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Master's thesis

# Stability Theory

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# Contents

# Introduction

Stability is a property possessed by certain complete first-order theories. Intuitively, a theory is stable if it is impossible to encode the natural numbers as a totally ordered set in it. It is well-known that a theory is stable if it does not have too many complete types, i.e. if the cardinality of the set of complete types $S(A)$ over any set of parameters $A$ of a certain cardinality $\kappa$ does not exceed $\kappa$.

Stability theory was introduced by Saharon Shelah in [17]. It was used as a dividing line on the class of first-order theories, intended to prove that if $\kappa < \lambda$ are uncountable cardinals, then for any theory $T$, the number of models of $T$ of cardinality $\kappa$ is at most the number of models of $T$ of cardinality $\lambda$ (both up to isomorphism). This was a generalization of a famous theorem by Morley, stating that if a countable theory has up to isomorphism only 1 model of cardinality $\kappa$ for some uncountable $\kappa$, then this is the case for every uncountable cardinal.

Ever since stability was first introduced by Shelah in the 1970's, it has been rapidly developing into a whole new field of mathematical logic. Many more classes of first-order theories have been defined (simple, superstable, strongly minimal, etc.), and the relation between these classes has been studied in detail. The most fruitful development has been the introduction of the so-called stable groups. This notion opened up the possibility to apply model theory to algebra and algebraic geometry, and it lead to the now active field of research called geometric stability theory. The most spectacular result of these developments was Hrushovski's proof of the Mordell-Lang conjecture for function fields in arbitrary characteristic. (this is explained in [2]).

This thesis is intended to serve as a set of lecture notes for a course on stability theory in the Dutch mastermath system. As such, there are a number of exercises throughout the thesis, intended for students to serve as practice. The solutions to these exercises are in the appendix (which is not intended to become publicly available). In writing this thesis, we have mainly used [5] as our guideline, but we have also drawn from other sources, such as [15], [9] and [17]. No original research has been done in this thesis, and no new ideas have been introduced. However, many proofs have been worked out in more detail, and the solutions to most exercises are original work.

The remainder of this introduction will be devoted to outlaying the contents of the thesis.

In chapter 1, we will treat some preliminary notions on model theory and set theory. Most of these are treated in basic courses on model theory and set theory. The reader who has done such courses and is therefore already familiar with the material is still encouraged to quickly read this part, since it will outlay a number of conventions on notation and terminology.

In chapter 2, the notion of stability is defined, as well as a few other classes of first-order theories. We will consider some examples of stable and unstable theories, and give a number of equivalent definitions of a theory being stable.

In chapter 3, we will take a look at a few rank functions. The definitions in this chapter are self-contained, but some of the results require definitions from chapter 2.

In chapter 4, we will investigate the important notions of dividing and forking. We will also take a look at some extensions of types, and introduce the concept of simple theories. We will see that these simple theories behave very nicely with respect to forking and dividing, and we will see that all stable theories are simple. The first three sections of this chapter are independent of chapter 3. In fact, they draw heavily on section 2.3, but are independent of the rest of chapter 2.

In chapter 5, we will take a look at a part of the classification picture. This is a picture of all known classes of first-order theories, displaying the various inclusions. Although the classification picture is in its full generality far too much for this thesis, we will take a look at a few of the inclusions relating to the classes of theories that where introduced in chapters 2 and 4. This material is independent of chapter 3.

In chapter 6, we will take a closer look at how forking behaves in stable theories, and also define what is known as the forking calculus. In order to do this, we will first consider some properties of automorphisms, and extend our theory to one in many-sorted logic. In the fourth section of chapter 6, we will connect chapters 3 and 4 to each other by showing how forking can be characterized in terms of a rank function. Chapter 6 is independent of chapter 5, but draws heavily on all the other chapters.

In chapter 7, we will take a short look at stable groups and the Mordell-Lang conjecture. The first two sections of this chapter are independent of the chapters 3 up to and including 6.

*Chapter 1*

# Preliminaries

## 1.1 Basic model theory

In this chapter we will treat some preliminary notions which will be used throughout the thesis. Proofs of statements will mostly be omitted. We will assume that the reader is familiar with some basic notions in model theory, such as the compactness theorem, quantifier elimination, the Löwenheim-Skolem theorems, and some examples of theories, such as Peano arithmetic (PA), the theory of dense linear orders without endpoints (DLO) and the theory of algebraically closed fields in characteristic $p$ ($\mathrm{ACF}_p$). If the reader is not yet familiar with the notions discussed in these preliminaries, then brushing up on basic model theory might be wise. This material can be found in standard books on model theory, such as [13] or [18]. These books are also filled with useful exercises, which are highly recommended to anyone who really wants to understand the material.

Throughout this thesis, we will only consider theories of first-order logic with equality, we will usually denote such theories by $T$. Our theories are assumed to be written in a language $\mathcal{L}$. We let $|T|$ denote the cardinality of the theory $T$, and we let $|\mathcal{L}|$ denote the cardinality of the set of $\mathcal{L}$-formulas. If we expand our language with a set of elements $A$ from a model, we will call this newly obtained language $\mathcal{L}(A)$. Formulas in the language $\mathcal{L}$ will be denoted by $\phi(x)$, $\phi(x, y)$, $\phi(x, y, z)$, etc. *We will never distinguish between variables and tuples of variables.* So if we write $\phi(x)$, this $x$ could be a single variable or a tuple of variables. Formulas in the language $\mathcal{L}(A)$ will often be denoted by $\phi(x, a)$, so we use $x, y, z$ for (tuples of) variables and $a, b, c$, for (tuples of) parameters. If $x$ and $y$ are tuples of variables $(x_0, ..., x_n)$ and $(y_0, ..., y_k)$ (possibly of length 1), then by $xy$ we mean the tuple of variables $(x_0, ..., x_n, y_0, ..., y_k)$. We use the same notation for parameters. If $M$ is a model of $T$, $\phi(x)$ is an $\mathcal{L}$-formula and $A \subseteq M$, then we use $\phi(A)$ for the set $\{a \in A \mid M \models \phi(a)\}$.

We will always assume that our theories are consistent, complete, and have only infinite models. Note that these assumptions are not very strong for the following reasons: if a theory $T$ is not complete, we can just extend it to a complete theory by taking the theory of a model of $T$. Also, if some complete theory $T$ has an infinite model, then it has only infinite models. Indeed, we can express in the empty language that there are more then $n$ elements for every natural number $n$, and since these sentences are true in the infinite model and $T$ is complete, they must be part of the theory. And if $T$ has a finite model, then all of its models are finite, and in fact they are all isomorphic (since in that case, the theory completely specifies the structure of the model). So the complete theories admitting only infinite models are really the only interesting theories to consider. So from now on, whenever we say 'let $T$ be a theory', we mean 'let $T$ be a consistent and complete first-order theory admitting only infinite models'. We will sometimes want to prove that a certain theory is complete, for which we will often use the Łoś-Vaught test:

**Theorem 1.1.** *Suppose $T$ is a consistent theory admitting only infinite models, but for which we do not assume completeness. If $T$ is $\kappa$-categorical for some $\kappa \geq |\mathcal{L}|$, then $T$ is complete.*

We will make extensive use of the notion of *types*.

**Definition 1.2.** Let $T$ be a theory, let $M$ be a model of $T$ and let $A \subseteq M$. Let $T'$ be the theory of $M$ in the language $\mathcal{L}(A)$. We say that a set of $\mathcal{L}(A)$-formulas $p(x)$ (so with the tuple of variables $x$ specified) is a *partial type* if $p \cup T'$ is consistent. We say that $p$ is a *complete type* if it is a partial type, and for every formula $\phi(x, a) \in \mathcal{L}(A)$, either $\phi(x, a) \in p$ or $\neg\phi(x, a) \in p$.

If $a$ is an element of a model $M$ and $B \subseteq M$ a set of parameters, then we can consider the *type of a over B*. This is $\text{tp}(a/B) = \{\phi(x, b) \mid \phi(x, y) \in \mathcal{L},\ b \in B,\ M \models \phi(a, b)\}$. Note that this is indeed a complete type over $B$. If $A$ is a set, then $\text{tp}(A/B)$ is the set of those $\mathcal{L}(B)$-formulas $\phi(x, b)$ such that $M \models \phi(a, b)$ for every $a \in A$. If $p$ is a type, then the set of parameters of formulas occurring in $p$ is called the *domain of p*. Note that if $p$ is a complete type over $A$, then $\text{dom}(p) = A$. We will denote the space of complete types over a set of parameters $A$ by $S(A)$. Note that the term 'space' is correct here, since we can equip the set $S(A)$ with a topological structure by letting the sets of the form $[\phi(x, a)] = \{p \in S(A) \mid \phi(x, a) \in p\}$ be the basic opens. With this topology, $S(A)$ is a *Stone space* (a compact Hausdorff space with a basis of clopens). We can also restrict ourselves to a finite set of formulas $\Delta$, and do the same thing. So suppose again we have a theory $T$ written in a language $\mathcal{L}$, we have a model $M$ of $T$ and a set of parameters $A \subseteq M$. Now if $\Delta$ is a finite set of $\mathcal{L}$-formulas of the form $\phi(x, y)$, then a complete $\Delta$-type is a maximal consistent set $p$ of Boolean combinations of formulas of the form $\phi(x, a)$ with $\phi(x, y) \in \Delta$ and $a \in A$. The space of complete $\Delta$-types over $A$ will be denoted by $S_\Delta(A)$, and can be given the structure of a topological space in the same way as $S(A)$. If the set $\Delta$ contains only one formula $\phi(x, y)$, then we write $S_\phi(A)$ instead of $S_\Delta(A)$.

Given a partial type $p$, we say that $p$ is *realized* in a model $M$ if there is some $a \in M$ such that for every $\phi(x) \in p$, we have $M \models \phi(a)$. If this is not the case, we say that $M$ *omits* the type $p$. We say that a type $p \in S(A)$ is *definable* if for every $\phi(x, y) \in \mathcal{L}$ there is some $\mathcal{L}(A)$-formula $d_p\phi(y)$ such that for every $a \in A$, we have that $\phi(x, a) \in p$ if and only if $M \models d_p\phi(a)$. If $B \subseteq A$ then we say that $p$ is $B$-definable if every formula $d_p\phi(y)$ is an $\mathcal{L}(B)$-formula.

Once we have specified a theory, it would be of great help if we could fix some very large model of this theory, and always work within this model. And in fact it turns out that we can actually do this.

**Definition 1.3.** Let $\kappa$ be some infinite cardinal, and let $M$ be a model of a complete theory $T$. Then we say that $M$ is *$\kappa$-saturated* if for any set of parameters $A \subseteq M$ such that $|A| < \kappa$, every partial type over $A$ can be realized in $M$. We say that $M$ is saturated if it is $|M|$-saturated.

We can use the concept of saturation to formulate a very useful test for quantifier elimination. We first need the following definition:

**Definition 1.4.** Let $M, N$ be models of a theory $T$ in a language $\mathcal{L}$, and let $A \subset M$ be a finite set. A map $f : A \to N$ is called a *local isomorphism* if for every quantifier-free $\mathcal{L}$-formula $\phi(x)$ and for every $a \in A$ we have $M \models \phi(a)$ if and only if $N \models \phi(f(a))$.

**Lemma 1.5.** *Let $T$ be a theory, then $T$ has quantifier elimination if and only if for every infinite cardinal $\kappa$ and for every pair of models $M, N$ of $T$ such that $N$ is $\kappa$-saturated, if $A \subseteq M$ is finite and $f : A \to N$ a local isomorphism, then for every element $m \in M$ there is a local isomorphism $f_m : A \cup \{m\} \to N$ extending $f$.*

The following is Theorem 4.3.20 from [13].

**Theorem 1.6.** *Let $T$ be a theory, and let $M$ and $N$ be saturated models of $T$ of the same cardinality. Then they are isomorphic.*

**Definition 1.7.** Let $\kappa$ be some infinite cardinal, and let $M$ be a model of a complete theory $T$. Then we say that $M$ is *$\kappa$-homogeneous* (sometimes called *strongly $\kappa$-homogeneous*) if any partial elementary map on $M$ with domain strictly smaller then $\kappa$ can be extended to an automorphism of $M$.

The automorphisms of a model $M$ are often under consideration. We denote the set of automorphisms of $M$ by $\mathrm{Aut}(M)$. If $A \subseteq M$ is a set of parameters in $M$, then $\mathrm{Aut}(M/A)$ is the set of automorphisms of $M$ which fix $A$ pointwise.

**Theorem 1.8.** *Let $T$ be a theory and let $\kappa$ be some infinite cardinal. Then there exists a $\kappa$-saturated and $\kappa$-homogeneous model $M$ of $T$.*

And in fact, if we assume the generalized continuum hypothesis, we can go a little but further. This is Corollary 4.3.13 from [13].

**Theorem 1.9.** *Assume GCH, let $T$ be a theory and let $\kappa$ be some infinite cardinal. Then there is a saturated model of cardinality $\kappa^+$ of $T$.*

These results enable us to do the following: if we are given a complete theory $T$, we can fix some very big cardinal number $\kappa$, and fix a model of $T$ which is $\kappa$-saturated and $\kappa$-homogeneous. We will call this the *monster model* of $T$ (sometimes we will just say 'the monster'). We will denote the monster model by $\mathbb{M}$, and consider every model $M$ of $T$ as an elementary substructure of $\mathbb{M}$. Given a complete theory $T$, we will assume that such a monster model is already fixed, and that models, elements and sets of parameters are all taken in this monster model. We will say that a model or a set of parameters is *small* if its cardinality is strictly less then $\kappa$. We will abbreviate $\mathbb{M} \models \phi(a)$ by $\models \phi(a)$. One of the advantages of working in a monster model is that we can give our compactness arguments in a nice way. Normally, whenever we use compactness we have to say that there is some elementary extension of our model which models a larger theory (usually with new constants added), but now we can do this inside the monster, and just say that the larger theory holds in the monster model (and there is an interpretation for the new constants in the monster).

A type can have the entire monster model as its set of parameters. A type $p \in S(\mathbb{M})$ will be called a *global type*. Note that global types need not be realized. For example, if we have a monster model $\mathbb{M}$ of DLO, and we look at $p(x) = \{a < x \mid a \in \mathbb{M}\}$, then we see that this is a type because every finite subset is consistent, but this type is not realized, because an element realizing it would be an endpoint.

We say that a subset $A \subseteq \mathbb{M}$ is *definable* if there is some $\mathcal{L}(\mathbb{M})$-formula $\phi(x)$ such that for every $a \in \mathbb{M}$, we have $\models \phi(a)$ if and only if $a \in A$. If this formula is an $\mathcal{L}(B)$-formula with $B$ some set of parameters, then we say that $A$ is $B$-definable. Given a formula $\phi(x)$, we will sometimes not distinguish between the formula and the definable subset that it defines. We notice that by taking $A$ to be a one-element set, we can talk about the elements that are $B$-definable. The set of all $B$-definable elements is called the *definable closure* of $B$, and denoted $\mathrm{dcl}(B)$. If $a$ is an element such that $\models \phi(a)$ with $\phi(x)$ an $\mathcal{L}(B)$ formula with only finitely many solutions, then we say that $a$ is *algebraic* over $B$. The set of all algebraic elements over $B$ is called the *algebraic closure* of $B$, and denoted $\mathrm{acl}(B)$.

## 1.2 Basic set theory

Since model theory is very pinned down on ZFC, we will often need to use some set-theoretical results and definitions. For those who want to take a closer look at the set-theoretical concepts involved, [10] contains everything there is to know on set theory, and more. Note that this part on set theory will not really be a coherent story, but rather we will be summing up all the things that we need at some point. Also note that we will only treat the basics here. Some more set-theoretic tools will be treated in more detail throughout the thesis. In this part however, proofs will again be omitted, but can usually be found in [10].

A very useful piece of combinatorial set theory is *Ramsey's theorem*.

**Theorem 1.10.** *Let $n, k \in \omega$, and suppose $\{X_1, ..., X_k\}$ is a partition of all the $n$-element subsets of $\omega$. Then there is some infinite $A \subseteq \omega$ such that there is some $i \in \{1, ..., k\}$ such that every $n$-element subset of $A$ is in $X_i$.*

Intuitively, this means that if we color every pair of natural numbers red or blue, then there is some infinite set $A$ of natural numbers such that the elements of $A^2$ all have the same color. But obviously, the theorem generalizes pairs to $n$-element subsets and red and blue to $k$ colors.

Let $\kappa$ be some (possibly finite) cardinal number. Then the *beth-function* is defined as follows:

- $\beth_0(\kappa) = \kappa$.
- $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$.
- $\beth_\lambda(\kappa) = \bigcup_{\mu<\lambda} \beth_\mu(\kappa)$.

Here $\alpha$ is an ordinal and $\lambda$ is a nonzero limit ordinal. We sometimes write $\beth_\alpha$ instead of $\beth_\alpha(\aleph_0)$ Note that the generalized continuum hypothesis can be formulated as 'for every ordinal $\alpha$, we have $\aleph_\alpha = \beth_\alpha$'.

**Lemma 1.11.** *For all ordinals $\alpha, \beta$ and cardinals $\kappa$: $\beth_{\alpha+\beta}(\kappa) = \beth_\beta(\beth_\alpha(\kappa))$.*

One can prove this by induction on $\beta$.

**Lemma 1.12.** *If $\alpha$ is a cardinal and $\lambda \in \alpha^+$ then $\beth_{\alpha^+}(\aleph_0) > \beth_\lambda(\lambda)$.*

The following theorem is known as the Erdős-Rado theorem.

**Theorem 1.13.** *Let $\kappa$ be some infinite cardinal and let $n \in \omega$. Let $X$ be a set such that $|X| > \beth_n(\kappa)$. Now let $I$ be some index set such that $|I| \leq \kappa$ and let $(C_i)_{i \in I}$ be sets such that $\{A \subseteq X \mid |A| = n+1\} \subseteq \bigcup_{i \in I} C_i$. Then there is some $Y \subseteq X$ and some $i \in I$ such that $|Y| > \kappa$ and $\{A \subseteq Y \mid |A| = n+1\} \subseteq C_i$.*

**Definition 1.14.** Let $\alpha$ be a limit ordinal and let $\beta \subseteq \alpha$. Then we say that $\beta$ is *cofinal* in $\alpha$ if for all $a \in \alpha$ there is some $b \in \beta$ such that $a \leq b$.

The notion of cuts in linear orders should be introduced. If $(I, <)$ is a linear order, then a *cut* in $I$ is a pair $(A, B)$ of nonempty subsets of $I$ such that $A \cup B = I$ and such that $a < b$ for every $a \in A$ and $b \in B$. In the following, we will only consider infinite cardinals, so whenever we say that $\kappa$ is a cardinal, we will mean that $\kappa$ is an infinite cardinal.

**Definition 1.15.** Let $\kappa$ be some cardinal, then the *Dedekind number* of $\kappa$, denoted $\mathrm{ded}(\kappa)$, is the maximal cardinal $\lambda$ such that there is a linear order of size $\kappa$ with at least $\lambda$ many cuts.

**Lemma 1.16.** *For all cardinals $\kappa$, we have $\kappa < ded(\kappa)$.*

*Proof.* Let $\mu$ be the smallest cardinal such that $2^\mu > \kappa$. Then consider the set $2^{<\mu} = \bigcup_{\lambda < \mu} 2^\lambda$. We order this set lexicographically, so if $f, g \in 2^{<\mu}$ we say that $f < g$ if for the smallest $\alpha$ such that $f(\alpha) \neq g(\alpha)$, we have $f(\alpha) < g(\alpha)$. And if $f(\alpha) = g(\alpha)$ for every $\alpha$ in the domain of both $f$ and $g$, then we say that $f < g$ if $\mathrm{dom}(f) < \mathrm{dom}(g)$. Notice that $\mu \leq \kappa$, since $\kappa < 2^\kappa$, and also notice that if $\lambda < \mu$, then we have that $2^\lambda \leq \kappa$. It follows that $|2^{<\mu}| \leq \kappa \cdot \kappa = \kappa$. So we have a linear order of size at most $\kappa$, but every element of $2^\mu$ defines a cut in this linear order, so we have at least $2^\mu > \kappa$ many cuts in this order. Hence $\kappa < \mathrm{ded}(\kappa)$. □

Notice that in order to prove that the cardinality of a set $A$ is at least $\mathrm{ded}(\kappa)$ for some cardinal $\kappa$, we would have to prove that for any linear order $I$ with $|I| = \kappa$, the number of cuts in $I$ is at most as large as the cardinality of $A$. But in fact it suffices to prove that this is the case for *dense* linear orders. For suppose $I$ is a linear order with $|I| = \kappa$ and the number of cuts in $I$ is $\mathrm{ded}(\kappa)$. Then consider the linear order $I'$, which is $I$ but with a copy of $\mathbb{Q}$ added between every pair of elements $i < j$ such that $j$ is the successor of $i$. Notice that $|I'| \leq \kappa \cdot |\mathbb{Q}| = \kappa$. Also notice that $I'$ is a dense linear order, and if we denote the number of cuts in $I'$ by $C(I')$, then we find that $\mathrm{ded}(\kappa) \leq C(I') \leq \mathrm{ded}(\kappa) \cdot |\mathbb{R}| = \mathrm{ded}(\kappa)$. Here the second inequality is due to the fact that $\mathbb{R}$ is exactly the set of cuts in $\mathbb{Q}$, and the last equality is due to the fact that $\kappa \geq |\mathbb{Q}|$ and hence $\mathrm{ded}(\kappa) \geq \mathrm{ded}(|\mathbb{Q}|) = |\mathbb{R}|$.

**Definition 1.17.** A *Boolean algebra* is a structure $\langle B, \vee, \wedge, \neg, 0, 1 \rangle$ where $B$ is called the *universe* of this algebra, 0 and 1 are two distinct elements of $B$, $\vee$ and $\wedge$ are two binary operations on $B$ and $\neg$ is a unary operation on $B$ such that for all $a, b, c \in B$ we have:

| | |
|---|---|
| $a \vee b = b \vee a$ | $a \wedge b = b \wedge a$ |
| $a \vee (b \vee c) = (a \vee b) \vee c$ | $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ |
| $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ | $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ |
| $0 \vee a = a$ | $1 \wedge a = a$ |
| $a \vee \neg a = 1$ | $a \wedge \neg a = 0$ |

We shall denote a Boolean algebra $\langle B, \vee, \wedge, \neg, 0, 1 \rangle$ by its universe $B$. We call $\vee$ the *join* on $B$, $\wedge$ the *meet* on $B$ and $\neg$ the *inverse* on $B$. We can define a partial order on Boolean algebras by letting $a \leq b$ if and only if $a = a \wedge b$. We say that an element $b \in B$ is an *atom* if for every $a \in B$ if $a \leq b$, then $a = 0$ or $a = b$. If $B$ is a Boolean algebra, then an ideal in $B$ is a set $I \subseteq B$ such that $0 \in B$ and $1 \notin B$, $I$ is closed under joins and if $a \leq b$ and $b \in I$, then $a \in I$. If $I$ is an ideal in $B$, then we can define $B/I = \{a \wedge I \mid a \in B\}$. Note that the $\mathcal{L}(A)$-formulas (with $A$ some set of parameters) of a given theory $T$ form a Boolean algebra with the meet being conjunction, the join being disjunction, and the inverse being negation. Here 0 is $\perp$ and 1 is $\top$. We will give some special attention to ideals in such a Boolean algebra.

**Definition 1.18.** Let $A$ be a set of parameters, and let $\Phi(x)$ be a nonempty collection of $\mathcal{L}(A)$-formulas (equivalently, a nonempty collection of $A$-definable sets) in a fixed tuple of variables $x$. Then $\Phi(x)$ is called an *ideal* if:

- If $B \subseteq C$ and $C \in \Phi(x)$, then $B \in \Phi(x)$, that is, if $\phi(x) \models \psi(x)$ and $\psi(x) \in \Phi(x)$, then $\phi(x) \in \Phi(x)$.
- If $\phi(x), \psi(x) \in \Phi(x)$, then $\phi(x) \vee \psi(x) \in \Phi(x)$.
- $\top \notin \Phi(x)$.

Note that if $\Phi(x)$ is an ideal, then $\emptyset \in \Phi(x)$. this is because $\Phi(x)$ is nonempty, so there is some $B \in \Phi(x)$, and $\emptyset \subseteq B$. If $\Psi(x)$ is a collection of formulas, then we can consider the ideal generated by $\Psi(x)$, which is the smallest ideal containing all the formulas in $\Psi(x)$. This ideal will be denoted by $(\Psi(x))$. If $\Phi(x)$ is an ideal over a set of parameters $A$, and $B$ is some other set of parameters, then we say that $\Phi(x)$ is *$B$-invariant* if whenever $\phi(x, a) \in \Phi(x)$ and $a \equiv_B b$, we have $\phi(x, b) \in \Phi(x)$. Here $a \equiv_B b$ means that $\models \phi(a) \leftrightarrow \phi(b)$ for every $\mathcal{L}(B)$-formula $\phi$.

The notion of an ideal also has a dual, which is known as a filter.

**Definition 1.19.** Let $A$ be a set of parameters, and let $\Psi(x)$ be a nonempty collection of $\mathcal{L}(A)$-formulas (equivalently, a nonempty collection of $A$-definable sets) in a fixed tuple of variables $x$. Then $\Psi(x)$ is called a *filter* if:

- If $B \subseteq C$ and $B \in \Psi(x)$, then $C \in \Psi(x)$, that is, if $\phi(x) \models \psi(x)$ and $\phi(x) \in \Psi(x)$, then $\psi(x) \in \Psi(x)$.
- If $\phi(x), \psi(x) \in \Psi(x)$, then $\phi(x) \wedge \psi(x) \in \Psi(x)$.
- $\perp \notin \Psi(x)$.

Note that if $A$ is a set of parameters and $\Phi(x)$ is a nonempty collection of $\mathcal{L}(A)$-formulas, then $\Phi(x)$ is an ideal if and only if the set $\neg\Phi(x) = \{\neg\phi(x) \mid \phi(x) \in \Phi(x)\}$ is a filter.

A filter that is maximal w.r.t. inclusion is called an *ultrafilter*.

**Lemma 1.20.** *Every filter can be extended to an ultrafilter.*

Note that a filter of $\mathcal{L}(A)$-formulas is exactly the same as a partial $A$-type, and an ultrafilter of $\mathcal{L}(A)$-formulas is an element of $S(A)$.

*Chapter 2*

# Stable theories

## 2.1 Stable formulas and stable theories

In this section, we will start the investigation of the notion of a *stable theory*. Most of this section is taken from [5].

**Definition 2.1.** Let $k \in \omega$. A formula $\phi(x, y)$ has the *k-order property* if there are sequences $(a_i)_{i<k}$ and $(b_i)_{i<k}$ in the monster model $\mathbb{M}$ such that $\models \phi(a_i, b_j)$ if and only if $i < j$. If a formula has the $k$-order property for all $k \in \omega$, we call it *unstable*, otherwise it is called *stable*. A theory $T$ is called stable if all $\mathcal{L}$-formulas are stable in the monster model of $T$, and is called unstable otherwise.

**Lemma 2.2.** *A formula $\phi(x, y)$ is unstable if and only if there are sequences $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ such that for all $i, j \in \omega$: $\models \phi(a_i, b_j)$ if and only if $i < j$.*

*Proof.* If there are sequences $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ such that for all $i, j \in \omega$: $\models \phi(a_i, b_j)$ if and only if $i < j$, then clearly $\phi(x, y)$ has the $k$-order property for every $k$, by taking the first $k$ elements of these sequences. So suppose that $\phi(x, y)$ has the $k$-order property for every $k \in \omega$. Now consider two new sets of constants: $\{c_i \mid i \in \omega\}$ and $\{d_i \mid i \in \omega\}$, and consider axioms $\phi(c_i, d_j)$ for all $i, j \in \omega$ such that $i < j$ and $\neg\phi(c_i, d_j)$ for all $i, j \in \omega$ such that $i \geq j$. Any finite subset of these axioms is consistent with $T$ because we can take a sufficiently large $k$ and a sequence witnessing that $\phi(x, y)$ has the $k$-order property. So we have interpretations $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ of these constants, and hence we have $\models \phi(a_i, b_j)$ if and only if $i < j$. $\square$

The alternative characterization of stability provided in Lemma 2.2 will very often be used as the definition of unstable formulas. Before we continue, we will give the definition of a stable theory some thought. One could wonder in what sense stability is actually a property of a theory, rather then a property of the monster model that we picked. But notice that if some formula is unstable in some monster model $\mathbb{M}$, then we can add new constants to our language for the sequences witnessing this, and sentences telling us that these constants are sequences witnessing this. The theory we obtain is consistent, for $\mathbb{M}$ is a model. By the downward Löwenheim-Skolem theorem there is some model $M$ of this theory with a smaller cardinality (say, a countable model). But any small model will be seen as a substructure of our monster model. Hence $M$ is contained in any monster model that we pick, and hence these sequences will be contained in every monster model. So if $T$ would be unstable given some choice of monster model, then it is unstable given every choice of monster model. This means that $T$ will either always be stable or always be unstable. So stability is truly a property of the theory, and independent of what monster model we pick.

**Exercise 1.** Show that a theory $T$ is stable if and only if there is no $\mathcal{L}$-formula $\phi(x, y)$ and sequence $(c_i)_{i\in\omega}$ in $\mathbb{M}$ such that $\models \phi(c_i, c_j)$ if and only if $i < j$.

Before we continue working with the notion of stability, we will take a look at a number of examples. In fact, we can just consider any theory we like, and ask ourselves whether or not it is stable. It turns out that most known theories are unstable, so we start with some examples of those.

**Example 2.3.**

- The theory of a model of ZF(C) is unstable. To see this, consider the formula $x \in y$, and consider the sequence $(a_i)_{i\in\omega}$ given by the finite ordinals. We see that for every $i, j \in \omega$ we have $a_i \in a_j$ if and only if $i < j$. And hence we find that this formula is unstable, so ZFC is unstable.

- Let $R$ be an ordered field, that is, a field equipped with a linear order $\leq$ such that for all $a, b, c \in R$ we have that if $a \leq b$, then $a + c \leq b + c$, and for all $a, b \in R$ we have that if $0 \leq a$ and $0 \leq b$, then $0 \leq a \cdot b$. Then the theory of $R$ is unstable. This is easy to see since any such field will contain a strictly increasing sequence $(a_i)_{i\in\omega}$ (since any ordered field is infinite), and we see that if $\phi(x, y)$ is the formula $x \leq y \wedge x \neq y$, we find that $\models \phi(a_i, a_j)$ if and only if $i < j$. This implies in particular that the theory of the real numbers (considered as ordered field) is unstable.

- A *monoid* is a set $X$ equipped with a binary operation $\cdot : X \times X \to X$ such that this operation is associative (so for every $a, b, c \in X$ we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$) and there is an identity element $e \in X$ such that for every $a \in X$ we have that $e \cdot a = a \cdot e = a$. Notice that if every element of a monoid $X$ would have an inverse element, then $X$ would in fact be a group. We say that a monoid $X$ has the *cancellation property* if we have for all $a, b, c \in X$ that if $a \cdot b = a \cdot c$, then $b = c$, and if $b \cdot a = c \cdot a$ then $b = c$. We say that a monoid $X$ is *commutative* if for every $a, b \in X$ we have that $a \cdot b = b \cdot a$.

  Now let $X$ be a commutative monoid with the cancellation property which is not a group, so there is an element which does not have an inverse. We claim that the theory of this monoid is unstable. To see this, notice that the formula $\exists x \neg \exists y (x \cdot y = y \cdot x = e)$ is contained in our theory, because this is true in $X$. This means that there is an element $a$ in the monster model of this theory which has no inverse. Now consider the sequence $(a^n)_{n\in\omega}$, and let $\phi(x, y)$ be the formula

  $$\exists z (z \cdot x = x \cdot z = y \wedge x \neq y)$$

  We notice that if $n < m$, then $a^{m-n} \cdot a^n = a^n \cdot a^{m-n} = a^m$ while $a^n \neq a^m$. This is the case because if $a^n = a^m$ then by the cancellation property we would find that $a^{m-n} = e$, and hence $a^{m-n-1}$ would be an inverse to the element $a$.

  If $n = m$, then clearly we see that $a^m = a^n$.

  Suppose that $n > m$, then we find that if there is some element $z$ such that $z \cdot a^n = a^n \cdot z = a^m$, then by cancellation we find that $a^{n-m} \cdot z = z \cdot a^{n-m} = e$, and hence we find that $a^{n-m-1} \cdot z$ is an inverse to $a$.

So we can conclude that $\models \phi(a^n, a^m)$ if and only if $n < m$. So this formula is unstable, and hence this theory is unstable.

**Exercise 2.** Prove that the theory DLO (Dense Linear Orders without endpoints) and the theory of a model of PA (Peano arithmetic) are unstable.

So it turns out that it is relatively easy to show that certain theories are unstable. However, proving stability is much harder, and can not be done in such a naïve way. So before we take a look at a few examples of theories which are stable, we will first take a closer look at the theoretical side of stability. First, stability of some formula can tell us a lot about stability of other formulas, as is described in the following lemma.

**Lemma 2.4.** *Let $\phi(x, y)$ and $\psi(x, z)$ be stable formulas.*

1. *The formula $\chi(y, x)$, defined by $\chi(y, x) := \phi(x, y)$ is also stable.*
2. *The formula $\neg\phi(x, y)$ is also stable.*
3. *The formula $\chi(x, yz) := \phi(x, y) \wedge \psi(x, z)$ is also stable.*
4. *The formula $\chi(x, yz) := \phi(x, y) \vee \psi(x, z)$ is also stable.*
5. *If $y = uv$ and $c$ is some constant, then the formula $\chi(x, u) := \phi(x, uc)$ is also stable.*

*Proof.* We prove parts 1 and 4, and leave the rest as an exercise. To prove 1, we will prove that if $\chi(y, x)$ is unstable, then $\phi(x, y)$ is also unstable. So suppose $\chi(y, x)$ is unstable, and let $k \in \omega$. Then we know that there are sequences $(a_i)_{i<k}$ and $(b_j)_{j<k}$ such that for all $i, j < k$ we have: $\models \chi(a_i, b_j) \Leftrightarrow i < j$. Now define new sequences $(c_i)_{i<k}$ and $(d_j)_{j<k}$ by $c_i = b_{k-i}$ for all $i < k$ and $d_j = a_{k-j}$ for all $j < k$. We now find:

$$\begin{aligned} \models \phi(c_i, d_j) &\text{ iff } \models \chi(d_j, c_i) \\ &\text{ iff } \models \chi(a_{k-j}, b_{k-i}) \\ &\text{ iff } k - j < k - i \\ &\text{ iff } i < j. \end{aligned}$$

So $\phi(x, y)$ also has the $k$-order property for every $k \in \omega$, and hence $\phi(x, y)$ is unstable. So it follows that if $\phi(x, y)$ is stable, then $\chi(x, y)$ is also stable.
To prove 4, we will use Ramsey's theorem. So suppose the formula $\chi(x, yz)$ is unstable, then by Lemma 2.2 there are sequences $(a_i)_{i\in\omega}$ and $(b_i c_i)_{i\in\omega}$ such that $\models \chi(a_i, b_j c_j)$ iff $i < j$. Now let $A = \{(i, j) \in \mathbb{N}^2 \mid i < j\}$. We notice that for every $(i, j) \in A$ we have $\models \phi(a_i, b_j)$ or $\models \psi(a_i, c_j)$. Now by Ramsey's theorem there is an infinite subset $I \subseteq \mathbb{N}$ such that we have for all $(i, j) \in A \cap I^2$ that $\models \phi(a_i, b_j)$, or for all $(i, j) \in A \cap I^2$ that $\models \psi(a_i, c_j)$. In the first case we see that $\phi(x, y)$ is unstable, using the sequences $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$, and in the second case we see that $\psi(x, y)$ is unstable, using the sequences $(a_i)_{i\in I}$ and $(c_i)_{i\in I}$. We conclude that $\chi(x, yz)$ has to be a stable formula. □

**Exercise 3.** Prove parts 2, 3 and 5 of Lemma 2.4.

The notion of stability of a theory might already be familiar, but under a different definition. In literature, one sometimes finds the definition that a theory $T$ is $\kappa$-stable (with $\kappa$ some cardinal) if for every set of parameters $A$ of size at most $\kappa$, there are at most $\kappa$ complete types over $A$, and stable if it is $\kappa$-stable for some $\kappa$. And in fact this indeed turns out to be an equivalent definition of stability. To prove this we first need the following theorem from combinatorial set theory, which is due to Erdős and Makkai.

**Lemma 2.5.** *Let $A$ be an infinite set and let $F \subseteq \mathcal{P}(A)$ be such that $|A| < |F|$. Then there are sequences $(a_i)_{i\in\omega}$ in $A$ and $(S_i)_{i\in\omega}$ in $F$ such that one of the following holds:*

1. *For all $i, j \in \omega$, we have $a_i \in S_j \Leftrightarrow j < i$.*
2. *For all $i, j \in \omega$, we have $a_i \in S_j \Leftrightarrow i < j$.*

*Proof.* Since the cardinality of the set of pairs $(B, C)$ where $B$ and $C$ are finite subsets of $A$ is equal to the cardinality of $A$, we can choose a subset $F' \subseteq F$ such that $|F'| = |A|$, and if two finite subsets of $A$ can be separated by an element of $F$ (so if there is some element of $F$ which contains one of the finite subsets and is disjoint from the other), they can also be separated by an element of $F'$. Now let $S \in F$ be such that $S$ is not a Boolean combination of elements of $F'$. Such a set exists since $|F'| < |F|$. Now we will construct sequences $(b_i)_{i\in\omega}$, $(c_i)_{i\in\omega}$ and $(S_i)_{i\in\omega}$ such that the following conditions are satisfied:

- For every $i \in \omega$, we have $S_i \in F'$, $b_i \in S$ and $c_i \in A \backslash S$.
- For every $i \in \omega$, the set $S_i$ separates the sets $\{b_0, ..., b_i\}$ and $\{c_0, ..., c_i\}$.
- For every $n \in \omega$ and every $i < n$, we have $b_n \in S_i$ iff $c_n \in S_i$.

We will construct these sequences by induction. First choose $b_0 \in S$ and $c_0 \in A\backslash S$ arbitrarily, which is possible since $S$ is not empty and $S \neq A$. Notice that the sets $\{b_0\}$ and $\{c_0\}$ are being separated by the set $S \in F$, and hence we can choose a set $S_0 \in F'$ which separates them.

Now suppose that for some $n \geq 1$ we have already constructed the sequences $(b_i)_{i<n}$, $(c_i)_{i<n}$ and $(S_i)_{i<n}$. Since $S_0, ..., S_{n-1} \in F'$ we know that $S$ is not a Boolean combination of $S_0, ..., S_{n-1}$. We claim that from this it follows that there are elements $b_n \in S$ and $c_n \in A\backslash S$ such that for all $i < n$, we have $b_n \in S_i$ if and only if $c_n \in S_i$. To prove this we define the function $f : A \to 2^n$ by $(f(a))_k = 1$ if and only if $a \in S_k$, where $(f(a))_k$ is the $k$th coordinate of the vector $f(a)$. Now suppose that there are two elements $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. Then we notice that we are done if one of them is in $S$ and the other is not. So suppose that for any $a_1, a_2 \in A$ we have either $a_1, a_2 \in S$ or $a_1, a_2 \notin S$. This way we find that for every vector $v \in 2^n$ we have $f^{-1}(v) \subseteq S$ or $f^{-1}(v) \subseteq A\backslash S$. And since we know that $A = \bigcup_{v\in 2^n} f^{-1}(v)$, we find that there are elements $v_0, ..., v_{k-1} \in 2^n$ such that $\bigcup_{i<k} f^{-1}(v_i) = S$. But since by definition of $f$ any set $f^{-1}(v)$ for $v \in 2^n$ is a Boolean combination of the sets $S_0, ..., S_{n-1}$, we find that $S$ is also such a Boolean combination. We arrive at a contradiction, hence we can conclude that there are two elements $b_n, c_n \in A$ such that $b_n \in S$ and $c_n \notin S$ and $f(b_n) = f(c_n)$. Now we can choose $S_n$ as any set in $F'$ separating the sets $\{b_0, ..., b_n\}$ and $\{c_0, ..., c_n\}$, which exists because these two sets are already separated by the set $S \in F$.

