Utrecht University

Bachelor Thesis

Lattice walks and elliptic functions

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1 Introduction

Before we can state the problem at hand, we will need the following definitions

Definition A *step set* is any $S \subset \mathbb{Z}^2 \setminus \{(0,0)\}$.

Definition Given a step set S, an *allowed step function* is any $A\colon \mathbb{Z}^2 \to \mathcal{P}(S)$.

We call the value $A(i,j)$ the *allowed step set at* (i,j).

Definition Given a step set S and allowed step function A. We define a (restricted) lattice walk starting at (x,y) to be any sequence $\{a_k\}_{k\le n}$, such that $a_0 \in A(x,y)$ and for $i > 0$ we have $a_i \in A(\sum_{k=0}^{i-1} a_i)$. The number n is called the *length* and the point $\sum_{k=0}^{n} a_k$ is called the *endpoint*.

So, a restricted lattice walk can be seen as a sequence of steps, where each step a_i is taken from the allowed step set at the current position. An example of restricted lattice walks are positive lattice walks. Positive lattice walks are lattice walks which are confined to the positive quadrant. The main interest of this thesis will lie in this type of walks, in particular with the so called Gessel walks.

Definition A positive lattice walk is a lattice walk with allowed step function

$$A(i,j) = \{(n,m) \in S \,|\, (i,j) + (n,m) \in \mathbb{Z}^2_{\ge 0}\}$$

Definition Gessel walks are positive lattice walks with step set $S = \{(1,1),(1,0),(-1,1),(-1,0)\}$.

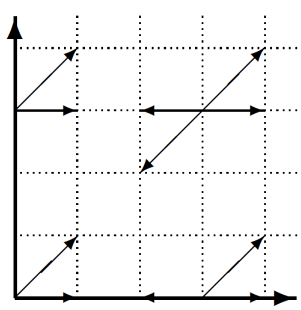


Figure 1: Here one can see a visualization of the step set of Gessel walks. Note that since it is a positive walk, there are restrictions when you are at the axis.

A common problem in combinatorics concerning lattice walks is the question of how many walks there are of a given length. For this type of question we need the following definitions.

Definition Given a step set and allowed step function, we define

$$q(i,j;n) = \#\{\text{lattice walks of length n starting at } (0,0) \text{ and ending at } (i,j)\}$$

and the generating function

$$Q(x,y;z) = \sum_{i,j,n\geq 0} q(i,j;n)x^i y^j z^n \tag{1}$$

Gessel walks have been puzzling the combinatorics community since 2001. One problem was to determine whether or not the generating function of Gessel walks (1) was algebraic. This was first obtained by Bostan and Kauers [2], using computer algebra techniques. Later Bostan, Kurkova and Raschel gave the first readable prove in [3]. This last paper was the inspiration for this thesis. Our goal will be to prove the algebraicity of (1) inspired by the method given in [3].

2 Difficulties surrounding Gessel walks

In order to understand why Gessel walks are an interesting case, we will first present a brief history of the difficulties they gave. This will be a brief summary of the introduction in [3].

After 2001, many approaches to treat walks in the quarter plane appeared. This can mostly be credited to Mishna and Bousquet-Mélou who studied positive walks with small steps (where the steps are a subset of the eight nearest neighbors). Mishna [8] first considered the case of step sets of cardinality three and presented a complete classification of the generation functions with respect to the classes algebraic, transcendental holonomic and non-holonomic. After this Mishna and Bousquet-Mélou [4] considered all 79 inherently different small step sets (by first reducing the problem for the in total $2^8 = 256$ small step sets). This eventually led to the study of a group of birational transformations of $\mathbb{C}^2$ resulting from a functional equation. In 23 cases this group turns out to be finite, and the corresponding functional equation were solved in 22 out of these 23 cases. The remaining case were Gessel walks. In 2010, Bostan and Kauers showed, using heavy computer algebra techniques, that the generating function of this last case was algebraic [2]. The first human readable proof was finally presented in [3].

