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An analysis on credit-adjusted corporate hedging strategies

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1 Introduction

This thesis evolves around corporate hedging strategies related to interest- and foreign exchange rate derivatives. Any corporate is somehow exposed to financial risk. Such risk comes in different forms, originating from movement in stock prices, the fluctuation in commodity valuations, changing interest rates or volatile exchange rates. In general, a hedging strategy refers to the treasury policy of a corporation aiming to manage financial risk to an acceptable level. Here we will focus on the risk induced by interest-bearing liabilities. To finance its activities, a corporate is required to attract funding. It is therefore common to find debt among the liabilities of a corporate's balance sheet. The interest rate risk, induced by moving interest rates associated to loans, are therefore a relevant concern to corporates.

Consider a corporate that entered a floating rate loan and is therefore exposed to interest rate risk. If rates go up, so do his interest rate costs and vice versa. High fluctuations in a corporates cost-profile are usually undesired. It could occur that the company's results appear very promising over one term, but disappointing over the next. On top of that, there is a liquidity risk to consider. If costs are uncertain, a corporate is forced to have a cash reserve at hand to absorb any sudden shocks arising from the money market. Holding such a reserve is costly on its own. For this reason a corporate will pursue a hedging strategy that mitigates financial risk and stabilizes earnings.

A straight-forward alternative to the floating rate loan would be a simple fixed rate loan, where the bank and the client agree upon a fixed interest rate for the whole tenor. Such a loan is however not always desirable. A floating rate loan implies that the lender will continuously pay a market implied interest plus perhaps a small loan margin. The loan has therefore no "intrinsic value" in terms of interest rates. This is not necessarily the case for a fixed rate loan. If a corporate has agreed on a constant rate of 1% and all of sudden the actual rates go up to 2%, this loan transformed into a very favorable deal with a positive market value. This property complicates for example a potential early repayment of the loan or transferring the loan to another bank as part of a syndication process.

Floating rate loans are therefore often preferred over fixed rate loans. Indeed, this preference induces an interest rate risk, but fortunately this can be hedged. A common financial instrument to do so is the *interest rate swap*. The interest rate swap (IRS) is a derivative contract in which a sequence of payment exchanges is agreed between two parties, benchmarked to some interest rate index and notional amount. In its most typical form, one party will periodically make a floating payment to another party based on the prevailing interest rate at that time. This party will in return receive a fixed interest payment based on a constant rate that was agreed upon up front. The combination of a floating rate loan together with an IRS is a classic example of an interest rate hedge: the corporate uses the received floating IRS coupons to pay the interest of the loan and effectively only pays fixed rate coupons itself.

A well-known downside to such a hedge is that it can be quite costly, especially for long tenors. Since an IRS is a risky deal for the party that pays the floating coupons (in our case the bank), the corresponding fixed rate will often turn out to be higher than the average realized floating rates. We say that the client pays a so called *risk premium*, which tends to be higher for longer tenors. Another contribution to the costs is that banks will charge their client a credit spread on top of the fixed rate. This is to compensate for the risk that the corporate might default during the trade. Also this charge tends to be disproportionally higher for longer tenors.

Contemporary discussions between corporates and banks therefore evolve around the question whether it would be beneficial to apply a partial hedge to safe costs. Take as an example a corporate that is about to take a 10-year loan. Instead of hedging the interest rate risk with a 10-year swap, he could consider a "rolling strategy" of 5 consecutive 2-year swaps. The benefit of this approach is that the tenor of each IRS is significantly smaller, from which we might expect that the hedge is less costly. A downside is that this approach implies a "roll-over risk": if the money market moves against the corporate, a new IRS after two years might be more costly than accounted for at first. In other words, the application of a rolling strategy induces a trade-off between expected gains and roll-over risk. For a corporate that regards a limited amount of risk acceptable, such an alternative strategy could be an interesting consideration. The questions that directly arise are the following: how do we estimate the potential gains and risks and how do we judge which roll-count is feasible?

A similar problem arises for a corporate that seeks to attract funds in a foreign currency. Consider as a second example a company that is stationed in the Netherlands, but requires a funding in US Dollars for some investment abroad. For a Dutch corporate with limited access to the American money market, it can be difficult to attract a Dollar loan under good terms. Therefore he may instead enter into a Euro loan. Not only is the corporate better known among the domestic banks, he also generates earnings in Euros. This assures that he will be able to pay the Euro interest and repay the Euro principal without any exposure to risk induced by moving exchange rates. This loan will therefore be relatively cheap. To still obtain the Dollar funding, the corporate could enter a *cross-currency swap* (CCS). The CCS works as follows: bank and client will at inception swap the Dollar versus the Euro notional according to the current spot exchange rate. Subsequently, throughout the tenor of the swap, the parties exchange the Dollar and Euro interest. At maturity, the notionals are exchanged back according to the same initial exchange rate, allowing the corporate to pay-off its loan. For a cross-currency swap it is not uncommon that the corporate pays a fixed rate, but receives a floating rate from the bank. The result is that the corporate lent Dollars against attractive rates, while being unexposed to interest- or exchange rate risk.

For similar reasons as the previous example, this hedge can be relatively costly. Also in this situation a client can count on a risk premium and credit charges, which disproportionally increase with the length of the tenor. In an attempt to reduce the costs, the corporate could again consider a rolling hedging strategy, involving multiple consecutive cross-currency swaps with shorter tenors. However, once again the question follows: what are the risks induced by a rolling strategy and what are the expected gains?

Our research objective is to quantify the potential gains and risks that are associated to rolling hedging strategies related to interest rate- and cross-currency swaps. We perform our investigation using a multi-currency interest rate model in combination with a foreign exchange rate model. Through Monte Carlo methods we simulate potential future market scenarios which allows us to compose risk-profiles associated to different rolling strategies.

In chapter 3 we will formally introduce the terminology and instruments that play a central role to the research. We will discuss the main assumptions for our general market model and treat some basic techniques in pricing interest rate derivatives. The interest rate model we use is a one-factor Hull-White model. This belongs to the class of affine term-structure models, which have as key feature that they provide an analytical formula for zero-coupon bond prices. Zero-coupon bonds can be considered fundamental quantities in the pricing procedure of a large range of interest rate derivatives. We construct a multi-currency framework, by considering multiple Hull-White processes for different currencies. The corresponding exchange rates are simulated by a Garman-Kohlhagen model, which simulates the exchange rate as a geometric Brownian motion. The details of our multi-currency framework will be the topic of chapter 4.

Our natural expectation is that the application of a rolling hedging strategy will for both instruments induce a risk. This risk will be higher for each additional roll we add to the scheme. However, in return we expect a drop in the average credit charges associated to the hedges. The credit spread is formally known as the *credit valuation adjustment* (CVA). We will introduce the notion of CVA in more detail in chapter 5.

CVA is in fact nothing more than the market-price of the credit risk related to a given derivative or portfolio. The literature provides a large range of techniques to compute CVA charges related to different financial instruments. See for example Green [2016], Glasserman [2004] or Brigo et al. [2013]. A widely applicable approach is a Monte Carlo method to estimate the expected loss at a potential event of a defaulting counterparty. One objective of our research is to construct risk-profiles of potential future CVA charges, using our interest- and FX rate model. Ideally we would therefore compute CVA along each simulation path to compose a distribution of potential credit spreads. Here we encounter a complication, as this would imply nested simulations. To solve this problem we step away from the Monte Carlo methods to estimate CVA charges and apply alternative techniques provided in the literature to derive analytical formulas instead. The analytical approach of CVA is treated in chapter 6 for an IRS and in chapter 7 for a CCS. We show that most formulas can in fact be derived in terms of our model parameters. This solves the problem of nested simulations and allows us to analytically compute CVA along the path in a model consistent manner.

Chapter 9 provides a detailed description of the hedging strategies that we investigate and states the assumptions under which we do so. We end the thesis by a thorough impact analysis of several alternative strategies to IRS and CCS hedges. We show how expected hedging costs can be reduced by considering a rolling strategy or lowering the ratio of the hedge. We additionally attempt to quantify the risk that is induced by entering these strategies. Throughout the analysis, our main goal is to provide insight in the benefits and risk of several IRS and CCS related hedges that deviate from the classical approach.

2 Mathematical prerequisites

In this chapter we will provide a brief summary of the relevant mathematical notions and theorems that we will apply throughout this thesis.

2.1 Brownian motion and Itô calculus

Brownian motions can be considered as the continuous-time equivalent of a random-walk. Throughout this thesis we will consider stochastic processes to model quantities that are exposed to randomness. The Brownian motion is a continuous process with convenient properties, that is often used to introduce "randomness" to a model. We will start with a formal definition.

Definition 1. *One-dimensional Brownian motion:* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Suppose that $W : \Omega \times [0, \infty) \to \mathbb{R}$ is a continuous stochastic process such that $\forall_{\omega \in \Omega} : W(0) = 0$. Then $W(t)$ is a Brownian motion if for all $0 = t_0 < \ldots < t_m$ the increments $W(t_1) - W(t_0), \ldots, W(t_m) - W(t_{m-1})$ are independent and normally distributed such that

$$W(t_i) - W(t_{i-1}) \sim N(0, t_i - t_{i-1})$$

The notion of a Brownian motion naturally extends to a multi-dimensional setting. We consider

$$\mathbf{W}(t) = (W_1(t), \ldots, W_d(t))^\top$$

to be a *d-dimensional Brownian motion* on the space $(\Omega, \mathcal{F}, \mathbb{P})$ if for all $i \in \{1, \ldots, d\}$ the process $W_i(t)$ is a one-dimensional Brownian motion and if $W_i(t)$ and $W_j(t)$ are independent whenever $i \neq j$. The Brownian motion restricted to a finite time-horizon $[0, T]$ is a text-book example of a Martingale.

Definition 2. *Martingale:* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, T a fixed positive constant and $\mathcal{F}_t$ a filtration of $\mathcal{F}$ for $0 \leq t \leq T$. Let $M(t)$ be an adapted stochastic process relative to $\mathcal{F}_t$ for $0 \leq t \leq T$. Then $M(t)$ is a Martingale if for all $0 \leq s \leq t \leq T$ we have that

$$\mathbb{E}\left(M(t) | \mathcal{F}_s \right) = M(s) \quad a.s.$$

The stochastic processes that we will consider for our models mostly take the form of an Itô process $X(t)$ adapted to the filtration $\mathcal{F}_t$. This means that X can be written as

$$X(t) = X(0) + \int_0^t \Theta(u) du + \int_0^t \Delta(u) dW(u)$$

where $\Theta(t)$ and $\Delta(t)$ are adapted, integrable stochastic processes. Often we will write the dynamics of a process X in its differential form, given by

$$dX(t) = \Theta(t)dt + \Delta(t)dW(t)$$

The term $\int_0^t \Delta(u)dW(u)$ is known as the Itô integral. There are several properties to this integral that we will use later on.

Theorem 3. ***(Properties of the Itô-integral)*** *Let T be a positive constant, $t \in [0, T]$ and $\Delta(t)$ some stochastic process adapted to $\mathcal{F}_t$. Assume that*

$$\mathbb{E}\left(\int_0^T \Delta^2(t) dt \right) < \infty$$

Then the Itô-integral $I(t) := \int_0^t \Delta(s)dW(s)$ has the following properties:

1. **(Continuity)** $I(t)$ has continuous paths w.r.t. the variable t.
2. **(Adaptivity)** $I(t)$ is $\mathcal{F}_t$-measurable.

3. **(Martingale)** $I(t)$ is a Martingale.
4. **(Itô-isometry)** $\mathbb{E}\left(I^2(t)\right) = \mathbb{E}\left(\int_0^t \Delta^2(s)ds\right)$

For a proof we refer to Shreve [2004]. If $\Delta(t)$ is a deterministic process, we have additionally the following result

Theorem 4. ***(Itô-integral of a deterministic integrand)*** *Let $\Delta(t)$ be a deterministic function of time. Then for all $t \geq 0$*

$$\int_0^t \Delta(u)dW(u) \sim N\left(0, \int_0^t \Delta^2(u)du\right).$$

See Shreve [2004] for a proof. A measure for the volatility of an Itô process is given by its quadratic variation. First consider the more general notion of cross-variation:

Definition 5. *Cross-variation:* Let $X(t)$, $Y(t)$ be two Itô processes on $[0,T]$, $\Pi = \{t_0, \dots, t_n\}$ some partition of $[0,t]$ for $0 < t \leq T$ and denote by $\|\Pi\|$ the mesh of Π. Then the cross-variation process of X and Y up to t is given by

$$[X,Y](t) = \lim_{\|\Pi\| \to 0} \sum_{j=1}^{n-1} \left(X(t_j) - X(t_{j-1})\right)\left(Y(t_j) - Y(t_{j-1})\right)$$

If X and Y in the definition above are the same process, then $[X,X](t)$ is referred to as the *quadratic variation* of X. In particular we have a.s. the following relations

$$\begin{aligned}
[X,X](s) &= \int_0^s \Delta^2(u)du \\
[W,W](s) &= s \\
[W,t](s) &= 0 \\
[t,t](s) &= 0
\end{aligned}$$

Often we capture the relations above in their differential form, in which they yield:

$$dX(t)dX(t) = \Delta^2(t)dt, \quad dW(t)dW(t) = dt, \quad dW(t)dt = 0, \quad dtdt = 0$$

See Shreve [2004] for details. An important general result, which we will use to evaluate stochastic differential equations, is known as Itô's lemma or the Itô-Doeblin formula.

Theorem 6. ***(Itô's lemma)*** *Let $f(t, x_1, \dots, x_d)$ denote a continuous function $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ with well-defined continuous partial derivatives f_t, f_{x_i} and $f_{x_i x_j}$ for all $i, j \in \{1, \dots, d\}$. Let $X_1(t), \dots, X_d(t)$ denote Itô processes on the space $(\Omega, \mathcal{F}, \mathbb{P})$. Then $Y(t) = f(t, X_1(t), \dots, X_d(t))$ defines an Itô process that satisfies the stochastic differential*

$$\begin{aligned}
dY(t) &= f_t(t, X_1(t), \dots, X_d(t)) + \sum_{i=1}^d f_{x_i}(t, X_1(t), \dots, X_d(t))\, dX_i(t) \\
&\quad + \frac{1}{2} \sum_{i,j=1}^d f_{x_i x_j}(t, X_1(t), \dots, X_d(t))\, dX_i(t)dX_j(t)
\end{aligned}$$

A proof for the case $d = 1$ can be found in Shreve [2004]. We will frequently apply a special case of Itô's lemma, known as the Itô product rule. Let $X(t)$ and $Y(t)$ denote two Itô processes, then we have the following relation

$$d(X(t)Y(t)) = X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t)$$

This result is a direct consequence of Itô's lemma. Lastly we will pose a theorem that provides an alternative characterization of a Brownian motion.

Theorem 7. ***(Lévy, one-dimension)*** *Let $M(t)$ be a Martingale relative to the filtration $\mathcal{F}_t$. Suppose that $M(0) = 0$, M(t) has continuous paths and $[M,M](t) = t$ for all $t \geq 0$. Then $M(t)$ is a Brownian motion.*

A sketch of the proof is provided in Shreve [2004].

2.2 Equivalent measures

Definition 8. *Equivalent measures:* Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mathbb{P}$, $\tilde{\mathbb{P}}$ denote two probability measures on this space. Then $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent if they agree on all null-sets, i.e. for all $A \in \mathcal{F}$ we have

$$\mathbb{P}(A) = 0 \Leftrightarrow \tilde{\mathbb{P}}(A) = 0$$

Two equivalent measures are related through a unique random variable, typically referred to as the Radon-Nikodym derivative. The following theorem is an important result, which allows us to construct equivalent measures on a common measurable space.

Theorem 9. ***(Radon-Nikodym)*** *Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mathbb{P}$, $\tilde{\mathbb{P}}$ denote two probability measures on this space. Then there exists an a.s. positive random variable Z, such that $\mathbb{E}Z = 1$ and for every $A \in \mathcal{F}$*

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$$

We refer to Shreve [2004] for a proof. The variable Z is called the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$. By convention we usually write $Z \equiv \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$. Related to Z is the Radon-Nikodym derivative process $Z(t)$, which is defined as

$$Z(t) = \mathbb{E}\left(Z | \mathcal{F}_t \right)$$

By the tower-property of conditional expectation, it can easily be shown that $Z(t)$ defines a Martingale.

Theorem 10. ***(Girsanov's theorem)*** *Let $\Theta(t) = (\Theta_1(t), \ldots, \Theta_d(t))^\top$ be a d-dimensional process adapted to $\mathcal{F}_t$ and $W(t)$ a d-dimensional Brownian motion on $[0, T]$. Define*

$$\begin{aligned} Z(t) &= \exp\left\{ -\int_0^t \Theta^\top(u) \cdot dW(u) - \frac{1}{2} \int \|\Theta(u)\|^2 \, du \right\} \\ \tilde{W}(t) &= W(t) + \int_0^t \Theta(u) du \end{aligned}$$

such that

$$\mathbb{E}\left(\int_0^T \|\Theta(u)\|^2 Z^2 du \right) < \infty.$$

Define the random variable $Z := Z(T)$. Then $\tilde{W}(t)$ is a d-dimensional Brownian motion under the measure $\tilde{\mathbb{P}}$, defined as

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) \, d\mathbb{P}(\omega)$$

See Shreve [2004] for a proof. Girsanov's theorem provides a tool to transform the dynamics of an Itô process under one measure to its dynamics under an equivalent measure. Say that a process $X(t)$ is captured by the SDE

$$dX(t) = \mu(t)dt + \sigma(t)dW(t)$$

where W is a Brownian motion under the measure $\mathbb{P}$. We can define any equivalent measure $\tilde{\mathbb{P}}$ through Radon-Nikodym's theorem by choosing a suitable adapted process $\Theta(t)$. Then Girsanov's theorem implies that the same SDE can be be written as

$$dX(t) = (\mu(t) - \sigma(t)\Theta(t)) \, dt + \sigma(t) d\tilde{W}(t)$$

where $\tilde{W}$ is a Brownian motion under the measure $\tilde{\mathbb{P}}$. In other words, Girsanov's theorem says that we can transform the drift of an Itô process to almost any other drift as long as we regard its dynamics under a suitable equivalent probability measure. Also note that changing the measure has only an impact on the drift of an Itô process, but not on its volatility. This effect is referred to as the *diffusion invariance principle* [Anderson and Piterbarg, 2010a].

3 Financial background: theory and definitions

This section serves as an overview of the relevant financial background that we will use throughout the thesis. In the first part we will treat a brief summary of the fundamental results from no-arbitrage theory. The absence of arbitrage is the fundamental assumption for the economic models that are presented here and dictates that there exists no investment strategy that is free of cost today, but has a positive probability of generating a non-zero profit in the future.

In the subsequent part we will discuss the change of numeraire technique. We will require this technique to perform some pricing routines, which would otherwise be difficult to compute. Lastly we will introduce some relevant definitions and terminology related to interest rate derivatives and option pricing.

3.1 A no-arbitrage economy

We start with a brief, but general description of a no-arbitrage economy. We largely follow the set-up described in Filipovic [2009] and Brigo and Mercurio [2007]. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $T > 0$ be some date in the future defining a finite time horizon $[0, T]$. Let $\{\mathcal{F}_t | \, 0 \leq t \leq T\}$ be a filtration on $\mathcal{F}$, such that $\mathcal{F}_T = \mathcal{F}$. Defined on this probability space we consider a multi-dimensional financial market, that consists of a finite number m of risky assets. We assume that the price of these assets can be captured by stochastic Itô processes, which we will denote as $X_1(t), \dots, X_m(t)$. We assume these assets to be continuously tradable on the market and let their price-process satisfy the following SDE

$$dX_i(t) = \mu_i \left(X_i(t), t\right) dt + \sigma_i \left(X_i(t), t\right) dW_i(t), \quad i = 1, \dots, m$$

Both $\mu_i(x,t)$ (the "drift") and $\sigma_i(x,t)$ (the "volatility") denote continuous functions which are adapted to $\mathcal{F}_t$. With W_i we denote standard one-dimensional Brownian motions, which are typically correlated to one another. On top of these assets, we consider a *money-market account* B which satisfies

$$\frac{dB(t)}{B(t)} = r(t)dt, \quad B(0) = 1$$

The process B is often also referred to as *the bank account*. It models the value of a unit of currency if it were invested in the money market, where the value accumulates according to the continuously compounded, short-term interest rate $r(t)$. An expression for B is given by:

$$B(t) = e^{\int_0^t r(s)ds}$$

The process r is usually referred to as the *short-rate*. As our research evolves around securities directly related to interest rates, we will throughout this thesis consider a stochastic interest rate economy. We will therefore assume that r is a stochastic process, predictable with respect to the filtration $\mathcal{F}_t$.

Since each asset is continuously tradable on the market, a potential investor is free to compose a portfolio that contains any number of assets. Let for $\phi_i(t)$ denote the quantity of asset X_i the investor holds in his portfolio for $i \in \{1, \dots, m\}$ and $\phi_0(t)$ the amount of currency that he invested in the money-market at time t. Given that each function ϕ_i is bounded and predictable on $[0, T]$, then the $(m + 1)$-dimensional process

$$\phi(t) = \left(\phi_0(t), \dots, \phi_m(t)\right), \quad 0 \leq t \leq T$$

defines a *trading strategy*, which at time t corresponds to a portfolio-value of

$$V(\phi, t) = \sum_{i=0}^{m} \phi_i(t) X_i(t)$$

The strategy ϕ is additionally called a *self-financing* strategy if any change in $V(\phi, t)$ over time is solely induced by changes in the asset-values. In other words, once the portfolio is composed, no cash is added or extracted from it until time T. In that case we can write

$$dV\left(\phi, t\right) = \sum_{i=0}^{m} \phi_i(t) dX_i(t)$$

Now that we illustrated the setting to model the economy, we will formulate the *fundamental theorems of asset pricing*. Before we do so, we require some additional definitions. We start with that of an arbitrage opportunity. Let ϕ be a self-financing strategy. We speak of *arbitrage* if at time zero we have $V(\phi, 0) = 0$, but at time $T > 0$ we have

$$\mathbb{P}(V(\phi, T) \geq 0) = 1 \text{ and } \mathbb{P}(V(\phi, T) > 0) > 0$$

The strategy ϕ would hence denote an investment that is free of charge, induces zero risk of making a loss, but has positive probability of gaining profit.

Secondly we consider the definition of a risk-neutral measure. Let $\mathbb{Q}$ be a probability measure defined on the measurable space $(\Omega, \mathcal{F})$. Then we call $\mathbb{Q}$ *risk-neutral* if:

i) $\mathbb{P}$ and $\mathbb{Q}$ are equivalent (i.e. $\forall_{A\in\mathcal{F}}: \mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0$).

ii) For each $i = 0, \ldots, m$, the process $D(0,t)X_i(t)$ is a martingale under $\mathbb{Q}$.

Through an application of the multi-dimensional Girsanov's Theorem, it can be shown that within our market model, such a risk-neutral measure indeed exists. This is shown for example in Shreve [2004]. A direct consequence of the definition of $\mathbb{Q}$ is that the value of a self-financing portfolio is, just like the discounted asset-price, a Martingale under this measure. This follows from an application of the Itô product rule as illustrated below. For any $t < T$ we have

$$\begin{aligned} d\left(D(0,t)V(\phi,t)\right) &= D(0,t)dV(\phi,t) + V(\phi,t)dD(0,t) \\ &= D(0,t)\sum_{i=0}^{m}\phi_i(t)dX_i(t) - \left(\sum_{i=0}^{m}\phi_i(t)X_i(t)\right)r(t)D(0,t)dt \\ &= \sum_{i=0}^{m}\phi_i(t)\left(D(0,t)dX_i(t) - r(t)D(0,t)X_i(t)dt\right) \\ &= \sum_{i=0}^{m}\phi_i(t)d\left(D(0,t)X_i(t)\right) \end{aligned}$$

Since each ϕ_i is predictable and $D(0,t)X_i(t)$ is a Martingale under $\mathbb{Q}$, we see that $D(0,t)V(\phi,t)$ is a Martingale under $\mathbb{Q}$.

This result is particularly useful for an agent that aims to hedge a derivative security. Say that we have a contract that at maturity T pays out $V(T)$. The agent would then be interested in a self-financing strategy ϕ such that

$$V(\phi, T) = V(T) \quad a.s.$$

If such a strategy exists, the contract can be *hedged*. Then according to the Martingale property, we know that under the measure $\mathbb{Q}$:

$$D(0,t)V(\phi,t) = \mathbb{E}^Q\left(D(0,T)V(\phi,T)|\,\mathcal{F}_t\right) = \mathbb{E}^Q\left(D(0,T)V(T)|\,\mathcal{F}_t\right)$$

Let ϕ be self-financing hedging strategy. Then note that $V(\phi,t)$ denotes the cash that is required if one is to compose a portfolio at time t in line with the strategy ϕ. Therefore, if we set $V(t) \equiv V(\phi,t)$, then the funding required to hedge a derivative with pay-off $V(T)$ can be calculated as follows:

$$V(t) = \mathbb{E}^Q\left(D(t,T)V(T)|\,\mathcal{F}_t\right)$$

We refer to this result as *the risk-neutral pricing formula*. It allows one to calculate the "fair price" of a derivative security, given that it can be hedged according to a self-financing strategy. A market model is called *complete* if for each derivative security, a self-financing strategy ϕ can be determined to hedge it.

Finally we can state the two fundamental theorems of asset pricing. Formal statement and proofs can for example be found in Shreve [2004]. The first one concerns the absence of arbitrage:

Theorem 11. *If a market model admits a risk-neutral measure $\mathbb{Q}$, then there doesn't exist a strategy ϕ that imposes an opportunity of arbitrage.*

This results indicates that it is sufficient to mathematically show the existence of a risk-neutral measure. Once this is done, we immediately satisfy an important condition for a realistic model, namely that it doesn't admit arbitrage. The second theorem concerns risk-neutral pricing:

Theorem 12. *Assume that a market model admits at least one risk-neutral measure. Then this measure is unique if and only if the the model is complete.*

From this theorem we can conclude that the fair value, which we compute through the risk-neutral pricing formulas, are in fact unique. This is an important result for pricing derivative securities. Given such an instrument there exists exactly one price for which an agent is able to hedge it.

3.2 Changing the numeraire

We have seen that a portfolio process denominated by the bank-account $V(\phi,t)/B(t)$ is a Martingale under the risk-neutral measure. This property can be generalized if we introduce the notion of a *numeraire.*

Definition 13. *Numeraire:* A continuously tradable, positively priced asset that is free of transaction costs and dividend payments.

So far we have used the bank-account as numeraire and used the Martingale property to construct the risk-neutral pricing formula. However, pricing a derivative or portfolio can sometimes be more convenient under a different numeraire. It appears that this is possible by application of a measure change. A portfolio process denominated by any numeraire N defined in the market model is still a Martingale under the risk-neutral measure $\mathbb{Q}^N$ associated to that numeraire.

To see why such a measure indeed always exists, we follow the arguments of Geman et al. [1995]. First of all let N be any numeraire. Secondly, consider the random variable Z, which is given by

$$Z = \frac{N(T)}{N(0) \cdot B(T)}$$

By their definition we have that $N(T)$, $N(0)$ and $B(T)$ are non-negative random variables and $\mathbb{E}^Q\left(\frac{N(T)}{N(0)\cdot B(T)}\right) = \frac{N(0)}{N(0)\cdot B(0)} = 1$. For any $A \in \mathcal{F}$ it therefore follows that the measure

$$\mathbb{Q}^N(\omega) \equiv \int_A Z(\omega) d\mathbb{Q}(\omega)$$

defines a probability measure on the measurable space $(\Omega, \mathcal{F})$. Furthermore, it follows directly from its definition that $\mathbb{Q}^N$ is equivalent to $\mathbb{Q}$, from which we can conclude that Z is in fact the Radon-Nikodym derivative $\frac{d\mathbb{Q}^N}{d\mathbb{Q}}$.

Now, we only need to show that the price of any security X_i denominated by the numeraire N is in fact a Martingale under $\mathbb{Q}^N$. To do so, consider the Radon-Nikodym derivative process given by $Z(T) := Z$ and for $0 \leq t < T$

$$Z(t) = \mathbb{E}^Q\left(\frac{d\mathbb{Q}^N}{d\mathbb{Q}}\middle| \mathcal{F}_t\right)$$

Then according to Shreve [2004], lemma 5.2.2, we have for any $\mathcal{F}$-measurable random variable Y and $0 \leq s \leq t \leq T$ the following relation:

$$\mathbb{E}^{Q^N}(Y|\mathcal{F}_s) = \frac{1}{Z(s)}\mathbb{E}^Q(Y \cdot Z(t)|\mathcal{F}_s)$$

Since both X_i and N are adapted to $\mathcal{F}_t$, we know that $\frac{X_i(t)}{N(t)}$ is $\mathcal{F}_t$-measurable. Furthermore, by the definition of the risk-neutral measure, the processes $\frac{N(t)}{B(t)}$ and $\frac{X_i(t)}{B(t)}$ are martingales under $\mathbb{Q}$. Therefore we have

$$Z(s) = \mathbb{E}^Q\left(\frac{d\mathbb{Q}^N}{d\mathbb{Q}}\middle|\mathcal{F}_s\right) = \frac{1}{N(0)}\mathbb{E}^Q\left(\frac{N(T)}{B(T)}\middle|\mathcal{F}_s\right) = \frac{N(s)}{N(0)\cdot B(s)}$$

and

$$\mathbb{E}^Q\left(\frac{X_i(t)}{N(t)}\cdot Z(t)\middle|\mathcal{F}_s\right) = \mathbb{E}^Q\left(\frac{X_i(t)}{N(t)}\cdot\frac{N(t)}{N(0)\cdot B(t)}\middle|\mathcal{F}_s\right) = \frac{X_i(s)}{N(0)\cdot B(s)}$$

Using these two results and setting $Y = \frac{X_i(t)}{N(t)}$ in the relation above, we find that

$$\mathbb{E}^{Q^N}\left(\frac{X_i(t)}{N(t)}\middle|\mathcal{F}_s\right) = \frac{1}{\frac{N(s)}{N(0)\cdot B(s)}}\cdot\frac{X_i(s)}{N(0)\cdot B(s)} = \frac{X_i(s)}{B(s)}$$

from which we conclude that X_i discounted by N is indeed a martingale. We summarize this result in the proposition below, as also formulated in Geman et al. [1995].

Proposition 14. *Let $\mathbb{Q}$ be the risk-neutral measure, such that $\frac{X_i(t)}{B(t)}$ is a Martingale for all $i \in \{1,\dots,m\}$ under $\mathbb{Q}$ and let N be any numeraire. Then there exists a probability measure $\mathbb{Q}^N$ equivalent to $\mathbb{Q}$ such that $\frac{X_i(t)}{N(t)}$ is a Martingale under $\mathbb{Q}^N$. The Radon-Nikodym derivative is given by $\frac{d\mathbb{Q}^N}{d\mathbb{Q}} = \frac{N(T)}{N(0)\cdot B(T)}$.*

A direct consequence of the proposition is that it expands our toolbox in pricing derivative securities. This follows from the Martingale property of a portfolio process that is denominated by the numeraire. Let N be a numeraire, $\mathbb{Q}^N$ the associated measure and $V(\phi, t)$ the value process of a strategy ϕ maturing at T. Then we have the following relations

$$\begin{aligned}\frac{V(t)}{B(t)} &= \mathbb{E}^Q\left(\frac{V(T)}{B(T)}\middle|\mathcal{F}_t\right)\\ \frac{V(t)}{N(t)} &= \mathbb{E}^N\left(\frac{V(T)}{N(T)}\middle|\mathcal{F}_t\right)\end{aligned}$$

Hence, it follows that the fair price of a derivative is invariant to the underlying numeraire N and can equivalently be computed according to the formula

$$V(t) = N(t)\cdot\mathbb{E}^N\left(\frac{V(T)}{N(T)}\middle|\mathcal{F}_t\right)$$

3.3 Interest- and exchange rate definitions

Here we will introduce standard terminology related to interest rate modeling and define some basic interest rate derivatives that will play a central role throughout the thesis. Additionally we will briefly discuss the risk-neutral price of some of the instruments that are put forward. We do so by applying the no-arbitrage arguments and the risk-neutral pricing formula. Most of the terminology that is introduced in this section follows the formulation presented in Brigo and Mercurio [2007].

3.3.1 Single-currency market

We start by considering a market model where each asset is denoted in a single currency. In the subsequent section we will generalize the concept to a multi-currency setting.

Definition 15. *Zero-coupon bond:* A contract that guarantees the buyer 1 unit of currency at maturity T.

We denote the value of a zero-coupon bond at time $t < T$ by $P(t,T)$. By its definition, we have $P(T,T) = 1$. For $t < T$ we have that

$$P(t,T) = \mathbb{E}^Q\left(D(t,T)P(T,T)\middle|\mathcal{F}_t\right) = \mathbb{E}^Q\left(e^{-\int_t^T r(s)ds}\middle|\mathcal{F}_t\right)$$

The zero-coupon bond maturing at T is a common alternative to the bank-account as a choice in numeraire when pricing instruments. This is due to the convenient characteristic that $P(T,T) = 1$. In the pricing formula for derivative securities this implies

$$V(t) = P(t,T)\cdot\mathbb{E}^{Q^T}\left(\frac{V(T)}{P(T,T)}\middle|\mathcal{F}_t\right) = P(t,T)\cdot\mathbb{E}^{Q^T}\left(V(T)\middle|\mathcal{F}_t\right)$$

We denote the risk-neutral measure associated with a zero-coupon bond by the T-forward measure.

Definition 16. *T-forward measure:* The probability measure $\mathbb{Q}^T$ under which any asset-price process denominated by the zero-coupon bond price process $t \mapsto P(t,T)$ becomes a Martingale.

From the definition of a zero-coupon bond we can easily move to interest rates. We distinguish two types of interest rates: the continuously compounded and the simply compounded. In practice we will mostly use the latter.

Definition 17. *Continuously compounded interest rate* The constant rate $R(t,T)$ prevailing at time t at which an investment of $P(t,T)$ units of currency is required to grow if one is to obtain 1 unit of currency at time T, given that the investment accrues continuously.

Let Δt denote the year-fraction between time t and T. Then by its definition, the continuously-compounded rate $R(t,T)$ can be written as

$$e^{R(t,T)\Delta t} P(t,T) = 1$$

Or equivalently

$$R(t,T) = -\frac{\log(P(t,T))}{\Delta t}$$

Definition 18. *Simply compounded interest rate:* The constant rate $L(t,T)$ prevailing at time t at which an investment of $P(t,T)$ units of currency is required to grow if one is to obtain 1 unit of currency at time T, given that the accruing is proportional to the investment time.

A well known example of a simply-compounded rate is the LIBOR (the London InterBank Offered Rate). For multiple maturity dates T, the LIBOR is quoted on the market and updated on a daily basis. The LIBOR is inferred from a collection of interest rates, which are quoted by a panel of prominent banks. Let Δt denote the year-fraction between time t and T. Then by its definition, the simply-compounded rate $L(t,T)$ can be written as

$$(1 + L(t,T)\Delta t) \cdot P(t,T) = 1$$

Or equivalently

$$L(t,T) = \frac{1 - P(t,T)}{\Delta t \cdot P(t,T)}$$

LIBOR rates over future time intervals, say $[S,T]$ for $S > t$, are not known today. Now consider an investor that would like to fix the LIBOR today for a future time instant. Hypothetically, this could be done through a contract, called a prototypical *forward rate agreement* (FRA). This is a contract in which a fixed interest rate K is settled today (time t) over some notional amount N. Subsequently, at maturity T (when the LIBOR $L(S,T)$ is already known) the two parties exchange $N \cdot \Delta t \cdot L(S,T)$ and $N \cdot \Delta t \cdot K$. The rate K that at time t sets the risk-neutral price of a FRA to zero is called the *forward rate*. In reality, only a generalization of the FRA is traded in the market, namely the interest rate swap (which will be treated in section 3.5.1).

Definition 19. *Simply compounded forward rate* The fixed rate $F(t,S,T)$ for which a prototypical FRA expiring at S and maturing at T has risk-neutral price 0 at inception date t.

The forward rate can be considered as the current expectation of the future realization of the LIBOR. We can derive a formula for the forward rate in terms of zero-coupon bond prices. There are several arguments that can be followed to do so. We choose to apply a measure change. According to the definition, $F(t,S,T)$ should satisfy the relation

$$\begin{aligned} \mathbb{E}^Q\left(D(t,T)\left(L(S,T) - F(t,S,T)\right) \middle| \mathcal{F}_t\right) &= 0 \\ \Rightarrow \qquad \frac{\mathbb{E}^Q\left(D(t,T)L(S,T) \middle| \mathcal{F}_t\right)}{\mathbb{E}^Q\left(D(t,T) \middle| \mathcal{F}_t\right)} &= F(t,S,T) \end{aligned}$$

The denominator is by definition equal to $P(t,T)$. The numerator can be computed by changing to the T-forward measure.

$$\begin{aligned}
\mathbb{E}^{Q}\left(D(t,T)L(S,T)\middle|\, \mathcal{F}_t\right) &= P(t,T)\cdot \mathbb{E}^{Q^T}\left(L(S,T)\middle|\, \mathcal{F}_t\right) \\
&= P(t,T)\cdot \mathbb{E}^{Q^T}\left(\frac{P(S,S)-P(S,T)}{\Delta t\cdot P(S,T)}\middle|\, \mathcal{F}_t\right) \\
&= P(t,T)\cdot \frac{P(t,S)-P(t,T)}{\Delta t\cdot P(t,T)}
\end{aligned}$$

The last equality follows from the fact that $P(t,S)$ and $P(t,T)$ are both tradable assets, implying that denominated by the numeraire $P(S,T)$, they are Martingales under $\mathbb{Q}^T$. Therefore we find that in terms of zero-coupon bonds the value of the forward rate can be written as

$$F(t,S,T) = \frac{P(t,T)\cdot \frac{P(t,S)-P(t,T)}{\Delta t\cdot P(t,T)}}{P(t,T)} = \frac{1}{\Delta t}\left(\frac{P(t,S)}{P(t,T)}-1\right)$$

We will often use the above expression for the forward rate. Note that by the definition of the T-forward measure, the forward rate itself

$$F(t,S,T) = \frac{1}{\Delta t}\left(\frac{P(t,S)}{P(t,T)}-1\right) = \frac{P(t,S)-P(t,T)}{\Delta t\cdot P(t,T)}$$

must be a Martingale under $\mathbb{Q}^T$. If we choose $t=S$, then it follows by the fact that $P(S,S)=1$ that we can write

$$F(t,S,T) = \mathbb{E}^{Q^T}\left(\frac{P(S,S)-P(S,T)}{\Delta t\cdot P(S,T)}\middle|\, \mathcal{F}_t\right) = \mathbb{E}^{Q^T}\left(L(S,T)\middle|\, \mathcal{F}_t\right)$$

And thus we conclude that the expected value of the LIBOR under the T-forward measure in fact yields the forward rate.

Considering the forward rate for infinitesimal accrual periods $[T, T+dt]$, one arrives at the *instantaneous forward rate.*

Definition 20. *Instantaneous forward rate* $f(t,T) = \lim_{S\nearrow T} F(t,S,T)$

Computation of the limit yields.

$$\begin{aligned}
f(t,T) &= \lim_{S\nearrow T}\frac{1}{T-S}\left(\frac{P(t,S)}{P(t,T)}-1\right) \\
&= \frac{1}{P(t,T)}\lim_{S\nearrow T}\left(\frac{P(t,S)-P(t,T)}{T-S}\right) \\
&= -\frac{\partial}{\partial T}\log\left(P(t,T)\right)
\end{aligned}$$

Finally, if we let the maturity of the instantaneous forward rate approach today, we obtain the instantaneous spot rate.

Definition 21. *Instantaneous spot rate* $r(t) = \lim_{T\searrow t} f(t,T)$

This variable is often called the *short rate* and corresponds to the drift of the money-market account.

3.3.2 Multi-currency market

The setting of a no-arbitrage market model can be extended to model assets in multiple currencies. We will for now consider two currencies which we denote by *domestic* and *foreign*. By introducing a foreign currency, we are also required to distinguish the domestic and foreign asset processes, which we will denote by the superscripts d and f. In particular we introduce the money-market process, at which foreign capital grows with the continuously compounded foreign short-rate r^f, given by

$$B^f(t) = e^{\int_0^t r^f(u)du}$$

In accordance with its domestic counterpart, we can similarly define a foreign zero-coupon bond which guarantees the buyer one unit of foreign currency at maturity. The price of a such a bond, in foreign currency, is given by

$$P^f(t,T) = \mathbb{E}^{Q^f}\left(e^{-\int_t^T r^f(s)ds} \middle| \mathcal{F}_t \right)$$

The measure $\mathbb{Q}^f$ used in the expression above is called the foreign risk-neutral measure.

Definition 22. *Foreign risk-neutral measure:* The probability measure $\mathbb{Q}^f$ equivalent to $\mathbb{P}$ such that for any foreign asset, the foreign discounted asset price $\frac{X_i^f(t)}{B^f(t)}$ is a Martingale under $\mathbb{Q}^f$.

For clarity we will in the rest of this paragraph denote the domestic risk-neutral measure as $\mathbb{Q}^d$. In other paragraphs the superscript d will be omitted if it is clear from the context. In line with its domestic counterpart, we can define the *foreign* T-forward measure:

Definition 23. *Foreign T-forward measure:* The probability measure $\mathbb{Q}^{f,T}$ under which any foreign asset-price process denominated by the foreign zero-coupon bond price process $t \mapsto P^f(t,T)$ becomes a Martingale.

A multi-currency framework increases the complexity of the model in a sense that we cannot directly compare foreign and domestic prices. To translate prices in the foreign currency to the domestic currency, we require the *foreign exchange rate*, which we often abbreviate as the FX rate. The FX rate, denoted by $\varphi(t)$, stands for *the amount of domestic currency for one unit of foreign currency at time t.* By the stochastic nature of φ, future FX rates are not known today. In line with the definition of the forward rate, we therefore introduce the notion of *forward exchange rate.* Its definition is based on an FX forward contract. This is a contract in which an FX rate Q is settled today (time t) and at maturity T the buyer exchanges Q units of domestic currency to one unit of foreign currency with the issuer.

Definition 24. *Forward FX rate:* The exchange rate $\Phi(t,T)$ for which an FX forward contract maturing at T has risk-neutral price 0 at inception date t.