Now that we have these sequences, we can apply Ramsey's theorem. We know that for every pair $(k, r)$ with $k < r$, either $b_r \in S_k$ or $b_r \notin S_k$. So by Ramsey's theorem there is an infinite subset $I \subseteq \mathbb{N}$ such that exactly one of the following holds:

1. For all $i, n \in I$, we have $i < n \Rightarrow b_n \in S_i$.
2. For all $i, n \in I$, we have $i < n \Rightarrow b_n \notin S_i$.

Suppose we are in the first case. We also notice that we have:

$$\forall n \in I(\{b_0, ..., b_n\} \subseteq S_n \ \vee \ \{c_0, ..., c_n\} \subseteq S_n)$$

So there must be some infinite subset $J \subseteq I$ such that we either have for all $n \in J$: $\{b_0, ..., b_n\} \subseteq S_n$, or for all $n \in J$ we have: $\{c_0, ..., c_n\} \subseteq S_n$. So by restricting the sequences $(b_i)_{i\in\omega}$, $(c_i)_{i\in\omega}$ and $(S_i)_{i\in\omega}$ to their subsequences indexed by $J$, we now find if $\forall n \in J(\{b_0, ..., b_n\} \subseteq S_n)$, then $\forall i, j \in J(i < j \Rightarrow b_j \in S_i)$, and hence we also have:

$$\forall i, j \in J(i < j \Rightarrow c_j \in S_i).$$

$$\forall i, j \in J(i \geq j \Rightarrow c_j \notin S_i).$$

So we can take the sequence $(a_i)_{i\in\omega}$ to be the sequence $(c_j)_{j\in J}$.
In the case that we have $\forall n \in J(\{c_0, ..., c_n\} \subseteq S_n)$, we are in a very similar situation, and can take $(a_i)_{i\in\omega}$ to be the sequence $(b_j)_{j\in J}$.
Now suppose we are in the second case. So we know that $\forall i, n \in I(i < n \Rightarrow b_n \notin S_i)$. We again find some infinite subset $J$ of $I$ such that either for all $n \in J$ we have $\{b_0, ..., b_n\} \subseteq S_n$, or for all $n \in J$ we have: $\{c_0, ..., c_n\} \subseteq S_n$. So by again restricting the sequences $(b_i)_{i\in\omega}$, $(c_i)_{i\in\omega}$ and $(S_i)_{i\in\omega}$ to their subsequences indexed by $J$, we now find in the case that $\forall n \in J(\{b_0, ..., b_n\} \subseteq S_n$ that we have: $\forall i, j \in J(i < j \Rightarrow b_j \notin S_i)$, and we also have $\forall i, j \in J(i \geq j \Rightarrow b_j \in S_i)$. So now we define $a_i = b_{j_i+1}$ for all $i \in \omega$, (where $J = \{j_0, j_1, j_2, ...\}$) and we have found a sequence with the desired properties. Likewise, if we are in the case that $\forall n \in J(\{c_0, ..., c_n\} \subseteq S_n$, we can define $a_i = c_{j_i+1}$ for all $i \in \omega$. □

Now we are ready to prove that the two different notions of stability of a theory actually coincide.

**Theorem 2.6.** *Let $T$ be an unstable theory. Then for any infinite cardinal $\kappa$, there is a model $M$ of $T$ such that $|M| = \kappa$ and such that $|S(M)| > \kappa$.*

*Proof.* Let $\kappa$ be some cardinal, and let $\phi(x, y)$ be a formula in $T$ which has the $k$-order property for every $k \in \omega$. Let $I$ be a dense linear order such that $|I| = \kappa$. We claim that there is a model $M \models T$ and sequences $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ in $M$ such that for all $i, j \in I$, we have $M \models \phi(a_i, b_j) \ \Leftrightarrow i < j$. To prove this we extend our language $\mathcal{L}$ to a language $\mathcal{L}' = \mathcal{L} \cup \{c_i \mid i \in I\} \cup \{d_i \mid i \in I\}$ where the $c_i$ and $d_i$ are new constants. We now extend the $\mathcal{L}$-theory $T$ to the $\mathcal{L}'$-theory $T'$, defined by

$$T' = T \cup \{\phi(c_i, d_j) \mid i < j\} \cup \{\neg\phi(c_i, d_j) \mid i \geq j\}.$$

We will use the compactness theorem to prove that the theory $T'$ is consistent. For let $T''$ be a finite subtheory of $T'$, then we notice that any model of $T$ can be made into a model of $T''$, since $\phi$ has the $k$-order property for every $k \in \omega$. This means that $T'$ is consistent, hence it has a model $M$, which is also a model of $T$, and contains the desired sequences. By using the downward Löwenheim-Skolem theorem we may assume that $|M| = \kappa$. Now let $C = (A, B)$ be a cut in $I$, and let $\Phi_C$ be the following set of formulas:

$$\Phi_C = \{\phi(x, b_j) \mid j \in B\} \cup \{\neg\phi(x, b_i) \mid i \in A\}.$$

By compactness we see that $\Phi_C$ is a consistent set of formulas, since every finite subset of $\Phi_C$ is realized by some $a_n$ which lies below all the elements of the finite subset of $B$, but above all the elements of the finite subset of $A$. Now let $P_C \in S(M)$ be a complete type extending $\Phi_C$. We notice that if $C$ and $C'$ are different cuts, then $P_C \neq P_{C'}$. So since every cut in $I$ defines a complete type in $S(M)$, we find that $|S(M)|$ is at least as large as the number of cuts in $I$. And since $I$ was an arbitrary dense linear order of size $\kappa$, we can conclude that $|S(M)| \geq \text{ded}(\kappa) > \kappa$. □

The other direction of our equivalence is almost a direct consequence of Lemma 2.5.

**Theorem 2.7.** *Let $\phi(x, y)$ be a formula and suppose $|S_\phi(A)| > |A|$ for some infinite set of parameters $A$. Then $\phi(x, y)$ is unstable.*

*Proof.* Let for any parameter $b \in \mathbb{M}$ the set $S_b$ be given by $S_b = \{a \in A \mid \models \phi(b, a)\}$. So for every $b \in \mathbb{M}$ we see that $S_b \subseteq A$. Now let $F = \{S_b \mid b \in \mathbb{M}\}$. We notice that $|A| < |F|$, since $|S_\phi(A)| > |A|$. So we can apply Lemma 2.5 to $A$ and $F$ to find a sequence $(a_i)_{i\in\omega}$ in $A$ and $(b_i)_{i\in\omega}$ in $\mathbb{M}$ such that one of the following holds:

1. For all $i, j \in \omega$ we have $a_i \in S_{b_j}$ iff $j < i$.

2. For all $i, j \in \omega$ we have $a_i \in S_{b_j}$ iff $i < j$.

In the first case we see that $\phi(x, y)$ is unstable by Lemma 2.2, and in the second case it is unstable by Lemma 2.4 and Lemma 2.2. □

We can now combine Theorem 2.6 and Theorem 2.7 into the following corollary:

**Corollary 2.8.** *Let $T$ be a theory, Then $T$ is stable if and only if for some cardinal $\kappa$ and for every set of parameters $A$ with $|A| \leq \kappa$, we have $|S(A)| \leq |A|$.*

**Exercise 4.** Prove Corollary 2.8

Now we are ready to consider examples of stable theories, because now we have Theorem 2.6, which tells us that if the number of complete types over a model $M$ can never exceed the cardinality of the model itself, then the corresponding theory is in fact stable.

## 2.2 Examples of stable theories

In this section, we will consider three examples of stable theories. The second of these three will be given in the form of exercises. The first two examples are from [5], and the last is from [18] and [14].

### The theory of Algebraically closed fields

Our first example of a stable theory is the theory of algebraically closed fields of characteristic 0. It is well-known that this theory has quantifier elimination, so any complete type is completely determined by the quantifier-free formulas in that type. So let $K$ be a small algebraically closed field, viewed as a subfield of a monster model $\mathbb{M}$. Since $\mathbb{M}$ is $|K|$-saturated, we know that any complete type over $K$ is realized by some element $a \in \mathbb{M}$. So any complete type over $K$ is determined by a complete set of quantifier-free formulas (which in this case means a set of polynomial equalities and inequalities with coefficients in $K$) which are satisfied by some $a \in \mathbb{M}$. Now there are two options, the first is that $a$ is transcendental over $K$, which means that it is not the zero of any such a polynomial, which completely determines the type. The other option is that it is algebraic over $K$, meaning that it is in fact an element of $K$, because $K$ is algebraically closed. So the number of complete types over $K$ is at most the number of elements of $K$ (and 1 more for the transcendental elements, but this has no influence on the cardinality), which is $|K|$. So we see that this theory is indeed stable.

### A theory of equivalence relations

Our next example of a stable theory is a theory of equivalence relations. Specifically, our language $\mathcal{L}_E$ is a set of relation symbols $\{E_n(x, y) \mid n \in \omega\}$ and our theory $T_E$ consists of axioms telling us that each $E_n$ is an equivalence relation, each relation $E_n$ has an infinite number of classes and for every $n \in \omega$ every class of $E_n$ is the union of an infinite number of classes of $E_{n+1}$. We can prove that this theory has quantifier elimination using the test that we saw in the preliminaries:

**Exercise 5.** Use Lemma 1.5 to prove that the theory $T_E$ has quantifier elimination.

Now that we know that this theory has quantifier elimination, we know that the complete types are completely determined by the quantifier-free formulas that they contain. So the number of complete types over a set of parameters $A$ is at most the number of maximal consistent sets of quantifier-free formulas.

**Exercise 6.** Show that the number of complete types over a set of parameters $A$ is at most $|A|^{\aleph_0}$, and conclude that $T_E$ is stable.

### The theory of modules

Our last example of a stable theory will be the theory of a module. This example will be rather lengthy and involved, and we will need to assume some knowledge of basic group theory.

We will first give the definition of a module. Intuitively, a module is to a ring what a vector space is to a field.

**Definition 2.9.** Let $R$ be a ring. An *R-module* $(M, 0, +, r)_{r\in R}$, also denoted $M$, is an abelian group $(M, 0, +)$ together with for every $r \in R$ a map $M \to M$, also denoted by $r$, such that we have for all $x, y \in M$ and $r, s \in R$:

- $r(x + y) = rx + ry$.
- $(r + s)x = rx + sx$.

- $(rs)x = r(sx)$.
- $1x = x$.

Note that we use $rx$ as shorthand for $r(x)$, just as we do with scalar multiplication for vector spaces. In this section, we will show that the theory of any infinite $R$-module is stable. In our previous examples, it was a big help that our theories had quantifier elimination, because this gave us a good idea of how many formulas, up to equivalence, there were. However, this theory does not have quantifier elimination. But we do have quantifier elimination up to a certain class of formulas, called the *positive primitive* formulas. Note that we will be working in the language of $R$-modules, consisting of the constant element 0, the binary operation $+$, and a unary function symbol $r$ for every element $r \in R$. The inverse of an element $m \in M$ under the operation $+$ will be denoted by $-m$, and the expression $a+(-b)$ will be abbreviated by $a - b$. Throughout this section, whenever we say 'formula', we will mean a formula in the language of $R$-modules, unless specified otherwise.

**Definition 2.10.** Let $\phi(x_0, ..., x_n)$ be a formula, then we say that $\phi$ is an *equation* if it is of the form

$$r_0x_0 + ... + r_nx_n = 0$$

for certain $r_0, ..., r_n \in R$.

Note that any atomic formula is equivalent to an equation, since any atomic formula will (after possibly applying some of the axioms) be of the form $r_0x_0 + ... + r_nx_n = s_0y_0 + ... + s_ky_k$ for some $n, k \in \mathbb{N}$ and some $r_0, ..., r_n, s_0, ..., s_k \in R$. And this formula is equivalent to $r_0x_0 + ... + r_nx_n - s_0y_0 - ... - s_ky_k = 0$.

**Definition 2.11.** Let $\phi(x)$ be a formula, then we say that $\phi$ is *positive primitive* (or: $\phi$ is a pp-formula) if it is of the form

$$\exists y(\psi_0(x, y) \wedge ... \wedge \psi_n(x, y))$$

where $\psi_0, ..., \psi_n$ are equations.

Notice that the class of pp-formulas is closed under $\wedge$ and $\exists$. Also notice that any equation is also equivalent to a pp-formula, since we can just quantify over a dummy variable. Our claim is that for every $R$-module $M$, under the assumption of the theory of $M$, any formula will be equivalent to a pp-formula. In order to prove this claim, we will first have to do some group theory.

**Exercise 7.** Let $\phi(x_0, ..., x_n)$ be a pp-formula. Prove that $\phi(M^{n+1})$ is a subgroup of the Abelian group $M^{n+1}$.

From now on we will no longer be bothered about the number of variables in formulas. So $\phi(x_0, ..., x_n)$ will just be denoted $\phi(x)$, and $\phi(M^{n+1})$ will just be denoted $\phi(M)$.

**Lemma 2.12.** *Let $\phi(x, y)$ be a pp-formula, and let $a \in M$. Then $\phi(M, a)$ is either empty or a coset of $\phi(M, 0)$.*

*Proof.* First note that this lemma makes sense since $\phi(x, 0)$ is itself a pp-formula, and hence $\phi(M, 0)$ is a group. Suppose $\phi(M, a)$ is nonempty. Then there is some element $b$ such that $M \models \phi(b, a)$. We will prove that $M \models \forall x(\phi(x, 0) \leftrightarrow \phi(b + x, a)$. So let $c \in M$ and suppose $M \models \phi(c, 0)$. We know that $\phi(x, y)$ is a positive primitive formula, so it will be of the form

$$\exists z(\psi_0(x, y, z) \wedge ... \wedge \psi_n(x, y, z))$$

where the $\psi_i$ are equations. So we notice that

$$M \models \exists z(\psi_0(c, 0, z) \wedge ... \wedge \psi_n(c, 0, z)).$$

And we also know by definition of $b$ that

$$M \models \exists w(\psi_0(b, a, w) \wedge ... \wedge \psi_n(b, a, w)).$$

So if we let $s$ and $v$ be witnesses to this, respectively, then we see that $M \models \psi_i(c, 0, s) \wedge \psi_i(b, a, v)$ for every $i \leq n$. But from this and the structure of equations, we see that $M \models \psi_i(b + c, a, s + v)$ for every $i \leq n$, so we find that

$$M \models \exists u(\psi_0(b + c, a, u) \wedge ... \wedge \psi_n(b + c, a, u)).$$

So we indeed see that $M \models \forall x(\phi(x, 0) \rightarrow \phi(b + x, a))$.
For the other direction, we suppose that $c$ is such that $M \models \phi(b + c, a)$. So we know that

$$M \models \exists y(\psi_0(b + c, a, y) \wedge ... \wedge \psi_n(b + c, a, y)).$$

And we also know that $M \models \phi(b, a)$, so we also know that

$$M \models \exists z(\psi_0(b, a, z) \wedge ... \wedge \psi_n(b, a, z)).$$

So if we let $u$ and $v$ be witnesses to this, respectively, then we see that

$$M \models \psi_0(c, 0, u - v) \wedge ... \wedge \psi_n(c, 0, u - v))$$

So we indeed find that $M \models \forall x(\phi(x, 0) \leftrightarrow \phi(b + x, a))$. So $\phi(M, a)$ is indeed either empty or a coset of $\phi(M, 0)$. □

As an immediate corollary to this, we see that if $a, b \in M$, then either $\phi(M, a) = \phi(M, b)$, or $\phi(M, a) \cap \phi(M, b) = \emptyset$, because cosets of $\phi(M, 0)$ are disjoint.

**Lemma 2.13.** *Let $(G, +, 0)$ be a group and let $H_0, ..., H_n$ be subgroups of $G$ and $g_0, ..., g_n \in G$ such that $G = \bigcup_{i \leq n}(g_i + H_i)$. Then at least one of the groups $H_0, ..., H_n$ has finite index in $G$.*

*Proof.* We will use induction on the number of subgroups to prove this. So suppose $n = 0$, so there is only one subgroup $H_0$, and $G = g_0 + H_0$ for some $g_0 \in G$. Then clearly $H_0$ has finite index in $G$.
So now suppose the lemma is true for some natural number $n$, we will prove it for $n + 1$. For this, consider the subgroup $H_{n+1}$. We will assume that the groups $H_0, ..., H_m$ are all different from $H_{n+1}$, while $H_{m+1} = ... = H_n = H_{n+1}$, for some $m \leq n$. Now there are two possible cases to consider. If $G = \bigcup_{i=m+1}^{n+1}(g_i + H_i)$, then we see that $H_{n+1}$ has finite index in $G$. So

now suppose that there is some $g \in G$ such that $g \notin \bigcup_{i=m+1}^{n+1}(g_i + H_i)$. Since $g + H_{n+1}$ is a coset of $H_{n+1}$ in $G$ and cosets of $H_{n+1}$ are disjoint, we now find that

$$g + H_{n+1} \cap \bigcup_{i=m+1}^{n+1} (g_i + H_i) = \emptyset.$$

So from this we find that $g + H_{n+1} \subseteq \bigcup_{i \leq m}(g_i + H_i)$, so

$$H_{n+1} \subseteq \bigcup_{i \leq m} (-g + g_i + H_i).$$

So $H_{n+1}$ is a finite union of cosets of the other subgroups. So any coset of $H_{n+1}$ will also be a finite union of cosets of the other subgroups, and hence $G$ is a finite union of cosets of $H_0, ..., H_n$. It follows from the induction hypothesis that one of these subgroups has finite index in $G$. □

**Lemma 2.14.** *Suppose $H, K$ are subgroups of $G$ with finite index in $G$. Then $H \cap K$ also has finite index in $G$.*

*Proof.* First note that two elements $x, y$ are in the same left coset of a subgroup $H$ of $K$, if and only if $-x + y \in H$. This is because if $-x + y \in H$, then $y \in x + H$, and obviously $x \in x + H$. And if $x, y \in a + H$ for some $a \in G$, then $x = a + h$ and $y = a + h'$ for some $h, h' \in H$. It follows that $-x + y = -h - a + a + h' = h + h' \in H$.

Now we use this observation to see that if $x, y$ are in the same left coset of $H$ and in the same left coset of $K$, then they are in the same left coset of $H \cap K$. For if $-x + y \in H$ and $-x + y \in K$, then $-x + y \in H \cap K$. And from this we see that if $x$ and $y$ are in different left cosets of $H \cap K$, then they are either in different left cosets of $H$ or in different left cosets of $K$. Hence if $H$ has index $n$ and $K$ has index $m$, then $H \cap K$ has index at most $nm$. □

**Lemma 2.15.** *Let $(G, +, 0)$ be a group and let $H_0, ..., H_n$ be subgroups of $G$ and $g_0, ..., g_n \in G$ such that $G = \bigcup_{i \leq n}(g_i + H_i)$. Suppose the groups $H_0, ..., H_n$ are ordered such that for some $m \leq n$, the groups $H_0, ..., H_m$ have finite index in $G$ and the groups $H_{m+1}, ..., H_n$ have infinite index. Then $G = \bigcup_{i \leq m}(g_i + H_i)$.*

*Proof.* Let $D = \bigcap_{i \leq m} H_i$. By Lemma 2.14, the group $D$ has finite index in $G$, and hence it also has finite index in every $H_i$ for $i \leq m$. This means that there are elements $d_0, ..., d_r \in G$ for some $r \in \omega$ such that $\bigcup_{i \leq r}(d_i + D) = \bigcup_{i \leq m}(g_i + H_i)$. So we find that

$$G = \bigcup_{i \leq r}(d_i + D) \cup \bigcup_{m+1 \leq i \leq n} (g_i + H_i).$$

Now suppose that $G \neq \bigcup_{i \leq r}(d_i + D)$, then there is some coset of $D$, say $g + D$, which is contained in $\bigcup_{m+1 \leq i \leq n}(g_i + H_i)$, since different cosets of $D$ are disjoint. This means that $D \subseteq \bigcup_{m+1 \leq i \leq n}(-g + g_i + H_i)$. Since $D$ has finite index in $G$, this means that there are elements $h_0, ..., h_l$ for some $l \in \omega$ such that $G = \bigcup_{j \leq l} \bigcup_{m+1 \leq i \leq n}(h_j - g + g_i + H_i)$. It follows from Lemma 2.13 that at least one of the groups $H_{m+1}, ..., H_n$ should have finite index in $G$. This contradicts our assumption, and hence we can conclude that $G = \bigcup_{i \leq m}(g_i + H_i)$. □

As an immediate corollary to this, we see the following:

**Corollary 2.16.** *Let $(G, +, 0)$ be a group and let $H_0, ..., H_n$ be subgroups of $G$ such that for some $k < n$ we have that $H_0/(H_0 \cap H_i)$ is infinite for all $i > k$. If $g_0, ..., g_n \in G$ are such that $g_0 + H_0 \subseteq \bigcup_{i=1}^n g_i + H_i$, then $g_0 + H_0 \subseteq \bigcup_{i=1}^k g_i + H_i$.*

Now we just need one more combinatorial result to finally be able to prove that the theory of $M$ has quantifier elimination up to pp-formulas.

**Lemma 2.17.** *Let $A_0, ..., A_k$ be sets, where $A_0$ is finite. Then $A_0 \subseteq \bigcup_{i=1}^k A_i$ if and only if*

$$\sum_{B \subseteq \{1,...,k\}} (-1)^{|B|} \left| A_0 \cap \bigcap_{i \in B} A_i \right| = 0.$$

*Proof.* First suppose that $A_0 \subseteq \bigcup_{i=1}^k A_i$. Notice that we can assume that $A_0 = \bigcup_{i=1}^k A_i$, since the elements in the $A_i$ that are not contained in $A_0$ will never have an influence on the mentioned sum. Now we know from the inclusion-exclusion principle from combinatorics:

$$|A_0| = \sum_{\emptyset \neq B \subseteq \{1,...,k\}} (-1)^{|B|+1} \left| \bigcap_{i \in B} A_i \right|.$$

So by subtracting the right side from both sides and taking the intersection with $A_0$ (which will have no effect), we find:

$$\sum_{B \subseteq \{1,...,k\}} (-1)^{|B|} \left| A_0 \cap \bigcap_{i \in B} A_i \right| = 0.$$

For the other direction, we will assume that $A_0 \not\subseteq \bigcup_{i=1}^k A_i$, so let $\{a_0, ..., a_n\} \subseteq A_0$ be $A_0 \backslash \bigcup_{i=1}^k A_i$. Note that we can assume that $\bigcup_{i=1}^n A_i \subseteq A_0$, because the elements in this union that are not contained in $A_0$ have no influence on the situation. Now we find using the inclusion-exclusion principle that

$$|A_0 \backslash \{a_0, ..., a_n\}| = \sum_{\emptyset \neq B \subseteq \{1,...,k\}} (-1)^{|B|+1} \left| \bigcap_{i \in B} A_i \right|.$$

And from this we find that

$$\sum_{B \subseteq \{1,...,k\}} (-1)^{|B|} \left| A_0 \cap \bigcap_{i \in B} A_i \right| = n + 1 \neq 0.$$

□

Now we have acquired all the ingredients for proving quantifier elimination up to pp-formulas.

**Theorem 2.18.** *If $R$ is a ring and $M$ is an infinite $R$-module, then in any model of the theory of $M$, every formula is equivalent to a Boolean combination of pp-formulas.*

*Proof.* We will use induction on the complexity of formulas. Notice that the class of Boolean combinations of pp-formulas is closed under conjunction, disjunction and negation. So it is enough to prove that this class is also closed under universal quantification, since we know it contains all the quantifier-free formulas. So suppose $\psi(x, y)$ is equivalent to a Boolean combination of pp-formulas, we will show that $\forall x \psi(x, y)$ is also equivalent to a Boolean combination of pp-formulas.
Since $\psi(x, y)$ is equivalent to a Boolean combination of pp-formulas, we see that it is equivalent to a conjunction of formulas of the form

$$\phi_0(x, y) \to \phi_1(x, y) \vee ... \vee \phi_n(x, y)$$

where the $\phi_i$ are all pp-formulas. It suffices to consider the case where $\psi(x, y)$ is itself of this form, since universal quantification distributes over conjunction. We now define the subgroup $H_i = \phi_i(M, 0)$ for every $i \leq n$. So for every parameter $y$, we see that $\phi_i(M, y)$ is either empty or a coset of $H_i$, by Lemma 2.12. So if $\phi_i(M, y)$ is nonempty, we will write $\phi_i(M, y) = y_i + \phi(M, 0) = y_i + H_i$. We can assume w.l.o.g. that the groups $H_0, ..., H_n$ are ordered such that for some $k \leq n$ we have that $H_0/(H_0 \cap H_i)$ is finite for all $i \leq k$ and $H_0/(H_0 \cap H_i)$ is infinite for all $i > k$. We now see by Corollary 2.16:

$$\begin{aligned} M \models \forall x \psi(x, y) \text{ iff } & M \models \forall x \left( \phi_0(x, y) \to \bigvee_{i=1}^{n} \phi_i(x, y) \right) \\ \text{iff } & \phi_0(M, y) \subseteq \bigcup_{i=1}^{n} \phi_i(M, y) \\ \text{iff } & y_0 + H_0 \subseteq \bigcup_{i=1}^{n} (y_i + H_i) \\ \text{iff } & y_0 + H_0 \subseteq \bigcup_{i=1}^{k} (y_i + H_i) \\ \text{iff } & M \models \forall x \left( \phi_0(x, y) \to \bigvee_{i=1}^{k} \phi_i(x, y) \right). \end{aligned}$$

So we see that we need to prove that $\phi_0(M, y) \subseteq \bigcup_{i=1}^{k} \phi_i(M, y)$ if and only if some pp-formula is true in $M$. For this we will first prove that this is the case if and only if

$$\phi_0(M, y)/(H_0 \cap ... \cap H_k) \subseteq \bigcup_{i=1}^{k} \phi_i(M, y)/(H_0 \cap ... \cap H_k).$$

We notice that the left to right direction is clear, so suppose we have an element in $\phi_0(M, y)$, which must be of the form $y_0 + h_0$, with $h_0 \in H_0$. Since we know that $y_0 + h_0 + (H_0 \cap ... \cap H_k) \in \bigcup_{i=1}^{k} (y_i + H_i)/(H_0 \cap ... \cap H_k)$, we see that there must be some $i$ with $1 \leq i \leq k$, some $h_i \in H_i$ and some $h \in H_0 \cap ... \cap H_k$ such that $y_0 + h_0 = y_i + h_i + h$. But since $h_i + h \in H_i$, we find that $y_0 + h_0 \in y_i + H_i$, so we indeed see that $\phi_0(M, y) \subseteq \bigcup_{i=1}^{k} \phi_i(M, y)$.
Now we define for every $i \leq k$ the set $A_i = \phi_i(M, y)/(H_0 \cap ... \cap H_k)$, and notice that $A_0$ is finite, since $H_0 \cap ... \cap H_k$ has finite index in $H_0$. So we can now apply Lemma 2.17 to find that $M \models \forall x \psi(x, y)$ if and only if $A_0 \subseteq \bigcup_{i=1}^{k} A_i$ if and only if

$$\sum_{B \subseteq \{1, ..., k\}} (-1)^{|B|} S_B = 0$$

where $S_B = 0$ if $\bigcap_{i\in B}\phi(M,y)=\emptyset$, and

$$S_B = \left| \left( H_0 \cap \bigcap_{i\in B} H_i \right) / (H_0 \cap ... \cap H_k) \right|$$

otherwise. This is the case because every $\phi_i(M,y)$ is either empty or a coset of $H_i$, so it is either empty or a number of cosets of $H_0 \cap ... \cap H_k$, and we see that $\phi_0(M,y)\cap\bigcap_{i\in B}\phi_i(M,y)$ consists of the same number of cosets of $H_0 \cap ... \cap H_k$ as $H_0 \cap \bigcap_{i\in B} H_i$ does. Now we see that $S_B$ is just something we can compute for every set $B$, and we see that it is not equal to 0 if and only if $M \models \exists x(\phi_0(x,y) \wedge \bigwedge_{i\in B}\phi(x,y))$, which is a pp-formula. So we only have to compute $S_B$ in these cases, and hence we find that $M \models \forall x\psi(x,y)$ if and only if some Boolean combination of pp-formulas holds. □

Now it turns out that this relative quantifier elimination result is by far the most important ingredient in proving that the theory of any $R$-module is stable. So we now find relatively quickly:

**Theorem 2.19.** *Let $R$ be a ring and let $M$ be an infinite $R$-module. Let $\kappa$ be a cardinal such that $\kappa^{|R|+\aleph_0} = \kappa$, then the theory of $M$ is $\kappa$-stable, and hence stable.*

*Proof.* Let $\mathbb{M}$ be our chosen monster-model of the theory of $M$. Let $B \subset \mathbb{M}$ be such that $|B| \leq \kappa$. Since our monster-model is $\kappa$-saturated, we know that every type over $B$ is satisfied by some element in $\mathbb{M}$. And the type of an element $a \in \mathbb{M}$ over $B$ is completely determined by the pp-formulas that $a$ satisfies. So let $\Phi(B)$ be this set of pp-formulas, then we see that for every pp-formula $\phi(x,y)$, if $b, b' \in B$, then only one of $\phi(x,b)$ and $\phi(x,b')$ can be in $\Phi$ (up to equivalence), because $\phi(M,b)$ and $\phi(M,b')$ are either the same or disjoint. So the type of $a$ is determined by a function $f$ mapping every pp-formula (of which there are at most $|R|+\aleph_0$ many) to an element of $B$. It follows that

$$|S(B)| \leq |B|^{|R|+\aleph_0} \leq \kappa^{|R|+\aleph_0} = \kappa.$$

So we indeed see that the theory of $M$ is $\kappa$-stable for a suitable $\kappa$, and hence this theory is stable. □

## 2.3 Indiscernibles

A notion that will be important in the future and that also gives us an alternative characterization of stable theories is the notion of a *sequence of indiscernibles*, which we will investigate in this section. This material is again taken from [5].

**Definition 2.20.** Let $I$ be some linearly ordered index set and $(a_i)_{i\in I}$ a sequence ordered by $I$. We say that $(a_i)_{i\in I}$ is a *sequence of indiscernibles* over a set of parameters $A$ (also called an $A$-indiscernible sequence) if $a_{i_0}a_{i_1}...a_{i_n} \equiv_A a_{j_0}a_{j_1}...a_{j_n}$ for all $i_0 < i_1 < ... < i_n$ and $j_0 < j_1 < ... < j_n$ from $I$ and $n \in \omega$.

So for any $\mathcal{L}(A)$-formula $\phi(x_0,...,x_n)$, the truth of $\phi(a_{i_0},...,a_{i_n})$ depends only on the order of the sequence $i_0,...,i_n$.

**Exercise 8.** For the following sequences, determine whether they are indiscernible over a set of parameters $A$:

1. A constant sequence.
2. A subsequence of an $A$-indiscernible sequence.
3. A sequence in the theory of equality.
4. A sequence in a linear order.

**Exercise 9.** Let $T$ be the theory of the structure $(\mathbb{N}, 0, S, <, +, \equiv_2, \equiv_3, \equiv_4, ...)$. Here $S$ is the successor function, and $\equiv_n$ stands for congruence modulo $n$. You can use without proof that this theory has quantifier elimination (a proof of this fact can be found in [8]). Give an example of a non-constant $\emptyset$-indiscernible sequence indexed by $\omega$ in the theory of a suitable model of $T$.

Given a sequence in the monster model, it will always be possible to find an indiscernible sequence closely resembling it. This is made more precise in the following definition.

**Definition 2.21.** Let $I, J$ be linear orders and let $(a_i)_{i \in I}$ be a sequence in $\mathbb{M}$, and let $A$ be a set of parameters. We say that a sequence $(b_j)_{j \in J}$ in $\mathbb{M}$ is *based on* $(a_i)_{i \in I}$ (relative to $A$) if for any $j_0 < ... < j_n$ in $J$ and any finite set of $\mathcal{L}(A)$-formulas $\Phi$, there is some sequence $i_0 < ... < i_n$ in $I$ such that for all $\phi \in \Phi$ we have

$$\models \phi(a_{i_0}, ..., a_{i_n}) \quad \text{iff} \quad \models \phi(b_{j_0}, ..., b_{j_n}).$$

Our claim is that for any sequence in $\mathbb{M}$, we can find an indiscernible sequence in $\mathbb{M}$ based on it. To prove this we need the notion of the *Ehrenfeucht-Mostowski type* of a sequence.

**Definition 2.22.** Let $I$ be a linear order and let $\overline{a} = (a_i)_{i \in I}$ be a sequence of elements in $\mathbb{M}$. The *Ehrenfeucht-Mostowski type* of $\overline{a}$, denoted $\text{EM}(\overline{a}/A)$ is the following collection of formulas:

$$\text{EM}(\overline{a}/A) = \{\phi(x_0, ..., x_n) \mid n \in \omega,\ \phi \text{ an } \mathcal{L}(A)\text{-formula },\ \forall i_0 < ... < i_n : \models \phi(a_{i_0}, ..., a_{i_n})\}.$$

We say that a sequence $\overline{b}$ realizes $\text{EM}(\overline{a}/A)$ if $\text{EM}(\overline{a}/A) \subseteq \text{EM}(\overline{b}/A)$. Note that $\overline{a}$ and $\overline{b}$ need not be indexed by the same set.

**Exercise 10.** Let $I, J$ be linear orders, $A$ a set of parameters, $\overline{a} = (a_i)_{i \in I}$ and $\overline{b} = (b_j)_{j \in J}$ sequences of elements in $\mathbb{M}$ such that $\overline{b}$ is indiscernible. Show that $\overline{b}$ is based on $\overline{a}$ if and only if $\overline{b}$ realizes $\text{EM}(\overline{a}/A)$.

**Theorem 2.23.** *Let $I, J$ be infinite linear orders and $A$ a set of parameters. If $\overline{a} = (a_i)_{i \in I}$ is a sequence in $\mathbb{M}$, then there is an $A$-indiscernible sequence $\overline{b} = (b_j)_{j \in J}$ based on $\overline{a}$.*

*Proof.* We will add a sequence of constants $\overline{c} = (c_j)_{j \in J}$ to our language, and we will extend our theory $T$ to a theory $T'$ containing a set of formulas $\Phi$ telling us that the interpretation of $\overline{c}$ realizes $\text{EM}(\overline{a}/A)$:

$$\Phi = \{\phi(c_{j_0}, ..., c_{j_n}) \mid n \in \omega,\ j_0 < ... < j_n \in J,\ \phi \in \text{EM}(\overline{a}/A)\}$$

and a set of formulas $\Psi$ telling us that the interpretation of $\overline{c}$ is $A$-indiscernible:

$$\Psi = \{\psi(c_{i_0}, ..., c_{i_n}) \leftrightarrow \psi(c_{j_0}, ..., c_{j_n}) \mid i_0 < ... < i_n \in J,\ j_0 < ... < j_n \in J,\ \psi(x_0, ..., x_n) \text{ an } \mathcal{L}(A)\text{-formula}\}.$$

By Exercise 10 it is enough to show that this theory is consistent. The interpretation of the sequence $\bar{c}$ will be the desired sequence. So let $T''$ be a finite subtheory of $T'$. We see that every formula in $\Phi$ is true if we interpret $\bar{c}$ as some subsequence of $\bar{a}$. Now suppose that $T''$ contains $k$ formulas from $\Psi$. Call this set $\Psi'$. Note that we may assume that every formula in $\Psi'$ has length $n+1$. Now define a relation on tuples of elements from $\bar{a}$ of length $n+1$ as follows: two tuples $w, v$ are equivalent if and only if $\psi(w) \leftrightarrow \psi(v)$ for every $\psi(c_{j_0}, ..., c_{j_n}) \leftrightarrow \psi(c_{i_0}, ..., c_{i_n})$ in $\Psi'$. Note that this is indeed an equivalence relation, and that it has at most $2^k$ many equivalence classes. This means that by Ramsey's theorem, there is some infinite subsequence $\bar{a}_K = (a_k)_{k \in K}$ of $(a_i)_{i \in I}$ such that every increasing tuple from $\bar{a}_K$ of length $n+1$ is in the same equivalence class. This means that we have found an interpretation for those elements of $\bar{c}$ which occur in $T''$, in the form of the sequence $\bar{a}_K$. So $T''$ is satisfied by the monster model, and hence $T'$ is consistent. □

The notion of a sequence of indiscernibles can obviously be generalized to a notion where the truth of a tuple of elements is only dependent on whether or not these elements are different from each other. To be precise:

**Definition 2.24.** Let $I$ be some index set and $(a_i)_{i \in I}$ a sequence indexed by $I$. We say that $(a_i)_{i \in I}$ is a *totally indiscernible sequence* over a set of parameters $A$ if $a_{i_0} a_{i_1} ... a_{i_n} \equiv_A a_{j_0} a_{j_1} ... a_{j_n}$ for all $i_0, i_1, ..., i_n$ and $j_0, j_1, ..., j_n$ from $I$ and $n \in \omega$ as long as $i_r \neq i_k$ and $j_r \neq j_k$ for all distinct $k, r \leq n$.

Note that we no longer need the $a_i$ to really be a sequence, which is why we no longer desire $I$ to be a linear order. So we can also speak about a *totally indiscernible set*.

**Exercise 11.** Consider the language of groups, and consider the free group $G$ on a set of generators $X$. Show that the set $X$ is a totally indiscernible set over $\emptyset$.

We can prove that stable theories are exactly the class of theories in which the notions of indiscernibility and total indiscernibility coincide. To do this, we will first take a look at what we know about a sequence if we know its Ehrenfeucht-Mostowski type.

**Exercise 12.** Let $I$ be a linear order and $A$ a set of parameters. Prove that a sequence $\bar{a} = (a_i)_{i \in I}$ is $A$-indiscernible if and only if $\mathrm{EM}(\bar{a}/A)$ is complete (so for every $\mathcal{L}(A)$-formula $\phi$, we have $\phi \in \mathrm{EM}(\bar{a}/A)$ or $\neg\phi \in \mathrm{EM}(\bar{a}/A)$).

Now suppose that $\bar{a} = (a_i)_{i \in I}$ and $\bar{b} = (b_i)_{i \in I}$ are both $A$-indiscernible sequences, indexed by the same linear order $I$. If they have the same EM-type over $A$, then in fact they have the same type over $A$. For suppose $\phi(x_0, ..., x_n)$ is an $\mathcal{L}(A)$-formula and $i_0, ..., i_n \in I$ are such that $\models \phi(a_{i_0}, ..., a_{i_n})$. Now we can rearrange the elements $i_0, ..., i_n$ such that they are in increasing order, say $i'_0 < ... < i'_n$. Now define a new formula $\psi(y_0, ..., y_n)$ which is just $\phi$, but with the variables arranged in this increasing order. This means that $\models \psi(a_{i'_0}, ..., a_{i'_n})$, and since $\bar{a}$ is $A$-indiscernible, this means that $\psi \in \mathrm{EM}(\bar{a}/A)$. This means that $\psi \in \mathrm{EM}(\bar{b}/A)$, and hence we see that $\models \phi(b_{i_0}, ..., b_{i_n})$. This means that $\bar{a} \equiv_A \bar{b}$.