3 The functional equation and the approach

From now on, $Q(x, y; z)$ will denote the generating function for Gessel walks.

Proposition 3.1. *$Q(x, y; z)$ converges for $z \in]0, 1/4[$, $|x| < 1$ and $|y| < 1$*

Proof. This follows immediately from the bound $q(i, j; n) \leq 4^n$, which is the number of walks of length n for an unbounded lattice walk. □

For this reason we will for now fix $z \in]0, 1/4[$. Next we will state the functional equation obtained by Bousquet-Mélou and Mishna which is valid for any $(x, y; z)$ with $|x| < 1$ and $|y| < 1$. This is a special case of Lemma 4 in [4].

Lemma 3.2. *The generating function $Q(x, y; z)$ is characterized by the following functional equation:*

$$K(x, y; z)xyQ(x, y; z) = zQ(x, 0; z) + z(1 + y)Q(0, y; z) - zQ(0, 0; z) - xy \quad (2)$$

where,

$$K(x, y; z) = 1 - z \sum_{(i,j)\in S} x^i y^j = 1 - z\left(xy + x + \frac{1}{xy} + \frac{1}{x}\right) \quad (3)$$

is called the kernel of the walk.

Our goal is to prove the algebraicity of $Q(x, y; z)$. In order to do so we will first prove the algebraicity of $Q(x, 0; z)$ and $Q(0, y; z)$ in their respective variables. From this and (2) the final result follows. In order to do this we will start by studying the curve $K(x, y; z) = 0$. This will be useful, since on this curve the left side of (2) will vanish. This allows us to study $Q(0, y; z)$ and $Q(x, 0; z)$ respectively, while still being able to use the functional equation.

4 The curve $K(x, y; z) = 0$

Let's define the affine curve T_z by

$$T_z = \{(x, y) \in (\mathbb{C}^\times)^2 \,|\, K(x, y; z) = 0\}. \tag{4}$$

This affine curve can be completed by adding the points where either $x \in \{0, \infty\}$ or $y \in \{0, \infty\}$. We do this by looking at the parametrization of $K(x, y; z)$ around these points.

Proposition 4.1. *T_z can be completed by adding the points $(0, \infty)$, $(0, -1)$, $(\infty, -1)$ and $(\infty, 0)$.*

Proof. When x is close to 0 we can use x as a local coordinate and by expanding the solutions for y we find

$$(x, y) = (x, -1/x^2 + \mathcal{O}(1/x)) = (0, \infty)$$

and

$$(x, y) = (x, -1 + \mathcal{O}(x)) = (0,\, -1).$$

When x is close to ∞ we use $t = 1/x$ as local coordinate and find

$$(x, y) = (1/t, -t^2 + \mathcal{O}(t^3)) = (\infty,\, 0)$$

and

$$(x, y) = (1/t, -1 + \mathcal{O}(t)) = (\infty,\, -1).$$

This concludes the completion of T_z. □

We denote the completion of T_z by $\overline{T_z}$.

Lemma 4.2. *The curve* $\overline{T_z}$ *is birationally equivalent to* $\overline{T_z'}$*, where*

$$T_z' = \{(r,s) \in (\mathbb{C}^\times)^2 \mid s^2 = 4r^3 - g_2 r - g_3\} \tag{5}$$

with

$$g_2 = \frac{1}{12}\left(16 + \frac{1}{z^4} - \frac{16}{z^2}\right), \quad g_3 = -\frac{1 - 24z^2 + 120z^4 + 64z^6}{216z^6} \tag{6}$$