By its definition, the forward FX rate can be expressed using the risk-neutral pricing formula. The method that we illustrate here is similar to that of the forward interest rate. By definition of the FX forward we should have

$$\begin{aligned}
\mathbb{E}^{Q^d}\left(D(t,T)\cdot(\varphi(T)-\Phi(t,T)) \middle| \mathcal{F}_t \right) &= 0 \\
\Rightarrow \quad \frac{\mathbb{E}^{Q^d}\left(D(t,T)\cdot\varphi(T) \middle| \mathcal{F}_t \right)}{\mathbb{E}^{Q^d}\left(D(t,T) \middle| \mathcal{F}_t \right)} &= \Phi(t,T)
\end{aligned}$$

From the denominator we know that $\mathbb{E}^{Q^d}\left(D(t,T) \middle| \mathcal{F}_t \right) = P^d(t,T)$. For evaluation of the numerator we do a change of measure, by switching from the domestic bank account as numeraire to the foreign zero-coupon bond as numeraire. For the Radon-Nikodym derivative we need the price of a foreign zero-coupon bond expressed in the domestic currency. Therefore, this measure change is similar to changing the numeraire from $B^d(t)$ to $\varphi(t)P^f(t,T)$. The Radon-Nikodym derivative process of these two measures is then given by

$$Z(t) = \mathbb{E}^{Q^{f,T}}\left(\frac{d\mathbb{Q}^d}{d\mathbb{Q}^{f,T}} \middle| \mathcal{F}_t \right) = \frac{B(t)}{B(0)} \cdot \frac{\varphi(0)P^f(0,T)}{\varphi(t)P^f(t,T)}$$

Application of the Radon-Nikodym derivative allows us to evaluate the expectation in the numerator, which yields

$$\begin{aligned}
\mathbb{E}^{Q^d}\left(D(t,T)\cdot\varphi(T) \middle| \mathcal{F}_t \right) &= \mathbb{E}^{Q^{f,T}}\left(D(t,T)\cdot\varphi(T)\cdot\frac{Z(T)}{Z(t)} \middle| \mathcal{F}_t \right) \\
&= \mathbb{E}^{Q^{f,T}}\left(D(t,T)\cdot\varphi(T)\cdot\frac{B(T)}{B(t)}\cdot\frac{\varphi(t)P^f(t,T)}{\varphi(T)P^f(T,T)} \middle| \mathcal{F}_t \right) \\
&= \mathbb{E}^{Q^{f,T}}\left(\varphi(t)P^f(t,T) \middle| \mathcal{F}_t \right) \\
&= \varphi(t)P^f(t,T)
\end{aligned}$$

Plugging this result back in the original equation, lets us formulate a compact expression for the forward exchange rate:

$$\Phi(t,T) \quad = \quad \frac{\varphi(t)P^f(t,T)}{P^d(t,T)}$$

Note that by the definition of the T-forward measure, the forward exchange rate must be a Martingale under the domestic forward measure $\mathbb{Q}^{d,T}$, just like the forward interest rate. As a consequence we find the relation

$$\Phi(t,T) = \mathbb{E}^{Q^{d,T}} \left(\frac{\varphi(T)P^f(T,T)}{P^d(T,T)} \middle| \mathcal{F}_t \right) = \mathbb{E}^{Q^T} \left(\varphi(T) | \mathcal{F}_t \right)$$

3.4 Option pricing

Highly relevant in the field of interest rate and FX modeling are the methodologies to price options. An option is a contract that gives the owner the right to buy or sell an instrument for a specific price at a future time instant. The value of this contract therefore depends on the development of the underlying instrument. Throughout this document we will mainly consider instruments that are related to the stochastic interest- or foreign exchange rate. Here we will introduce two models that allow us to evaluate the risk-neutral value of such an option, by an analytical option-price formula.

As a guiding example we will consider a European call option on a prototypical FRA with notional $N = 1$. Such a contract is called a *caplet*. Although this derivative is in reality not traded on the market, the theoretical treatment will provide a clear insight in the methodology of option pricing. Let t denote the time today, such that $t < T_1 < T_2$. Recall that the pay-off of a FRA at time T_2 is given by

$$\Delta t \cdot (L(T_1, T_2) - K) = \Delta t \cdot (F(T_1, T_1, T_2) - K)$$

The pay-off for a caplet is only executed if the value at maturity is positive. Therefore it should be clear that the risk-neutral value of such a contract is given by

$$\mathbb{E}^Q \left(D(t, T_2) \cdot \Delta t \cdot (F(T_1, T_1, T_2) - K)^+ \middle| \mathcal{F}_t \right) = P(t, T_2) \cdot \Delta t \cdot \mathbb{E}^{Q^{T_2}} \left((F(T_1, T_1, T_2) - K)^+ \middle| \mathcal{F}_t \right)$$

where we obtained the expression on the right by changing the measure $\mathbb{Q}$ to the T_2-forward measure. Note that the value is only driven by the stochastic process that describes the forward rate $F(t, T_1, T_2)$. Therefore, the evaluation of the expectation depends on the dynamics that we assume on the forward rate. In the following paragraphs we will treat two frameworks:

- *Black's model*: assume log-normal dynamics for F
- *Bachelier's model*: assume Gaussian dynamics for F

3.4.1 Black's model

In his 1976 paper, Black examined the valuation of options on future contracts. It appears that his methodology has general applications in option pricing. In his framework, he considered an asset X and a *future price* for this asset at some future time instant T. We shall denote this price as $f(t) := f_X(t, T)$. Furthermore he assumed that the evolution of this future price satisfies the SDE

$$df(t) = \mu(t)f(t)dt + \sigma(t)f(t)dW(t)$$

Here $\mu(t)$ and $\sigma(t)$ denote real functions of time and W a standard Brownian motion. Now, lets translate this setting to that of a caplet: an option on a FRA. The forward price of a FRA at time t is in fact given by the forward rate $F(t) := F(t, T_1, T_2)$. We have seen in section 3.3 that $F(t)$ is a Martingale under the $\mathbb{Q}^{T_2}$ measure. If we furthermore assume log-normal dynamics for the forward rate, we arrive at the following SDE for F:

$$dF(t) = v(t)F(t)dW(t)$$

where $v(t)$ denotes the instantaneous volatility of F and $W(t)$ a one-dimensional Brownian motion under the measure $\mathbb{Q}^{T_2}$. Note that the dynamics define a generalized geometric Brownian motion with zero drift. An expression for F under this assumption is therefore given by

$$F(t) = F(0) \exp\left\{ \int_0^t v(s)dW(s) - \frac{1}{2}\int_0^t v^2(s)ds \right\}$$

Following the arguments provided by Black, allows us to analytically compute the value of a caplet. The only thing we need to do is evaluate the conditional expectation in the caplet-price. Note that we can write

$$\begin{aligned}
\mathbb{E}^{Q^{T_2}}\left((F(T_1) - K)^+ \middle| \mathcal{F}_t \right) &= \mathbb{E}^{Q^{T_2}}\left((F(T_1) - K)\, \mathbb{1}_{F(T_1)>K} \middle| \mathcal{F}_t \right) \\
&= \mathbb{E}^{Q^{T_2}}\left(F(T_1)\, \mathbb{1}_{F(T_1)>K} \middle| \mathcal{F}_t \right) - \mathbb{E}^{Q^{T_2}}\left(K \mathbb{1}_{F(T_1)>K} \middle| \mathcal{F}_t \right)
\end{aligned}$$

These expectations can be evaluated using the expression for F and the Gaussian properties of the Brownian motion. We will omit the details for now, but they can for example be found in Musiela and Rutkowski [2005]. The expectations yield

$$\begin{aligned}
\mathbb{E}^{Q^{T_2}}\left(F(T_1)\, \mathbb{1}_{F(T_1)>K} \middle| \mathcal{F}_t \right) &= F(t) \cdot \mathcal{N}\left(\frac{\log\left(\frac{F(t)}{K}\right) + \frac{1}{2}\int_t^{T_1} v^2(s)ds}{\sqrt{\int_t^{T_1} v^2(s)ds}} \right) \\
\mathbb{E}^{Q^{T_2}}\left(K \mathbb{1}_{F(T_1)>K} \middle| \mathcal{F}_t \right) &= K \cdot \mathcal{N}\left(\frac{\log\left(\frac{F(t)}{K}\right) - \frac{1}{2}\int_t^{T_1} v^2(s)ds}{\sqrt{\int_t^{T_1} v^2(s)ds}} \right)
\end{aligned}$$

where $\mathcal{N}(\cdot)$ denotes the cumulative distribution function of a standard normal random variable, which is given by

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y}{2}} dy$$

Summarizing, we have found a closed-form expression for the caplet price $V(t)$ namely:

$$V(t) = P(t, T_2) \cdot \Delta t \left[F(t) \cdot \mathcal{N}\left(d_+ \left(K, F(t), \int_t^{T_2} v^2(s)ds \right) \right) - K \cdot \mathcal{N}\left(d_- \left(K, F(t), \int_t^{T_2} v^2(s)ds \right) \right) \right]$$

where

$$d_\pm(K, F, u) = \frac{\log\left(\frac{F}{K}\right) - \frac{1}{2}u}{\sqrt{u}}$$

In general, the formula

$$c(K, F, u) = F \cdot \mathcal{N}(d_+(K, F, u)) - K \cdot \mathcal{N}(d_-(K, F, u))$$

is referred to as *Black's formula*.

3.4.2 Bachelier's model

For some assets, it is more natural to consider a Gaussian behavior, rather than log-normal dynamics. Forward rates are in fact a good example. In the current economy it is not uncommon to observe negative interest rates. This phenomenon is not reflected by modeling the forward rate as a geometric Brownian motion, which is necessarily non-negative. The Gaussian framework, first examined by Bachelier in 1900, is therefore a good alternative. He considered a market where the price of an asset X satisfies the SDE

$$dX(t) = \mu(t)dt + \sigma(t)dW(t)$$

where $\mu(t)$ and $\sigma(t)$ denote real functions of time and W a standard Brownian motion. We will now assume that the forward rate has Gaussian dynamics instead of log-normal. We should still have that F is a Martingale under $\mathbb{Q}^{T_2}$. Combining these two properties, we arrive at the following SDE for F:

$$dF(t) = v(t)dW(t)$$

where $v(t)$ denotes the instantaneous (Gaussian) volatility. In other words, this corresponds to the Bachelier framework for an asset with zero drift. It should be clear that an expression for F is this time given by:

$$F(T) = F(0) + \int_0^T v(s)dW(s)$$

Bachelier showed that within this setting a closed-form expression for an option on F (i.e. a caplet) can be derived. The steps are somewhat similar to that of Black's model. We again need to evaluate the following conditional expectation

$$\begin{aligned}\mathbb{E}^{Q^{T_2}}\left(\left(F\left(T_1\right)-K\right)^+\middle|\mathcal{F}_t\right) &= \mathbb{E}^{Q^{T_2}}\left(F\left(T_1\right)\mathbb{1}_{F(T_1)>K}\middle|\mathcal{F}_t\right)-\mathbb{E}^{Q^{T_2}}\left(K\mathbb{1}_{F(T_1)>K}\middle|\mathcal{F}_t\right)\\ &= \mathbb{E}^{Q^{T_2}}\left(\int_t^{T_1} v(s)dW(s)\mathbb{1}_{F(T_1)>K}\middle|\mathcal{F}_t\right)-\mathbb{E}^{Q^{T_2}}\left(\left(K-F(t)\right)\mathbb{1}_{F(T_1)>K}\middle|\mathcal{F}_t\right)\end{aligned}$$

We will omit the details in evaluating these expectation, as they can be found in Musiela and Rutkowski [2005]. They yield

$$\begin{aligned}\mathbb{E}^{Q^{T_2}}\left(\int_t^{T_1} v(s)dW(s)\mathbb{1}_{F(T_1)>K}\middle|\mathcal{F}_t\right) &= \sqrt{\int_t^{T_1} v^2(s)ds}\cdot n\left(\frac{F(t)-K}{\sqrt{\int_t^{T_1} v(s)ds}}\right)\\ \mathbb{E}^{Q^{T_2}}\left(\left(K-F(t)\right)\mathbb{1}_{F(T_1)>K}\middle|\mathcal{F}_t\right) &= \left(K-F(t)\right)\cdot\mathcal{N}\left(\frac{F(t)-K}{\sqrt{\int_t^{T_1} v(s)ds}}\right)\end{aligned}$$

where $n(\cdot)$ denotes the probability density function of a standard normal random variable, given by

$$n(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

Summarizing, we have found a closed-form expression for the caplet price $V(t)$ namely:

$$V(t) = P\left(t,T_2\right)\cdot\Delta t\left[\int_t^{T_2} v^2(s)ds\cdot n\left(d\left(K,F(t),\int_t^{T_2} v^2(s)ds\right)\right)+\left(F(t)-K\right)\cdot\mathcal{N}\left(d\left(K,F(t),\int_t^{T_2} v^2(s)ds\right)\right)\right]$$

where

$$d\left(K,F,u\right) = \frac{F-K}{\sqrt{u}}$$

In general, the formula

$$c\left(K,F,u\right) = u\cdot n\left(d\left(K,F,u\right)\right)+\left(F-K\right)\cdot\mathcal{N}\left(d\left(K,F,u\right)\right)$$

is referred to as *Bachelier's formula*.

3.5 Swaps

Here we introduce the two interest rate derivatives that play a central role throughout this thesis, namely the interest rate swap and the cross-currency swap. We treat the set-up of the instruments and illustrate how their risk-neutral price is evaluated. Additionally we will introduce the notion of a swaption, which will use in later later chapters. The definitions we present here are largely based on Brigo and Mercurio [2007].

3.5.1 The interest rate swap

Recall that a FRA is a contract that allows the buyer to fix a future interest rate. A generalization of such an instrument is the interest rate swap (IRS). It is one of the most commonly traded OTC derivatives and there exist multiple variations to it. We discuss here the floating-fix version of the IRS.

An IRS is a contract that settles a sequence of cash exchanges between two parties. The instrument specifies several properties.

- A set $\mathcal{T}$ of future dates $t < T_0 < T_1 < \ldots < T_m$. The time-instants are usually equidistant, meaning that $\Delta t_i = T_i - T_{i-1}$ is equal for each $i \in \{1, \ldots, m\}$. Typical accrual periods Δt are 1 month, 3 months, 6 months or 1 year.
- A notional amount N.
- A fixed rate K.

One sequence of payments is referred to as the fixed leg as it is associated with the fixed rate K. At each date $T_i \in \{T_1, \ldots, T_m\}$, this leg pays out an amount

$$N \cdot K \cdot \Delta t_i$$

In return the floating leg, associated to the floating LIBOR, pays out

$$N \cdot L\left(T_{i-1}, T_i\right) \cdot \Delta t_i$$

We assume that the fixed and floating cashflows are each time exchanged at the same date. Note that the interest rate $L\left(T_{i-1}, T_i\right)$ is fixed at date T_{i-1}, which corresponds to the end-date of the previous coupon, but is only paid out at T_i, the end date of the current coupon. This induces a natural time-lag, which will turn out to be convenient in the following pricing routine.

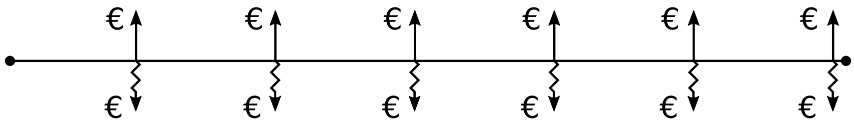

Figure 3.1: Graphic illustration of the cashflows of a floating-fix IRS.

Since the IRS is a derivative security with multiple pay-offs, we will treat it as a sequence of single payments trades. This allows us to calculate its fair price by the risk-neutral pricing formula. Denote the value of the swap at time $t < T_0$ by $V(t)$, then we have

$$\begin{aligned} V(t) &= \mathbb{E}^{Q}\left(\sum_{i=1}^{m} D\left(t, T_i\right) N \Delta t_i \left(L\left(T_{i-1}, T_i\right) - K\right) \middle| \mathcal{F}_t\right) \\ &= N \sum_{i=1}^{m} \mathbb{E}^{Q}\left(D\left(t, T_i\right) \Delta t_i \left(\frac{1 - P\left(T_{i-1}, T_i\right)}{\Delta t_i \cdot P\left(T_{i-1}, T_i\right)} - K\right) \middle| \mathcal{F}_t\right) \end{aligned}$$

Switch to the T_i-forward measure, recall that $P\left(T_{i-1}, T_{i-1}\right) = 1$ and use the Martingale property of assets denominated by $P\left(T_{i-1}, T_i\right)$ under $\mathbb{Q}^{T_i}$ to see

$$\begin{aligned} V(t) &= N \sum_{i=1}^{m} P\left(t, T_i\right) \mathbb{E}^{Q^{T_i}}\left(\Delta t_i \left(\frac{P\left(T_{i-1}, T_{i-1}\right) - P\left(T_{i-1}, T_i\right)}{\Delta t_i \cdot P\left(T_{i-1}, T_i\right)} - K\right) \middle| \mathcal{F}_t\right) \\ &= N \sum_{i=1}^{m} P\left(t, T_i\right) \Delta t_i \left(\frac{P\left(t, T_{i-1}\right) - P\left(t, T_i\right)}{\Delta t_i \cdot P\left(t, T_i\right)} - K\right) \\ &= N \sum_{i=1}^{m} \left(P\left(t, T_{i-1}\right) - P\left(t, T_i\right) - P\left(t, T_i\right) \Delta t_i K\right) \\ &= N \left(P\left(t, T_0\right) - P\left(t, T_m\right) - K \sum_{i=1}^{m} P\left(t, T_i\right) \Delta t_i\right) \end{aligned}$$

Note that we have now decomposed the value of the IRS in terms of zero-coupon bonds and some scalars. We will later see that this is very convenient for simulation purposes.

It is common practice to enter an IRS deal *at par*. This means that at inception of the trade, the fixed rate is chosen such that the contract has zero value at that time. This particular rate K is called the *swap rate*. Hence, by its definition, at any time t the swap rate $S(t)$ is defined by the following equation:

$$\begin{aligned} N\left(P\left(t,T_0\right)-P\left(t,T_m\right)-S(t)\sum_{i=1}^{m}P\left(t,T_i\right)\Delta t_i\right) &= 0 \\ \Rightarrow \qquad \frac{P\left(t,T_0\right)-P\left(t,T_m\right)}{\sum_{i=1}^{m}P\left(t,T_i\right)\Delta t_i} &= S(t) \end{aligned}$$

3.5.2 The cross-currency swap

The cross-currency swap (CCS) can be considered an IRS with one leg in a foreign currency, in combination with two FX forward contracts. Just like the IRS there exist multiple variations to this instrument. Here we will treat the *floating-fixed* version with a notional exchange at the start and the maturity of the trade.

The CCS is a contract that settles a sequence of payments between two parties, in two different currencies. We will denote the currency associated with the floating leg as the *domestic* currency and that of the fixed leg as the *foreign* currency. The following properties are specified in the contract:

- A set $\mathcal{T}$ of future dates $t < T_0 < T_1 < \ldots < T_m$. We assume the accrual periods to be equidistant, denoted by $\Delta t_i = T_i - T_{i-1}$ for each $i \in \{1,\ldots,m\}$.
- A domestic notional amount N^d and a foreign notional amount N^f.
- A fixed rate K.

At date T_0, the fixed-rate payer transfers the domestic notional amount N^d to the floating-rate payer. In return he receives the foreign notional amount N^f. It is common practice to let the ratio of the two notionals be equal to the FX spot rate between the two currencies at that time. This means that

$$N^d = \varphi\left(T_0\right)N^f$$

As a consequence, the net present value of the notional exchange at inception is equal to zero. At each subsequent date $T_i \in \{T_1,\ldots,T_m\}$ the the fixed-rate payer pays out

$$N^f \cdot K \cdot \Delta t_i$$

in the foreign currency. We will refer to these payments as the fixed leg or foreign leg. In return, at each corresponding date T_i the other party pays out

$$N^d \cdot L\left(T_{i-1},T_i\right)\cdot \Delta t_i$$

in the domestic currency. Naturally, L denotes here the LIBOR that corresponds to the domestic currency. This sequence is often referred to as the floating leg or domestic leg. Finally at the maturity date T_m, the notionals are exchanged back, meaning that N^f is transferred to the floating-rate payer and N^d to the fixed-rate payer. It is very likely that at this point in time $N^d \neq \varphi\left(T_m\right)N^f$. Therefore, the exchange at maturity typically does have a non-zero value.

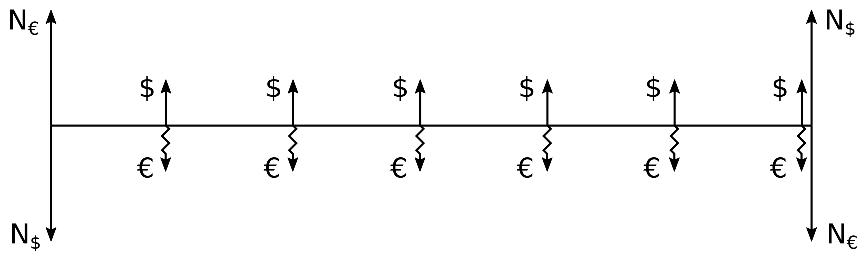

Figure 3.2: Graphic illustration of the cashflows of a floating-fix CCS.

Using the risk-neutral pricing formulas we find for the fair value of a CCS contract at time $t < T_0$ expressed in domestic currency:

$$\begin{aligned} V(t) &= \mathbb{E}^{Q}\left(\sum_{i=1}^{m} D\left(t,T_i\right)\varphi\left(T_i\right)N^{f}\Delta t_i K - \sum_{i=1}^{m} D\left(t,T_i\right)L\left(T_{i-1},T_i\right)N^{d}\Delta t_i\right. \\ &\qquad \left. D\left(t,T_0\right)\left(\varphi\left(T_0\right)N^{f}-N^{d}\right)+D\left(t,T_m\right)\left(N^{d}-\varphi\left(T_m\right)N^{f}\right)\right|\mathcal{F}_t\Bigg) \end{aligned}$$

As mentioned, we assume that $N^d = \varphi\left(T_0\right)N^f$, which allows the initial notional exchange to drop out of the equation. Hence we can rewrite

$$\begin{aligned} V(t) &= N^f K\sum_{i=1}^{m}\Delta t_i\mathbb{E}^{Q}\left(D\left(t,T_i\right)\varphi\left(T_i\right)\middle|\mathcal{F}_t\right) \\ &\quad -N^d\sum_{i=1}^{m}\Delta t_i\mathbb{E}^{Q}\left(D\left(t,T_i\right)L\left(T_{i-1},T_i\right)\middle|\mathcal{F}_t\right) \\ &\quad +N^d\mathbb{E}^{Q}\left(D\left(t,T_m\right)\middle|\mathcal{F}_t\right)-N^f\mathbb{E}^{Q}\left(D\left(t,T_m\right)\varphi\left(T_m\right)\middle|\mathcal{F}_t\right) \end{aligned}$$

The above expression shows that the value of the CCS is composed of a superposition of payments, scheduled at the future dates $T_i \in \mathcal{T}$. We proceed by changing the measure of each expectation to its corresponding domestic T_i-forward measure, T_i being the pay-date of each individual cashflow. We then find

$$\begin{aligned} V(t) &= N^f K\sum_{i=1}^{m}\Delta t_i P\left(t,T_i\right)\mathbb{E}^{Q^{T_i}}\left(\varphi\left(T_i\right)\middle|\mathcal{F}_t\right) \\ &\quad -N^d\sum_{i=1}^{m}\Delta t_i P\left(t,T_i\right)\mathbb{E}^{Q^{T_i}}\left(L\left(T_{i-1},T_i\right)\middle|\mathcal{F}_t\right) \\ &\quad +N^d P\left(t,T_m\right)\mathbb{E}^{Q^{T_m}}\left(1\middle|\mathcal{F}_t\right)-N^f P\left(t,T_m\right)\mathbb{E}^{Q^{T_m}}\left(\varphi\left(T_m\right)\middle|\mathcal{F}_t\right) \end{aligned}$$

Finally, recall that $\mathbb{E}^{Q^T}\left(L\left(S,T\right)\middle|\mathcal{F}_t\right) = F(t,S,T)$ and $\mathbb{E}^{Q^T}\left(\varphi\left(T\right)\middle|\mathcal{F}_t\right) = \Phi\left(t,T\right)$. We conclude that the fair value of the CCS can be expressed as

$$\begin{aligned} V(t) &= \sum_{i=1}^{m}\Delta t_i P\left(t,T_i\right)\left(N^f K\Phi\left(t,T_i\right)-N^d F\left(t,T_{i-1},T_i\right)\right) \\ &\quad +N^d P\left(t,T_m\right)-N^f P\left(t,T_m\right)\Phi\left(t,T_m\right) \end{aligned}$$

3.5.3 The swaption

Lastly we consider an interest rate derivative called the swaption. A swaption is a contract that gives the holder the right to enter an IRS at a future time instant for a specific fixed rate K (physical settlement) or to receive the cash value of the IRS (cash settlement). A swaption thus defines a European option written on a swap. Consider for the underlying IRS the following properties

- A set $\mathcal{T}$ of future dates $t < T_0 < T_1 < \ldots < T_m$., with equidistant accrual periods, denoted by $\Delta t_i = T_i - T_{i-1}$ for each $i \in \{1,\ldots,m\}$.
- A notional amount N.
- A fixed rate K.

Time T_0 then denotes the expiry date of the swaption, meaning that the holder will exercise the option if it has positive value at that time. We know that the value of an IRS at time T_0 is given by

$$\begin{aligned} V(T_0) &= N\left(P(T_0,T_0) - P(T_0,T_m) - K\sum_{i=1}^{m} P(T_0,T_i)\,\Delta t_i\right) \\ &= N\left(\sum_{i=1}^{m} P(T_0,T_j)\,\Delta t_j\right)\cdot\left(\frac{P(T_0,T_0)-P(T_0,T_m)}{\sum_{j=1}^{m} P(T_0,T_j)\,\Delta t_k} - K\right) \end{aligned}$$

Recall that the swap rate at time T_0 was defined as

$$S(T_0) = \frac{P(T_0,T_0)-P(T_0,T_m)}{\sum_{j=1}^{m} P(T_0,T_j)\,\Delta t_j}$$

To obtain a more convenient expression for the value of a swaption, we introduce the notion of an *annuity*. We will denote an annuity corresponding to the time schedule $\mathcal{T}$ as $A^{0,m}(t)$. It is defined as

$$A^{0,m}(t) = \sum_{i=1}^{m} P(t,T_i)\,\Delta t_i$$

Note that an annuity is nothing more that the superposition of a finite number of zero-coupon bonds. As zero-coupon bonds are freely tradable, positively priced assets, so are annuities. Therefore, $A^{0,m}(t)$ is in fact a well-defined numeraire. Using this notation, the value of a swaption at expiry can be written as

$$V(T_0) = N\cdot A^{0,m}(T_0)\cdot(S(T_0)-K)^+$$

Now we are interested in the fair price of a swaption at some time $t < T_0$. Naturally this is given by the risk-neutral pricing formula, which yields

$$\begin{aligned} V(t) &= \mathbb{E}^Q\left(D(t,T_0)\,V(T_0)\middle|\,\mathcal{F}_t\right) \\ &= \mathbb{E}^Q\left(D(t,T_0)\cdot N\cdot A^{0,m}(T_0)\cdot(S(T_0)-K)^+\middle|\,\mathcal{F}_t\right) \end{aligned}$$

Since the annuity $A^{0,m}(t)$ is a numeraire, we know that there exist a measure $\mathbb{Q}^{A^{0,m}}$ such that any asset denominated by $A^{0,m}(t)$ becomes a Martingale under this measure. By changing to the annuity-measure, we see that we can therefore rewrite

$$\begin{aligned} V(t) &= A^{0,m}(t)\,\mathbb{E}^{A^{0,m}}\left(\frac{N\cdot A^{0,m}(T_0)\cdot(S(T_0)-K)^+}{A^{0,m}(T_0)}\middle|\,\mathcal{F}_t\right) \\ &= N\cdot A^{0,m}(t)\,\mathbb{E}^{A^{0,m}}\left((S(T_0)-K)^+\middle|\,\mathcal{F}_t\right) \end{aligned}$$

Note that this expression is now only dependent on the stochastic behavior of the swap rate. We can therefore evaluate the swaption price by assuming appropriate dynamics on the swap rate process. The assumption that S follows a Gaussian process would imply that we can price a swaption using Bachelier's formula. Likewise, the assumption that S follows a geometric Brownian motion would imply that we can use Black's formula.

4 A multi-currency framework for interest- and FX rates

Interest rate models are widely used to price interest rate related derivatives and financial risk-management in general. In the past decades many different models have been treated in the academic literature. Here we will treat the one-factor model proposed by John Hull and Alan White in 1990 [Hull and White, 1990]. The model is an extension of the 1977 Vasicek model and allows for a perfect fit of today's term-structure of interest rates. Due to the Gaussian character of the Hull-White state variables we can derive explicit pricing formulas for a large range of interest rate derivatives. On top of the Hull-White model, we will construct a multi-currency framework that allows us to model interest rates in different currencies together with their corresponding foreign exchange rates (FX). For the exchange rates we consider a Garman-Kohlhagen model, which assumes that the FX follows a log-normal process. We will introduce the mathematical background for a multi-currency setting and treat some properties of the exchange rate. We conclude this chapter by describing the numerical methods that are applied in standard valuation procedures.

4.1 The one-factor Hull-White model

A common approach to interest rate modeling is the simulation of a mathematical variable $r(t)$, which we call the short-rate. This one-dimensional instantaneous spot rate is in reality is not observed in the market. This variable corresponds to the drift-term of the money-market account, which we have seen in the previous chapter. As our research evolves around instruments directly related to interest rates, the consideration of a suitable interest rate model is key. We will introduce the one-factor Hull-White short-rate model, which can be categorized as an affine term-structure model. The main characteristic of such a model is that the continuously compounded spot interest rate $R(t,T)$ is an affine function (linear term plus constant) of the short-rate $r(t)$ [Brigo and Mercurio, 2007]:

$$R(t,T) = \alpha(t,T) + \beta(t,T)r(t)$$

An important implication of this definition is that a zero-coupon bond price can be written in the following form:

$$P\left(t,T\right) = A\left(t,T\right)e^{-B(t,T)r(t)}$$

Recall that the continuously compounded interest rate is defined through the relation

$$R\left(t,T\right) = -\frac{\log\left(P\left(t,T\right)\right)}{\Delta t}$$

We hence arrive at the above relation for P if we set

$$A\left(t,T\right) = e^{-\alpha(t,T)\Delta t}, \quad B\left(t,T\right) = \beta\left(t,T\right)\Delta t$$

For the pricing procedure of interest rate derivatives, this is a convenient property. We have seen that the model-independent definition of the zero-coupon bond price is given by

$$P(t,T) = \mathbb{E}^{Q}\left(e^{-\int_t^T r(s)ds}\middle|\mathcal{F}_t\right)$$

Depending on which model you use, the above expression can be difficult to compute. Since the Hull-White model belongs to the class of affine-term structure models, we will see that a convenient closed-form formula can be derived for $P\left(t,T\right)$. Zero-bonds are often considered the fundamental quantities in interest rate modeling. The risk-neutral value of many interest rate derivatives can be expressed in terms of $P\left(t,T\right)$. This implies that the model offers efficient pricing routines for these instrument, which is particularly important for calibration purposes. Additionally, its implementation is easy and efficient in comparison to other interest rate models. All together, the Hull-White model is still a popular tool in risk-management for financial institutions.

4.2 The dynamics of the short-rate

Hull and White have examined several variations of the short-rate process. Here we will treat the extension of the Vasicek model that considers the following short-rate dynamics under the risk-neutral measure $\mathbb{Q}$:

$$dr(t) = (\theta(t) - a \cdot r(t))\, dt + \sigma dW(t)$$

Here a, the mean-reversion rate, and σ, the volatility are deterministic scalars. W is a standard one-dimensional Brownian motion under $\mathbb{Q}$. $\theta(t)$ is a deterministic function of time, which is calibrated such that the corresponding yield curve matches the currently observed term-structure of interest rates in the market. The dynamics of $r(t)$ follow an Ornstein-Uhlenbeck process. This yields a mean-reverting character, which is a desired property as this behavior is also observed in reality. As the short-rate is driven by a single one-dimensional Brownian motion, this model is referred to as the *one-factor* Hull-White model.

Given its dynamics, an expression for $r(t)$ can be derived by an application of the one-dimensional Itô-Doeblin formula [Sterling and Hári, 2008]. Let $f(t,r) = r \cdot e^{a \cdot t}$, it then follows that

$$\begin{aligned} df\,(t, r(t)) &= ar(t)e^{at}dt + e^{at}dr(t) \\ &= ar(t)e^{at}dt + e^{at}\left[(\theta(t) - a \cdot r(t))\, dt + \sigma dW(t)\right] \\ &= e^{at}\theta(t)dt + e^{at}\sigma dW(t) \end{aligned}$$

Hence we see that for any $0 \leq s \leq t$, we have

$$r(t) \cdot e^{at} = r(s) \cdot e^{as} + \int_s^t e^{au}\theta(u)du + \int_s^t e^{au}\sigma dW(u)$$

which can equivalently be rewritten as

$$r(t) = r(s)e^{a(s-t)} + \int_s^t e^{a(u-t)}\theta(u)du + \int_s^t e^{a(u-t)}\sigma dW(u)$$

What we know from the properties of an Itô integral with deterministic integrand that $\int_s^t e^{a(u-t)}\sigma dW(u)$ is Gaussian with mean zero. Its variance is computed by the application of Itô isometry. Due to the deterministic nature of θ and $r(s)$ conditioned on $\mathcal{F}_s$, we find that $r(t)$ is normally distributed with moments:

$$\begin{aligned} \mathbb{E}^Q\left(r(t)|\,\mathcal{F}_s\right) &= r(s)e^{a(s-t)} + \int_s^t e^{a(u-t)}\theta(u)du \\ \mathrm{Var}\left(r(t)|\,\mathcal{F}_s\right) &= \int_s^t \left(e^{a(u-t)}\sigma\right)^2 du = \frac{\sigma^2}{2a}\left(1 - e^{2a(s-t)}\right) \end{aligned}$$

4.3 Pricing a zero-coupon bond

The Gaussian nature of short-rate has as a convenient consequence that not only $r(t)$ itself, but also $\int_s^t r(u)du$ conditioned on $\mathcal{F}_s$ is normally distributed. Considering the integrated short-rate we find that

$$\begin{aligned} \int_s^t r(u)du &= \int_s^t r(s)e^{a(s-u)}du + \int_s^t \int_s^u e^{a(v-u)}\theta(v)dvdu \\ &\quad + \int_s^t \int_s^u e^{a(v-u)}\sigma dW(v)du \end{aligned}$$

By the positive integrability of $e^{a(v-u)}\sigma$ and $\theta(v)$ on $[0,T]$, we can apply Fubini's theorem to change the order of integration of the last two terms (see Anderson and Piterbarg [2010b]), yielding

$$\begin{aligned}\int_s^t r(u)du &= \int_s^t r(s)e^{a(s-u)}du + \int_s^t\int_v^t e^{a(v-u)}\theta(v)dudv \\ &\quad + \int_s^t\int_v^t e^{a(v-u)}\sigma dudW(v) \\ &= \frac{r(s)}{a}\left(1-e^{a(s-t)}\right) + \int_s^t \frac{1}{a}\left(1-e^{a(v-t)}\right)\theta(v)dv \\ &\quad + \int_s^t \frac{1}{a}\left(1-e^{a(v-t)}\right)\sigma dW(v)\end{aligned}$$

As a result we find that $\int_s^t r(u)du$ is indeed Gaussian. To shorten the notation, define $B(S,T) = \frac{1}{a}\left(1-e^{a(S-T)}\right)$. The moments of the integrated short-rate can be obtained by application of Itô isometry, so that

$$\begin{aligned}\mathbb{E}^Q\left(\int_s^t r(u)du\middle|\mathcal{F}_s\right) &= r(s)B(s,t) + \int_s^t B(v,t)\theta(v)dv \\ \mathrm{Var}\left(\int_s^t r(u)du\middle|\mathcal{F}_s\right) &= \int_s^t \left(\frac{1}{a}\left(1-e^{a(v-t)}\right)\sigma\right)^2 dv \\ &= \frac{\sigma^2}{a^2}\left(t-s+\frac{2}{a}e^{a(s-t)}-\frac{1}{2a}e^{2a(s-t)}-\frac{3}{2a}\right)\end{aligned}$$

Now we have come to a point where we can write an expression for a zero-coupon bond price. Recall that its fair value under the risk-neutral measure is defined by $\mathbb{E}^Q\left(e^{-\int_t^T r(s)ds}\middle|\mathcal{F}_t\right)$. We now know that $\int_t^T r(s)ds$ is normally distributed. As a consequence, $e^{-\int_t^T r(s)ds}$ is a lognormal random variable. The moments of the lognormal distribution are known. Let X be a Gaussian random random variable with parameters μ and σ^2. Then the mean of $Y=\exp\{X\}$ is given by $\mathbb{E}(Y) = \exp\left\{\mu + \sigma^2/2\right\}$. Using this given property lets us compute

$$\begin{aligned}P(s,t) &= \mathbb{E}^Q\left(e^{-\int_s^t r(u)du}\middle|\mathcal{F}_s\right) \\ &= \exp\left\{-\mathbb{E}^Q\left(\int_s^t r(u)du\middle|\mathcal{F}_s\right) - \frac{1}{2}\mathrm{Var}\left(\int_s^t r(u)du\middle|\mathcal{F}_s\right)\right\} \\ &= A(s,t)\,e^{-B(s,t)r(s)}\end{aligned}$$

Where A and B are both deterministic functions of time, defined as

$$\begin{aligned}A(s,t) &= \exp\left\{-\int_s^t B(u,t)\theta(u)du \right.\\ &\quad \left. -\frac{\sigma^2}{2a^2}\left(t-s+\frac{2}{a}e^{a(s-t)}-\frac{1}{2a}e^{2a(s-t)}-\frac{3}{2a}\right)\right\} \\ B(s,t) &= \frac{1}{a}\left(1-e^{a(s-t)}\right)\end{aligned}$$

As a result we have derived a tractable expression for a zero-coupon bond price, being a deterministic function of time and $r(s)$. What remains to be done, is to compute $\theta(t)$. We do so by a fitting procedure, which we will treat in the following paragraph.

4.4 Fitting to the current market

A strong improvement of the Hull-White model compared to the Vasicek model, is that the time-dependence of $\theta(t)$ allows for a perfect fit of the model to the currently observed term-structure of zero-coupon bonds. This is for example shown in Sterling and Hári [2008], of which we show the main steps here. Our starting point is therefore to find an expression for θ, such that that our modeled zero-coupon prices $P(0,T)$ match the bond prices observed in the market $P^M(0,T)$. In other words: $\forall_{T>0}\, P^M(0,T) = P(0,T)$. Mathematically speaking, the calibration procedure it is easier if we fit θ to the term-structure of instantaneous forward rates $f^M(0,T)$. Recall that an expression for f is given by

$$f(t,T) = -\frac{\partial \log\left(P(t,T)\right)}{\partial T}$$

With the model consistent expression for a zero-coupon bond from the previous section, we can substitute P in the relation above. We write $f^M(0,T)$ for the instantaneous forward that would correspond to the current market. Substitution of P yields

$$\begin{aligned}
f^M(0,T) &= \frac{-\partial\left(\log\left(A(0,T)\right) - B(0,T)\,r(0)\right)}{\partial T} \\
&= \frac{\partial}{\partial T}\left(\frac{1}{a}\int_0^T \left(1 - e^{a(u-T)}\right)\theta(u)du\right) \\
&\quad + \frac{\partial}{\partial T}\frac{\sigma^2}{2a^2}\left(T + \frac{2}{a}e^{-aT} - \frac{1}{2a}e^{-2aT} - \frac{3}{2a}\right) \\
&\quad - \frac{\partial}{\partial T}\frac{1}{a}\left(1 - e^{-aT}\right)r(0) \\
&= e^{-aT}\int_0^T e^{au}\theta(u)du - \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2 + e^{-aT}r(0)
\end{aligned}$$

In order to isolate $\theta(T)$ from the expression, we differentiate f^M a second time, so that we obtain

$$\begin{aligned}
\frac{\partial f^M(0,T)}{\partial T} &= -ae^{-aT}\int_0^T \theta(u)e^{au}du + \theta(T) \\
&\quad -\frac{\sigma^2}{a}e^{-aT} + \frac{\sigma^2}{a}e^{-2aT} - ae^{-aT}r(0)
\end{aligned}$$

Note that we have the relation $-ae^{-aT}\int_0^T \theta(u)e^{au}du - ae^{-aT}r(0) = -a\cdot f^M(0,T) - \frac{\sigma^2}{2a}\left(1-e^{-aT}\right)^2$. We substitute the relation into the expression above and do some rewriting. By doing so we end up with a compact expression for θ.

$$\begin{aligned}
\frac{\partial f^M(0,T)}{\partial T} &= -a\cdot f^M(0,T) - \frac{\sigma^2}{2a}\left(1-e^{-aT}\right)^2 + \theta(T) - \frac{\sigma^2}{a}e^{-aT} + \frac{\sigma^2}{a}e^{-2aT} \\
\Rightarrow \quad \theta(T) &= \frac{\partial f^M(0,T)}{\partial T} + a\cdot f^M(0,T) + \frac{\sigma^2}{2a}\left(1 - e^{-2aT}\right)
\end{aligned}$$

If we substitute this result into our formula for a zero-coupon bond, we arrive at the following expression

$$P(s,t) = \frac{P^M(0,t)}{P^M(0,s)}\exp\left\{B(s,t)f^M(0,s) - \frac{\sigma^2}{4a}B^2(s,t)\left(1-e^{-2as}\right) - B(s,t)r(s)\right\}$$

4.5 The shifted short-rate process

Note that our formula for pricing a zero-coupon bond requires the instantaneous forward rate f^M as an input (next to $P^M(0,s)$ and $P^M(0,t)$). Although this parameter is mathematically well-defined, it can in reality not be observed in the market. Therefore it would be convenient to remove this term from the expression. It appears we can do so if we consider a related zero-mean process $x(t)$, (see also Sterling and Hári [2008]), of which the dynamics are defined as

$$dx(t) = -a\cdot x(t)dt + \sigma dW(t), \quad x(0) = 0$$

By a simple application of Itô's lemma on the function $f(t,x) = x \cdot e^{at}$, it can be shown that an expression for $x(t)$, conditioned on $\mathcal{F}_s$ is given by

$$x(t) = x(s)e^{a(s-t)} + \int_s^t e^{a(u-t)}\sigma dW(u)$$

If we compare this formula to that of $r(t)$, we see that for each $t > 0$, $r(t)$ can be constructed from the shifted short-rate process $x(t)$ through the relation $r(t) = x(t) + \alpha(t)$, where $\alpha(t) = f^M(0,t) + \frac{\sigma^2}{a^2}\left(1 - e^{-at}\right)^2$. If we substitute $x(t) + \alpha(t)$ in the formula for a zero-coupon bond, we find that

$$\begin{aligned} P(s,t) &= \frac{P(0,t)}{P(0,s)} \exp\left\{B(s,t)f^M(0,s) - \frac{\sigma^2}{4a}B^2(s,t)\left(1 - e^{-2as}\right) - B(s,t)\left(x(s) + \alpha(s)\right)\right\} \\ &= \frac{P(0,t)}{P(0,s)} \exp\left\{-\frac{\sigma^2}{4a}B^2(s,t)\left(1 - e^{-2s}\right) - \frac{\sigma^2}{a^2}B(s,t)\left(1 - e^{-as}\right)^2 - B(s,t)x(s)\right\} \end{aligned}$$

For simulation purposes this is an important result. For the computation of a zero-coupon bond, all we have to consider are the shifted short-rate process $x(t)$ and the current term-structure of bonds.

