nabla_v\nabla_u - \nabla_{[u,v]}.$$

In local coordinates the term $\nabla_{[u,v]}$ vanishes and is usually omitted in physics literature. However due to this term, the curvature tensor doesn't acts like a differential operator but acts tensorial.

At last we introduce the exterior derivative. This derivative makes use of the fact that antisymmetric tensor fields forms a Grassmann algebra where fields on T^*M are the Grassmann numbers. In mathematics this algebra is more commonly known as the **exterior algebra on forms** and it is defined as follows: Let ϕ be am antisymmetric $(0,k)$-tensor field and let ψ be am antisymmetric $(0,m)$-tensor field. The **wedge product** $\phi\wedge\psi$ is an antisymmetric $(0,k+m)$-tensor field. Up to an constant it is

[18]That is $\mathcal{L}_u g(v,w) = g(\nabla_u v, w) + g(v, \nabla_u w)$ for all $u, v, w \in \Gamma(TM)$

[19]That is $\nabla_u v - \nabla_v u = [u,v]$ for all $u, v \in \Gamma(TM)$.

the graded sum of all permutation of the indices. For example if ϕ is a $(0,2)$-tensor field and ψ is a $(0,1)$-tensor field, then

$$(\phi \wedge \psi)_{\mu\nu\rho} = \frac{1}{2}(\phi_{\mu\nu}\psi_\rho + \phi_{\nu\rho}\psi_\mu + \phi_{\rho\mu}\psi_\nu - \\ -\phi_{\nu\mu}\psi_\rho - \phi_{\rho\nu}\psi_\mu - \phi_{\mu\rho}\psi_\nu).$$

By direct calculation we can check that $\phi \wedge \psi$ is antisymmetric in its indices and that $\phi \wedge \psi = (-1)^k \psi \wedge \phi$. So the wedge product defines a Grassmann algebra on the antisymmetric tensor fields. Usually the space of antisymmetric $(0,k)$-tensors is denoted by $\Gamma(\Lambda^k T^* M)$ or $\Omega^k(M)$. Antisymmetric $(0,k)$-tensors are also called k-**forms**.

Notice that $\Omega^0(M)$ is just the space of smooth functions and $\Omega^1(M)$ is the space of vector fields. The **exterior derivative** for a function $f \in \Omega^0(M)$ is the unique dual vector field $\mathrm{d}\, f \colon \Gamma(TM) \to C^\infty(M)$ such that $\mathrm{d}\, f(v) = \frac{\partial f}{\partial v}$ (Compare this to the definition of the variational derivative). Hence, d is a map from $\Omega^0(M)$ to $\Omega^1(M)$. We extend d to a map from $\Omega^k(M)$ to $\Omega^{k+1}(M)$ using the following rules:

- $\mathrm{d} \circ \mathrm{d} \colon \Omega^k(M) \to \Omega^{k+2}(M) = 0$ $\qquad \forall k \in \mathbb{N}$
- $\mathrm{d}(\alpha \wedge \beta) = (\mathrm{d}\, \alpha) \wedge \beta - (-1)^k \alpha \wedge (\mathrm{d}\, \beta)$ $\qquad \forall \alpha \in \Omega^k(M), \beta \in \Omega^m(M).$

The exterior derivative is related to the Lie derivative by the use of the **interior product**. This is a map $\iota \colon \Gamma(TM) \times \Omega^k(M) \to \Omega^{k-1}(M)$ which reduces a form with one degree. For a $(0,k)$-tensor field T it is defined as

$$(\iota_\mu T)_{\nu\rho\sigma\ldots} = T_{\mu\nu\rho\sigma\ldots}.$$

Cartan showed that the Lie derivative can be expressed as

$$\mathcal{L}_\mu = \iota_\mu \mathrm{d} + \mathrm{d}\, \iota_\mu.$$

This result is called **Cartans magic formula**.

Topology of vector bundles

As mathematicians we often ask the question: "How can we tell two spaces apart?". One of the main theorems in differential geometry is that the exterior derivative can be used to discriminate different manifolds. First we recall this result and then we explain how this is relevant for physics.