Proof. The method of this proof was inspired by [5] section 3.3. Let $(x,y) \in T_z$. Then $xy \neq 0$ and thus by looking at the polynomial

$$\frac{xy}{z}K(x,y;z) = \frac{xy}{z} - (1 + y + x^2y + x^2y^2) = 0 \tag{7}$$

it follows from the quadratic formula, that

$$\left(-2(y^2+y)x + \frac{y}{z}\right)^2 = (y/z)^2 - 4(y^2+y)(y+1). \tag{8}$$

So, by the substituting $s = 2(y^2+y)x - \frac{y}{z}$ and $r = \frac{1}{12z^2} - \frac{2}{3} - y$ this becomes

$$s^2 = 4r^3 - \frac{1}{12}\left(16 + \frac{1}{z^4} - \frac{16}{z^2}\right)r + \frac{1 - 24z^2 + 120z^4 + 64z^6}{216z^6} \tag{9}$$

This shows that T_z is birationally equivalent to T_z', where the resulting map becomes

$$\psi\colon T_z \to T_z'; (x,y) \mapsto \left(\frac{1}{12z^2} - \frac{2}{3} - y, 2x(y^2+y) - \frac{y}{z}\right). \tag{10}$$

To show how ψ extends between $\overline{T_z}$ and $\overline{T_z'}$ we use Lemma 4.1

$$\begin{aligned}
\psi(0,\infty) &= \psi(x, -1/x^2 + \mathcal{O}(1/x)) = (1/x^2 + \mathcal{O}(1/x), -2/x^3 + \mathcal{O}(1/x^2)) \\
&= (\infty,\infty), \\
\psi(0,-1) &= \psi(x, -1 + \mathcal{O}(x)) = \left(\frac{4z^2+1}{12z^2} + \mathcal{O}(x), \frac{1}{z} + \mathcal{O}(x)\right) \\
&= \left(\frac{4z^2+1}{12z^2}, \frac{1}{z}\right) \\
\psi(\infty,0) &= \psi(1/t, -t^2 + \mathcal{O}(t^3)) = \left(\frac{1-8z^2}{12z^2} + \mathcal{O}(t^2), -2t + \mathcal{O}(t^2)\right), \\
&= \left(\frac{1-8z^2}{12z^2}, 0\right) \\
\psi(\infty,-1) &= \psi(1/t, -1 + \mathcal{O}(t)) = \left(\frac{4z^2+1}{12z^2} + \mathcal{O}(t), -\frac{1}{z} + \mathcal{O}(t)\right), \\
&= \left(\frac{4z^2+1}{12z^2}, -\frac{1}{z}\right).
\end{aligned}$$

□

The form of (5) is commonly known as the Weierstrass normal form and is parameterized by the Weierstrass elliptic function $\wp$ and its derivative $\wp'$ as $(r(t), s(t)) = (\wp(t), \wp'(t))$. The function $\wp$ is an elliptic function with periods w_1 and w_2. These periods follow from the values of g_2 and g_3. For some common definitions and theorems for elliptic functions we refer to the appendix. If one is not familiar with this subject it is advised to first take a look at these. We will use the results from the appendix the remainder of this section. From now we denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.

Lemma 4.3. *We have $w_1 \in \mathbb{R}$ and $w_2 \in i\mathbb{R}$ and*

$$\wp(t) \in \overline{\mathbb{R}} \iff t = iq + \frac{n}{2}w_1 \vee t = q + \frac{n}{2}w_2, \quad q \in \mathbb{R}, n \in \mathbb{Z}. \tag{11}$$

Furthermore, $\wp$ is strictly decreasing on $]0, 1/2w_1[$ and strictly increasing on $]1/2w_1, w_1[$.

Proof. From (6) it follows that $g_2^3 - 27g_3^2 = \frac{1}{z^4} - \frac{16}{z^2} > 0$. The result then follows from Theorem 10.7 in the Appendix. □

Next, we define $p\colon t \mapsto \psi^{-1}(\wp(t), \wp'(t))$. This map then defines a parameterization of $\overline{T_z}$. In particular, the map $[t] \mapsto p(t)$ defines an isomorphism between $\mathbb{C}/\Lambda$ and $\overline{T_z}$. Here Λ denotes the lattice generated by w_1 and w_2. We conclude this section with the following Lemma that will be crucial for the next section. Let's first define $\Pi_\wp = \{\alpha w_1 + \beta w_2 \,|\, 0 \le \alpha, \beta < 1\}$.