**Lemma 2.25.** *Suppose $I \subseteq J$ are linear orders (where $I \subseteq J$ means that the order on $J$ extends the order on $I$), $A$ is a small set of parameters and $\bar{a} = (a_i)_{i \in I}$ is an $A$-indiscernible sequence. Then there is an $A$-indiscernible sequence $\bar{b} = (b_j)_{j \in J}$ such that $b_i = a_i$ for all $i \in I$.*

*Proof.* Consider an $A$-indiscernible sequence $\overline{c} = (c_j)_{j \in J}$ based on $\overline{a}$. Then by the consideration above we see that $\overline{a} \equiv_A (c_i)_{i \in I}$. This means that there is a partial elementary map $f : A \cup \{c_i \mid i \in I\} \to \mathbb{M}$ fixing $A$ and such that $f(c_i) = a_i$ for every $i \in I$. Since the monster model is $\kappa$-homogeneous for some sufficiently large $\kappa$, this means that we can extend $f$ to an automorphism $\sigma : \mathbb{M} \to \mathbb{M}$. Now we define $\overline{b}$ by $b_j = \sigma(c_j)$ for every $j \in J$. Since $\overline{c}$ is $A$-indiscernible, we see that $\overline{b}$ is also $A$-indiscernible, and $b_i = a_i$ for every $i \in I$. □

Now we have all the tools we need for our main result.

**Theorem 2.26.** *A theory $T$ is stable if and only if every indiscernible sequence over a small set of parameters $A$ is totally indiscernible over $A$.*

*Proof.* First suppose $T$ is unstable. Then by Exercise 1 there is a formula $\phi(x, y)$ and a sequence $\overline{a} = (a_i)_{i \in \omega}$ such that $\models \phi(a_i, a_j)$ if and only if $i < j$. Now let $\overline{b} = (b_i)_{i \in \omega}$ be an $A$-indiscernible sequence based on $\overline{a}$. Now if $i < j$ then there are $k < r$ such that $\models \phi(b_i, b_j)$ if and only if $\models \phi(a_k, a_r)$. And we know that $\models \phi(a_k, a_r)$ is true since $k < r$, and hence we find that $i < j \Rightarrow \models \phi(b_i, b_j)$. Using the same method, we see that $i \geq j \Rightarrow \models \neg\phi(b_i, b_j)$. So this sequence is indiscernible but not totally indiscernible.

Suppose on the other hand that there is a sequence $\overline{a} = (a_i)_{i \in I}$ for some linear order $I$ which is $A$-indiscernible for some set of parameters $A$, but is not totally indiscernible over $A$. Because of Lemma 2.25, we can assume that $I$ is a dense linear order. For otherwise, we could just extend $I$ to a dense linear order $J$, and find a new sequence indexed by $J$, which will also be $A$-indiscernible but not totally $A$-indiscernible. Now there is some $\mathcal{L}(A)$-formula $\phi(x_0, ..., x_n)$ and some $r_0 < ... < r_n$ and some $\sigma \in S_{n+1}$ such that

$$\models \phi(a_{r_0}, ..., a_{r_n}) \wedge \neg\phi(a_{\sigma(r_0)}, ..., a_{\sigma(r_n)}).$$

Since every element of $S_{n+1}$ can be written as a product of transpositions of neighboring elements, we know that there are some elements $l_0, ..., l_n$ such that for some element $j < n$, we have:

$$\models \phi(a_{l_0}, ..., a_{l_n}) \wedge \neg\phi(a_{l_0}, ..., a_{l_{j-1}}, a_{l_{j+1}}, a_{l_j}, a_{l_{j+2}}, ..., a_{l_n}).$$

Now w.l.o.g. we can assume that $l_j < l_{j+1}$, otherwise we could consider the formula $\neg\phi$ and follow the same line of reasoning. Now we define the formula

$$\psi(x_0...x_n, y_0...y_n) := \phi(x_0, ..., x_{j-1}, x_j, y_{j+1}, ..., y_n)$$

and we define the sequence $(b_i)_{l_j < i < l_{j+1}}$ (note that there are infinitely many such $i$) by

$$b_i = a_{l_0}...a_{l_{j-1}} a_i a_i a_{l_{j+2}}...a_{l_n}.$$

Now since $\overline{a}$ is $A$-indiscernible, we see that $\models \psi(b_i, b_k)$ if and only if $i < k$. By Exercise 1 it follows that $T$ is unstable. □

## 2.4 Unstable theories are IP or SOP

In this section we will investigate an alternative characterization of stable theories: we prove that a theory is stable exactly when it satisfies neither of two properties called the strict order property and the independence property. The material in this section is taken from [6].

**Definition 2.27.** A formula $\phi(x, y)$ has the *strict order property* (or SOP), if there is a sequence $(b_i)_{i\in\omega}$ such that for all $i \in \omega$, $\phi(\mathbb{M}, b_i) \subsetneq \phi(\mathbb{M}, b_{i+1})$. A theory $T$ has SOP if there is some $\mathcal{L}$-formula which has SOP. We will also denote the class of all theories which have SOP by SOP. The class of theories which are not in SOP will be denoted by NSOP.

Note that a formula having SOP or not depends on the monster model, and therefore it depends on the theory.

**Lemma 2.28.** *If a formula $\neg\phi(x, y)$ has SOP, then $\phi(x, y)$ has SOP.*

*Proof.* Suppose that $(b_i)_{i\in\omega}$ is a sequence such that for all $i \in \omega$ we have that $\neg\phi(\mathbb{M}, b_i) \subsetneq \neg\phi(\mathbb{M}, b_{i+1})$. Now let $k \in \omega$ be given, and define for all $i \leq k$: $d_i = b_{k-i}$. Then we notice that for all $i < k$ we have: $\phi(\mathbb{M}, d_i) \subsetneq \phi(\mathbb{M}, d_{i+1})$. Now we can use the compactness theorem. Consider a new set of constants $(c_i)_{i\in\omega}$ and axioms $\forall x(\phi(x, c_i) \to \phi(x, c_{i+1}))$ for all $i \in \omega$ and $\exists x(\phi(x, c_{i+1}) \wedge \neg\phi(x, c_i))$ for all $i \in \omega$. Consider a finite subset of these axioms, then there is some $n \in \omega$ such that no constant $c_i$ occurring in this subset has an index larger than $n$. Now we notice that this subtheory is consistent with T by interpreting the sequence $(c_i)_{i\leq n}$ with the sequence $(d_i)_{i\leq n}$ as defined above. And hence the entire theory is consistent, so we can find an interpretation of the sequence $(c_i)_{i\in\omega}$ in $\mathbb{M}$ such that the axioms above are satisfied. It follows that $\phi(x, y)$ has SOP. □

**Definition 2.29.** A formula $\phi(x, y)$ has the *independence property* (or IP), if there are sequences $(b_i)_{i\in\omega}$ and $(a_s)_{s\subseteq\omega}$ such that for all $i \in \omega$ and $s \subseteq \omega$: $\models \phi(a_s, b_i)$ iff $i \in s$. A theory $T$ has IP if there is some formula which has IP. We will also denote the class of all theories which have IP by IP. The class of theories which are not in IP will be denoted by NIP.

**Lemma 2.30.** *Let $\phi(x, y)$ be a formula. If for some sequence $(b_i)_{i\in\omega}$ and for all $n \in \omega$ and $\mu \in 2^n$ we have:*

$$\models \exists x \left( \bigwedge_{\mu(i)=1,\ i<n} \phi(x, b_i) \quad \wedge \bigwedge_{\mu(i)=0,\ i<n} \neg\phi(x, b_i) \right),$$

*then $\phi(x, y)$ has IP.*

*Proof.* Consider a set of new constants $\{c_s \mid s \subseteq \omega\}$ and for all $s \subseteq \omega$ and $i \in \omega$ an axiom $\phi(c_s, b_i)$ if $i \in s$ and $\neg\phi(c_s, b_i)$ if $i \notin s$. Any finite subset of this infinite set of axioms is consistent with $T$, since we only have a finite number of axioms containing $c_s$, for any $s$. So for every $s$, let $n_s + 1$ be maximal such that $\phi(c_s, b_{n_s})$ or $\neg\phi(c_s, b_{n_s})$ occurs in this finite subtheory, then we can simply choose a suitable $\mu \in 2^{n_s}$, and since we have a witness to

$$\models \exists x \left( \bigwedge_{\mu(i)=1,\ i<n_s+1} \phi(x, b_i) \quad \wedge \bigwedge_{\mu(i)=0,\ i<n_s+1} \neg\phi(x, b_i) \right),$$

we obtain an interpretation for $c_s$ in $\mathbb{M}$. If follows by compactness that $\phi(x, y)$ has IP. □

**Lemma 2.31.** *Let $\phi(x, y)$ be a formula. If there is an $n \in \omega$ and a $\mu \in 2^n$ such that the formula $\psi(x, y_0, ..., y_{n-1})$ defined by*

$$\psi(x, y_0, ..., y_{n-1}) = \bigwedge_{\mu(i)=1,\ i<n} \phi(x, y_i) \quad \wedge \bigwedge_{\mu(i)=0,\ i<n} \neg\phi(x, y_i)$$

*has SOP, then $\phi(x, y)$ has SOP.*

*Proof.* Since the formula $\psi$ has the strict order property, we have sequences $(b_i^k)_{i\in\omega}$ for every $k < n$ such that for every $i \in \omega$ we have that $\psi(\mathbb{M}, b_i^0, ..., b_i^{n-1}) \subsetneq \psi(\mathbb{M}, b_{i+1}^0, ..., b_{i+1}^{n-1})$. This means that for every $i \in \omega$ there is a number $k < n$ such that there is some $x_i \in \phi(\mathbb{M}, b_i^k)\backslash\phi(\mathbb{M}, b_{i+1}^k)$ or $x_i \in \neg\phi(\mathbb{M}, b_i^k)\backslash\neg\phi(\mathbb{M}, b_{i+1}^k)$. It now follows from the pigeonhole principle that there is some infinite subset $A \subseteq \omega$ such that we either have for all $i \in A$: there is some $k < n$ such that there is some $x_i \in \phi(\mathbb{M}, b_i^k)\backslash\phi(\mathbb{M}, b_{i+1}^k)$, or for every $i \in A$ there is some $k < n$ such that there is some $x_i \in \neg\phi(\mathbb{M}, b_i^k)\backslash\neg\phi(\mathbb{M}, b_{i+1}^k)$. Because of Lemma 2.4, we can say without loss of generality that we are in the first case. Since $n$ is finite, it follows by the pigeonhole principle that there is some $k < n$ such that there is an infinite number of $i \in A$ such that $\phi(\mathbb{M}, b_i^k) \subsetneq \phi(\mathbb{M}, b_{i+1}^k)$. This infinite subsequence of $(b_i)_{i\in\omega}$ is a witness to the statement that $\phi(x, y)$ has SOP. □

**Theorem 2.32.** *A formula $\phi(x, y)$ is unstable iff $\phi \in IP \cup SOP$.*

*Proof.* Suppose first that $\phi$ has SOP, say that the sequence $(b_i)_{i\in\omega}$ is a witness to this. Now let $k \in \omega$ be given. For every $i < k$, choose an element $a_i \in \phi(\mathbb{M}, b_{i+1})\backslash\phi(\mathbb{M}, b_i)$. Then we notice that we have sequences $(a_i)_{i<k}$ and $(b_i)_{i<k}$ such that for all $i, j < k$ we have $\models \phi(a_i, b_j)$ iff $i < j$. So $\phi(x, y)$ has the $k$-order property for every $k \in \omega$, hence it is unstable.
Now suppose that $\phi$ has IP, say that the sequences $(b_i)_{i\in\omega}$ and $(a_s)_{s\subseteq\omega}$ are witnesses to this. Now let $k \in \omega$ be given and define for every $i < k$: $c_i = a_{\{i+1,i+2,...,k\}}$. Then we notice that for all $i, j \in \omega$ we have $\models \phi(c_i, b_j)$ iff $i < j$, so $\phi(x, y)$ has the $k$-order property, hence $\phi$ is unstable.

Now we will prove the converse to this statement, so suppose $\phi$ is unstable, so by Lemma 2.2 there are sequences $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ such that for all $i, j \in \omega$ we have $\models \phi(a_i, b_j)$ iff $i < j$. By Lemma 2.23 we can assume these sequences to be indiscernible. If this formula has IP we are done, so we will assume that it does not have IP, and prove that it has SOP. So by Lemma 2.30 there are $n \in \omega$ and $\mu \in 2^n$ such that

$$\models \neg\exists x \left( \bigwedge_{\mu(i)=1,\ i<n} \phi(x, b_i) \ \wedge \bigwedge_{\mu(i)=0,\ i<n} \neg\phi(x, b_i) \right). \tag{2.1}$$

Now define the set $X_0 = \{i \mid \mu(i) = 1\}$, and let $m = |X_0|$. We notice that $0 < m < n$, for suppose $m = 0$, then we would have $\models \neg\exists x \bigwedge_{i<n} \neg\phi(x, b_i)$, which is impossible since $a_n$ is a witness to the statement $\models \exists x \bigwedge_{i<n} \neg\phi(x, b_i)$. And if $m = n$, then we would have $\models \neg\exists x \bigwedge_{i<n} \phi(x, b_i)$. However, we notice that $a_0$ is a witness to the statement $\models \exists x \bigwedge_{0<i\leq n} \phi(x, b_i)$, and since the sequence $(b_i)_{i\in\omega}$ is indiscernible we know that $\models \exists x \bigwedge_{i<n} \phi(x, b_i)$ must also have a witness.
Now we will define sets $X_1, ..., X_N$ for some $N \geq 1$ such that for the sequence of sets $X_0, X_1, ..., X_N$ we have that for every $k \leq N$:

- For every $k \leq N$: $X_k \subset \{0, ..., n-1\}$ and $|X_k| = m$,
- $X_N = \{n - m, n - m + 1, ..., n - 1\}$,
- for all $k < N$ there is some $l \in X_k$ such that $X_{k+1} = (X_k\backslash\{l\}) \cup \{l + 1\}$.

This can be done in the following way: if we have already constructed $X_p$ for some $p < N$, then write $X_p = \{l_1, ..., l_m\}$ with $l_1 < l_2 < ... < l_m$, and notice that $l_i \leq n - 1 - m + i$. Now

let $i$ be maximal such that $l_i < n-1-m+i$ and let $X_{p+1} = X_p \backslash \{l_i\} \cup \{l_i+1\}$.
Now we can translate (2.1) to:

$$\models \neg\exists x \left( \bigwedge_{i \in X_0} \phi(x, b_i) \quad \wedge \bigwedge_{i \notin X_0,\ i<n} \neg\phi(x, b_i) \right).$$

And we also notice that we have $a_{n-m-1}$ as a witness to the statement:

$$\models \exists x \left( \bigwedge_{i \in X_N} \phi(x, b_i) \quad \wedge \bigwedge_{i \notin X_N,\ i<n} \neg\phi(x, b_i) \right).$$

Now from these two observations and the fact that $N < \omega$ it follows that there must be some $k < N$ such that

$$\models \neg\exists x \left( \bigwedge_{i \in X_k} \phi(x, b_i) \quad \wedge \bigwedge_{i \notin X_k,\ i<n} \neg\phi(x, b_i) \right) \tag{2.2}$$

and

$$\models \exists x \left( \bigwedge_{i \in X_{k+1}} \phi(x, b_i) \quad \wedge \bigwedge_{i \notin X_{k+1},\ i<n} \neg\phi(x, b_i) \right). \tag{2.3}$$

Now let $l \in X_k$ be the element such that $X_{k+1} = X_k \backslash \{l\} \cup \{l+1\}$, and define the following formula:

$$\psi(x, y, y_0, ..., y_{l-1}, y_{l+2}, ..., y_{n-1}) = \phi(x, y) \wedge \bigwedge_{i \in X_k \backslash \{l\}} \phi(x, y_i) \quad \wedge \bigwedge_{i \notin X_{k+1} \cup \{l\}} \neg\phi(x, y_i).$$

Now define for every $r \in \omega$ the sequence $\overline{b_r} = (b_0, ..., b_{l-1}, b_{l+2+r}, ..., b_{n-1+r})$. Then (2.3) can be rewritten as $\models \exists x(\psi(x, b_{l+1}, \overline{b_0}) \wedge \neg\phi(x, b_l))$. Since the sequence $(b_i)_{i\in\omega}$ is indiscernible it follows for all $r \in \omega$ and for all $i, j \in \omega$ such that $l \leq i < j < l+2+r$:

$$\models \exists x(\psi(x, b_j, \overline{b_r}) \wedge \neg\phi(x, b_i)). \tag{2.4}$$

And in much the same way we can translate (2.2) to $\models \neg\exists x(\psi(x, b_l, \overline{b_0}) \wedge \neg\phi(x, b_{l+1}))$ and hence we find using the same argument that for all $i, j \in \omega$ with $l \leq i < j < l+2+r$:

$$\models \neg\exists x(\psi(x, b_i, \overline{b_r}) \wedge \neg\phi(x, b_j)). \tag{2.5}$$

Now we notice that $\models \forall x(\psi(x, b_i, \overline{b_r}) \rightarrow \phi(x, b_i))$ and hence $\models \forall x(\neg\phi(x, b_i) \rightarrow \neg\psi(x, b_i, \overline{b_r}))$. From this and (2.4) we can obtain for all $r \in \omega$ and $l \leq i < j < l+2+r$:

$$\models \exists x(\psi(x, b_j, \overline{b_r}) \wedge \neg\psi(x, b_i, \overline{b_r})).$$

Now we find from (2.5):

$$\models \forall x(\psi(x, b_i, \overline{b_r}) \rightarrow \phi(x, b_j)).$$

And from the definition $\psi$, we also know that

$$\models \forall x, y, y', \overline{y}(\psi(x, y, \overline{y}) \wedge \phi(x, y') \rightarrow \psi(x, y', \overline{y})).$$

And from this we find

$$\models \forall x(\psi(x, b_i, \overline{b_r}) \to \psi(x, b_j, \overline{b_r})).$$

Now consider a new set of constants $\{c_i \mid i \in \omega\}$ and a sequence of new constants $B$, and consider the set of axioms:

$$\{\exists x(\psi(x, c_j, B) \wedge \neg\psi(x, c_i, B))\} \cup \{\forall x(\psi(x, c_i, B) \to \psi(x, c_j, B))\}$$

for all $i, j \in \omega$ with $i < j$. Notice that any finite subset of this set of axioms is consistent with $T$, by interpreting $c_i$ by $b_{l+i}$ and $B$ by $\overline{b_r}$ for sufficiently large $r$. So by compactness the entire set is consistent with $T$, with interpretation $\{c'_i \mid i \in \omega\}$ and $B'$ in $\mathbb{M}$. Now define for all $i \in \omega$: $d_i = (c'_i, B')$. Then we find that for all $i \in \omega$ we have $\psi(\mathbb{M}, d_i) \subsetneq \psi(\mathbb{M}, d_{i+1})$. So $\psi$ has SOP, and by Lemma 2.31 it follows that $\phi(x, y)$ has SOP, concluding our proof. $\square$

**Corollary 2.33.** *A theory $T$ is stable iff $T \in \mathrm{NIP} \cap \mathrm{NSOP}$.*

*Chapter 3*

# Ranks

## 3.1 Cantor-Bendixson rank

There are several notions of ranks occurring in model theory. In essence, a rank is a map, assigning an ordinal to a formula, set of formulas or a (complete) type. We will investigate several notions of ranks, and see how they connect to each other. In a later stage we will see how these ranks connect to stability and forking. The first notion of rank will be motivated from topology. The material in this section is mostly taken from [5], [19] and [3].

**Definition 3.1.** Let $X$ be a compact Hausdorff topological space. We define for every ordinal $\alpha$ a subspace $X_\alpha$ of $X$ as follows:

- $X_0 = X$.
- $X_{\alpha+1} = X_\alpha - \{p \in X_\alpha \mid p \text{ is isolated in } X_\alpha\}$.
- $X_\lambda = \bigcap_{\alpha<\lambda} X_\alpha$ if $\lambda$ is a limit ordinal.

Note that one can prove by induction that every $X_\alpha$ is closed in $X$. Indeed, $X$ is clearly closed in itself. If $X_\alpha$ is closed in $X$ for some ordinal $\alpha$, then we notice that $\{p \in X_\alpha \mid p \text{ is isolated in } X_\alpha\}$ is a union of opens in $X_\alpha$, hence it is open in $X_\alpha$. We find that $X_{\alpha+1}$ is closed in $X_\alpha$, and hence it is also closed in $X$. Furthermore, if $\lambda$ is a limit ordinal, then we see that $X_\lambda$ is an intersection of closed sets in $X$ (by the induction hypothesis), and hence it is itself closed in $X$. It also follows that $X_\alpha$ is compact for every ordinal $\alpha$. Now we will define the Cantor-Bendixson rank of a point $x \in X$, denoted $\mathrm{CB}(x)$, as follows:

**Definition 3.2.** Let $X$ be a compact Hausdorff topological space, and $x \in X$. The *Cantor-Bendixson rank* of $x$ is defined as

$$\mathrm{CB}(x) = \max\{\alpha \mid x \in X_\alpha\}.$$

And we say that $\mathrm{CB}(x) = \infty$ if $x \in X_\alpha$ for all ordinals $\alpha$. If $\mathrm{CB}(x) = \alpha$ for some ordinal $\alpha$, we will also say that $\mathrm{CB}(x) < \infty$, and we will call $x$ a *ranked* element.

Note that since any topological space $X$ is itself a set, there must be some ordinal $\alpha$ such that $X_\alpha = X_{\alpha+1}$, for otherwise, we could find an element $x_\alpha \in X_\alpha - X_{\alpha+1}$ for every ordinal $\alpha$, and the cardinality of $X$ would exceed the ordinals. Note that if $X_\alpha = X_{\alpha+1}$, then we see that $X_\alpha = X_\beta$ for all ordinals $\beta \geq \alpha$. So if $x \in X_\alpha$, then $\mathrm{CB}(x) > \beta$ for all ordinals $\beta$. Now suppose that every $x \in X$ is ranked, then it immediately follows that if $X_\alpha = X_{\alpha+1}$, we must have $X_\alpha = \emptyset$. So now let $\alpha$ be the smallest ordinal such that $X_\alpha = X_{\alpha+1}$. By Compactness, any nested sequence of nonempty closed subspaces of a compact space has a nonempty intersection, and hence $\alpha$ cannot be a limit ordinal. So there is some ordinal $\beta$ such that $X_\beta \neq \emptyset$ and $X_{\beta+1} = \emptyset$. It follows that $X_\beta$ must be a finite set of points (since it is compact). Now from these considerations, the following definition follows:

**Definition 3.3.** Let X be a non-empty compact Hausdorff topological space such that every element of $X$ is ranked. The largest ordinal $\beta$ such that $X_\beta$ is non-empty is called the *Cantor-Bendixson rank of the space* $X$, and the *Cantor-Bendixson multiplicity of* $X$ is the cardinality of the set $\{x \in X \mid \mathrm{CB}(x) = \beta\}$.

To get a feeling for the notion of Cantor-Bendixson rank, we will consider a few examples in the Euclidean topology.

**Example 3.4.** Consider the space $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ with the Euclidean topology. Note that this is a compact Hausdorff space. The points $\frac{1}{n}$ are all isolated, hence they have Cantor-Bendixson rank 0. The point 0 is not isolated, so this has a larger CB-rank. However, if we remove all points with CB-rank 0 from the space $X$, we are left with just the point 0, which has therefore become isolated, so it has CB-rank 1, which is also the CB-rank of the space $X$. The CB-multiplicity of $X$ is 1.

**Exercise 13.** Let $n, k$ be natural numbers. Give an example of a topological space $X$ with CB-rank $n$ and CB-multiplicity $k$.

**Example 3.5.** Consider the space $Y = [-1, 0] \cup X$, where $X$ is the space from Example 3.4. We again see that all the points $\frac{1}{n}$ have CB-rank 0, but if we remove them we will be left with the closed interval $[-1, 0]$. This interval contains no isolated points, and hence the CB-rank of every point of this interval is $\infty$. The CB-rank of $Y$ is therefore undefined.

The CB-rank satisfies some basic properties, which are usually proved by induction on the ordinals.

**Lemma 3.6.** *Let $X$ be a compact Hausdorff topological space.*

- *If $Y \subseteq X$ is a closed subspace, then $Y_\alpha \subseteq X_\alpha$ for all ordinals $\alpha$.*
- *If $Y, Z \subseteq X$ are closed subspaces, then $Y_\alpha \cup Z_\alpha = (Y \cup Z)_\alpha$.*

*Proof.*

- We use induction on $\alpha$. If $\alpha = 0$ the statement is trivial. So suppose $Y_\alpha \subseteq X_\alpha$ for some ordinal $\alpha$. Now if $y \in Y_\alpha$ is isolated in $X_\alpha$, then clearly it is also isolated in $Y_\alpha$. It follows that $Y_{\alpha+1} \subseteq X_{\alpha+1}$. Now suppose that $\lambda$ is a limit ordinal and for all ordinals $\alpha < \lambda$ we have that $Y_\alpha \subseteq X_\alpha$. Then clearly $\bigcap_{\alpha<\lambda} Y_\alpha \subseteq \bigcap_{\alpha<\lambda} X_\alpha$, so $Y_\lambda \subseteq X_\lambda$.
- Note that by the previous part, it is clear that $Y_\alpha \subseteq (Y \cup Z)_\alpha$ and $Z_\alpha \subseteq (Y \cup Z)_\alpha$. So we only have to prove that $(Y \cup Z)_\alpha \subseteq Y_\alpha \cup Z_\alpha$ for all $\alpha$. This is again done by induction on $\alpha$. For $\alpha = 0$ the statement is clear. So now suppose that $(Y \cup Z)_\alpha = Y_\alpha \cup Z_\alpha$ for some ordinal $\alpha$. Let $x \in (Y \cup Z)_{\alpha+1}$, so $x$ is not isolated in $Y_\alpha \cup Z_\alpha$. Since $x \in Y_\alpha \cup Z_\alpha$, we can w.l.o.g. assume that $x \in Y_\alpha$. Now suppose that $x$ is isolated in $Y_\alpha$, then there is some open $V$ in $X$ such that $V \cap Y_\alpha = \{x\}$. But since $x$ is not isolated in $Y_\alpha \cup Z_\alpha$,

it follows that $V \cap Z_\alpha \neq \{x\}$. And in fact, if $U$ is an open neighborhood of $x$, then $U \cap Z_\alpha \neq \{x\}$. It follows since $Z_\alpha$ is closed that $x \in Z_\alpha$ and $x$ is not isolated in $Z_\alpha$, hence $x \in Z_{\alpha+1}$. So we indeed find that $(Y \cup Z)_{\alpha+1} \subseteq Y_{\alpha+1} \cup Z_{\alpha+1}$.

Now suppose $\lambda$ is a limit ordinal and for all $\alpha < \lambda$ we have that $(Y \cup Z)_\alpha = Y_\alpha \cup Z_\alpha$. Then we find:

$$\begin{aligned}(Y \cup Z)_\lambda &= \bigcap_{\alpha<\lambda} (Y \cup Z)_\alpha \\ &= \bigcap_{\alpha<\lambda} (Y_\alpha \cup Z_\alpha) \\ &= \bigcap_{\alpha<\lambda} Y_\alpha \cup \bigcap_{\alpha<\lambda} Z_\alpha \\ &= Y_\lambda \cup Z_\lambda.\end{aligned}$$

Where the first step is due to the induction hypothesis and the second due to the fact that the $Y_\alpha$ and $Z_\alpha$ form downwards chains. We can now conclude that $Y_\alpha \cup Z_\alpha = (Y \cup Z)_\alpha$ for all ordinals $\alpha$. □

We find as an immediate corollary:

**Corollary 3.7.** *Let $X$ and $Y$ be ranked compact Hausdorff spaces.*

- *If $X$ is a subspace of $Y$, then* $\mathrm{CB}(X) \leq \mathrm{CB}(Y)$.
- $\mathrm{CB}(X \cup Y) = \max(\mathrm{CB}(X), \mathrm{CB}(Y))$

Based on the results above, one might also expect a result like $(X \cap Y)_\alpha = X_\alpha \cap Y_\alpha$ or $\mathrm{CB}(X \cap Y) = \min(\mathrm{CB}(X), \mathrm{CB}(Y))$ to hold, but it turns out that this is not the case. For consider the spaces $X = \{0\} \cup \{\frac{1}{n} \mid n \in \omega\}$ and $Y = \{0\} \cup \{-\frac{1}{n} \mid n \in \omega\}$ with the Euclidean topology. Then we notice that $X \cap Y = \{0\}$, so $(X \cap Y)_1 = \emptyset$, but $X_1 = Y_1 = \{0\}$.

Now we will start focussing specifically on type spaces. So we consider the space of complete $n$-types over a set of parameters $A$ as a Stone space, and define the Cantor-Bendixson rank of a complete type $p$ as the Cantor-Bendixson rank of this point in the space $S_n(A)$. This is in fact not the only possible way to define Cantor-Bendixson rank. Another way would be by first defining the Cantor-Bendixson rank of a formula, and then use that to define the CB-rank of a type. This is done inductively as follows:

**Definition 3.8.** Let $\phi$ be an $\mathcal{L}(A)$-formula, with $A$ a set of parameters. If $\phi$ is inconsistent we define $\mathrm{CB}(\phi) = -1$. Now if $\alpha$ is an ordinal, we define $\Psi_\alpha$ to be the set of $\mathcal{L}(A)$-formulas with CB-rank smaller then $\alpha$. Now we define $\mathrm{CB}(\phi) = \alpha$ if the set $\{p \in S_n(A) \mid \phi \in p \text{ and } \neg\psi \in p \text{ for all } \psi \in \Psi_\alpha\}$ is non-empty and finite.

Of course in the definition above, we assume that $-1$ is an element that we add to the ordinals with the property that $-1 < 0$. Now we find the following alternative definition of the CB-rank of a type:

**Lemma 3.9.** *Let $p \in S_n(A)$. Then* $\mathrm{CB}(p) = \inf\{\mathrm{CB}(\phi) \mid \phi \in p\}$.

*Proof.* We will prove by induction on the ordinal $\alpha$ that $\text{CB}(p) = \alpha$ iff $\inf\{\text{CB}(\phi) \mid \phi \in p\} = \alpha$. First we notice that if $\text{CB}(p) = 0$, then $p$ is an isolated point in $S_n(A)$, so there is some formula $\phi$ such that $[\phi] = \{p\}$. It follows that $\text{CB}(\phi) = 0$ and $\phi \in p$, so we indeed have that $\inf\{\text{CB}(\phi) \mid \phi \in p\} = 0$. Now suppose that $\inf\{\text{CB}(\phi) \mid \phi \in p\} = 0$, then there is some $\phi \in p$ such that $\text{CB}(\phi) = 0$. It follows from the definition of $\text{CB}(\phi)$ that the set of complete types containing $\phi$ is non-empty and finite, say this set is $\{q_0, ..., q_n\}$. It follows that there are formulas $\psi_0, ..., \psi_n$ such that for all $i \leq n$ we have $\psi_i \in q_i$ and if $i \neq j$, then $\psi_i \notin q_j$. It follows that $[\phi \wedge \psi_i] = \{q_i\}$ for each $i \leq k$. So the type $p$, being one of the $q_i$, is an isolated point because $\{p\} = [\phi \wedge \psi_i]$ for some $i$, and $[\phi \wedge \psi_i]$ is open. It follows that $\text{CB}(p) = 0$.

Now we will provide the induction step. So suppose that $\beta > 0$ is an ordinal and for all $\alpha < \beta$ we know that $\text{CB}(p) = \alpha$ iff $\inf\{\text{CB}(\phi) \mid \phi \in p\} = \alpha$. Now first suppose that $\text{CB}(p) = \beta$. By the induction hypothesis we see that $\inf\{\text{CB}(\phi) \mid \phi \in p\} \geq \beta$, because if $\inf\{\text{CB}(\phi) \mid \phi \in p\} = \alpha < \beta$, then by IH we see that $\text{CB}(p) = \alpha$, a contradiction. So we have to show that there is some formula $\phi \in p$ such that $\text{CB}(\phi) = \beta$. Notice that $p$ is isolated in the space of types which have CB-rank at least $\beta$, so there is some formula $\phi \in p$ such that $\{p\} = [\phi] \cap \{q \in S_n(A) \mid \text{CB}(q) \geq \beta\}$. We notice that the set of types containing $\phi$ and the negations of all formulas with CB-rank less then $\beta$ is non-empty and finite, because $p$ is the only type satisfying this property, by assumption. So by definition we see that $\text{CB}(\phi) = \beta$, and hence $\inf\{\text{CB}(\phi) \mid \phi \in p\} = \beta$.

Now suppose that $\inf\{\text{CB}(\phi) \mid \phi \in p\} = \beta$, then there is some $\phi \in p$ such that $\text{CB}(\phi) = \beta$. This means that there is only a finite number of types containing $\phi$ and the negations of all formulas with CB-rank less then $\beta$, let's say that this set is $\{q_0, ..., q_n\}$. Using the same method as in the base case, we can now find a formula $\psi$ such that $p$ is the only type containing $\psi$ and such that any formula in $p$ has CB-rank at least $\beta$. By IH this means that $p$ is isolated in the space of types which have CB-rank at least $\beta$, so $\text{CB}(p) = \beta$. This concludes our proof. □

Another possible way to define Cantor-Bendixson rank is by using Boolean algebras. We know that the clopens of a topological space $X$ form a Boolean algebra (under the usual intersection, union and set-theoretic complement), so we can define a notion of rank on Boolean algebras which will turn out to coincide with the notion of CB-rank on the clopens of a Stone space.

**Definition 3.10.** Let $B$ be a Boolean algebra. We assign to each ordinal $\alpha$ an ideal $I_{<\alpha}$ and an ideal $I_\alpha$ of $B$ as follows:

1. $I_{<0} = \{0\}$.
2. If $I_{<\alpha}$ has been defined for some ordinal $\alpha$, we let $I_\alpha$ be the ideal of $B$ such that $I_{<\alpha} \subseteq I_\alpha$ and such that the image of $I_\alpha$ under the mapping that sends $a \in B$ to $a/I_{<\alpha} \in B/I_{<\alpha}$ is exactly $(\text{at}(B/I_{<\alpha}))$, the ideal generated by the atoms of $B/I_{<\alpha}$.
3. If $\alpha > 0$ and $I_\beta$ has been defined for all $\beta < \alpha$, we set $I_{<\alpha} = \bigcup_{\beta<\alpha} I_\beta$.

Notice that we have $I_\alpha \subseteq I_{\alpha+1}$ for every ordinal $\alpha$. We can characterize in a somewhat easier way the elements that are in $I_\alpha$ for some $\alpha$.

**Lemma 3.11.** *Let $B$ be a Boolean algebra, $b \in B$ and $\alpha$ an ordinal. If $b \notin I_\alpha$, then there is a subset $\{a_i \mid i \in \omega\} \subset B$ such that for each $i \in \omega$ we have $a_i < b$ and $a_i \notin I_{<\alpha}$ and for each $i, j \in \omega$ with $i \neq j$ we have $a_i \wedge a_j = 0$.*

*Proof.* First suppose that there is an infinite set $A \subset B/I_{<\alpha}$ of atoms in $B/I_{<\alpha}$, such that for each $a \in A$ we have $a < b/I_{<\alpha}$. Now let $\{a_i \wedge I_{<\alpha} \mid i \in \omega\} \subseteq A$ be a countably infinite subset of $A$. Pick for the element $a_i \wedge I_{<\alpha} \in A$ a representative $a_i \wedge b_i \in B$. Now the sequence $\{a_i \wedge b_i \wedge \bigwedge_{j<i} \neg b_j \mid i \in \omega\}$ will meet the requirements.

So suppose that the set of atoms $A$ in $B/I_{<\alpha}$ below $b/I_{<\alpha}$ is only finite, say $A = \{a_1, ..., a_n\}$. Then we can consider the element $c = b/I_{<\alpha} \wedge \neg a_1 \wedge ... \wedge \neg a_n$. There are no more atoms smaller then this element, and we also notice that $c \notin I_\alpha$. For suppose $c \in I_\alpha$, then since $I_\alpha$ is an ideal containing all the $a_i$, we find that

$$\left(b/I_{<\alpha} \wedge \neg \bigvee_{i \leq n} a_i\right) \vee \bigvee_{i \leq n} a_i \in I_\alpha$$

and hence we find that $b/I_{<\alpha} \vee \bigvee_{i \leq n} a_i \in I_\alpha$, and since $I_\alpha$ is downwards closed we would find that $b/I_{<\alpha} \in I_\alpha$, a contradiction. So we can assume without loss of generality that there are no such atoms, but $b/I_{<\alpha}$ is not an atom so in that case we can pick some $b_0 < b$, $b_1 < b \wedge \neg b_0$, $b_2 < b \wedge \neg b_0 \wedge \neg b_1$, etc. In this case the set $\{b_i \mid i \in \omega\}$ will meet the requirements. □

Using the ideals $I_\alpha$, we can define a rank notion on elements of a Boolean algebra in a natural way.

**Definition 3.12.** Let $B$ be a Boolean algebra. We define the *Cantor rank* of an element $b \in B$, denoted by $\mathrm{CR}(b)$, by $\mathrm{CR}(0) = -1$ and if $b \neq 0$ and $\alpha$ is the smallest ordinal such that $b \in I_\alpha$, then $\mathrm{CR}(b) = \alpha$. If there is no such ordinal, we define $\mathrm{CR}(b) = \infty$.

Note that using our previous lemma, we quickly find the following corollary:

**Corollary 3.13.** *Let $B$ be a Boolean algebra, $b \in B$ and $\alpha$ an ordinal. Then $\mathrm{CR}(b) > \alpha$ iff there is an infinite set $\{a_i \mid i \in \omega\} \subset B$ such that for each $i < \omega$ we have $a_i < b$ and $\mathrm{CR}(a_i) \geq \alpha$ and for all $i, j \in \omega$ with $i \neq j$ we have $a_i \wedge a_j = 0$.*

As mentioned before, this notion of rank on a Boolean algebra coincides with the notion of CB-rank on the clopens of a Stone space.

**Theorem 3.14.** *Let $A$ be a nonempty clopen subset of a Stone space $X$. Then $\mathrm{CB}(A) = \mathrm{CR}(A)$.*

*Proof.* We will first prove by induction that if $\mathrm{CB}(A) = \alpha$, then $\mathrm{CR}(A) \geq \alpha$. We first notice that since $A$ is nonempty, we always have that $\mathrm{CR}(A) \geq 0$. Suppose that $\mathrm{CB}(A) = \alpha$ and for all ordinals $\beta < \alpha$ we know that if $A'$ is a clopen subset and $\mathrm{CB}(A') = \beta$, we have that $\mathrm{CR}(A') \geq \beta$. Now let $\beta < \alpha$ be arbitrary, then there is some $a \in A$ such that $a$ is not isolated in the space of points of $A$ which have CB-rank at least $\beta$. So in this space we can find a sequence of elements $(a_i)_{i \in \omega}$ such that there are clopens $(c_i)_{i \in \omega}$ with $c_i \cap c_j = \emptyset$ for all $i, j \in \omega$ with $i \neq j$, and $a_i \in c_i$ for all $i \in \omega$, and $\mathrm{CB}(c_i) \geq \beta$ for every $i \in \omega$. By the induction hypothesis we have $\mathrm{CR}(c_i) \geq \beta$ for all $i \in \omega$, and hence we find by Corollary 3.13 that $\mathrm{CR}(A) > \beta$. So we find that $\mathrm{CR}(A) \geq \alpha$.