4.6 A multi-currency framework

In this section we expand the described Hull-White framework to fit a multi-currency market. Although the concepts that are presented here apply to an economy with many currencies, we will for now consider a simplified setting with only two. We will denote these two currencies like before as *domestic* and *foreign*. For each currency there is a risk-free bank account process, both with a drift corresponding to their respective short-rate. The dynamics of the short-rates conveniently generalize to the new setting, in which we now consider two processes:

$$\begin{aligned} dr^d(t) &= \left(\theta^d(t) - a^d \cdot r^d(t)\right)dt + \sigma^d dW^d(t) \\ dr^f(t) &= \left(\theta^f(t) - a^f \cdot r^f(t)\right)dt + \sigma^f dW^f(t) \end{aligned}$$

Although the notation might look straight-forward, we should keep in mind that the processes are both defined under the risk-neutral measure of their own currency. Thus we have that W^d is a standard brownian motion under $\mathbb{Q}^d$ and W^f a standard brownian motion under $\mathbb{Q}^f$.

In addition to interest rates, we need to consider the process for the foreign exchange rate φ. The dynamics of FX rate can be captured by a stochastic process, which we will define under the domestic risk-neutral measure. From a domestic perspective, the process φ represents the current value of one unit of foreign currency. Clearly, this value can never be negative. For this reason, it is common practice to model the FX process as a geometric Brownian motion:

$$d\varphi(t) = \mu(t)\varphi(t)dt + \sigma^\varphi(t)\varphi(t)dW^\varphi(t)$$

where we choose W^φ to be a standard brownian motion under the domestic risk-neutral measure. Unlike the short-rate process where the volatility is modeled as a scalar, σ^φ is considered a deterministic, continuous function of time. The drift $\mu(t)$ can be determined by a no-arbitrage argument, described for example in Shreve [2004]. The arguments is as follows.

Under $\mathbb{Q}^d$, investing in the domestic money market should yield the same mean rate of return as investing in the foreign money market and subsequently exchanging it to the domestic currency. Indeed we have for the domestic money market account:

$$dB^d(t) = r^d(t)B^d(t)dt$$

Since foreign currency can be considered a freely tradable asset, risk-neutral pricing theory should apply. By the definition of $\mathbb{Q}^d$, the discounted process $D(0,t)B^f(t)\varphi(t)$ must hence be a Martingale. From this we can deduce that its differential should be given by

$$d\left(D(0,t)B^f(t)\varphi(t)\right) = D(0,t)B^f(t)\varphi(t)\sigma^\varphi(t)dW^\varphi(t)$$

The differential of the undiscounted process $B^f(t)\varphi(t)$, can now be derived by multiplying the discounted process with the domestic bank account process, as $B^d(t) \cdot D(0,t) = 1$. Subsequently apply Itô's product rule to compute

$$\begin{aligned} d\left(B^f(t)\varphi(t)\right) &= d\left(B^d(t)D(0,t)B^f(t)\varphi(t)\right) \\ &= B^d(t) \cdot d\left(D(0,t)B^f(t)\varphi(t)\right) \\ &\quad + D(0,t)B^f(t)\varphi(t) \cdot dB^d(t) \\ &= B^f(t)\varphi(t)\left(r^d(t)dt + \sigma^\varphi(t)dW^\varphi(t)\right) \end{aligned}$$

As a final step we recover the differential for φ by multiplying the undiscounted process given above with the foreign discount process. Recall that $dD^f(0,t) = -r^f(t)D^f(0,t)dt$. By applying Itô's product rule one more time and substituting with the result above we find

$$\begin{aligned} d\varphi(t) &= d\left(D^f(0,t)B^f(t)\varphi(t)\right) \\ &= D^f(0,t) \cdot d\left(B^f(t)\varphi(t)\right) \\ &\quad + B^f(t)\varphi(t) \cdot dD^f(0,t) \\ &= \varphi(t)\left(r^d(t) - r^f(t)\right)dt + \varphi(t)\sigma^\varphi(t)dW(t) \end{aligned}$$

What we conclude is that the mean rate of change of the FX rate process is the difference between the foreign and domestic short-rate. The φ dynamics hence follow a geometric Brownian motion with adapted, stochastic drift given by $\mu(t) = r^d(t) - r^f(t)$ under the domestic risk-neutral measure.

In line with the analysis of the generalized Brownian motion in Shreve [2004], we can now explicitly write the formula for the foreign exchange rate, which is given by

$$\varphi(t) = \varphi(0) \cdot \exp\left\{\int_0^t \left(r^d(s) - r^f(s) - \frac{1}{2}\sigma^\varphi(s)^2\right)ds + \int_0^t \sigma^\varphi(s)dW^\varphi(s)\right\}$$

To see why this is the case, consider the Itô process Y, that is defined as.

$$Y(t) = \int_0^t \left(r^d(s) - r^f(s) - \frac{1}{2}\sigma^\varphi(s)^2\right)ds + \int_0^t \sigma^\varphi(s)dW^\varphi(s)$$

Clearly, the dynamics of process Y are described by the SDE for which we have

$$\begin{aligned} dY(t) &= \left(r^d(t) - r^f(t) - \frac{1}{2}\sigma^\varphi(t)^2\right)ds + \sigma^\varphi(t)dW^\varphi(t) \\ dY(t)dY(t) &= \sigma^\varphi(t)^2 dt \end{aligned}$$

Now apply Itô's lemmo to the function $f(y) = \varphi(0) \cdot \exp\{y\}$, to find

$$d\varphi(t) = df\left(Y(t)\right) = \varphi(t)\left(r^d(t) - r^f(t)\right)dt + \varphi(t)\sigma^\varphi(t)dt$$

which corresponds with our original dynamics for φ.

4.7 Numerical simulation methods

In this section we will discuss the transition from exact continuous SDE's to discretized numerical approximations. Within the framework of short-rate models, the *Monte Carlo* method is a powerful tool. Although its convergence is sometimes slow, it is applicable to a broad range of products of which the payout is path-dependent and relies on future values of an underlying variable. The method simulates future scenarios, by propagating relevant risk-factors (interest rate, FX rate) through time, by considering the variable's time-dependent transition probabilities. The core mathematical ingredient of Monte Carlo is the Strong Law of Large Numbers. In the risk-neutral setting, pricing a contract often comes down to the computation of an expectation. Through simulation, a large number of independent realizations of a variable is computed. For each particular realization j, the product's value $V_j(t)$ is determined.

The result is an independent sample of product valuations, of which the sample mean is an estimator of today's risk-neutral price of the product

$$V(t) = \frac{1}{N} \sum_{j=1}^{N} V_j(t)$$

Simply said, the Monte Carlo method which we will apply comes down to the following steps:

- Simulate discretized paths for the relevant risk-factors, according to its risk-neutral dynamics.
- Compute the fair value of the product or portfolio, using the closed-form risk-neutral pricing formulas on each path.
- Compute the mean of the sampled valuations.

4.7.1 The short-rate process

Within the framework of Hull-White most of the interest rate derivatives that we consider can be priced once we know the future values of zero-coupon bonds. We have seen that for the computation of a zero-coupon bond price, it suffices to consider the the mean-zero process $x(t)$. We will therefore apply the Monte Carlo method to numerically simulate realizations of $x(t)$, so that we can subsequently approximate bond prices and thus related product valuations. The dynamics of the process x are our starting point:

$$dx(t) = -a \cdot x(t)dt + \sigma dW(t)$$

We will consider a discretized Euler set of time instants. Let today be time zero, assume our product of interest matures at time T, and let $\mathcal{T}$ denote a set of sampling times $0 = t_0 < t_1 < \ldots < t_n = T$. Denote the year-fraction in-between each future time instant by $\Delta t_i = t_{i+1} - t_i$. Our aim is to path-wise approximate $x(t_i)$ for each $t_i \in \mathcal{T}$. We do so by applying the Euler forward method on the SDE of the short-rate, where we utilize our knowledge on the transition distributions introduced by the Brownian motion. To obtain simulated scenarios, we consider the following Euler scheme:

$$r(t_{i+1}) = r(t_i) + (\theta(t_i) - a \cdot r(t_i))\Delta t_i + \sigma \Delta W(t_i)$$

The dynamics of W are constructed by the simulation of a random walk. From the properties of the Brownian motion, we know that the increments of W are independent and normally distributed. The transition densities of W are hence Gaussian and we have $\Delta W(t_i) = (W(t_{i+1}) - W(t_i)) \sim \mathcal{N}(0, t)$. Let $Z_1, \ldots, Z_n$ denote independent standard normal random variables. Then we can rewrite $\sigma \Delta W(t_i) = \sigma \sqrt{\Delta t_i} Z_i$. Through utilization of a pseudo-random number generator, we can sample independent realizations of Z_i. This subsequently allows us to simulate scenarios for x, from which path-specific bond and product valuations can be derived [Glasserman, 2004].

The reader should keep in mind that the Euler approximation does impose discretization errors. The simulation mean and variance will therefore not exactly match the moments introduced by the model, even for a large number of paths. The errors will be more severe for large values of Δt_i, but small values of Δt_i will come with higher computational cost. In practice we let the time increments be approximately one month, with smaller steps during the first 30 days.

4.7.2 The FX process

In the simulation of paths for the FX process, we use the same Euler discretization $\mathcal{T}$ as we considered for the short-rate process. Given the SDE for the FX rate, it would be natural to consider the following Euler scheme

$$\varphi(t_{i+1}) = \varphi(t_i) + \varphi(t_i)\left(r^d(t_i) - r^f(t_i)\right)\Delta t_i + \varphi(t_i)\sigma^{\varphi}(t_i)\Delta W^{\varphi}(t_i)$$

Application of this scheme however, would force us to first evaluate $r^d(t_i)$ and $r^f(t_i)$ for each $t_i \in \mathcal{T}$ on each path. Given the parallel computing set-up of our simulation engine, it is however more convenient to start with a related zero-mean process ϕ instead. Consider for this zero-mean process the following SDE

$$d\phi(t) = \sigma^{\varphi}(t)dW(t), \quad \phi(0) = 0$$

Then clearly

$$\phi(t) = \int_0^t \sigma^{\varphi}(s)dW^{\varphi}(s)$$

We will see that simulation of $\phi(t)$ suffices to deduce any realization of φ in a later stage. To see why this is the case, first recall the formula for φ, which we justified in the previous paragraph:

$$\begin{aligned}\varphi(t) &= \varphi(0) \cdot \exp\left\{\int_0^t \left(r^d(s) - r^f(s) - \frac{1}{2}\sigma^{\varphi}(s)^2\right) ds + \int_0^t \sigma^{\varphi}(s)dW^{\varphi}(s)\right\} \\ &= \varphi(0) \exp\left\{\int_0^t r^d(s)ds\right\} \exp\left\{-\int_0^t r^f(s)ds\right\} \exp\left\{-\int_0^t \frac{1}{2}\sigma^{\varphi}(s)^2 ds\right\} \exp\{\phi(t)\}\end{aligned}$$

From this expression for the FX rate, $\int_0^t \frac{1}{2}\sigma^{\varphi}(s)^2 ds$ is deterministic and can be evaluated up front. The integrals of the foreign and domestic short rates can be approximated by Riemann sums, using simulated realizations of the short-rate:

$$\int_0^{t_i} r(s)ds \approx \sum_{j=0}^{i-1} \frac{1}{2}\left(r\left(t_i\right) + r\left(t_{i+1}\right)\right) \cdot \Delta t_i$$

Together with a realization of $\phi(t)$, this lets us compute the FX rate. For simulation of the process ϕ we apply an Euler scheme

$$\phi\left(t_{i+1}\right) = \phi\left(t_i\right) + \sigma^{\varphi}\left(t_i\right) \Delta W^{\varphi}(t_i)$$

where pathwise approximation of $\Delta W^{\varphi}(t_i)$ is done in a similar way as for the short-rate process.

4.7.3 Multi-currency simulation

Within our two-currency Hull-White setting, we have now seen the methods to approximate scenarios for the two short-rates and the corresponding FX rate. In a realistic economy, these processes are typically not independent. If we are to model realistic scenarios, we are hence forced to work with correlated Brownian motions and simulate accordingly. A second complication comes from the fact that the dynamics of the two short-rates are defined under different measures. For simulation purposes, it is desirable to work under one single measure. Here we will elaborate the numerical approach that allows us to simulate each process under the risk-neutral measure and show how to numerically introduce correlations.

For the scenario simulation in our model, we need to approximate the dynamics of three correlated one-dimensional Brownian motions

$$\left\{\left(W^d_{\mathbb{Q}^d}(t), W^f_{\mathbb{Q}^f}(t), W^{\varphi}_{\mathbb{Q}^d}(t)\right)\middle| 0 \le t \le T\right\}$$

For computational simplicity it is preferred to describe all the variable's under the risk-neutral measure. For r^d and φ this is by definition the case, but r^f is defined under the foreign risk-neutral measure $\mathbb{Q}^f$. We will later on see that it is possible to describe the dynamics of r^f with a Brownian motion under $\mathbb{Q}^d$ by properly rewriting its SDE. For now assume we have already done this and wish to simulate three correlated Brownian motions under the common measure $\mathbb{Q}^d$, namely:

$$\left\{\left(W^d_{\mathbb{Q}^d}(t), W^f_{\mathbb{Q}^d}(t), W^{\varphi}_{\mathbb{Q}^d}(t)\right)\middle| 0 \le t \le T\right\}$$

Numerically, it is complicated to generate paths for correlated Brownian motions. We therefore aim to rewrite these processes in terms of three independent Brownian motions, as independent samples are in practice easily generated. Say that we want to compute $dW^x_{\mathbb{Q}^d}(t_i)$, then we use the following approximation

$$\Delta W^x_{\mathbb{Q}^d}\left(t_i\right) = \sqrt{\Delta t_i} Z_i$$

with Z_i a standard normal random variable. Now we need to incorporate the correlation between two Brownian motions through the relation $dW^x_{\mathbb{Q}^d} dW^y_{\mathbb{Q}^d} = \rho_{xy} dt$. Here $\rho_{xy} \in [-1, 1]$ denotes the correlation coefficient. In the two currency setting this introduces a 3×3 correlation matrix

$$\Sigma = \begin{pmatrix} 1 & \rho_{df} & \rho_{d\varphi} \\ \rho_{df} & 1 & \rho_{f\varphi} \\ \rho_{d\varphi} & \rho_{f\varphi} & 1 \end{pmatrix}$$

Given that this matrix is positive semi-definite, it allows a representation of the form $\Sigma = AA^\top$. Then by the Gaussian nature of a Brownian motion, the increments $dW^x_{\mathbb{Q}^d}$ can be approximated by

$$\begin{pmatrix} \Delta W^d_{\mathbb{Q}}(t_i) \\ \Delta W^f_{\mathbb{Q}^d}(t_i) \\ \Delta W^\varphi_{\mathbb{Q}^d}(t_i) \end{pmatrix} = \sqrt{\Delta t_i} A \cdot \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}$$

where Z_1, Z_2 and Z_3 are independent standard normal random variables. In practice it is common to consider the Cholesky factorization of Σ, which is possible given that Σ is positive-definite [Glasserman, 2004]. The Cholesky factorization of Σ is unique and lets A be a lower triangular matrix, so that

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

This is numerically convenient as it reduces the number of computations per simulated increment. We can thus represent the correlated increments as

$$\begin{aligned} \Delta W^d_{\mathbb{Q}^d}(t_i) &= \sqrt{\Delta t_i} a_{11} Z_1 \\ \Delta W^f_{\mathbb{Q}^d}(t_i) &= \sqrt{\Delta t_i} a_{21} Z_1 + \sqrt{\Delta t_i} a_{22} Z_2 \\ \Delta W^\varphi_{\mathbb{Q}^d}(t_i) &= \sqrt{\Delta t_i} a_{31} Z_1 + \sqrt{\Delta t_i} a_{32} Z_2 + \sqrt{\Delta t_i} a_{33} Z_3 \end{aligned}$$

or in other words we can write

$$d\mathbf{W}(t) = A d\widetilde{\mathbf{W}}(t)$$

where $\mathbf{W}(t) = \left(W^d_{\mathbb{Q}^d}(t), W^f_{\mathbb{Q}^d}(t), W^\varphi_{\mathbb{Q}^d}(t)\right)^\top$ is the 3-dimensional Brownian motion with correlated entries and $\widetilde{\mathbf{W}}(t) = \left(\tilde{W}^d_{\mathbb{Q}^d}(t), \tilde{W}^f_{\mathbb{Q}^d}(t), \tilde{W}^\varphi_{\mathbb{Q}^d}(t)\right)^\top$ the 3-dimensional Brownian motion with independent entries.

What remains to be shown is that the SDE of r^f can be rewritten so that the process is driven by a Brownian motion under $\mathbb{Q}^d$ instead of $\mathbb{Q}^f$. Therefore we will do a change of numeraire from the foreign to the domestic bank account and subsequently apply Girsanov's theorem in three dimensions. We follow the approach of Sterling and Hári [2010]. As both numeraires must be expressed in the same currency, we in fact change from $\varphi(t)B^f(t)$ to $B^d(t)$, which implies the following Radon-Nikodym derivative process

$$Z(t) = \mathbb{E}^{Q^d}\left(\frac{d\mathbb{Q}^f}{d\mathbb{Q}^d}\bigg|\mathcal{F}_t\right) = \frac{\varphi(t)B^f(t)}{\varphi(0)B^f(0)} \cdot \frac{B^d(0)}{B^d(t)} = \frac{\varphi(t)B^f(t)}{\varphi(0)B^d(t)}$$

The formulas for the bank accounts and FX process are all known and treated in previous paragraphs. If we plug their expressions into the result above we find

$$\begin{aligned} Z(t) &= \frac{\varphi(0) \cdot \exp\left\{\int_0^t \left(r^d(s) - r^f(s) - \frac{1}{2}\sigma^\varphi(s)^2\right) ds + \int_0^t \sigma^\varphi(s) dW^\varphi_{\mathbb{Q}^d}(s)\right\} \cdot \exp\left\{\int_0^t r^f(s) ds\right\}}{\varphi(0) \cdot \exp\left\{\int_0^t r^d(s) ds\right\}} \\ &= \exp\left\{\int_0^t \sigma^\varphi(s) dW^\varphi_{\mathbb{Q}^d}(s) - \frac{1}{2}\int_0^t \sigma^\varphi(s)^2 ds\right\} \end{aligned}$$

Note that we can alternatively write $dW^{\varphi}_{\mathbb{Q}^d}(s) = A_3 \cdot d\widetilde{\mathbf{W}}(s)$ in the expression for $Z(t)$, where A_i denotes the i^{th} row vector of the Cholesky matrix A. By definition we have that $\|A_3\|^2 = 1$. We therefore see that Z can equally be written as

$$Z(t) = \exp\left\{\int_0^t \sigma^{\varphi}(s) A_3 \cdot d\widetilde{\mathbf{W}}(s) - \frac{1}{2}\int_0^t \|\sigma^{\varphi}(s)A_3\|^2 \, ds\right\}$$

Using the process above, we can apply Girsanov's theorem in three dimensions. By its deterministic nature, $-\sigma^{\varphi}(t)$ is adapted to $\mathcal{F}_t$ on $[0,T]$. Furthermore, by an application of Tonnelli's theorem we have that

$$\begin{aligned}
\mathbb{E}^{Q^d}\left(\int_0^T (\|\sigma^{\varphi}(s)A_3\| \cdot Z(s))^2 \, ds\right) &= \int_0^T \sigma^{\varphi}(s)^2 \mathbb{E}^{Q^d}\left(Z(s)^2\right) ds \\
&= \int_0^T \sigma^{\varphi}(s)^2 \exp\left\{-\frac{1}{2}\int_0^s \sigma^{\varphi}(u)^2 du\right\} \\
&\quad \cdot \mathbb{E}^{Q^d}\left(\exp\left\{2\int_0^s \sigma^{\varphi}(u) dW^{\varphi}_{\mathbb{Q}^d}(u)\right\}\right) ds \\
&< \infty
\end{aligned}$$

The inequality follows from that fact that σ^{φ} is deterministic and bounded on $[0,T]$ and that due to the nature of the Itô integral, $\exp\left\{2\int_0^s \sigma^{\varphi}(u) dW^{\varphi}_{\mathbb{Q}^d}(u)\right\}$ is log-normally distributed and has a finite mean. Al together we satisfy the conditions of Girsanov's theorem. By the same theorem it follows that the three-dimensional process $\widehat{\mathbf{W}}$ defined as

$$\widehat{\mathbf{W}}(t) = \widetilde{\mathbf{W}}(t) - \int_0^t \sigma^{\varphi}(s) A_3^{\top} ds$$

is a three-dimensional Brownian motion under the measure $\mathbb{Q}^f$ with independent entries. Our aim was to rewrite the SDE for r^f. Now note that we can recover $W^f_{\mathbb{Q}^f}$ and $W^f_{\mathbb{Q}^d}$ from the equation above by multiplying both sides with A_2. Also note that $A_2 \cdot A_3^{\top} = \rho_{f\varphi}$. We find:

$$\begin{aligned}
A_2 \cdot \widehat{\mathbf{W}}(t) &= A_2 \cdot \widetilde{\mathbf{W}}(t) - \int_0^t \sigma^{\varphi}(s) A_2 \cdot A_3^{\top} ds \\
W^f_{\mathbb{Q}^f}(t) &= W^f_{\mathbb{Q}^d}(t) - \int_0^t \sigma^{\varphi}(s)\rho_{f\varphi} ds
\end{aligned}$$

In differential form this equation yields:

$$dW^f_{\mathbb{Q}^f}(t) = dW^f_{\mathbb{Q}^d}(t) - \sigma^{\varphi}(t)\rho_{f\varphi} dt$$

Using this result, we can reformulate the dynamics of r^f, this time in terms of a Brownian motion under the risk-neutral measure:

$$\begin{aligned}
dr^f(t) &= \left(\theta^f(t) - a^f \cdot r^f(t)\right) dt + \sigma^f \left(dW^f_{\mathbb{Q}^d}(t) - \sigma^{\varphi}(t)\rho_{f\varphi} dt\right) \\
&= \left(\theta^f(t) - \sigma^f \sigma^{\varphi}(t)\rho_{f\varphi} - a^f \cdot r^f(t)\right) dt + \sigma^f dW^f_{\mathbb{Q}^d}(t)
\end{aligned}$$

Hence we have shown that the dynamics of the processes r^d, r^f and φ can be represented in terms of independent Itô processes on a common measure space. In the generation of scenarios, the transition probabilities can thus be simulated with the use of a standard normal pseudo-random number generator, which is numerically convenient an computationally tractable.

5 Counterparty Credit Risk and CVA

In this section we will introduce the notion of counterparty credit risk and discuss how this notion leads to the concept of credit valuation adjustment. *Counterparty credit risk* or *default risk* is defined as the risk that a corporate engaged in a financial agreement defaults prior to the contract's maturity and therefore fails to make the remaining payments. The occurrence of default can have different underlying reasons, of which a corporate's bankruptcy is probably the simplest example. The concept of credit risk is particularly relevant for institutions that are involved in trading Over The Counter (OTC) derivatives. We speak of OTC derivatives if the two parties directly trade with one another, without a third party as intermediary. Such an intermediary could for example be an exchange or a central-clearing house. Usually they take standardized security measures to averse credit risk, such as mandatory collateral posting. In most occasions this allows them to recover the value of an outstanding trade in case of a default. This security is absent if two parties trade over the counter. A direct bilateral trade can be desirable if a party requires a customized agreement, which is the reason that currently a major share of the derivatives are traded over-the-counter. Interest rate products such as interest rate swaps and foreign exchange products such as cross-currency swaps are the most common trade types among OTC derivatives. Derivatives have a tendency to develop a non-zero exposure, which is defined as the expected fair market value of the contract at a future instant in time. A positive exposure implies a potential loss in an event of default, which in itself induces a risk. The development of exposure can be driven by one or more underlying risk-factors, like the propagation of interest rates or the foreign exchange rate. Also the deterioration of the counterparty's credit-worthiness can be an important driver of default risk [Gregory, 2010].

Methods in managing and quantifying counterparty risk have over the last decade received a large amount of attention within the financial industry. Prior to the last global credit crisis, which started in 2008, the threat of financial institutions defaulting on their obligations was often neglected or just assumed to be non-existent. The bankruptcy of Lehman Brothers and many other financial institutions over a short period of time, proved that entering a bilateral financial trade always introduces a default risk, no matter the counterparty. In a response, many financial institutions decided to add an additional charge to OTC trades to cover potential losses imposed by defaulting counterparties. This charge is know as the *Credit Valuation Adjustment*, often abbreviated as CVA. In essence, by the computation of CVA, one quantifies the value change imposed by default risk to outstanding trades. From a banks's point of view, pricing this risk is important for setting realistic transaction charges to clients. In addition it is a relevant quantity to report on the balance sheet [Gregory, 2010].

In the following section the general framework for CVA calculation will be illustrated. We will treat a derivation of the CVA charge and discuss an approximation method to quantify the credit-worthiness of a given counterparty.

5.1 Defining unilateral CVA

Here we will consider unilateral credit valuation adjustment to an OTC derivative contract. A transaction's CVA is defined as the difference between the value of the instrument, given that the counterparty might default and the value of the same instrument if it were traded with a hypothetical counterparty that is free of default risk. With the term *unilateral* we mean to specify that we only take into consideration the default risk of the counterparty. The credit worthiness of the reporting bank is in this context hence ignored. Here we will show how an analytical formula can be derived for the calculation of CVA. Before we do so, we will introduce some relevant terminology.

5.1.1 Marked-to-market

The marked-to-market (MtM) of a contract depicts its present value given the current market conditions. It is the risk-neutral value of all the payments a counterparty still is required to make, minus the payments it is required to receive. In other words, the MtM reflects the price of the contract if one were to replace it in the current market. This is therefore a continuously changing quantity that strongly depends on the present state of underlying risk-factors. In the context of modeling, pricing marked-to-market indicates the model parameters are calibrated to instrument prices that are presently observed in the market. In this case, a derivative's MtM corresponds to the fair value computed by the risk-neutral pricing formula. This is in contrast with historically implied parameters, where the model is calibrated such that the parameters match historically observed behavior. We will denote a contract's MtM at time t with $V(t)$, where a positive MtM indicates the contract has a value in favor of the bank and a negative MtM a value in favor of the counterparty.

5.1.2 Exposure profile

The MtM of an instrument today can be calculated through market observations. Since future states of the market are unknown, it is impossible to deterministically compute future values of a derivative. Through simulations of the market, we can however compute potential realizations of future MtM's as a function of time. The average of these realizations at a given time in the future provides an estimate for the contract's value at this date. The term-structure of these estimates are denoted as the expected exposure profile, hence defined as

$$T \mapsto EE(t,T) = \mathbb{E}^{Q}\left(D\left(t,T\right) V(T) \middle| \mathcal{F}_t\right)$$

The risk of a counterparty defaulting is only relevant for trades that have a positive MtM, since from the bank's perspective no money is lost if the counterparty defaults on a trade with negative value. Therefore, in the computation of CVA it is relevant to consider the *expected positive exposure (EPE) profile*. We define the positive MtM as $V(t)^{+} = \max\left\{V(t), 0\right\}$. In line we define the EPE-profile as:

$$T \mapsto EPE(t,T) = \mathbb{E}^{Q}\left(D\left(t,T\right) V(T)^{+} \middle| \mathcal{F}_t\right)$$

5.1.3 Recovery rate and loss-given-default

In the event that a corporate defaults, it is unlikely that there are precisely zero funds left (cash or activa) from the defaulting party. Hence, the remaining capital is distributed among claim holders. Usually claim holders are categorized by seniority. Within a category, capital is typically distributed in accordance with a *pari passu* clause. This means that there is no preference in compensating the creditors within this group and each one is paid by ratio to the size of the claim that they hold. For this reason it is sensible to consider a *recovery rate* (RR), which is defined as the percentage of a claim that is recovered in case of a default [Gregory, 2010]. In our model we consider $RR \in [0, 1]$ to be a deterministic scalar. Of course, the real recovery rate on an obligation will not be known until an actual event of default. Nonetheless, estimates are provided by credit agencies, based on historical data. A directly related quantity is the *loss given default* (LGD), which is defined as the fraction of the claim that is lost in the event of a defaulting counterparty. Clearly we have

$$LGD = 1 - RR$$

5.1.4 Probability of default

Obviously, the time of default of a given counterparty is unknown prior to the event. We can however represent this unknown as a random variable; a stochastic stopping time in fact, which highly depends on the propagation of the underlying market variables. We denote the time of default as $\tau\left(\omega\right) \in (0, \infty]$, or just simply as τ. The *probability of default* is subsequently defined as the risk-neutral probability that a corporate defaults at a given time t, which we will denote by $PD\left(t\right)$. Hence we have

$$PD(t) = \mathbb{Q}\left(\left\{\omega \in \Omega \middle| \tau\left(\omega\right) = t\right\}\right)$$

In a later paragraph we will discuss how PD is modeled in general.

5.1.5 An analytical formula for CVA

Let us repeat the definition of CVA for an OTC derivative: it is the difference between the value of a contract that includes counterparty risk and the value of the same contract without this risk. Following the analysis of Green [2016], we can derive an expression for CVA. Denote the MtM of the contract unadjusted to credit risk with V (given by the risk-neutral pricing formula) and the credit adjusted value with $\hat{V}$. By pure intuition, V is necessarily larger than $\hat{V}$. We can write CVA at time t as below:

$$CVA(t) = V(t) - \hat{V}(t)$$

We assume the instrument of interest has a maturity T. The value of a risky contract depends on the time of default τ, the values of the cashflows that are scheduled before τ and the value of the contract at τ. We will denote by $C(s,u)$ the present day value (i.e. risk-neutral price, discounted to time t $(< s < u)$) of the cashflows of the contract that are scheduled to occur during the time interval $[s,u]$. Using the indicator $\mathbb{1}_{\tau<T}$ we distinguish between the event that default occurs before or after maturity. Should default occur before maturity, then the contract terminates prematurely. In this case there are two possibilities: The contract has a positive MtM and the bank receives the outstanding value multiplied by the recovery rate (RR). Or the contract has a negative MtM, in which case the bank is obliged to directly pay-out the outstanding value to the defaulting party. Given this information, we can now formulate the value of a risky contract at time t:

$$\begin{aligned} \hat{V}(t) &= \mathbb{E}^Q\left(\left[C(t,\tau) + RR\cdot C(\tau,T)^+ + C(\tau,T)^-\right]\mathbb{1}_{\tau<T}\middle|\mathcal{F}_t\right) \\ &\quad + \mathbb{E}^Q\left(C(t,T)\,\mathbb{1}_{\tau\geq T}\middle|\mathcal{F}_t\right) \end{aligned}$$

Note that we have the relations $C(s,u)^+ + C(s,u)^- = C(s,u)$ and $C(s,u)\,\mathbb{1}_{\tau<T} + C(s,u)\,\mathbb{1}_{\tau\geq T} = C(s,u)$. It should be clear that by the risk-neutral pricing formula $V(t) = \mathbb{E}^Q\left(C(t,T)\middle|\mathcal{F}_t\right)$. Furthermore, we let $V(\tau) = 0$ whenever $\tau > T$, which is a natural assumption given that all cashflows have been exchanged after maturity. Then we can rewrite

$$\begin{aligned} \hat{V}(t) &= \mathbb{E}^Q\left(\left[C(t,\tau) + (RR-1)\,C(\tau,T)^+ + C(\tau,T)^+ + C(\tau,T)^-\right]\mathbb{1}_{\tau<T}\middle|\mathcal{F}_t\right) \\ &\quad + \mathbb{E}^Q\left(C(t,T)\,\mathbb{1}_{\tau\geq T}\middle|\mathcal{F}_t\right) \\ &= \mathbb{E}^Q\left(\left[C(t,\tau) + C(\tau,T)\right]\mathbb{1}_{\tau<T}\middle|\mathcal{F}_t\right) + (RR-1)\,\mathbb{E}^Q\left(C(\tau,T)^+\,\mathbb{1}_{\tau<T}\middle|\mathcal{F}_t\right) \\ &\quad + \mathbb{E}^Q\left(C(t,T)\,\mathbb{1}_{\tau\geq T}\middle|\mathcal{F}_t\right) \\ &= \mathbb{E}^Q\left(C(t,T)\middle|\mathcal{F}_t\right) + (RR-1)\,\mathbb{E}^Q\left(C(\tau,T)^+\,\mathbb{1}_{\tau<T}\middle|\mathcal{F}_t\right) \\ &= V(t) + (RR-1)\,\mathbb{E}^Q\left(D(t,\tau)\,V(\tau)^+\middle|\mathcal{F}_t\right) \end{aligned}$$

Recall that $LGD = 1 - RR = -(RR-1)$. We can therefore conclude that by its definition, an expression for CVA is given by [Green, 2016]:

$$CVA(t) = LGD\cdot\mathbb{E}^Q\left(D(t,\tau)\,V(\tau)^+\middle|\mathcal{F}_t\right)$$

The above result should underline the relevance of EPE profiles in the context of quantifying credit risk.

5.1.6 A numerical approach for CVA

Typically, in a realistic setting, the random variable τ and the risk-neutral price of a derivative $V(t)$ are correlated quantities. This is only natural: the state of the market affects the value of a derivative and simultaneously the creditworthiness of a corporation. A correlation of this kind can be unfavorable from a bank's perspective, if the counterparty is more likely to default when its exposure is high. We call this dependency *wrong-way-risk*, which can cause a significant increase in credit risk and hence the CVA. Numerically it can be quite challenging to model the influence of wrong-way-risk, especially for complex portfolios. Its impact is very sensitive to model assumptions and for this reason there is no general consensus on how it should be incorporated to CVA computations. Throughout this thesis we will ignore the implications of wrong-way-risk, which is not an uncommon practice. This will keep our numerical methods in approximating CVA tractable. Although it should be noted that by doing so we are making an assumption, that does not necessarily reflect reality.

Now that we assume exposure and time of default to be independent, we can continue to simplify the expression for CVA and propose an approximation formula. First we apply the law of total conditioning to find

$$\begin{aligned} CVA(t) &= LGD \cdot \mathbb{E}^Q \left(D(t,\tau) V(\tau)^+ \middle| \mathcal{F}_t \right) \\ &= LGD \cdot \mathbb{E}^Q \left(\mathbb{E}^Q \left(D(t,\tau) V(\tau)^+ \middle| \tau \right) \middle| \mathcal{F}_t \right) \\ &= LGD \cdot \mathbb{E}^Q \left(\int_t^T D(t,s) V(s)^+ d\mathbb{Q}(\tau = s) \middle| \mathcal{F}_t \right) \end{aligned}$$

By the independence of τ and V^+ and the bounded, positive nature of V^+ on $[t,T]$, we can apply Tonelli's theorem to change the order of integration, so that we can write

$$CVA(t) = LGD \cdot \int_t^T \mathbb{E}^Q \left(D(t,s) V(s)^+ \middle| \mathcal{F}_t \right) d\mathbb{Q}(\tau = s)$$

We will approximate this integral by a finite sum. Consider a discretization scheme $t = t_0 < t_1 < \ldots < t_m = T$. Denote by $PD(s,t) = \int_s^t dPD(u) = \int_s^t d\mathbb{Q}(\tau = u)$. Then a discrete approximation of the expression above is given by

$$\begin{aligned} CVA(t) &\approx LGD \cdot \sum_{i=1}^m \mathbb{E}^Q \left(D(t,t_i) V(t_i)^+ \middle| \mathcal{F}_t \right) \cdot \mathbb{Q}(\{t_{i-1} \leq \tau < t_i\}) \\ &= LGD \cdot \sum_{i=1}^m EPE(t,t_i) \cdot PD(t_{i-1},t_i) \end{aligned}$$

From the above result, $EPE(t,t_i)$ is approximated using Monte Carlo simulations. An approximation for PD will be discussed in the following paragraph.

5.2 Probabilities of default

From the previous section it should be clear that estimating the probability that a counterparty will default during a future time interval is crucial to the calculation of CVA. Modeling default probabilities is common practice in credit risk management and there exist different methods. One approach is to consider historical market data, in which case survival probabilities are calibrated to the historical performance of corporations operating within a common sector. Another approach is to compute market implied default probabilities based on *credit default swap* quotes, which are insurance protections against defaulting parties. We will discuss the latter, as nowadays it is the industry standard in quantifying counterparty risk. This is due to some advantages over an historical calibration [Green, 2016], for example:

- Exposure profiles are based on market-implied prices, which therefore fit the no-arbitrage principles. The same holds for the probabilities of default if they are market implied.
- The method corresponds to the approach suggested in the Basel III accord.

The standard method to model the time until default of a given party is the use of a survival function. The random variable τ is considered to have an exponential distribution under $\mathbb{Q}$ with non-constant rate λ. Conditioned on the fact that a counterparty has not defaulted yet at time t, one can compute the probability it will still be alive at time $T > t$. The corresponding survival probability is defined as

$$S(t,T) = \mathbb{Q}(\tau > T | \tau > t) = \exp\left\{ -\int_t^T \lambda(s) ds \right\}$$

As a consequence we find for the probability that $\tau \in [S,T)$ that

$$\begin{aligned} PD(S,T) &= \mathbb{Q}(S \leq \tau < T | \tau > t) \\ &= \exp\left\{ -\int_t^T \lambda(s) ds \right\} - \exp\left\{ -\int_t^S \lambda(s) ds \right\} \end{aligned}$$

The variable λ is often called the *hazard rate* and is modeled to be a positive real function of time.

5.2.1 Credit default swap

A term structure for the hazard rate is implied from the market by quotes of *credit default swaps* (CDS). A CDS is an insurance derivative that is traded and quoted in the market. The buyer of a CDS contract receives protection against the default of a reference institution to an underlying obligation (e.g. a bond). The contract defines a standardized procedure. Over a fixed tenor (6 months, 1 year, 3 years, ...), the buyer pays the issuer a periodic premium. This premium is usually quoted as yearly rate over the notional value of the underlying and referred to as the *CDS spread*. The buyer pays the premium until maturity of the contract, unless an event of default occurs. In that case, the buyer pays the remaining premium that accrued between the last payment and the time of default to the issuer. In return the issuer pays out the notional value of the underlying obligation multiplied with one minus the recovery rate $(1 - RR)$. Clearly the value of a CDS contract highly depends on the reference party it is written to. If the corporate has a low rating and has an elevated likelihood to default, the CDS spreads will be higher. The premiums are therefore based on a reference index, which reflects the creditworthiness of the underlying counterparty.

Given the structure of a CDS, one can compute the fair value of such a contract. In principle, the instrument consists of three legs:

- The premium leg, which consists of the regular periodic premium cashflows to the issuer.
- The accrued premium leg, this is the fraction of the premium build up between the last payment and the potential default.
- The protection leg, the pay-out of $(1 - RR)$ times the notional value in case of a default.

Therefore the face value of a CDS derivative is given by [Green, 2016]:

$$V(t) = V_{premium}(t) + V_{accrued}(t) - V_{protection}(t)$$

Let κ denote the CDS spread, N the notional value of the underlying obligation and Δt_i the year-fraction between two consecutive premium payments. Assume that there are m premium payments in total until maturity time T. Recall that we denote the risk-neutral discount factor by $D(s,t) = \exp\left\{-\int_s^t r(u)du\right\}$. Then using the risk-neutral pricing formulas, the value of each leg at time t can be calculated by

$$\begin{aligned}
V_{premium}(t) &= \mathbb{E}^Q\left(\sum_{i=1}^m D(t,t_i)\,\kappa\Delta t_i N \mathbb{1}_{\tau>t_i}\middle|\,\mathcal{F}_t\right)\\
V_{accrued}(t) &= \mathbb{E}^Q\left(\sum_{i=1}^m D(t,\tau)\,\kappa\,(\tau-t_i)\,N \mathbb{1}_{t_{i-1}\le\tau<t_i}\middle|\,\mathcal{F}_t\right)\\
V_{protection}(t) &= \mathbb{E}^Q\left(\sum_{i=1}^m D(t,\tau)\,(1-RR)\,N \mathbb{1}_{\tau<T}\middle|\,\mathcal{F}_t\right)
\end{aligned}$$

Using the law of total conditioning, the expectations above can be evaluated using the default probability distributions. Assuming that these probabilities and thus the hazard rate is known, one can compute the fair price of a CDS. In the calibration of the hazard rate, one follows the inverse procedure: using quoted CDS spreads, the market implied hazard rate is computed for a specific time interval.

Contract duration	6M	1Y	3Y	5Y	7Y	10Y	RR
CDS spread (bps)	27.8	45.5	99.2	140.8	280.9	330.1	40%

Typically CDS contracts are only available for a limited number of standardized maturities. A CDS curve, in which by convention the spreads are quoted in base-points (1 bps $\equiv$ 1/100 %), may therefore look something like the table presented above. An approximation of the hazard rate term-structure can subsequently be obtained by a bootstrapping procedure applied to the tenors provided by the CDS quotes. It is common practice to consider a piecewise constant parametrization for λ. In the calibration routine of the above example, the hazard rate would then be considered constant in over the time intervals 0-6M, 6M-1Y, 1Y-3Y, etc. The computation of λ then reduces to a limited number of root solving problems, which are numerically tractable. As a result we can generate a profile of default probabilities, that are in practice easy to work with. An example of a piecewise constant term-structure for λ is shown in figure 5.1, together with the corresponding cumulative probabilities of default.

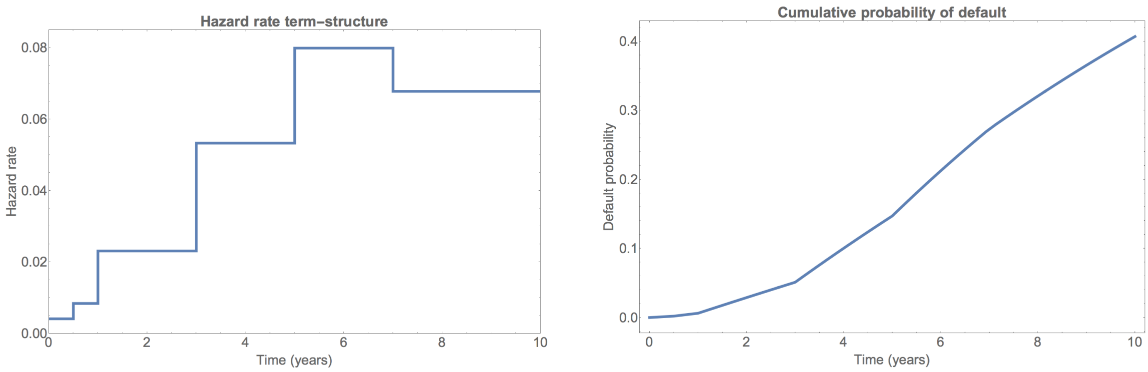

Figure 5.1: (a) Term-structure of hazard rates for a piecewise constant parametrization of λ and (b) the corresponding cumulative probabilities of default.

A typical property of hazard rates implied by CDS quotes, is that they are relatively low in the near future, but tend to increase if we move further in time. For a corporation that has a healthy balance sheet today, it will be unlikely that it defaults instantly. Even if the corporation has to endure severe set-backs, due to a low economy or perhaps mismanagement, it will still take some time until it can no longer meet its obligations. For this reason, hazard rates tend to be low at first. If we move further into the future, this will become uncertain. The credit-worthiness of the corporation might well have decreased. Due to this uncertainty, hazard rates are typically higher further away in time. In the long-end it is not uncommon to observe a small decrease in the hazard rates again, due to the survivorship bias: if the corporate has not defaulted until now, the stability will likely endure.