Lemma 4.4. *We have that $p(0) = (0, \infty)$, $p(1/2w_1) = (\infty, 0)$, $p(3w_1/4) = (0, -1)$ and $p(w_1/4) = (\infty, 0)$.*

Proof. First of all we clearly have $p(0) = \psi^{-1}(\wp(0), \wp'(0)) = \psi^{-1}(\infty, \infty) = (0, \infty)$. Next we have $p(w_1/2) = \psi^{-1}(\wp(w_1/2), \wp'(w_1/2)) = \psi^{-1}(e_1, 0)$, where e_1 is a zero of (5). Because g_2 and g_3 satisfy the condition in Theorem 10.7, it follows from Lemma 10.8 that e_1 is in fact its largest zero. All zeros of (5) are given by

$$\frac{1 - 8z^2}{12z^2}, \quad \frac{-z^2 + 8z^4 + 3\sqrt{z^4 - 16z^6}}{24z^4}, \quad \frac{-z^2 + 8z^4 - 3\sqrt{z^4 - 16z^6}}{24z^4} \tag{12}$$

so, since $z \in]0, 1/4[$, we have $e_1 = \frac{1-8z^2}{12z^2}$ and thus $p(w_1/2) = (\infty, 0)$.

For the last part, take $t_0 \in \Pi_\wp$ with $p(t_0) = (0, -1)$. Then $(\wp(t_0), \wp'(t_0)) = \psi(0, -1) = \left(\frac{4z^2+1}{12z^2}, \frac{1}{z}\right)$. Since we have $\frac{4z^2+1}{12z^2} > \frac{1-8z^2}{12z^2} = e_1$ and $\wp'(t_0) > 0$ it follows from Lemma 10.8 and Lemma 4.3 that $t_0 \in]w_1/2, w_1[$. Finally, we can use the duplication formula for the Weierstrass function to show that $(\wp(2t_0), \wp'(2t_0)) = \left(\frac{1-8z^2}{12z^2}, 0\right)$. We thus conclude that $t_0 = 3w_1/4$. Similar we find that $p(w_1/4) = (\infty, 0)$. □

Note here that in this section we have made a different approach than most other material surrounding this subject (like [3] and [5]). In these the choice was always made to look at (7) as a polynomial in y instead of x. We hope to show here that our approach is (as expected) also possible and in some parts maybe even preferable, but we leave this up for the reader to decide.

5 The automorphisms η and ξ

We define the functions η and ξ on $(\mathbb{C}^\times)^2$ by

$$\eta(x, y) = \left(\frac{1}{xy}, y\right), \quad \xi(x, y) = \left(x, \frac{1}{x^2 y}\right). \tag{13}$$

Lemma 5.1. *The functions η and ξ define automorphisms on T_z.*

Proof. We will start by showing that η and ξ map T_z to itself and are injective. Suppose we have a $(x, y) \in T_z$. It then follows from (3) that

$$K\left(\frac{1}{xy}, y; z\right) = 1 - z\left(\frac{1}{x} + \frac{1}{xy} + x + xy\right) = K(x, y; z) = 0. \tag{14}$$

$$K\left(x, \frac{1}{x^2 y}; z\right) = 1 - z\left(\frac{1}{xy} + x + xy + \frac{1}{x}\right) = K(x, y; z) = 0, \tag{15}$$

This shows that $\eta(x, y) \in T_z$ and $\xi(x, y) \in T_z$.