Now we will prove that if $\mathrm{CB}(A) = \alpha$, we have that $\mathrm{CR}(A) \leq \alpha$. So suppose that $\mathrm{CB}(A) = \alpha$ and $\mathrm{CR}(A) > \alpha$, and for all $\beta < \alpha$ and clopens $A'$ we know that if $\mathrm{CB}(A') = \beta$ then $\mathrm{CR}(A') = \beta$. Since $\mathrm{CR}(A) > \alpha$ we know by Corollary 3.13 that there must be clopens $(c_i)_{i \in \omega}$ such that each $c_i$ is contained in $A$ and they are pairwise disjoint and all have CR-rank at least $\alpha$. By the induction hypothesis, they must also have CB-rank at least $\alpha$, so by taking a point with CB-rank at least $\alpha$ from each of these clopens, we find an infinite set of points in $A$ which have CB-rank at least $\alpha$, which is impossible as this set should be finite, since $\mathrm{CB}(A) = \alpha$. So by contradiction we find that $\mathrm{CR}(A) \leq \alpha$.

We conclude that $\mathrm{CB}(A) = \mathrm{CR}(A)$ for all clopens $A$. □

We also find that the CB-rank of formulas as defined in Definition 3.8 can easily be reformulated in terms of CR-rank, because $\mathrm{CB}(\phi) = \mathrm{CR}([\phi])$ for every formula $\phi$. This is easily seen because

$$\begin{aligned}\mathrm{CR}([\phi]) &= \mathrm{CB}([\phi]) \\ &= \max\{\mathrm{CB}(p) \mid \phi \in p\} \\ &= \max\{\inf\{\mathrm{CB}(\psi) \mid \psi \in p\} \mid \phi \in p\} \\ &\leq \mathrm{CB}(\phi).\end{aligned}$$

And on the other hand, if $\mathrm{CB}(\phi) = \alpha$, then there must be some type $p$ such that $\phi \in p$ and if $\mathrm{CB}(\psi) < \alpha$ we have $\psi \notin p$. Hence we also find $\mathrm{CB}(\phi) \leq \max\{\inf\{\mathrm{CB}(\psi) \mid \psi \in p\} \mid \phi \in p\}$.

These equivalences always allow one to choose one's favorite notion of rank.

We will now focus on local types. So let $\Delta$ be some finite set of formulas $\phi_1(x, y), ..., \phi_n(x, y)$. We consider the space $S_\Delta(\mathbb{M})$ of complete global $\Delta$-types. If we assume stability, the CB-rank of this space in fact turns out to be very well-behaved in the following sense:

**Lemma 3.15.** *If all formulas in $\Delta$ are stable, then for all $p \in S_\Delta(\mathbb{M})$ we have that $\mathrm{CB}(p) < \omega$.*

*Proof.* We abbreviate the topological space $S_\Delta(\mathbb{M})$ by $X$. Assume that there is some $p \in X$ such that $\mathrm{CB}(p) \geq \omega$, and fix some $n \in \omega$. We notice that $p$ is not isolated in the space $\{q \in X \mid \mathrm{CB}(q) \geq n\}$. This means that there must be at least two different points $q_1, q_2 \in X$ with $\mathrm{CB}(q_1), \mathrm{CB}(q_2) \geq n$, and since these are complete $\Delta$-types, there is some $\delta(x, y) \in \Delta$ and some $a \in \mathbb{M}$ such that $\delta(x, a) \in q_1$ and $\neg\delta(x, a) \in q_2$. Now since $\mathrm{CB}(q_1) \geq n$ we know that $q_1$ is not isolated in the space $\{r \in X \mid \mathrm{CB}(r) \geq n - 1\}$. Since $[\delta(x, a)]$ is a basic clopen, it follows that $q_1$ cannot be the only point in $[\delta(x, a)] \cap \{r \in X \mid \mathrm{CB}(r) \geq n - 1\}$, so there must be two different elements $r_1, r_2$ in this space. We see in the same way that there are two different elements $s_1, s_2 \in [\neg\delta(x, a)] \cap \{r \in X \mid \mathrm{CB}(r) \geq n - 1\}$. We continue in this fashion to build a finite binary tree of formulas starting with $x = x$ at the root, and if we have some formula $\phi(x, a)$ in the tree, then the next formulas are by construction going to be of the form $\phi(x, a) \wedge \psi(x, b)$ and $\phi(x, a) \wedge \neg\psi(x, b)$ for some $\psi(x, y) \in \Delta$ and $b \in \mathbb{M}$. Each path in this tree has length $n + 2$, because we stop at that point, since we have arrived at the types with CB-rank at least 0 but not necessarily larger. Also, we notice that any path in this tree is a consistent set of formulas, since any path is a subset of some complete $\Delta$-type. It follows by Compactness that there is also an infinite binary tree of formulas with the property that if we have some formula $\phi(x, a)$ in the tree, then the next formulas are of the form $\phi(x, a) \wedge \psi(x, b)$ and $\phi(x, a) \wedge \neg\psi(x, b)$ for some $\psi(x, y) \in \Delta$ and $b \in \mathbb{M}$, and such that each formula in the tree is consistent. It follows that every path in this tree is a partial $\Delta$-type. Now let $\mathcal{L}'$ be a countable sublanguage of our language $\mathcal{L}$ such that every formula in $\Delta$ is an $\mathcal{L}'$-formula, and let $M_0$ be a countable $\mathcal{L}'$-elementary substructure of $\mathbb{M}$ such that all formulas in our infinite tree are $\mathcal{L}(M_0)$-formulas. Remember that this is possible since this tree has cardinality $|2^{<\omega}|$, so it is countable. So we have some countable set of parameters which are contained in these formulas, and hence we can use the downward Löwenheim-Skolem theorem to find $M_0$. Now we obtain an uncountable number of types (one for each path in our tree) over the countable submodel $M_0$, which is impossible. Hence there is no $p \in X$ such that $\mathrm{CB}(p) \geq \omega$. □

Using the Cantor-Bendixson rank of types, we can actually define another important rank on sets of formulas, called the $\Delta$-rank. There is a treatment of this in [15] and [5], but they seem to make a mistake in the definition of the $\Delta$-rank. They define for any finite set of formulas $\Delta$ and for every set of formulas $\Phi(x)$ with small domain:

$$R_\Delta(\Phi(x)) = \mathrm{CB}(\{q \in S_\Delta(\mathbb{M}) \mid q \cup \Phi(x) \text{ is consistent}\}).$$

Then they claim (without proof) that $R_\Delta(\Phi(x)) = \min(R_\Delta(\Phi'(x)) \mid \Phi'(x) \subseteq \Phi(x) \text{ finite})$. However, the following example, which is based on the example provided in [11], seems to dispute this.

Our language consists of a single binary relation symbol $R(x, y)$. Our theory will contain for all natural numbers $n, k$ the axiom

$$\exists y_0, ..., y_n, z_0, ..., z_k \forall x \left( \bigwedge_{i<n} \bigwedge_{j \leq k} (R(x, y_{i+1}) \to R(x, y_i) \wedge R(z_j, y_i) \wedge \neg R(z_j, y_{i+1})) \right)$$

Now using the compactness theorem, there is some sequence $(a_i)_{i\in\omega}$ such that $R(\mathbb{M}, a_{i+1}) \subseteq R(\mathbb{M}, a_i)$ for every $i \in \omega$, and such that for every $i \in \omega$ there is an infinite set of elements $b$ such that $\models R(b, a_i) \wedge \neg R(b, a_{i+1})$. Now we consider the set of formulas $\Phi(x) = \{R(x, a_i)\}$. If $\Delta = \{R(x, y), x = y\}$, then we find that the space of types consistent with $\Phi(x)$ consists of realized types (note that these always have rank 0) and one type which is not realized, and hence $R_\Delta(\Phi(x)) = 1$. Now suppose $\Phi'(x) \subseteq \Phi(x)$ is finite, then we note that there is some $i \in \omega$ such that $R_\Delta(\Phi'(x)) = R_\Delta(\{R(x, a_i)\})$. Now we note that this is consistent with $\{R(x, a_j) \mid j \leq k\} \cup \{\neg R(x, a_j) \mid j > k\}$ for all $k \geq i$. And since for every $k \geq i$ this set of formulas is consistent iwth an infinite number of types, one of which is not realized, we see that it is consistent with an infinite number of types with rank 1. So $R_\Delta(\Phi'(x)) = 2$. This contradicts the statement that there should be some finite $\Phi'(x) \subseteq \Phi(x)$ with $R_\Delta(\Phi(x)) = R_\Delta(\Phi'(x))$.

We will now give a treatment of the $\Delta$-rank based on the definition provided in [11].

**Definition 3.16.** Let $\phi(x)$ be a formula, and let $\Delta$ be a finite set of formulas. Then the $\Delta$-rank of $\phi$, denoted by $R_\Delta(\phi)$, is defined by

$$R_\Delta(\phi) = \mathrm{CB}(\{q(x) \in S_\Delta(\mathbb{M}) \mid q(x) \cup \{\phi(x)\} \text{ is consistent}\}).$$

Now if $\Phi(x)$ is a set of formulas, we define the $\Delta$-rank of $\Phi$ as follows:

$$R_\Delta(\Phi(x)) = \min \left\{ R_\Delta \left( \bigwedge_{\phi \in \Phi'} \phi(x) \right) \mid \Phi' \subseteq \Phi \text{ finite} \right\}.$$

From the definition of CB-rank and using Lemma 3.15, we see that if $\Delta$ is a set of stable formulas, then $R_\Delta(\Phi(x))$ is finite for any set of formulas $\Phi(x)$. Using known properties of the CB-rank, we can now prove certain other properties of the $\Delta$-rank, which will turn out to have a clear connection to forking, an important notion that we shall encounter in the next chapter.

**Lemma 3.17.** *Let $\Phi(x)$ and $\Psi(x)$ be sets of formulas over small sets of parameters $A$ and $B$ respectively, and let $\phi(x)$ and $\psi(x)$ be formulas. Then we have:*

1. *(monotonicity) If $\Psi(x) \models \Phi(x)$, then $R_\Delta(\Psi) \leq R_\Delta(\Phi)$.*
2. *$R_\Delta(\phi(x) \vee \psi(x)) = \max(R_\Delta(\phi(x)), R_\Delta(\psi(x)))$.*
3. *$R_\Delta$ is invariant under automorphisms of $\mathbb{M}$.*
4. *Let $\phi(x)$ be a $\Delta$-formula and $\{\phi_i(x) \mid i \in \omega\}$ a pairwise contradictory set of $\Delta$-formulas such that $\phi_i(x) \models \phi(x)$ for every $i \in \omega$ and $R_\Delta(\phi_i(x)) \geq n$ for all $i \in \omega$ and some $n \in \omega$. Then $R_\Delta(\phi(x)) \geq n + 1$.*

*Proof.*

1. Suppose $R_\Delta(\Phi(x)) = \alpha$, then there is some finite subset $\Phi' \subseteq \Phi$ such that $\phi(x) = \bigwedge_{\phi' \in \Phi'} \phi'(x)$ and such that $R_\Delta(\phi(x)) = \alpha$. Now we see that there is some finite $\Psi' \subseteq \Psi$ such that $\Psi'(x) \models \phi(x)$. Now let $\psi'(x)$ be the conjunction over the formulas in $\Psi'$, then we see that if $p \in S_\Delta(\mathbb{M})$ and $p \cup \{\psi'(x)\}$ is consistent, then $p \cup \{\phi'(x)\}$ is also consistent. Hence we see that

$$\{p \in S_\Delta(\mathbb{M}) \mid p \cup \{\psi'\} \text{ is consistent}\} \subseteq \{p \in S_\Delta(\mathbb{M}) \mid p \cup \{\phi'\} \text{ is consistent}\},$$

and by monotonicity of Cantor-Bendixson rank, we see that $R_\Delta(\psi') \leq R_\Delta(\phi') = \alpha$. And hence we conclude that $R_\Delta(\Psi) \leq R_\Delta(\Phi)$.

2. We notice that

$$\begin{aligned}&\{q(x) \in S_\Delta(\mathbb{M}) \mid q(x) \cup \{\phi(x) \vee \psi(x)\} \text{ is consistent}\} = \\ &\{q(x) \in S_\Delta(\mathbb{M}) \mid q(x) \cup \{\phi(x)\} \text{ is consistent}\} \cup \\ &\{q(x) \in S_\Delta(\mathbb{M}) \mid q(x) \cup \{\psi(x)\} \text{ is consistent}\},\end{aligned}$$

so the result immediately follows from Corollary 3.7.

3. We will prove this for formulas only, notice that the result immediately follows for sets of formulas. We will only prove that $R_\Delta(\phi(x)) \leq R_\Delta(\phi(f(x)))$ for all $f \in \text{Aut}(\mathbb{M})$, notice that the other direction follows, because if $f \in \text{Aut}(\mathbb{M})$, then $f^{-1} \in \text{Aut}(\mathbb{M})$. So now suppose $p \in S_\Delta(\mathbb{M})$ and $p \cup \{\phi(x)\}$ is consistent. Then for every finite subset $p' \subseteq p$, there is some $a \in \mathbb{M}$ such that $\models p'(a) \cup \{\phi(a)\}$. Now if $f \in \text{Aut}(\mathbb{M})$, we see that $\models p'(a) \cup \{\phi(f(a))\}$, and hence we see that $p(x) \cup \{\phi(f(x))\}$ is consistent. And hence we see that

$$\{p \in S_\Delta(\mathbb{M}) \mid p(x) \cup \{\phi(x)\} \text{ is consistent}\} \subseteq \{p \in S_\Delta(\mathbb{M}) \mid p(x) \cup \{\phi(f(x))\} \text{ is consistent}\}$$

And hence by monotonicity of the Cantor-Bendixson rank, we conclude that $R_\Delta(\phi(x)) \leq R_\Delta(\phi(f(x)))$.

4. For every $i \in \omega$ there is a type $q_i$ such that $\text{CB}(q_i) \geq n$ and $q_i \cup \phi_i(x)$ is consistent. It follows that there is an infinite number of different types (because the $\phi_i$ are pairwise contradictory) which are consistent with $\phi(x)$ and have CB-rank at least $n$. It follows that $R_\Delta(\phi) \geq n + 1$. □

The very first (historically speaking) notion of a rank in model theory is actually a special case of the $\Delta$-rank, and is called the *Morley rank*. This is the same as the $\Delta$-rank, but now with $\Delta$ being the set of all formulas. This also induces the concept of *Morley degree*, as well as a natural analogue to Lemma 3.17.

## 3.2 Shelah's local rank and stability

Another notion of rank that is historically rather old is *Shelah's local rank*. Like most ranks, it has an inductive definition. The material in this section is taken from [5] and [17]. However, the proof of Theorem 3.20 was incomplete in [5], and finishing it is our own work.

**Definition 3.18.** Let $\Delta$ be a set of formulas, $p(x)$ a partial type and $\alpha \geq 2$ a cardinal. Then we define Shelah's local $\alpha$-rank of $p(x)$, denoted $R_\alpha(p, \Delta)$ as follows:

- $R_\alpha(p(x), \Delta) \geq 0$ if and only if $p(x)$ is consistent.
- For any ordinal $\beta$, $R_\alpha(p(x), \Delta) \geq \beta + 1$ if and only if for every finite $r \subseteq p$ there is a set of partial $\Delta$-types $\{q_i \mid i \in \alpha\}$ such that:
  1. For all $i \in \alpha$ we have that $R_\alpha(r \cup q_i, \Delta) \geq \beta$.
  2. For all $i, j \in \alpha$ such that $i \neq j$ we have that $q_i \cup q_j \models \perp$.
- For any limit ordinal $\lambda$, $R_\alpha(p(x), \Delta) \geq \lambda$ if and only if $R_\alpha(p(x), \Delta) \geq \beta$ for every $\beta < \lambda$.

If $\Delta = \{\phi\}$ for some formula $\phi$, we will just write $R_\alpha(p(x), \phi)$ for this rank. And if $p(x) = \{\psi(x)\}$ for some formula $\psi$, we will write $R_\alpha(\psi(x), \Delta)$ for this rank.

**Exercise 14.** Let $p(x), q(x)$ be two partial types such that $p(x) \models q(x)$, let $\Delta$ be some set of formulas and let $\alpha$ be a cardinal.

a Prove that if $r \subseteq q$ is finite, then there is some finite $r' \subseteq p$ such that $r' \models r$.

b Prove that $R_\alpha(p(x), \Delta) \leq R_\alpha(q(x), \Delta)$.

It follows from this exercise that if $p(x)$ is some partial type, then $R_\alpha(p(x), \Delta) \leq R_\alpha(x = x, \Delta)$.

Note that for all partial types $p$ and for all cardinals $\alpha \leq \beta$ such that $\alpha \geq 2$ we have that $R_\beta(p(x), \Delta) \leq R_\alpha(p(x), \Delta)$. This follows directly from the definition, since a set $\{q_i \mid i \in \beta\}$ can be restricted to a set $\{q_i \mid i \in \alpha\}$. Note that it follows that for all cardinals $\alpha \geq 2$ and for all partial types $p(x)$ and sets of formulas $\Delta$ we have that $R_\alpha(p(x), \Delta) \leq R_2(p(x), \Delta)$. It turns out that Shelah's local 2-rank, being an upper bound for every $\alpha$-rank, is our main point of interest, for it will give us an alternative characterization of stable formulas.

We note first that the case of $R_2$ is somewhat more easy then the general definition in the following sense: If $p \in S_\Delta$ is finite then $R_2(p(x), \Delta) \geq \alpha + 1$ for some ordinal $\alpha$ if and only if there is some formula $\phi(x, a) \in \Delta$ such that $R_2(p(x) \cup \{\phi(x, a)\}, \Delta) \geq \alpha$ and $R_2(p(x) \cup \{\neg\phi(x, a)\}, \Delta) \geq \alpha$. For the first direction: if $R_2(p(x), \Delta) \geq \alpha + 1$, then there are two different partial $\Delta$-types $q$ and $q'$ such that $R_2(p(x) \cup q(x), \Delta) \geq \alpha$ and $R_2(p(x) \cup q'(x), \Delta) \geq \alpha$. This means that there must be some $\Delta$-formula $\phi(x, a)$ such that $\phi(x, a) \in q$ and $\neg\phi(x, a) \in q'$ and by monotonicity we see that $R_2(p(x) \cup \{\phi(x, a)\}, \Delta) \geq \alpha$ and $R_2(p(x) \cup \{\neg\phi(x, a)\}, \Delta) \geq \alpha$. For the other direction, note that formulas are in particular partial types.

**Lemma 3.19.** *A formula $\phi(x, y)$ is stable if and only if $R_2(x = x, \phi) < \omega$.*

*Proof.* First suppose $\phi(x, y)$ is unstable, hence it has the $k$-order property for every $k \in \omega$. Now we extend our language with a set of constants $\{c_i \mid i \in [0,1]\} \cup \{d_i \mid i \in [0,1]\}$ and we extend our theory with axioms $\phi(c_i, d_j)$ for every pair $(i,j) \in [0,1]^2$ such that $i < j$, and $\neg\phi(c_i, d_j)$ for every pair $(i,j) \in [0,1]^2$ such that $i \geq j$. Notice that we obtain a consistent theory by the compactness theorem, the fact that every finite subtheory is consistent easily follows from unstability of $\phi$. Now let $\{a_i \mid i \in [0,1]\}$ be the interpretation of $\{c_i \mid i \in [0,1]\}$ and let $\{b_i \mid i \in [0,1]\}$ be the interpretation of $\{d_i \mid i \in [0,1]\}$. Now we see that $\phi(x, b_{\frac{1}{2}})$ and $\neg\phi(x, b_{\frac{1}{2}})$ are both consistent. We also see that $\phi(x, b_{\frac{1}{2}})$ is consistent with $\phi(x, b_{\frac{1}{4}})$ and with $\neg\phi(x, b_{\frac{1}{4}})$. And $\neg\phi(x, b_{\frac{1}{2}})$ is consistent with $\phi(x, b_{\frac{3}{4}})$ and with $\neg\phi(x, b_{\frac{3}{4}})$. We can continue in this fashion to find that $R_2(x = x, \phi) \geq n$ for every $n \in \omega$, hence $R_2(x = x, \phi) \geq \omega$.

Now suppose that $R_2(x = x, \phi) \geq \omega$. We extend our language with a set of constants $C = \{c_s \mid s \in 2^{<\omega}\}$, and with a set of constants $D = \{d_f \mid f \in 2^\omega\}$. Notice that for every $s : n+1 \to 2$ with $n \in \omega$, the formula $\bigwedge_{i=0}^{n} \phi^{s(i)}(x, c_{s|i})$, where $\phi^1$ is just $\phi$ and $\phi^0$ is $\neg\phi$ is consistent, because $R_2(x = x, \phi) \geq k$ for every suitable $k \in \omega$. Now we add sentences $\phi^{f(i)}(d_f, c_{f|_i})$ for every $f \in 2^\omega$ and $i \in \omega$. By compactness, we obtain a consistent theory. And hence we see that every element $d_f$ has a different $\phi$-type over $C$, with $|C| = |2^{<\omega}| = \aleph_0$. It follows that $|S_\phi(C)| > |C|$, and hence by Theorem 2.7 the formula $\phi(x, y)$ is unstable. □

**Exercise 15.** Prove that if $p, \Delta_1, \Delta_2$ are sets of formulas, $p$ is finite and $R_2(p, \Delta_1 \cup \Delta_2) \geq \omega$, then $R_2(p, \Delta_1) \geq \omega$ or $R_2(p, \Delta_2) \geq \omega$. Conclude that for every finite set of formulas $\Delta$, we have that $R_2(x = x, \Delta) < \omega$ if and only if $R_2(x = x, \phi) < \omega$ for every $\phi \in \Delta$.

In fact we can do much more than this. Shelah's local 2-rank allows us to prove the following useful theorem.

**Theorem 3.20.** *Let $\phi(x, y)$ be a formula. Then the following are equivalent:*

1. *$\phi(x, y)$ is stable.*
2. *Every $\phi$-type is definable.*
3. *For every $\kappa \geq |\mathcal{L}|$ and $M \models T$ with $|M| = \kappa$, $|S_\phi(M)| \leq \kappa$.*
4. *There is some $\kappa$ such that for every $M \models T$ with $|M| = \kappa$, we have $|S_\phi(M)| < \mathrm{ded}(\kappa)$.*

*Proof.* We will leave (1)⇒(2) for last, because it turns out to be the hardest to prove. For (2)⇒(3) we see that if $|\mathcal{L}| \leq \kappa$ and $|M| = \kappa$, then there are at most $\kappa$ many $\mathcal{L}(M)$-formulas, and since every type in $S_\phi(M)$ is definable, this means that $|S_\phi(M)| \leq \kappa$. For (3)⇒(4) we see that this is a direct result from Lemma 1.16. (4)⇒(1) follows directly from the proof of Theorem 2.6.

Now we will prove (1)⇒(2). So suppose $\phi(x, y)$ is stable, then by Lemma 3.19 we see that $R_2(x = x, \phi) < \omega$. Let $A$ be some set of parameters and let $p \in S_\phi(A)$. Now let $p' \subseteq p$ be a subtype over a set $A' \subseteq A$. We say that $p'$ is *one-element minimal* if $R_2(p', \phi) = R_2(q, \phi)$ for every $q$ such that $p' \subseteq q \subseteq p$ and such that $q$ is a type over $A' \cup \{a\}$ for some $a \in A$. We now claim that for any $p \in S_\phi(A)$, there is some one-element minimal $p' \subseteq p$ such that $|\operatorname{dom}(p')| \leq R_2(x = x, \phi)$. We can prove this claim by constructing $p'$ as follows: Start with $p_0 = \emptyset$, and given $p_i$, let $p_{i+1}$ be any one-element extension of $p_i$ such that $R_2(p_{i+1}, \phi) < R_2(p_i, \phi)$, if such an extension exists. If it doesn't, then clearly we have found a one-element minimal subtype of $p$. And since $R_2(x = x, \phi) < \omega$, we know that this process will terminate. So now we can fix for any type $p \in S_\phi(A)$ a one-element minimal subtype $p'$ such that $|\operatorname{dom}(p')| \leq R_2(x = x, \phi)$. Now we claim that for any $a \in A$, we have that $\phi(x, a) \in p$ if and only if $R_2(p'(x) \wedge \phi(x, a), \phi) = R_2(p'(x), \phi)$. To see this, note that if $\phi(x, a) \in p$, then $R_2(p'(x) \wedge \phi(x, a), \phi) = R_2(p'(x), \phi)$ since $p'$ is one-element minimal. And if $\phi(x, a) \notin p$, then $\neg\phi(x, a) \in p$, and hence $R_2(p'(x) \wedge \neg\phi(x, a), \phi) = R_2(p'(x), \phi)$. So now if we would have $R_2(p'(x) \wedge \phi(x, a), \phi) = R_2(p'(x), \phi)$, then we have found a contradiction from the definition of this rank. This is because by definition, if $R_2(p'(x) \wedge \neg\phi(x, a), \phi) = R_2(p'(x) \wedge \phi(x, a), \phi) = \alpha$ for some $\alpha$, then $R_2(p'(x), \phi) \geq \alpha + 1$. So we see that we are done if we can express the following into a formula where $y$ is the only variable:

$$R_2(p'(x) \wedge \phi(x, y), \phi) = R_2(p'(x), \phi).$$

We notice that this rank is bounded from above by $R_2(x = x, \phi)$, which is some finite number. So suppose $R_2(x = x, \phi) = n$. Then we see that we can express the equality above as:

$$\bigvee_{i=0}^{n} R_2(p'(x) \wedge \phi(x, y), \phi) = R_2(p'(x), \phi) = i.$$

This can be expressed as follows:

$$\bigvee_{i=0}^{n} (R_2(p'(x) \wedge \phi(x, y), \phi) \geq n \wedge \neg(R_2(p'(x) \wedge \phi(x, y), \phi) \geq i + 1)$$
$$\wedge\, R_2(p'(x), \phi) \geq n \wedge \neg(R_2(p'(x), \phi) \geq i + 1)).$$

So all that remains to be done is prove that for any $\phi$-type $q(x, y)$ with finite domain $A$ and any $n \in \omega$, we can express $R_2(q(x, y), \phi) \geq n$ in a formula where $y$ is the only variable (note that this would imply that we can express $R_2(q(x), \phi) \geq n$ in a sentence). We do this by induction on $n$. For $n = 0$, we let $A = A_1 \cup A_2$ where we have that $a \in A_1 \Rightarrow \phi(x, y, a) \in q$ and $a \in A_2 \Rightarrow \phi(x, y, a) \notin q$. Now we simply take the formula

$$\exists z( \bigwedge_{a \in A_1} \phi(x, y, a) \wedge \bigwedge_{a \in A_2} \neg\phi(x, y, a).$$

We notice that this is indeed a formula, since $q$ has finite domain, so the conjunctions are finite. Now suppose that we have such a formula for every type and every natural number at most $n$. Then we see that $R_2(q(x, y), \phi) \geq n + 1$ can be expressed in the formula

$$\exists z(R_2(q(x, y) \wedge \phi(x, z), \phi) \geq n \wedge R_2(q(x) \wedge \neg\phi(x, z), \phi) \geq n).$$

And hence we know that there is some formula $\psi(y)$ such that $\phi(x, a) \in p$ if and only if $\models \psi(a)$. So $p$ is indeed definable. $\square$

*Chapter 4*

# Forking

## 4.1 Ideals and dividing

A key notion in Stability Theory is the notion of *forking*. In this chapter, we will investigate this notion, and show its connection to the notions of stability and ranks that we have seen before. There are of course different equivalent ways of introducing forking, but our definition will be that a formula forks over a set of parameters if it belongs to some ideal. Hence we will first take a closer look at ideals. The material in this section is taken from [5] and [9]

**Definition 4.1.** Let $\Phi(x)$ be an ideal over a set of parameters $A$, and let $\kappa$ be a cardinal. We say that $\Phi(x)$ is *$\kappa$-prime* if for any family $(\phi_i)_{i<\kappa}$ of $\mathcal{L}(A)$-formulas such that $\phi_i \wedge \phi_j \in \Phi(x)$ for all $i < j < \kappa$, there is some $i < \kappa$ such that $\phi_i \in \Phi(x)$. We say that $\Phi(x)$ is *generically prime* if it is $\kappa$-prime for some $\kappa$.

Notice that an ideal $\Phi(x)$ is 2-prime iff for all formulas $\phi(x)$ and $\psi(x)$, if $\phi(x)\wedge\psi(x) \in \Phi(x)$, then $\phi(x) \in \Phi(x)$ or $\psi(x) \in \Phi(x)$. Ideals with this property are usually just called prime.

**Lemma 4.2.** *Let $\Phi(x)$ be an $A$-invariant ideal of $\mathcal{L}(\mathbb{M})$-formulas, with $A$ some set of parameters. Then the following are equivalent:*

1. *$\Phi(x)$ is generically prime.*
2. *If $(b_i)_{i\in\omega}$ is an $A$-indiscernible sequence, $\phi(x,y)$ is an $\mathcal{L}(A)$-formula and $\phi(x,b_0) \notin \Phi(x)$ then $\phi(x,b_0) \wedge \phi(x,b_1) \notin \Phi(x)$.*

*Proof.* First we will prove (1) $\Rightarrow$ (2). So suppose that $(a_i)_{i\in\omega}$ is an $A$-indiscernible sequence such that $\phi(x,a_0) \notin \Phi(x)$ and $\phi(x,a_0) \wedge \phi(x,a_1) \in \Phi(x)$. Now let $\kappa$ be any cardinal number. We will show that $\Phi(x)$ is not $\kappa$-prime, thus showing that $\Phi(x)$ is not generically prime. We will use the compactness theorem for this, so we add to our language a set of constants $(b_i)_{i\in\kappa}$ and to our theory a set of axioms telling us that $\phi(x,b_i) \notin \Phi(x)$ for all $i \in \kappa$, so for example an axiom $\exists x(\phi(x,b_i) \wedge \neg\psi(x,b_i))$ for every $i \in \kappa$ and every $\psi(x,a) \in \Phi(x)$, and (using the same strategy on the complement of $\Phi$) axioms telling us that $\phi(x,b_i) \wedge \phi(x,b_j) \in \Phi(x)$ for every $i,j \in \kappa$ with $i \neq j$. Since we know that $\phi(x,a_i) \notin \Phi(x)$ by indiscernibility of $(a_i)_{i\in\omega}$, invariance of $\Phi(x)$ and the fact that $\phi(x,a_0) \notin \Phi(x)$, and the fact that $\phi(x,a_i) \wedge \phi(x,a_j) \in \Phi(x)$ for all $i,j \in \omega$ such that $i \neq j$ (using the same argument), we see that the theory we obtain is in fact consistent. Hence we find that $\Phi(x)$ is not $\kappa$-prime.

Now we will prove (2) $\Rightarrow$ (1). Suppose that $\Phi(x)$ is not generically prime, so there is no cardinal $\kappa$ such that it is $\kappa$-prime. This means that for every cardinal $\kappa$ we can find some sequence of definable sets $(\phi_i(x, a_i))_{i \in \kappa}$ such that $\phi_i(x, a_i) \notin \Phi(x)$ for any $i \in \kappa$ and $\phi(x, a_i) \wedge \phi(x, a_j) \in \Phi(x)$ for any $i, j \in \kappa$ with $i \neq j$. So by taking $\kappa$ large enough (at least larger than the number of formulas in our language), we can find a sequence $(\phi(x, a_i))_{i \in \lambda}$ with these properties, so with the formula fixed, for every cardinal $\lambda$. Now it follows using Theorem 2.23 that we can find an indiscernible sequence with the desired properties. $\square$

Now that some background concerning ideals is clear, we can move on to dividing and forking.

**Definition 4.3.** Let $B$ be a set of parameters and $\phi(x, a)$ some formula. We say that $\phi(x, a)$ *divides* over $B$ if there is some $B$-indiscernible sequence $(a_i)_{i \in \omega}$ such that $a_0 = a$ and such that $\{\phi(x, a_i) \mid i \in \omega\}$ is inconsistent. We say that a (partial) type $p$ divides over $B$ if $p$ implies a formula which divides over $B$.

Note that $\phi(x, a)$ is in general *not* an $\mathcal{L}(B)$-formula. In fact we have the following:

**Exercise 16.** Let $\phi(x, a)$ be a consistent $\mathcal{L}(B)$-formula. Show that $\phi(x, a)$ does not divide over $B$.

In many sources, one will not find the above definition, but in fact a definition in which $\{\phi(x, a_i) \mid i \in \omega\}$ is $k$-inconsistent for some $k \in \omega$, instead of inconsistent. We say that $\{\phi(x, a_i) \mid i \in \omega\}$ is $k$-inconsistent if for every $i_1, ..., i_k \in \omega$ we have $\models \neg \exists x \bigwedge_{1 \leq j \leq k} \phi(x, a_{i_j})$, so any set of $k$ of these formulas is inconsistent. We note that if there is some $k \in \omega$ such that $\{\phi(x, a_i) \mid i \in \omega\}$ is $k$-inconsistent, then obviously $\{\phi(x, a_i) \mid i \in \omega\}$ is inconsistent. If the sequence $(a_i)_{i \in \omega}$ is indiscernible, the converse also holds. This is the case because if $\{\phi(x, a_i) \mid i \in \omega\}$ is inconsistent, then by compactness there is some $k \in \omega$ and some $i_1, ..., i_k \in \omega$ such that $\models \neg \exists x \bigwedge_{1 \leq j \leq k} \phi(x, a_{i_j})$, and hence by indiscernibility of $(a_i)_{i \in \omega}$, every $k$-long sequence of elements in $(a_i)_{i \in \omega}$ satisfies this formula, and hence $\{\phi(x, a_i) \mid i \in \omega\}$ is $k$-inconsistent.

**Exercise 17.** Suppose $\phi(x, a)$ divides over $A$ and $a \equiv_A b$. Prove that $\phi(x, b)$ also divides over $A$.

We can find the following equivalence of dividing:

**Theorem 4.4.** *Let $a$ and $b$ be two parameters, and $A$ a set of parameters. Then the following are equivalent:*

1. $\text{tp}(a/Ab)$ *does not divide over $A$.*

2. *If $I$ is some infinite linearly ordered index set and $(b_i)_{i \in I}$ is an $A$-indiscernible sequence such that $b = b_i$ for some $i \in I$, then there is some $a'$ such that $a' \equiv_{Ab} a$ and such that $(b_i)_{i \in I}$ is $Aa'$-indiscernible.*

3. *If $I$ is some infinite linearly ordered index set and $(b_i)_{i \in I}$ is an $A$-indiscernible sequence such that $b = b_i$ for some $i \in I$, then there is some linearly ordered index set $J$ and $Aa$-indiscernible sequence $(b_j)_{j \in J}$ such that $(b_i)_{i \in I} \equiv_{Ab} (b_j)_{j \in J}$.*

*Proof.* We will save the most difficult proof for last, so we start with (2) $\Rightarrow$ (3). Let $f$ be an automorphism of $\mathbb{M}$ fixing $Ab$ and such that $f(a) = a'$. Let $J = I$ and consider the sequence $(f(b_i))_{i \in I}$. We see that $(f(b_i))_{i\in I} \equiv_{Ab} (b_i)_{i\in I}$, and this sequence is $Aa$-indiscernible.

We now prove (3) $\Rightarrow$ (1) by contraposition. So suppose that $\text{tp}(a/Ab)$ does divide over $A$, then there is some $\mathcal{L}(A)$-formula $\phi(x, y)$ such that $\models \phi(a, b)$ and some $A$-indiscernible sequence $(b_i)_{i\in\omega}$ with $b_0 = b$ such that $\{\phi(x, b_i) \mid i \in \omega\}$ is inconsistent. However, it is not possible that there is a sequence $(c_i)_{i\in\omega}$ such that $(c_i)_{i\in\omega} \equiv_{Ab} (b_i)_{i\in\omega}$ and such that $(c_i)_{i\in\omega}$ is $Aa$-indiscernible, because in this case we would have that $\models \phi(a, c_i)$ for every $i \in \omega$, because $c_0 = b$ and the sequence is $Aa$-indiscernible. This means that $\models \exists x \bigwedge_{i<n} \phi(x, c_i)$ for all $n \in \omega$, which means that $\models \exists x \bigwedge_{i<n} \phi(x, b_i)$ for each $n \in \omega$, since $(b_i)_{i\in\omega} \equiv_{Ab} (c_i)_{i\in\omega}$. But this means that $\{\phi(x, b_i) \mid i \in \omega\}$ is consistent, which is not the case. So we find that there is no such sequence $(c_i)_{i\in\omega}$, hence the statement is proven.

We now prove (1) $\Rightarrow$ (2).

Let $(b_i)_{i\in I}$ be an $A$-indiscernible sequence such that there is some $i \in I$ with $b = b_i$, say w.l.o.g. $b = b_0$. We denote $\text{tp}(a/Ab)$ by $p(x)$. Now we let $\Phi(x, A, (b_i)_{i\in I})$ be a set of formulas expressing that the sequence $(b_i)_{i\in I}$ is indiscernible over $A \cup \{x\}$. Now consider the set $p(x) \cup \Phi(x, A, (b_i)_{i\in I})$. If this set of formulas is consistent, then it is realized by some $a'$, since the monster model is $\kappa$-saturated for sufficiently large $\kappa$. We notice that we will then have that $a' \equiv_{Ab} a$ and $(b_i)_{i\in I}$ is $Aa'$-indiscernible. In order to do this, we first consider the set of formulas $q(x) = \bigcup_{i\in I} p(x, b_i)$, where $p(x, b_i)$ is the type $p(x)$, but with the parameter $b$ replaced everywhere by $b_i$. We will prove that $q(x)$ is consistent. For suppose that $q(x)$ would be inconsistent, then by compactness there would be some finite set of formulas in $q(x)$ which is inconsistent. So let's denote this set by $\{\phi_0(x, b_{i_0}), ..., \phi_n(x, b_{i_n})\}$. Now consider the formula $\bigwedge_{j\leq n} \phi_j(x, b)$. Notice that this is an element of $p(x)$, let us call this formula $\psi(x, b)$. We now see that $\bigwedge_{j\leq n} \psi(x, b_{i_j}) \models \bigwedge_{j\leq n} \phi_j(x, b_{i_j})$, and since $\bigwedge_{j\leq n} \phi_j(x, b_{i_j})$ is inconsistent, so is $\bigwedge_{j\leq n} \psi(x, b_{i_j})$. So now we see that $\psi(x, b)$ is a formula in $p(x)$ and we have some $A$-indiscernible sequence $(b_i)_{i\in\omega}$ telling us that $p(x)$ divides over $A$, which we assumed was not the case. And hence we find that $q(x)$ is consistent.

So let $c$ be an element realizing $q(x)$. We will prove using compactness that $q(x) \cup \Phi(x, A, (b_i)_{i\in I})$ is consistent, from which it immediately follows that $p(x) \cup \Phi(x, A, (b_i)_{i\in I})$ is consistent. So let $\Psi(x, A, (b_i)_{i\in I})$ be some finite subset of $\Phi(x, A, (b_i)_{i\in I})$. Notice that there is only a finite number of elements of our sequence $(b_i)_{i\in I}$ involved in $\Psi$, so from now on we can w.l.o.g. assume that $I = \omega$.

**Claim:** There is an order-preserving function $f : \omega \to \omega$ such that $\models \Psi(c, A, (b_{f(i)})_{i\in\omega})$.