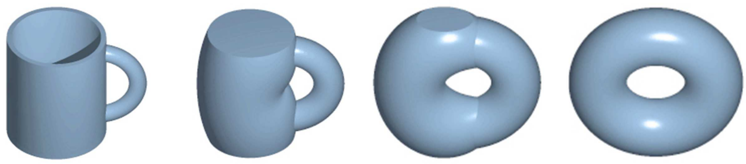


Figure 8: How to transform a coffee mug into a donut (Commons [2007])

Before we know the difference between manifold we need to know when they are the same. There is a saying that for a mathematician a donut and a coffee mug are equal. How weird as this sounds but if the mug was not baked then the mag can be continuously molded into a donut. That is, it can be molded *without tearing, gluing, creating holes or closing holes*. This is shown in figure 8. We say that if two spaces are equivalent up to continuous deformation they are **homotopy equivalent**.

Look at figure 9. We added a subscript to the exterior derivative to keep track of the vector spaces it acts on. So d_{k-1} is a map from $\Omega^{k-1}(M)$ into $\Omega^k(M)$. It doesn't necessary mean that the whole space $\Omega^k(M)$ is reached. We call the subspace in $\Omega^k(M)$ that is is reached the **image** of d_{k-1} and is denoted by $\mathrm{Im}\, d_{k-1}$. In figure 9 the images are depicted as red squares. The blue square represent the **kernel** of d_k. It is the subspace in $\Omega^k(M)$ that is mapped to zero by d_k and we denote it by $\ker \mathrm{d}_k$. Because $d_k d_{k-1} = 0$, the image always lies inside of the kernel. That is, the red square lies inside the blue square. The part of the blue that is not covered by the red square is called the k-th cohomology class. We denote this by $H^k_{dR}(M)$. The collection of all $H^k_{dR}(M)$ is called the **de Rham cohomology**. Formally $H^k_{dR}(M)$ is a subspace of $\ker \mathrm{d}_k$ that is orthogonal to $\mathrm{Im}\, \mathrm{d}_{k-1}$.

One of the main theorem in differential geometry is that cohomology is invariant under continuous deformations. That means that if two spaces are homotopy equivalent, then they must have the same cohomology classes. So if the cohomology classes differ, then the manifolds must differ.

Also vector bundles can be characterized using differential operators. For this we don't use the exterior derivative but the curvature tensor. The idea is that from the curvature tensor we create elements in $H^\bullet_{dR}(M)$ called **characteristic classes**. These classes don't depend on the choice of connection and so it only depends on the shape of the vector bundle. This plays an important role in the study of anomalies. Indeed, anomalies are given in terms of characteristic classes. For example, the abelian

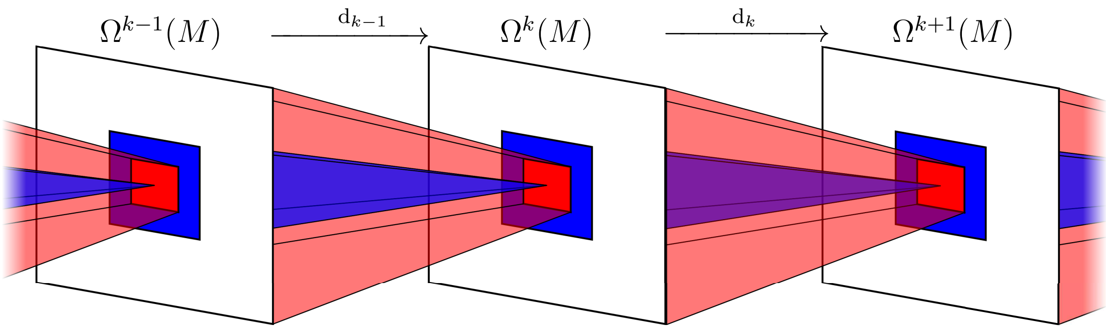


Figure 9: *In this image the red squares represents the image of the exterior derivative. The blue squares represent the kernel of* d. *Because* $\mathrm{d}^2 = 0$, *the image always lies inside the kernel.*

anomaly is proportional to

$$\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}).$$

This is proportional to the second Chern class. So the shape of the vector bundle breaks the classical conservation laws. More details on characteristic classes can be found in chapter 6.