Now suppose we have $(x, y), (x', y') \in T_z$ with $\eta(x, y) = \eta(x', y')$. From the definition of η we immediately get that $x = x'$ and then it also follows that $y = y'$. A similar argument holds for ξ. Since both η and ξ are rational, we conclude that they are indeed automorphisms. □

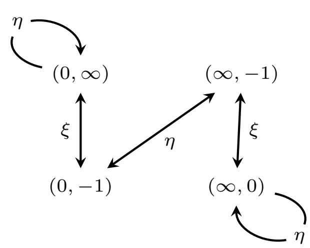
We will also need the following lemma

Lemma 5.2. *Both η and ξ have at least one fixed point.*

Proof. Using the quadratic formula, it is easy to check that $K(x, 1/x; z)$ and $K(1/y, y^2; z)$ both have solutions in $\mathbb{C}^\times$. Let x_0 resp. y_0 be such a solution. Then it follows immediately that $\xi(x_0, 1/x_0) = (x_0, 1/x_0)$ and $\eta(1/y_0, y_0^2) = (1/y_0, y_0^2)$. □

In order to extend η and ξ to the completion $\overline{T_z}$ we will look at the behavior near the completion points.

Proposition 5.3. *We have for the points $(0,\infty)$, $(\infty, 0)$, $(0,-1)$ and $(\infty,-1)$:*



Proof. We use the results from Lemma 4.1 to show that

$$\xi(0,\infty) = \xi(x, -1/x^2 + \mathcal{O}(1/x)) = \left(x, \frac{1}{-1+\mathcal{O}(x)}\right) = (x, -1+\mathcal{O}(x)) = (0,-1)$$

$$\xi(0,-1) = \xi(x, -1+\mathcal{O}(x)) = \left(x, \frac{1}{-x^2+\mathcal{O}(x^3)}\right) = (x, -1/x^2 + \mathcal{O}(1/x)) = (0,\infty)$$

$$\xi(\infty,-1) = \xi(1/t, -1+\mathcal{O}(t)) = \left(1/t, \frac{1}{-1/t+\mathcal{O}(1)}\right) = (1/t, -t+\mathcal{O}(t^2)) = (\infty, 0)$$

$$\xi(\infty,0) = \xi(1/t, -t^2+\mathcal{O}(t^3)) = \left(1/t, \frac{1}{-1+\mathcal{O}(t)}\right) = (1/t, -1+\mathcal{O}(t)) = (\infty,-1)$$

$$
\begin{aligned}
\eta(0,\infty) &= \eta\left(x,-1/x^2+\mathcal{O}(1/x)\right) = \left(\frac{1}{-1/x+\mathcal{O}(1)},-1/x^2+\mathcal{O}(1/x)\right) \\
&= (-x+\mathcal{O}(x^2),-1/x^2+\mathcal{O}(1/x)) = (0,\infty) \\
\eta(0,-1) &= \eta(x,-1+\mathcal{O}(x)) = \left(\frac{1}{-x+\mathcal{O}(x^2)},-1+\mathcal{O}(x)\right) \\
&= (-1/x+\mathcal{O}(1),-1+\mathcal{O}(x)) = (\infty,-1) \\
\eta(\infty,-1) &= \eta(1/t,-1+\mathcal{O}(t)) = \left(\frac{1}{-1/t+\mathcal{O}(1)},-1+\mathcal{O}(t)\right) \\
&= (-t+\mathcal{O}(t^2),-1+\mathcal{O}(t)) = (0,-1) \\
\eta(\infty,0) &= \eta(1/t,-t^2+\mathcal{O}(t^3)) = \left(\frac{1}{-t+\mathcal{O}(t^2)},-t^2+\mathcal{O}(t^3)\right) \\
&= (-1/t+\mathcal{O}(1),-t^2+\mathcal{O}(t^3)) = (\infty,0)
\end{aligned}
$$

from which Proposition 5.3 follows. □

Corollary 5.4. *The automorphisms η and ξ can be extended to automorphisms on $\overline{T_z}$.*

Next we will look at the group generated by η and ξ.