**Proof of claim:** We will use Ramsey's theorem to prove this claim. First observe that $\Phi(c, A, (b_i)_{i\in\omega})$ consists of a formula $\phi(x, b_{i_0}, ..., b_{i_n}) \leftrightarrow \phi(x, b_{j_0}, ..., b_{j_n})$ for every $\mathcal{L}(A \cup \{c\})$-formula $\phi$ and tuples $i_0 < ... < i_n$ and $j_0 < ... < j_n$. It follows that $\Psi$ consists of $k$ formula's of this form. So we will denote this set by

$$\left\{\phi_j\left(x, b_{i^j_0}, ..., b_{i^j_{n_j}}\right) \leftrightarrow \phi_j\left(x, b_{l^j_0}, ..., b_{l^j_{n_j}}\right) \mid j < k\right\}.$$

Now consider the formulas in $\Psi$ containing the longest tuple $b_{i_0}, ..., b_{i_r}$, and say that there are $s$ formulas like this, say $\phi_1, ..., \phi_s$. Now we will colour each tuple containing $r + 1$ elements of $\omega$ in a colour, our colours will be the elements of the set $2^s$. Specifically, if we denote the truth value of a sentence $\phi$ by $\|\phi\|$ (where the truth value of a sentence is 0 if it is false and 1 if it is true), then we give $\{i_0, ..., i_{n-1}\}$ the colour

$$(\|\phi_1(c, b_{i_0}, ..., b_{i_{n-1}})\|, ..., \|\phi_s(c, b_{i_0}, ..., b_{i_{n-1}})\|).$$

Now by Ramsey's theorem there is an infinite subset $I_n \subseteq \omega$ such that all $n$-tuples of elements in $I_n$ have the same colour. Now we will repeat this process within $I_n$, but for the formulas where the tuples have length $n-1$. By continuing this way we obtain an infinite $I_1 \subseteq \omega$ such that for all formulas $\phi \in \Psi$ and for all tuples $\overrightarrow{b_1}, \overrightarrow{b_2}$ with indices in $I_1$, we have that $\models \phi(c, \overrightarrow{b_1}) \leftrightarrow \phi(c, \overrightarrow{b_2})$ as long as $\overrightarrow{b_1}, \overrightarrow{b_2}$ are in the same order. Now we write $I_1 = \{i_0, i_1, ...\}$, and we let $f : \omega \to \omega$ be given by $f(r) = i_r$. Then this $f$ suffices, and the claim is proven.

Now we take an automorphism $\sigma$ of the monster model such that $\sigma(b_{f(i)}) = b_i$ for every $i$. It follows that $\models q(\sigma(c)) \cup \Psi(\sigma(c), A, (b_i)_{i\in I})$. Hence this set of formulas is consistent and the proof is complete. □

The following corollary to Theorem 4.4 is often called the left transitivity lemma, or the pairs lemma:

**Corollary 4.5.** *Let $a$, $b$ and $c$ be three parameters, and let $A$ be a set of parameters. If* $\mathrm{tp}(a/Ab)$ *does not divide over $A$ and* $\mathrm{tp}(c/Aab)$ *does not divide over $Aa$, then* $\mathrm{tp}(ac/Ab)$ *does not divide over $A$.*

*Proof.* Since $\mathrm{tp}(a/Ab)$ does not divide over $A$, if $I$ is some infinite index set and $(b_i)_{i\in I}$ is an $A$-indiscernible sequence with $b = b_i$ for some $i \in I$, then there is some index set $J$ and some $Aa$-indiscernible sequence $(b_j)_{j\in J}$ such that $(b_j)_{j\in J} \equiv_{Ab} (b_i)_{i\in I}$. Since $b = b_i$ for some $i \in I$ and $(b_j)_{j\in J} \equiv_{Ab} (b_i)_{i\in I}$, we know that $b = b_j$ for some $j \in J$. Now since $\mathrm{tp}(c/Aab)$ does not divide over $Aa$, it follows that there is some index set $K$ and some $Aac$-indiscernible sequence $(a_k)_{k\in K}$ such that $(a_k)_{k\in K} \equiv_{Aab} (b_j)_{j\in J}$. And hence we find that $\mathrm{tp}(ac/Ab)$ does not divide over $A$. □

**Definition 4.6.** Let $\mathbf{F}(B)$ be the ideal in the set of $\mathbb{M}$-definable sets generated by the formulas dividing over $B$. We say that a formula $\phi(x, a)$ *forks* over $B$ if $\phi(x, a) \in \mathbf{F}(B)$. We say that a (partial) type $p$ forks over $B$ if it implies a formula which forks over $B$.

Note that it follows that $\phi(x, a)$ forks over $B$ iff there is a finite set of formulas $\{\psi_i(x, b_i) \mid i < n\}$ such that each $\psi_i(x, b_i)$ divides over $B$, and $\models \forall x(\phi(x, a) \to \bigvee_{i<n} \psi_i(x, b_i))$.

**Lemma 4.7.** *Let $A$ and $B \subseteq C$ be sets of parameters and let $p \in S(B)$ be a type such that $p$ does not fork over $A$. Then there is some $q \in S(C)$ such that $q$ does not fork over $A$ and $q|_B = p$.*

*Proof.* Consider the set of formulas $\Gamma$ given by

$$\Gamma = \{\neg\psi(x, c) \mid c \in C,\ \psi(x, c) \text{ forks over } A\}.$$

We will use compactness to prove that $p \cup \Gamma$ is consistent. For suppose it would be inconsistent, then there is a finite set $\{\neg\psi_i(x, c_i) \mid i < n\} \subseteq \Gamma$ such that $p \cup \{\neg\psi_i(x, c_i) \mid i < n\}$ is inconsistent. This means that $p(x) \models \bigvee_{i<n} \psi_i(x, c_i)$, but each of the $\psi_i$ forks over $A$. This means that each of these formulas implies a finite disjunction of formulas, all of which divide over $A$. But this means in particular that this formula itself implies a finite disjunction of formulas, all of which divide over $A$. So $\bigvee_{i<n} \psi_i(x, c_i)$ forks over $A$, and hence $p$ forks over $A$. This is a contradiction, so we find that $p \cup \Gamma$ is consistent.

Now we expand $p \cup \Gamma$ to a complete type $q \in S(C)$. If $q$ would fork over $A$ then it implies a formula $\phi(x, c)$ with $c \in C$ such that $\phi(x, c)$ forks over $A$. But this means that $\neg\phi(x, c) \in \Gamma$, hence $\neg\phi(x, c) \wedge \phi(x, c) \in q$. So we have found a contradiction, and conclude that $q$ does not fork over $A$. □

**Exercise 18.** Let $T$ be DLO, and let $a \in \mathbb{M}$. Show that the formula $a < x$ does not divide over $\emptyset$, but $a < x < b$ does divide over $\emptyset$. Does the formula $a < x$ fork over $\emptyset$?

**Lemma 4.8.** *Let $\Phi(x)$ be a generically prime $B$-invariant ideal. Then $\mathbf{F}(B) \subset \Phi(x)$.*

*Proof.* Let $\phi(x, a)$ be some formula dividing over $B$ (so $\phi(x, a)$ is a generator of the ideal $F(B)$). It is enough to show that $\phi(x, a) \in \Phi(x)$. Since the formula $\phi(x, a)$ divides over $B$ there is some $B$-indiscernible sequence $(a_i)_{i \in \omega}$ such that $a_0 = a$ and $\{\phi(x, a_i) \mid i \in \omega\}$ is inconsistent. If $\phi(x, a_0) \in \Phi(x)$ we are done, so suppose $\phi(x, a_0) \notin \Phi(x)$, then by Lemma 4.2 we must also have that $\phi(x, a_0) \wedge \phi(x, a_1) \notin \Phi(x)$. Now since $(a_i)_{i \in \omega}$ is a $B$-indiscernible sequence, we see that $(a_{2i}a_{2i+1})_{i \in \omega}$ is also a $B$-indiscernible sequence. And since $\phi(x, a_0) \wedge \phi(x, a_1) \notin \Phi(x)$, it follows that we also have that $\bigwedge_{i<4} \phi(x, a_i) \notin \Phi(x)$. By repeating this procedure inductively, we find that for all $k \in \omega$, we have that

$$\bigwedge_{i<2^k} \phi(x, a_i) \notin \Phi(x).$$

Since $\Phi(x)$ is an ideal, we know that $\emptyset \in \Phi(x)$, and it follows that the set $\{\phi(x, a_i) \mid i < 2^k\}$ is consistent for every $k \in \omega$. By compactness if follows that $\{\phi(x, a_i) \mid i \in \omega\}$ is consistent, which is a contradiction. Hence we find that $\phi(x, a) \in \Phi(x)$, so $\mathbf{F}(B) \subseteq \Phi(x)$. □

**Definition 4.9.** Let $p \in S(\mathbb{M})$ be a type and $A$ a small set of parameters. We say that $p$ is *$A$-invariant* if for every $a, b \in \mathbb{M}$ with $a \equiv_A b$ and for every $\mathcal{L}$-formula $\phi(x, y)$ we have that $\phi(x, a) \in p$ iff $\phi(x, b) \in p$.

**Exercise 19.** Let $p \in S(\mathbb{M})$ be a global type and $A$ a small set of parameters. Show that the following are equivalent:

1. $p$ is $A$-invariant.
2. For every $f \in \text{Aut}(\mathbb{M}/A)$ and for every formula $\phi(x, y)$ and $b \in \mathbb{M}$, we have $\phi(x, b) \in p$ if and only if $\phi(x, f(b)) \in p$.
3. Let $f \in \text{Aut}(\mathbb{M}/A)$. Then $f(p) = p$, meaning that if $B$ is some definable set, then $B \in p$ if and only if $f(B) \in p$.

**Exercise 20.** Let $q \in S(\mathbb{M})$ be an $A$-invariant global type. Show that $q$ doesn't fork over $A$.

## 4.2 Heirs and coheirs

Let $p$ be a type over a set of parameters $A$. Then if $A \subseteq B$, it is always possible to extend $p$ to a complete type over $B$. In this section we consider some extensions of types with special properties, called heirs and coheirs. The material in this section was taken from [5].

**Definition 4.10.** Suppose $A \subseteq B$ are sets of parameters and $q \in S(B)$. Let $p = q|_A \in S(A)$ be the complete type $q$ restricted to the set of parameters $A$. We say that $q$ is an *heir* of $p$ if for every $\mathcal{L}(A)$-formula $\phi(x, y)$, if there is some $b \in B$ such that $\phi(x, b) \in q$, then there is some $a \in A$ such that $\phi(x, a) \in p$.

**Exercise 21.** Let $A \subseteq B$ be sets of parameters such that $A$ is a model of $T$. Show that if $q \in S(B)$ is definable over $A$, then it is an heir of $q|_A$.

**Definition 4.11.** Let $A \subseteq B$ be sets of parameters and $q \in S(B)$ be a complete type over $B$. We say that $q$ is a *coheir* over $A$ (or: a coheir of $q|_A$) if for any $\phi(x,b) \in q$ there is some $a \in A$ such that $\models \phi(a,b)$.

**Exercise 22.** Suppose $A \subseteq B$ are sets of parameters. Let $q \in S(B)$ be a coheir over $A$, and let $b, b' \in B$ be such that $b \equiv_A b'$. Show that for every $\mathcal{L}(A)$-formula $\phi(x,y)$ we have $\phi(x,b) \in q$ iff $\phi(x,b') \in q$ (we also say that $q$ *splits over* $A$ if it satisfies this property). Conclude that if $B = \mathbb{M}$, then $q$ is $A$-invariant. Show that this is also the case if $q$ is not a coheir over $A$, but $q$ is definable over $A$.

**Exercise 23.** Let $M$ be a small model. Show that $\text{tp}(a/Mc)$ is an heir of $\text{tp}(a/M)$ iff $\text{tp}(c/Ma)$ is a coheir over $M$.

**Exercise 24.** Consider $M = (\mathbb{Q}, <)$ as a small model of DLO. Let the type $p(x) \in S(M)$ be given by $p(x) = \{a < x \mid a \in M\}$. Now consider two global extensions $q$ and $r$ of $p$, defined as follows:

- $q(x) = \{a < x \mid a \in \mathbb{M}\}$.
- $r(x) = p(x) \cup \{x < b \mid b \in \mathbb{M},\ m < b \text{ for all } m \in M\}$.

Show that $q(x)$ is an heir of $p(x)$, but not necessarily a coheir over $M$. Also show that $r(x)$ is not an heir of $p(x)$, but that it is a coheir over $M$.

One could now ask the question whether every type $p \in S(A)$ with $A$ some set of parameters always has a global heir (that is, an heir $q \in S(\mathbb{M})$) or a global coheir (that is, a coheir $q \in S(\mathbb{M})$). And in fact it turns out that this is the case, if we assume that the set of parameters is itself a small model:

**Theorem 4.12.** *Let $M$ be a small model, and let $p(x) \in S(M)$ be given. Then $p(x)$ has both a global heir and a global coheir.*

*Proof.* First we will prove that $p(x)$ has a global coheir. Since $p(x)$ defines a filter on $\mathcal{P}(M)$, we can extend it to an ultrafilter $U$ on $\mathcal{P}(M)$. Now we use this ultrafilter $U$ to define the global type $q_U$ by $\phi(x) \in q_U$ iff $\{a \in M \mid\ \models \phi(a)\} \in U$, for every $\mathcal{L}(\mathbb{M})$-formula $\phi$. Note that every formula in $q_U$ can be realized (since $\emptyset \notin U$), and also note that since $U$ is an ultrafilter, we know that $\{a \in M \mid\ \models \phi(a)\} \in U$ or $\{a \in M \mid\ \models \phi(a)\}^c \in U$ for every $\mathcal{L}(\mathbb{M})$-formula $\phi$. This means that $\{a \in M \mid\ \models \phi(a)\} \in U$ or $\{a \in M \mid\ \models \neg\phi(a)\} \in U$ for every $\mathcal{L}(\mathbb{M})$-formula $\phi$, so $q_U$ is indeed a complete type. We also notice that if $\phi(x) \in q_U$, then there is some $a \in M$ such that $\models \phi(a)$ (since $\emptyset \notin U$), so $q_U$ is indeed a global coheir over $M$.

Now we will prove that $p(x)$ has a global heir. For this we only have to prove that the following collection of formulas is in fact consistent:

$$s(x) = p(x) \cup \{\phi(x,c) \mid \phi(x,y) \text{ an } \mathcal{L}(M)\text{-formula},\ c \in \mathbb{M},\ \forall m \in M(\phi(x,m) \in p(x))\}$$

For if $s(x)$ is consistent, we can extend it to a complete type $r(x)$, and this $r(x)$ will be an heir of $p(x)$. For if $\phi(x,c) \in r(x)$ but for every $m \in M$ we have that $\phi(x,m) \notin p(x)$, then since $p(x)$ is a complete type we know that $\neg\phi(x,m) \in p(x)$ for every $m \in M$, and hence $\neg\phi(x,c) \in r(x)$ by definition. So we find that $\phi(x,c) \wedge \neg\phi(x,c) \in r(x)$, a contradiction since

$r(x)$ is a type. So if $s(x)$ is consistent then $p(x)$ indeed has a global heir.
To prove that $s(x)$ is consistent, we first note that the collection of formulas

$$\{\phi(x,c) \mid \phi(x,y) \text{ an } L(M)\text{-formula },\ c \in \mathbb{M},\ \forall m \in M(\phi(x,m) \in p(x))\}$$

is closed under taking conjunctions, since $p(x)$ is a complete type. By compactness it follows that if $s(x)$ would be inconsistent, there would be a formula $\phi(x,c) \in p(x)$ and $\psi(x,d) \in s(x)\backslash p(x)$ such that $\models \neg(\phi(x,c) \wedge \psi(x,d))$. So from this we find that $\models \phi(x,c) \to \neg\psi(x,d)$, and hence

$$\models \exists y \forall x(\phi(x,c) \to \neg\psi(x,y)).$$

From this we find that $M \models \exists y \forall x(\phi(x,c) \to \neg\psi(x,y))$, and hence since $\phi(x,c) \in p(x)$ we find that $\exists y \neg\psi(x,y) \in p(x)$. But since $\psi(x,d) \in s(x)$ and we know that $\psi(x,d) \notin p(x)$, it follows that $\psi(x,m) \in p(x)$ for every $m \in M$, by definition of $s(x)$. But this means that $p(x)$ is inconsistent, a contradiction. So we find that $s(x)$ is consistent, as desired. □

Note that it follows that if $M \subseteq A$, then $p(x)$ also has an heir and a coheir in $S(A)$. Because we see that the restrictions of the global heir and coheir suffice. We could now ask whether we have some information about heirs and coheirs of types for which we have some additional information. It turns out that this is indeed the case.

**Theorem 4.13.** *Let $M$ be a small model and $p \in S(M)$ a complete type which is definable. If $M \subseteq A$, then the heir of $p$ in $S(A)$ is unique and definable over $M$.*

*Proof.* First we will prove that $p$ has an $M$-definable extension in $S(A)$. Note that since $p$ is definable, for every $\mathcal{L}$-formula $\phi(x,y)$ there is some $\mathcal{L}(M)$-formula $d_p\phi(y)$ such that $\phi(x,a) \in p(x)$ iff $\models d_p\phi(a)$ for all $a \in M$. Now consider the following collection of formulas:

$$q(x) = \{\phi(x,a) \mid \phi(x,y) \text{ an } L\text{-formula},\ a \in A,\ \models d_p\phi(a)\}.$$

We will use compactness to prove that this collection of formulas is in fact consistent. For if it is not, then is has some finite inconsistent subset. So there must be some $n \in \omega$ such that there are $\mathcal{L}$-formula $\phi_i(x,y)$ and constants $a_i \in A$ for all $i < n$ such that $\models d_p\phi_i(a_i)$ for all $i < n$ but $\models \forall x(\neg \bigwedge_{i<n} \phi_i(x,a_i))$. So we find:

$$\models \exists y_0,...,y_{n-1} \forall x \left( \neg \bigwedge_{i<n} \phi_i(x,y_i) \wedge \bigwedge_{i<n} d_p\phi_i(y_i) \right).$$

And hence this must also be true in $M$, which means that there are $m_0,...,m_{n-1}$ in $M$ such that

$$M \models \forall x(\neg \bigwedge_{i<n} \phi_i(x,m_i) \wedge \bigwedge_{i<n} d_p\phi_i(m_i)).$$

So since we have $M \models \bigwedge_{i<n} d_p\phi_i(m_i)$, it follows by definition of the $d_p\phi_i$ and the fact that $M$ is an elementary substructure of $\mathbb{M}$ that $\phi(x,m_i) \in p$ for all $i < n$, and since we have $M \models \forall x \neg \bigwedge_{i<n} \phi_i(x,m_i)$, we obtain a contradiction. So $q(x)$ is indeed consistent.

This means that $q(x)$ is a partial $A$-type extending $p(x)$. We also notice that $q(x)$ is complete, for if $\phi(x,a)$ is some $\mathcal{L}(A)$-formula, then we know that $\models d_p\phi(a)$ or $\models \neg d_p\phi(a)$, and from this it follows that either $\phi(x,a) \in q(x)$ or $\neg\phi(x,a) \in q(x)$. Hence $q(x)$ is indeed complete. We also see by definition that $q(x)$ is $M$-definable, and hence by Exercise 21 it is an heir of $p$.

Now suppose that $q' \neq q$ is another $A$-type extending $p$, then we will show that is is not an heir of $p$. Since $q' \neq q$ there must be some formula $\phi(x, b) \in q'$ such that $\neg\phi(x, b) \in q$, and hence $\models \neg d_p\phi(b)$. It follows that $\phi(x, b) \wedge \neg d_p\phi(b) \in q'$, but there is no $m \in M$ such that $\phi(x, m) \wedge \neg d_p\phi(m) \in p$, so $q'$ is not an heir of $p$. □

## 4.3 Simple theories and Morley sequences

In this section we will investigate the notion of a *simple theory*, which is a class of theories containing the stable theories (we will prove this later in this chapter). It turns out that this class of theories satisfies some nice properties, specifically, that forking and dividing coincide. Hence we will be able to conclude this chapter with the result that forking and dividing coincide in stable theories. The material in this section was taken from [9].

**Definition 4.14.** A theory $T$ is *simple* if for every set of parameters $A$ and every $p \in S(A)$, there is some $A_0 \subseteq A$ with $|A_0| \leq |T|$ such that $p$ does not fork over $A_0$.

In order to prove some properties of simple theories, we will need the notion of a *Morley sequence*.

**Definition 4.15.** Let $A \subseteq B$ be two sets of parameters and let $p(x) \in S(B)$. Let $I$ be some infinite linearly ordered set, and let $\overline{a} = (a_i)_{i \in I}$ be a sequence. Then $\overline{a}$ is called a *Morley sequence* for $p$ over $A$ if the following properties are satisfied:

1. $\overline{a}$ is $A$-indiscernible.

2. For every $i \in I$ we have $\models p(a_i)$.

3. For every $i \in I$, the type $\text{tp}(a_i/B \cup \{a_j \mid j < i\})$ does not divide over $A$.

If $A = B$, then we simply say that $\overline{a}$ is a Morley sequence for $p$.

**Exercise 25.** Let $A$ be a set of parameters, let $p \in S(A)$ and let $\overline{a} = (a_i)_{i \in I}$ be a Morley sequence for $p$. Prove that if $f$ is an automorphism of $\mathbb{M}$ fixing $A$ pointwise, then $f(\overline{a}) = (f(a_i))_{i \in I}$ is also a Morley sequence for $p$.

If $\overline{a} = (a_i)_{i \in I}$ is some sequence and $J \subseteq I$, then we use $\overline{a}_J$ to denote the set $\{a_j \mid j \in J\}$.

**Exercise 26.** Suppose $A \subseteq B$ are sets of parameters and $p \in S(B)$. If $\overline{a} = (a_i)_{i \in I}$ is a Morley sequence for $p$ over $A$, and $X, Y \subset I$ are such that for every $x \in X$ and $y \in Y$ we have $x < y$, then $\text{tp}(a_Y/B \cup a_X)$ does not divide over $A$.

One could wonder, does every type always have a Morley sequence? It turns out that this is not the case. For example, if $p$ is a global type which is not realized, then it can never have a Morley sequence. An example of such a type is $\{a < x \mid a \in \mathbb{M}\}$ where $\mathbb{M}$ is a monster model for DLO. Clearly every finite subset of this is consistent, hence it is a type, but any realizer of this set would be an endpoint, which doesn't exist. However, there are several situations in which we can ensure the existence of Morley sequences.

**Lemma 4.16.** *Let $I$ be a linear order, and let $(a_j)_{j\in\beta}$ be a sequence, where $\beta$ is the ordinal $\beth_{(2^{|T|})^+}$. Then there is some indiscernible sequence $(b_i)_{i\in I}$ such that for every natural number $n$ and $i_0 < ... < i_n \in I$ there are ordinals $j_0 < ... < j_n$ such that*

$$\mathrm{tp}(b_{i_0}, ..., b_{i_n}/\emptyset) = \mathrm{tp}(a_{j_0}, ..., a_{j_n}/\emptyset).$$

*Proof.* Let $n \geq 1$ be a natural number, and define

$$\Gamma_n = \{p(x_1, ..., x_n) \in S(\emptyset) \mid \exists j_1 < ... < j_n \in \beth_{(2^{|T|})^+} \ (\models p(a_{j_1}, ..., a_{j_n}))\}.$$

We will use the compactness theorem to find a sequence $(b_i)_{i\in I}$ with the desired properties. So we add a sequence of constants $(c_i)_{i\in I}$ to our language. Now we will construct for every natural number $n \geq 1$ a type $p_n(x_1, ..., x_n) \in \Gamma_n$ such that the set $\{p_n(c_{i_1}, ..., c_{i_n}) \mid n \in \omega,\ i_1 < ... < i_n \in I\}$ is consistent with $T$. If these types exist, then we let $(b_i)_{i\in I}$ be the interpretation of $(c_i)_{i\in I}$ in the monster model. Now if there is some sequence $i_1 < ... < i_n \in I$, then $\models p_n(b_{i_1}, ..., b_{i_n})$, and since $p_n \in \Gamma_n$, there are also $j_1 < ... < j_n \in \beth_{(2^{|T|})^+}$ Such that $\models p(a_{j_1}, ..., a_{j_n})$. And hence we see that $\mathrm{tp}(b_{i_1}, ..., b_{i_n}/\emptyset) = \mathrm{tp}(a_{j_0}, ..., a_{j_n}/\emptyset)$.

In order to find these types, we will build the following by induction on $n$:

- A set $\{F_n \mid n \in \omega\}$ of cofinal subsets of $(2^{|T|})^+$.
- A set $\{X_{\alpha,n} \mid \alpha \in F_n,\ n \in \omega\}$ of subsets of $\beth_{(2^{|T|})^+}$.
- A set $\{p_n(x_1, ..., x_n) \mid n \in \omega,\ p_n \in \Gamma_n\}$ of complete types.

We want to build these sets and types in such a way that the following properties are satisfied:

1. For every $n \in \omega$, we have $F_{n+1} \subseteq F_n$.
2. For every $n \in \omega$, if $\gamma$ is the order-type of $F_n$ with bijection $f : \gamma \to F_n$, then for every $\delta \in \gamma$, we have $|X_{f(\delta),n}| > \beth_\delta(2^{|T|})$.
3. For every $n \in \omega$, if $i_1 < ... < i_n \in X_{\alpha,n}$, then $\models p_n(a_{i_1}, ..., a_{i_n})$.

Note that if we construct the $p_n$ satisfying these properties, then we are done. Because with compactness, we only have to consider some finite part of $I$, and there is a maximal number $k \in \omega$ that we have to consider. And $X_{\alpha,k}$ will be large enough to incorporate this.

So let us give the construction. For $n = 0$, we take $F_0 = (2^{|T|})^+$, which is clearly cofinal in itself, and for every $\alpha \in F_0$, we let $X_{\alpha,0} = \beth_{(2^{|T|})^+}$. We indeed see that $\beth_{(2^{|T|})^+} > \beth_\lambda(2^{|T|})$ if $\lambda \in (2^{|T|})^+$.

Now suppose that for some natural number $n$, the sets $X_{\alpha,n}$ and $F_n$ have been defined for every $\alpha \in F_n$. Let $\beta$ be the order type of $F_n$, and let $g : \beta \to F_n$ be the order-isomorphism. Define $G_n = \{g(\lambda + n) \mid \lambda \in \beta\}$, and notice that since $F_n$ is cofinal in $(2^{|T|})^+$, $G_n$ also has this property. Now suppose $\alpha = g(\lambda + n)$, then we know that $|X_{\alpha,n}| > \beth_{\lambda+n}(2^{|T|})$. Now we define for every $\alpha \in F_n$ a map $\sigma_\alpha : \{(a_{i_1}, ..., a_{i_n}) \mid i_1, ..., i_n \in X_{\alpha,n}\} \to S_n(\emptyset)$ by mapping the tuple $(a_{i_1}, ..., a_{i_n})$ to the type $\mathrm{tp}(a_{i_1}, ..., a_{i_n}/\emptyset)$. Since there are at most $2^{|T|}$ many complete types over $\emptyset$ (for every complete type $p$ and for every $\mathcal{L}$-formula $\phi(x)$ we know that either $\phi(x) \in p$ or $\neg\phi(x) \in p$), this induces a partition of $[X_{\alpha,n}]^n$ in $2^{|T|}$ many classes. And since $|X_{\alpha,n}| > \beth_{\lambda+n}(2^{|T|}) = \beth_n(\beth_\lambda(2^{|T|}))$, we see using the Erdős-Rado theorem that there is some subset $X_{\alpha,n+1} \subseteq X_{\alpha,n}$ such that if $i_1, ..., i_n, j_1, ..., j_n \in X_{\alpha,n+1}$

then $\mathrm{tp}(a_{i_1}, ..., a_{i_n}/\emptyset) = \mathrm{tp}(a_{j_1}, ..., a_{j_n}/\emptyset)$, and such that $|X_{\alpha,n+1}| > \beth_\lambda(2^{|T|})$. So let us for every $\alpha \in G_n$ define the type $p_{\alpha,n+1} = \mathrm{tp}(a_{i_1}, ..., a_{i_n}/\emptyset)$ where $i_1, ..., i_n \in X_{\alpha,n+1}$. Now since there are only $2^{|T|}$ many complete types over $\emptyset$ and $G_n$ is cofinal in $(2^{|T|})^+$, we see that there must be some $F_{n+1} \subseteq G_n$ which is cofinal in $G_n$ and such that for every $\alpha, \gamma \in F_{n+1}$ we have that $p_{\alpha,n+1} = p_{\gamma,n+1}$. So now we have found the desired $X_{\alpha,n+1}$, $F_{n+1}$ and $p_n$, concluding the construction. □

Now we can prove the following.

**Lemma 4.17.** *Let $A \subseteq B$ be two small sets of parameters, and let $p \in S(B)$ be such that $p$ does not fork over $A$. Then for every linear order $I$ there is a Morley sequence for $p$ over $A$, indexed by $I$.*

*Proof.* First note that we can view $T$ as an $\mathcal{L}(B)$-theory, and use the same monster model. We will do this throughout this proof, so if we write $|T|$ we mean the cardinality of $T$ viewed as an $\mathcal{L}(B)$-theory.

First we will construct a sequence $\{a_j \mid j \in \beth_{(2^{|T|})^+}\}$ such that for every $j \in \beth_{(2^{|T|})^+}$ we have that $\models p(a_j)$ and $\mathrm{tp}(a_j/B \cup \{a_k \mid k < j\})$ does not fork over $A$. First we let $a_0$ be any element realizing $p$. Then $\mathrm{tp}(a_0/B) = p$, which indeed does not fork over $A$. Now suppose that for some $j \in \beth_{(2^{|T|})^+}$ we have constructed $\{a_k \mid k < j\}$. Then we note that by Lemma 4.7, there is some type $q \in S(B \cup \{a_k \mid k < j\})$ such that $q|_B = p$ and such that $q$ does not fork over $a$. Let $a_j$ be any element realizing $q$ (which exists because $B$ is small).

Now we use Lemma 4.16 to see that there is some $B$-indiscernible sequence (since $B$ is part of our language now) $\bar{b} = (b_i)_{i \in I}$ such that for every $i_0, ..., i_n \in I$ there are $j_0, ..., j_n \in \beth_{(2^{|T|})^+}$ such that $\mathrm{tp}(a_{j_0}, ..., a_{j_n}/B) = \mathrm{tp}(b_{i_0}, ..., b_{i_n}/B)$. We claim that $\bar{b}$ is a Morley sequence for $p$ over $A$. In fact, since $\bar{b}$ is $B$-indiscernible, it is also $A$-indiscernible. And since $a_0 \models p$ we see that $b_i \models p$ for every $i \in I$. So suppose $i \in I$ and $\mathrm{tp}(b_i/B \cup \{b_k \mid k < i\})$ divides over $A$. Then there is some formula $\phi(x, b_i, b_{k_0}, ..., b_{k_n})$ in this type which divides over $A$, where $b \in B$ and $k_0 < ... < k_n < i$. Now by construction of $\bar{b}$ there are $j_0 < ... < j_n < j \in \beth_{(2^{|T|})^+}$ such that

$$\mathrm{tp}(b_i, b_{k_0}, ..., b_{k_n}/B) = \mathrm{tp}(a_j, a_{j_0}, ..., a_{j_n}/B).$$

Now it follows from Exercise 17 that $\phi(x, a_j, a_{j_0}, ..., a_{j_n})$ divides over $A$. From this it follows that $\mathrm{tp}(a_j/B \cup \{a_k \mid k < j\})$ divides over $A$, and hence it forks over $A$. This is impossible, and hence we find that $\bar{b}$ is indeed a Morley sequence for $p$ over $A$. □

Note that if $p = \mathrm{tp}(b/B)$, then since $b_0 \models p$, we have $b \equiv_B b_0$. So we can choose some $f \in \mathrm{Aut}(\mathbb{M}/B)$ such that $f(b_0) = b$, and by Exercise 25 the sequence $(f(b_i))_{i \in I}$ will also be a Morley sequence for $p$ over $A$, starting with $b$. So we can choose our Morley sequence such that $b_0 = b$. We can apply this lemma to global invariant types.

**Corollary 4.18.** *Let $A \subseteq B \subseteq C \subseteq \mathbb{M}$ be sets of parameters, and let $p \in S(\mathbb{M})$ be $A$-invariant. Then there is a Morley sequence for $p|_C$ over $B$.*

*Proof.* Since $p$ is $A$-invariant, it follows from Exercise 20 that $p$ does not fork over $A$. Now we see that if $p$ forks over $B$, then it would divide over $B$, by the definition of forking and the fact that $p$ is a global type. But by the definition of dividing, if $p$ would divide over $B$, it would also divide over $A$. And since this is not the case, we see that $p$ does not fork over $B$. It follows that $p|_C$ also doesn't fork over $B$, for if it would, it would contain a formula that forks over $B$, and hence this formula would also be an element of $p$. And hence $p|_C$ does not fork over $B$. So from Lemma 4.17 it follows that there is a Morley sequence for $p|_C$ over $B$. □

**Exercise 27.** Let $T$ be a simple theory. Prove that if $A$ is a set of parameters and $p \in S(A)$, then $p$ does not fork over $A$. Also show that there exists a Morley sequence for $p$ over $A$.

Now we will prove a famous characterization of dividing in simple theories.

**Lemma 4.19.** *Let $T$ be a simple theory. Then the following are equivalent:*

1. *$\phi(x, b)$ divides over $A$.*
2. *If $(a_i)_{i\in I}$ is a Morley sequence for $\text{tp}(b/A)$, then $\{\phi(x, a_i) \mid i \in I\}$ is inconsistent.*
3. *For some Morley sequence $(a_i)_{i\in I}$ for $\text{tp}(b/A)$, the set $\{\phi(x, a_i) \mid i \in I\}$ is inconsistent.*

*Proof.* We see that (2)⇒(3) follows from Lemma 4.17 and exercise 27. And (3)⇒(1) follows immediately from the definition of dividing. So we prove (1)⇒(2).

**Claim:** If $\overline{a} = (a_i)_{i\in I}$ is a Morley sequence for $\text{tp}(b/A)$ and $J \subseteq I$ is such that $\inf(J)$ exists and $i \in I$ is such that $i < \inf(J)$, then $\phi(x, a_i)$ divides over $A \cup \{a_j \mid j \in J\}$.
**Proof of claim:** Let $q(x, y) = \text{tp}(a_i, \overline{a_J}/A)$. We know that $\phi(x, b)$ divides over $A$, and we know that $b \equiv_A a_i$. So there is some automorphism $f$ of $\mathbb{M}$ interchanging $b$ and $a_i$ and leaving $A$ fixed. And since $\phi(x, b)$ divides over $A$, we can conclude that $\phi(x, a_i)$ divides over $A$. So let $\overline{c} = (c_k)_{k\in\omega}$ be a sequence witnessing this. Now we consider $q(a_i, y)$. Notice that from Exercise 26 it follows that $q(a_i, y)$ does not divide over $A$. Now from Theorem 4.4 it follows that there is some $a'$ such that $a' \equiv_{Aa_i} a_J$ and such that $\overline{c}$ is $Aa'$-indiscernible. Hence it follows that there is an automorphism $g$ of the monster model leaving $Aa_i$ fixed and interchanging $a'$ and $a_J$. Now we can quickly see that the sequence $(f(c_k))_{k\in\omega}$ witnesses that $\phi(x, a_i)$ divides over $Aa_J$. So the claim is proven.

So now we can prove the lemma. So let $\overline{a} = (a_i)_{i\in I}$ be a Morley sequence for $\text{tp}(b/A)$, and suppose that $\{\phi(x, a_i) \mid i \in I\}$ would be consistent. Since $T$ is a simple theory, we know that for every type $p \in S(B)$ there is some subset $B_0 \subseteq B$ of size at most $|T|$ such that $p$ does not fork over $B_0$. So let $\kappa \leq |T|$ be minimal such that for every type $p \in S(B)$ there is such a set of size at most $\kappa$. Now let $(J, <_J)$ be a linear order with $J = (\kappa + \aleph_0)^+$ such that $\alpha <_J \beta$ if and only if $\alpha > \beta$. So we just take the reverse order on $(\kappa + \aleph_0)^+$. Now using Lemma 4.16 (and the same reasoning as in the proof of Lemma 4.17), we see that there is an $A$-indiscernible sequence $\overline{c} = (c_j)_{j\in J}$ such that if we have a finite tuple in $\overline{c}$ indexed by an increasing sequence in $J$, then there is some increasing sequence in $I$ such that the corresponding tuple in $\overline{a}$ realizes the same complete $A$-type as the one in $\overline{c}$. Using the same reasoning as in the proof of Lemma 4.17, we see that $\overline{c}$ is actually a Morley sequence for $\text{tp}(b/A)$. Now using compactness and the assumption that $\{\phi(x, a_i) \mid i \in I\}$ is consistent, we can show that $\{\phi(x, c_j) \mid j \in J\}$ is consistent. This involves adding constants for every element in $A$ and every $c_j$, and an additional constant $c$. We also add every $Ac$-sentence which is true in $\mathbb{M}$ and has the parameters from $Ac$ replaced by the corresponding constants. And we add $\phi(c, c_j)$ for every $j \in J$. Now the consistency of $\{\phi(x, c_j) \mid j \in J\}$ indeed follows from the consistency of $\{\phi(x, a_i) \mid i \in I\}$. Now let $m$ be an element realizing $\{\phi(x, c_j) \mid j \in J\}$.

Now since $T$ is simple there are $A' \subseteq A$ and $J' \subseteq J$ such that $|A'J'| \leq \kappa$ and such that $\text{tp}(m/AJ)$ does not fork over $A'\overline{c_{J'}}$. Now choose $i \in J$ such that $i < j$ for every $j \in J'$. This is possible since $|J'| \leq \kappa$ and $|J| \geq \kappa^+$. Now the claim tells us that $\phi(x, c_i)$ divides over $A\overline{c}$. But since $m$ realizes this formula we see that $\phi(x, c_i) \in \text{tp}(m/A\overline{c})$. This means that $\text{tp}(m/A\overline{c})$ divides over $A\overline{c}$, but not over $A'\overline{c_{J'}} \subseteq A\overline{c}$. This is impossible, so we have found a contradiction. We conclude that $\{\phi(x, a_i) \mid i \in I\}$ must be inconsistent. □

One of the most useful properties of simple theories is that in simple theories, forking and dividing are equivalent. Of course, if a formula divides over a set of parameters $A$, then it also forks over $A$. So we only have to prove the other direction.