Appendix: Source code

In this chapter we add the source codes we used to do the nasty calculations. We performed the calculations using FORM from Vermaseren [2000].

```
*#-;

*
* Define global variables
*

Indices m1, ..., m10;
Indices x1, ..., x10;

Functions g5, g;
CFunction ep(a);

Symbol i;
Symbol z1,..., z10;

.global

*
*       Give an explicit expression for D^2.
*       The results are hardcoded later.
*

Symbol M;
Functions D1, D2, d;
Functions Pp, Pm;

Functions V,A, dA, dV;
CFunctions k;

* D
Local expr1 = 1/M^2 * (g(m1) * D1(m1) * g(m2) * D1(m2));

* Dbar
Local expr2 = 1/M^2 * (g(m1) * D2(m1) * g(m2) * D2(m2));
```

```
* Give definition of D and D Bar
Id D1(x1?) = d(x1) - i * V(x1) + A(x1) * g5;
Id D2(x1?) = d(x1) - i * V(x1) - A(x1) * g5;

* Reorder Clifford Multiplication
repeat;
Id d(x1?) * g(x2?) = g(x2) * d(x1);
Id V(x1?) * g(x2?) = g(x2) * V(x1);
Id A(x1?) * g(x2?) = g(x2) * A(x1);

Id g(x1?) * g5 = - g5 * g(x1);
Id A(x1?) * g5 = g5 * A(x1);
Id V(x1?) * g5 = g5 * V(x1);
Id d(x1?) * g5 = g5 * d(x1);

Id g5 * g5 = 1;
endrepeat;

* Pull e^{ikx} through expression
Id d(x1?) = d(x1) + i * k(x1) * M;
Id i^2 = -1;

* Write clifford multiplication using commutator and anticommutator
Id g(x1?)*g(x2?) = d_(x1,x2) + 1/2 * (g(x1)*g(x2) - g(x2)*g(x1));
Id g(x1?)*g(x2?)*d(x2?)*d(x1?) = g(x1)*g(x2)*d(x1)*d(x2);

* Take derivatives
repeat;
Id d(x1?) * A(x2?) = dA(x1,x2) + A(x2) * d(x1);
Id d(x1?) * V(x2?) = dV(x1,x2) + V(x2) * d(x1);
endrepeat;

* Print answer
Sum m1, ..., m10;
Bracket M,g, g5;
Print;
```

```
.store
Symbols s,t M;

*
* Calculate the Bardeen Anomaly
*

Functions Xa, Xb, Ya, Yb;

Set spaceIndOdd: m1,m3,m5,m7,m9;
Set spaceIndEven: m2,m4,m6,m8,m10;

Functions Xa1, ..., Xa4;
Functions Ya1, ..., Ya4;
Functions Xb1, ..., Xb4;
Functions Yb1, ..., Yb4;

Set gFields:
        Xa1, ..., Xa4,
        Xb1, ..., Xb4,
        Ya1, ..., Ya4,
        Yb1, ..., Yb4;
Functions gField;

Functions V, A, d;
Functions dA, dV;

Functions F,G;
CFunctions k;

*
* Expand Exp (Xa/M^2 + Ya/M) + Exp (Xb/M^2 + Yb/M)
*

Global expr = t/2 * g5 * M^4 * (1+
#do a = 1, 5
+ 1
#do b = 1, 'a'
```

```
* (Xa(spaceIndOdd['b'], spaceIndEven['b'])/M^2
        + Ya(spaceIndOdd['b'], spaceIndEven['b'])/M)
#enddo
/ fac_('a')
#enddo
) +
t/2 * g5 * M^4 * (1+
#do a = 1, 5
+ 1
#do b = 1, 'a'
* (Xb(spaceIndOdd['b'], spaceIndEven['b'])/M^2
        + Yb(spaceIndOdd['b'], spaceIndEven['b'])/M)
#enddo
/ fac_('a')