Proposition 5.5. *The functions η and ξ generate a group of order 8 under composition, which is isomorphic to D_4, the symmetry group of the square.*

Proof. We will show that ξ and $\xi \circ \eta$ satisfy the relations of D_4

$$D_4 = \langle s, r \,|\, r^4 = s^2 = e, srs^{-1} = r^{-1} \rangle \tag{16}$$

with $\xi \leftrightarrow s$ and $\xi \circ \eta \leftrightarrow r$. We start with

$$(\xi \circ \xi)\colon (x,y) \xmapsto{\xi} \left(x, \frac{1}{x^2 y}\right) \xmapsto{\xi} (x,y). \tag{17}$$

So we conclude that $\xi^2 = id$ and thus that $\xi^{-1} = \xi$. Secondly, we have

$$\xi \circ \eta\colon (x,y) \xmapsto{\eta} \left(\frac{1}{xy}, y\right) \xmapsto{\xi} \left(\frac{1}{xy}, x^2 y\right). \tag{18}$$

Using this, we can show that

$$(\xi \circ \eta)^4\colon (x,y) \xmapsto{\xi\circ\eta} \left(\frac{1}{xy}, x^2 y\right) \xmapsto{\xi\circ\eta} \left(\frac{1}{x}, \frac{1}{y}\right) \xmapsto{\xi\circ\eta} \left(xy, \frac{1}{x^2 y}\right) \xmapsto{\xi\circ\eta} (x,y). \tag{19}$$

So we have verified that $(\xi \circ \eta)^4 = id$ and we can see that

$$(\xi \circ \eta)^{-1} \colon (x, y) \mapsto \left(xy, \frac{1}{x^2 y} \right). \tag{20}$$

The last thing to check is

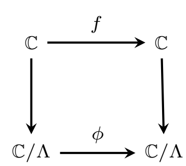
$$\xi \circ (\xi \circ \eta) \circ \xi^{-1} = \xi^2 \circ \eta \circ \xi = \eta \circ \xi \colon (x, y) \xmapsto{\xi} \left(x, \frac{1}{x^2 y} \right) \xmapsto{\eta} \left(xy, \frac{1}{x^2 y} \right). \tag{21}$$

So indeed we have $\xi \circ (\xi \circ \eta) \circ \xi^{-1} = (\xi \circ \eta)^{-1}$. □

We would now like to lift these automorphisms to $\mathbb{C}$. In order to do so, we first recall that the map $[t] \mapsto p(t)$ defines an isomorphism between $\mathbb{C}/\Lambda$ and $\overline{T_z}$. Thus we can speak of the automorphisms $\tilde{\eta}$ and $\tilde{\xi}$ on $\mathbb{C}/\Lambda$. Next we state the following theorem regarding holomorphic maps $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$.

Theorem 5.6. *Given a holomorphic function $\phi \colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$. Then there exist a lift $f \colon \mathbb{C} \to \mathbb{C}$ of the form $f \colon t \mapsto at + b$ where $a \in \mathbb{C}$ and $b \in \phi(0) \cap \Pi_\wp$.*

Proof. The proof of this theorem is largely taken from the proof of Theorem 4.1 in [9].
Let $\phi \colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ be holomorphic. Since $\mathbb{C}$ is simply connected we can lift ϕ to a holomorphic map $f \colon \mathbb{C} \to \mathbb{C}$ with $[f(0)] = \phi(0)$ so that the following diagram commutes:



and thus

$$f(t + w) \equiv f(t) \mod \Lambda, \quad \text{for all } w \in \Lambda, t \in \mathbb{C}. \tag{22}$$

Using the discreteness of Λ we can conclude that the difference $f(t + w) - f(t)$ is constant. Differentiating, we find that

$$f'(t + w) = f'(t), \quad \text{for all } w \in \Lambda, t \in \mathbb{C}. \tag{23}$$

We thus conclude that f' is a holomorphic elliptic function. As a consequence of Liouville's theorem it follows that f' must be constant. Thus $f\colon t \mapsto at + b$ for some $a, b \in \mathbb{C}$. From $[f(0)] = \phi(0)$ it follows that $b \in \phi(0)$, so we can choose $b = \phi(0) \cap \Pi_\wp$. □