6 Approximating CVA for an IRS

Here we shall provide a closed form approximation of the CVA for an interest rate swap (IRS). Throughout the section we will consider the *floating-fixed* type as discussed in section 3.5.1. Recall that an IRS is a contract that settles a sequence of payments between two parties. We assume that the deal itself is settled today, at time t. Consider a schedule of future, equidistant time instants $t < T_0 < T_1 < ... < T_m$. The date T_0 denotes the inception of the swap. $T_1, \ldots, T_m$ denotes the payment dates, at which the one party pays the fixed rate coupons and the other pays the floating rate coupons.

We will consider a client that is exposed to default risk and we aim to derive an expression for the counterparty's expected exposure at a potential time of default. We approximate this expression by composing the client's expected positive exposure profile (EPE) between inception of the contract and the time of maturity. What we will see is that the EPE of a counterparty that enters an IRS can be rewritten as the price of a European call option on an IRS. By making some assumptions on the dynamics of the swap rate and the underlying forward rates, we will be able to derive a formula for the CVA that can be fully captured by Hull-White parameters. This turns out to be convenient for simulation purposes.

The analysis and derivation we show here is partly based on Brigo and Mercurio [2007]. There the volatility of the swap rate is estimated within the setting of a LIBOR market model. We perform a similar analysis within the Hull-White framework, assuming Gaussian dynamics of the swap-rate instead of log-normal, which is assumed by Brigo and Mercurio [2007].

Before we start, it is important that we clarify some of the key assumptions we do while approximating the CVA. First of all, we choose to approximate *unilateral* CVA. This hence implies that we assume that the counterparty is the only party that potentially might default. If we were to also include the possibility of the bank to default, we would end up with the bilateral CVA. In that case we would have to evaluate the exposure profiles of both parties. For now we will omit this step.

Our second assumption is that of independence between the credit-worthiness of the client and the current condition of the underlying market. In other words, we are neglecting the presence of wrong-way risk, by which we mean the adverse correlation between the credit quality of a counterparty and its exposure. It is not unlikely for wrong-way risk to play a role in the valuation of credit risk in this context. In times of recession for example, the floating interest rates could drop, which results in an increased exposure for the client, that pays the fixed rate. Simultaneously the client experiences a drop in its credit quality due to the poor current market conditions. Neglecting this form of systematic risk, allows us to by-pass a level of complexity in the computations of the CVA.

The analytical approach that is presented in this chapter has been numerically tested against CVA computations based on Monte Carlo methods. A selection of the test-results are shown in the appendix, section A.

6.1 Unilateral CVA for an IRS

We consider an IRS derivative security, that is specified by the following properties

$$\begin{aligned} N \quad &- \quad \text{The notional amount} \\ K \quad &- \quad \text{The fixed rate paid by the client} \\ \mathcal{T} \quad &- \quad \text{The coupon schedule } (T_i)_{1\leq i\leq m}, \\ & \qquad \text{denoting equidistant future time instants} \\ \Delta t_i \quad &- \quad \text{The } i^{th} \text{ accrual period } (T_i - T_{i-1}) \end{aligned}$$

We denote by the random variable τ the time of default of the client and assume that it has an exponential distribution with parameter $\lambda(t)$. The hazard rate $\lambda(t)$ is considered to be a positive function of time. Let $LGD = (1 - RR) \in [0, 1]$ be a scalar denoting the loss-given-default of the client. For a derivative that has risk-neutral value $V(t)$ at time t, we have seen that the unilateral CVA is defined as follows:

$$CVA(t) = LGD \cdot \mathbb{E}^Q \left(D(t, \tau) V(\tau)^+ \middle| \mathcal{F}_t \right)$$

Also, we have seen that if we assume independence between the exposure profile $t \mapsto V(t)$ and the time of default τ, then the CVA charge can be approximated by a discretization of the expected positive exposure and the probabilities of default. This allowed us to write

$$CVA(t) \approx LGD \cdot \sum_{i=1}^{m} EPE\left(t, T_i\right) \cdot PD\left(T_{i-1}, T_i\right)$$

We choose to let the coupon schedule $\mathcal{T}$ coincide with the time scheme that is used for the discretization of the CVA. This is a natural choice, since most of the movement in the exposure profile happens directly after a cashflow.

6.1.1 The positive exposure profile

In the expression for the CVA charge we observe three relevant quantities: LGD, PD and EPE. LGD is a constant, which is estimated by the bank. We therefore assume it to be known. A term-structure of default probabilities $T_i \mapsto PD\left(T_{i-1}, T_i\right)$ is implied from quotes of related credit-default swaps. These quotes can be obtained from the market and will therefore also be assumed to be known. What remains to be determined is the positive exposure profile $T_i \mapsto \mathbb{E}^Q\left(D\left(t, T_i\right) V\left(T_i\right)^+ \middle| \mathcal{F}_t\right)$.

Let $T_i \in \{T_1, \ldots, T_m\}$ and let $V(t)$ denote the risk-neutral value of the IRS at time t. We have seen in section 3.5.1 that if $t < T_0$ we can write

$$V(t) = N\left(P\left(t, T_0\right) - P\left(t, T_m\right) - K \sum_{i=1}^{m} P\left(t, T_i\right) \Delta t_i\right)$$

Now instead of the value at time t, we consider the contracts value at time T_i, just after the i^{th} cashflow. Following the same argument as provided in section 3.5.1, it should be clear that

$$\begin{aligned} V\left(T_i\right) &= \mathbb{E}^Q\left(\sum_{j=i+1}^{m} D\left(T_i, T_j\right) N \Delta t_j \left(L\left(T_{j-1}, T_j\right) - K\right) \middle| \mathcal{F}_{T_i}\right) \\ &= N\left(P\left(T_i, T_i\right) - P\left(T_i, T_m\right) - K \sum_{j=i+1}^{m} P\left(T_i, T_j\right) \Delta t_j\right) \\ &= N \cdot A^{i,m}\left(T_i\right) \cdot \left(S^{i,m}\left(T_i\right) - K\right) \end{aligned}$$

Recall that $A^{i,m}$ denotes an annuity, which was defined as

$$A^{i,m}(t) = \sum_{j=i+1}^{m} P\left(t, T_j\right) \Delta t_j$$

Also recall that the *swap rate* S, corresponding to an IRS with coupon schedule $\mathcal{T}$ was defined as the fixed rate for which the risk-neutral value of the IRS is set to zero. With $S^{i,m}(t)$ used in the formula above, we denote the swap rate at time t corresponding to an IRS with coupon schedule $\mathcal{T}^{i,m} = \{T_{i+1}, \ldots, T_m\}$. An expression for $S^{i,m}$ is then given by

$$S^{i,m}(t) = \frac{P\left(t, T_i\right) - P\left(t, T_m\right)}{\sum_{j=i+1}^{m} P\left(t, T_j\right) \Delta t_j}$$

Now that we have a tractable expression for the exposure, we will proceed by considering the expected positive exposure. We find that

$$\begin{aligned} EPE\left(t, T_i\right) &= \mathbb{E}^Q\left(D\left(t, T_i\right) V\left(T_i\right)^+ \middle| \mathcal{F}_t\right) \\ &= \mathbb{E}^Q\left(D\left(t, T_i\right) \cdot N \cdot A^{i,m}\left(T_i\right) \cdot \left(S^{i,m}\left(T_i\right) - K\right)^+ \middle| \mathcal{F}_t\right) \\ &= N \cdot A^{i,m}\left(t\right) \mathbb{E}^{A^{i,m}}\left(\left(S^{i,m}\left(T_i\right) - K\right)^+ \middle| \mathcal{F}_t\right) \end{aligned}$$

where in the last step we changed to the annuity measure $\mathbb{Q}^{A^{i,m}}$, just as we did in section 3.5.3. Take a closer look at the expression for the exposure profile. Note that the EPE written in this form resembles the value of a European call option written on an IRS with fixed rate K and coupon schedule $\mathcal{T}^{i,m}$. Such a derivative is known as a *swaption*, as described in section 3.5.3. We therefore conclude that we can compute the EPE profile by pricing a sequence of swaptions.

6.1.2 Pricing EPE as a swaption

In order to price a swaption, we need to consider an appropriate assumption on the dynamics of the swap rate. Black's model might appear to be a logical choice, but in that framework we would assume log-normal dynamics for the swap rate. In the current economy, many interest rates are close to zero or even negative. The same therefore goes for the swap rate. This property is not reflected by the Black's model. A more suitable framework is therefore Bachelier's model, under which the dynamics of $S^{i,m}(t)$ are Gaussian. We know that under the annuity measure, the process $S^{i,m}(t)$ is a Martingale. Therefore we arrive at the following Gaussian dynamics

$$dS^{i,m}(t) = v_{S^{i,m}}(t) dW_{S^{i,m}}(t)$$

where $W_{S^{i,m}}(t)$ is a standard Brownian motion under the annuity-measure $\mathbb{Q}^{A^{i,m}}$. Then, in accordance with the Bachelier framework, the risk-neutral price $c(t)$ of a call-option on $S^{i,m}$ with strike price K and expiry T is given by

$$c(K, S^{i,m}(t), \sigma_{S^{i,m}}(t)) = \sigma_{S^{i,m}}(t) \cdot n\left(\frac{S^{i,m}(t) - K}{\sigma_{S^{i,m}}(t)}\right) + \left(S^{i,m}(t) - K\right) \cdot \mathcal{N}\left(\frac{S^{i,m}(t) - K}{\sigma_{S^{i,m}}(t)}\right)$$

where n and $\mathcal{N}$ respectively denote the probability density function and the cumulative distribution function of a standard normal random variable. The parameter $\sigma_{S^{i,m}}$ is given by

$$\sigma_{S^{i,m}}(t) = \sqrt{\int_t^{T_i} \left(v_{S^{i,m}}(u)\right)^2 du}$$

Now, summarizing the earlier steps of this section, we can approximate the CVA charge for an IRS as follows

$$CVA(t) \approx LGD \cdot \sum_{i=1}^{m} EPE(t, T_i) \cdot PD(T_{i-1}, T_i)$$

where

$$EPE(t, T_i) = N \cdot A^{i,m}(t) \cdot c\left(K, S^{i,m}(t), \sigma_{S^{i,m}}(t)\right)$$

So far we have found a very general relation to compute the CVA on an IRS without making any assumption on the model we work with. The only assumption so far is that the swap rate evolves according to a Gaussian process, which allows us to apply Bachelier's formula. Apart from that assumption, our relation is model-independent. What we aim to do now, is to incorporate this relation in the Hull-White model. This allows us to compute CVA realizations that are consistent with the Hull-White simulations of the short-rate. In the EPE-formula, both $A^{i,m}(t)$ and $S^{i,m}(t)$ can be expressed in terms of zero-coupon bond prices by definition. What remains to be done is finding an expression for the volatility of the swap rate $v_{S^{i,m}}$ in terms of Hull-White parameters. In the next section we will therefore derive a formula that approximates $v_{S^{i,m}}$. What we will see is that this firstly requires an expression for the volatility of the forward rate. Therefore, before we work out the volatility of the swap rate, we will do an analysis of the dynamics of the forward rate in the Hull-White framework.

6.2 Forward rate volatility

To find an expression for the volatility of the forward rate, we will work out its differential. For simplicity we adopt the notation $F_i(t) = F(t, T_{i-1}, T_i)$. We know that an expression for the forward rate is given by

$$F_i(t) := F(t, T_{i-1}, T_i) = \frac{1}{\Delta t_i}\left(\frac{P(t, T_{i-1})}{P(t, T_i)} - 1\right)$$

We have also seen that within the Hull-White model, $P(t,T)$ can alternatively be written as

$$P(t,T) = A(t,T) \cdot e^{-B(t,T)r(t)}$$

where

$$\begin{aligned} A(t,T) &= \frac{P(0,T)}{P(0,t)} \exp\left\{B(t,T)f^M(0,t) - \frac{\sigma}{4a}\left(1-e^{-2at}\right)B(t,T)^2\right\} \\ B(t,T) &= \frac{1}{a}\left(1-e^{-a(T-t)}\right) \end{aligned}$$

which allows us to also rewrite F_i as follows:

$$F_i(t) = \frac{1}{\Delta t_i}\left(\frac{A(t,T_{i-1})}{A(t,T_i)} e^{(B(t,T_i)-B(t,T_{i-1}))r(t)} - 1\right)$$

We will use the above expression, to apply Itô's lemma and evaluate the SDE for F_i. To do so, recall that the dynamics of $r(t)$ in the Hull-White context are given by:

$$dr(t) = (\theta(t) - a \cdot r(t))\, dt + \sigma dW(t)$$

As a first step, define the function

$$f(t,x) = \frac{1}{\Delta t_i}\left(\frac{A(t,T_{i-1})}{A(t,T_i)} e^{(B(t,T_i)-B(t,T_{i-1}))x} - 1\right)$$

Then according to Itô's lemma, we find that

$$\begin{aligned} dF_i(t) &= df(t,r(t)) \\ &= \left(f_t(t,r(t)) + f_x(t,r(t))(\theta(t) - a\cdot r(t)) + \frac{1}{2}f_{xx}(t,r(t))\sigma^2\right)dt \\ &\quad + f_x(t,r(t))\sigma dW_i(t) \\ &= \mu_i(t)\,dt + f_x(t,r(t))\sigma dW_i(t) \end{aligned}$$

where $\mu_i(t)$ is some stochastic, adapted drift-process. For the partial derivative of f we find.

$$\begin{aligned} f_x(t,r(t)) &= \frac{\partial}{\partial x}\frac{1}{\Delta t_i}\left(\frac{A(t,T_{i-1})}{A(t,T_i)} e^{(B(t,T_i)-B(t,T_{i-1}))x} - 1\right)\Bigg|_{x=r(t)} \\ &= \frac{1}{\Delta t_i}(B(t,T_i) - B(t,T_{i-1}))\frac{A(t,T_{i-1})}{A(t,T_i)} e^{(B(t,T_i)-B(t,T_{i-1}))r(t)} \\ &= \frac{1}{\Delta t_i}(B(t,T_i) - B(t,T_{i-1}))\frac{P(t,T_{i-1})}{P(t,T_i)} \end{aligned}$$

We do not work out the drift-component μ_i as we will not need it. If we substitute the partial derivative in the expression obtained by Itô's lemma, we find that the differential of F_i is given by

$$dF_i(t) = \mu_i(t)\,dt + \frac{1}{\Delta t_i}(B(t,T_i) - B(t,T_{i-1}))\frac{P(t,T_{i-1})}{P(t,T_i)}\sigma dW_i(t)$$

From the above result we conclude that the volatility of the forward rate can be written in terms of Hull-White parameters as follows:

$$v_i(t) = \frac{\sigma}{\Delta t_i}(B(t,T_i) - B(t,T_{i-1}))\frac{P(t,T_{i-1})}{P(t,T_i)}$$

6.3 Swap rate volatility

We proceed by approximating the swap rate volatility using the result of the previous section. Recall that the swap rate $S^{i,m}(t)$ evolves according to a Gaussian process. Since $S^{i,m}$ is a Martingale under the annuity-measure, it follows that the dynamics are given by

$$dS^{i,m}(t) = v_{S^{i,m}}(t)dW_{S^{i,m}}(t)$$

Where $W_{S^{i,m}}(t)$ is a Brownian motion under $\mathbb{Q}^{A^{i,m}}$ associated with the underlying swap. The swap rate can be expressed as the weighted sum of forward rates [Brigo and Mercurio, 2007]. To see how this is done, we start with the definition of the swap rate:

$$\begin{aligned}
S^{i,m}(t) &= \frac{P(t,T_i) - P(t,T_m)}{\sum_{j=i+1}^{m} \Delta t_j P(t,T_j)} \\
&= \frac{\sum_{k=i+1}^{m} \left(P(t,T_{k-1}) - P(t,T_k)\right)}{\sum_{j=i+1}^{m} \Delta t_j P(t,T_j)} \\
&= \frac{\sum_{k=i+1}^{m} \Delta t_k P(t,T_k) \frac{1}{\Delta t_k}\left(\frac{P(t,T_{k-1})}{P(t,T_k)} - 1\right)}{\sum_{j=i+1}^{m} \Delta t_j P(t,T_j)} \\
&= \sum_{k=i+1}^{m} \frac{\Delta t_k P(t,T_k)}{\sum_{j=i+1}^{m} \Delta t_j P(t,T_j)} F_k(t) \\
&= \sum_{k=i+1}^{m} \omega_k(t) F_k(t)
\end{aligned}$$

A first approximation is done by freezing the weights at $t = 0$, which is a suggestion that is also proposed in Anderson and Piterbarg [2010b]. By doing so the weights become deterministic and are given by

$$\omega_k \equiv \omega_k(0) = \frac{\Delta t_k P(0,T_k)}{\sum_{j=i+1}^{m} \Delta t_j P(0,T_j)}$$

This allows us to write the volatility of the swap rate in terms of volatilities of the forward rates, which we derived in the previous section. First note that we have the relation

$$dS^{i,m}(t)dS^{i,m}(t) = v_{S^{i,m}}(t)dW_{S^{i,m}}(t)v_{S^{i,m}}(t)dW_{S^{i,m}}(t) = v_{S^{i,m}}(t)^2 dt$$

To find an explicit expression for $v_{S^{i,m}}(t)$, we substitute our approximation of $S^{i,m}(t)$ in the relation above. By doing so we will substitute the differential of F_i which we derived in section 6.2. We should remark that the drift of F_i might be different under the annuity-measure. The volatility-coefficient will however be the same, due to the diffusion invariance principle [Anderson and Piterbarg, 2010b]. As the drift-term will drop out in the derivation below, this does not present any problem.

$$\begin{aligned}
v_{S^{i,m}}(t)^2 dt &\approx d\left(\sum_{j=i+1}^{m} \omega_j F_j(t)\right) d\left(\sum_{k=i+1}^{m} \omega_k F_k(t)\right) \\
&= \sum_{j=i+1}^{m} \sum_{k=i+1}^{m} \omega_j \omega_k dF_j(t) dF_k(t) \\
&= \sum_{j=i+1}^{m} \sum_{k=i+1}^{m} \omega_j \omega_k \left(\mu_j(t)dt + v_j(t)dW_j(t)\right) \cdot \left(\mu_k(t)dt + v_k(t)dW_k(t)\right) \\
&= \sum_{j=i+1}^{m} \sum_{k=i+1}^{m} \omega_j \omega_k \rho_{jk} v_j(t) v_k(t) dt
\end{aligned}$$

Here ρ_{jk} denotes the instantaneous correlation between $W_j(t)$ and $W_k(t)$. Under the one-factor Hull-White dynamics, forward rates are fully correlated. This is due to the fact that under the model assumptions, $F_i(t)$ only stochastically depends on $r(t)$ for each $i \in \{1, \ldots, m\}$. Thus as expected we see that:

$$dF_j(t)dF_k(t) = v_j(t)dW_j(t)v_k(t)dW_k(t) = v_j(t)v_k(t)dt$$

Although this “perfect correlation” could be considered a short-coming of the Hull-White model, it does simplify the expression above, as it lets us set $\rho_{jk} = 1$. If we subsequently substitute the forward rate volatilities as we found them in the previous section, we are left with the Gaussian volatility of the swap rate. This allows us to deterministically compute the Gaussian swap rate variance, seen from time t, which can be applied in Bachelier's formula for pricing a swaption.

$$\begin{aligned}
\sigma^2_{S^{i,m}}(t) &= \int_t^{T_i} v_{S^{i,m}}(u)^2 du \\
&= \sum_{j=i+1}^{m} \sum_{k=i+1}^{m} \omega_j \omega_k \int_t^{T_i} v_j(u) v_k(u) du
\end{aligned}$$

Apart from the $P(t,T_{i-1})/P(t,T_i)$ term in the expression for $v_i(t)$, this variance is deterministic. To make it fully deterministic and hence suitable for simulation purposes, we approximate by replacing the zero-bond ratio with $P(0,T_{i-1})/P(0,T_i)$. It follows

$$\begin{aligned}
\int_t^{T_i} v_j(u)v_k(u)du &\approx \frac{\sigma^2}{\Delta t_j \Delta t_k} \frac{P(0,T_{j-1})\,P(0,T_{k-1})}{P(0,T_j)\,P(0,T_k)} \\
&\quad \cdot \int_t^{T_i} \left(B(u,T_j) - B(u,T_{j-1})\right)\left(B(u,T_k) - B(u,T_{k-1})\right) du
\end{aligned}$$

Where

$$\begin{aligned}
&\int_t^{T_i} \left(B(u,T_j) - B(u,T_{j-1})\right)\left(B(u,T_k) - B(u,T_{k-1})\right) du \\
=\; &\frac{1}{a^2} \int_t^{T_i} \left(e^{-a(T_{j-1}-u)} - e^{-a(T_j-u)}\right)\left(e^{-a(T_{k-1}-u)} - e^{-a(T_k-u)}\right) du \\
=\; &\frac{1}{a^2} \left(e^{-aT_{j-1}} - e^{-aT_j}\right)\left(e^{-aT_{k-1}} - e^{-aT_k}\right) \int_t^{T_i} e^{2au} du \\
=\; &\frac{1}{2a^3} \left(e^{-aT_{j-1}} - e^{-aT_j}\right)\left(e^{-aT_{k-1}} - e^{-aT_k}\right)\left(e^{2aT_i} - e^{2at}\right)
\end{aligned}$$

Hence we conclude

$$\begin{aligned}
\sigma^2_{S^{i,m}}(t) &\approx \frac{\sigma^2}{2a^3} \sum_{j=i+1}^{m} \sum_{k=i+1}^{m} \left[\frac{\omega_j \omega_k}{\Delta t_j \Delta t_k} \frac{P(0,T_{j-1})\,P(0,T_{k-1})}{P(0,T_j)\,P(0,T_k)} \cdot \right. \\
&\qquad \left. \left(e^{-aT_{j-1}} - e^{-aT_j}\right)\left(e^{-aT_{k-1}} - e^{-aT_k}\right)\left(e^{2aT_i} - e^{2at}\right) \right] \\
&= \frac{\sigma^2}{2a^3} \sum_{j=i+1}^{m} \sum_{k=i+1}^{m} \left[\frac{P(0,T_{j-1})\,P(0,T_{k-1})}{\left(\sum_{l=i+1}^{m} \Delta t_l P(0,T_l)\right)^2} \cdot \right. \\
&\qquad \left. \left(e^{-aT_{j-1}} - e^{-aT_j}\right)\left(e^{-aT_{k-1}} - e^{-aT_k}\right)\left(e^{2aT_i} - e^{2at}\right) \right]
\end{aligned}$$

7 Approximating CVA for a CCS

In this section we shall provide a closed form approximation of the CVA for a cross-currency swap (CCS). The type of CCS we consider is the *floating-fixed* CCS. We assume a market model that entails two currencies, which we denote by *domestic* and *foreign*. Once again we denote the time at which the contact is settled by t and consider a time schedule $t < T_0 < \ldots < T_m$. The contract settles a sequence of transactions between two parties, in our case a client and the bank.

- At time T_0, the parties exchange two notional amounts. The client pays to the bank an amount N^d in the domestic currency. Simultaneously the client receives an amount N^f from the bank in the foreign currency.
- Over the subsequent tenor $T_1, \ldots, T_m$, the bank will pay floating rate payments to the client over the domestic notional, referred to as the floating or domestic leg. In return the client pays fixed rate coupons over the foreign notional, which is called the fixed or foreign leg.
- At T_m, on top of the last coupon, the client pays back N^f in the foreign currency to the bank. In return he receives N^d in the domestic currency.

Here we will approximate the CVA charge associated with a CCS contract. The method we follow is partly similar to that of the previous chapter for an IRS. The current setting comes with a few extra difficulties. First of all we deal with two currencies instead of one, which forces us to take two short-rate processes into account: the domestic and the foreign. Next to that, the FX process should be considered, which we assume to be a geometric Brownian motion with stochastic drift and deterministic volatility as treated in section 4.6. A second difference between the CCS case and IRS case are the notional exchanges at the start and end of the trade and the corresponding FX risk it brings forth.

Our aim is to derive an expression for the *cross-currency swap rate*. This is equivalent to the *swap rate* in the IRS setting: it is the fair domestic rate that would set the net-present value of the CCS to zero. Secondly we derive an expression for the *effective cross-currency swap rate*, which reflects the net present value of the trade, given the current market. Once we obtained these quantities, quantifying the EPE profile becomes similar to pricing a European put option on a CCS. The method we present is largely based on Brigo et al. [2013]. There, a general approach is given for quantifying the CVA charge for a fixed-fixed CCS. Brigo also provides a suggestion on how the result can be adjusted to match a CCS with floating leg coupon. We will use his approach and apply it on a floating-fixed CCS. In addition we will show how the result that we obtained can be expressed in Hull-White parameters, so that it matches the multi-currency Hull-White and FX framework in which we perform our market simulations.

We will adopt the same assumptions as for the IRS case. The effect of potential wrong-way risk is ignored and we only consider an approach for *unilateral* CVA. An additional assumption we do is that of ignoring the presence of a basis. In fact we ignore two types of bases, which both play a role in modern derivative pricing:

- The basis between rates over different tenors
- The cross-currency basis

The first basis is an effect of a non-flat yield-curve. Loans over a short period of time are in practice less risky than loans over a longer tenor. That is why yields on the yield curve often increase with the maturity (1 Month LIBOR $<$ 6 Month LIBOR). The second type is a consequence of supply and demand of different currencies. We have seen that theoretically the forward FX rate can be expressed in terms of the spot exchange rate, a domestic- and a foreign zero-coupon bond. In practice however, this theoretical value of a forward FX often does not match with the value that is observed in the market. The mismatch between the market-implied and the theoretical forward FX rate is referred to as the cross-currency basis. One way to account for these bases in a modeling-environment is by switching from a single to a dual curve framework. In a dual curve-framework one distinguishes between curves used for discounting and curves used for computing forward rates. For now we neglect the (cross-currency) basis and assume a single-curve framework. In chapter 8 we will propose a correction to this method, applicable in a dual curve Hull-White and FX framework.

The analytical approach that is presented in this chapter has been numerically tested against CVA computations based on Monte Carlo methods. A selection of the test-results are shown in the appendix, section A.

7.1 Unilateral CVA for a CCS

We consider an CCS derivative security, that is specified by the following properties

$$\begin{aligned} N^d \quad & - \quad \text{The domestic notional amount} \\ N^f \quad & - \quad \text{The foreign notional amount} \\ K \quad & - \quad \text{The fixed rate paid by the client} \\ \mathcal{T} \quad & - \quad \text{The coupon schedule } (T_i)_{1\le i\le m}, \\ & \quad\quad \text{denoting equidistant future time instants} \\ \Delta t_i \quad & - \quad \text{The } i^{th} \text{ accrual period } (T_i - T_{i-1}) \end{aligned}$$

Just like before we denote by the random variable τ the time of default and assume it has a similar distribution. LGD denotes the loss-given-default of the client. For the unilateral CVA we can again write:

$$CVA(t) = LGD \cdot \mathbb{E}^Q \left(D(t,\tau) V(\tau)^+ \middle| \mathcal{F}_t \right)$$

and by the independence assumption between the exposure and τ, we can again approximate the CVA by

$$CVA(t) \approx LGD \cdot \sum_{i=1}^{m} EPE(t,T_i) \cdot PD(T_{i-1},T_i)$$

Also for the CCS we choose to let the coupon schedule $\mathcal{T}$ coincide with the time scheme that is used for the discretization of the CVA.

7.1.1 The positive exposure profile

In the formula for CVA, the LGD is a constant between zero and one of which an estimate is made by the bank. A term-structure for $PD(s,t)$ can be obtained by CDS-quotes from the market as discussed in section 5.1. We will therefore focus on the EPE profile. Let $T_i \in \{T_1,\dots,T_m\}$ and let $V(T_i)$ denote the risk-neutral value of the remaining cashflows of the CCS at time T_i. We have seen that at $t < T_0$ the value of a CCS is given by

$$\begin{aligned} V(t) \quad = \quad & \sum_{i=1}^{m} \Delta t_i P(t,T_i) \left(N^f K \Phi(t,T_i) - N^d F(t,T_{i-1},T_i) \right) \\ & - N^d P(t,T_m) + N^f P(t,T_m) \Phi(t,T_m) \end{aligned}$$

Therefore it should be straight-forward that it follows that

$$\begin{aligned} V(T_i) \quad = \quad & \sum_{j=i+1}^{m} \Delta t_j P(T_i,T_j) \left(N^f K \Phi(T_i,T_j) - N^d F(T_i,T_{j-1},T_j) \right) \\ & - N^d P(T_i,T_m) + N^f P(T_i,T_m) \Phi(T_i,T_m) \end{aligned}$$

Recall the definition of the simply-compounded forward rate, which is given by

$$F(T_i,T_{j-1},T_j) = \frac{1}{\Delta t_j} \left(\frac{P(T_i,T_{j-1})}{P(T_i,T_j)} - 1 \right)$$

Now lets take a closer look at one part of the expression for $V(T_i)$. The contribution of the floating coupons are given by $N^d \sum_{j=i+1}^{m} P(T_i,T_j) \Delta t_j F(T_i;T_{j-1},T_j)$. If we substitute the above relation for the forward rate into this part, we will see that the overall exposure can be simplified.

$$\begin{aligned}N^d \sum_{j=i+1}^{m} \Delta t_j P(T_i, T_j) F(T_i; T_{j-1}, T_j) &= N^d \sum_{j=i+1}^{m} \Delta t_j P(T_i, T_j) \frac{1}{\Delta t_j} \left(\frac{P(T_i, T_{j-1})}{P(T_i, T_j)} - 1 \right) \\ &= N^d \sum_{j=i+1}^{m} \left(P(T_i, T_{j-1}) - P(T_i, T_j) \right) \\ &= P(T_i, T_i) N^d - P(T_i, T_m) N^d\end{aligned}$$

Note that the last term on the right cancels out against the repayment of the domestic notional. Therefore we find the following result, once we plug back in the expression for the floating leg:

$$V(T_i) = N^f \sum_{j=i+1}^{m} \Delta t_j P(T_i, T_j) K \Phi(T_i, T_j) - N^d P(T_i, T_i) + N^f P(T_i, T_m) \Phi(T_i, T_m)$$

In a last simplifying step we define a new deterministic foreign rate sequence $(K_j)_{j \in \{i+1,\ldots,m\}}$. The rates are defined as follows [Brigo et al., 2013].

$$K_j = \begin{cases} \frac{K \cdot N^f}{N^d} & j = i+1, \ldots, m-1 \\ \frac{K \cdot N^f + N^f / \Delta t_j}{N^d} & j = m \end{cases}$$

If we use this set of rates, we can rewrite the exposure in two separate terms corresponding to the foreign and domestic leg. Eventually this will allow us to factor out the notional N^d. We end up with a compact representation for the exposure as below.

$$V(T_i) = \underbrace{N^d \sum_{j=i+1}^{m} \Delta t_j P(T_i, T_j) \Phi(T_i, T_j) K_j}_{\text{foreign leg}} - \underbrace{N^d P(T_i, T_i)}_{\text{domestic leg}}$$

7.1.2 The effective CCS rates

In the case of the IRS, we have seen that the exposure of the counterparty is similar to the value of a swaption. Now that we have derived a convenient representation for the exposure above, we will proceed by applying a similar approach to the CCS. Difficulty remains that the two legs are defined under different currencies, which makes us dependent on the stochastic forward FX rate. We will see that by doing some extra simplifications, this problem can be tackled. For this purpose we will derive three essential quantities that allow us to conveniently rewrite the exposure. The first two are the *effective domestic rate* $\widetilde{K}_i$ and the *effective foreign rate* $\widetilde{K}_i^f$.

Consider a CCS of which the domestic leg has inception date T_i and coupon payments at $T_{i+1}, \ldots, T_m$ with a given fixed rate $\widetilde{K}_i$. By choosing $\widetilde{K}_i$ such that this artificial fixed leg has the same net present value of the actual domestic leg of our original CCS, we can rewrite the exposure using this effective rate [Brigo et al., 2013]. We solve as below to find an expression for $\widetilde{K}_i$:

$$\begin{aligned}N^d P(T_i, T_i) &= \widetilde{K}_i(T_i) N^d \sum_{j=i+1}^{m} \Delta t_j P(T_i, T_j) \\ \Rightarrow \quad \widetilde{K}_i(T_i) &= \frac{P(T_i, T_i)}{\sum_{j=i+1}^{m} \Delta t_j P(T_i, T_j)}\end{aligned}$$

In a similar way we can define an *effective foreign rate* $\widetilde{K}_i^f$. This is a hypothetical constant fixed rate that leaves the net present value of the foreign leg unchanged [Brigo et al., 2013]. Recall that $\Phi_i(t) = {}^{P^f(t,T_i) \cdot \varphi(t)} / {}_{P(t,T_i)}$. We solve as below to find an expression for $\widetilde{K}_i^f$:

$$N^d \sum_{j=i+1}^{m} \Delta t_j P(T_i, T_j) \Phi(T_i, T_j) K_j = \widetilde{K}_i^f(T_i) N^d \sum_{j=i+1}^{m} \Delta t_j P(T_i, T_j) \Phi(T_i, T_j)$$

$$\begin{aligned}
\Rightarrow \quad \widetilde{K}_i^f(T_i) &= \sum_{j=i+1}^{m} \frac{\Delta t_j P(T_i,T_j)\,\Phi(T_i,T_j)}{\sum_{k=i+1}^{m} \Delta t_k P(T_i,T_k)\,\Phi(T_i,T_k)} K_j \\
&= \sum_{j=i+1}^{m} \frac{\Delta t_j P(T_i,T_j)\,\frac{P^f(T_i,T_j)\varphi(T_i)}{P(T_i,T_j)}}{\sum_{k=i+1}^{m} \Delta t_k P(T_i,T_k)\,\frac{P^f(T_i,T_j)\varphi(T_i)}{P(T_i,T_j)}} K_j \\
&= \sum_{j=i+1}^{m} \frac{\Delta t_j P^f(T_i,T_j)}{\sum_{k=i+1}^{m} \Delta t_k P^f(T_i,T_k)} K_j
\end{aligned}$$

If we introduce a notation for the weight factors of K_j in the expression above, $\widetilde{K}_i^f$ can represented in an even more compact way. Let $\omega_j^f(t) = \frac{\Delta t_j P^f(t,T_j)}{\sum_{k=i+1}^{m} \Delta t_k P^f(t,T_k)}$ and we obtain for the effective foreign rate

$$\widetilde{K}_i^f(T_i) = \sum_{j=i+1}^{m} \omega_i^f(T_i)\,K_j$$

7.1.3 The fair CCS rate

Lastly we introduce a *fair domestic rate* K_i^{eq}. Its definition is almost similar to that of the effective domestic rate. It represents a constant fixed rate for coupons paid at the dates $T_{i+1},\ldots,T_m$. Except, this fixed rate would set the net present value of contract at zero, given that its inception date is T_i. In comparison to the IRS case, $K_i^{eq}(T_i)$ is the cross-currency equivalent to the swap rate. The fair rate represents the fixed rate that would be settled on, if a new contract were entered at par on T_i [Brigo et al., 2013]. We solve for $K_i^{eq}(T_i)$ to obtain the following

$$N^d \sum_{j=i+1}^{m} \Delta t_j P(T_i,T_j)\,\Phi(T_i,T_j)\,K_j - K_i^{eq}(T_i)\,N^d \sum_{j=i+1}^{m} \Delta t_j P(T_i,T_j) = 0$$

$$\begin{aligned}
\Rightarrow \quad K_i^{eq}(T_i) &= \frac{\sum_{j=i+1}^{m} \Delta t_j P(T_i,T_j)\,\Phi(T_i,T_j)\,K_j}{\sum_{k=i+1}^{m} \Delta t_k P(T_i,T_k)} \\
&= \widetilde{K}_i^f(T_i) \sum_{j=i+1}^{m} \frac{\Delta t_j P(T_i,T_j)}{\sum_{k=i+1}^{m} \Delta t_k P(T_i,T_k)} \Phi(T_i,T_j)
\end{aligned}$$

For a more compact expression we will also introduce domestic weight factors. As before let $\omega_j(t) = \frac{\Delta t_j P(t,T_j)}{\sum_{k=i+1}^{m} \Delta t_k P(t,T_k)}$, which conveniently simplifies the above representation to

$$K_i^{eq}(T_i) = \widetilde{K}_i^f(T_i) \sum_{j=i+1}^{m} \omega_j(T_i)\,\Phi(T_i,T_j)$$

7.1.4 The EPE in terms of CCS rates

So far we have derived a *fair* rate and a domestic and foreign *effective* rate. Now we reach the point where the effort pays off. If we return to the exposure, which we aimed to rewrite in the first place, we will see that it can now be written in terms of $\widetilde{K}_i(t)$, $\widetilde{K}_i^f(t)$ and $K_i^{eq}(t)$. If we go one step further and consider the expected positive exposure at T_i, we find a quite familiar expression.

$$\begin{aligned}
EPE(t,T_i) &= \mathbb{E}^Q\left(D(t,T_i)\,V(T_i)^+ \middle| \mathcal{F}_t \right) \\
&= \mathbb{E}^Q\left(D(t,T_i)\,N^d \sum_{j=i+1}^{m} \Delta t_j P(T_i,T_j) \left(\widetilde{K}_i(T_i) - K_i^{eq}(T_i) \right)^+ \middle| \mathcal{F}_t \right)
\end{aligned}$$

As a last step, we change the measure to $\mathbb{Q}^{A^{i,m}}$, which is the risk-neutral measure associated with the annuity numeraire

$$A^{i,m}(t) = \sum_{j=i+1}^{m} \Delta t_j P(T_i, T_j)$$

The EPE can then be written as

$$EPE(t, T_i) = N^d A^{i,m}(t) \mathbb{E}^{A^{i,m}} \left(\left(\widetilde{K}_i(T_i) - K_i^{eq}(T_i) \right)^+ \middle| \mathcal{F}_t \right)$$

The EPE now resembles the value of an option. The only problem at this point is that both $\widetilde{K}_i(t)$ and $K_i^{eq}(t)$ are stochastic functions of time. This is in contrast with the IRS case where the fixed rate K is a deterministic scalar. We solve this last issue by freezing the weights. We take the stochastic weights $\omega_j(t)$ and $\omega_j^f(t)$ and eliminate their time-dependency by fixing them at their time-zero value. Thus we simplify by setting $\omega_j^f(t) \equiv \omega_j^f(0) = \omega_j^f$ and $\omega_j(t) \equiv \omega_j(0) = \omega_j$. As a consequence, $\widetilde{K}_i$ and $\widetilde{K}_i^f$ become deterministic quantities. $K_i^{eq}(t)$ can be treated as a linear combination of forward FX rates. Freezing the weights is clearly an approximation, but it is justified by the weights' low variability through time [Brigo et al., 2013]. Most importantly however, it allows us to evaluate the expectation using familiar option pricing techniques as we can write

$$EPE(t, T_i) = N^d A^{i,m}(t) \mathbb{E}^{A^{i,m}} \left(\left(\widetilde{K}_i - K_i^{eq}(T_i) \right)^+ \middle| \mathcal{F}_t \right)$$

7.1.5 Pricing EPE as an option

We proceed by considering a suitable model to capture the dynamics of the cross-currency swap rate K_i^{eq}. We do so by assuming that $K_i^{eq}(t)$ follows a geometric Brownian Motion. This is a justified assumption since $K_i^{eq}(t)$ is a linear combination of forward exchange rates, which within our framework are modeled by log-normal Wiener processes. Under the annuity measure, the CCS-rate is a Martingale. Hence it follows that the dynamics of this rate are given by

$$dK_i^{eq}(t) = v_i^{eq}(t) K_i^{eq}(t)\, dW_i^{eq}(t)$$

where W_i^{eq} is a standard Brownian motion under $\mathbb{Q}^{A^{i,m}}$. Note that these dynamics match the assumptions of Black's model as we described them in section 3.4.1. Then, in accordance with this framework, we can compute the expectation in the expression for the EPE by using Black's formula. The risk-neutral price $p(t)$ of a put-option on $K_i^{eq}(T_i)$ with strike price $\widetilde{K}_i$ and expiry T_i at time t is given by

$$p\left(\widetilde{K}_i, K_i^{eq}(t), \sigma_i^{eq}(t)\right) = \widetilde{K}_i \cdot \mathcal{N}\left(\frac{-\log\left(\frac{K_i^{eq}(t)}{\widetilde{K}_i}\right) + \frac{1}{2}\sigma_i^{eq}(t)^2}{\sigma_i^{eq}(t)} \right) - K_i^{eq}(t) \cdot \mathcal{N}\left(-\frac{\log\left(\frac{K_i^{eq}(t)}{\widetilde{K}_i}\right) + \frac{1}{2}\sigma_i^{eq}(t)^2}{\sigma_i^{eq}(t)} \right)$$

where $\mathcal{N}$ denotes the the cumulative distribution function of a standard normal random variable. The parameter σ_i^{eq} is defined as

$$\sigma_i^{eq}(t) = \sqrt{\int_t^{T_i} \left(v_i^{eq}(u)\right)^2 du}$$

Now, summarizing what we have seen throughout this section, we can approximate the CVA charge for an CCS by

$$CVA(t) \approx LGD \cdot \sum_{i=1}^{m} EPE(t, T_i) \cdot PD(T_{i-1}, T_i)$$

where

$$EPE(t, T_i) = N^d \cdot A^{i,m}(t) \cdot p\left(\widetilde{K}_i, K_i^{eq}(t), \sigma_i^{eq}(t)\right)$$

What remains to be done is finding an expression for the volatility of the cross-currency swap rate $v_{S^{i,m}}$. We would like to express this variable in terms of parameters of the Hull-White and FX model. Doing so allows us to calculate the CVA charge along the simulation paths of our model, which is one of our objectives. In the next section we will show how this can be done. At first we will require an expression for the volatility of the forward exchange rate Φ. Therefore, before we work out the volatility of the cross-currency swap rate, we will focus on the dynamics of the forward exchange rate.