**Lemma 4.20.** *Let $T$ be a simple theory and $\phi(x, b)$ some formula. If $\phi(x, b)$ forks over a set of parameters $A$, then it also divides over $A$.*

*Proof.* By the definition of forking, we know that there is some natural number $n$ and formulas $\phi_i(x, a_i)$ such that each $\phi_i(x, a_i)$ divides over $A$ and such that $\models \forall x(\phi(x, b) \rightarrow \bigvee_{i \leq n} \phi_i(x, a_i))$. Now let $\overline{a} = (a_0, ..., a_n)$. Because of Exercise 27 there is a Morley sequence for $\text{tp}(\overline{a}b/A)$, which we will denote by $(\overline{c}_i d_i)_{i \in \omega}$, where $\overline{c}_i d_i = (c^i_0, ..., c^i_n, d_i)$. Because of Exercise 25, we can assume that $(c^0_0, ..., c^0_n, d_0) = (a_0, ..., a_n, b)$. Since $(\overline{c}_i d_i)_{i \in \omega}$ is $A$-indiscernible, we see that $(d_i)_{i \in \omega}$ is also $A$-indiscernible. So it suffices to show that $\{\phi(x, d_i) \mid i \in \omega\}$ is inconsistent. So suppose that this set would be consistent, then it is a partial type over a countable set of parameters, which means that is is realized by some element $e$. Now since we know that for all $i \in \omega$ we have that $\text{tp}(\overline{a}b/A) = \text{tp}(\overline{c}_i d_i/A)$, we see that for all $i \in \omega$:

$$\models \forall x \left( \phi(x, d_i) \rightarrow \bigvee_{j \leq n} \phi_j(x, c^i_j) \right).$$

And since we know that $\models \phi(e, d_i)$ for all $i \in \omega$, we see that $\models \bigvee_{j \leq n} \phi_j(e, c^i_j)$ for all $i \in \omega$. This means that using the pigeonhole principle, there is some infinite set $S \subseteq \omega$ such that for some $j \leq n$ and for all $i \in S$ we have $\models \phi_j(e, c^i_j)$. Now we notice that $(a^i_j)_{i \in S}$ is a Morley sequence for $\text{tp}(a_j/A)$, and from the above it follows that $\{\phi_j(x, a^i_j) \mid i \in S\}$ is consistent. So now it follows from Lemma 4.19 that $\phi_j(x, a_j)$ does not divide over $A$. This is a contradiction, and hence we conclude that $\phi(x, b)$ divides over $A$. □

Lemma 4.19 and the equivalence of forking and dividing allows us to prove that forking is symmetric in simple theories.

**Lemma 4.21.** *Let $T$ be a simple theory, and let $A$ be a set of parameters and let $a, b$ be two parameters. Then* $\text{tp}(a/Ab)$ *forks over $A$ if and only if* $\text{tp}(b/Aa)$ *forks over $A$.*

*Proof.* We only prove one direction, one can see that the other direction can be proved in the same way. So we assume that $\text{tp}(a/Ab)$ forks over $A$. So let $\phi(x, c, b) \in \text{tp}(a/Ab)$, where $c \in A$, and we have chosen this formula such that $\phi(x, c, b)$ forks over $A$. Then the equivalence of forking and dividing tells us that $\phi(x, c, b)$ divides over $A$. Now suppose that $\text{tp}(b/Aa)$ does not fork over $A$. Then it follows from Lemma 4.17 that there exists a Morley sequence $\overline{b} = (b_i)_{i \in \omega}$ for $\text{tp}(b/Aa)$ over $A$ such that $b_0 = b$.

Now we know that $\models \phi(a, c, b)$, and hence we know that $\models \phi(a, c, b_i)$ for every $i \in \omega$. Hence the set $\{\phi(x, c, b_i) \mid i \in \omega\}$ is consistent. Now notice that $(cb_i)_{i \in \omega}$ is a Morley sequence for $\text{tp}(cb/A)$ over $A$, all the properties are easily checked using that $\bar{b}$ is a Morley sequence. But now it follows from Lemma 4.19 that $\{\phi(x, c, b_i) \mid i \in \omega\}$ is inconsistent. So we have found a contradiction, from which it follows that $\text{tp}(b/Aa)$ forks over $A$. □

**Corollary 4.22.** *Let $T$ be a simple theory, and let $\phi(x, b)$ be some formula. Then $\phi(x, b)$ forks over a set of parameters $A$ if and only if it divides over $A$.*

## 4.4 Stable theories are simple

In this section, we will prove that stable theories are simple. This means that we can use all the properties of simple theories (most notably the equivalence between forking and dividing) for stable theories. But in order to do this, we will have to capture the notion of simplicity in terms of a certain rank function being bounded. This will require a detailed study of this specific rank function. The material in this section was taken from [9]. We will start with the definition of what we shall call the $D$-rank.

**Definition 4.23.** Let $p(x)$ be a set of formulas, $\Delta$ a set of formulas and $k \in \omega$. The $D$-rank of $p$ with respect to $\Delta$ and $k$, denoted $D(p, \Delta, k)$, is inductively defined as follows:

- $D(p, \Delta, k) \geq 0$ if $p$ is consistent.
- For any ordinal $\alpha$, $D(p, \Delta, k) \geq \alpha + 1$ if for every finite $r \subseteq p$ there is some $\phi(x, y) \in \Delta$ and a set of elements $\{a_i \mid i \in \omega\}$ such that $D(r \cup \{\phi(x, a_i)\}, \Delta, k) \geq \alpha$ for every $i \in \omega$, and such that $\{\phi(x, a_i) \mid i \in \omega\}$ is $k$-contradictory. This means that for any $i_0, ..., i_{k-1} \in \omega$, we have $\bigwedge_{j<k} \phi(x, a_{i_j}) \models \perp$.
- If $\lambda$ is a limit ordinal, then $D(p, \Delta, k) \geq \lambda$ if $D(p, \Delta, k) \geq \beta$ for every $\beta < \lambda$.

For any ordinal $\alpha$, we say that $D(p, \Delta, k) = \alpha$ if $D(p, \Delta, k) \geq \alpha$ but $D(p, \Delta, k) \not\geq \alpha + 1$. We say that $D(p, \Delta, k)$ is undefined if $D(p, \Delta, k) \geq \alpha$ for every ordinal $\alpha$.

**Exercise 28.** a Suppose $p_1 \models p_2$, $\Delta_1 \subseteq \Delta_2$ and $k_1 \leq k_2$. Then $D(p_1, \Delta_1, k_1) \leq D(p_2, \Delta_2, k_2)$. (This is called *monotonicity*).

b Show that for every $p$, $\Delta$ and $k$, there is some finite $r \subseteq p$ such that $D(p, \Delta, k) = D(r, \Delta, k)$. (This is called *finite character*).

c Show that if $f : \mathbb{M} \to \mathbb{M}$ is an automorphism, then $D(p, \Delta, k) = D(f(p), \Delta, k)$.

The following property of the $D$-rank is called the *ultrametric property.*

**Lemma 4.24.** *Let $p, \Delta$ be sets of formulas, and let $k, n \in \omega$. If $\{\phi_l(x, a_l) \mid l < n\}$ is a set of formulas, then*

$$D\left(p \cup \left\{\bigvee_{l<n} \phi_l(x, a_l)\right\}, \Delta, k\right) = \max_{l<n} D(p \cup \{\phi_l(x, a_l)\}, \Delta, k).$$

*Proof.* The fact that

$$\max_{l<n} D(p \cup \{\phi_l(x, a_l)\}, \Delta, k) \leq D\left(p \cup \left\{\bigvee_{l<n} \phi_l(x, a_l)\right\}, \Delta, k\right)$$

follows directly from monotonicity. For the other direction, we will prove by induction on the ordinal $\alpha$ that for every set of formulas $p$, if $D(p \cup \{\bigvee_{l<n} \phi_l(x, a_l)\}, \Delta, k) \geq \alpha$, then $\max_{l<n} D(p \cup \{\phi_l(x, a_l)\}, \Delta, k) \geq \alpha$. For the base case we simply note that if $p \cup \{\bigvee_{l<n} \phi_l(x, a_l)\}$ is consistent, then for some $l < n$ we see that $p \cup \{\phi_l(x, a_l)\}$ is consistent. The case where $\alpha$ is a limit ordinal follows directly from the induction hypothesis and the definition of the $D$-rank.

So now suppose that $D(p \cup \{\bigvee_{l<n} \phi_l(x, a_l)\}, \Delta, k) \geq \alpha + 1$, but for every $l < n$ we have that $D(p \cup \{\phi(x, a_l)\}, \Delta, k) \leq \alpha$. Then we can choose (using finite character) for every $l < n$ a finite subset $r_l \subseteq p$ such that $D(r_l \cup \{\phi(x, a_l)\}, \Delta, k) \leq \alpha$. Now we let $r = \bigcup_{l<n} r_l$, and note that $r$ is a finite union over finite sets, meaning that it is finite. Now since $D(p \cup \{\bigvee_{l<n} \phi_l(x, a_l)\}, \Delta, k) \geq \alpha + 1$, it follows that there is some $\phi(x, y) \in \Delta$ and $\{b_i \mid i \in \omega\}$ such that $\{\phi(x, b_i) \mid i \in \omega\}$ is $k$-contradictory and for every $i \in \omega$ we have: $D(r \cup \{\bigvee_{l<n} \phi_l(x, a_l)\} \cup \{\phi(x, b_i)\}, \Delta, k) \geq \alpha$. Now using the induction hypothesis we see that $\max_{l<n} D(r \cup \{\phi_l(x, a_l)\} \cup \{\phi(x, b_i)\}, \Delta, k) \geq \alpha$ for every $i \in \omega$. So for every $i \in \omega$ there is some $l_i < n$ such that $D(r \cup \{\phi_{l_i}(x, a_{l_i})\} \cup \{\phi(x, b_i)\}, \Delta, k) \geq \alpha$. With the pigeonhole principle there is some $l' < n$ such that for some infinite $S \subseteq \omega$ we have that for all $i \in S$:

$$D(r \cup \{\phi_{l'}(x, a_{l'})\} \cup \{\phi(x, b_i)\}, \Delta, k) \geq \alpha.$$

It now follows that $D(r \cup \{\phi_{l'}(x, a_{l'})\}, \Delta, k) \geq \alpha + 1$, and since $r \models r_{l'}$ it follows with monotonicity that $D(r_{l'} \cup \{\phi_{l'}(x, a_{l'})\}, \Delta, k) \geq \alpha + 1$. This is a contradiction, and hence we conclude that $\max_{l<n} D(p \cup \{\phi_l(x, a_l)\}, \Delta, k) \geq \alpha + 1$. This completes the induction. □

**Lemma 4.25.** *Let $p$ be a partial type, let $\Delta$ and $\Phi$ be sets of formulas, and let $k \in \omega$. Then for every set of parameters $B$ there is some $q \in S_\Phi(B)$ such that $D(p, \Delta, k) = D(p \cup q, \Delta, k)$.*

*Proof.* First note that complete types are closed under conjunction, which means that we can assume that $\Phi$ is closed under conjunction. Now consider:

$$\Gamma = \{\neg\phi(x, b) \mid b \in B,\ \phi(x, y) \in \Phi,\ D(p \cup \{\phi(x, b)\}, \Delta, k) < D(p, \Delta, k)\}.$$

We claim that $p \cup \Gamma$ is consistent. For if it is not consistent, then by the compactness theorem there is some finite $\Gamma' = \{\neg\phi_i(x, b_i) \mid i < n\} \subseteq \Gamma$ such that $p \cup \Gamma'$ is inconsistent. This means that $p \models \bigvee_{i<n} \phi_i(x, b_i)$. Now with monotonicity and the ultrametric property (Lemma 4.24), we see:

$$D(p, \Delta, k) \leq D\left(p \cup \left\{\bigvee_{i<n} \phi_i(x, b_i)\right\}, \Delta, k\right) = \max_{i<n} D(p \cup \{\phi_i(x, b_i)\}, \Delta, k).$$

But now we have found a contradiction with the definition of $\Gamma$. Now we let $q \in S_\Phi(B)$ extend $\Gamma$. If $D(p \cup q, \Delta, k) < D(p, \Delta, k)$, then using finite character and the fact that $\Phi$ is closed under conjunction there is some $\phi(x, b) \in q$ such that $D(p \cup \{\phi(x, b)\}, \Delta, k) < D(p, \Delta, k)$. This means that $\neg\phi(x, b) \in \Gamma \subseteq q$, so $\phi(x, b) \wedge \neg\phi(x, b) \in q$. This is impossible, and hence we conclude that $D(p \cup q, \Delta, k) = D(p, \Delta, k)$. □

**Lemma 4.26.** *Let $A$ be a set of parameters, $p \in S(A)$ and suppose $D(p, \Delta, k)$ is defined for every finite $\Delta$ and every $k \in \omega$. If $\phi(x, a)$ is a formula which forks over $A$, then there is some finite $\Delta_0$ and some $k_0 \in \omega$ such that for all finite $\Delta \supseteq \Delta_0$ and $k \geq k_0$:*

$$D(p \cup \{\phi(x, a)\}, \Delta, k) < D(p, \Delta, k).$$

*Proof.* First suppose that $\phi(x, a)$ divides over $A$. Then there exists an $A$-indiscernible sequence $(a_i)_{i \in \omega}$ and some $k_0 \in \omega$ such that $a_0 = a$ and $\{\phi(x, a_i) \mid i \in \omega\}$ is $k$-contradictory. Now let $\Delta_0 = \{\phi(x, y)\}$. Suppose $\Delta \supseteq \Delta_0$ is finite and $k \geq k_0$, and suppose that $D(p \cup \{\phi(x, a)\}, \Delta, k) \geq D(p, \Delta, k)$. Then it follows from monotonicity that in fact $D(p \cup \{\phi(x, a)\}, \Delta, k) = D(p, \Delta, k)$. Now using finite character we see that there is some finite $r \subseteq p$ such that $D(r, \Delta, k) = D(p, \Delta, k)$, say $D(r, \Delta, k) = \alpha$. It follows from monotonicity that $D(r \cup \{\phi(x, a)\}, \Delta, k) = \alpha$. We know that for every $i \in \omega$ there exists some $f_i \in \text{Aut}(\mathbb{M}/A)$ interchanging $a$ and $a_i$. Now since the $D$-rank is invariant under automorphisms it follows that $D(f_i(r) \cup \{\phi(x, f_i(a))\}, \Delta, k) = \alpha$ for every $i \in \omega$. And since $r \subseteq p \in S(A)$, it follows that $D(r \cup \{\phi(x, a_i)\}, \Delta, k) = \alpha$ for every $i \in \omega$. However, we know that $\phi(x, y) \in \Delta$ and we know that $\{\phi(x, a_i) \mid i \in \omega\}$ is $k$-contradictory, since it is $k_0$-contradictory, and $k \geq k_0$. So from this it follows that $D(r, \Delta, k) \geq \alpha + 1$, which we know is not true. So we have found a contradiction, concluding that the lemma must be true if $\phi(x, a)$ divides over $A$.

So now suppose that $\phi(x, a)$ does not divide over $A$. Since we know that it does fork over $A$, we know that there are some $n \in \omega$ and formulas $\phi_i(x, b_i)$ for every $i < n$ such that $\phi(x, a) \models \bigvee_{i<n} \phi_i(x, b_i)$ and such that every $\phi_i(x, b_i)$ divides over $A$. Using the argument above, we see that for every formula $\phi_i(x, b_i)$, there are some finite $\Delta_i$ and some $k_i \in \omega$ such that for every finite $\Delta \supseteq \Delta_i$ and for every $k \geq k_i$ we have $D(p \cup \{\phi_i(x, b_i)\}, \Delta, k) < D(p, \Delta, k)$. We take $\Delta' = \bigcup_{i<n} \Delta_i$ and $k' = \max_{i<n} k_i$. Now suppose that $\Delta \supseteq \Delta'$ is finite and $k \geq k'$. Then we see using monotonicity and the ultrametric property (Lemma 4.24):

$$\begin{aligned} D(p \cup \{\phi(x, a)\}, \Delta, k) &\leq D\left(p \cup \left\{\bigvee_{i<n} \phi_i(x, b_i)\right\}, \Delta, k\right) \\ &= \max_{i<n} D(p \cup \{\phi_i(x, b_i)\}, \Delta, k) \\ &< D(p, \Delta, k). \end{aligned}$$

□

**Lemma 4.27.** *Suppose $D(x = x, \Delta, k)$ is defined for every finite $\Delta$ and every $k \in \omega$. Then $T$ is simple.*

*Proof.* Suppose $A$ is a set of parameters, and $p(x) \in S(A)$. Since $p(x) \models x = x$ we see that $D(p, \Delta, k)$ is defined for every finite $\Delta$ and $k \in \omega$. Using finite character, for every finite $\Delta$ and $k \in \omega$ there is some finite $r_{\Delta,k} \subseteq p$ such that $D(r_{\Delta,k}, \Delta, k) = D(p, \Delta, k)$. Now we let $q$ be the union over all these $r_{\Delta,k}$, and we let $B$ be the domain of $q$. Since $B$ is bounded from above by the number of finite sets of formulas, we see that $|B| \leq |T|$. We also see that $D(p, \Delta, k) = D(q, \Delta, k)$ for every finite $\Delta$ and every $k \in \omega$. We will now show that $p$ does not fork over $B$, from which it would follow from the definition of simple theories that $T$ is in fact simple.

So suppose $p$ forks over $B$. Then by definition there is some $\phi(x, a) \in p$ such that $\phi(x, a)$ forks over $B$. By monotonicity we see:

$$D(p, \Delta, k) \leq D(q \cup \{\phi(x, a)\}, \Delta, k) \leq D(q, \Delta, k).$$

And hence we conclude that $D(q \cup \{\phi(x,a)\}, \Delta, k) = D(q, \Delta, k)$ for every finite $\Delta$ and $k \in \omega$. However, this contradicts Lemma 4.26, and hence we conclude that $p$ forks over $B$. □

**Theorem 4.28.** *If $T$ is stable, then $T$ is simple.*

*Proof.* If $T$ is stable, then every formula is stable, and hence $R_2(x = x, \phi) < \omega$ for every formula $\phi$. This means that $R_\omega(x = x, \Delta) < \omega$ for every finite set of formulas $\Delta$. So if we can prove that $D(x = x, \Delta, k) \leq R_\omega(x = x, \Delta)$ for every finite set of formulas $\Delta$ and every $k \in \omega$, then we are done by Lemma 4.27. So now we will use induction on $\alpha$ to show for every type $p$ that if $D(p, \Delta, k) \geq \alpha$, then $R_\omega(p, \Delta) \geq \alpha$. Note that the base case and the limit case are trivial, because these two ranks are defined in the same way for these cases. So suppose $\alpha$ is some ordinal and $D(p, \Delta, k) \geq \alpha + 1$. Let $r \subseteq p$ be finite, then there is some set $A = \{a_i \mid i \in \omega\}$ and some $\phi(x, y) \in \Delta$ such that $\{\phi(x, a_i) \mid i \in \omega\}$ is $k$-contradictory and $D(r \cup \{\phi(x, a_i)\}, \Delta, k) \geq \alpha$ for all $i \in \omega$. It follows from Lemma 4.25 that for every $i \in \omega$ there is some $q_i \in S_\Delta(A)$ such that

$$D(r \cup \{\phi(x, a_i)\} \cup q_i, \Delta, k) = D(r \cup \{\phi(x, a_i)\}, \Delta, k) \geq \alpha.$$

Now it follows from the induction hypothesis that $R_\omega(r \cup \{\phi(x, a_i)\} \cup q_i, \Delta) \geq \alpha$. Since $\phi(x, a_i) \in q_i$ for every $i \in \omega$ (since otherwise $\neg\phi(x, a_i) \in q_i$, which would mean that $q_i \cup \{\phi(x, a_i)\}$ is inconsistent and therefore does not have rank at least 0), we can omit the $\{\phi(x, a_i)\}$ from this. We now use that $\{\phi(x, a_i) \mid i \in \omega\}$ is $k$-contradictory to see that every $k$-element subset of $\{q_i \mid i \in \omega\}$ contains two types which are contradictory. Now one can prove by induction on $k$ that there is an infinite subset $S \subseteq \omega$ such that for every $i, j \in S$ with $i \neq j$ we have that $q_i$ and $q_j$ are contradictory. Hence it follows that $R_\omega(r \cup q_i, \Delta) \geq \alpha$ for every $i \in S$, and hence $R_\omega(r, \Delta) \geq \alpha + 1$. And since this is the case for every finite $r \subseteq p$, we find that $R_\omega(p, \Delta) \geq \alpha + 1$. This concludes the induction. □

At first sight, it might seem that the proof of Theorem 4.28 is a little overkill. We only require $D(x = x, \Delta, k)$ to be defined for every finite $\Delta$, but we prove that it is finite. However, it follows from the compactness theorem that if $D(p, \Delta, k)$ is defined for finite $\Delta$, then it is finite. It suffices to prove this for finite $p$ (by finite character), but the proof is still very tedious, and hence we omit it.

*Chapter 5*

# The classification picture

## 5.1 Simple theories are NSOP

We have already seen a few possible properties that first-order theories can have. They can be stable, simple, NSOP or NIP, and we have already seen some implications between these properties. Specifically, we know that stable theories are simple and that stable theories are the intersection between the NSOP theories and the NIP theories. However, we currently have no knowledge of these other properties in relation to each other. So in this section, we will prove some implications and show that they are strict, to obtain what is sometimes called the *classification picture*. Note that we will only obtain a small part of the picture, since many more properties of first-order theories can be defined, and the relations between all of these properties have been studied extensively. A much larger version of the classification picture can be found online, including some open problems which could provide interesting topics for future research. The material in this section was taken from [9].

Let us start with the first implication we are going to consider: we will prove that simple theories do not obey the strict order property (they are NSOP). We will do this by proving that theories that have the strict order property also satisfy what is called the *tree property*, and that theories which satisfy the tree property are not simple.

**Definition 5.1.** A theory $T$ has the *tree property* if there is a formula $\phi(x, y)$ and a set of parameters $\{a_r \mid r \in \omega^{<\omega}\}$ such that:

1. For every $f \in \omega^\omega$, the set $\{\phi(x, a_{f|_l}) \mid l \in \omega\}$ is consistent.
2. There is some $k \in \omega$ such that for every $r \in \omega^{<\omega}$, the set $\{\phi(x, a_{r^\wedge l}) \mid l \in \omega\}$ is $k$-contradictory.

Note that the term 'tree property' is suggestive of the idea of this formula and set of parameters. The formulas $\phi(x, a_r)$ indeed form a tree, and the two properties in the definition tell us that any path downward through the tree is consistent, and that if we take $k$ formulas on the same level, they are inconsistent. This intuition will be important for understanding the proof of the next lemma.

**Lemma 5.2.** *Let $T$ be a theory with the strict order property. Then $T$ also has the tree property.*

*Proof.* Since $T$ has the strict order property, there is some formula $\phi(x, y)$ and some set of parameters $(b_i)_{i\in\omega}$ such that $\phi(\mathbb{M}, b_i) \subsetneq \phi(\mathbb{M}, b_{i+1})$ for every $i \in \omega$. Note that this means that $\neg\phi(\mathbb{M}, b_i) \supsetneq \neg\phi(\mathbb{M}, b_{i+1})$ for every $i \in \omega$. Now let $\phi(\mathbb{M}, b_i) = B_i$ for every $i \in \omega$, and define the formula $\psi(x, yz)$ by $\psi(x, yz) := \neg\phi(x, y) \wedge \phi(x, z)$. So we have

$$B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq B_3 \subsetneq ...$$

We will prove that $T$ has the tree property with this formula $\psi$ and with $k = 2$. We will do this by using the compactness theorem, so we add a constant $a_r$ for every $r \in \omega^{<\omega}$ to our language, and we add formulas telling us that we have a tree. This means that we add $\exists x \bigwedge_{i\leq l} \psi(x, a_{f|_{n_i}})$ for every $f \in \omega^\omega$ and for every $n_1, ..., n_l \in \omega$ with $l \in \omega$ (downward paths are consistent). And we also add $\neg\psi(x, a_r) \vee \neg\psi(x, a_{r'})$ for every distinct $r, r'$ of equal length (sets of formulas at the same level are 2-inconsistent). We now see that in order to prove that this theory is consistent, we need to be able to build any finite tree, and the result would follow from compactness. Note that $\psi(\mathbb{M}, b_ib_j) = B_i^c \cap B_j$ for every $i, j \in \omega$. The idea of building finite trees is as follows. We first assume that we have a connected tree (so a tree with a single root), if we can make this we can also make a finite number of these. Now look at how deep the tree has to be (formally the length of the longest path containing the root as one of its endpoints), and expand the tree to make sure that every node in the tree is part of such a path. We can also look at the node with the highest degree, and expand the tree to make sure that every node has this degree. So we now have a tree which is $n$ nodes deep and every node splits into $m$ new nodes. We will first work out a few small cases. If $n = 1$ we only have one node, so we just take $B_0^c \cap B_1$. If $n = 2$ we let the root be $B_0^c \cap B_m$, and the $m$ nodes below will be $B_0^c \cap B_1$, $B_1^c \cap B_2, ..., B_{m-1}^c \cap B_m$. Note that this indeed meets the requirements. So in this case we interpret $a_0$ as $b_0b_m$ and $a_{0i}$ as $b_ib_{i+1}$ for every $i \in \{0, ..., m-1\}$. If $n = 3$ we let the root be $B_0^c \cap B_{m^2}$, and the $m$ nodes underneath the root will be $B_0^c \cap B_m$, $B_m^c \cap B_{2m}, ...., B_{(m-1)m}^c \cap B_{m^2}$. And under the node $B_{i\cdot m}^c \cap B_{(i+1)\cdot m}$ with $i \in \{0, ..., m-1\}$, we have the nodes $B_{i\cdot m}^c \cap B_{i\cdot m+1}$, $B_{i\cdot m+1}^c \cap B_{i\cdot m+2}, ..., B_{i\cdot m+m-1}^c \cap B_{(i+1)k}$. We now notice that in full generality, if $l \in \{1, ..., n\}$ and $i_1, ..., i_l \in \omega$, then we can interpret $a_{i_1...i_l}$ as follows:

$$a_{i_1...i_l} = b_{i_1\cdot m^n+...+i_l\cdot m^{n+1-l}}b_{i_1\cdot m^n+...+i_{l-1}\cdot m^{n+2-l}+(i_l+1)\cdot m^{n+1-l}}.$$

It follows from compactness that $T$ indeed has the tree property. □

Now all we need to do is prove that any theory which has the tree property is not simple. We will do this by proving the following lemma.

**Lemma 5.3.** *Let $T$ be a theory with the tree property, and let $\kappa$ be some infinite cardinal. Then there is a type $p$ which forks over all subsets of cardinality $\kappa$ of its domain.*

*Proof.* Since $T$ has the tree property, there is some formula $\phi(x, y)$ and some $k \in \omega$ witnessing this. Now let $\kappa$ be given, and let $\lambda$ be strictly larger then the number of complete types over a subset of size $\kappa$ can be, so $\lambda \geq (2^{\kappa+|T|})^+$. Now using compactness and the tree property we quickly see that there is a set of parameters $\{a_f \mid f \in \lambda^{<\kappa^+}\}$ such that:

1. For every $g \in \lambda^{\kappa^+}$, the set $\{\phi(x, a_{g|_\alpha}) \mid \alpha \in \kappa^+\}$ is consistent.

2. For every $f \in \lambda^{<\kappa^+}$, the set $\{\phi(x, a_{f^\wedge\beta}) \mid \beta \in \lambda\}$ is $k$-inconsistent.

Now suppose $\alpha < \kappa^+$, meaning that the cardinality of $\alpha$ is at most $\kappa$. If $f : \alpha \to \lambda$ is a function, then we can consider the set $A = \{a_h \mid h = f|_\beta,\ \beta \in \alpha\}$, and notice that $A$ has at most cardinality $\kappa$. Now since $|\{a_{f^\wedge\beta} \mid \beta \in \lambda\}| = \lambda$ and $|A| \leq \kappa$, we notice by the pigeonhole principle that there must be countably many elements of the form $a_{f^\wedge\beta}$ with the same complete type over $A$. So using the compactness theorem we find that there is a set of parameters $\{b_f \mid f \in \omega^{<\kappa^+}\}$ such that:

1. For every $g \in \omega^{\kappa^+}$, the set $\{\phi(x, b_{g|_\alpha}) \mid \alpha \in \kappa^+\}$ is consistent.
2. For every $f \in \omega^{<\kappa^+}$, the set $\{\phi(x, b_{f^\wedge\beta}) \mid \beta \in \omega\}$ is $k$-inconsistent.
3. For every $f \in \omega^{<\kappa^+}$ and for every $n \in \omega$, we have $\text{tp}(b_{f^\wedge 0}/\{a_h \mid h = f|_\beta,\ \beta \in \text{dom}(f)\}) = \text{tp}(b_{f^\wedge n}/\{a_h \mid h = f|_\beta,\ \beta \in \text{dom}(f)\})$.

So now let $g \in \omega^{\kappa^+}$ and consider the set $p = \{\phi(x, b_{f|_\alpha}) \mid \alpha \in \kappa^+\}$, which is consistent (by the first item in the above enumeration), and therefore a partial type. If $A$ is a subset of the domain of $p$ of cardinality $\kappa$, then $p|_A \subseteq \{\phi(x, b_{f|_\beta}) \mid \beta \in \alpha\}$ for some $\alpha \in \kappa^+$. We now see by the second and third item in the above enumeration that $p$ divides over $A$. □

**Corollary 5.4.** *Let $T$ be a simple theory. Then $T$ does not have the strict order property.*

## 5.2 NSOP and NIP are incomparable

We already know that the class of stable theories is exactly the class of theories which do not have the independence property, and do not have the strict order property, so STAB=NIP∩NSOP. However, we do not yet know whether the inclusions STAB⊆NIP and STAB⊆NSOP are strict. We will see in this section that in fact both of these inclusions are strict. We will do this by giving two examples of theories, one in SOP∩NIP and one in NSOP∩IP. The material in this section was taken from [17]. For the first example, we will need the following lemma:

**Lemma 5.5.** *Let $\phi(x, y)$ be a formula with the independence property. Then there is some indiscernible sequence $(c_i)_{i\in\omega}$ and some $b \in \mathbb{M}$ such that $\models \phi(b, c_i)$ if and only if $i$ is even.*

*Proof.* Since $\phi(x, y)$ has the independence property, there are $(a_s)_{s\subseteq\omega}$ and $(b_i)_{i\in\omega}$ such that for all $i \in \omega$ and $s \subseteq \omega$ we have $\models \phi(a_s, b_i)$ if and only if $i \in s$. Now let $(c_i)_{i\in\omega}$ be an indiscernible sequence based on $(b_i)_{i\in\omega}$, which exists by Theorem 2.23. Now we will add a constant $b$ to the language, and we add $\phi(b, c_i)$ for every even $i$ and $\neg\phi(b, c_i)$ for every odd $i$. So suppose we have some finite subset of these axioms. Then we need to prove that this finite set is consistent, but this follows from the fact that $(c_i)_{i\in\omega}$ is based on $(b_i)_{i\in\omega}$, and from the fact that

$$\models \exists x \left( \bigwedge_{i\in I} \phi(x, b_i) \wedge \bigwedge_{j\in J} \neg\phi(x, b_j) \right).$$

for every two finite disjoint subsets $I, J \subset \omega$, since $a_I$ is a witness to this statement. So the existence of such $b$ and $c_i$ now follows from compactness. □

**Lemma 5.6.** *The theory DLO has the strict order property but not the independence property.*

*Proof.* It is easy to see that DLO has the strict order property, by considering an increasing sequence of parameters $a_0 < a_1 < a_2 ....$ Because if $\phi(x, y)$ is the formula $x < y$, then $\phi(\mathbb{M}, a_0) \subsetneq \phi(\mathbb{M}, a_1) \subsetneq \phi(\mathbb{M}, a_2) \subsetneq ....$ Now suppose that DLO has the independence property, then there is some formula which has the independence property, and since DLO has elimination of quantifiers, there is some quantifier-free formula which has the independence property. This means that there is some Boolean combination $\phi(x, y)$ of formulas of the form $v < w$ which has the independence property. However, that means that for this formula, there is some indiscernible sequence $(c_i)_{i\in\omega}$ and some $b \in \mathbb{M}$ such that $\models \phi(b, c_i)$ if and only if $i$ is even. However, since in DLO any indiscernible sequence is strictly increasing, strictly decreasing or constant, we see that this is impossible. Hence DLO does not have the independence property. □

Now we need to prove that there is a theory which has the independence property but not the strict order property. Unfortunately, the examples of such theories are far less natural and well-known then DLO.

**Lemma 5.7.** *There is a theory which has the independence property but not the strict order property.*

*Proof.* We will define the theory $T_{ind}$ in the language $\mathcal{L}_{ind} = \{P, E\}$, where $P$ is a predicate symbol and $E$ a binary relation symbol. The axioms are:

1. $\forall xy(xEy \rightarrow \neg P(x) \wedge P(y))$.
2. $\exists x_1...x_n(\bigwedge_{i\neq j} x_i \neq x_j)$.
3. $\forall x_1...x_n y_1...y_n \left(\bigwedge_{k,l\leq n}(x_k \neq y_l \wedge P(x_k) \wedge P(y_l)) \rightarrow \exists z \bigwedge_{k\leq n}(zEx_k \wedge \neg zEy_k)\right)$.
4. $\forall x_1...x_n y_1...y_n \left(\bigwedge_{k,l\leq n}(x_k \neq y_l \wedge \neg P(x_k) \wedge \neg P(y_l)) \rightarrow \exists z \bigwedge_{k\leq n}(x_kEz \wedge \neg y_kEz)\right)$.

Here the last three axioms are actually axiom schemes, so these axioms are in our theory for every natural number $n \geq 1$. First we need to prove that this theory is consistent. We do this by building a model for it. The underlying set of this model will be $2 \times \omega^2$. The elements satisfying $P$ will be exactly the elements of the form $(1, a, b)$ with $a, b \in \omega$. Now we will use that the set of finite subsets of $\omega$ is countable. So for every finite subset $B \subset \omega$, we choose an element $a_0 \in \omega$ and let $(0, 0, a_0)E(1, n, b)$ if and only if $n = 0$ and $b \in B$. Now we also choose an element $a_1 \in \omega$ such that $(0, n, b)E(1, 1, a_1)$ if and only if $n = 0$ and $b \in B$. Now we choose an element $a_2 \in \omega$ such that $(0, 1, a_2)E(1, n, b)$ if and only if $n = 1$ and $b \in B$. We can continue in this fashion to find a model which clearly satisfies the first axiom and is infinite, and also satisfies the last two axiom schemes.

Now we take the theory of this model to be our new theory $T^*_{ind}$, so $T^*_{ind}$ is consistent and complete. We will prove that it has quantifier elimination using Lemma 1.5. So suppose that $M, N$ are two models of $T^*_{ind}$ and $f : A \rightarrow N$ is a local isomorphism from some finite subset $A$ of $M$ to $N$. Now let $m \in M$ be some other element, say w.l.o.g. that $M \models P(m)$. Now we can split the finite set $A$ into three pairwise disjoint subsets $A_1, A_2, A_3$ such that $A_1 = \{a \in A \mid M \models P(a)\}$, $A_2 = \{a \in A \mid M \models aEm\}$ and $A_3 = \{a \in A \mid M \models \neg aEm \wedge \neg P(a)\}$. Now we consider the set of elements $f(A) \subset N$. We know from the axioms of $T_{ind}$ that there is some $b \in N$ such that $N \models aEb$ for every $a \in f(A_2)$ and $N \models \neg a'Eb$ for every $a' \in f(A_3)$. So if we define $f(m) = b$ then we have extended $f$ to a local isomorphism from $A \cup \{m\}$ to $N$. And hence any such local isomorphism can be extended, so it follows from Lemma 1.5 that $T^*_{ind}$ has quantifier elimination.

We can use quantifier elimination of $T^*_{ind}$ to show that this theory does not have the strict order property. For suppose it does, then there is some quantifier-free formula with the strict order property. However, this formula should be a Boolean combination of formulas of the form $x = y$, $xEy$ and $P(x)$, since these are the only atomic formulas. And we clearly see that such a formula does not have the strict order property. On the other hand, we can show that the formula $xEy$ has the independence property. For let $(b_i)_{i\in\omega}$ be a set of constants added to the language with axioms $P(b_i)$ also added. Now we also add constants $c_s$ for every $s \in \omega$ and axioms $c_sEb_i$ for every $i \in s$ and $\neg c_sEb_i$ for every $i \notin s$. We let $(b_i)_{i\in\omega}$ be interpreted by an arbitrary sequence of elements. Now suppose we have a finite subset of these axioms. Then there is some finite subset $A \subset \omega$ such that if $b_i$ occurs in one of these axioms, then $i \in A$. Now for every $c_s$ occurring in these axioms, consider $s \cap A$. We know that there is some element $a_s$ such that $a_sEb_i$ for every $i \in s \cap A$, and $\neg a_sEb_i$ for every $i \in A\backslash s$. So we let $a_s$ be the interpretation of $c_s$. So we see that this finite subset of formulas is consistent, and hence by compactness the entire theory is consistent. And hence we see that $xEy$ indeed has the independence property. $\square$

*Chapter 6*

# More on Forking

## 6.1 Automorphisms of the monster model

In this section we will prove some properties of automorphisms of the monster model which will be useful later on. This is taken from [5].

**Lemma 6.1.** *Let $A, B \subset \mathbb{M}$ be small, and let $C$ be a $B$-definable set. Then $C$ is $A$-definable if and only if every $f \in \mathrm{Aut}(\mathbb{M}/A)$ fixes $C$ setwise.*

*Proof.* Suppose $C$ is $A$-definable, so $C = \{c \mid\ \models \psi(c, a)\}$, where $\psi$ is an $\mathcal{L}$-formula and $a \in A$. Then because $f$ is an automorphism we notice:

$$\begin{aligned} c \in C &\Leftrightarrow \models \psi(c, a) \\ &\Leftrightarrow \models \psi(f(c), f(a)) \\ &\Leftrightarrow \models \psi(f(c), a) \\ &\Leftrightarrow f(c) \in C. \end{aligned}$$

Now we will prove the other direction. Since $C$ is $B$-definable, we can write $C = \{c \mid\ \models \phi(c, b)\}$ with $\phi$ an $\mathcal{L}$-formula and $b \in B$. Let $p(y) = \mathrm{tp}(b/A)$. We will prove the following:

$$p(y) \models \forall x(\phi(x, y) \leftrightarrow \phi(x, b)).$$

So let $b'$ be such that $\models p(b')$. Then since $p(y) = \mathrm{tp}(b/A)$ we see that $\mathrm{tp}(b/A) = \mathrm{tp}(b'/A)$. This means that there is some $f \in \mathrm{Aut}(\mathbb{M}/A)$ such that $f(b) = b'$, and we have assumed that this $f$ fixes $C$ setwise. We now see for any element $m \in \mathbb{M}$:

$$\begin{aligned} \models \phi(m, b) &\Leftrightarrow m \in C \\ &\Leftrightarrow f^{-1}(m) \in C \\ &\Leftrightarrow \models \phi(f^{-1}(m), b) \\ &\Leftrightarrow \models \phi(m, b'). \end{aligned}$$

And hence we indeed find: $p(y) \models \forall x(\phi(x, y) \leftrightarrow \phi(x, b))$. It follows from the Compactness theorem that there must be a formula $\psi(y) \in p(y)$ such that:

$$\psi(y) \models \forall x(\phi(x, y) \leftrightarrow \phi(x, b)).$$

Now define the formula $\theta(x)$ as $\exists y(\psi(y) \wedge \phi(x,y))$. Since $\phi(x,y)$ is an $\mathcal{L}$-formula and $\psi(y)$ is an $\mathcal{L}(A)$ formula, we see that $\theta(x)$ is an $\mathcal{L}(A)$ formula. So we just have to prove that $C = \{c \mid \models \theta(c)\}$. Notice that if $c \in C$, then $\models \phi(c,b)$, and we always have $\models \psi(b)$, since $\psi \in \mathrm{tp}(b/A)$. So $\models \theta(c)$. Now suppose $\models \theta(c)$ for some $c$. Then $\models \psi(a) \wedge \phi(c,a)$ for some $a$. This means that $\models \forall x(\phi(x,a) \leftrightarrow \phi(x,b))$, and hence $\models \phi(c,b)$. So $c \in C$. This concludes our proof. □

A nice application of this lemma is the following:

**Theorem 6.2.** *Let $A$ be a small set of parameters and let $p(x) \in S(\mathbb{M})$ be a global $A$-invariant type. If $p$ is definable, then it is definable over $A$.*

*Proof.* Let $\phi(x,y)$ be an $\mathcal{L}$-formula, and let $d_p\phi(y)$ be the formula defining it. Consider the set $B = \{a \in \mathbb{M} \mid \models d_p\phi(a)\}$. If $f$ is an $\mathbb{M}$-automorphism fixing $A$ pointwise, then we see using Exercise 19:

$$\begin{aligned} a \in B &\text{ iff } \models d_p\phi(a) \\ &\text{ iff } \phi(x,a) \in p(x) \\ &\text{ iff } \phi(x,f(a)) \in p(x) \\ &\text{ iff } \models d_p\phi(f(a)) \\ &\text{ iff } f(a) \in B. \end{aligned}$$

So $a \in B$ iff $f(a) \in B$. So as a set, $B$ is invariant under these automorphisms. It follows from Lemma 6.1 that $B$ is $A$-definable, and hence $p$ is $A$-definable. □

Note that by taking $C$ to be a singleton set, we can use Lemma 6.1 to prove that for any set of parameters $A$ and any $b \in \mathbb{M}$, $b \in \mathrm{dcl}(A)$ if and only if $f(b) = b$ for every $f \in \mathrm{Aut}(\mathbb{M}/A)$. We could wonder if we also have a generalization which is suited for the algebraic closure of $A$, instead of the definable closure. And indeed there is.