Corollary 5.7. *Given a holomorphic function $\phi\colon \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$, such that $\phi \neq id$ and $\phi^2 = id$. Moreover, assume ϕ has a fixed point $[t_0]$. Then there exist a lift $f\colon \mathbb{C} \to \mathbb{C}$ of the form $f\colon t \mapsto -t + b$, with $b \in \phi(0) \cap \Pi_\wp$.*

Proof. From Theorem 5.6 it follows that there exists a lift $f\colon t \mapsto at + b$ of ϕ. From $\phi^2 = id$ it follows then that $f^2(t) \equiv t \mod \Lambda$, and thus that $a = \pm 1$. If $a = 1$ we find for the fixed point $[t_0]$ that $f(t_0) = t_0 + b \equiv t_0 \mod \Lambda$. But then $b \in \Lambda$ and thus $\phi([t]) = [f(t)] = [t]$. So we must have $a = -1$. □

We will use Corollary 5.7 in order to prove the following

Theorem 5.8. *The maps defined by*

$$\eta^*\colon t \mapsto -t, \quad \xi^*\colon t \mapsto -t + 3w_1/4 \tag{24}$$

are lifts of the automorphisms η and ξ to $\mathbb{C}$.

Proof. We know from Lemma 4.4 that $p(0) = (0, \infty)$. Combined with the fact that $\eta(0, \infty) = (0, \infty)$ we conclude that $\tilde{\eta}(0) = 0$. So by applying Corollary 5.7, we thus have a lift $\eta^*\colon t \mapsto -t$.

Next, we know that $\xi(0, \infty) = (0, -1)$ so, using Lemma 4.4 we conclude that $\tilde{\xi}(0) = [3/4w_1]$. So by applying Corollary 5.7, we thus have a lift $\eta^*\colon t \mapsto -t + 3w_1/4$. □

6 The regions where $|x| < 1$ and $|y| < 1$

We define $x(t)$ and $y(t)$ to be the coordinate functions of $\overline{T_z}$, i.e. $p(t) = (x(t), y(t))$. We have the following

Lemma 6.1. *The functions $x(t)$ and $y(t)$ are elliptic with periods w_1 and w_2 and are both of order two. Moreover, $y(t)$ has a double pole at 0 and double zero* *at* $1/2w_1$ *and* $x(t)$ *has two simple poles at* $w_1/4$ *and* $w_1/2$ *and two simple zeros at* 0 *and* $3w_1/4$*.*

Proof. From the definition of ψ (and thus p) both are rational function in $\wp$ and $\wp'$ and are thus elliptic. Secondly, given any x (or y), there are at most two solution to $K(x, y; z) = 0$, from which we can conclude that both must be of order two. From Lemma 4.1 we conclude that in $\Pi_\wp$, $y(t)$ only has one pole/zero (and they are thus double) and $x(t)$ has two simple poles/zeros. The last part then follows from Lemma 4.4. $\square$

Lemma 6.2. *We have*

$$x(-t + 3w_1/4) = x(t), \quad x(-t) = \frac{1}{x(t)y(t)}$$
$$y(-t + 3w_1/4) = \frac{1}{x(t)^2 y(t)}, \quad y(-t) = y(t)$$

Proof. This follows immediately by noting that

$$(p \circ \eta^*)(t) = (\eta \circ p)(t) = \eta(x(t), y(t)) = \left(\frac{1}{x(t)y(t)}, x(t)\right)$$
$$(p \circ \xi^*)(t) = (\xi \circ p)(t) = \xi(x(t), y(t)) = \left(x(t), \frac{1}{x(t)^2 y(t)}\right)$$

and the definition of η^* and ξ^*. $\square$

The first thing we will look at is where $x(t), y(t) \in \overline{\mathbb{R}}$.