7.2 Forward exchange rate volatility

Our starting point is the known expression for the forward FX rate. Let $i \in \{1, \dots, m\}$. For simplicity we adopt the notation $\Phi_i(t) \equiv \Phi(t, T_i)$. In section 3.3 it was shown that we have the following relation:

$$\Phi_i(t) = \frac{\varphi(t) P^f(t, T_i)}{P^d(t, T_i)}$$

Both the foreign and domestic zero-coupon bond prices can be expressed in terms of Hull-White parameters:

$$\begin{aligned} P^d(t,T) &= A^d(t,T) \cdot e^{-B^d(t,T) r^d(t)} \\ P^f(t,T) &= A^f(t,T) \cdot e^{-B^f(t,T) r^f(t)} \end{aligned}$$

Here it is important to distinguish between the foreign and domestic parameters. It allows us to rewrite the forward exchange rate as follows

$$\Phi_i(t) = \varphi(t) \frac{A^f(t, T_i)}{A^d(t, T_i)} e^{B^d(t,T_i) r^d(t) - B^f(t,T_i) r^f(t)}$$

In order to isolate the volatility of the forward FX, we will work out the differential of $\Phi_i(t)$. Before we do so, note that Φ_i depends on three random variables, namely r^d, r^f and φ. For all three, the dynamics within our multi-currency Hull-White and FX model are known:

$$\begin{aligned} dr^f(t) &= \left(\theta^f(t) - a^f \cdot r^f(t)\right) dt + \sigma^f dW^f(t) \\ dr^d(t) &= \left(\theta^d(t) - a^d \cdot r^d(t)\right) dt + \sigma^d dW^d(t) \\ d\varphi(t) &= \left(r^d(t) - r^f(t)\right) \varphi(t) dt + \sigma^\varphi(t) \varphi(t) dW^\varphi(t) \end{aligned}$$

In the above expressions, W^d and W^φ are by definition Brownian motions under the domestic risk-neutral measure $\mathbb{Q}^d$. W^f is a Brownian motion under the foreign risk-neutral measure $\mathbb{Q}^f$. However, we have seen that the SDE of r^f can equivalently be expressed under the domestic risk-neutral measure, such that

$$dr^f(t) = \left(\theta^f(t) - \sigma^f \sigma^\varphi(t) \rho_{f\varphi} - a^f \cdot r^f(t)\right) dt + \sigma^f dW^f(t)$$

Where this time, W^f denotes a standard Brownian motion under $\mathbb{Q}^d$. Now that we have the dynamics of r^d, r^f and φ under a common measure, we continue to apply the 3-dimensional Itô-Doeblin formula. To do so, first define the function:

$$f(t, X_1, X_2, X_3) = X_3 \frac{A^f(t, T_i)}{A^d(t, T_i)} e^{B^d(t,T_i) X_1 - B^f(t,T_i) X_2}$$

Itô-Doeblin then states that

$$\begin{aligned} df(t, X_1, X_2, X_3) &= f_t(t, X_1, X_2, X_3) dt + \sum_{j=1}^{3} f_{x_j}(t, X_1, X_2, X_3) dX_j \\ &\quad + \frac{1}{2} \sum_{j=1}^{3} \sum_{k=1}^{3} f_{x_j x_k}(t, X_1, X_2, X_3) dX_j dX_k \end{aligned}$$

Now first write the dynamics of each Itô process $X_j(t)$ in the general form $dX_j(t) = \Theta_j(t)dt + \Delta_j(t)dW_j(t)$, where Θ_j and Δ_j are adapted stochastic processes. We allow the Brownian motions to be correlated, which implies $dW_j(t)dW_k(t) = \rho_{jk}dt$ when $j \neq k$. This lets us rewrite the expression above as follows (for now omitting the time arguments of each function where they are clear from the context):

$$\begin{aligned} df\left(t, X_1, X_2, X_3\right) &= f_t dt + \sum_{j=1}^{3} f_{x_j}\Theta_j dt + \sum_{j=1}^{3} f_{x_j}\Delta_j dW_j \\ &\quad + \sum_{j\neq k} f_{x_j x_k}\Delta_j\Delta_k\rho_{jk}dt + \frac{1}{2}\sum_{j=1}^{3} f_{x_j x_j}\Delta_j^2 dt \\ &= \mu_i(t)dt + \sum_{j=1}^{3} f_{x_j}\Delta_j dW_j \end{aligned}$$

where $\mu_i(t)$ is an adapted, stochastic process. For the partial derivatives of the function f we find:

$$\begin{aligned} f_{x_1}(t) &= B^d\left(t, T_i\right) X_3 \frac{A^f\left(t, T_i\right)}{A^d\left(t, T_i\right)} e^{B^d(t,T_i)X_1 - B^f(t,T_i)X_2} \\ f_{x_2}(t) &= -B^f\left(t, T_i\right) X_3 \frac{A^f\left(t, T_i\right)}{A^d\left(t, T_i\right)} e^{B^d(t,T_i)X_1 - B^f(t,T_i)X_2} \\ f_{x_3}(t) &= \frac{A^f\left(t, T_i\right)}{A^d\left(t, T_i\right)} e^{B^d(t,T_i)X_1 - B^f(t,T_i)X_2} \end{aligned}$$

Using the results above we will now substitute $(X_1, X_2, X_3)(t) = \left(r^d(t), r^f(t), \varphi(t)\right)$ and $(\Delta_1, \Delta_2, \Delta_3)(t) = \left(\sigma^d, \sigma^f, \sigma^\varphi(t)\varphi(t)\right)$. This results in the following expression for the FX forward differential:

$$\begin{aligned} d\Phi_i(t) &= df\left(t, r^d(t), r^f(t), \varphi(t)\right) \\ &= \mu_i(t)\Phi_i(t)dt + B^d\left(t, T_i\right)\varphi(t)\frac{A^f\left(t, T_i\right)}{A^d\left(t, T_i\right)} e^{B^d(t,T_i)r^d(t) - B^f(t,T_i)r^f(t)}\sigma^d dW^d(t) \\ &\quad -B^f\left(t, T_i\right)\varphi(t)\frac{A^f\left(t, T_i\right)}{A^d\left(t, T_i\right)} e^{B^d(t,T_i)r^d(t) - B^f(t,T_i)r^f(t)}\sigma^f dW^f(t) \\ &\quad +\frac{A^f\left(t, T_i\right)}{A^d\left(t, T_i\right)} e^{B^d(t,T_i)r^d(t) - B^f(t,T_i)r^f(t)}\varphi(t)\sigma^\varphi(t)dW^\varphi(t) \\ &= \mu_i(t)\Phi_i(t)dt + \Phi_i(t)\left[B^d\left(t, T_i\right)\sigma^d dW^d(t) - B^f\left(t, T_i\right)\sigma^f dW^f(t) + \sigma^\varphi(t)dW^\varphi(t)\right] \end{aligned}$$

We do not work out the drift term μ_i as we will not need it. We see that the diffusion term of the process $\Phi_i(t)$ is a superposition of three correlated Brownian motions, namely W^d, W^f and W^φ. We assume their correlations to be deterministic, given by the parameters ρ_{df}, $\rho_{d\varphi}$ and $\rho_{f\varphi}$. We assume that these correlation coefficients are scalars, such that $\rho_{df}, \rho_{d\varphi}, \rho_{f\varphi} \in [-1, 1]$. As a next step we would like to rewrite the diffusion term as a process driven by a single Brownian motion. We will show that this is possible by an application of Lévy's theorem. Therefore, define a new process $\widetilde{W}_i$ with dynamics as stated below

$$d\widetilde{W}_i(t) = \frac{1}{\sqrt{Y_i(t)}}\left[B^d\left(t, T_i\right)\sigma^d dW^d(t) - B^f\left(t, T_i\right)\sigma^f dW^f(t) + \sigma^\varphi(t)dW^\varphi(t)\right]$$

where $\widetilde{W}_i(0) = 0$ and the adapted, deterministic process Y_i is defined as

$$\begin{aligned} Y_i(t) &= \left(B^d\left(t, T_i\right)\sigma^d\right)^2 + \left(B^f\left(t, T_i\right)\sigma^f\right)^2 + \sigma^\varphi(t)^2 + 2B^d\left(t, T_i\right)\sigma^d\sigma^\varphi(t)\rho_{d\varphi} \\ &\quad -2B^f\left(t, T_i\right)\sigma^f\sigma^\varphi(t)\rho_{f\varphi} - 2B^d\left(t, T_i\right)B^f\left(t, T_i\right)\sigma^d\sigma^f\rho_{df} \end{aligned}$$

We can show by Lévy's theorem in one dimension that the process $\widetilde{W}_i$ is actually a Brownian motion. The first requirement is to show that $\widetilde{W}_i$ is a Martingale with continuous paths. Therefore note that

$$\widetilde{W}_i(t) = \int_0^t \frac{B^d\left(u, T_i\right)\sigma^d}{\sqrt{Y_i(u)}}dW^d(u) - \int_0^t \frac{B^f\left(u, T_i\right)\sigma^f}{\sqrt{Y_i(u)}}dW^f(u) + \int_0^t \frac{\sigma^\varphi(t)}{\sqrt{Y_i(u)}}dW^\varphi(u)$$

By the properties of the Itô integral, we know that $\widetilde{W}_i$ has continuous paths and is a Martingale relative to $\mathcal{F}_t$. The second requirement concerns the quadratic variation of $\widetilde{W}_i$. We see that

$$\begin{aligned}
d\widetilde{W}_i(t)d\widetilde{W}_i(t) &= \frac{1}{Y_i(t)}\left[B^d(t,T_i)\,\sigma^d dW^d(t) - B^f(t,T_i)\,\sigma^f dW^f(t) + \sigma^{\varphi}(t)dW^{\varphi}(t)\right]^2 \\
&= \frac{1}{Y_i(t)}\Big[\left(B^d(t,T_i)\,\sigma^d\right)^2 dt + \left(B^f(t,T_i)\,\sigma^f\right)^2 dt + \sigma^{\varphi}(t)^2 dt \\
&\quad +2B^d(t,T_i)\,\sigma^d\sigma^{\varphi}(t)\rho_{d\varphi}dt - 2B^f(t,T_i)\,\sigma^f\sigma^{\varphi}(t)\rho_{f\varphi}dt \\
&\quad -2B^d(t,T_i)\,B^f(t,T_i)\,\sigma^d\sigma^f\rho_{df}dt\Big] \\
&= \frac{1}{Y_i(t)}Y_i(t)dt = dt
\end{aligned}$$

In other words, $\left[\widetilde{W}_i, \widetilde{W}_i\right](t) = t$. By this we satisfy the conditions for Lévy's theorem and it follows that $\widetilde{W}_i$ is a standard Brownian motion. As a result, we can rewrite the differential of Φ_i as follows

$$\begin{aligned}
d\Phi_i(t) &= \mu_i(t)\Phi_i(t)dt + \Phi_i(t)\left[B^d(t,T_i)\,\sigma^d dW^d(t) - B^f(t,T_i)\,\sigma^f dW^f(t) + \sigma^{\varphi}(t)dW^{\varphi}(t)\right] \\
&= \mu_i(t)\Phi_i(t)dt + \Phi_i(t)\sqrt{Y_i(t)}d\widetilde{W}_i(t)
\end{aligned}$$

We conclude that the instantaneous volatility of the forward FX rate can be expressed in model parameters as follows:

$$\begin{aligned}
v_i(t) &= \sqrt{Y_i(t)} \\
&= \sqrt{\begin{array}{c}\left(B^d(t,T_i)\,\sigma^d\right)^2 + \left(B^f(t,T_i)\,\sigma^f\right)^2 + \sigma^{\varphi}(t)^2 + 2B^d(t,T_i)\,\sigma^d\sigma^{\varphi}(t)\rho_{d\varphi} \\ -2B^f(t,T_i)\,\sigma^f\sigma^{\varphi}(t)\rho_{f\varphi} - 2B^d(t,T_i)\,B^f(t,T_i)\,\sigma^d\sigma^f\rho_{df}\end{array}}
\end{aligned}$$

7.3 Forward exchange rate correlations

In our analysis of the forward interest rate dynamics, we noted that the correlation between two forward rates F_i and F_j is always equal to one. This is a consequence of the Hull-White framework in which we work. This "perfect correlation" does not hold for forward exchange rates. What therefore remains to be determined is the correlation coefficient ρ_{ij}, that denotes the instantaneous correlation between two FX forward processes. Let $i,j \in \{1,\dots,m\}$ such that $i \neq j$. Consider two corresponding FX forward processes of which we assumed the dynamics are given by

$$\begin{aligned}
d\Phi_i(t) &= \mu_i(t)\Phi_i(t)dt + v_i(t)\Phi_i(t)dW_i(t) \\
d\Phi_j(t) &= \mu_j(t)\Phi_j(t)dt + v_j(t)\Phi_j(t)dW_j(t)
\end{aligned}$$

For two correlated Brownian motions we have in general that

$$dW_i(t)dW_j(t) = \rho_{ij}(t)dt$$

We would like to express ρ_{ij} in model parameters. We will therefore use the formula for $\widetilde{W}_i$ that we found in section 7.2. Computation of $dW_i(t)dW_j(t)$ then yields

$$\begin{aligned}
d\widetilde{W}_i(t)d\widetilde{W}_j(t) &= \frac{1}{\sqrt{Y_i(t)Y_j(t)}}\left[B^d(t,T_i)\,\sigma^d dW^d(t) - B^f(t,T_i)\,\sigma^f dW^f(t) + \sigma^{\varphi}(t)dW^{\varphi}(t)\right] \\
&\quad \times\left[B^d(t,T_j)\,\sigma^d dW^d(t) - B^f(t,T_j)\,\sigma^f dW^f(t) + \sigma^{\varphi}(t)dW^{\varphi}(t)\right] \\
&= \frac{1}{\sqrt{Y_i(t)Y_j(t)}}\Big[B^d(t,T_i)\,B^d(t,T_j)\left(\sigma^d\right)^2 dt \\
&\quad +B^d(t,T_j)\,\sigma^d\sigma^{\varphi}(t)\rho_{d\varphi}dt - B^d(t,T_i)\,B^f(t,T_j)\,\sigma^d\sigma^f\rho_{fd}dt \\
&\quad +B^f(t,T_i)\,B^f(t,T_j)\left(\sigma^f\right)^2 dt - B^f(t,T_j)\,\sigma^f\sigma^{\varphi}(t)\rho_{f\varphi}dt \\
&\quad +B^d(t,T_i)\,\sigma^d\sigma^{\varphi}(t)\rho_{d\varphi}dt - B^f(t,T_i)\,\sigma^f\sigma^{\varphi}(t)\rho_{f\varphi}dt \\
&\quad -B^f(t,T_i)\,B^d(t,T_j)\,\sigma^f\sigma^d\rho_{fd}dt + \sigma^{\varphi}(t)^2 dt\Big]
\end{aligned}$$

We conclude that

$$\begin{aligned}\rho_{ij}(t) &= \frac{1}{\sqrt{Y_i(t)Y_j(t)}}\Big[B^d(t,T_i)\,B^d(t,T_j)\left(\sigma^d\right)^2 \\ &+B^d(t,T_j)\,\sigma^d\sigma^\varphi(t)\rho_{d\varphi} - B^d(t,T_i)\,B^f(t,T_j)\,\sigma^d\sigma^f\rho_{fd} \\ &+B^f(t,T_i)\,B^f(t,T_j)\left(\sigma^f\right)^2 - B^f(t,T_j)\,\sigma^f\sigma^\varphi(t)\rho_{f\varphi} \\ &+B^d(t,T_i)\,\sigma^d\sigma^\varphi(t)\rho^\varphi - B^f(t,T_i)\,\sigma^f\sigma^\varphi(t)\rho^\varphi \\ &-B^f(t,T_i)\,B^d(t,T_j)\,\sigma^f\sigma^d\rho_{fd} + \sigma^\varphi(t)^2\Big]\end{aligned}$$

7.4 Cross-currency swap rate volatility

We proceed by approximating the cross-currency swap rate volatility using the results of sections 7.2 and 7.3. Recall that $K_i^{eq}(t)$ evolves according to a geometric Brownian motion, of which the dynamics are given by

$$dK_i^{eq}(t) = v_i^{eq}(t)K_i^{eq}(t)\,dW_i^{eq}(t)$$

where W_i^{eq} is a Brownian motion under $\mathbb{Q}^{A^{i,m}}$ [Brigo et al., 2013]. In section 7.1 we have seen that the cross-currency swap rate can be expressed as the weighted sum of forward exchange rates. By freezing the weights we obtained the following approximation:

$$K_i^{eq}(t) \approx \widetilde{K}_i^f \sum_{j=i+1}^{m} \omega_j \Phi_j(t)$$

We will isolate $v_i^{eq}(t)$ using the SDE above. Note that we can write

$$dK_i^{eq}(t)\,dK_i^{eq}(t) = v_i^{eq}(t)^2 K_i^{eq}(t)^2\,dW_i^{eq}(t)dW_i^{eq}(t) = v_i^{eq}(t)^2 K_i^{eq}(t)^2\,dt$$

Therefore we have

$$\begin{aligned}v_i^{eq}(t)^2 dt &= \frac{dK_i^{eq}(t)\,dK_i^{eq}(t)}{\left(K_i^{eq}(t)\right)^2} \\ &= \frac{d\left(\widetilde{K}_i^f\sum_{j=i+1}^m \omega_j\Phi_j(t)\right)d\left(\widetilde{K}_i^f\sum_{k=i+1}^m \omega_k\Phi_k(t)\right)}{\left(\widetilde{K}_i^f\sum_{l=i+1}^m \omega_l\Phi_l(t)\right)^2} \\ &= \frac{\widetilde{K}_i^f\sum_{j=i+1}^m \omega_j d\Phi_j(t)\cdot\widetilde{K}_i^f\sum_{k=i+1}^m \omega_k d\Phi_k(t)}{\left(\widetilde{K}_i^f\sum_{l=i+1}^m \omega_l\Phi_l(t)\right)^2}\end{aligned}$$

Recall that we had

$$d\Phi_j(t) = \mu_j(t)\Phi_j(t)dt + v_j(t)\Phi_j(t)dW_j(t)$$

In this differential W_i is a Brownian motion under $\mathbb{Q}^d$. By an application of Girsanov's theorem we can rewrite the differential so that the diffusion term is driven by a Brownian motion under the annuity-measure. Doing so will affect the drift-term μ_i, but will leave the volatility untouched due to the diffusion invariance principle:

$$d\Phi_j(t) = \mu_j^{A^{i,m}}(t)\Phi_j(t)dt + v_j(t)\Phi_j(t)dW_j^{A^{i,m}}(t)$$

For our application, this is convenient as the drift-term will drop out. Denote the instantaneous correlation between W_i and W_j with ρ_{ij}. Now substitute the differential of $\Phi_j(t)$ and $\Phi_k(t)$ in the expression for v_i^{eq}.

The volatility can then be written as follows [Brigo et al., 2013]:

$$\begin{aligned} v_i^{eq}(t)^2 dt &= \frac{\widetilde{K}_i^f \sum_{j=i+1}^m \omega_j \left(\mu_j^{A^{i,m}}(t)\Phi_j(t)dt + v_j(t)\Phi_j(t)dW_j^{A^{i,m}}(t) \right)}{\left(\widetilde{K}_i^f \sum_{l=i+1}^m \omega_l \Phi_l(t) \right)^2} \times \\ & \quad \widetilde{K}_i^f \sum_{k=i+1}^m \omega_k \left(\mu_k^{A^{i,m}}(t)\Phi_k(t)dt + v_k(t)\Phi_k(t)dW_k^{A^{i,m}}(t) \right) dt \\ &= \sum_{j=i+1}^m \sum_{k=i+1}^m \frac{\omega_j \omega_k \Phi_j(t)\Phi_k(t)\rho_{jk} v_j(t) v_k(t)}{\left(\sum_{l=i+1}^M \omega_l \Phi_l(t) \right)^2} dt \end{aligned}$$

If we subsequently substitute the forward exchange rate volatilities and correlations as we found them in section 7.2 and 7.3, we are left with the log-normal volatility of the cross-currency swap rate. This allows us to compute variance σ_i^{eq} for K_i^{eq}, which we can apply in Black's formula for pricing an option. The formula for v_i^{eq} is almost deterministic, apart from the $\Phi_j(t)$ terms. We make a fully deterministic approximation by freezing $\Phi_j(t)$ at their value today at $t = 0$. This will be convenient for simulation purposes. As a result we can compute

$$\begin{aligned} \sigma_i^{eq}(t)^2 &= \int_t^{T_i} v_i^{eq}(t)^2 dt \\ &\approx \sum_{j=i+1}^m \sum_{k=i+1}^m \frac{\omega_j \omega_k \Phi_j(0)\Phi_k(0)}{\left(\sum_{l=i+1}^m \omega_l \Phi_l(0) \right)^2} \int_t^{T_i} \rho_{jk}(u) v_j(u) v_k(u) du \end{aligned}$$

With closed-form expressions for ρ_{jk}, v_k and v_j at hand we are ready to evaluate the integral above. First however we will further work out the expression $\rho_{jk}(t)v_j(t)v_k(t)$. To do so, recall that the formula for B as used in ρ_{jk}, v_k and v_j is given by

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$$

Plugging B in the results for the instantaneous correlations and volatilities of the FX forward rate found in the previous sections provides us with the following expression

$$\begin{aligned} \rho_{jk}(t)v_j(t)v_k(t) &= \left(\frac{\sigma^d}{a^d}\right)^2 \left(1 - e^{-a^d(T_j-t)}\right)\left(1 - e^{-a^d(T_k-t)}\right) \\ & + \left(\frac{\sigma^f}{a^f}\right)^2 \left(1 - e^{-a^f(T_j-t)}\right)\left(1 - e^{-a^f(T_k-t)}\right) + \sigma^\varphi(t)^2 \\ & - \frac{\sigma^f\sigma^d\rho_{fd}}{a^f a^d}\left(1 - e^{-a^f(T_j-t)}\right)\left(1 - e^{-a^d(T_k-t)}\right) \\ & - \frac{\sigma^f\sigma^d\rho_{fd}}{a^f a^d}\left(1 - e^{-a^d(T_j-t)}\right)\left(1 - e^{-a^f(T_k-t)}\right) \\ & - \frac{\sigma^f\sigma^\varphi(t)\rho_{f\varphi}}{a^f}\left(1 - e^{-a^f(T_j-t)}\right) - \frac{\sigma^f\sigma^\varphi(t)\rho_{f\varphi}}{a^f a^\varphi}\left(1 - e^{-a^f(T_k-t)}\right) \\ & + \frac{\sigma^d\sigma^\varphi(t)\rho_{d\varphi}}{a^d}\left(1 - e^{-a^d(T_j-t)}\right) + \frac{\sigma^d\sigma^\varphi(t)\rho_{d\varphi}}{a^d}\left(1 - e^{-a^d(T_k-t)}\right) \end{aligned}$$

$$
\begin{aligned}
= \quad & \left(\frac{\sigma^d}{a^d}\right)^2 + \left(\frac{\sigma^f}{a^f}\right)^2 - 2\frac{\sigma^f\sigma^d\rho_{fd}}{a^f a^d} + 2\frac{\sigma^d\sigma^{\varphi}(t)\rho_{d\varphi}}{a^d} - 2\frac{\sigma^f\sigma^{\varphi}(t)\rho_{f\varphi}}{a^f} + \sigma^{\varphi}(t)^2 \\
& + \left(\frac{\sigma^f\sigma^d\rho_{fd}}{a^f a^d} - \left(\frac{\sigma^d}{a^d}\right)^2 - \frac{\sigma^d\sigma^{\varphi}(t)\rho_{d\varphi}}{a^d}\right)\left(e^{-a^d T_j} + e^{-a^d T_k}\right)e^{a^d t} \\
& + \left(\frac{\sigma^f\sigma^d\rho_{fd}}{a^f a^d} - \left(\frac{\sigma^f}{a^f}\right)^2 + \frac{\sigma^f\sigma^{\varphi}(t)\rho_{f\varphi}}{a^f}\right)\left(e^{-a^f T_j} + e^{-a^f T_k}\right)e^{a^f t} \\
& + \left(\frac{\sigma^d}{a^d}\right)^2 e^{-a^d(T_j+T_k)}e^{2a^d t} + \left(\frac{\sigma^f}{a^f}\right)^2 e^{-a^f(T_j+T_k)}e^{2a^f t} \\
& - \frac{\sigma^f\sigma^d\rho_{fd}}{a^f a^d}\left(e^{-a^d T_j - a^f T_k} + e^{-a^f T_j - a^d T_k}\right)e^{\left(a^d+a^f\right)t}
\end{aligned}
$$

And thus we conclude

$$
\begin{aligned}
\sigma_i^{eq}(t) \quad \approx \quad & \sum_{j=i+1}^{m}\sum_{k=i+1}^{m} \frac{\omega_j\omega_k\Phi_j(t)\Phi_k(t)}{\left(\sum_{l=i+1}^{m}\omega_l\Phi_l(t)\right)^2}\int_t^{T_i}\rho_{jk}(t)v_j(t)v_k(t)dt \\
= \quad & \sum_{j=i+1}^{m}\sum_{k=i+1}^{m} \frac{\Delta t_j\Delta t_k P^f(t,T_j)P^f(t,T_k)}{\left(\sum_{l=i+1}^{m}\Delta t_l P^f(t,T_l)\right)^2} \times \left[\int_t^{T_i}\sigma^{\varphi}(u)^2du\right. \\
& \left(\left(\frac{\sigma^d}{a^d}\right)^2 + \left(\frac{\sigma^f}{a^f}\right)^2 - 2\frac{\sigma^f\sigma^d\rho_{fd}}{a^f a^d}\right)(T_i - t) + 2\left(\frac{\sigma^d\rho_{d\varphi}}{a^d} - \frac{\sigma^f\rho_{f\varphi}}{a^f}\right)\int_t^{T_i}\sigma^{\varphi}(u)du \\
& - \frac{\sigma^d\rho_{d\varphi}}{a^d}\left(e^{-a^d T_j} + e^{-a^d T_k}\right)\int_t^{T_i}\sigma^{\varphi}(u)e^{a^d u}du + \frac{\sigma^f\rho_{f\varphi}}{a^f}\left(e^{-a^f T_j} + e^{-a^f T_k}\right)\int_t^{T_i}\sigma^{\varphi}(u)e^{a^f u}du \\
& + \left(\frac{\sigma^f\sigma^d\rho_{fd}}{a^f a^d} - \left(\frac{\sigma^d}{a^d}\right)^2\right)\left(e^{-a^d T_j} + e^{-a^d T_k}\right)\left(e^{a^d T_i} - e^{a^d t}\right) \\
& + \left(\frac{\sigma^f\sigma^d\rho_{fd}}{a^f a^d} - \left(\frac{\sigma^f}{a^f}\right)^2\right)\left(e^{-a^f T_j} + e^{-a^f T_k}\right)\left(e^{a^f T_i} - e^{a^f t}\right) \\
& + \left(\frac{\sigma^d}{a^d}\right)^2 e^{-a^d(T_j+T_k)}\left(e^{2a^d T_i} - e^{2a^d t}\right) + \left(\frac{\sigma^f}{a^f}\right)^2 e^{-a^f(T_j+T_k)}\left(e^{2a^f T_i} - e^{2a^f t}\right) \\
& \left. - \frac{\sigma^f\sigma^d\rho_{fd}}{a^f a^d}\left(e^{-a^d T_j - a^f T_k} + e^{-a^f T_j - a^d T_k}\right)\left(e^{\left(a^d+a^f\right)T_i} - e^{\left(a^d+a^f\right)t}\right)\right]
\end{aligned}
$$

8 CVA under a dual curve framework

The CVA approximation of a CCS presented in chapter 7 is based on the assumption of an underlying single-curve interest rate framework. In the standard model calibration procedure, market-implied swap rates are bootstrapped to construct a forward curve. Forward rates of different tenors and different maturities, should theoretically be consistent. By this we mean to say that for example the product of 2 consecutive 3M forward rates should yield the 6M forward rate, without a basis in between them. The modeled forward rates should therefore be independent of the underlying set of instruments used for calibration. As a consequence, all zero-coupon bond prices can be evaluated on the same underlying curve. Before the credit crunch of 2008, the basis between forward rates was practically negligible. LIBOR rates were used as a proxy for forward rates and simultaneously the risk-free discounting rate. After the crisis the basis could no longer be ignored and the sole use of LIBOR was no longer accurate. Bonds prices used for discounting are by their definition implied by rates of an infinitesimal tenor. Bonds that appear in the evaluation of the derivatives are to compute simply-compounded rates of a much longer tenor. For this reason it is now common practice to calibrate the discount curve based on instruments with the shortest possible tenors. Typically we use *Overnight Indexed Swaps* (OIS). The pay-off of these instruments are based on daily compounded interest rates, also called overnight rates. Henceforth we will throughout this chapter distinguish between the *overnight curve* implied by OIS rates and the *index curve* implied by swap rates. This distinction should improve the match between theoretical and market-observed evaluations.

In the following section we will illustrate some adjustments to the multi-currency Hull-White framework, under the introduction of a dual curve setting. Subsequently we will describe how this affects the analytical CVA for a cross-currency swap and derive an adjusted formula to approximate the EPE profile.

8.1 Modeling the short-rate

For a given currency, the distinction between overnight and index curves has an effect on modeling the short-rate. As usual we assume the dynamics of the short rate to be modeled by the one-factor Hull-White model. In the single-curve setting, we considered only a single short-rate $r(t)$, modeled according to an Ornstein-Uhlenbeck process. For calibration purposes, the short-rate $r(t)$ is often represented as the sum of a stochastic zero-mean process $x(t)$ and a deterministic process $\alpha(t)$. Recall that the dynamics of these processes under $\mathbb{Q}$ are given by

$$\begin{aligned} dx(t) &= -ax(t)dt + \sigma dW(t) \\ \alpha(t) &= f^M(0,t) + \frac{\sigma^2}{2a^2}\left(1 - e^{-at}\right)^2 \end{aligned}$$

such that $r(t) = \alpha(t) + x(t)$. Recall that $f^M(0,T)$ denotes the instantaneous forward rate at time T.

Now to make the switch to a dual-curve framework, we expand the setup as described above, by considering two stochastic short-rates. One will be associated with the index curve: $l(t)$. The other will be associated with the overnight curve: $r(t)$. We assume the stochastic dynamics of both processes to be similar. This allows us to model the short-rates by simulating only one common stochastic zero-mean process x. Under this assumption, the short-rates are subsequently given by

$$\begin{aligned} r(t) &= \alpha(t) + x(t) \\ l(t) &= \beta(t) + x(t) \end{aligned}$$

Here the deterministic perturbations α and β are characterized by the different term-structures of f_r^M and f_l^M associated with the discount and index curves.

8.2 Zero-coupon bonds

Having defined the Hull-White short-rate processes for the dual-curve framework, we can continue by considering an expression for the zero-coupon bond price. Here we will now need to distinguish between bond prices associated to the overnight curve and the index curve. As before, under the risk-neutral measure $\mathbb{Q}$, the zero-bond prices are given by

$$\begin{aligned} P^r(t,T) &= \mathbb{E}^Q\left(e^{-\int_t^T r(s)ds}\middle|\mathcal{F}_t\right) \\ P^l(t,T) &= \mathbb{E}^Q\left(e^{-\int_t^T l(s)ds}\middle|\mathcal{F}_t\right) \end{aligned}$$

As both short-rates share a common process for their stochastic dynamics, we can express the zero-bond price P^l in terms of P^r. According to our assumptions, $\alpha(t)$ and $\beta(t)$ are deterministic and hence they are in particular $\mathcal{F}_t$-measurable processes. By application of the "taking out what is known" property of conditional expectations, we can rewrite the zero-bond prices as follows:

$$\begin{aligned} P^r(t,T) &= \mathbb{E}^Q\left(e^{-\int_t^T(\alpha(s)+x(s))ds}\middle|\mathcal{F}_t\right) = e^{-\int_t^T\alpha(s)ds}\mathbb{E}^Q\left(e^{-\int_t^T x(s)ds}\middle|\mathcal{F}_t\right) \\ P^l(t,T) &= \mathbb{E}^Q\left(e^{-\int_t^T(\beta(s)+x(s))ds}\middle|\mathcal{F}_t\right) = e^{-\int_t^T\beta(s)ds}\mathbb{E}^Q\left(e^{-\int_t^T x(s)ds}\middle|\mathcal{F}_t\right) \end{aligned}$$

Hence allowing us to write

$$\begin{aligned} P^l(t,T) &= e^{\int_t^T(\alpha(s)-\beta(s))ds}P^r(t,T) \\ &= \delta(t,T)\,P^r(t,T) \end{aligned}$$

where $\delta(t,T) = e^{\int_t^T(\alpha(s)-\beta(s))ds}$, denotes the deterministic process of the continuously compounded short-rate basis. We can consider δ to be a deterministic correction coefficient, that should be applied, whenever a zero-coupon bond price is used in the context of LIBOR rates, rather than discounting. Using this coefficient, we will propose a correction to the fair cross-currency swap-rate as derived in the previous section.

8.3 Approximating the EPE profile

For a given $T_i \in \{T_1,\dots,T_m\}$ and $t \le T_0$, we aim to approximate the expected positive exposure for a cross-currency swap deal at time T_i. To do so we will use a similar approach as presented for the single-curve setting. Eventually this will lead to an analytic expression for the CVA on a CCS deal, adjusted to a multi-curve setting. As before, an expression for the exposure is given as below.

$$\begin{aligned} V(T_i) &= \sum_{j=i+1}^{m}\Delta t_j P(T_i,T_j)\left(N^f K\Phi(T_i,T_j) - N^d F(T_i,T_{j-1},T_j)\right) \\ &\quad -N^d P(T_i,T_m) + N^f P(T_i,T_m)\,\Phi(T_i,T_m) \end{aligned}$$

Henceforth, until the end of this chapter, we will denote P^r simply as P. In other words $P \equiv P^r$. We therefore implicitly assume that $P \neq P^l$ and thus allow for a basis spread between the discount and index curve.

8.3.1 The foreign leg

The assumptions above have no direct consequences for our expression of the foreign leg: each P included in this fixed leg is a zero-bond price associated with the OIS curve. The effective foreign rate $\widetilde{K}_i^f$, as derived in the previous section, remains therefore unchanged.

$$\widetilde{K}_i^f(T_i) = \sum_{j=i+1}^{m}\omega_j^f(T_i)\,K_j$$

With weights ω_j^f given by

$$\omega_j^f(T_i) = \frac{P^f(T_i,T_j)\,\Delta t_j}{\sum_{k=i+1}^{m}P^f(T_i,T_k)\,\Delta t_k}$$

and adjusted fixed rates

$$K_j = \begin{cases} \frac{K}{\varphi(T_0)} & j = i+1,\dots,m-1 \\ \frac{K+1/\Delta t_j}{\varphi(T_0)} & j = m \end{cases}$$

8.3.2 The domestic leg

The domestic leg on the other hand calls for a revision. The forward rate, which is required to evaluate the floating flows, is calculated using zero-coupon bond prices associated with the index curve. By its definition we have

$$F(t, T_{i-1}, T_i) = \frac{1}{\Delta t_i}\left(\frac{P^l(t, T_{i-1})}{P^l(t, T_i)} - 1\right) = \frac{1}{\Delta t_i}\left(\frac{P(t, T_{i-1})\,\delta(t, T_{i-1})}{P(t, T_i)\,\delta(t, T_i)} - 1\right)$$

Note that $\delta(t,T_{i-1})/\delta(t,T_i) = e^{\int_{T_{i-1}}^{T_i}(\beta(s)-\alpha(s))ds}$. For this deterministic expression we will use the following short-hand notation

$$\hat{\delta}(S, T) = e^{\int_S^T(\beta(s)-\alpha(s))ds} = \frac{1}{\delta(T, S)}$$

At this point we can rewrite the floating leg of the CCS as follows

$$\begin{aligned} N^d \sum_{j=i+1}^{m} P(T_i, T_j)\,\Delta t_j F^d(T_i; T_{j-1}, T_j) &= N^d \sum_{j=i+1}^{m} P(T_i, T_j)\left(\frac{P(T_i, T_{j-1})}{P(T_i, T_j)}\hat{\delta}(T_{j-1}, T_j) - 1\right) \\ &= N^d \sum_{j=i+1}^{m-1} P(T_i, T_j)\left(\hat{\delta}(T_j, T_{j+1}) - 1\right) \\ &\quad + P(T_i, T_i)\,\hat{\delta}(T_i, T_{i+1})\,N^d - P(T_i, T_m)\,N^d \end{aligned}$$

Now that we have the expression above, we will define a *domestic rate*, which allows us to rewrite the floating leg. Define K_j^d as follows:

$$K_j^d = \begin{cases} \frac{\hat{\delta}(T_i, T_{i+1})}{\Delta t_i} & j = i \\ \frac{\hat{\delta}(T_j, T_{j+1}) - 1}{\Delta t_j} & j = i+1, \ldots, m-1 \end{cases}$$

As a result we have

$$V(T_i) = \underbrace{N^d \widetilde{K}_i^f \sum_{j=i+1}^{m} \Delta t_j P(T_i, T_j)\,\Phi_j(T_i)}_{\text{foreign leg}} - \underbrace{N^d \sum_{j=i}^{m-1} \Delta t_j P(T_i, T_j)\,K_j^d}_{\text{domestic leg}}$$

As a next step we look for a new *effective* domestic rate. This is the rate $\widetilde{K}_i^d$ that corresponds with common fixed rate payments at times $T_{i+1}, \ldots, T_m$, that leaves the value of the domestic leg at T_i unchanged. It will provide a convenient expression for the net present value of the contract. For the effective rate we obtain

$$\begin{aligned} \widetilde{K}_i^d N^d \sum_{j=i+1}^{m} \Delta t_j P(T_i, T_j) &= N^d \sum_{j=i}^{m-1} \Delta t_j P(T_i, T_j)\,K_j^d \\ \Rightarrow \quad \widetilde{K}_i^d &= \frac{\sum_{j=i}^{m-1} \Delta t_j P(T_i, T_j)\,K_j^d}{\sum_{j=i+1}^{m} \Delta t_j P(T_i, T_j)} \end{aligned}$$

In addition to the effective domestic rate, one can derive a *fair* domestic rate. This is the rate K_i^{eq} that corresponds with common fixed rate payments at times $T_{i+1}, \ldots, T_m$, that sets the contract value at time T_i to zero. The derivation of K_i^{eq} is similar to that in the single-curve framework. Given that the value of the foreign leg is unchanged since we moved to the dual-curve framework, also our expression for the fair domestic rate remains the same. Hence recall that

$$K_i^{eq}(T_i) = \widetilde{K}_i^f(T_i) \sum_{j=i+1}^{m} \omega_j(T_i)\,\Phi_j(T_i)$$

with weights given by

$$\omega_j(T_i) = \frac{P(T_i, T_j)\,\Delta t_j}{\sum_{k=i+1}^{m} P(T_i, T_k)\,\Delta t_k}$$

8.4 Adjusted CVA for a CCS

With the new notations at hand, we can again rewrite expression for the positive exposure at T_i in such a way, it resembles the value of a put option. To simplify the expression, we perform the same approximation as before. The weights are frozen at time zero, $\omega_j^f \equiv \omega_j^f(0)$ and $\omega_j \equiv \omega_j(0)$, which leads to deterministic quantities $\widetilde{K}_i^f$ and $\widetilde{K}_i^d$. For the positive exposure we obtain

$$(V(T_i))^+ = N^d \sum_{j=i+1}^{m} \Delta t_j P(T_i, T_j) \left(\widetilde{K}_i^d - K_i^{eq}(T_i) \right)^+$$

The expected value of the above expression is similar to that of a European option with strike K_i^d. It can be evaluated Black's formula for a put option. The process of $K_i^{eq}(t)$ is similar as in the single-curve framework, allowing us to use the same approximation of its volatility. Concluding, we can determine the EPE of a CCS adjusted to the dual-curve framework in a similar way as before, just by applying a minor correction to the exposure expression of the contract. The CVA can then be computed as follows:

$$CVA(t) \approx LGD \cdot \sum_{i=1}^{m} EPE(t, T_i) \cdot PD(T_{i-1}, T_i)$$

8.5 Dual-curve or single-curve?

Throughout this chapter we have seen that we can fine-tune the analytical approximation of a CVA for a CCS. This approach should provide a better match with alternative CVA computations obtained by Monte Carlo methods under a dual-curve framework. In general this approach should yield a more accurate approximation. In a similar way, we should be able to construct an adjustment for the CVA on an IRS.

We have numerically tested the adjustment against numerous Monte Carlo CVA computations to test its accuracy. Our observation was that the difference between the CVA approximation with and without a dual-curve adjustment was relatively small. By our judgement, the average adjustment is in the same order of magnitude as the approximation errors induced by the discretization of the EPE, freezing the weights of the fair CCS rate and freezing the effective CCS rates. For this reason we decided to apply the single-curve analysis for the remainder of our research. A selection of the numerical test-results can be found in the appendix, section B.

9 The hedging strategies

The aim of this research is to quantify the potential gains and risks of several alternative strategies applied to interest rate and FX hedges. Here we will describe in detail the strategies that we investigate and discuss their implementation.

9.1 IRS related strategies

Consider a corporate institution that we will refer to as *the client*. Additionally consider a financial institution that we will refer to as *the bank*. Our base assumption will be that the client is to enter a floating rate loan, which he intends to hedge. We assume the loan is characterized by the following properties:

$$\begin{aligned} N &\quad - \quad \text{The notional volume of the loan} \\ T_0 &\quad - \quad \text{Inception date of the loan} \\ T_m &\quad - \quad \text{Maturity date of the loan} \\ (T_i)_{1\leq i\leq m} &\quad - \quad \text{Payment dates of the accrued interest} \\ \Delta t_i &\quad - \quad \text{The } i^{th} \text{ accrual period } (T_i - T_{i-1}) \end{aligned}$$

The time today will be denoted by t. Assume that the coupon schedule contains equidistant time instants, such that Δt_i is close to constant for each $i \in \{1, \ldots, m\}$. At each coupon date T_i, the client is required to pay the financial institution the simply-compounded interest rate that accrued over the time interval Δt_i, which is equal to

$$N \cdot L\left(T_{i-1}, T_i\right) \cdot \Delta t_i$$

Since the interest rate is stochastic, the LIBOR $L\left(T_{i-1}, T_i\right)$ will not be known until T_{i-1}. This means that the corporate is exposed to interest rate risk: if rates go up, so do his interest rate costs and vice versa. A classic way to hedge this risk is by entering a floating-fix interest rate swap (IRS). Let the notional amount N of the swap match the notional volume of the loan, let the coupon schedule $\mathcal{T}$ of the swap match the interest schedule of the loan and let K denote the fixed swap rate settled at T_0. Then at each payment date $T_i \in \mathcal{T}$, the client pays $N \cdot L\left(T_{i-1}, T_i\right) \cdot \Delta t_i$ as interest for the loan and receives a floating IRS coupon of $N \cdot L\left(T_{i-1}, T_i\right) \cdot \Delta t_i$ from the bank. These payments hence cancel each other out. Simultaneously the client pays a fixed IRS coupon of $N \cdot K \cdot \Delta t_i$ to the bank. From the client's perspective by entering the IRS, the sequence of floating rate interest payments have been transformed to a sequence of fixed rate payments. The client is therefore no longer exposed to interest rate risk and we say that the loan is "fully hedged".

From a client's perspective there can be several reasons to prefer the combination *floating-rate loan and IRS* to a simple fixed-rate loan. The main advantage of a floating-rate loan is that it is by definition *at par*. By this we mean to say that it has no market value. This is not necessarily the case for a fixed rate loan. Such a loan has an intrinsic value that can be either favorable or unfavorable for the client, depending on the intermediate movement of the current rates. This market-value has an influence when the client for example wants to prematurely terminate or refinance the loan. In that case, banks usually charge repay costs if the rates moved in their favor, but typically are not willing to pay-out in the opposite situation. Also transferring the debt to another bank is complicated if the loan is not at par. These complications do not apply to a floating-rate loan in combination with an IRS, making it a popular alternative to the fixed-rate loan.

In the subsequent paragraphs we will describe two hedging strategies that form an alternative to the classic IRS hedge described above, namely:

- The rolling strategy
- The hedging ratio strategy

With these alternatives, the loan will only be partly hedged, meaning that the client will still be exposed to some risk. Details of these strategies and a combination of the two are provided below.

9.1.1 The rolling strategy

Instead of entering an IRS of which the tenor matches the tenor of the loan, the client enters an IRS that matches only a fraction of the loan's tenor. When this IRS matures, the client "rolls" into a new IRS, of which the tenor is equal to the previous one. Subsequently, the client repeatedly enters new IRS trades, until the underlying loan matures. This means that during its lifespan, the loan is only "partly hedged" in terms of the tenor. A degree of freedom to this strategy is the number of rolls n.