**Lemma 6.3.** *Let $C \subseteq \mathbb{M}$ be definable, and let $A \subseteq \mathbb{M}$ be a set of parameters. Then the following are equivalent:*

1. *There is an $A$-formula $E(x,y)$ which defines an equivalence relation with finitely many classes on $\mathbb{M}$, such that $C$ is a union of $E$-classes (we call $C$* almost $A$-definable *in this case).*
2. *The set $\{f(C) \mid f \in \mathrm{Aut}(\mathbb{M}/A)\}$ is finite.*
3. *The set $\{f(C) \mid f \in \mathrm{Aut}(\mathbb{M}/A)\}$ is small.*

*Proof.* For (1)⇒(2), note that if $f \in \mathrm{Aut}(\mathbb{M}/A)$, then $\models E(a,b)$ if and only if $\models E(f(a),f(b))$, since $E$ is an $\mathcal{L}(A)$-formula. So $a$ and $b$ are equivalent if and only if $f(a)$ and $f(b)$ are equivalent, which means that any $f \in \mathrm{Aut}(\mathbb{M}/A)$ permutes the classes of $E$. Since $E$ has only finitely many classes and $C$ is a union of these classes, we see that $\{f(C) \mid f \in \mathrm{Aut}(\mathbb{M}/A)\}$ is indeed finite.

(2)⇒(3) is trivial, any finite set is small. So now we will prove (3)⇒(1). So suppose $C = \phi(\mathbb{M},b)$ with $b \in \mathbb{M}$ and $\phi(x,y) \in \mathcal{L}$, and let $p = \mathrm{tp}(b/A)$. Now we note that if $f \in \mathrm{Aut}(\mathbb{M}/A)$, then $f(C) = \phi(\mathbb{M},f(b))$, and hence we see that there is some small set of parameters $B$ with $b \in B$ such that $\models p(b')$ for every $b' \in B$, and such that for every $f \in \mathrm{Aut}(\mathbb{M}/A)$ there is some $b' \in B$ such that $f(C) = \phi(\mathbb{M}, b')$. As before, we see that if $\models p(c)$ for some element $c$, then there is some $f \in \mathrm{Aut}(\mathbb{M}/A)$ such that $f(b) = c$, and hence $f(C) = \phi(\mathbb{M}, c)$, so we see that $p(y) \cup \{\neg\forall x(\phi(x, y) \leftrightarrow \phi(x, b')) \mid b' \in B\}$ is inconsistent (because $A \cup B$ is small, and any partial type over a small set should be realized in $\mathbb{M}$). So by the compactness theorem there is some formula $\psi(y) \in p$ and some $b_0, ..., b_k \in B$ with $b_0 = b$ such that $\psi(y) \models \bigvee_{i \leq k} \forall x(\phi(x, y) \leftrightarrow \phi(x, b_i))$. Now we define $E(x_1, x_2)$ to be the following formula:

$$\forall y(\psi(y) \rightarrow (\phi(x_1, y) \leftrightarrow \phi(x_2, y))).$$

Note that this is in fact an $\mathcal{L}(A)$-formula. We also note that it defines an equivalence relation on $\mathbb{M}$, and we will show that this equivalence relation has finitely many classes. If $\models E(a_1, a_2)$, then $\models \forall y(\psi(y) \rightarrow (\phi(a_1, y) \leftrightarrow \phi(a_2, y)))$, and since we know that $\models \psi(b_i)$ for every $i \leq k$, we see that $\models \phi(a_1, b_i) \leftrightarrow \phi(a_2, b_i)$ for every $i \leq k$. Now suppose that $\models \neg E(a_1, a_2)$. Then we have

$$\models \exists y(\psi(y) \wedge \neg(\phi(a_1, y) \leftrightarrow \phi(a_2, y))),$$

so let $m$ be a witness to this. Then we have $\models \psi(m)$, and hence there must be some $i \leq k$ such that $\models \forall x(\phi(x, m) \leftrightarrow \phi(x, b_i))$. However, we also have $\models \neg(\phi(a_1, m) \leftrightarrow \phi(a_2, m))$, and hence we find $\models \neg(\phi(a_1, b_i) \leftrightarrow \phi(a_2, b_i))$. So two elements $a_1, a_2 \in \mathbb{M}$ are in the same $E$-class if and only if they agree on $\phi(x, b_i)$ for every $i \leq k$, which means that this equivalence relation has exactly $2^{k+1}$ many equivalence classes.

Now we note that if $a_1 \in C = \phi(\mathbb{M}, b_0)$, then we see that if $\models E(a_1, a_2)$, then the elements $a_1$ and $a_2$ must agree on $\phi(x, b_0)$, and hence $\models \phi(a_2, b_0)$, so $a_2 \in C$. This means that if some element of $\mathbb{M}$ is in $C$, then the entire $E$-class of this element is in $C$, so $C$ is a union of $E$-classes. □

From this lemma it follows that if $A$ is some set of parameters and $b \in \mathbb{M}$, then $b \in \mathrm{acl}(A)$ if and only if the orbit of $b$ under the automorphism group $\mathrm{Aut}(\mathbb{M}/A)$ is finite, if and only if the orbit of $b$ under the automorphism group $\mathrm{Aut}(\mathbb{M}/A)$ is small. Using these observations, we can also prove the following useful lemma.

**Lemma 6.4.** *Let $A \subseteq \mathbb{M}$ be small, then* $\mathrm{acl}(A) = \bigcap\{M \supseteq A \mid M \prec \mathbb{M}\}$.

*Proof.* Suppose $a \in \mathrm{acl}(A)$ and $M$ is an elementary substructure of $\mathbb{M}$ with $A \subseteq M$. Then there is some $\mathcal{L}(A)$-formula $\phi(x)$ such that $\models \phi(a)$, and for some natural number $n$, $\phi$ has only $n$ solutions in $\mathbb{M}$. But since we can express in an $\mathcal{L}$-sentence that $\phi$ has exactly $n$ solutions, and this sentence must be true in $\mathbb{M}$, it must also be true in $M$. So all of these solutions must lie in $M$, and therefore $a \in M$. And hence $a \in \bigcap\{M \supseteq A \mid M \prec \mathbb{M}\}$.

Now suppose $a \notin \mathrm{acl}(A)$. Then the $\mathrm{Aut}(\mathbb{M}/A)$-orbit of $a$ is not small. So suppose $M \supseteq A$ is a small model, then there is some $f \in \mathrm{Aut}(\mathbb{M}/A)$ such that $f(a) \notin M$. But this means that $f^{-1}(M)$ is a small model of $T$ which contains $A$ but does not contain $a$. So $a \notin \bigcap\{M \supseteq A \mid M \prec \mathbb{M}\}$. □

## 6.2 Elimination of imaginaries

In this section, we will take a look at the concept of imaginaries. Elimination of imaginaries is an assumption that we will have to make to prove certain results in the next section. The material in this section is mostly taken from [16].

It is sometimes useful to be able to assume that definable sets can be identified with single elements. It turns out that this assumption can be safely made by assigning to every theory $T$ a theory $T^{eq}$ in a *many-sorted* language. This means that we have a set of *sorts* $\{S_i \mid i \in I\}$ with $I$ some index set, and any structure in our language will be partitioned into sorts. For every sort we have a countably infinite number of variables of this sort. In addition, every quantifier will specify in which sort the variable over which we quantify lives, and every function symbol will have as domain a product of sorts, and as codomain a single sort. Also every relation symbol will be a relation on specified sorts, and the constant symbols are also given the sort in which they live. This distinction between sorts can be very useful, as the following example will make clear.

**Example 6.5.** We will use many-sorted logic to give a language and theory for vector fields. The language will consist of two sorts, a sort $V$ for vectors and a sort $S$ for scalars. Our language consists of a constant $\mathbf{0}$ of sort $V$ (the zero vector) and constants $0, 1$ of sort $S$. Furthermore, we have functions $\cdot_S$ and $+_S$ with domain $S^2$ and codomain $S$ (multiplication and addition in the scalar field), and a function $+_V$ with domain $V^2$ and codomain $V$ (addition of vectors). We also have a function $\cdot_V$ with domain $S \times V$ and domain $V$ (scalar multiplication). We can now use these symbols to give all the axioms of a vector space, including axioms telling us that the elements of sort $S$ form a field. We will omit these details.

Now we will take a look at how many-sorted logic is going to help in our situation.

Suppose we have a language $\mathcal{L}$, and an $\mathcal{L}$-theory $T$ with monster model $\mathbb{M}$. Let $E$ be the set of $\mathcal{L}$-formulas $\phi(x, y)$ such that in every model of $T$, the formula $\phi$ defines an equivalence relation (note that the formula $x = y$ is an example of such a formula). Then we define for every $\phi \in E$:

- A *sort* $S_\phi$, and
- A function symbol $f_\phi : S_= \to S_\phi$.

We use $S_=$ instead of $S_{x=y}$. Now we define the language $\mathcal{L}^{eq}$ as the language having sorts $S_\phi$ and functions symbols $f_\phi$ for every formula $\phi \in E$. Furthermore, every constant, function symbol and relation symbol of $\mathcal{L}$ will be added to $\mathcal{L}^{eq}$, but now everything is living in the sort $S_=$. We define the theory $T^{eq}$ in the language $\mathcal{L}^{eq}$ as follows:

- $T^{eq}$ contains all the sentences of $T$, but with everything living in $S_=$.
- For every $\mathcal{L}$-formula $\phi(x, y) \in E$, we add the sentence
  $\forall x, y \in S_=(\phi(x, y) \leftrightarrow f_\phi(x) = f_\phi(y))$.
- For every $\mathcal{L}$-formula $\phi(x, y) \in E$, we add the sentence saying that the map $f_\phi : S_= \to S_\phi$ is surjective. So we add the sentence $\forall x \in S_\phi \exists y \in S_=(f_\phi(y) = x)$.

Given a model $M$ of $T$, we can build a model $M^{eq}$ of $T^{eq}$ in the following way: $S_=(M^{eq})$ is the underlying set of $M$, and every element of $\mathcal{L}$ will be interpreted in $S_=(M^{eq})$ as it was in $M$. Now for every formula $\phi(x, y) \in E$, we define $S_\phi(M^{eq}) = M/\phi(x, y)$. This means that the sort $S_\phi$ is exactly the set of equivalence classes of the relation $\phi(x, y)$ on $M$. Now for every formula $\phi(x, y) \in E$, we define $f_\phi(a) = [a]_\phi$, where $[a]_\phi$ is the equivalence class of the element $a \in M$ under the relation $\phi$. Of course every element of $M$ has its own class if our equivalence relation is equality, so in this case we identify every element with its own class.

Note that if we have a model $M$ of $T^{eq}$, then $M = (S_=(M))^{eq}$, meaning that every model of $T^{eq}$ is of the form $M^{eq}$ for some model $M$ of $T$. Also note that if $M, N$ are isomorphic models of $T$, then $M^{eq}$ and $N^{eq}$ are also isomorphic, because we can extend the isomorphism between $M$ and $N$.

Since we are only studying *complete* theories $T$, we should wonder whether $T^{eq}$ is complete. In **??** and **??** it is stated without proof that $T^{eq}$ is complete. We will give a proof based on the proof in **??**, which uses the generalized continuum hypothesis.

**Theorem 6.6.** (GCH) *The theory $T^{eq}$ is complete.*

*Proof.* Suppose $T^{eq}$ is not complete. Then since any model of $T^{eq}$ is of the form $M^{eq}$ for some model $M$ of $T$, this means that there are models $M, N$ of $T$ and an $\mathcal{L}^{eq}$-sentence $\phi$ such that $M^{eq} \models \phi$ and $N^{eq} \models \neg\phi$. Let $\kappa$ be large enough such that for every $\lambda < \kappa$, we have $\lambda^{|\mathcal{L}|} < \kappa$. Now we know that there are models $M'^{eq} \succ M^{eq}$ and $N'^{eq} \succ N^{eq}$ such that $|M'^{eq}| = |N'^{eq}| = \kappa$ and such that $M'^{eq}$ and $N'^{eq}$ are saturated (this follows from Theorem 1.9. This means that we require $\kappa$ to be a successor cardinal, but that is fine. Once we have found a sufficiently large cardinal, we can just take the successor of that cardinal to be $\kappa$). We now notice that by our choice of $\kappa$, the models $M'$ and $N'$ must also have cardinality $\kappa$. Also notice that these models must be saturated, because $M'^{eq}$ and $N'^{eq}$ are, and hence it follows from Theorem 1.6 that $M'$ and $N'$ are isomorphic. And hence it follows that $M'^{eq}$ and $N'^{eq}$ must also be isomorphic, but this is impossible because of the sentence $\phi$. We conclude that $T^{eq}$ must be complete. □

The following theorem will be the one telling us that given a theory $T$, we can safely assume that we are working in the theory $T^{eq}$.

**Theorem 6.7.** *Let $\phi(x_1, ..., x_n)$ be an $\mathcal{L}^{eq}$-formula, with $x_i$ of sort $S_i$. Then there is some $\mathcal{L}$-formula $S\phi(y_1, ..., y_n)$ such that*

$$T^{eq} \models \forall y_1, ..., y_n \in S_=(S\phi(y_1, ..., y_n) \leftrightarrow \phi(f_{S_1}(y_1), ..., f_{S_n}(y_n))).$$

*Proof.* For convenience of notation, we will assume that $n = 1$. Note that this does not affect the proof.

We consider the restriction map $\pi : S_{S_=}(\emptyset) \to S(\emptyset)$, from the space of complete types (in $T^{eq}$) with variable in the sort $S_=$, to the space of complete types (in $T$). The author apologizes for the inconvenience of this notation. Unfortunately, "$S$" is standard notation for both type spaces and sorts, it is best not to deviate from this. Now, The map $\pi$ sends a complete $\mathcal{L}^{eq}$-type $p$ to its restriction to $\mathcal{L}$. Now if $\phi(x)$ is some $\mathcal{L}$-formula, then the inverse image of $[\phi(x)]$ is $[\phi(x)]$ where in this case $x$ is a variable of sort $S_=$. So the inverse image of an open is open, meaning that $\pi$ is continuous. Since it can be seen that $\pi$ is also bijective, it is a homeomorphism (since any continuous bijective map from a compact space into a Hausdorff space is a homeomorphism).

Now suppose we have some $\mathcal{L}^{eq}$-formula $\phi(x)$, with $x$ of sort $S$. Note that if $y$ is a variable of sort $S_=$, then $\pi([\phi(f_S(y))])$ is the image of a clopen set under a homeomorphism, and therefore clopen. This means that it is a union of basic clopens, and by compactness (we have a closed space in a compact space, so this is compact with the induced topology) this means that it is a finite union of clopens. So there are $\mathcal{L}$-formulas $\phi_1(x), ..., \phi_n(x)$ such that

$$\pi([\phi(f_S(y))]) = [\phi_1(y)] \cup ... \cup [\phi_n(y)] = [\phi_1(y) \vee ... \vee \phi_n(y)].$$

So now we take $S\phi(y) = \phi_1(y) \vee ... \vee \phi_n(y)$, and the result follows. □

Now suppose that $X$ is some definable subset of $\mathbb{M}$, so $X = \phi(\mathbb{M}, b)$. Consider the equivalence relation $E(y_1, y_2)$ defined by $\forall x(\phi(x, y_1) \leftrightarrow \phi(x, y_2))$. Note that $[b]_E$ is an element of the sort $S_E$. Note that $X$ is $[b]_E$-definable via the formula $\exists y(f_E(y) = [b]_E \wedge \phi(x, y))$. And in fact $[b]_E$ is the only element of sort $S_E$ such that $X$ is defined via this formula. We call $[b]_E$ a *code* for $X$ in $\mathbb{M}^{eq}$.

**Definition 6.8.** A theory $T$ has *elimination of imaginaries* if for every definable set $X \subseteq \mathbb{M}$ there is some formula $\phi(x, y)$ and some element $a \in \mathbb{M}$ (the code of X) such that $X = \phi(\mathbb{M}, a)$ and if $X = \phi(\mathbb{M}, a')$ then $a = a'$.

Now the strength of working in $T^{eq}$ is summarized in the following exercise:

**Exercise 29.** Show that for any theory $T$, the theory $T^{eq}$ has elimination of imaginaries.

In the following section, we will assume elimination of imaginaries whenever we need it.

## 6.3 Forking in stable theories

In this section, we will investigate the notion of forking under the assumption of stability. We will also use this opportunity to define the notion of *forking independence*. The material in this section was taken from [5].

**Lemma 6.9.** *Let $T$ be a stable theory, $M$ a small model of $T$, and let $p \in S(\mathbb{M})$ be a global type which does not divide over $M$. Then $p$ is $M$-definable.*

*Proof.* We will show that $p$ is an heir of $p|_M$. Note that this will be enough, since $T$ is stable, and hence we can invoke Theorem 3.20 to see that $p|_M$ is in fact a definable type, so with Theorem 4.13 it will follow that $p$ is definable over $M$.

In order to show that $p$ is an heir of $p|_M$, we need to show that if $\phi(x, y)$ is an $\mathcal{L}(M)$-formula and $a \in \mathbb{M}$ such that $\phi(x, a) \in p$, then there is some $a' \in M$ such that $\phi(x, a') \in p|_M$. In order to do this, consider the type $\text{tp}(a/M)$, which we will call $q$. Now let $q'$ be a global coheir of $q$, which exists because of Theorem 4.12. We notice that $q'$ is $M$-invariant because of Exercise 22, and hence it does not fork over $M$ because of Exercise 20. Hence we see by Lemma 4.17 that there is a Morley sequence $(a_i)_{i\in\omega}$ for $q'$ over $M$, and we know that we can choose this sequence such that $a_0 = a$.

Now we let $b \models p|_{Ma}$, and note that $p|_{Ma}$ does not divide over $M$, since $p$ does not. So now we use Theorem 4.4 to see that there is some $b'$ such that $b' \equiv_{Ma} b$ and such that $(a_i)_{i\in\omega}$ is $Mb'$-indiscernible. So since $\text{tp}(b/Ma) = \text{tp}(b'/Ma)$, we can assume that we chose $b$ such that $(a_i)_{i\in\omega}$ is $Mb$-indiscernible. Now since $\phi(x, a) \in p$ is an $\mathcal{L}(Ma)$-formula, it follows that $\models \phi(b, a)$, and since $a_0 = a$ and $\phi(b, y)$ is an $\mathcal{L}(Mb)$-formula, we see that $\models \phi(b, a_i)$ for every $i \in \omega$.

Now we consider the type $r = \text{tp}(b/M(a_i)_{i\in\omega})$. It follows from Theorem 3.20 that this type is definable. So for every $\mathcal{L}(M(a_i)_{i\in\omega})$-formula $\psi(x, y)$ there is some $\mathcal{L}(M(a_i)_{i\in\omega})$-formula $d_r\psi(y)$ such that for all $c \in M(a_i)_{i\in\omega}$ we have $\psi(x, c) \in r$ if and only if $\models d_r\psi(c)$. Now let $n$ be a natural number such that all the parameters of $d_r\phi(y)$ are contained in $M \cup \{a_0, ..., a_{n-1}\}$. Note that $\models \phi(b, a_i)$ for every $i \in \omega$, and hence $\phi(x, a_i) \in r$ for every $i \in \omega$, so $\models d_r\phi(a_i)$ for every $i \in \omega$. We also notice that $\models q'(a_n)$, and hence $\text{tp}(a_n/Ma_{<n}) = q'|_{Ma_{<n}}$. And we know that $q'$ is a global coheir of $q$, and hence $\text{tp}(a_n/Ma_{<n})$ is also a coheir of $q$. But this together

means that there must be some $a' \in M$ such that $\models d_r\phi(a')$. But this means that $\models \phi(b, a')$, and hence $\phi(x, a') \in \mathrm{tp}(b/M) = p|_M$. So we indeed see that $p$ is an heir of $p|_M$. □

**Theorem 6.10.** *Let $T$ be a stable theory, $M$ a small model, $p \in S(M)$ and $A \supseteq M$ a small set of parameters. Then if $q \in S(A)$ is an extension of $p$, the following properties are equivalent:*

1. *$q$ does not divide over $M$.*
2. *$q$ does not fork over $M$.*
3. *$q$ is definable over $M$.*
4. *$q$ is an heir of $p$.*
5. *$q$ is a coheir of $p$.*

*Furthermore, $p$ has a unique extension $q$ with these properties.*

*Proof.* The equivalence of (1). and (2). follows from the fact that stable theories are always simple, and hence forking and dividing are equivalent. For (2).⇒(3). we use Lemma 4.7 to see that we can extend $q$ to a global type $r$ such that $r$ does not fork over $M$. Now we use Lemma 6.9 to see that $r$ is definable over $M$, and hence $q$ is definable over $M$. (3).⇒(4). follows from Exercise 21. (4).⇒(1). follows from Theorem 4.13, since $q$ is the unique heir of $p$ in $S(A)$, and hence it must be the restriction of the global heir of $p$. This global heir is $M$-definable by Theorem 4.13, and hence it is $M$-invariant by Exercise 22, and hence it doesn't fork over $M$ by Exercise 20. And since this global heir does not fork, we see that any restriction of it also doesn't fork. So now we only have to prove that (5). is equivalent to the others. For this we will write $q = \mathrm{tp}(a/Mb)$ (where $Mb = A$ and $a$ is an element realizing $q$), and we use the equivalences of (1). up to (4). together with Exercise 23 and the symmetry of forking (Lemma 4.21) to note the following: $\mathrm{tp}(a/Mb)$ is an heir of $\mathrm{tp}(a/M)$ if and only if it does not fork over $M$, which is the case if and only if $\mathrm{tp}(b/Ma)$ does not fork over $M$, which is true if and only if $\mathrm{tp}(b/Ma)$ is an heir of $\mathrm{tp}(a/M)$, which holds if and only if $\mathrm{tp}(a/Mb)$ is a coheir of $\mathrm{tp}(a/M)$. □

Notice that the first 4 of these statements are also equivalent to each other if the set of parameters $A$ is not small. Only the equivalence with the fifth statement requires that every type over $A$ is realized, which we can only ensure for types over small sets of parameters.

A natural question to ask at this point is what we know about forking if the set of parameters is not itself a model. We will investigate this notion further in this section.

**Definition 6.11.** Let $p \in S(\mathbb{M})$ be a global type. Then a set of parameters $A$ is called a *canonical base* of $p$ if for any $f \in \mathrm{Aut}(\mathbb{M})$, $f(p) = p$ if and only if $f|_A$ is the identity function on $A$.

**Exercise 30.** Let $p$ be a global type and let $A$ and $B$ be canonical bases of $p$. Prove that $A$ and $B$ have the same definable closure, and that this is also a canonical base of $p$.

The exercise above shows that if a global type $p$ has a canonical base, then it has a unique definably closed canonical base, which we will denote by $\mathrm{cb}(p)$.

**Lemma 6.12.** *If $T$ has elimination of imaginaries, then any definable global type $p$ has a canonical base, and $\mathrm{cb}(p)$ is the smallest set over which $p$ is defined which is definably closed.*

*Proof.* Let $\phi(x,y)$ be an $\mathcal{L}$-formula, and let $d_p\phi(y)$ be the formula defining it. Consider $d_p\phi(\mathbb{M})$, which is a definable set, and hence by elimination of imaginaries it has a code, which we will denote by $c_\phi$. Now let $C = \{c_\phi \mid \phi(x,y) \in \mathcal{L}\}$. Now we notice that $f(p) = p$ if and only if for all $\mathcal{L}$-formulas $\phi(x,y)$ and for all $a \in \mathbb{M}$ we have: $\phi(x,a) \in p \Leftrightarrow \phi(x, f(a)) \in p$, which is the case if and only if $\models d_p\phi(a) \Leftrightarrow \models d_p\phi(f(a))$. This is the case if and only if for all $\mathcal{L}$-formulas $\phi(x,y)$ we have $f(d_p\phi(\mathbb{M})) = d_p\phi(\mathbb{M})$ setwise, which is the case if and only if $c_\phi = f(c_\phi)$ for all such $\phi$, which is the case if and only if $f(C) = C$ pointwise. So we see that $C$ is a canonical base of $p$. So $\mathrm{dcl}(C) = \mathrm{cb}(p)$.

Now we have to show that $\mathrm{cb}(p)$ is the smallest definably closed set over which $p$ is definable. So first we have to show that $p$ is in fact definable over $\mathrm{cb}(p)$. Using Theorem 6.2 we see that it is enough to show that $p$ is $\mathrm{dcl}(C)$-invariant. So suppose $f \in \mathrm{Aut}(\mathbb{M}/\mathrm{dcl}(C))$, then we know that $f(C) = C$ pointwise, because $C \subseteq \mathrm{dcl}(C)$, which means that $f(p) = p$. So for any formula $\phi$ and parameter $a$, we see that $\phi(x,a) \in p$ if and only if $\phi(x, f(a)) \in p$, and hence $p$ is indeed $\mathrm{dcl}(C)$-definable. Now suppose that $B$ is a definably closed set and $p$ is $B$-definable. Then we see that for every $\mathcal{L}$-formula $\phi(x,y)$ there is some $\mathcal{L}(B)$-formula $d_p\phi(y)$ such that for all $a \in \mathbb{M}$ we have $\phi(x,a) \in p$ if and only if $\models d_p\phi(a)$. Now let $c_\phi$ be a code for this element, so there is some $\mathcal{L}$-formula $\psi(x,y)$ such that $c_\phi$ is the unique element such that $\psi(\mathbb{M}, c_\phi) = d_p\phi(\mathbb{M})$. So we notice that $c_\phi$ is the unique solution of the $\mathcal{L}(B)$-formula given by $\forall x(\psi(x,y) \leftrightarrow d\phi(x))$. And hence we see that $c_\phi \in \mathrm{dcl}(B) = B$, so $C \subseteq B$, and hence $\mathrm{cb}(p) = \mathrm{dcl}(C) \subseteq B$. So $\mathrm{cb}(p)$ is indeed the smallest definable set over which $p$ is definable. □

**Definition 6.13.** Let $A$ be a small set of parameters and let $p \in S(A)$ be a definable type which is defined by the schema $D = \{d_p\phi(y) \mid \phi(x,y) \in \mathcal{L}\}$. Then we say that $D$ is a *large* schema if it defines a global type $q \in S(\mathbb{M})$ extending $p$.

**Exercise 31.** Let $D$ be a definition schema for a type $p \in S(A)$, with $A$ small. Show that $D$ is large if and only if it is a definition schema for some type $q$ over a model $M$ containing $A$.

**Lemma 6.14.** *Suppose $T$ is a stable theory, $A \subseteq B$ are sets of parameters such that $A$ is small, and $p \in S(B)$. Then $p$ does not fork over $A$ if and only if $p$ has a large definition over $\mathrm{acl}^{eq}(A)$.*

*Proof.* Suppose that $p$ does not fork over $A$. Then using Lemma 4.7 there is some global extension $p'$ of $p$ such that $p'$ does not fork over $A$. Now let $M$ be some model containing $A$, then $p'$ does not fork over $M$, and hence, by Theorem 6.10, it is definable over $M$. This means that it is also definable over $M^{eq}$, and since this is a definably closed set, we see that $\mathrm{cb}(p') \subseteq M^{eq}$. And hence we see that

$$\mathrm{cb}(p') \subseteq \bigcap_{M \supseteq A} M^{eq} = \mathrm{acl}^{eq}(A)$$

where the equality is due to Lemma 6.4. So we see that $p'$ is $\mathrm{acl}^{eq}(A)$-definable, so it has a definition schema $D$ over $\mathrm{acl}^{eq}(A)$. This schema is automatically a definition schema for $p$, and hence it is a large definition schema for $p$.

Now suppose $p$ has a large definition schema over $\mathrm{acl}^{eq}(A)$. Then there is some global extension $p'$ of $p$ which is $\mathrm{acl}^{eq}(A)$-definable. Hence using Exercises 22 and 20, we see that $p'$ does not fork over $\mathrm{acl}^{eq}(A)$, and hence $p'$ also doesn't fork over $A$. And since $p'$ is an extension of $p$, we see that $p$ also doesn't fork over $A$. □

**Lemma 6.15.** *Let $T$ be a stable theory and let $p(x)$ and $q(y)$ be global types. Now let $\phi(x,y) \in \mathcal{L}$ be a formula such that there are definitions $d_p\phi(y)$ and $d_q\phi(x)$ for $\phi$ such that for all $a, b \in \mathbb{M}$ we have: $\models d_p\phi(b)$ if and only if $\phi(x,b) \in p(x)$, and $\models d_q\phi(a)$ if and only if $\phi(a,y) \in q(y)$. Then $d_p\phi(y) \in q$ if and only if $d_q\phi(x) \in p$.*

*Proof.* First notice that since $T$ is stable, there are small sets such that $p$ and $q$ don't fork over these sets. It follows from Theorem 6.10 that there are small models over which they are definable. We can take the union of these to find a small set over which they are both definable. So let $A$ be some small set of parameters such that $p$ and $q$ are both definable over $A$. We will build a sequence $(a_i, b_i)_{i\in\omega}$ recursively. First we set $b_0 \models q|_A$ and $a_0 \models p|_{Ab_0}$. We can always choose such elements because the restrictions of $p$ and $q$ are types over small sets of parameters, and hence these restrictions are realized by saturation. Now given $(a_i, b_i)_{i<n}$, we choose $b_n \models q|_{Aa_0 \ldots a_{n-1}}$ and $a_n \models p|_{Ab_0 \ldots b_n}$. Now suppose $i < j$, then we see:

$$\begin{aligned} \models \phi(a_i, b_j) &\Leftrightarrow \phi(a_i, y) \in q \\ &\Leftrightarrow \models d_q\phi(a_i) \\ &\Leftrightarrow d_q\phi(x) \in p. \end{aligned}$$

And if $i \geq j$, then we see:

$$\begin{aligned} \models \phi(a_i, b_j) &\Leftrightarrow \phi(x, b_j) \in p \\ &\Leftrightarrow \models d_p\phi(b_j) \\ &\Leftrightarrow d_p\phi(y) \in q. \end{aligned}$$

Now since $T$ is stable we know that $\phi$ is stable, so it cannot be the case that $\models \phi(a_i, b_j)$ if and only if $i < j$. So suppose $d_p\phi(y) \in q(y)$, then $\models \phi(a_i, b_j)$ for all $i \geq j$. And by stability this means that there must be some $i < j$ such that $\models \phi(a_i, b_j)$, and hence we see that $d_q\phi(x) \in p(x)$. The other direction is analogous. □

**Definition 6.16.** Let $A$ be a set of parameters and $p \in S(A)$. Then we say that $p$ is *stationary* if it has a unique global extension $q$ such that $q$ does not fork over $A$.

**Lemma 6.17.** *Let $T$ be a stable theory and let $A \subseteq \mathbb{M}$ be small such that $A = \mathrm{acl}^{eq}(A)$. Then every $p \in S(A)$ is stationary.*

*Proof.* Let $p \in S(A)$ be such a type. In order to prove that $p$ is stationary, we need to prove existence and uniqueness of such a global non-forking extension. For existence, we notice that since $T$ is stable, it is simple, and hence there is some $A_0 \subseteq A$ such that $p$ does not divide over $A_0$, and since dividing and forking are equivalent in these theories, we see that $p$ does not fork over $A_0$. So according to Lemma 4.7 there is some global extension $q$ of $p$ such that $q$ does not fork over $A_0$, and hence $q$ does not fork over $A$.

To prove uniqueness, suppose $p', p''$ are two such types, let $\phi(x, b) \in \mathcal{L}(\mathbb{M})$ be any formula, and let $q$ be a global non-forking extension of $\mathrm{tp}(b/A)$. Then by Lemma 6.14 and the assumption that $A = \mathrm{acl}^{eq}(A)$ the types $p', p''$ and $q$ are all definable over $A$. Then using

Lemma 6.15 and the fact that $d_q\phi(x) \in \mathcal{L}(A)$ we now see:

$$\begin{aligned}
\phi(x,b) \in p' &\Leftrightarrow \models d_{p'}\phi(b)\\
&\Leftrightarrow d_{p'}\phi(y) \in q\\
&\Leftrightarrow d_q\phi(x) \in p'\\
&\Leftrightarrow d_q\phi(x) \in p\\
&\Leftrightarrow d_q\phi(x) \in p''\\
&\Leftrightarrow d_{p''}\phi(y) \in q\\
&\Leftrightarrow \models d_{p''}\phi(b)\\
&\Leftrightarrow \phi(x,b) \in p''.
\end{aligned}$$

And hence we see that $p' = p''$. □

In order to be able to use this lemma, we will in the rest of this section assume that $T$ has elimination of imaginaries.

**Definition 6.18.** We define the notion of *forking independence*, denoted by $\mathop{\smile\!\!\!\!\!\!|}$, as follows: if $a, b, c$ are small, then $a \mathop{\smile\!\!\!\!\!\!|}_c b$ if $\mathrm{tp}(a/bc)$ does not fork over $c$.

The properties of forking independence depend on the theory that we are working in. So we will start with the properties that are valid in any theory, and then take a look at the more specialized cases of simple and stable theories.

**Theorem 6.19.** *Let $T$ be an arbitrary theory. Then forking independence satisfies the following properties:*

1. *(Invariance): If $f \in \mathrm{Aut}(\mathbb{M})$ then for any $a, b, c$, we have $a \mathop{\smile\!\!\!\!\!\!|}_c b$ if and only if $f(a) \mathop{\smile\!\!\!\!\!\!|}_{f(c)} f(b)$.*
2. *(Finite character): For any $a, b, c$, if $a \not\mathop{\smile\!\!\!\!\!\!|}_c b$ then there are some finite $a' \subseteq a$ and $b' \subseteq b$ such that $a' \not\mathop{\smile\!\!\!\!\!\!|}_c b'$.*
3. *(Monotonicity): For any $a, a', b, b', c$ if $aa' \mathop{\smile\!\!\!\!\!\!|}_c bb'$, then $a \mathop{\smile\!\!\!\!\!\!|}_c b$.*
4. *(Base monotonicity): For any $a, b, b', c$ if $a \mathop{\smile\!\!\!\!\!\!|}_c bb'$, then $a \mathop{\smile\!\!\!\!\!\!|}_{cb'} b$.*
5. *(Right extension): For any $a, b, c$ if $a \mathop{\smile\!\!\!\!\!\!|}_c b$ then for every $d$ there is some $d' \equiv_{bc} d$ such that $a \mathop{\smile\!\!\!\!\!\!|}_c bd'$.*
6. *(Left transitivity): For any $a, a', b, c$ if $a \mathop{\smile\!\!\!\!\!\!|}_c b$ and $a' \mathop{\smile\!\!\!\!\!\!|}_{ac} b$, then $aa' \mathop{\smile\!\!\!\!\!\!|}_c b$.*

*Proof.* 1. Suppose $a \not\mathop{\smile\!\!\!\!\!\!|}_c b$, so $\mathrm{tp}(a/bc)$ forks over $c$. Then there is some formula $\phi(x,b,c) \in \mathrm{tp}(a/bc)$ such that $\phi$ forks over $c$. So there are $\psi_0(x,d_0), ..., \psi_n(x,d_n)$ all dividing over $c$ such that $\phi(x,b,c) \models \bigvee_{i\leq n} \psi_i(x,d_i)$. So now we see that if $f \in \mathrm{Aut}(\mathbb{M})$ then $\models \phi(f(a), f(b), f(c))$ since $\models \phi(a,b,c)$. And we also see that $\phi(x, f(b), f(c)) \models \bigvee_{i\leq n} \psi_i(x, f(d_i))$, and we know that the $\psi_i(x, f(d_i))$ all divide over $f(c)$. So we conclude that $\phi(x, f(b), f(c))$ forks over $f(c)$, so $\mathrm{tp}(f(a)/f(b)f(c))$ forks over $f(c)$, and hence $f(a) \not\mathop{\smile\!\!\!\!\!\!|}_{f(c)} f(b)$. So this means that if $f(a) \mathop{\smile\!\!\!\!\!\!|}_{f(c)} f(b)$, then $a \mathop{\smile\!\!\!\!\!\!|}_c b$. The other direction is clear, because from the previous result we see that if $f(a) \not\mathop{\smile\!\!\!\!\!\!|}_{f(c)} f(b)$, then by taking $f^{-1} \in \mathrm{Aut}(\mathbb{M})$ we have $a \not\mathop{\smile\!\!\!\!\!\!|}_c b$. So we conclude that if $a \mathop{\smile\!\!\!\!\!\!|}_c b$, then

$f(a) \mathop{\smile\!\!\!\!\!|}_{f(c)} f(b)$. So we indeed have for any $f \in \mathrm{Aut}(\mathbb{M})$ that $a \mathop{\smile\!\!\!\!\!|}_c b$ if and only if $f(a) \mathop{\smile\!\!\!\!\!|}_{f(c)} f(b)$.

2. Suppose $a \not\mathop{\smile\!\!\!\!\!|}_c b$, then $\mathrm{tp}(a/bc)$ forks over $c$, so there is some formula in this type which forks over $c$. Since a formula contains only a finite number of variables and parameters we see that this means that there are some finite $a' \subseteq a$ and $b' \subseteq b$ such that there is some $\phi(x, b', c) \in \mathrm{tp}(a/bc)$ such that $\models \phi(a', b', c)$ and such that $\phi(x, b', c)$ forks over $c$. So $\mathrm{tp}(a'/b'c)$ forks over $c$, and hence $a' \not\mathop{\smile\!\!\!\!\!|}_c b'$.

3. Suppose that $a \not\mathop{\smile\!\!\!\!\!|}_c b$, so $\mathrm{tp}(a/bc)$ forks over $c$, then it contains some formula $\phi(x, b, c)$ which forks over $c$. Now define the formula $\psi(x, y, b, c) := \phi(x, b, c)$, then we see that $\psi(x, y, b, c) \in \mathrm{tp}(aa'/bb'c)$ and we clearly see that it forks over $c$. So we conclude that $\mathrm{tp}(aa'/bb'c)$ forks over $c$, so $aa' \not\mathop{\smile\!\!\!\!\!|}_c bb'$. So we conclude that if $aa' \mathop{\smile\!\!\!\!\!|}_c bb'$, then $a \mathop{\smile\!\!\!\!\!|}_c b$.