#enddo
);

* Consider only constant order in M
Id M^{-1} = 0;


* Write X = X1 + g5 * X2 + g(mu) g(nu) X3(mu,nu)
*         + g5 g(mu)g(nu) X4(mu,nu). Same for Y
Id Xa(x1?, x2?) = Xa1(x1,x2) + g5 * Xa2(x1,x2)
        + g(x1) * g(x2) * Xa3(x1,x2)
        + g5 * g(x1) * g(x2) * Xa4(x1, x2);
Id Ya(x1?, x2?) = Ya1(x1,x2) + g5 * Ya2(x1,x2)
        + g(x1) * g(x2) * Ya3(x1,x2)
        + g5 * g(x1) * g(x2) * Ya4(x1, x2);

Id Xb(x1?, x2?) = Xb1(x1,x2) + g5 * Xb2(x1,x2)
        + g(x1) * g(x2) * Xb3(x1,x2)
        + g5 * g(x1) * g(x2) * Xb4(x1, x2);
Id Yb(x1?, x2?) = Yb1(x1,x2) + g5 * Yb2(x1,x2)
        + g(x1) * g(x2) * Yb3(x1,x2)
        + g5 * g(x1) * g(x2) * Yb4(x1, x2);

* Order gamma matrices
```

```
repeat;
Id gField?gFields * g5 = g5 * gField;
Id gField?gFields(x1?) * g5 = g5 * gField(x1);
Id gField?gFields(x1?, x2?) * g5 = g5 * gField(x1, x2);
Id g(x1?) * g5 = - g5 * g(x1);
Id g5 * g5 = 1;

Id gField?gFields * g(x1?) = g(x1) * gField;
Id gField?gFields(x1?) * g(x2?) = g(x2) * gField(x1);
Id gField?gFields(x1?, x2?) * g(x3?) = g(x3) * gField(x1, x2);
endrepeat;

* Fill in earlier results for Y
Id Ya1(x1?,x2?) = 2*d(x1)*k(x1)*i + 2*V(x1)*k(x1);
Id Ya2(x1?,x2?) = 0;
Id Ya3(x1?,x2?) = 0;
Id Ya4(x1?,x2?) = - A(x1)*k(x2)*i + A(x2)*k(x1)*i;


Id Yb1(x1?,x2?) = 2*d(x1)*k(x1)*i + 2*V(x1)*k(x1);
Id Yb2(x1?,x2?) = 0;
Id Yb3(x1?,x2?) = 0;
Id Yb4(x1?,x2?) = A(x1)*k(x2)*i - A(x2)*k(x1)*i;

Id i^2 = -1;

* Perform K integration
Id k(x1?) * k(x2?) * k(x3?) * k(x4?)
        = (d_(x1,x2)* d_(x3,x4)
        + d_(x1,x3) * d_(x2,x4)
        + d_(x1,x4)* d_(x2,x3))/4;
Id k(x1?) * k(x2?) * k(x3?) = 0;
Id k(x1?) * k(x2?) = d_(x1,x2)/2;
Id k(x1?) = 0;

* Group gamma functions and remove odd gamma fields
Id g5 * g(x1?) * g(x2?) * g(x3?) * g(x4?)
        * g(x5?) * g(x6?) * g(x7?) * g(x8?) =  g5(x1,x2,x3,x4,x5,x6,x7,x
```

```
Id g(x1?) * g(x2?) * g(x3?) * g(x4?) * g(x5?)
        * g(x6?) * g(x7?) * g(x8?) = g(x1,x2,x3,x4,x5,x6,x7,x8);
Id g5 * g(x1?) * g(x2?) * g(x3?) * g(x4?)
        * g(x5?) * g(x6?) * g(x7?) = 0;
Id g(x1?) * g(x2?) * g(x3?) * g(x4?) * g(x5?)
        * g(x6?) * g(x7?) = 0;
Id g5 * g(x1?) * g(x2?) * g(x3?)
        * g(x4?) * g(x5?) * g(x6?) = g5(x1,x2,x3,x4,x5,x6);;
Id g(x1?) * g(x2?) * g(x3?) * g(x4?)
        * g(x5?) * g(x6?) = g(x1,x2,x3,x4,x5,x6);
Id g5 * g(x1?) * g(x2?) * g(x3?) * g(x4?) * g(x5?) = 0;
Id g(x1?) * g(x2?) * g(x3?) * g(x4?) * g(x5?) = 0;
Id g5 * g(x1?) * g(x2?) * g(x3?) * g(x4?)  = g5(x1,x2,x3,x4);