Let n be a positive integer such that $n \in \{1, \ldots, m\}$. Let k be an integer such that $k = \left\lceil \frac{m}{n+1} \right\rceil$. We define the n^{th} rolling strategy as follows:

1. At inception of the loan, T_0, enter a floating-fix IRS with notional amount N and coupon schedule $\mathcal{T}_0 = \{T_1, \ldots, T_k\}$. Let the trade be entered *at par*, meaning that the fixed rate K_0 is set such that the MtM of the swap initially equals zero.
2. While $j < n$, at each subsequent j^{th} *roll-date* $T_{j \cdot k}$, enter a new floating-fix IRS with notional amount N and coupon schedule $\mathcal{T}_j = \left\{T_{j \cdot k+1}, \ldots, T_{(j+1) \cdot k}\right\}$. Let the trade be entered *at par*.
3. At the last roll-date $T_{n \cdot k}$ enter a new floating-fix IRS with notional amount N and coupon schedule $\mathcal{T}_n = \{T_{n \cdot k+1}, \ldots, T_m\}$. Let the trade be entered *at par*.

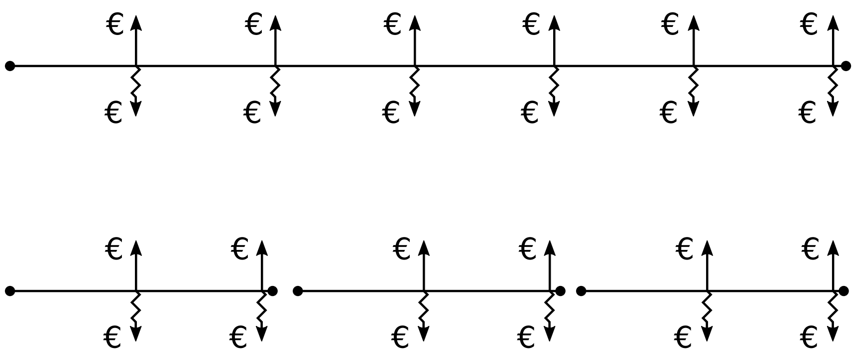

Figure 9.1: Graphic illustration of the cashflows under a regular IRS hedge (up) versus a rolling strategy with 2 rolls (down)

9.1.2 The hedging ratio strategy

Instead of entering an IRS of which the notional amount matches the notional volume of the loan, the client enters an IRS that matches only a fraction of the loan's notional. The start and maturity of the IRS will match the inception and maturity of the loan. This means that during its lifespan, the loan is only "partly hedged" in terms of the notional. A degree of freedom to this strategy is the hedging ratio q.

Let q be a real number such that $q \in (0, 1)$. We define the q-hedge ratio strategy as follows:

1. At inception of the loan, T_0, enter a floating-fix IRS with notional amount $q \cdot N$ and coupon schedule $\mathcal{T} = \{T_1, \ldots, T_k\}$. Let the trade be entered *at par*, meaning that the fixed rate K is set such that the MtM of the swap initially equals zero.

9.1.3 The combined rolling-hedge ratio strategy

Given the two strategies above, the client could as a third option consider a hybrid version of the two. This means that both the notional and the tenor are partly hedged. The definition of the n^{th} rolling q-hedge ratio strategy is similar to the definition on the n^{th}-rolling strategy, with as only difference that the notional amount of each IRS is chosen to be $q \cdot N$ instead of N.

9.2 Numerical method for IRS strategies

We will investigate the expected costs and risks associated to the hedging strategies described above through Monte Carlo simulation. Market simulations will be performed using the one-factor Hull-White model as described in section 4. The model allows us to generate realizations of the costs that the client potentially faces when entering either of the described strategies. Per scenario we compute from the clients perspective the costs that are associated with a strategy, which are composed of:

- The netted payable cashflows, related to the swap and loan
- The CVA charge related to the swap

For the simulation we consider an Euler discretization scheme that at least contains the coupon dates $\{T_0, \ldots, T_m\}$ of the hypothetical underlying loan. Assume that we investigate a strategy that has n rolls and a hedge ratio q. Let $n \in \{0, 1, \ldots, m\}$, where $n = 0$ if we consider a strategy without rolling. Let $q \in (0, 1]$, where $q = 1$ if we consider a strategy without hedge ratio. If $n > 0$, denote by $\{T_{i_1}, \ldots, T_{i_n}\} \subseteq \mathcal{T}$ the roll-dates corresponding to the strategy. Denote by K_j the fixed IRS rate that is settled at the corresponding roll-date T_{i_j}. A cost profile is computed as follows:

1. Sample short-rate scenarios using the one-factor Hull-White model.
2. Per scenario compute the netted cashflows $C(t)$ performed by the client at each coupon date $T_i \in \mathcal{T}$. For $T_{i_j} < T_k \leq T_{i_{j+1}}$, these are given by
$$C(T_k) = q \cdot N \cdot K_j \cdot \Delta t_k + (1 - q) \cdot N \cdot L(T_{k-1}, T_k) \cdot \Delta t_k$$
3. Per scenario compute the CVA charge at T_0 and (if relevant) each subsequent roll-date T_{i_j}. These charges are given by
$$CVA\left(T_{i_j}\right) = LGD \cdot \sum_{k=i_j+1}^{i_{j+1}} EPE\left(T_{i_j}, T_k\right) \cdot PD\left(T_{k-1}, T_k\right)$$
where $EPE(S, T)$ is analytically approximated as described in section 6.1.

The computations result in a profile of risk-neutral distributions that reflect the funding costs and CVA costs associated with the given strategy.

9.2.1 The reset rate $\mathbf{K_j}$

At inception date T_0 and, if $n > 0$, at the subsequent roll-dates, the fixed rate paid by the client resets to a new value. This rate K_j is settled such that the MtM of the IRS is set to zero at inception of the trade. By its definition, this means that K_j corresponds to the *swap rate* that prevails at T_{i_j}. We have seen the expression for the swap rate. Per scenario this means that K_j is given by

$$K_j = \frac{P\left(T_{i_j}, T_{i_j}\right) - P\left(T_{i_j}, T_{i_{j+1}}\right)}{\sum_{k=i_j+1}^{i_{j+1}} P\left(T_{i_j}, T_k\right) \Delta t_k}$$

9.3 CCS related strategies

Consider once again a client and a bank. Additionally consider a market with two different currencies, which we will denote by *domestic* (d) and *foreign* (f). Our base assumption will again be that the client has scheduled to enter a floating rate loan in the domestic currency. Additionally we will assume that the client is in need of an amount N^f in foreign currency and intends to swap the domestic cash for the foreign cash through a cross-currency swap with the bank. Let the domestic loan be characterized by the following properties:

$$\begin{aligned} N^d \quad &- \quad \text{The notional volume of the loan} \\ T_0 \quad &- \quad \text{Inception date of the loan} \\ T_m \quad &- \quad \text{Maturity date of the loan} \\ (T_i)_{1 \le i \le m} \quad &- \quad \text{Payment dates for the interest} \\ \Delta t_i \quad &- \quad \text{The } i^{th} \text{ accrual period } (T_i - T_{i-1}) \end{aligned}$$

The time today is denoted by t and let the coupon schedule be equidistant. A classic way of attracting funds in a foreign currency is by entering a floating-fix cross-currency swap (CCS) with a notional exchange at the start and maturity of the trade. Let the domestic notional amount of the swap N^d match the notional volume of the loan and let the coupon schedule $\mathcal{T}$ of the swap match the interest schedule of the loan. Choose the foreign notional of the swap such that $N^f = {}^{N^d}/_{\varphi(T_0)}$ and let K denote the fixed rate corresponding to the fixed leg of the swap. At inception of the trade, the client receives N^d from his loan. Simultaneously he pays N^d to the bank and receives in return N^f due to the notional exchange of the CCS. At each coupon date T_i, the client is required to pay the simply-compounded interest rate that accrued over the time interval Δt_i for his loan, which will be equal to

$$N^d \cdot L\left(T_{i-1}, T_i\right) \cdot \Delta t_i$$

He receives the same amount from the bank as a floating coupon payment from the CCS. These payments hence cancel out. Simultaneously the client pays a fixed coupon of $N^f \cdot K \cdot \Delta t_i$ to the bank in the foreign currency. Finally, at the end of the trade, the client pays back N^f to the bank and receives N^d due to the notional exchange at maturity. The client uses this N^d to repay his loan, which matured simultaneously. From the client's perspective, a domestic floating rate loan is transformed to a foreign fixed rate loan by entering the CCS. Moreover, the client is not exposed to interest rate risk and we say that the loan is "fully hedged".

A typical motivation for a corporate to enter a domestic loan in combination with a CCS is the requirement of funds in a foreign currency. Often corporates have limited access to foreign money-markets. If they were to directly take a foreign loan, they are likely confronted with unfavorable terms. The foreign banks will likely charge high credit spreads since they are unfamiliar with the corporation and fear the risk that the corporate defaults on interest payments or the repayment of the principal due to the moving FX rate. A domestic loan can typically be obtained under much better terms. The corporate likely has a sustainable relationship with a domestic bank and since its earnings are in the domestic currency, he is not exposed to FX risk. By entering a domestic loan in combination with a CCS, the corporate is hence able to obtain cash in a foreign currency, but still under favorable terms.

In the subsequent paragraphs we will describe two hedging strategies that form an alternative to the classic setting described above:

- The rolling strategy with fixed notional
- The rolling strategy with reset notional

For these alternatives, the loan will only be partly hedged, meaning that the client will still be exposed to some risk. Below we will provide details of both strategies

9.3.1 The rolling strategy with fixed notional

Instead of entering a CCS of which the tenor matches the tenor of the loan, the client enters a CCS that matches only a fraction of the loan's tenor. When this CCS matures, the client "rolls" into a new CCS, of which the tenor is equal to the previous one. Subsequently, the client repeatedly enters new CCS trades, until the underlying loan matures. The client chooses to let both the domestic and foreign underlying CCS notionals remain fixed for each subsequent trade. The notional exchanges that take place at the rolling dates will therefore cancel out. As a consequence the initial notional exchange of each new CCS might not be at par. The fixed rate the client needs to pay to set the MtM of the CCS to zero may therefore be higher or lower than usual. This effect is known as *blending the MtM*. Furthermore we should note that the loan is only "partly hedged" in terms of the tenor. A degree of freedom to this strategy is the number of rolls n.

Let n be a positive integer such that $n \in \{1, \ldots, m\}$. Let k be an integer such that $k = \left\lceil \frac{m}{n+1} \right\rceil$. We define the n^{th} rolling strategy with fixed notional as follows:

1. At inception of the loan, T_0, enter a floating-fix CCS with domestic notional N^d, foreign notional $N^f = {}^{N^d}/_{\varphi(T_0)}$ and coupon schedule $\mathcal{T}_0 = \{T_1, \ldots, T_k\}$. Let the trade be entered *at par*, meaning that the fixed rate K_0 is settled such that the MtM of the swap is zero at inception of the trade.

2. While $j < n$, at each subsequent j^{th} *roll-date* $T_{j \cdot k}$, enter a new floating-fix CCS with domestic notional N^d, foreign notional N^f and coupon schedule $\mathcal{T}_j = \left\{T_{j \cdot k+1}, \ldots, T_{(j+1) \cdot k}\right\}$. Let the trade be entered *at par*, meaning that the fixed rate K_j is settled such that the MtM of the swap is zero at inception of the trade.

3. At the last roll-date $T_{n \cdot k}$ enter a new floating-fix CCS with domestic notional N^d, foreign notional N^f and coupon schedule $\mathcal{T}_n = \{T_{n \cdot k+1}, \ldots, T_m\}$. Let the trade be entered *at par*, meaning that the fixed rate K_n is settled such that the MtM of the swap is zero at inception of the trade.

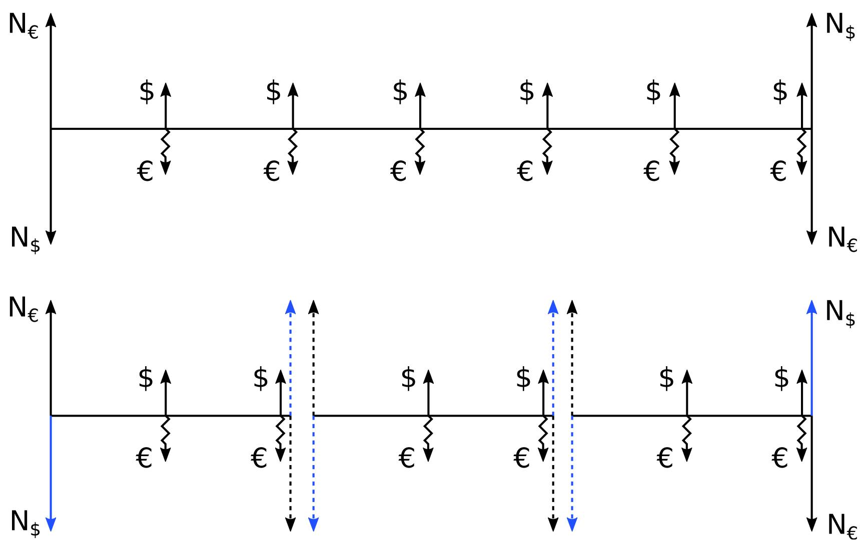

Figure 9.2: Graphic illustration of the cashflows under a regular CCS hedge (up) versus a fixed-notional rolling strategy with 2 rolls (down)

9.3.2 The rolling strategy with reset notional

This strategy is almost similar to the previous one. Key difference is that each new CCS is settled with a notional exchange that is once again at par at inception. This is in contrast to the previous strategy where the notional amounts were fixed, even if the spot exchange rate had moved. The client enters a CCS that matches only a fraction of the loan's tenor. When this CCS matures, the client "rolls" into a new CCS, of which the tenor is equal to the previous one. Subsequently, the client repeatedly enters new CCS trades, until the underlying loan matures. The client will again keep the domestic notional fixed, as he will require this amount to repay the domestic loan at maturity. The foreign notional will this time however be reset according to the prevailing spot FX rate at inception of the new trade. The notional exchanges that are scheduled at the rolling dates will not cancel out, implying that a transaction would normally take place to unwind the notionals. However, instead of unwinding, the fixed rate of the new CCS is set such that it matches the remaining MtM of the previous CCS without performing the notional exchange at maturity. Each final notional exchange is, so to speak, absorbed by the fixed leg of the subsequent CCS. More formally: the remaining MtM of the previous swap is blended into the new one. This will therefore strongly affect the fixed rate the client needs to pay. The reason that a client will likely prefer blending over unwinding is that a corporate has typically limited cash available. Big cash-outs due to notional resets are mostly undesirable. They induce a liquidity risk and holding large amounts of cash is moreover quite expensive.

Lastly, we should note that the loan is only "partly hedged" in terms of the tenor. A degree of freedom to this strategy is the number of rolls n. Let n be a positive integer such that $n \in \{1, \dots, m\}$. Let k be an integer such that $k = \left\lceil \frac{m}{n+1} \right\rceil$. We define the n^{th} rolling strategy with notional reset as follows:

1. At inception of the loan, T_0, enter a floating-fix CCS with domestic notional N^d, foreign notional $N_0^f = {}^{N^d}/_{\varphi(T_0)}$ and coupon schedule $\mathcal{T}_0 = \{T_1, \dots, T_k\}$. Let the trade be entered *at par*, meaning that the fixed rate K_0 is settled such that the MtM of the swap is zero at inception of the trade.

2. At maturity of the CCS, do *not* execute the final notional exchange, where N^d would be paid to the client and N_0^f would be paid to the bank.

3. While $j < n$, at each subsequent j^{th} *roll-date* $T_{j \cdot k}$, enter a new floating-fix CCS with domestic notional N^d, foreign notional $N_j^f = {}^{N^d}/_{\varphi(T_{j\cdot k})}$ and coupon schedule $\mathcal{T}_j = \{T_{j\cdot k+1}, \dots, T_{(j+1)\cdot k}\}$. Let the MtM of the trade match the value of the unexecuted notional exchange of the previous trade. This means that the fixed rate K_j is settled such that the MtM of the swap minus the remaining MtM of the previous swap equals zero at inception of the trade.

4. While $j < n$, at maturity of each CCS, do *not* execute the final notional exchange, where N^d would be paid to the client and N_j^f would be paid to the bank.

5. At the last roll-date $T_{n\cdot k}$ enter a new floating-fix CCS with domestic notional N^d, foreign notional N^f and coupon schedule $\mathcal{T}_n = \{T_{n\cdot k+1}, \dots, T_m\}$. Let the MtM of the trade match the value of the unexecuted notional exchange of the previous trade.

6. At maturity T_m, *do* execute the final notional exchange , where N^d is paid to the client and N_n^f is paid to the bank.

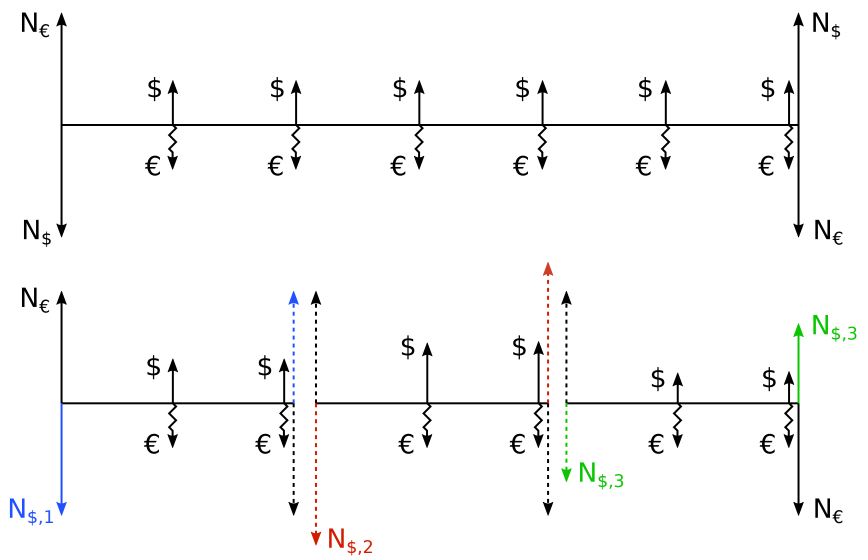

Figure 9.3: Graphic illustration of the cashflows under a regular CCS hedge (up) versus a rolling strategy with 2 rolls (down)

9.4 Numerical method for CCS strategies

Similar to the IRS cases we will investigate the hedging strategies described above through Monte Carlo simulation, this time using a multi-currency Hull-White and FX model. Per scenario we will compute a cost-profile consisting of:

- The netted payable cashflows, related to the CCS and the loan
- The CVA charge related to the CCS

Consider an Euler discretization scheme that at least contains the coupon dates $\{T_0, \ldots, T_m\}$ of the hypothetical underlying loan. Assume that we investigate a strategy that has n rolls. Let $n \in \{0, 1, \ldots, m\}$, where $n = 0$ if we consider a strategy without rolling. If $n > 0$, denote by $\{T_{i_1}, \ldots, T_{i_n}\} \subseteq \mathcal{T}$ the roll-dates corresponding to the strategy. Denote by K_j the fixed CCS rate that is settled at the corresponding roll-date T_{i_j}. Denote by N_j^f the foreign notional amount that is settled at T_{i_j}. In case of the fixed notional strategy, this means that $N_0^f = \ldots = N_n^f = {}^{N^d}/_{\varphi(T_0)}$. In case of the reset notional strategy, $N_j^f = {}^{N^d}/_{\varphi(T_{i_j})}$. A cost profile is computed as follows:

1. Sample domestic short-rate, foreign short-rate and FX spot rate scenarios using the multi-currency Hull-White and FX model.
2. Per scenario compute the netted cashflows $C(t)$ performed by the client at each coupon date $T_i \in \mathcal{T}$. For $T_{i_j} < T_k \leq T_{i_{j+1}}$, in the foreign currency, these are given by
$$C(T_k) = K_j \cdot N_j^f \cdot \Delta t_k$$
3. Per scenario compute the CVA charge at T_0 and (if relevant) each subsequent roll-date T_{i_j}. These charges are given by
$$CVA\left(T_{i_j}\right) = LGD \cdot \sum_{k=i_j+1}^{i_{j+1}} EPE\left(T_{i_j}, T_k\right) \cdot PD\left(T_{k-1}, T_k\right)$$
where $EPE(S,T)$ is analytically approximated as described in section 7.1.
4. Per scenario, compute the net value of the final notional exchange $V(T_m)$ at maturity. In the domestic currency this is given by
$$V(T_m) = \varphi(T_m) N_j^f - N^d$$

The computations result in a profile of risk-neutral distributions that reflect the funding costs and CVA costs associated with the given strategy.

9.4.1 The reset rate $\mathbf{K_j}$ for the fixed notional strategy

Assume $n > 0$, then at inception date T_0 and each subsequent roll-date, the fixed rate paid by the client resets to a new value. This rate K_j is settled such that the MtM of the CCS is set to zero at inception of the trade. We have seen that the MtM at T_{i_j} of a regular CCS starting at T_{i_j} and maturing at $T_{i_{j+1}}$ is given by

$$\begin{aligned} V\left(T_{i_j}\right) &= \sum_{k=i_j+1}^{i_{j+1}} \Delta t_k P\left(T_{i_j}, T_k\right)\left(N^f K_j \Phi\left(T_{i_j}, T_k\right) - N^d F\left(T_{i_j}, T_{k-1}, T_k\right)\right) \\ &\quad - N^f \Phi\left(T_{i_j}, T_{i_j}\right) + N^d \\ &\quad - N^d P\left(T_{i_j}, T_{i_{j+1}}\right) + N^f P\left(T_{i_j}, T_{i_{j+1}}\right) \Phi\left(T_{i_j}, T_{i_{j+1}}\right) \end{aligned}$$

Usually the term $N^f \Phi(T_i, T_i) - N^d$ reduces to zero since the foreign notional is chosen to be $N^f = {}^{N^d}/_{\varphi(T_i)} = {}^{N^d}/_{\Phi(T_i, T_i)}$. Here this is not the case as the foreign notional is kept fixed at ${}^{N^d}/_{\varphi(T_0)}$. By its definition, we obtain an expression for K_j by solving $V\left(T_{i_j}\right)=0$. We hence find

$$\begin{aligned} 0 &= \sum_{k=i_j+1}^{i_{j+1}} \Delta t_k P\left(T_{i_j},T_k\right)\left(N^f K_j \Phi\left(T_{i_j},T_k\right)-N^d F\left(T_{i_j},T_{k-1},T_k\right)\right) \\ &\quad -N^f \Phi\left(T_{i_j},T_{i_j}\right)+N^d \\ &\quad -N^d P\left(T_{i_j},T_{i_{j+1}}\right)+N^f P\left(T_{i_j},T_{i_{j+1}}\right)\Phi\left(T_{i_j},T_{i_{j+1}}\right) \\ \Rightarrow \quad K_j &= \frac{N^d \sum_{k=i_j+1}^{i_{j+1}} \Delta t_k P\left(T_{i_j},T_k\right) F\left(T_{i_j},T_{k-1},T_k\right)}{N^f \sum_{k=i_j+1}^{i_{j+1}} \Delta t_k P\left(T_{i_j},T_k\right)\Phi\left(T_{i_j},T_k\right)}+ \\ &\quad \frac{N^f\left(\varphi\left(T_{i_j}\right)-P\left(T_{i_j},T_{i_{j+1}}\right)\Phi\left(T_{i_j},T_{i_{j+1}}\right)\right)+N^d\left(P\left(T_{i_j},T_{i_{j+1}}\right)-1\right)}{N^f \sum_{k=i_j+1}^{i_{j+1}} \Delta t_k P\left(T_{i_j},T_k\right)\Phi\left(T_{i_j},T_k\right)} \end{aligned}$$

9.4.2 The reset rate $\mathbf{K_j}$ for the reset notional strategy

Also here, if $n>0$, at inception date T_0 and each subsequent roll-date, the fixed rate paid by the client resets to a new value. K_j is settled such that the MtM of the CCS is matches the remaining value of the un-executed notional exchange of the previous swap. This time, in the expression for the MtM of a CCS, the term $N^f\Phi\left(T_i,T_i\right)-N^d$ does reduce to zero since the foreign notional is constantly reset, so that $N_j^f={}^{N^d}\!/\!_{\varphi\left(T_{i_j}\right)}$. By its definition, we obtain an expression for K_j by solving $V\left(T_{i_j}\right)=N^d-N_{j-1}^f\varphi\left(T_{i_j}\right)$. We hence find

$$\begin{aligned} N_{j-1}^f\varphi\left(T_{i_j}\right)-N^d &= \sum_{k=i_j+1}^{i_{j+1}} \Delta t_k P\left(T_{i_j},T_k\right)\left(N_j^f K_j \Phi\left(T_{i_j},T_k\right)-N^d F\left(T_{i_j},T_{k-1},T_k\right)\right) \\ &\quad -N^d P\left(T_{i_j},T_{i_{j+1}}\right)+N_j^f P\left(T_{i_j},T_{i_{j+1}}\right)\Phi\left(T_{i_j},T_{i_{j+1}}\right) \\ \Rightarrow \quad K_j &= \frac{N_{j-1}^f\varphi\left(T_{i_j}\right)+N^d \sum_{k=i_j+1}^{i_{j+1}} \Delta t_k P\left(T_{i_j},T_k\right) F\left(T_{i_j},T_{k-1},T_k\right)}{N_j^f \sum_{k=i_j+1}^{i_{j+1}} \Delta t_k P\left(T_{i_j},T_k\right)\Phi\left(T_{i_j},T_k\right)} \\ &\quad +\frac{-N_j^f P\left(T_{i_j},T_{i_{j+1}}\right)\Phi\left(T_{i_j},T_{i_{j+1}}\right)+N^d\left(P\left(T_{i_j},T_{i_{j+1}}\right)-1\right)}{N_j^f \sum_{k=i_j+1}^{i_{j+1}} \Delta t_k P\left(T_{i_j},T_k\right)\Phi\left(T_{i_j},T_k\right)} \end{aligned}$$

9.5 Assumptions on the hazard rate term-structure

Consider a rolling hedging strategy, either IRS or CCS related, for which the number of rolls $n>0$. Let T_{i_j} denote a roll-date, so that $T_{i_j}>T_0$. There is one relevant question, concerning CVA modeling, that we have not discussed so far: *What does the term-structure of hazard rates look like at a future time-instant T_{i_j}?* A client that follows the rolling strategy enters a new trade at T_{i_j}. At this point in time, the new CVA charge will be computed by the bank. However, note that the bank will perform his CVA calculation, *conditioned on the fact that the counterparty has not defaulted until T_{i_j}*. This is straight-forward, since if the counterparty would have defaulted, it would not be able to enter the new trade in the first place.

So how do we calculate the probabilities of default used in the CVA computation, after a rolling date T_{i_j}? A natural choice would be to calculate the survival probabilities S in line with the known term-structure of hazard rates at $t=0$ and condition on the event $\tau>T_{i_j}$. It would follow that

$$S\left(t,T_{i_j}+\Delta t\right)\equiv S\left(T_{i_j},T_{i_j}+\Delta t\right)=\mathbb{Q}\left(\tau>T_{i_j}+\Delta t \middle|\, \tau>T_{i_j}\right)=\exp\left\{-\int_{T_{i_j}}^{T_{i_j}+\Delta t}\lambda(s)ds\right\}$$

For the computation of risk-neutral credit spreads, which are charged today, this is a perfect approach. We are however interested in the CVA spreads, given that they are charged in the future. Recall that the parameter λ used to model probabilities of default is implied from CDS quotes. Typically, these hazard rates are low in the near future, but tend to increase with time. This is due to the fact that a corporation which is financially healthy today, will not likely default very soon, whereas in the future this becomes more uncertain. Given that a client has not defaulted until a future roll-date, one might expect that the hazard rates at that point in time are once again low at first and increasing with time. For an impact analysis of a rolling strategy, we would ideally know today what the hazard rates will be at a future time-instant. Clearly the future CDS quotes to which the hazard rates are calibrated are not known today. Neither do we model future realizations of the CDS quotes as this falls outside the scope of this research. Hence if we aim to perform a realistic analysis, we are required to adopt an assumption on the development of the hazard rates through time. Below we will discuss three of such potential assumptions. For each we will illustrate its implications to an exemplary term-structure as seen in figure 5.1 for a strategy with two rolls. It should be noted that the second and third assumption imply a deviation of the risk-neutral framework, for the sake of a more realistic impact analysis.

1. *Following the curve:* Assume that the hazard rates exactly follow the curve of the term-structure, for example as in figure 9.4. Advantage of this assumption is that the probabilities of default are fully in line with the risk-neutral framework, as all the hazard rates are market-implied. Basically, the only assumption we make here is that the hazard rates will in expectation perfectly follow the term-structure that is observed today. In fact that is the same assumption we make for interest- and FX rates, as this is a direct consequence of the no-arbitrage principles in combination with the calibration routines of our models. For an impact analysis of potential future CVA charges, this might however not be the best starting point. Market practice shows that hazard-rates are typically low at first and higher later on. By fully ignoring this effect we could unnecessarily over-estimate the CVA charges.

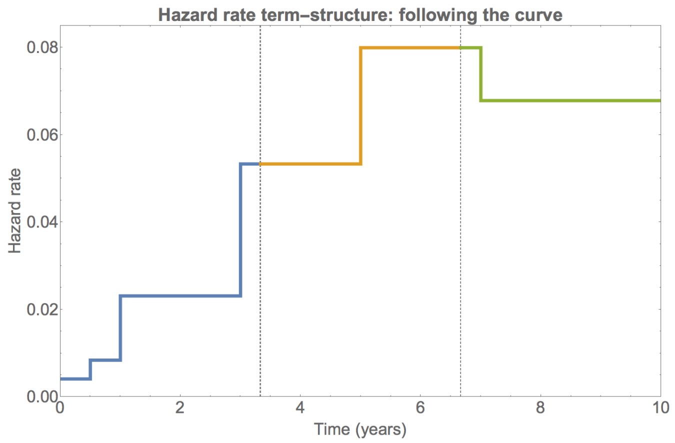

Figure 9.4: Hazard rate term-structure during a 10 year hedge for a 2 rolls-strategy under the assumption that the hazard rates *follow the curve*.

2. *Time-homogeneous:* Assume that at a roll-date in the future, the term-structure of hazard rates remained exactly the same and shifted forward to that roll-date. See for example figure 9.5. This assumption would be realistic given that the credit-worthiness of the counterparty remains perfectly unchanged through time. Clearly this is an assumption that has its limitations as we can not tell if the credit-worthiness of the counterparty is going to change or not. Should the credit-worthiness indeed deteriorate, then the projections on the CVA charges might be too optimistic. Nevertheless, since we do not model future realizations of the hazard rates, the term-structure observed today is maybe our best guess for the hazard rates in the future.

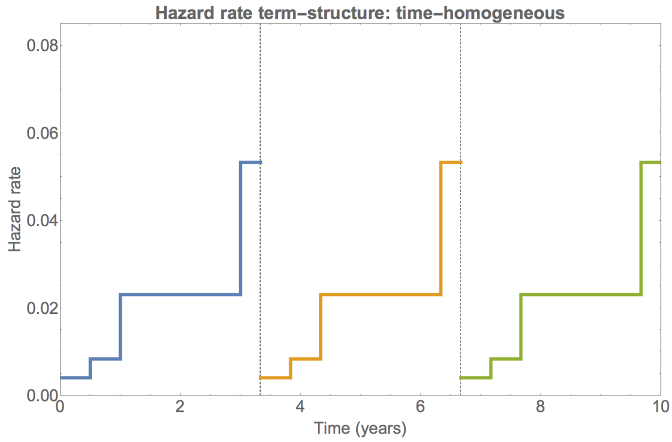

Figure 9.5: Hazard rate term-structure during a 10 year hedge for a 2 rolls-strategy under the assumption that the hazard rates are *time-homogeneous*.

3. *Flat:* Our last suggested assumption is that of a constant flat term-structure of hazard rates, like for example in figure 9.6. This will by definition not be a realistic assumption, but allows us to rule out any effect of the hazard rate on the simulation results. This can be a good thing if we want to isolate the impact of other risk-factors on the CVA charge, such as a shorter time-span of the trade or a notional reset. It additionally rules out any positive impact caused by an assumption on the hazard rate that in reality might be too optimistic. A downside to this assumption is that *any* positive effect induced by a rolling strategy is ruled out, even though it is reasonable to expect one.

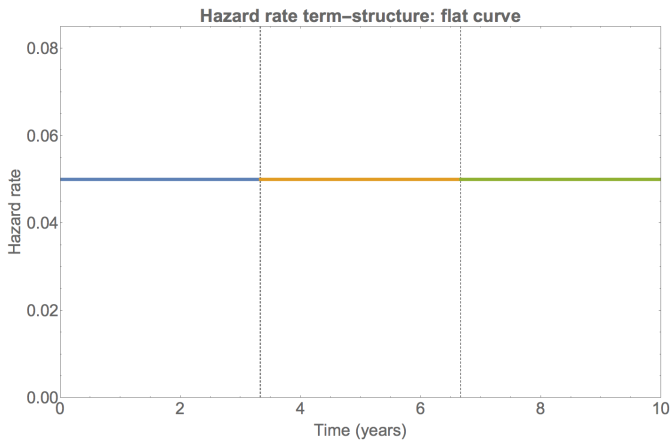

Figure 9.6: Hazard rate term-structure during a 10 year hedge for a 2 rolls-strategy under the assumption that the hazard rates are *flat*.

10 Results for the IRS strategies

We investigate the impact of the *rolling strategy* and *hedging ratio* strategy in terms of funding cost and CVA charges in comparison to a regular IRS hedge. In this section we will discuss cost-profiles that have been obtained by market simulations.

For our analysis we consider a client that is about to enter a 10 year floating-rate loan in Euros. We let the coupon schedule consist of monthly LIBOR payments. The market simulations are performed using an implementation of the one-factor Hull-White model as described in chapter 4. For the loan we consider the following specifications:

Notional	**Currency**	**Interest frequency**	**Start-date**	**End-date**	**Valuation date**
100 M	EUR	Monthly	11-Dec-2017	11-Dec-2027	08-Dec-2017

On top of the loan we will consider several variations to the IRS based hedging strategies as described in the previous section. The following strategies have been investigated:

100% IRS hedge with:
- no roll
- 1 roll
- 2 rolls
- 3 rolls
- 4 rolls

50% IRS hedge with:
- no roll
- 1 roll
- 2 rolls
- 3 rolls
- 4 rolls

10.1 Expected funding costs

Through market simulations we composed realizations for each strategy of the cashflows that a client would have paid in a no-arbitrage, risk-neutral setting. We will start with considering the cost-profiles associated by the combination of a loan and a rolling scheme of IRS trades. For a 100% hedge ratio, the notional amount of the IRS matches the notional volume of the loan. Therefore, only the fixed legs of the interest rate swaps effectively contribute to the cost-profile. The floating legs cancel out against the interest coupons of the loan. For the 50% hedge ratio, the cost-profile is a superposition of 50% floating coupons and 50% fixed coupons. As a point of reference, we first show in figure 10.1 the 1M forward rate curve, which correspond to the risk-neutral, floating payments the client is expected to make if he did not hedge the loan at all. In other words, it shows the term-structure $T \mapsto \mathbb{E}^Q\left(L\left(T, T+1M\right)\middle|\,\mathcal{F}_0\right)$.

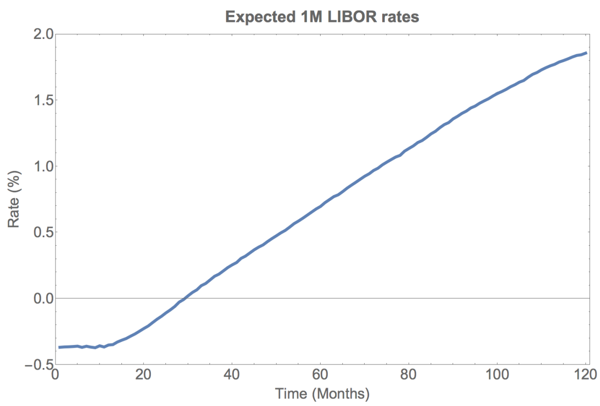

Figure 10.1: Expected risk-neutral floating-rates for the coming 10 years

Now, what happens if we apply the mentioned hedging strategies? In figure 10.2 we show the expected funding costs related to the 10 different rolling strategies. Since the underlying loan has a tenor of 10 years with monthly interest payments, there will in total be 120 coupon dates. The graph on the left illustrates the expected effective rates paid under a 100% hedge ratio for 0, 1, 2, 3 and 4 rolls, for each of the 120 coupons. With "effective rates" we mean the annualized costs as a percentage of the notional. Say that K_j denotes the effective rate paid at the j^{th} coupon, then the absolute costs at T_j will be $N \cdot K_j \cdot \Delta t_j$. In the 100% case the costs fully depend on the prevailing swap rate at time of a roll and are therefore constant between two consecutive rolling dates. On the right we see the expected effective rates for a 50% hedge ratio. Here the rates between two roll dates are not constant due to the superposition with the floating LIBOR.

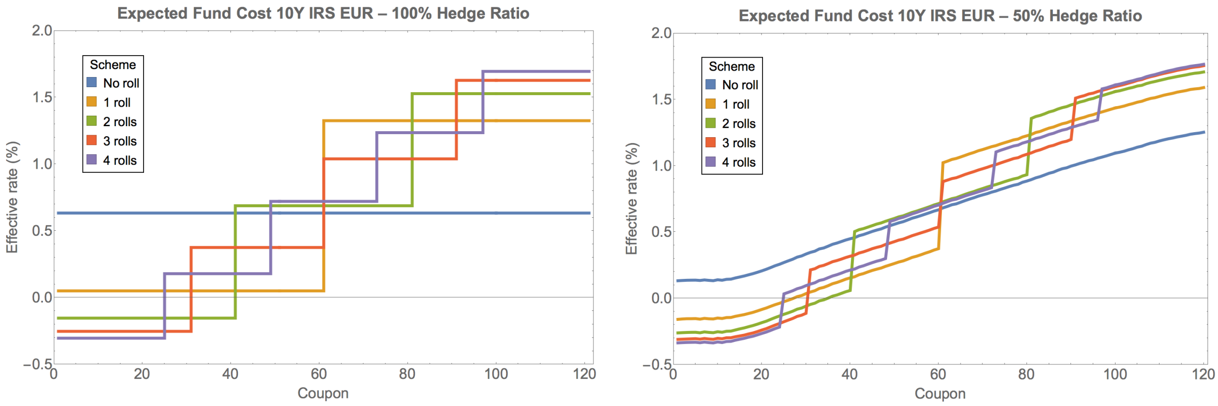

Figure 10.2: Expected effective rates for a 10Y hedge with (a) 0-4 rolling strategies with 100% hedge ratio and (b) 0-4 rolling strategies with 50% hedge ratio

An important first observation to make is that in a risk-neutral framework *no strategy has a preference in terms of expected funding costs.* By this we mean to say that the sum of the expected cashflows discounted to today is independent of the chosen strategy; fully hedged or partly hedged. This is a direct consequence of the no-arbitrage principles.

The main effect of adding a roll under a 100% hedge ratio is that the client is expected to pay a low rate at first and a higher rate after each consecutive roll. This is due to the forward rate curve shown in figure 10.1. In the current market the forward curve is increasing. This implies that if the client enters a shorter IRS now, it will face lower rates, but they are expected to be higher once he enters a new IRS trade afterwards. Note that increasing the number of rolls, causes the cost-profile to step-wise converge to the floating rate profile.

We observe a similar behavior for the 50% hedge-ratio strategy. By adding a roll to the strategy, the client can lock a lower rate at first, but is expected to face increased rates after each consecutive roll. For each roll-scheme, the effective rates are expected to continuously increase, due to the increasing forward rate curve, which in this case for 50% contributes to the cost-profile. Note here that also by decreasing the hedge ratio, the cost profile converges to the floating rate profile.

A natural question to pose, based on the results above, is the following: what would be a motivation for the client to enter a rolling strategy? If the application of a rolling scheme does not improve the cost-profile in expectation, one could argue that a classic IRS hedge, where the tenor of the IRS matches the tenor of the underlying loan would always be preferred. Under such a hedge the effective funding costs for the client are fully locked from the start and certain for the full tenor, which is a desirable situation for the treasurer. We can however distinguish two motivations to prefer a rolling strategy over a static IRS hedge.

1. *The expected CVA charge will be lower.* The impact of each strategy on the expected CVA spreads will be discussed in section 10.3.

2. *The expected payable risk premium will be lower.* We know that under the risk-neutral measure $\mathbb{Q}$, the expected realizations of the LIBOR perfectly match with the term-structure of the forward rates. In other words, for all time instants $t < S < T$ we find that:

$$\mathbb{E}^{Q}\left(D\left(t,T\right)F\left(t,S,T\right)\middle|\mathcal{F}_t\right)=\mathbb{E}^{Q}\left(D\left(t,T\right)L\left(S,T\right)\middle|\mathcal{F}_t\right)$$

In theory, it should therefore in expectation make no difference for an investor to settle on a fixed rate today through a prototypical FRA (forward rate agreement, see section 3.3) or wait and pay the realized floating LIBOR. This is a direct consequence of the no-arbitrage principles and the fact that the Hull-White model is perfectly calibrated to the currently observed yield curve. However, an important remark to make here is that this behavior does not fully reflect reality. Most investors know from market experience that the current forward rates typically do not materialize and in fact over-estimate the future LIBOR. This is due to the fact that a forward rate incorporates a so called *risk premium*. Generally speaking, a majority of the investors will be risk-averse. This means that if an investor can choose between two investment strategies, which in expectation yield the same profit, he will choose the least risky one. In the context of a swap, the party that pays the floating leg accepts to take a risk. Therefore, by the laws of supply and demand, an agent will require compensation before accepting such a risk: the risk premium. This is why the forward rates (which are calibrated to swaps) are in practice often slightly higher than the eventually realized LIBOR. As the uncertainty of the future LIBOR increases the further we look ahead in time, so does the risk associated with a swap and the corresponding risk premium added to the swap-rate. A client that intends to hedge his loan knows this and might therefore suspect that the prevailing swap-rate for a 10 year IRS is relatively high due to this risk-premium. If he instead enters a rolling strategy, he settles for a shorter IRS today and pays a lower rate. According to the risk-neutral expectation, this low rate will be set-off by a higher rate after rolling. However, the client has reason to believe that this higher rate will in fact not materialize as the forward rates today are likely to overestimate what the rates will turn out to be at the future roll-date. For this reason a client could prefer a rolling-strategy over a static hedge.

10.2 Distributions of the funding costs

The motivations mentioned in the previous paragraph might seem to indicate that an investor should in fact roll as often as possible or even not hedge at all. In expectation, this is indeed true, but not hedging or choosing a strategy that hedges only partly, obviously comes with a drawback. By either applying a rolling strategy or hedge-ratio strategy, the investor accepts a risk. A risk that could work in his favor, but might just as well work against him. Not hedging at all induces the highest degree of risk, due to the stochastic nature of the LIBOR. As a point of reference we illustrate the uncertainty of the future LIBOR in figure 10.3. It shows risk-neutral certainty intervals of what the 1M LIBOR will be be in the future on a monthly basis. In 10 years there is hence a 95% likelihood that the LIBOR will roughly be between -2% and +5.5%.