4. If $a \mathop{\smile\!\!\!\!\!|}_c bb'$, then $\mathrm{tp}(a/bb'c)$ does not fork over $c$, and hence it also does not fork over $cb'$, and hence we see that $a \mathop{\smile\!\!\!\!\!|}_{cb'} b$.

5. Suppose $a \mathop{\smile\!\!\!\!\!|}_c b$, so $\mathrm{tp}(a/bc)$ does not fork over $c$. Let $M$ be a small $|cb|^+$-saturated model containing $cb$. According to Lemma 4.7 there is some extension $p \in S(M)$ of $\mathrm{tp}(a/bc)$ which does not fork over $c$. Now let $d$ be arbitrary, then because $M$ is saturated enough, we see that $\mathrm{tp}(d/bc)$ is realized by some $d' \in M$. Now since $M$ is small we know that $p$ is realized by some $a' \in M$, so $a' \equiv_{bc} a$ and $a' \mathop{\smile\!\!\!\!\!|}_c M$. So with monotonicity it follows that $a' \mathop{\smile\!\!\!\!\!|}_c bd'$. Now since $a' \equiv_{bc} a$ we can take some $f \in \mathrm{Aut}(\mathbb{M}/bc)$ such that $f(a') = a$. With invariance it follows that $a \mathop{\smile\!\!\!\!\!|}_c bf(d')$. Since $f(d') \equiv_{bc} d' \equiv_{bc} d$, the desired result follows.

6. First we define $a \mathop{\smile\!\!\!\!\!|}_c^d b$ to mean that $\mathrm{tp}(a/bc)$ does not divide over $c$. Now suppose $a \mathop{\smile\!\!\!\!\!|}_c^d b$ and $a' \mathop{\smile\!\!\!\!\!|}_{ac}^d b$. This means that $\mathrm{tp}(a/bc)$ does not divide over $c$ and $\mathrm{tp}(a'/bac)$ does not divide over $ac$. Now it follows from Corollary 4.5 that $\mathrm{tp}(aa'/bc)$ does not divide over $c$, and hence $aa' \mathop{\smile\!\!\!\!\!|}_c^d b$. So the result is valid if we replace $\mathop{\smile\!\!\!\!\!|}_c$ by $\mathop{\smile\!\!\!\!\!|}_c^d$.

   Now suppose $M_1$ is a small model of $T$ such that $bc \subseteq M_1$ and such that $M_1$ contains every definable set which is defined by a formula forking over $c$. Now using Lemma 4.7, there is a nonforking extension $\mathrm{tp}(a_1/M_1)$ of $\mathrm{tp}(a/bc)$, so $a_1 \mathop{\smile\!\!\!\!\!|}_c M_1$. Now using invariance and the fact that $a_1 \equiv_{bc} a$, there is some $f \in \mathrm{Aut}(\mathbb{M}/bc)$ such that $a \mathop{\smile\!\!\!\!\!|}_c f(M_1)$. Now using right extension there is some $M_2 \equiv_{abc} f(M_1)$ such that $a' \mathop{\smile\!\!\!\!\!|}_{ac} bM_2$, and since $b \in M_2$ we find that $a' \mathop{\smile\!\!\!\!\!|}_{ac} M_2$. Now since $a \mathop{\smile\!\!\!\!\!|}_c f(M_1)$ and $f(M_1) \equiv_{abc} M_2$ we see with invariance that $a \mathop{\smile\!\!\!\!\!|}_c M_2$. So we now have $a \mathop{\smile\!\!\!\!\!|}_c M_2$ and $a' \mathop{\smile\!\!\!\!\!|}_{ac} M_2$. So it follows that $a \mathop{\smile\!\!\!\!\!|}_c^d M_2$ and $a' \mathop{\smile\!\!\!\!\!|}_{ac}^d M_2$, and hence we see that $aa' \mathop{\smile\!\!\!\!\!|}_c^d M_2$. By our choice of $M_1$ (containing every set which forks over $c$) and the fact that $M_2 \equiv_c M_1$, we conclude that $aa' \mathop{\smile\!\!\!\!\!|}_c M_2$, and hence we see that $aa' \mathop{\smile\!\!\!\!\!|}_c b$. □

We can of course specialize by making assumptions on our theory. For example, we can prove a lot more if our theory is simple:

**Exercise 32.** Let $T$ be a simple theory. Prove that $\mathop{\smile\!\!\!\!\!|}_c$ satisfies the following properties:

1. Local character: For any $a, b$ there is some $c \subseteq b$ such that $|c| \leq |T|$ and such that $a \mathop{\smile\!\!\!\!\!|}_c b$.

2. Symmetry: For any $a, b, c$ we have $a \mathop{\smile\!\!\!\!\!\!|}_c b$ if and only if $b \mathop{\smile\!\!\!\!\!\!|}_c a$.
3. Left extension: For any $a, b, c$ if $a \mathop{\smile\!\!\!\!\!\!|}_c b$ then for every $d$ there is some $d' \equiv_{ac} d$ such that $ad' \mathop{\smile\!\!\!\!\!\!|}_c b$.
4. Right transitivity: For any $a, b, b', c$, if $a \mathop{\smile\!\!\!\!\!\!|}_c b$ and $a \mathop{\smile\!\!\!\!\!\!|}_{bc} b'$, then $a \mathop{\smile\!\!\!\!\!\!|}_c bb'$.

We can specialize even further by assuming that our theory is stable.

**Theorem 6.20.** *Let $T$ be a stable theory. Then forking satisfies the following properties:*

1. *(Conjugacy): Let $A$ be a small set of parameters and let $p \in S(A)$. Then any two global non-forking extensions of $p$ are* conjugate *over $A$ (this means that if $p_1, p_2$ are these extensions, then there is some $f \in \mathrm{Aut}(\mathbb{M}/A)$ such that $f(p_1) = p_2$.*
2. *(Boundedness): Let $A$ be a small set of parameters and let $p \in S(A)$. Then $p$ has at most $2^{|T|}$ global non-forking extensions.*

*Proof.* 1. Suppose $p_1$ and $p_2$ are two global non-forking extensions of $p$. Then we notice that $p_1|_{\mathrm{acl}(A)}$ and $p_2|_{\mathrm{acl}(A)}$ are conjugate over $A$. So there is some $f \in \mathrm{Aut}(\mathbb{M}/A)$ such that $f(p_1|_{\mathrm{acl}(A)}) = p_2|_{\mathrm{acl}(A)}$. However, we know from Lemma 6.17 that $p_1|_{\mathrm{acl}(A)}$ and $p_2|_{\mathrm{acl}(A)}$ are both stationary, meaning that $p_1$ and $p_2$ are there unique global non-forking extensions. However, we notice that $f(p_1)$ is a global non-forking extension of $p_2|_{\mathrm{acl}(A)}$, meaning that $f(p_1) = p_2$. So $p_1$ and $p_2$ are conjugate over $A$.

2. Since $T$ is stable, it is in particular simple. This means that their is some $A_0 \subseteq A$ with $|A_0| \leq |T|$ such that $p$ does not fork over $A_0$. Now consider a global extension $p'$ of $p$. Then we see that $p'$ is in particular an extension of $p|_{\mathrm{acl}(A_0)}$, which is a stationary type. So the number of global extensions of $p$ is at most the number of extensions of $p|_{A_0}$ to $\mathrm{acl}(A_0)$. Now since $T$ is complete and $|A_0| \leq |T|$, we see that $|\mathcal{L}(\mathrm{acl}(A_0))| \leq |T|$, and since the number of extensions is at most $2^{|\mathcal{L}(\mathrm{acl}(A_0))|}$, we see that this number of extensions is indeed at most $2^{|T|}$. □

## 6.4 Forking and ranks

In this section, we will see how under the assumption of stability we can use the $\Delta$-rank of sets of formulas to determine when types are forking over algebraically closed sets. We will also be able to conclude that we can use the Morley-rank to determine when types are forking over arbitrary sets. The material in this section was taken from [15]. In order to prove our main result, we first have to take another look at definability of types in stable theories. We know from Theorem 3.20 that all types are definable in stable theories. However, we want to know a little bit about the formulas defining them.

**Lemma 6.21.** *Let $T$ be a stable theory, and let $M$ be a small model of $T$. Let $p(x) \in S(M)$ and let $\phi(x, y)$ be some $\mathcal{L}$-formula. Then $\phi(x, y)$ has a defining formula $d_p\phi(y)$ for $p(x)$, which is a positive Boolean combination of formulas of the form $\phi(c, y)$, with $c \in M$.*

*Proof.* Let $c'$ be an element realizing $p(x)$. Note that if $\models \phi(c, m)$ for all $m \in M$, then we take an empty conjunction as defining formula. We can take the empty disjunction if $\models \neg\phi(c, m)$ for all $m \in M$. This means we may assume that there are $d, e \in M$ such that $\models \phi(c', d) \wedge \neg\phi(c', e)$. Now we will define the following:

- A sequence of elements $(c_i)_{i\in\omega}$ in $M$;
- For every $i \in \omega \cup \{-1\}$ two sets $K_i, L_i \subseteq \mathcal{P}(\{0, ..., i\})$;
- An element $a^s_{i+1}$ in $M$ for every $i \in \omega \cup \{-1\}$ and $s \in K_i$;
- An element $b^t_{i+1}$ in $M$ for every $i \in \omega \cup \{-1\}$ and $t \in L_i$.

Here we have obviously added $-1$ to $\omega$ as an element with the property that $-1 + 1 = 0$. We now make our construction as follows: first let $c_0$ be arbitrary, and set $K_{-1} = L_{-1} = \{\emptyset\}$. Now suppose that for some $n \in \omega$, the elements $c_0, ..., c_n$, the sets $K_{-1}, ..., K_{n-1}$, the sets $L_{-1}, ..., L_{n-1}$, the elements $a^s_i$ (for all $i \in \{0, ..., n\}$ and $s \in K_i$) and the elements $b^t_i$ (for all $i \in \{0, ..., n\}$ and $t \in L_i$) have already been defined. We now proceed as follows:

- We define
$$K_n = \{s \subseteq \{0, ..., n\} \mid \exists a \in M \forall j \in s\ ( \models \phi(c_j, a) \wedge \neg\phi(c', a))\}.$$
- If $s \in K_n$, then we let $a^s_{n+1}$ be an element such that for all $j \in s$, we have $\models \phi(c_j, a^s_{n+1}) \wedge \neg\phi(c', a^s_{n+1})$. We let $A_n = \{a^s_{i+1} \mid i \leq n,\ s \in K_i\}$.
- We define
$$L_n = \{t \subseteq \{0, ..., n\} \mid \exists b \in M \forall j \in t\ ( \models \neg\phi(c_j, b) \wedge \phi(c', b))\}.$$
- If $t \in L_n$, then we let $b^t_{n+1}$ be an element such that for all $j \in t$, we have $\models \neg\phi(c_j, b^t_{n+1}) \wedge \phi(c', b^t_{n+1})$. We let $B_n = \{b^t_{i+1} \mid i \leq n,\ t \in L_i\}$.

Now we notice that $A_n \cup B_n$ is a finite subset of $M$. We claim that there is some $c \in M$ such that for all $d \in A_n \cup B_n$, we have $\models \phi(c, d)$ if and only if $\models \phi(c', d)$. To see this, consider the formula $\chi(x)$, defined by
$$\bigwedge_{b\in B_n} \phi(x, b) \wedge \bigwedge_{a\in A_n} \neg\phi(x, a).$$
We know that $\models \chi(c')$, and hence $\models \exists x\chi(x)$. So $M \models \exists x\chi(x)$, so there is some $c \in M$ such that $\models \chi(c)$. This $c$ satisfies, so we choose $c_{n+1}$ to be such an element in $M$.

We now claim that if we have natural numbers $i_0 < ... < i_n$ for some $n \in \omega$, and there is some $a \in M$ such that
$$\models \phi(c_{i_0}, a) \wedge ... \wedge \phi(c_{i_n}, a) \wedge \neg\phi(c', a),$$
then there are $d_0, ..., d_n \in M$ such that for all $j, r$ with $0 \leq j, r \leq n$ we have $\models \phi(c_{i_j}, d_r)$ if and only if $j < r$.

To prove this claim, we let $d_0 = a^s_{i_0}$ for some arbitrary $s \in K_{i_0-1}$. Then we know that if $0 \leq j \leq n$, then $\models \neg\phi(c_{i_j}, d_0)$. Now suppose $0 \leq k < n$. We know by assumption that the set $s = \{i_0, ..., i_k\} \in K_{i_k}$. Now we let $d_{k+1} = a^s_{i_k+1}$. Then we notice that $\models \phi(c_{i_0}, d_{k+1}) \wedge ... \wedge \phi(c_{i_k}, d_{k+1})$ and $\models \neg\phi(c_{i_j}, d_{k+1})$ if $k+1 \leq j \leq n$. And hence we have indeed constructed the desired elements $d_0, ..., d_n$.

In the same way we can prove that if we have $i_0 < ... < i_n \in \omega$ and for some $b \in M$ we have
$$\models \neg\phi(c_{i_0}, b) \wedge ... \wedge \neg\phi(c_{i_n}, b) \wedge \phi(c', b),$$

then there are $e_0, ..., e_n \in M$ such that for all $j, r$ with $0 \le j, r \le n$ we have $\models \neg\phi(c_{i_j}, e_r)$ if and only if $j < r$.

It follows from the first of these two observations that if $a \in M$ and $n \in \omega$, and $s \subset \{0, ..., 2n\}$ with $|s| = n$, and we have $\models \phi(c_i, a)$ for all $i \in s$, then $\models \phi(c', a)$. And in the same way (from the second observation) we see that if we have some $b \in M$ and some $t \subset \{0, ..., 2n\}$ with $|t| = n$, and we have $\models \neg\phi(c_i, b)$ for all $i \in t$, then $\models \neg\phi(c', b)$. Now we fix $n \in \omega$ to be a natural number such that there are no sequences $(a_i)_{i \le n}$ and $(b_i)_{i \le n}$ such that $\models \phi(a_i, b_j)$ if and only if $i < j$ for all $i, j \le n$.

We find that for any $a \in M$, the formula $\phi(x, a)$ is in $p(x)$ if and only if

$$\models \bigvee \left\{ \bigwedge \{\phi(c_i, a) \mid i \in s\} \mid s \subset \{0, ..., 2n\},\ |s| = n \right\}.$$

This gives us the desired formula $d_p\phi$. □

*Remark.* In the case of global types, this can be strengthened a little. If $p(x) \in S(\mathbb{M})$ and $A$ is a small set of parameters, then there is a sequence $(c_i)_{i \in \omega}$ such that for every $i \in \omega$, the element $c_{i+1}$ realizes $p|_{A \cup \{c_0, ..., c_i\}}$, and the formula $d_p\phi(y)$ is a positive Boolean combination of formulas of the form $\phi(c_i, y)$. This can be done by being careful about what elements $c_i$ are being picked, and by using saturation of the monster model.

Now we are ready to see how we can use the $\Delta$-rank to determine forking.

**Theorem 6.22.** *Let $T$ be a stable theory, $\Delta$ a finite set of formulas and $A$ an algebraically closed set. If $A \subseteq B$ and $p(x) \in S_\Delta(B)$, the $p(x)$ forks over $A$ if and only if $R_\Delta(p(x)) < R_\Delta(p|_A(x))$.*

*Proof.* Suppose $p(x)$ does not fork over $A$. From the definition of the $\Delta$-rank applied to complete types, we see that there must be some formula $\phi(x, b) \in p(x)$ such that $R_\Delta(\phi(x, b)) = R_\Delta(p(x))$. And since $p$ doesn't fork over $A$, we know that $\phi$ doesn't fork over $A$.

Now consider $\mathrm{tp}(b/A)$, which clearly doesn't fork over $A$ (since no type forks over its domain). Now since $A$ is algebraically closed, we see that this type has a large definition over $A$, by Lemma 6.14. So let $q(y) \in S(M)$ be an $A$-definable type over a small $|A|^+$-saturated model $M \supseteq A$ extending $\mathrm{tp}(b/A)$. Clearly $q(y)$ doesn't fork over $A$. Consider the $\mathcal{L}(A)$-formula $d_q\phi^*(x)$, where $\phi^*(y, x)$ is $\phi(x, y)$. By Lemma 6.21 and its remark, there is some $A$-indiscernible sequence $(c_i)_{i \in \omega}$ in $M$ such that $d_q\phi^*(x)$ is a positive Boolean combination of formulas of the form $\phi(x, c_i)$.

Now we notice that by Lemma 4.7, the type $p(x)$ can be extended to a type $p'(x) \in S_\Delta(M)$ which doesn't fork over $A$. Now we consider the formula $d_{p'}\phi(y)$. Since this formula is a positive Boolean combination of formulas of the form $\phi^*(d_i, y)$, and $\models d_{p'}\phi^*(b)$. We conclude that $d_{p'}\phi(y) \in q(y)$, and with Lemma 6.15, we see that $d_q\phi^*(x) \in p'(x)$. And since this is an $\mathcal{L}(A)$-formula, we see that $d_q\phi^*(x) \in p|_A(x)$. So there is some formula in $p|_A(x)$ which is equivalent to a positive Boolean combination of formulas of the form $\phi(x, c_i)$, where every $c_i$ has the same type over $A$ as $b$. Now it follows from Lemma 3.17 that

$$R_\Delta(p(x)) \le R_\Delta(p|_A(x)) \le R_\Delta(d_q\phi^*(x)) \le R_\Delta(\phi(x, b)) = R_\Delta(p(x))$$

So we indeed conclude that $R_\Delta(p(x)) = R_\Delta(p|_A(x))$.

Now for the other direction, suppose $p(x)$ does fork over $A$. Let $p'(x) \in S_\Delta(\mathbb{M})$ extend $p$ such that $p'$ doesn't fork over $B$. Then we know from the first part of this proof that

$R_\Delta(p) = R_\Delta(p')$. We also note that $p'$ forks over $A$, since $p$ does. This means that there is some formula $\phi(x, b) \in p'(x)$ which divides over $A$. From the definition of dividing, this means that the type $p'$ has infinitely many $A$-conjugates. Now we choose a pairwise inconsistent set of formulas from these $A$-conjugates of $p'$, and conclude from Lemma 3.17 part 4 that $R_\Delta(p') > R_\Delta(p|_A)$. □

If we are not considering local ranks but the Morley rank (so the $\Delta$-rank but with $\Delta$ being the set of all formulas), then we get an even better result. For in this case, we don't need $A$ to be algebraically closed. We can consider not the $\mathcal{L}(\mathrm{acl}(A))$-formula $d_q\phi^*(y)$, but the $A$-definable formula defined as the disjunction of the finitely many $A$-conjugates of $d_q\phi^*(y)$. This proof would not work in the case of local ranks, because this might no longer be an element of $p$, since the elements in $\mathrm{acl}(A)$ might not be given by the solutions of formulas in $\Delta$.

*Chapter 7*

# Stable groups

Stability theory has proven its worth with some applications to algebraic geometry. An application of model theory to algebra is via the concept of *stable groups*. This is currently an active area of research. In this chapter, we will give a short introduction to this concept.

## 7.1 Chain conditions

In this section, we will define stable groups, and take a look at the interplay between groups and model theory.

**Definition 7.1.** Let $T$ be a stable theory and let $G$ be a definable subset of $\mathbb{M}$. Then we say that $G$ is a *stable group* if there is some definable operation $\cdot : G \times G \to G$ such that $(G, \cdot)$ satisfies the axioms of a group.

As an easy example, we have seen that the theory of algebraically closed fields of characteristic 0 is a stable theory. Clearly, the entire monster model is a definable additive group, so this is an example of a stable group.

**Definition 7.2.** Let $G$ be a stable group. Then a *uniformly defined family of subgroups* of $G$ is a family of subgroups $(H_i)_{i \in I}$ (with $I$ some index set) of $G$ such that there is some $\mathcal{L}$-formula $\phi(x, y)$ and some family of parameters $(a_i)_{i \in I}$ such that for every $i \in I$: $H_i = \phi(\mathbb{M}, a_i)$.

**Lemma 7.3.** *Let $G$ be a definable group in a theory which is in NSOP. If $H_0 \subseteq H_1 \subseteq H_2 \subseteq ...$ is a chain of subgroups of $G$ uniformly defined by the formula $\phi(x, y)$, then there is some $n \in \omega$ such that $H_n = H_{n+1} = H_{n+2} = ....$*

*Proof.* Otherwise, the formula $\phi(x, y)$ would clearly have the strict order property. □

**Lemma 7.4.** *Let $G$ be a definable group in a theory which is in NIP. Then for every formula $\phi(x, y)$ there is some natural number $m$ such that if $I$ is a* finite *index set and $(H_i)_{i \in I}$ is a family of subgroups of $G$ uniformly defined by $\phi(x, y)$, then there is some $J \subseteq I$ such that $|J| \leq m$ and such that $\bigcap_{i \in I} H_i = \bigcap_{j \in J} H_j$.*

*Proof.* Suppose this is not the case. Then for every natural number $m$ there is a family of subgroups $(H_i)_{i \leq m}$ and parameters $(b_i)_{i \leq m}$ such that for every $i \leq m$ we have $H_i = \phi(\mathbb{M}, b_i)$

and $\bigcap_{j\le m} H_j \subsetneq \bigcap_{j\le m,\ j\neq i} H_j$. Now we can use the compactness theorem to show that from this it follows that $\phi(x,y)$ has the independence property, which would yield a contradiction since our theory is in NIP. So we add to our language constants $(a_s)_{s\subseteq\omega}$ and $(c_i)_{i\in\omega}$, and we add axioms $\phi(a_s,c_i)$ whenever $i\in s$ and $\neg\phi(a_s,c_i)$ whenever $i\notin s$. Now suppose that we have a finite subtheory, so we only have a finite set of these axioms, say that we only use the constants $c_0,...,c_m$. Now notice that there is a family of subgroups $(H_i)_{i\le m}$ and parameters $(b_i)_{i\le m}$ such that for every $i\le m$ we have $H_i=\phi(x,b_i)$ and $\bigcap_{j\le m} H_j \subsetneq \bigcap_{j\le m,\ j\neq i} H_j$. So we interpret $c_i=b_i$ for every $i\le m$. For every $i\le m$, there is some element $h_i\in G$ such that $h_i\in\bigcap_{j\le m,\ j\neq i} H_j$ but $h_i\notin\bigcap_{j\le m} H_j$. Now we interpret $a_s=\prod_{i\notin s,\ i\le m} h_i$, where the product is the group-theoretic product. Now suppose that $i\in s$. Then we see that if $j\notin s$ and $j\le m$, then $h_j\in H_i$, and hence we see that $\models\phi(h_j,b_i)$ for all of these elements $h_j$. And since $\phi(\mathbb{M},b_i)$ is a group and is therefore closed under multiplication we see that $a_s\in H_i$. So if $i\in s$, then $\models\phi(a_s,b_i)$. Now suppose that $i\notin s$. Then we see that if $a_s\in H_i$, then $\prod_{j\notin s,\ j\le m} h_j\in H_i$. We also know that $h_j\in H_i$ if $i\neq j$. This means that $h_i\in H_i$, because the product itself and all the other terms of the product are in $H_i$. But we know that $h_i\notin H_i$, and hence we conclude that $a_s\notin H_i$, so if $i\notin s$, then $\models\neg\phi(a_s,b_i)$.

So with compactness we conclude that there are $(a_s)_{s\subseteq\omega}$ and $(c_i)_{i\in\omega}$ such that $\models\phi(a_s,c_i)$ if and only if $i\in s$, hence we conclude that $\phi(x,y)$ has the independence property. Since we assumed that $T$ is in NIP, we conclude that there is some natural number $m$ such that if $I$ is a finite index set and $(H_i)_{i\in I}$ is a family of subgroups of $G$ uniformly defined by $\phi(x,y)$, then there is some $J\subseteq I$ such that $|J|\le m$ and such that $\bigcap_{i\in I} H_i=\bigcap_{j\in J} H_j$. □

**Theorem 7.5.** *Let $G$ be a stable group. Then for every formula $\phi(x,y)$ there is some natural number $k$ such that any descending chain of intersections of $\phi$-definable groups has length at most $k$.*

*Proof.* Since $G$ is a stable group and stable implies NIP, we can use Lemma 7.4. So consider an element of such a descending chain, which is an intersection of $\phi$-definable groups. By Lemma 7.4, these have to be finite intersections, since there is some $m\in\omega$ such that all finite intersections are at most $m$ big. But if we have an infinite intersection which is not at most $m$ big, then we can also make arbitrarily large finite intersections. Now since all the elements in this chain are at most $m$ large intersections, they are themselves uniformly definable by an $m$-large conjunction over formulas $\phi(x,y)$. Now it follows from Lemma 7.3 that such a chain has bounded length. □

This model-theoretic result about stable groups can be used to show a purely group-theoretic result about stable groups, which is left as an exercise.

**Exercise 33.** Let $G$ be a stable group and let $A\subseteq G$ be a subset of $G$. Then there is some finite $B\subseteq A$ such that $A$ and $B$ have the same centralizer (so $\{g\in G\mid\forall a\in A(g\cdot a=a\cdot g)\}=\{g\in G\mid\forall b\in B(g\cdot b=b\cdot g)\}$).

**Theorem 7.6.** *Let $G$ be a stable group and let $H\le G$ be an Abelian subgroup of $G$. Then there is some* definable *Abelian subgroup $H'\le G$ such that $H\le H'$.*

*Proof.* Consider the centralizer of $H$:

$$C_G(H)=\{g\in G\mid\forall h\in H(g\cdot h=h\cdot g)\}.$$

Since $H$ is an Abelian subgroup of $G$, we know that $H \leq C_G(H)$. Now we let $H'$ be the center of $C_G(H)$, so

$$H' = Z(C_G(H)) = \{g \in C_G(H) \mid \forall h \in C_G(H)(g \cdot h = h \cdot g)\}.$$

Suppose that we have some element $g \in H$, then $g \in C_G(H)$, and if $h \in C_G(H)$, then by definition of $C_G(H)$ we see that $g \cdot h = h \cdot g$, and hence $H$ is a subgroup of $Z(C_G(H))$. We also notice that $Z(C_G(H))$ is an Abelian subgroup of $G$, so if we can prove that $Z(C_G(A))$ is definable, we are done. For this, we notice that by Exercise 33, there is some finite $H_0 \subseteq H$ such that $C_G(H) = C_G(H_0)$. So we see that

$$\begin{aligned} Z(C_G(H)) &= Z(C_G(H_0)) \\ &= \{g \in C_G(H_0) \mid \forall h \in C_G(H_0)(g \cdot h = h \cdot g)\} \\ &= \{g \in G \mid g \in C_G(H_0) \ \wedge \ \forall h \in C_G(H_0)(g \cdot h = h \cdot g)\} \\ &= \{g \in G \mid \forall a \in H_0(g \cdot a = a \cdot g) \ \wedge \ \forall h(\forall b \in H_0(h \cdot b = b \cdot h) \to g \cdot h = h \cdot g)\} \\ &= \left\{ g \in G \mid \bigwedge_{a \in H_0} g \cdot a = a \cdot g \ \wedge \ \forall h \left( \left( \bigwedge_{b \in H_0} h \cdot b = b \cdot h \right) \to g \cdot h = h \cdot g \right) \right\}. \end{aligned}$$ □

## 7.2 Generics

Left- and right-genericity is a generalization of the notion of cosets. The goal of this section is to learn a little more about this notion. We have used [5] as a source here.

**Definition 7.7.** Let $G \subseteq \mathbb{M}$ be a stable group. A definable subset $A \subseteq \mathbb{M}$ is called *left-generic* if there is some $n \in \omega$ and there are some $g_0, ..., g_n \in G$ such that $G = \bigcup_{i \leq n} g_i \cdot (A \cap G)$. $A$ is called *right-generic* if there is some $n \in \omega$ and there are some $h_0, ..., h_n \in G$ such that $G = \bigcup_{i \leq n} (A \cap G) \cdot h_i$. $A$ is called *generic* if there is some $n \in \omega$ and there are some $g_0, ..., g_n, h_0, ..., h_n \in G$ such that $G = \bigcup_{i \leq n} g_i \cdot (A \cap G) \cdot h_i$.

**Exercise 34.** Let $A \subseteq \mathbb{M}$ be generic and $f \in \mathrm{Aut}(\mathbb{M})$. Show that $f(A)$ is also generic. Conclude that this is also the case for left-generic and right-generic.

**Lemma 7.8.** *Let $G$ be a stable group and let $A \subseteq \mathbb{M}$ be a definable set. Then either $A$ is left generic or $A^c$ is right-generic.*

*Proof.* Suppose that both is not the case. Since $A$ is not left-generic, we have for every $g_0, ..., g_n \in G$ some $g \in G$ such that $g \notin \bigcup_{i \leq n} g_i^{-1} \cdot (A \cap G)$, and hence we see that $g_i \cdot g \in A^c \cap G$ for every $i \leq n$. In much the same way, but using that $A^c$ is not right-generic, we see that for every $h_0, ..., h_n \in G$ there is some $h \in G$ such that $h \cdot h_i \in A \cap G$ for every $i \leq n$. Now we will inductively build a sequence $(a_i, b_i)_{i \in \omega}$ of pairs of elements of $G$ such that $b_n \cdot a_0, ..., b_n \cdot a_n \in A \cap G$ for every $n \in \omega$ and $b_0 \cdot a_{n+1}, ..., b_n \cdot a_{n+1} \in A^c \cap G$ for every $n \in \omega$. First choose $a_0$ arbitrarily and choose $b_0$ such that $b_0 \cdot a_0 \in A \cap G$. Now suppose that we already have $(a_i, b_i)_{i \leq n}$ for some $n \in \omega$. Now by the above considerations we see that we can choose $a_{n+1}$ such that $b_0 \cdot a_{n+1}, ..., b_n \cdot a_{n+1} \in A^c \cap G$, and $b_{n+1} \cdot a_0, ..., b_{n+1} \cdot a_{n+1} \in A \cap G$. Now that we have these elements, we see that if $i < j$, then $b_i \cdot a_j \notin A$. And if $i \geq j$, then $b_i \cdot a_j \in A$. But this means that $G$ is not stable, hence we have found a contradiction. We conclude that either $A$ is left-generic or $A^c$ is right-generic. □

**Lemma 7.9.** *Let $G$ be a stable group, then the definable sets that are not generic form an ideal in the family of definable sets.*

*Proof.* Suppose $B$ is definable but not generic and $A \subset B$ is definable. If $A$ would be generic, then there is some $n \in \omega$ and there are $g_0, ..., g_n, h_0, ..., h_n \in G$ such that $G = \bigcup_{i\leq n} g_i \cdot (A \cap G) \cdot h_i$. But since $A \subseteq B$ we notice that

$$G = \bigcup_{i\leq n} g_i \cdot (A \cap G) \cdot h_i \subseteq \bigcup_{i\leq n} g_i \cdot (B \cap G) \cdot h_i \subseteq G$$

and hence we conclude that if $A$ is generic then $B$ is also generic. So since $B$ is not generic, we conclude that $A$ is not generic.

Now suppose $A$ and $B$ are two definable sets such that $A \cup B$ is generic. We will prove that either $A$ is generic or $B$ is generic. So suppose that

$$G = \bigcup_{i\leq n} a_i((A \cup B) \cap G)b_i \subseteq \bigcup_{i\leq n} a_i(A \cap G)b_i \cup \bigcup_{i\leq n} a_i(B \cap G)b_i.$$

We know from Lemma 7.8 that either $\bigcup_{i\leq n} a_i(A \cap G)b_i$ is left-generic or its complement is right-generic. In the second case we see that $\left(\bigcup_{i\leq n} a_i(A \cap G)b_i\right)^c \cap G \subseteq \bigcup_{i\leq n} a_i(B \cap G)b_i$, and hence this set would also be right-generic. So we see that either $\bigcup_{i\leq n} a_i(A \cap G)b_i$ is left-generic or $\bigcup_{i\leq n} a_i(B \cap G)b_i$ is right-generic. In the first case we see that there is some $k \in \omega$ and there are some $g_0, ..., g_k \in G$ such that

$$G = \bigcup_{j\leq k} g_j \cdot \bigcup_{i\leq n} a_i(A \cap G)b_i \subseteq \bigcup_{j\leq k} \bigcup_{i\leq n} g_j \cdot a_i(A \cap G)b_i.$$

And hence we conclude from this that $A$ is generic. In much the same way we conclude that $B$ is generic if $\bigcup_{i\leq n} a_i(B \cap G)b_i$ is right-generic. And hence we conclude that if $A \cup B$ is generic, then either $A$ is generic or $B$ is generic.

Since the family of sets that are not generic is clearly non-empty (it contains the empty set, which is definable), we conclude that it is indeed an ideal. □

## 7.3 The Mordell-Lang conjecture

In this section, which is based on [2], we will quickly say something about one of the most striking applications of stability theory. In 1996, Hrushovski gave a proof of the Mordell-Lang conjecture for function fields in general characteristic, making heavy use of stable group theory. This proof is very involved, and requires not only knowledge of stable group theory, but also of algebraic geometry. Therefore, we shall not treat this proof, but we will state the theorem. However, even for this we will need a number of definitions from algebraic geometry. We advise the reader to take the appropriate time to let these definitions sink in.

**Definition 7.10.** Let $F$ be a field, $n \geq 1$ and let $A \subseteq F^n$. Then we say that $A$ is *Zariski closed* is there is a finite set of polynomials over $F$ such that $A$ is the intersection of the zero-sets of these polynomials.

It turns out (using Hilbert's basis theorem) that this notion of closed defines a topology, which is called the *Zariski topology*.

**Definition 7.11.** Let $K$ be an algebraically closed field. Then an *affine variety* is a Zariski closed subset of $K^n$, for some $n \geq 1$.

**Definition 7.12.** If $V \subseteq K^n$ and $W \subseteq K^m$ are affine varieties (with possibly $n \neq m$), then a *morphism* $f$ from $V$ to $W$ is a tuple of functions $f = (f_1, ..., f_m)$ such that for every $i \leq m$, the function $f_i : V \to K$ is an element of $K[V] = K[x_1, ..., x_n]/I(V)$, where $I(V)$ is the ideal generated by the polynomials which define $V$. If $f$ is bijective and its inverse is also a morphism, then $f$ is an *isomorphism*.

**Definition 7.13.** A *quasi-affine variety* is a Zariski open subset of an affine variety.

**Definition 7.14.** A *variety* is a set $V$ with a finite covering $V = \bigcup_{i \leq n} V_i$ such that for every $i \leq n$ there is some affine variety $U_i$ and a bijection $f_i : V_i \to U_i$ such that for all $i, j \leq n$, we have:

- The set $U_{ij} = f_i(V_i \cap V_j)$ is a Zariski open subset of $U_i$, hence it is a quasi-affine variety.
- $f_j \circ f_i^{-1}$ is an isomorphism between the quasi-affine varieties $U_{ij}$ and $U_{ji}$.

**Definition 7.15.** If $V, W$ are varieties, witnessed by $(V_i, f_i)_{i \leq n}$ and $(W_j, g_j)_{j \leq m}$ respectively, then a map $f : V \to W$ is a *morphism* if it is continuous and for all $i \leq n$ and $j \leq m$ we have $g_j \circ f|_{f^{-1}(W_j) \cap V_i} : f^{-1}(W_j) \cap V_i \to g_j(W_j)$ is a morphism of affine varieties.

**Definition 7.16.** A variety $V$ is called *complete* if for every variety $W$ the projection map $\pi_1 : V \times W \to W$ sends closed sets to closed sets.

**Definition 7.17.** An *algebraic group* is a variety $V$ with morphisms $\mu : V \times V \to V$ and $\rho : V \to V$ such that $\mu$ defines a group operation on $V$ and for every $x \in V$, the element $\rho(x)$ is the inverse to $x$ under this operation.

**Definition 7.18.** An *Abelian variety* is a connected algebraic group whose underlying variety is complete.

**Definition 7.19.** Let $A$ be an abelian variety with group operation $\mu$, and let $X$ be a subvariety of $A$. Then we define the *stabilizer* of $X$ as $\text{Stab}_X = \{a \in A \mid \mu(a, X) \subseteq X\}$.

**Definition 7.20.** We say that a group $G$ is of *finite rank* if there is a finitely generated subgroup $G_0 \subseteq G$ such that for every $g \in G$ there is some natural $n$ such that $g^n \in G_0$.

Now we have all the definitions we need to state the Mordell-Lang conjecture.

**Theorem 7.21.** *Let $K' \subsetneq K$ be algebraically closed fields, let $A$ be an Abelian variety defined over $K$ with group operation $\mu$, and let $X$ be an infinite subvariety of $A$, defined over $K$. Let $\Gamma$ be a subgroup of finite rank of $A$. Suppose $X \cap \Gamma$ is dense in $X$ (in the sense of the Zariski topology) and the stabilizer of $X$ in $A$ is finite. Then the following exist:*

- *An Abelian subvariety $B \subseteq A$,*
- *An Abelian variety $S$ defined over $K'$,*
- *A subvariety $X_0 \subset S$ defined over $K'$,*
- *A bijective morphism $h : B \to S$, where $S$ is viewed as a variety over $K$.*

*And these are such that for some $a_0 \in A$, we have $X = \mu(a_0, h^{-1}(X_0))$.*

This statement can be translated to a (partly) model-theoretic statement, using the following definition:

**Definition 7.22.** A stable theory $T$ is called *one-based* if in $T^{eq}$ we have $A \mathop{\smile\!\!\!\!\!\!\mid}_{A\cap B} B$ for all algebraically closed sets $A, B$.

It turns out that the Mordell-Lang conjecture is equivalent to saying that for every Abelian variety $A$ and subgroup $\Gamma$ of $A$ of finite rank, $\Gamma$ is a stable group and the theory of $K$ is one-based, or we can 'descend' to a smaller field, in the sense of the translation given in the statement.

# Conclusion

In this thesis, we have seen several equivalent definitions of stability. We have also studied the concept of forking, and the connection between stability and a number of rank functions. Finally, we have investigated a part of the classification picture and saw a short introduction to stable groups. There are several topics that could be studied after this point.

There have been applications of stability theory to classical model theory. An example of this is in [4], where a special case of Vaught's conjecture is proven by using stable group theory. Specifically, they consider theories $T$ which satisfy the property that for any $\kappa \geq 2^{|T|}$, if $A$ is a set of parameters such that $|A| \leq \kappa$, then $|S(A)| \leq \kappa$. Such theories are called *superstable theories*. The theorem they prove is that if a superstable theory has a finite rank of a specific kind, and has strictly less than $2^{\aleph_0}$ many countable models, then it only has countably many countable models.

There are several links between model theory and category theory. One of these links is studied in [12]. Here a certain class of commuting squares in a category $\mathcal{C}$ is called a *stable independence relation* if it satisfies a number of properties. It turns out that if $T$ is a stable theory and $\mathcal{C}$ is the abstract elementary class of algebraically closed sets in $T$, then a stable independence relation is induced by the relation $A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}_C^M B$, where $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.6em}}_C^M$ denotes independence over $C$ inside the model $M$. Since the properties of a stable independence relation are designed to resemble the properties of forking, this can be seen as a generalization of forking to arbitrary categories.

The connection between stability theory and algebraic geometry can be studied by someone who has both read this thesis (or is familiar with this material in some other way) and has a background in algebraic geometry. The strongest connection between these fields is of course the proof of the Mordell-Lang conjecture. An explanation of this proof is given in [2].

We have only seen a small fragment of the classification picture, there are still many more classes of first-order theories in existence, and the relation between all of these classes has been studied intensively. A comprehensive overview of this can be found on [7]. A few questions are still left open here (specifically, whether certain inclusions of classes are strict).

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# Index of symbols

# Index of terms