Id g(x1?) * g(x2?) * g(x3?) * g(x4?) = g(x1,x2,x3,x4);
Id g5 * g(x1?) * g(x2?) * g(x3?)  = 0;
Id g(x1?) * g(x2?) * g(x3?) = 0;
Id g5 * g(x1?) * g(x2?)  = 0;
Id g(x1?) * g(x2?) = g(x1,x2);
Id g5 * g(x1?)   = 0;
Id g(x1?) = 0;
Id g5   = 0;


* Now simplify gamma expressions
repeat;
Id g(?pl1, x1?,x1?, ?pl2) = 4 * g(?pl1, ?pl2);
Id g5(?pl1, x1?,x1?, ?pl2) = 4 * g5(?pl1, ?pl2);
Id g(?pl1, x1?,?pl2,x2?,x1?, ?pl3) = - g(?pl1, x1,?pl2,x1,x2, ?pl3)
        + 2 * g(?pl1, x2,?pl2,?pl3);
Id g5(?pl1, x1?,?pl2,x2?,x1?, ?pl3) = - g5(?pl1, x1,?pl2,x1,x2, ?pl3)
        + 2 * g5(?pl1, x2,?pl2,?pl3);
endrepeat;

repeat;
Id g5(x1?,x2?) = 0;
Id g5(x1?,x2?,x3?,x4?) = 4 * ep(x1,x2,x3,x4)/t;
endrepeat;
```

```
* Fill in earlier results for X
Id Xa1(x1?,x2?) = d(x1)*d(x1) - 2*V(x1)*d(x1)*i
        - V(x1)*V(x1) - A(x1)*A(x1) - dV(x1,x1)*i;
Id Xa2(x1?,x2?) = - V(x1)*A(x1)*i + A(x1)*V(x1)*i + dA(x1,x1);
Id Xa3(x1?,x2?) = - 1/2*V(x1)*V(x2) + 1/2*V(x2)*V(x1)
        - 1/2*A(x1)*A(x2) + 1/2*A(x2)*A(x1)
        - 1/2*dV(x1,x2)*i + 1/2*dV(x2,x1)*i;
Id Xa4(x1?,x2?) = - 1/2*V(x1)*A(x2)*i + 1/2*V(x2)*A(x1)*i
        - A(x1)*d(x2) + 1/2*A(x1)*V(x2)*i
        + A(x2)*d(x1) - 1/2*A(x2)*V(x1)*i
        + 1/2*dA(x1,x2) - 1/2*dA(x2,x1);

Id Xb1(x1?,x2?) = d(x1)*d(x1) - 2*V(x1)*d(x1)*i
        - V(x1)*V(x1) - A(x1)*A(x1)
        - dV(x1,x1)*i;
Id Xb2(x1?,x2?) = V(x1)*A(x1)*i - A(x1)*V(x1)*i - dA(x1,x1);
Id Xb3(x1?,x2?) = - 1/2*V(x1)*V(x2) + 1/2*V(x2)*V(x1)
        - 1/2*A(x1)*A(x2) + 1/2*A(x2)*A(x1)
        - 1/2*dV(x1,x2)*i + 1/2*dV(x2,x1)*i ;
Id Xb4(x1?,x2?) = 1/2*V(x1)*A(x2)*i - 1/2*V(x2)*A(x1)*i
        + A(x1)*d(x2) - 1/2*A(x1)*V(x2)*i
        - A(x2)*d(x1)+ 1/2*A(x2)*V(x1)*i
        - 1/2*dA(x1,x2) + 1/2*dA(x2,x1);

* Take derivatives
repeat;
Id d(x1?) * A(x2?) = dA(x1,x2) + A(x2) * d(x1);
Id d(x1?) * V(x2?) = dV(x1,x2) + V(x2) * d(x1);
Id d(x1?) * dA(x2?, x3?) = dA(x2,x3) * d(x1);
Id d(x1?) * dV(x2?, x3?) = dV(x2,x3) * d(x1);
endrepeat;
Id d(x1?) = 0;

* Rewrite in terms of Bardeen Curvatures
Id dA(x1?,x2?) = 1/2 * (G(x1,x2) + i * V(x1)*A(x2)
        - i *A(x2) * V(x1) + i*A(x1)*V(x2)
        - i *V(x2) * A(x1));
Id dV(x1?,x2?) = 1/2 * (F(x1,x2) + i *V(x1)*V(x2)
```

```
        - i * V(x2) * V(x1) -  i * A(x1)*A(x2)
        + i *A(x2) * A(x1));

Id i^2 = -1;

* Return result
Sum m1,...,m10;
Bracket ep, g, g5;
Print;

.store;
.end
```

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