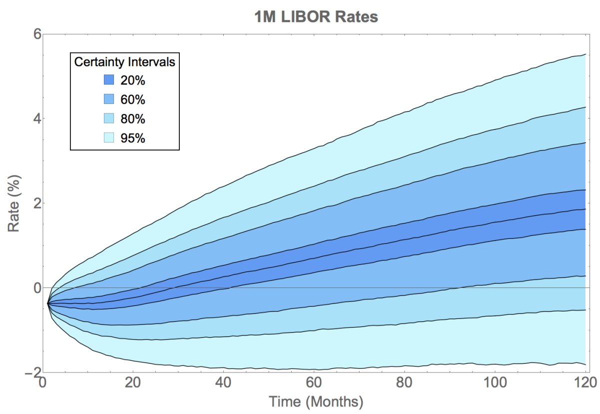

Figure 10.3: Risk-neutral distribution profile of floating-rate coupons on a 10Y loan

In figure 10.4 we see risk-neutral distributions of the effective rates that will be paid by the client for each corresponding strategy. Again on the left we find the strategies with a 100% hedge ratio and on the right the strategies with a 50% hedge ratio. Note that the 100% ratio strategy with no rolls induces zero risk, as this corresponds to a classic full IRS hedge. Each additional roll introduces additional roll-over risk. Clearly this is a consequence of the fact that the LIBOR is more uncertain on dates further away in the future. Similarly does a decrease in hedge ratio introduce additional risk due to the volatility of the LIBOR.

The graphs have been composed by Monte Carlo simulations of the short rate, based on 20,000 paths. Each figure should be interpreted as a sequence of 120 confidence intervals, that indicate the likelihood of what the effective rate for the client will be at each coupon date. These intervals have been obtained by sorting the simulated effective rates separately on each coupon date from small to large. Subsequently, for each coupon date, the 2.5, 10, 20, 40, 50, 60, 80, 90 and 97.5 percentiles are computed, from which the certainty intervals are composed. Although the procedure is simple, the graphs in figure 10.4 can be misleading. Say that for one scenario, the effective rate is in the top 2.5% range of the distribution after one roll, it does not necessarily mean that it will still be after a second roll. The intervals should hence be treated as 120 separate distributions. Although on the other hand it should be said that for the interest rate swap, the consecutive rates are highly correlated. Take as an example the 100% hedge ratio strategy with 2 rolls. There, the realized simulated rates after the first roll show a correlation coefficient of 0.7 with the rates after the second roll, which is relatively high.

The distributions serve as a quantification of the risk that is associated with the application of a rolling- or ratio-strategy in comparison to the classic IRS hedge. Given this information, a client can ask himself the question: how much additional risk am I willing to take, in order to decrease my average CVA charge and expected risk premium? In the next section we attempt to quantify the CVA reduction induced by the application of the given strategies.

10.2.1 Real-world or risk-neutral framework?

A fair critical question concerning the presented distributions would be: *Are the statistics provided by risk-neutral confidence intervals a representative measure for the real-world risk?* In general, a client would want to judge the risk of a given strategy based on the *actual* likelihood that he might face a certain rate. This means he would be interested in real-world probabilities under the measure $\mathbb{P}$ rather than risk-neutral probabilities under the measure $\mathbb{Q}$.

The model we work with, models market-implied risk-neutral scenarios. The application of a risk-neutral model has many advantages. A widely used characteristic is that under a risk-neutral model, the average realizations of a discounted risk-factor (interest rate, FX rate) coincide with the currently observed forward market-value. If risk-factors are historically calibrated, this is not necessarily the case. By market-implied calibration, we guarantee that the simulations of our model are performed under the risk-neutrality condition. The first fundamental theorem of asset-pricing then implies that the model is arbitrage-free [Shreve, 2004, Anderson and Piterbarg, 2010b]. For our analysis this is a very desirable property. Our objective is to assess the impact of several different hedging strategies in the absence of arbitrage. Since this property is not guaranteed under a real-world measure, it would complicate our analysis significantly. Any model-assumptions would highly influence the outcome of the simulations and hidden arbitrage opportunities could unknowingly point to strategy characteristics, that are in reality not there. Additionally, there is currently no consensus on which real-world model-assumptions provide accurate simulation results. Such consensus does exist for risk-neutral models, which are broadly covered in the literature. The application of a risk-neutral model hence reduces the influence of any model-assumptions, allowing us to isolate the impact of particular parameters to the projected risk- and cost-profiles.

A downside of the risk-neutral model is that the estimated probabilities do not exactly match the real-world probabilities. We therefore cannot claim that the confidence intervals we present here are a perfect reflection of real-world likelihoods. We can however reasonably assume that the relative impacts that we present are similar to those under a real-world measure. Theoretically, the volatility of the risk-factors even fully matches the real-world volatility, due to the diffusion invariance principle. Although it should be said that in practice the implied volatility slightly over-estimates the real-world volatility, see for example Flemings [1998]. Furthermore, we have seen under a risk-neutral model the expected rates are in general slightly overestimated due to the incorporated risk-premium. Al together it is reasonable to argue that the presented distributions are a conservative estimation of the real-world. For a potential client, this should be acceptable as an investor rather judges his risk on conservative than over-promising projections.

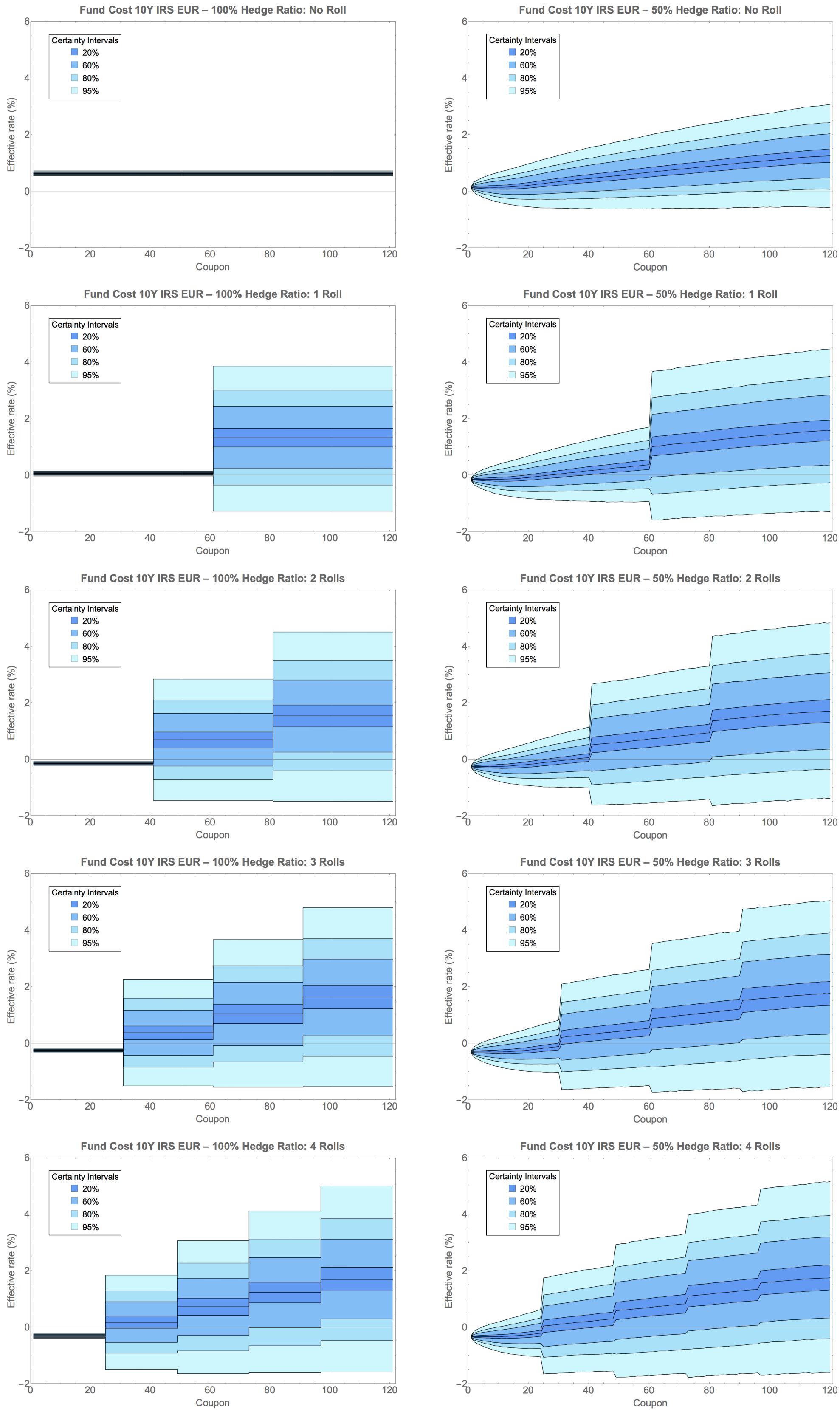

Figure 10.4: Risk-neutral distribution-profiles of the effective rates for a 10Y hedge with (left) 0-4 rolling strategies with 100% hedge ratio and (right) 0-4 rolling strategies with 50% hedge ratio

10.3 Expected CVA charges

Lastly we discuss the expected CVA that will be charged for each of the 0-4 rolling strategies. Under the assumption of a time-homogeneous hazard rate (section 9.5), the results are displayed in figure 10.5. The bar graphs show an estimation of the so called *running spread* that is expected for each strategy. CVA is usually expressed as a spread in basis-points (1/100%) which is charged on top of the fixed swap-rate. Take as example an IRS with inception time T_0 and maturity T_m. Given that at time T_0 we have computed the CVA charge, we subsequently approximate the spread κ as follows:

$$\begin{aligned} CVA\left(T_0\right) &= \mathbb{E}^Q\left(\sum_{j=1}^{m} D\left(T_0, T_j\right) \cdot N \cdot \Delta t_j \cdot \kappa \middle| \mathcal{F}_{T_0}\right) \\ &= \sum_{j=1}^{m} P\left(T_0, T_j\right) \cdot N \cdot \Delta t_j \cdot \kappa \\ \Rightarrow \quad \kappa &= \frac{CVA\left(T_0\right)}{N \sum_{j=1}^{m} P\left(T_0, T_j\right) \Delta t_j} \end{aligned}$$

In figure 10.5 we see the expected consecutive spreads per rolling scheme. This means that under a full hedge we expect a spread of 3.58 bps during the full tenor (blue), with one roll we expect 1.05 bps for the first 60 coupons and 1.18 for the subsequent 60 coupons (yellow), etc. Note that we make no distinction between the charge for a 100% ratio hedge or a 50% ratio hedge. This is because the charge is directly proportional to the notional amount. The CVA is expressed as a spread on top of the fixed rate. thereffore we will observe no difference between the CVA for a 50% ratio, a 100% ratio or any other ratio strategy. Clearly, changing the hedge ratio does affect the *absolute* expected CVA charge. It scales linearly with the notional size of the IRS, meaning that a 50% ratio will in total imply half the CVA costs compared to a 100% hedge ratio.

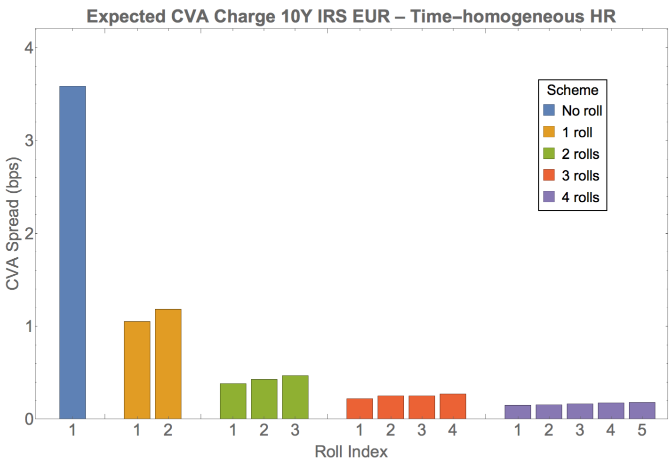

Figure 10.5: Expected CVA charges for a 10Y hedge with 0-4 rolling strategies under the assumption of a time-homogeneous hazard rate term-structure.

We will now focus on the impact of a rolling strategy on the CVA charge. First of all, observe that the application of a rolling scheme significantly decreases the spread compared to a classic IRS hedge with no roll. Each additional roll lowers the spread even further. This effect can be attributed to two factors:

1. *The hazard rate term-structure.* For the results in figure 10.5 we assumed a time-homogeneous hazard rate. This means that after a roll, the term-structure of hazard rates is reset to the time T_0 value (see figure 9.5). This has a positive impact on the computation of the CVA. Recall that the charge is calculated as follows

$$CVA(t) = LGD \cdot \sum_{i=1}^{m} EPE\left(t, T_i\right) \cdot PD\left(T_{i-1}, T_i\right)$$

Without a roll, the probabilities of default near maturity will be relatively high as they are implied by high hazard rates. Introducing a roll, resets the hazard-rate term structure at this roll-date. As a result, the default probabilities after a roll-date are again partially implied by low hazard rates, which will lower the CVA. It should be noted that this property is a direct consequence of our assumption on the development of the term-structure. Although we believe the assumption is realistic, it remains an assumption with its limitations. As a comparison, the CVA-profiles under the other two assumptions (as discussed in section 9.5) can be found in figure 10.7.

2. *Time-span of each consecutive IRS.* CVA is only charged for the trade that the client is currently in. For example, by switching from a strategy with zero rolls to one roll, the tenor of the IRS that starts at T_0 reduces from 10 years to 5 years. This has a major impact on the EPE-profile. At inception, $EPE\,(T_0) = 0$. This is by definition, since the swap is settled at par (i.e. $V\,(T_0) = 0$). If we move further in time, the market changes and for some scenarios the trade will gain value, inducing an increase in the EPE. When we approach the end of the trade the EPE will decrease again as more and more cashflows have been executed, until it is eventually zero at maturity. The larger the time-span of the trade, the more uncertain the last cashflows become. This is why the EPE of an IRS with short tenor will overall always be smaller than that of an IRS with long tenor. What the fixed rate for the second IRS will be, is not know today, but it is certain that it will be at par for each individual scenario. Therefore, also at inception of the second trade the EPE will start at zero and follow a similar profile as that of the first IRS. We illustrate this effect in figure 10.6, where we see that the overall EPE is significantly lower for a strategy with one roll compared to the strategy without a roll. As an immediate consequence the CVA charge under a rolling strategy will be lower.

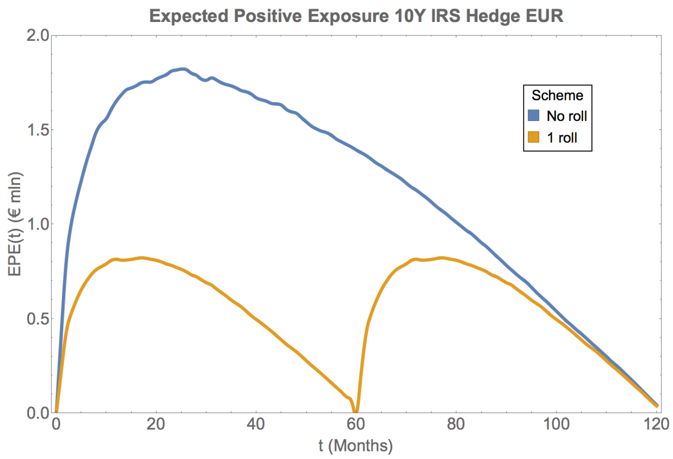

Figure 10.6: An exemplary expected positive exposure profile for a 10Y hedge with 0 and 1 roll.

Figure 10.7 shows the expected CVA charges under the two other assumptions on the hazard rate term-structure. On the left we see the profile for a curve-consistent term-structure, meaning that the hazard rates are not reset at a roll-date. Comparing graph 10.7 (a) to 10.5 shows that our assumption on the hazard rate has a significant impact on the spreads after the first roll. This should be kept in mind, as none of the assumptions are perfect. Observe that after the first roll the spreads tend to increase. This is due to the fact that these spreads are computed with higher hazard rates, in line with figure 9.4.

Consider graph 10.7 (b) to rule out the impact of the hazard rate term-structure. We still observe a clear decrease in the CVA charges for each additional roll. Under the assumption of a flat hazard rate, we can fully attribute this effect to the shorter time-span of the individual IRS trades and the lower EPE-profile they induce. This result provides confidence that no matter what the assumption on the hazard rate is, an application of a rolling strategy will always imply a reduction on the CVA charge.

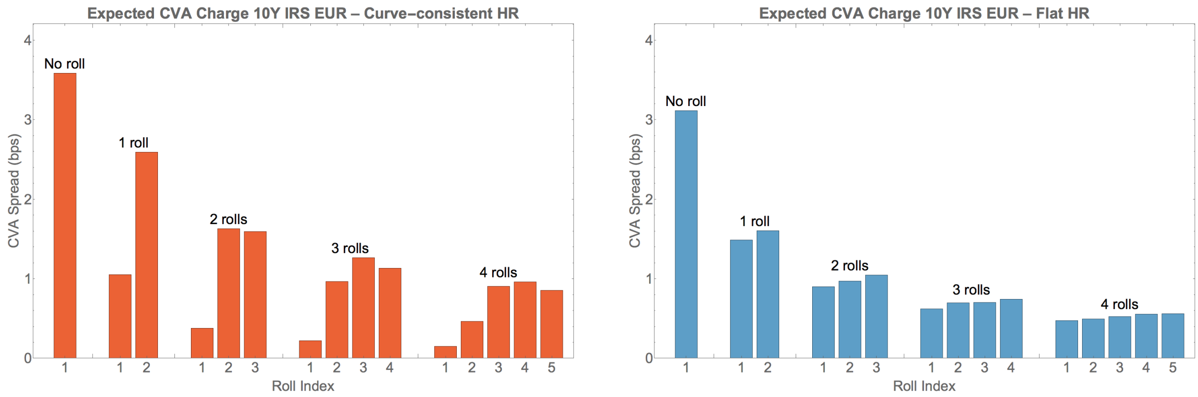

Figure 10.7: Expected CVA charges for a 10Y hedge with 0-4 rolling strategies for two alternative assumptions on the hazard rate (HR) term-structure. (a) Curve-consistent HR and (b) Flat HR.

11 Results for the CCS strategies

We investigate the impact of the *rolling strategy with fixed notional* and the *rolling strategy with reset notional* in terms of funding cost and CVA charges in comparison to a regular CCS hedge. In this section we will discuss cost-profiles that have been obtained by market simulations.

For our analysis we consider again a client that is about to enter a 5 year floating rate loan in Euros. We assume it has a coupon schedule with monthly LIBOR payments. The market simulations are performed using an implementation of the multi-currency Hull-White and FX model. For the loan we consider the following specifications:

Notional	Currency	Interest frequency	Start-date	End-date	Valuation date
100 M	EUR	Monthly	11-Dec-2017	11-Dec-2022	08-Dec-2017

On top of the loan we will consider several variations to the CCS based hedging strategies as described in section 9.3. We will compare results for two types of CCS, namely the EUR-USD (US Dollar) and the EUR-CHF (Swiss Franc). Reason that we choose these particular currencies as example is that in the current economy, the US Dollar interest rates are higher than that of the Euro, whereas the Swiss Franc interest rates are lower. We will see that this particular property is of strong influence of the hedge-related cost-profiles. The following strategies have been investigated:

Fixed notional EUR-USD CCS hedge:
- no roll
- 1 roll
- 2 rolls
- 3 rolls
- 4 rolls

Fixed notional EUR-CHF CCS hedge:
- no roll
- 1 roll
- 2 rolls
- 3 rolls
- 4 rolls

Reset notional EUR-USD CCS hedge:
- no roll
- 1 roll
- 2 rolls
- 3 rolls
- 4 rolls

Reset notional EUR-CHF CCS hedge:
- no roll
- 1 roll
- 2 rolls
- 3 rolls
- 4 rolls

11.1 Expected fund costs

Through market simulations we composed realizations for each strategy of the cashflows that a client would have paid in a no-arbitrage, risk-neutral setting. We will start with considering the costs that are implied by the netted flows associated with the CCS and the underlying loan. We assume that the CCS notional matches the notional volume of the loan. As a consequence, the floating interest rate payments fully cancel out against the floating leg of the CCS and only the fixed leg payments of the swap remain. In figure 11.1 we show the expected funding costs related to the 10 different rolling strategies associated to a EUR-USD swap (left) and a EUR-CHF swap (right).

This time, by "effective rates" we mean the annualized foreign costs as a percentage of the *original* foreign notional. Say that K_j denotes the effective rate paid at the j^{th} coupon, then the absolute costs at T_j will be $N_0^f \cdot K_j \cdot \Delta t_j$ expressed in the foreign currency. Carefully note that also in case of the notional reset strategy, the effective rates are expressed in terms of the foreign notional amount *before* the first roll. We do this to avoid misleading figures. For the reset strategy, the CCS notional will change after each roll, which would therefore have made it difficult to compare the funding costs for different roll schemes.

The funding costs of the client are effectively equal to the the fixed-rate coupons of the CCS. The monthly effective payments are therefore constant between two rolling dates. When no roll is applied, the client pays the same rate during the whole tenor. With a one-roll strategy, he will pay one fixed rate for the first 30 coupons and another for the last 30 coupons, etc. The fixed rate will by definition be similar for the fixed- and reset-notional strategies until the first roll. After the first roll, the foreign notional is reset in case of the reset-strategy, whereas it is kept constant in case of the fix-strategy. Therefore, the expected rates after the first roll is executed will differ for a reset- and a fix-strategy. We will discuss the funding cost profiles for the two different currencies in the foreign leg separately in the following paragraphs.

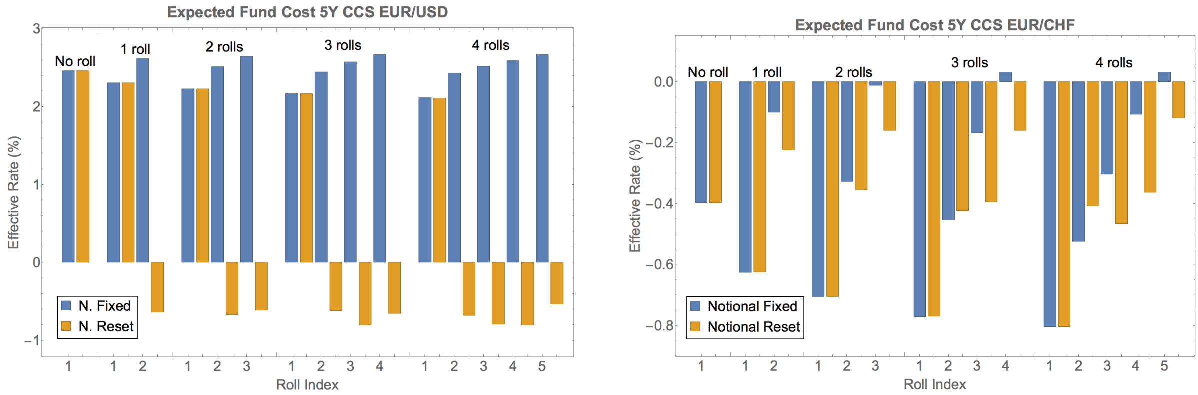

Figure 11.1: Expected effective rates for a 5Y CCS hedge with (a) 0-4 rolls for EUR-USD and (b) 0-4 rolls for CHF

11.1.1 The EUR-USD case

First of all, we will consider the profile for a *fixed-notional* strategy (the blue bars in figure 11.1 a). Observe that similar to the IRS case, there is no preference in roll-scheme in terms of expected funding costs, due to the no-arbitrage principles. For each additional roll, the rates will be lower at first, but tend to increase after each consecutive roll-date. This can be explained by the fact that also the USD forward rates are increasing over time.

Another important quantity to consider, apart form the rates, is the expected value of the final notional exchange at maturity. If the exchange rate is expected to increase during the trade (i.e. $\varphi(T_0) < \Phi_0(T_m)$), then this final exchange is expected to be in favor of the bank, since $\mathbb{E}^Q\left(\varphi(T_m)N^f - N^d \middle| \mathcal{F}_{T_0}\right) > 0$. If the FX is expected to decrease, it is the other way around. Recall that the SDE for the FX rate is given by

$$d\varphi(t) = \varphi(t)\underbrace{\left(r^d(t) - r^f(t)\right)}_{\text{the FX drift}}dt + \varphi(t)\sigma^\varphi(t)dW(t)$$

which indicates that we can expect a decreasing trend in φ if the foreign interest rates are higher than the domestic interest rates. For the US dollar market this is currently the case. That also explains the relatively high observed CCS rates for this currency in comparison to the Swiss franc case, where the intereset rates are currently lower than that of the Euro.

Now let us consider the *reset-notional* strategy (the orange bars in figure 11.1 a). Here we observe a completely different pattern in the development of the effective CCS rate. At first, the rate is positive, but after one roll the rate is expected to drop significantly. This is due to the nature of the reset-strategy. Consider a rolling strategy with $n > 0$. At the first roll-date, the unwinding notional exchange of the first CCS is supposed to take place. This unwinding is however not executed and the remaining MtM is absorbed in the rate of the new CCS, which is entered at this time (see section 9.3). Since the unwinding of the notional expected to be in favor of the client, this MtM will be negative. The new CCS rate is therefore expected to be significantly lower than the first one, even negative in our case.

Surely, the decreasing coupon rates come with a price. In case of the fixed notional strategy, the client could expect a large final transaction in his favor, with a value of roughly 11 million Euros. In case of the reset-strategy, the notional is reset at each roll-date according to the prevailing FX spot rate. As the spot rate is expected to decrease, the notional N_j^f amount is likely to be larger after each roll as we let $N_j^f = {}^{N^d}/_{\varphi(T_{i_j})}$. As a consequence, the final exchange will have a smaller value in expectation as $\mathbb{E}^Q\left(\varphi(T_m)N_n^f - N^d \middle| \mathcal{F}_{T_0}\right)$ is closer to zero than $\mathbb{E}^Q\left(\varphi(T_m)N_0^f - N^d \middle| \mathcal{F}_{T_0}\right)$. The more rolls are added to the scheme, the shorter the time will be between the last notional update and the final exchange at maturity. Hence the smaller the value of that cashflow is expected to be for the client. This is illustrated in figure 11.2, where the expected value of the national exchange at maturity in a reset-strategy is shown per roll-scheme. One could say that by entering a rolling-reset strategy, the client chooses to "spread out" the value of the final cashflow over the intermediate coupon payments. The charged coupon rate will therefore be lower, but in return he has to repay a larger notional amount at the final exchange compared to the classic CCS hedge.

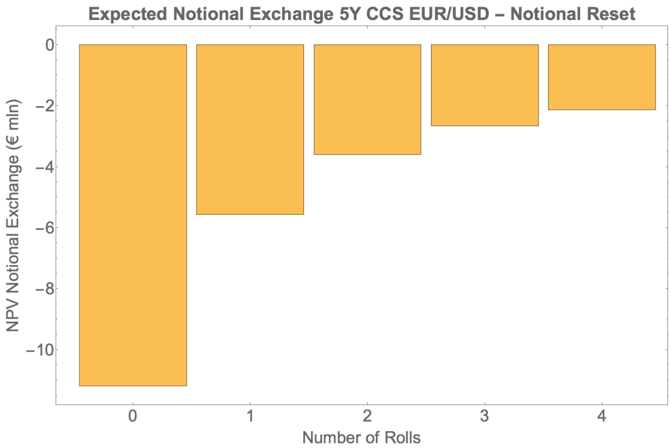

Figure 11.2: Expected risk-neutral value of the final notional exchange after a 5Y hedge with 0-4 rolling strategies with notional reset in USD

11.1.2 The EUR-CHF case

As a comparison we will now consider the cost profile of a CCS hedge with the Swiss Franc as the foreign currency. The results for the fixed- and reset-notional strategies are displayed in figure 11.1 b. We observe that most projected rates are negative, but this is nothing special. It is simply a consequence of the remarkably low CHF LIBOR that is currently prevailing (around -0.8% on average in 2017 for the 1M LIBOR).

For the *fixed-notional* strategy we in fact observe a similar behavior as for the Dollar case (the blue bars in figure 11.1 b). Under the risk-neutral setting there appears to be no preference in roll-scheme in terms of expected funding costs. By increasing the number of rolls, the rates will be lower at first, but tend to be higher after each consecutive roll-date. This is again due to the increasing development of the forward rates through time. In the current economy the CHF interest rate is smaller than that of EUR, implying a positive drift for the FX process. For this reason, we also have that the final notional exchange at maturity is this time expected to be in favor of the bank, rather than the client (as we have $\varphi(T_0) > \Phi_0(T_m)$ and thus $\mathbb{E}^Q\left(\varphi(T_m)\,N^f - N^d\middle|\,\mathcal{F}_{T_0}\right) > 0$). For the fixed-notional strategy, the final notional exchange is again independent of the number of rolls and expected to have an intrinsic value of around 2.3 million Euros at maturity.

Now, let us consider the *reset-notional* strategy (the orange bars in figure 11.1 b). As mentioned, we have a positive drift for the FX process, implying that each time the foreign notional is reset under a rolling strategy, it is expected to be lower than before. This induces that the final transaction will be less favorable for the bank, as again $\mathbb{E}^Q\left(\varphi(T_m)\,N_n^f - N^d\middle|\,\mathcal{F}_{T_0}\right)$ will be closer to zero for a larger number of rolls n. In figure 11.3, we indeed observe this pattern as the expected value of the final notional exchange decreases with number of rolls.

In the US Dollar case we observed that an *increase* in the expected value of the notional exchange due to additional rolls, resulted in a *decrease* in the average effective rate in the client's coupon payments. Naturally we would expect the opposite effect for the Swiss Franc case. However, take a closer look at figure 11.1 b and compare the rates for a fixed strategy (blue bars) with the rates for a reset strategy (orange bars). Note that also for this currency the average rates in a reset strategy are *lower* than those for the fixed-notional strategy. This rather counterintuitive phenomenon can be explained by the fact that the effective rates are expressed in terms of the foreign currency, in combination with *Siegel's paradox*. We will briefly treat this effect in the following paragraph.

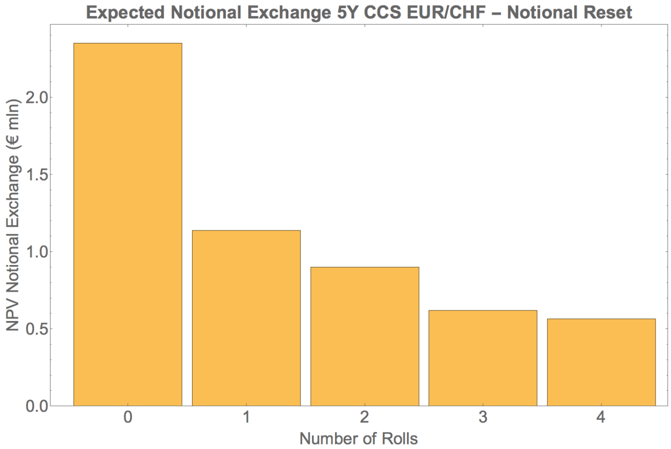

Figure 11.3: Expected risk-neutral value of the final notional exchange after a 5Y hedge with 0-4 rolling strategies with notional reset in CHF

11.1.3 Siegel's paradox

Let us consider an example of a CCS related hedge for the Swiss Franc, under the application of a rolling strategy with only one roll. The effective rate paid by the client during the first CCS before the roll, will be similar for both strategies by definition. After the roll, the client enters a new CCS with a new effective rate. Now compare the expression for the new rate of the CCS under the fix-strategy to the new rate of the reset strategy, as given in sections 9.4.1 and 9.4.2. We will denote the new rate under the fixed-notional strategy as $K_1 \equiv K^{fix}$ and the rate under the reset strategy as $K_1 \equiv K^{reset}$. If both rates are expressed in terms of the original foreign notional N_0^f, we find that

$$K^{fix} - K^{reset} = \left(N_1^f - N_0^f\right) P\left(T_{i_1}, T_m\right) \Phi\left(T_{i_1}, T_m\right)$$

We know that by definition $P\left(T_{i_1}, T_m\right)$ and $\Phi\left(T_{i_1}, T_m\right)$ are positive numbers. Hence we can say that $\mathbb{E}^Q\left(K^{reset} \middle| \mathcal{F}_{T_0}\right) > \mathbb{E}^Q\left(K^{fix} \middle| \mathcal{F}_{T_0}\right)$ if and only if $N_0^f > \mathbb{E}^Q\left(N_1^f \middle| \mathcal{F}_{T_0}\right)$. By intuition, this condition should hold. The original and the reset notional are defined as $N_0^f = {}^{N^d}/_{\varphi(T_0)}$ and $N_1^f = {}^{N^d}/_{\varphi(T_{i_1})}$. The FX rate φ is expected to grow due to its positive drift. Therefore we presume that $N_1^f < N_0^f$ and thus $K^{fix} < K^{reset}$ in expectation. However, figure 11.1 b shows *the exact opposite*.

To explain this, we should consider the dynamics of N_j^f or in fact the dynamics of ${}^1/_{\varphi(t)}$. We know that under $\mathbb{Q}$, the FX rate has drift $r^d(t) - r^f(t)$ and

$$d\varphi(t) = \varphi(t)\left(r^d(t) - r^f(t)\right)dt + \varphi(t)\sigma^\varphi(t)dW(t)$$

However, this does not mean that ${}^1/_{\varphi(t)}$ has a drift equal to $r^f(t) - r^d(t)$. This is due to the convexity of the function $f(t,x) = {}^1/_x$. The impact becomes clear once we work out the differential of ${}^1/_{\varphi(t)}$ using Itô's lemma:

$$\begin{aligned}
d\left(\frac{1}{\varphi(t)}\right) &= df\left(t, \varphi(t)\right) \\
&= f_t\left(t,\varphi(t)\right)dt + f_x\left(t,\varphi(t)\right)d\varphi(t) + \frac{1}{2}f_{xx}\left(t,\varphi(t)\right)d\varphi(t)d\varphi(t) \\
&= -\frac{1}{\varphi(t)^2}\left(\varphi(t)\left(r^d(t) - r^f(t)\right)dt + \varphi(t)\sigma^\varphi(t)dW(t)\right) + \frac{1}{2}\frac{2}{\varphi(t)^3}\varphi(t)^2\sigma^\varphi(t)^2dt \\
&= \frac{1}{\varphi(t)}\underbrace{\left(r^f(t) - r^d(t) + \sigma^\varphi(t)^2\right)}_{\text{The } {}^1/_\varphi \text{ drift}}dt - \frac{1}{\varphi(t)}\sigma^\varphi(t)dW(t)
\end{aligned}$$

Note that the drift is hence given by $r^f(t) - r^d(t) + \sigma^\varphi(t)^2$. This asymmetry compared to the drift of φ, is called *Siegel's paradox*. It appears that in our particular example, we have the special situation that on average $0 < \mathbb{E}^Q\left(r^d(t) - r^f(t)\middle|\mathcal{F}_{T_0}\right) < \sigma^\varphi(t)^2$, implying a positive drift for $1/\varphi(t)$ and hence N_j^f. This would justify that also for the EUR-CHF CCS hedge we observe that $K^{fix} < K^{reset}$ in expectation.

We tested the conjecture described above by rerunning the simulations after manually overriding the value of σ^φ. If this volatility is small enough we should have that $0 < \sigma^\varphi(t)^2 < \mathbb{E}^Q\left(r^d(t) - r^f(t)\middle|\mathcal{F}_{T_0}\right)$ and observe that indeed $K^{fix} > K^{reset}$ in expectation. We choose to set

$$\sigma^\varphi_{override}(t) := 0.5 \cdot \sigma^\varphi(t)$$

Our findings are displayed in figure 11.4. Here we indeed observe that the rates under a reset-strategy (orange bars) are after the first roll higher than the rates under the fix-strategy (blue bars). This is in line with our expectation and this time also our intuition.

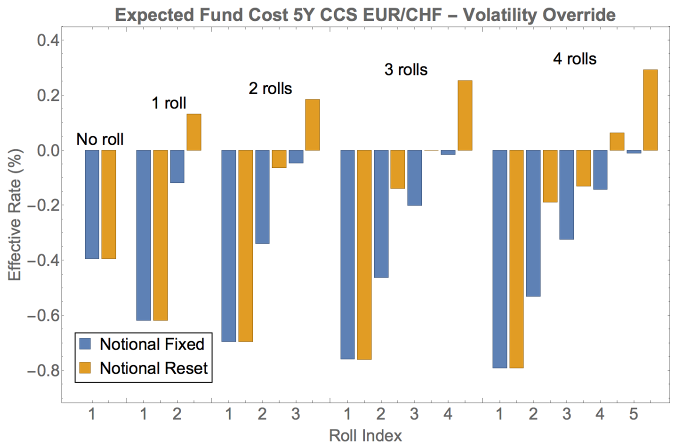

Figure 11.4: Expected effective rates for a 5Y CCS hedge with 0-4 rolls for EUR-CHF under a FX volatility override.

11.2 The expected CVA charges

Here we discuss the expected CVA charges that will be associated with the 0-4 rolling strategies for both the EUR-USD and the EUR-CHF hedge. We will assume a time-homogeneous term-structure of the hazard rates (see section 9.5), as we consider this the most realistic assumption. The results are displayed in figure 11.5. The bar graphs show an estimation of the running spread that is expected to be charged during each strategy. The spread is expressed in basis-points, which would be charged on top of the fixed rate. The domestic notional is kept fixed for both strategy types, which makes it easy to compare the charges (recall that in a notional reset-strategy only the foreign notional is updated).

First of all observe that also for a CCS hedge, a higher number of rolls generally implies a lower expected CVA charge. This appears to be the case for each strategy type and each currency. This general pattern can be attributed to *the hazard rate term-structure*. Similar to the IRS related hedge, are the CVA spreads implied by lower hazard rates on average. This is of course a direct consequence of our assumption on hazard rate term-structure.

The other factor that had a positive impact on CVA in the IRS case was *the shorter time-span of each consecutive trade*. However, this property does not necessarily contribute in the CCS case. This is because for a rolling IRS hedge, each new trade was defined to have a zero MtM at inception, independent of the market scenario. For the CCS related strategies this is no longer the case, as for both types each new CCS is likely to be in- or out-of-the-money at inception. In the fixed-strategy this is because the foreign notional is kept constant, even if the FX rate has moved. In the reset-strategy this is because the remaining MtM of the previous CCS is blend into the new trade. Therefore the EPE-profile of the first CCS in a rolling strategy might on average be lower, but the subsequent ones are likely higher, due to the absorbed MtM at each roll-date. We recognize this effect if we compare figure 11.5 to figure 10.5. Note that the average reduction of the CVA charge as result of a rolling scheme is significantly less pronounced in the CCS case than in the IRS case.

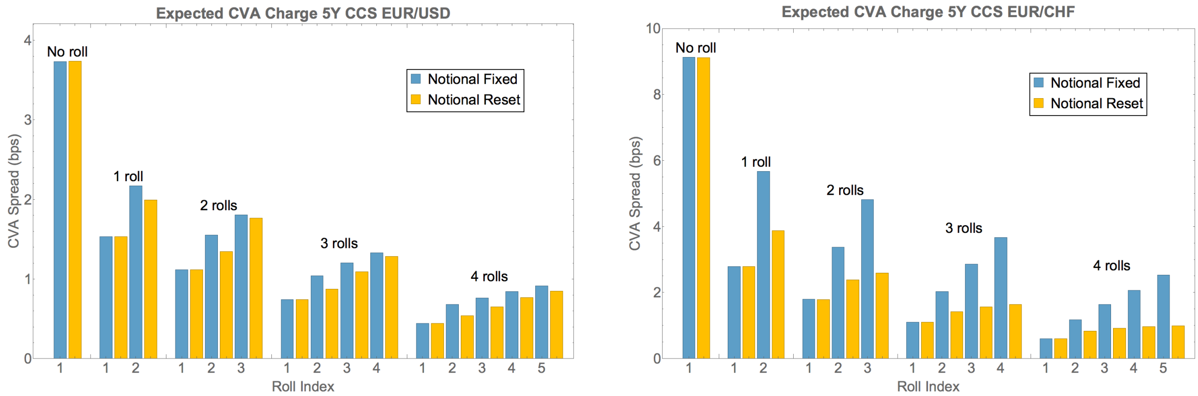

Figure 11.5: Expected CVA charges for a 5Y hedge with (a) 0-4 rolling strategies in USD and (b) 0-4 rolling strategies in CHF

11.2.1 The impact of the final notional exchange

For CCS related strategies there is another factor that highly affects the CVA charge, which did not play a role in the IRS hedges. This is the impact of the final notional exchange. Due to the stochastic nature of the exchange rate, this last cashflow of the CCS adds a large amount of uncertainty to the trade and has the potential to accrue a lot of value until maturity, either in favor of the bank or the client. For this reason we observe major differences in the CVA profiles associated to the different strategy types, but also the different currencies. We can distinguish the following patterns concerning the CCS related hedging strategies:

1. *If the interest rate of the foreign currency is higher than the domestic, the CVA will in general be lower.* Observe that the CVA spreads for the EUR-USD hedges are significantly lower than for the EUR-CHF. As mentioned before, we have that for the Swiss Franc in expectation $r^d > r^f$. Therefore the exchange rate is expected to grow and the final notional exchange will likely be in favor of the bank. This will imply that the EPE-profile associated to this trade will remain high until the scheduled date of the very last transaction. Needless to say, this induces a relatively high CVA charge. For the EUR-USD hedge it is the other way around. We expect that the notional exchange is in favor of the client. The exposure associated to this trade will therefore quickly drop as the final transaction approaches. A lower CVA charge will be the result.

2. *If the interest rate of the foreign currency is lower than the domestic, there is a distinct preference for a reset-notional strategy in terms of expected CVA charge.* Observe that for the EUR-CHF hedge, the expected CVA charges under a reset-strategy are significantly lower than under a fixed-strategy. This is in contrast with EUR-USD case, where the difference appears to be relatively small. This is because in the USD case, the final exchange has a positive impact on the exposure-profile, but in the CHF case a negative impact. An important property of the reset-strategy is that after a roll, the foreign notional is set to be at par again, and the value-difference between the previous and new notional is spread out over the coupons. We have seen that as a consequence, the value of the final exchange is expected to be closer to zero. In case of the EUR-CHF hedge, each additional roll will therefore substantially decrease the contribution of the notional exchange to the EPE-profile. This is in opposition to the fixed-strategy, where the large transaction at maturity is maintained and is independent of the roll-count. For the EUR-USD hedge we do not observe such a distinct difference between the two strategies because the contribution of the notional exchange to the EPE-profile is in this case far less pronounced.

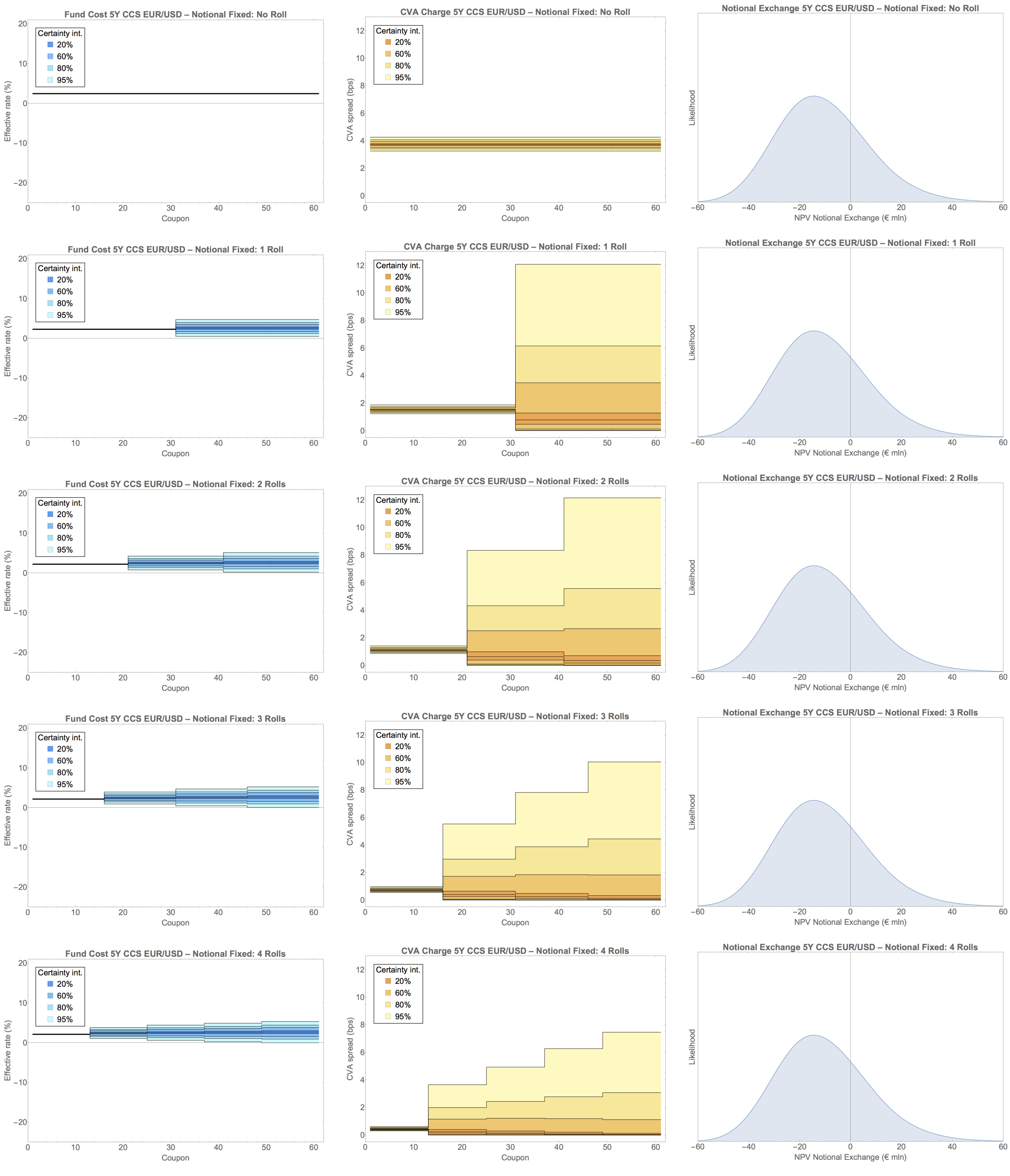

Figure 11.6: For a 5Y EUR-USD CCS hedge with 0-4 rolling strategies with fixed notional, the risk-neutral distributions of (left) the effective rates, (middle) CVA charges and (right) net value of the final notional exchange at maturity

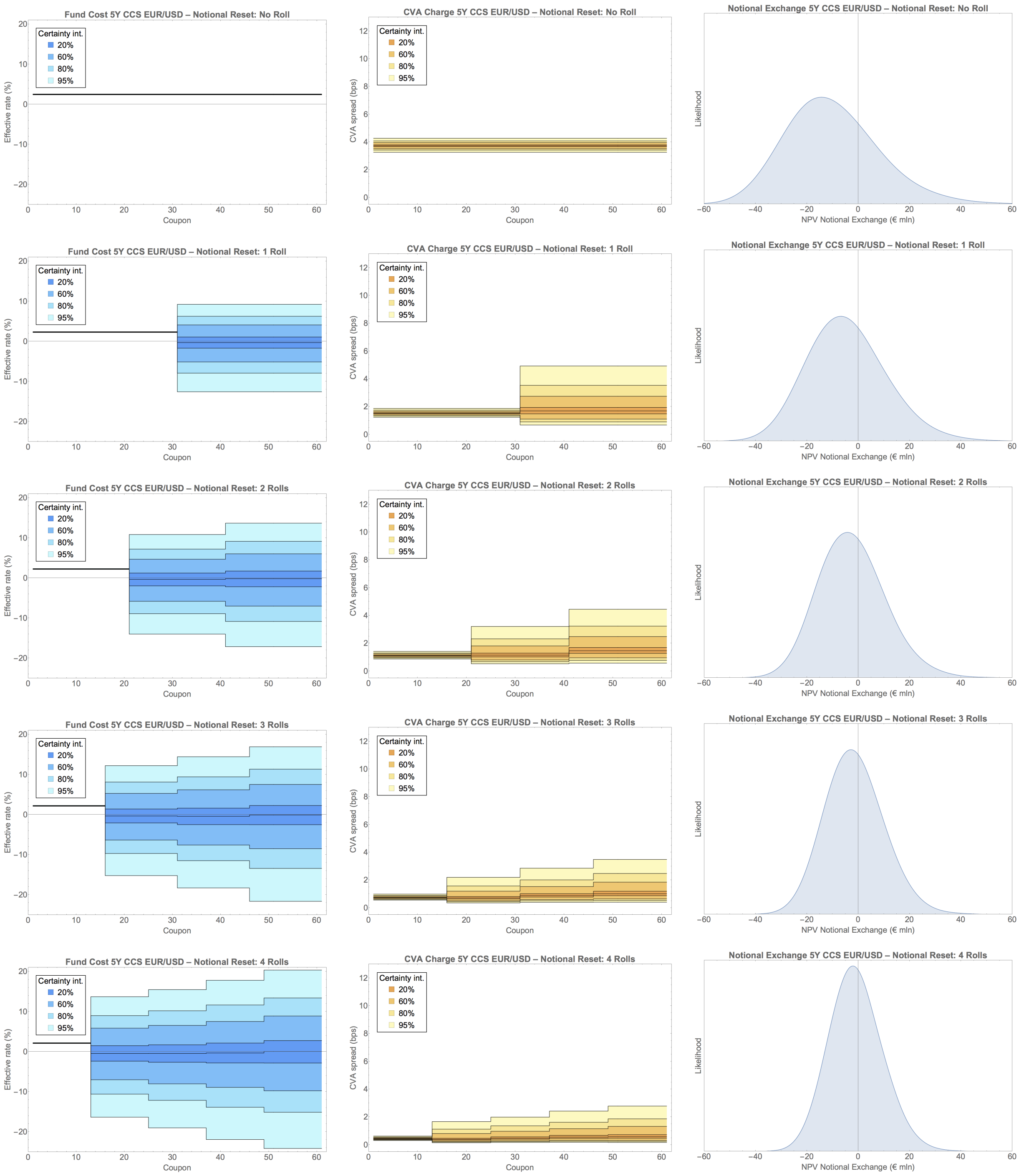

Figure 11.7: For a 5Y EUR-USD CCS hedge with 0-4 rolling strategies with reset notional, the risk-neutral distributions of (left) the effective rates, (middle) CVA charges and (right) net value of the final notional exchange at maturity.

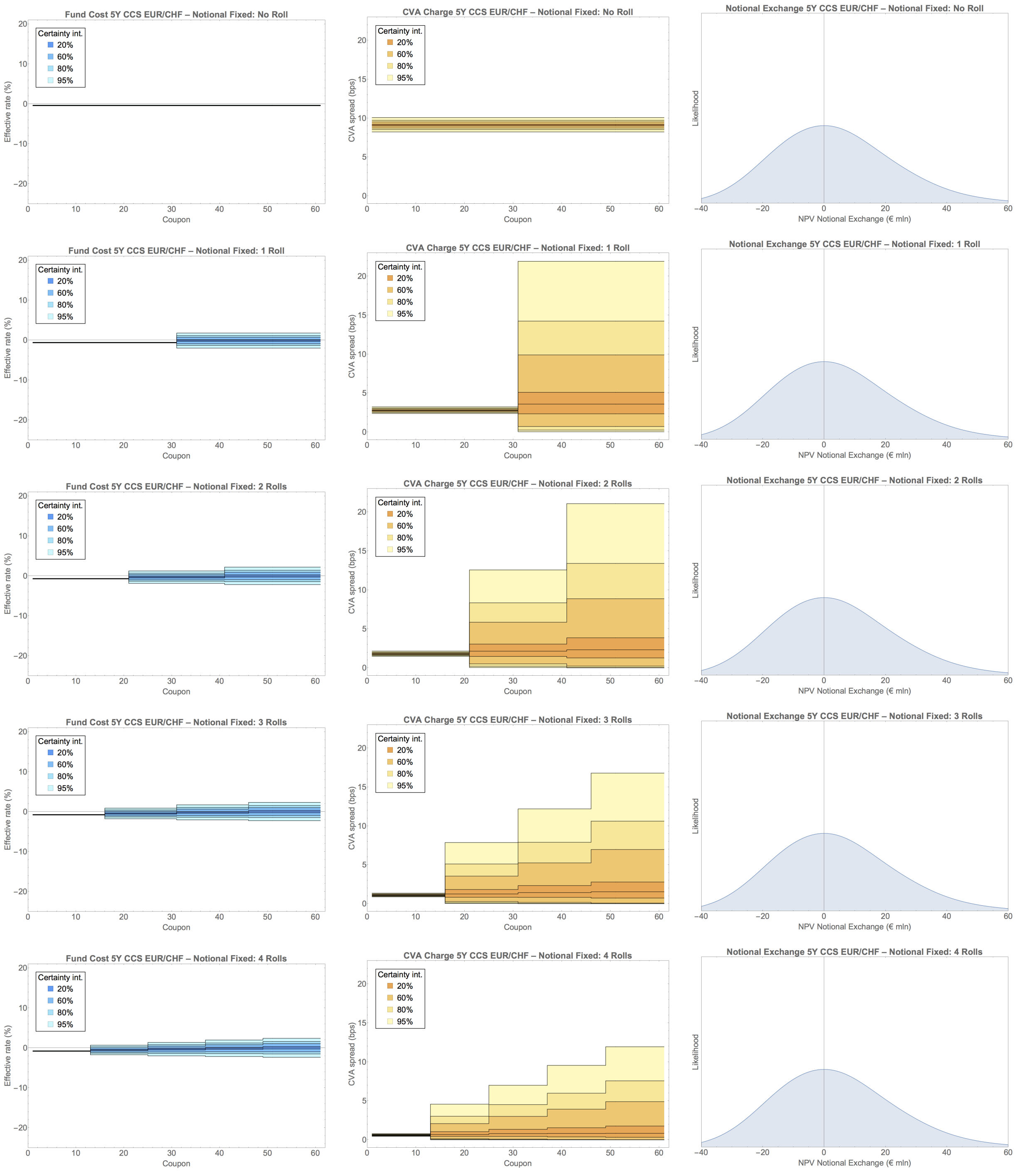

Figure 11.8: For a 5Y EUR-CHF CCS hedge with 0-4 rolling strategies with fixed notional, the risk-neutral distributions of (left) the effective rates, (middle) CVA charges and (right) net value of the final notional exchange at maturity.

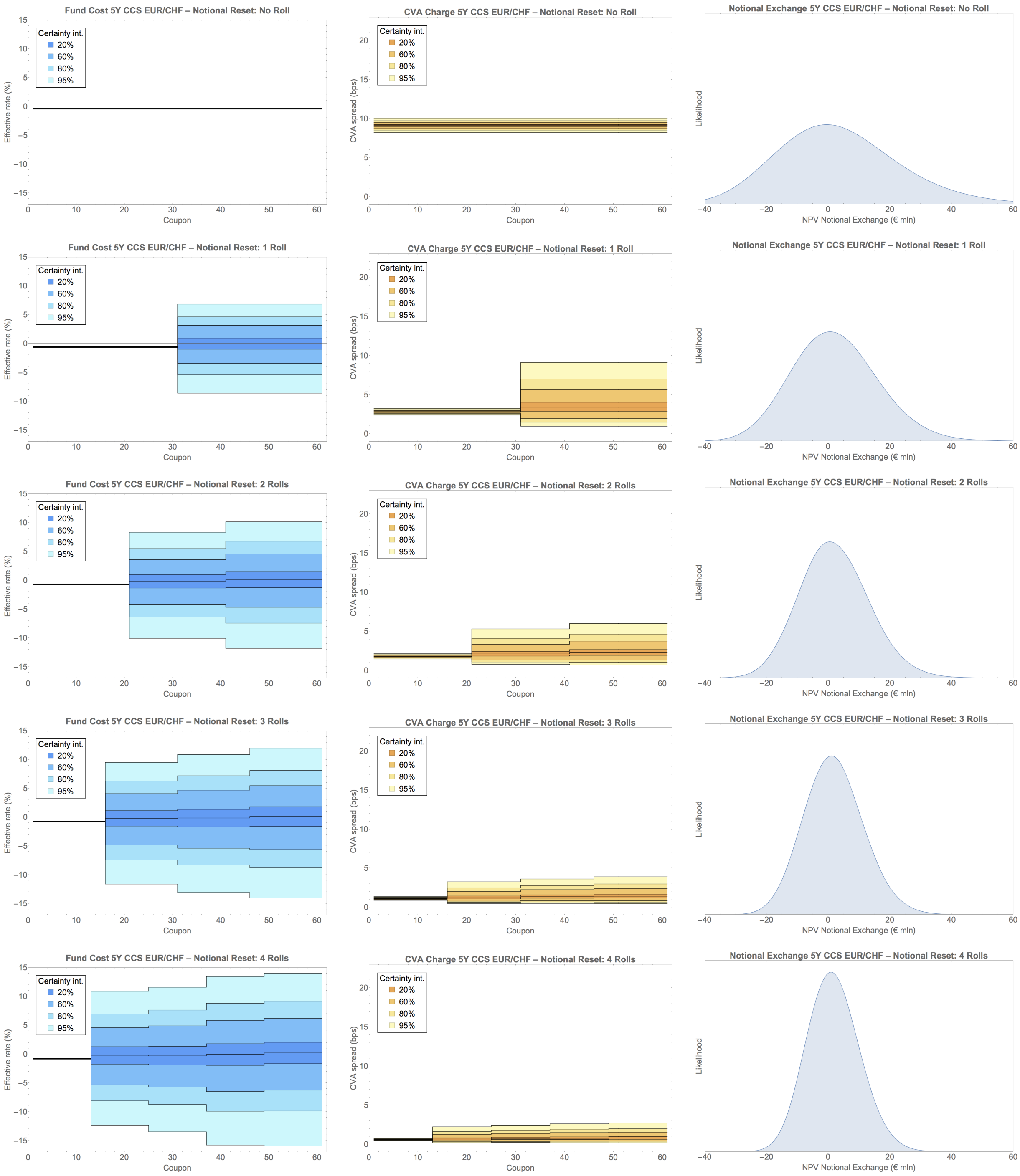

Figure 11.9: For a 5Y EUR-CHF CCS hedge with 0-4 rolling strategies with reset notional, the risk-neutral distributions of (left) the effective rates, (middle) CVA charges and (right) net value of the final notional exchange at maturity.

11.3 Distributions of the fund costs and CVA charges

Based on the results presented in the previous sections, the application of a rolling strategy appears to be a favorable option for the client. The CVA charges under a rolling scheme are expected to be significantly smaller and potential risk-premiums should be lower. However, similar to the IRS case, choosing for a rolling hedge has a drawback. Once again, the client must be willing to accept a risk. For the CCS hedge, there are several aspects to this risk. We distinguish the following risk-factors:

- The fixed CCS rate, which is updated after each roll-date
- The CVA spread, which is also be updated after each roll-date
- The risk-neutral value of the final notional exchange

Note that for a classic CCS hedge, where the tenor of the CCS matches the tenor of the underlying loan, both the fixed rate and the CVA charge are locked at inception and not exposed to any risk. The same holds for the final notional exchange, as the contract settles at inception at which forward FX rate this transaction will take place. However, what the realized FX rate will be at maturity is not known a priori, making it uncertain whether this forward FX is favorable for the client or not.

Based on market simulations, we have composed risk-neutral distributions of each of the three mentioned risk-factors. This is done for the two foreign currencies (EUR-USD and EUR-CHF), and the two strategy types (fixed notional and reset notional). The results are displayed as follows:

1. EUR-USD hedge with fixed-notional strategy: figure 11.6
2. EUR-USD hedge with reset-notional strategy: figure 11.7
3. EUR-CHF hedge with fixed-notional strategy: figure 11.8
4. EUR-CHF hedge with reset-notional strategy: figure 11.9

The figures have been composed by Monte Carlo simulations of the domestic short rate, the foreign short rate and the exchange rate using 20,000 paths. The graphs on the *left* represent a sequence of 60 confidence intervals, that indicate the likelihood of what the effective rate for the client will be at each coupon date. It should once more be said that the intervals should be treated as independent distributions. Given that the client would be in the top 2.5% range of the distribution after one roll, does again not necessarily mean that it still will be after the second roll. In contrast to the IRS case, the consecutive effective rates even show very little correlation this time. This is partly because the rate is now dependent on three stochastic processes instead of one (r^d, r^f and φ), but also because the MtM of the previous MtM is blended in the new CCS after a roll. This can on itself cause unstable behavior.

The graphs in the *middle* show risk-neutral confidence intervals of what the CVA spread for the client will be at each coupon date. Since the CVA charge is highly affected by the MtM of the new trade at a roll-date, we observe significant variation in the simulated CVA spreads per scenario. Also this is in contrast with the IRS case where each new trade started with a zero MtM by definition.

The graphs on the *right* show the risk-neutral distribution of the value of the final notional exchange at maturity. By this we mean $N^d - \varphi(T_m) N_n^f$, where N_n^f denotes the notional amount of the CCS after the last roll. Note that under the fixed-notional strategy, the foreign notional is never updated. Therefore we observe the exact same distribution for each roll scheme under the fixed-notional strategy.

The distributions together can again serve as a quantification of the risk that is associated with the application of each rolling-strategy. A potential client can use the data provided by such figures to judge which trade-off between additional roll-over-risk and reduction in overall costs in expectation he is willing to accept. Based on the specific results that are presented here, we can determine the following patterns:

- *The uncertainty in the future coupon rates becomes larger with each additional roll.* This is independent of the currency or strategy type. If a strategy involves many rolls, it means that some rates will be locked at a later date in the future. Each additional roll will therefore induce an increased uncertainty, as the variance in the short-rate and exchange rate conditioned on what is known today, increase with time.

- *Under a rolling reset-strategy, the risk in the coupon rates is relatively high, but the risk in the notional exchange is relatively low.* That the roll-over risk in the CCS rates is significantly higher under a reset-strategy is clearly illustrated if we compare the graphs on the left of figures 11.7 and 11.9 to those of figures 11.6 and 11.8. This is due to the nature of the reset-strategy. At a roll-date the remaining MtM of the unexecuted notional exchange of the previous swap is blended into the new trade. This remaining MtM can be relatively large and also relatively volatile, as it is dependent on the FX process. This effect in combination with the stochastic LIBOR can cause that in one scenario the new rate is remarkably high after one roll, but then very low after a second. We see this reflected in the broad confidence intervals, which in some cases can be a factor three to four larger than the fixed-strategy counterpart. In return we however observe that the uncertainty in the final notional exchange shrinks. Although the absolute amount that the client has to pay at the final exchange becomes uncertain by applying a rolling scheme, the MtM of the final cashflow will likely be closer to zero. We see this reflected in the distributions on the right in figures 11.7 and 11.9, which tend to become narrower around zero. Note that the effect described above intensifies with each extra roll.

- *Under a rolling fixed-strategy, the risk in the coupon rates is relatively low, but the risk in the notional exchange is relatively high.* We have already seen that the CCS rates under a fixed-strategy are significantly less uncertain than under a reset-strategy. This is due to the fact that under a fixed-strategy, we do not blend the MtM of the previous swap, which implies that the reset-rates are far less volatile. The trade-off is that it remains evenly uncertain what the MtM of the notional exchange will be at maturity, no matter how many rolls we add to the strategy.

- *The risk in the CVA charges is relatively high under a rolling fixed-strategy, but relatively low under a rolling reset-strategy.* Also this effect can be attributed to the final notional exchange. Its MtM throughout the trade largely contributes to the EPE-profile and therefore it directly affects the CVA charge. Under the reset strategy we know that after each roll, the exposure induced by the notional exchange is reduced as the foreign notional is reset to the par-value. The CVA spread will as a result be reduced. This is not the case in the fixed-strategy. Under this strategy there will be simulated scenarios under which the MtM of this final transaction has accumulated enormously. For these scenarios, the CVA charge will be quite large after a roll, thus contributing to the uncertainty in the CVA charge.

12 Conclusion and application

This thesis evolved around corporate hedging strategies related to interest rate swaps and cross-currency swaps. As a starting point of our research we considered the following two situations:

1. A corporate requiring domestic funding
2. A corporate requiring foreign funding

A common approach for obtaining domestic funds is by taking a floating rate loan. If the corporate combines the loan with a synchronized IRS, the interest rate risk is hedged. Foreign funds can be obtained with a domestic loan in combination with a CCS. This is particularly relevant for corporates with limited access to the foreign money-market. A CCS hedges not only the interest-, but also the foreign exchange-rate risk. Traditionally, the tenors of an IRS or CCS are matched with the tenor of the underlying loan, by which the risk is locked until maturity. A drawback of this approach is that hedges over a long tenor can be costly due to associated risk premiums and credit spreads, which tend to disproportionally increase with the duration of the contract.

We investigated several hedging strategies that in expectation yield lower hedging cost, but as a trade-off induce a risk. As an alternative to the classic IRS hedge we looked at

- *A rolling strategy:* a hedge consisting of several consecutive IRS's with shorter tenor.
- *A ratio strategy:* a hedge where the notional amount of the IRS matches only a fraction of the loan.

As an alternative to the classic CCS hedge we looked at

- *A rolling strategy with fixed notional*: a hedge consisting of several consecutive CCS's with shorter tenor and fixed notional amounts.
- *A rolling strategy with reset notional*: a hedge consisting of several consecutive CCS's with shorter tenor and a foreign notional amount that is reset at each roll-date.

To perform an impact analysis on each of these strategies we considered a multi-currency interest rate and foreign exchange rate modeling framework. For modeling the short-rates we used one-factor Hull-White term-structure models in different currencies. For the foreign exchange rates we used a Garman-Kohlhagen model, simulating the FX as a geometric Brownian motion.

To investigate the risk induced by each strategy, we composed fund-costs-profiles using Monte Carlo methods. The simulations have been performed under a risk-neutral framework. Risk-neutrality implies that the model is free of arbitrage, which is a desirable property. It allows us to isolate the impact of each particular strategy, without the influences of model-assumptions. A drawback of the risk-neutral framework is that the projected distributions of future risk-factor do not fully coincide with real-world probabilities. It is however reasonable to assume that the relative impacts are similar and that the risk-neutral cost-profiles are conservative estimators of the real-world profiles.

Apart from funding costs, we considered credit charges associated to the hedging strategies. We in particular focussed on unilateral CVA, which is charged to a client to compensate for the risk induced by a potential event of default. Typically additional charges will be presented to a client next to the credit spread. This so called *mark-up* is to compensate for capital and operational costs and to yield a profit for the bank. Although we have not treated the more general notion of a mark-up in our analysis, it is reasonable to assume that these charges scale linearly with CVA. The results and observations considering CVA can therefore be carried over to a general notion of hedging costs.

A fair share of our research has been dedicated to construct an efficient computation routine for CVA on an IRS and a CCS. CVA computation typically requires an estimation of default probabilities particular to the client and an exposure-profile particular to the financial instrument. The former is derived from CDS quotes observed in the market. The latter is commonly computed through Monte Carlo simulation. To compute a distribution of potential future CVA charges, we would hence be required to model an exposure-profile per simulated market-scenario. This would imply nested simulations, which is computationally inefficient.

We have shown that we can alternatively apply an analytical approach to estimate exposure profiles. For both instruments this approach is generally similar. First we discretize the EPE term-structure and apply some simplifications. This allowed us to approximate the positive exposure in terms of option pay-offs. The computation of the EPE-profile is then reduced to evaluating of a sequence of (cross-currency) swaptions. We subsequently showed that we could derive model-consistent parameters for the application of Black's and Bachelier's option-pricing formulas. The analytical approach induces some approximation and discretization errors, which tend to be more pronounced in the multi-currency case. We however performed numerical tests, which indicate that there exists a relatively good match with CVA approximations obtained by Monte Carlo methods, making the approximations suitable for our impact analyses. The method enabled us to directly compute future realizations of cashflows *and* CVA charges along the simulation paths, resulting in coherent funding- and credit cost-profiles.

An analysis of the IRS related hedges confirmed that within the risk-neutral framework there is no preference for any strategy. All yield in expectation a similar cost-profile, which is a direct consequence of the no-arbitrage principles. A rolling- or ratio strategy does however imply uncertainty in the costs, since the risk is no longer hedged for the full notional volume or full tenor. A benefit is that for both strategies the credit charge will be reduced in expectation. For a ratio-strategy, the CVA scales directly proportional with the hedge-ratio. For a rolling strategy the reduction is in expectation stronger. This is due to the constraining effect of shorter tenors on the EPE-profile and the hazard rates. Although it must be noted that the latter is a consequence of our assumption on the development of hazard rates through time.

For the CCS related hedges we considered two versions of a rolling strategy. Under a fixed-notional strategy, each new CCS is entered with the same fixed domestic and foreign notional, even if the FX spot-rate has moved. Under a notional-reset strategy, the domestic notional is kept constant, but the foreign notional is reset according to the prevailing FX rate. The remaining MtM of the notional exchanges at each roll-date are not cashed-out, but instead blended into the new CCS. We again observed that in terms of expected fund-costs, there is no preference for a given strategy. Under the fixed-notional strategy, the reduction of the expected credit charged is less pronounced as each new CCS is entered with non-zero MtM. Only the assumption on the hazard-rates appears to impact the average credit spread. Under the reset-notional strategy we observe a clear distinction between the cases where the foreign interest rate is higher than the domestic and vice versa. In the latter case, the notional exchange at maturity is expected to be in favor of the bank. Under the reset strategy, the MtM of the final exchange is partly spread out over the intermediate coupons, which suppresses the exposure and hence the CVA. If the foreign interest rate is higher than the domestic, this effect is absent. Both strategies imply uncertainty in the funding costs and credit charges. For the fixed-notional strategy we find a relatively small uncertainty in the effective CCS-rates, but a large variance in the CVA as there is a significant uncertainty in the MtM of the final notional exchange. For the reset-strategy we observe a relatively large variance in the CCS-rates as the MtM of the intermediate notional exchanges is blended into this rate. The variance in the credit spread is however smaller as the MtM of the final notional exchange is certain to be closer to zero with each additional roll.

12.1 An example case

The model set-up that has been discussed throughout this thesis, can be used to perform general impact analyses of the alternative strategies for an IRS or CCS related hedge. To illustrate an application of the set-up, we will treat a simplified example of a case below.

12.1.1 The set-up

Consider a mid-size corporation that is specialized in providing industrial ICT services. We assume the company has its headquarters stationed in The Netherlands and has additional offices in Belgium and Germany. To increase the market reach of the company, the corporate is planning to expand and intends to open an office in the UK. To finance the expansion, the company intends to attract foreign capital with an approximate value of 90 million British pound Sterling (GBP). At this moment the corporate has limited access to the British money-market, but has a sustainable relationship with a Dutch bank. He therefore decides to take a loan in Euros with monthly floating rate interest payments, which he means to hold for at least 5 years. The Euro notional he means to swap to British pounds, through a floating-fix cross-currency swap. The loan should eventually be repaid by earnings generated by the new British office.

12.1.2 The problem

For a 5 year cross-currency swap, the corporate is confronted with relatively high hedging costs. Assume that in the current market, a 5 year CCS with monthly coupons comes with the following terms:

- A fixed CCS-rate of 1.24%
- A CVA spread of 5.9 bps

The numbers are based on market-data of December 2017 and a hazard-rate term-structure similar to the one shown in figure 5.1.

To reduce the hedging costs, the corporate considers to apply a rolling strategy. By entering several consecutive swaps with a shorter tenor, he expects lower credit charges. He additionally believes that a fixed rate of 1.24% is relatively high and possibly contains a large risk-premium. Since 2013, the 1M GBP LIBOR has not been much above 0.5% and according to his personal view on the market, the current forward rates will unlikely materialize. The corporate is therefore willing to take some roll-over risk associated with a rolling strategy. Nevertheless, he does have a liquidity risk to consider. Large fluctuations in the monthly coupons are undesirable and he want to be certain that the rates remain relatively bounded. If during the 5 years, the CCS-rates go beyond 4%, his liquidity position will be in danger. For this reason, he wants a 95% confidence that the maximum rates associated to a given strategy will not exceed 4%.

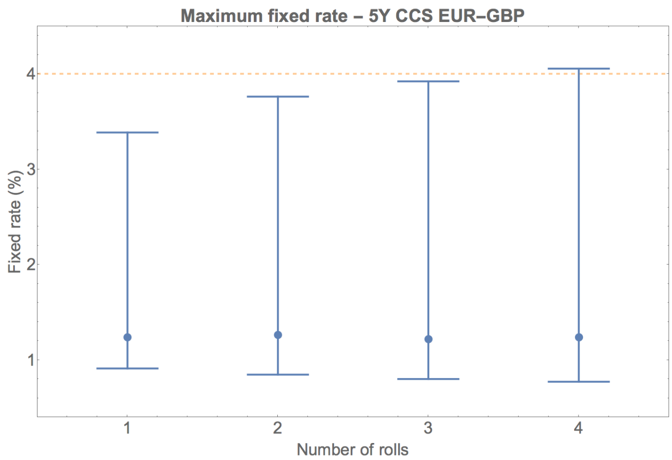

Figure 12.1: 90%-confidence intervals of the maximum CCS fixed rates for a *fixed-notional rolling strategy* with 1-4 rolls.

12.1.3 A potential solution

We perform an impact analysis on the application of *a rolling strategy with fixed notional.* We assume this strategy-type to be suitable because:

- The fluctuations in the CCS fixed rates are limited in comparison to a reset-strategy (see as a reference figure 11.6).
- The corporate expects to generate earnings in the foreign currency. A movement of the FX will therefore not influence the repayment of the foreign notional. Therefore a reset of the foreign notional will be less relevant.
- The current GBP interest rate is higher than the EUR interest rate. A reset-strategy will therefore unlikely yield a higher reduction in the credit charges than the fixed-national strategy.

We perform Monte Carlo market simulations of 5,000 paths using the multi-currency Hull-White and FX model. Under a rolling scheme, each scenario implies several CCS-rates, dependent on the number of rolls. As the corporate requires the maximum CCS rate to be constrained, we consider per scenario only the maximum rate. We perform the simulation for a strategy with 1, 2, 3 and 4 rolls. Figure 12.1 shows two-sided 90% confidence intervals for the maximum realized CCS-rate per rolling scheme. The dots mark the *average* of the expected CCS rates per scheme, which are due to the no-arbitrage principles similar for each strategy. We observe that the uncertainty in the rates increase with each additional roll, conform our expectation. The 95-percentiles of the simulations, correspond to the upper-limits of the confidence intervals. We note that for 3 rolls, this percentile is still under the 4%, whereas with 4 rolls this boundary is breached. Since the expected credit charges are reduced with each additional roll, we conclude that a *fixed-notional strategy with 3 rolls* offers an optimal alternative for the traditional CCS hedge.

With the application of a rolling scheme with three rolls, the corporate will yield an expected reduction in the credit charges of almost 75%. If we assume that the domestic notional amount is equal to 100 million Euros, this translates to a total expected cost reduction of approximately 200 thousand Euros in terms of CVA. It is reasonable to assume that the additional mark-up that is charged to the client is reduced with an equal factor, yielding an even larger reduction in costs.

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Appendix

A Test results for analytical CVA approximation

In chapters 6 and 7 we derived analytical formulas to compute the CVA charge for an IRS and a CCS. For both instruments we applied several approximation steps in order to write the EPE-profile as the sum of European option pay-offs. This allowed us to evaluate the exposures using respectively Bachelier's and Black's formula. Our aim was to evaluate CVA pathwise in a multi-currency Hull-White and FX simulation framework. For this reason we derived the option volatilities in terms of Hull-White and exchange rate parameters.

We tested the accuracy of the formulas against Monte Carlo computations of the CVA for several exemplary trades. Here we present a selection of the results and discuss some of the limitations of the CVA approximation method.

A.1 Results for an IRS

Figure A.1 displays the absolute CVA charge for a 5-year and a 10-year interest rate swap both as function of the fixed rate K. We observe an increasing curve, since the exposure to a client increases when the fixed rate is higher. After all, the receivable cashflows from the clients are bigger if the fixed rate is higher, and therefore also the potential loss in case of a default.

The blue line corresponds to the CVA charge computed by the analytical approximation derived in chapter 6. There are several approximation errors that affect the accuracy of the formula. These are:

- Freezing the weights and zero-coupon-bond prices in the estimation of the swap-rate volatility
- Discretization error in the estimation of the EPE-profile

There is no direct improvement at hand for the freezing of the weights, although Brigo and Mercurio [2007] argue that this technique in general has a relatively small impact on the accuracy of the volatility-estimate. The discretization error could possibly be reduced by applying a smaller step-size to the discretization scheme.

The red points of reference are obtained by a Monte Carlo procedure, based on 4,000 Hull-White simulation paths. The error-bars indicate a two standard-deviation window induced by the variance in the simulated sample-mean. It should be noted that also the Monte Carlo suffers from a discretization error. This error is not reflected in the error-bars, as it is difficult to measure. For the Monte Carlo estimation, the EPE-profile is discretized according to a standard grid with small steps at first and larger steps further away in time.

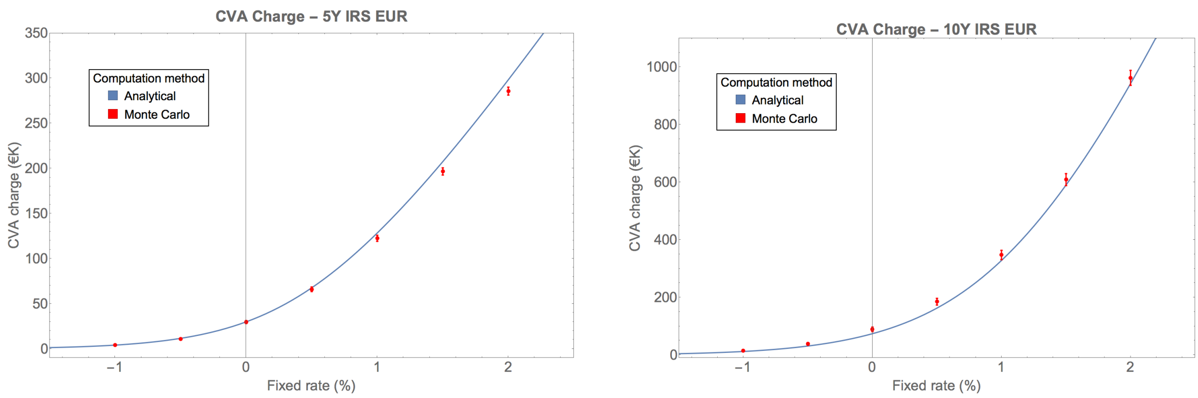

Figure A.1: CVA charge as function of the fixed rate K corresponding to (a) a 5 year IRS and (b) a 10 year IRS, both with a notional of 100 Million EUR.

A.2 Results for a CCS

Figure A.2 displays the absolute CVA charge for a 5-year and a 10-year EUR-GBP cross-currency swap both as function of the fixed rate K. Figure A.3 shows similar results only with USD as the foreign currency. The blue lines corresponds to the CVA charge computed by the analytical approximation derived in chapter 7. Also the approximation of the CCS related CVA is affected by structural errors, which are:

- Freezing the weights in the estimation of the domestic and foreign effective CCS-rate
- Freezing the weights in the estimation of the CCS-rate volatility
- Discretization error in the estimation of the EPE-profile

The red points of reference are again obtained by a Monte Carlo procedure, based on 4,000 multi-currency Hull-White and FX simulation paths. The error-bars indicate a two standard-deviation window induced by the variance in the simulated sample-mean.

The discretization error in the CVA appears to be more pronounced in the CCS multi-currency framework, where the charge is driven by three stochastic processes instead of one. We observed small, but significant differences in the Monte Carlo results for different exposure discretization schemes. The best match with the analytical estimation was found if the EPE-discretization schemes of the Monte Carlo and the analytical method coincided. For this case the results are displayed in the figures below. An improvement to the analytical method is possibly achieved by the application of smaller discretization steps.

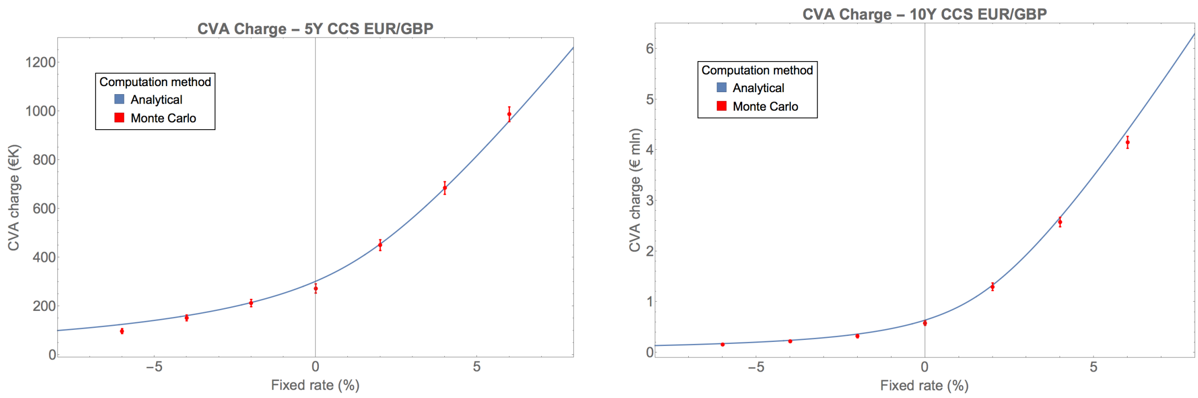

Figure A.2: CVA charge as function of the fixed rate K corresponding to (a) a 5 year EUR-GBP CCS and (b) a 10 year EUR-GBP CCS, both with a domestic notional of 100 Million EUR.

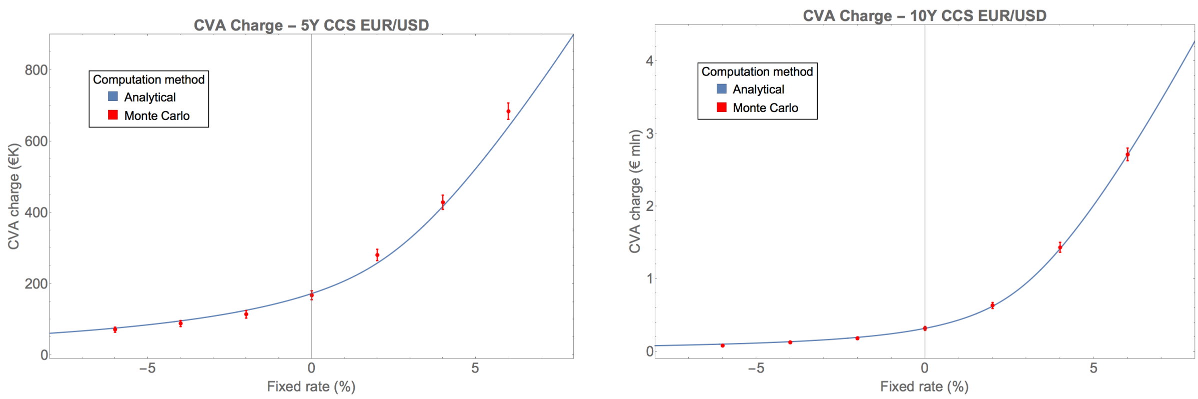

Figure A.3: CVA charge as function of the fixed rate K corresponding to (a) a 5 year EUR-USD CCS and (b) a 10 year EUR-USD CCS, both with a domestic notional of 100 Million EUR.

B Test results for CVA dual-curve adjustment on a CCS

In chapter 8 we presented an adjustment to the analytical approximation of the CVA for CCS. This adjustment is to increase the accuracy of the CVA approximation under a dual-curve framework. Figure B.1 shows the results for some numerical tests we performed. The graphs display the absolute CVA charge for a 5-year EUR-GBP cross-currency swap as function of the fixed rate K.

In the graph on the left we measure the impact of the dual-curve adjustment by comparing the analytical approximations with and without this adjustment. As a benchmark we show CVA computations obtained by a Monte Carlo procedure, based on 4,000 multi-currency Hull-White and FX simulation paths. The error-bars indicate a two standard-deviation window induced by the variance in the simulated sample-mean. For the Monte Carlo estimation, the EPE-profile is discretized according to a standard grid with small steps at first and larger steps further away in time. It appears that the adjustment indeed provides an estimation of the CVA that is slightly more accurate. We however believe that this is difficult to conclude from just the graph on the left. Both the Monte Carlo and the analytic computation suffer from discretization errors. The analytical computation additionally suffers from approximation errors induced by freezing weights and rates. The impact of the adjustment seems to be in the same order of magnitude of these errors.

To further test the effectiveness of the dual-curve adjustment, we performed a re-run of the simulations using customized market-data. We artificially created a large basis between the OIS and index curve to which the Hull-White model is calibrated. The results of this test are shown in the graph on the right. From this figure, it is reasonable to conclude that the adjustment indeed improves the accuracy of the analytical CVA approximation under a dual-curve framework.

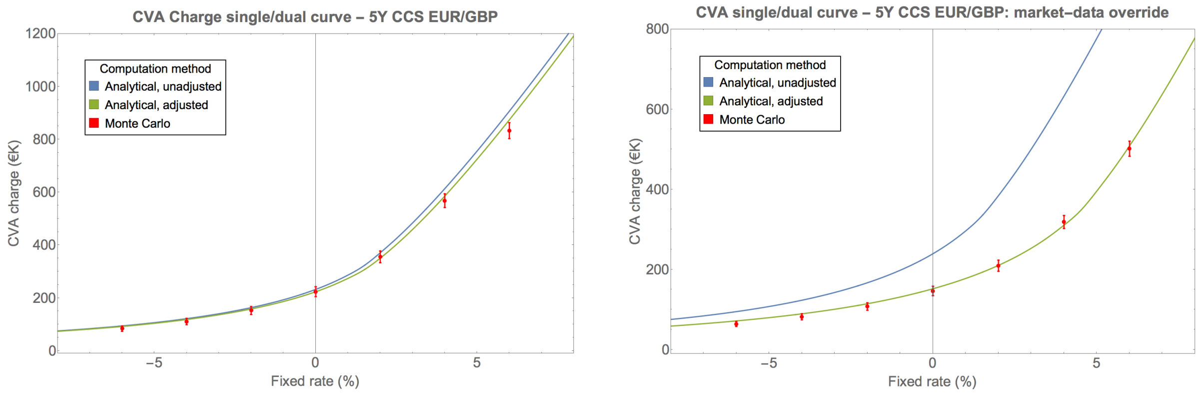

Figure B.1: CVA charge as function of the fixed rate K corresponding to (a) a 5 year EUR-GBP CCS and (b) a 5 year EUR-GBP CCS modeled under custom market-data calibration, both with a domestic notional of 100 Million EUR.