Lemma 6.3. *We have*

$$x(t) \in \overline{\mathbb{R}} \Longleftrightarrow t = iq + \left(\frac{n}{2} + \frac{3}{8}\right) w_1 \vee t = q + \frac{n}{2} w_2, \quad q \in \mathbb{R}, n \in \mathbb{Z}.$$

and

$$y(t) \in \overline{\mathbb{R}} \Longleftrightarrow t = iq + \frac{n}{2} w_1 \vee t = q + \frac{n}{2} w_2, \quad q \in \mathbb{R}, n \in \mathbb{Z}.$$

Proof. First of all, from ψ it follows that $y(t) \in \overline{\mathbb{R}} \Leftrightarrow \wp(t) \in \overline{\mathbb{R}}$. Next, define $v \colon t \mapsto t + 3/8w_1$. We then get that

$$(x \circ v)(t) = (x \circ \xi^* \circ v)(t) = x(-t + 3/8w_1) = (x \circ v)(-t) \tag{25}$$

Thus $(x \circ v)(t)$ is even. It has two simple poles, one at $w_1/8$ and one at $-w_1/8$ and two simple zeros, one at $3w_1/8$ and one at $-3w_1/8$. A common result from the theory of elliptic function now state that

$$(x \circ v)(t) = C \frac{\wp(t) - \wp(3w_1/8)}{\wp(t) - \wp(w_1/8)}, \quad C \in \mathbb{C}. \tag{26}$$

Furthermore, from ψ and Lemma 4.3 we find $t \in \mathbb{R} \Leftrightarrow \wp(t) \in \overline{\mathbb{R}} \Leftrightarrow y(t) \in \overline{\mathbb{R}} \Rightarrow x(t) \in \overline{\mathbb{R}} \Rightarrow C \in \mathbb{R}$. We thus conclude $(x \circ v)(t) \in \overline{\mathbb{R}} \Leftrightarrow \wp(t) \in \overline{\mathbb{R}}$. The results then follow from Lemma 4.3. $\square$

The next thing we will show is where $|x(t)| = 1$ and $|y(t)| = 1$. For this we first need

Proposition 6.4. *We have*

$$|x(t)| = 1 \wedge x(t) \notin \mathbb{R} \Longleftrightarrow x(-t) \in \mathbb{R} \wedge y(-t) \notin \overline{\mathbb{R}}$$

and

$$|y(t)| = 1 \wedge y(t) \notin \mathbb{R} \Longleftrightarrow y(-t + 3w_1/4) \in \mathbb{R} \wedge x(-t + 3w_1/4) \notin \overline{\mathbb{R}}$$

Proof. It this proof we will frequently use lemma 6.2 without explicit reference. We start with the first claim. Suppose $|x(t)| = 1$ and $x(t) \notin \mathbb{R}$. Then we must have $1/x(t) + x(t) \in (-2, 2)$ and thus from (3) it follows that

$$x(t)y(t) + \frac{1}{x(t)y(t)} = \frac{1}{z} - \frac{1}{x(t)} - x(t) > 2. \tag{27}$$

We thus conclude that $x(-t) = \frac{1}{x(t)y(t)} \in \mathbb{R}$. Since $x(t) \notin \mathbb{R}$ we conclude that $y(-t) = y(t) \notin \overline{\mathbb{R}}$.

Now suppose $\frac{1}{x(t)y(t)} = x(-t) \in \mathbb{R}$ and $y(t) = y(-t) \notin \overline{\mathbb{R}}$. Then we have $x(t) \notin \mathbb{R}$. From (3) it follows that $1/x(t) + x(t) \in \mathbb{R}$ and thus we conclude $|x(t)| = 1$.

For the second claim we use that from (3) it follows that

$$\left(\frac{1}{x(t)\sqrt{y(t)}} + x(t)\sqrt{y(t)} \right) \left(\frac{1}{\sqrt{y}} + \sqrt{y} \right) = \frac{1}{z} > 4. \tag{28}$$

Assume that $|y(t)| = 1$ and $y(t) \notin \mathbb{R}$. Then we also have that $|\sqrt{y(t)}| = 1$ and $\sqrt{y(t)} \notin \mathbb{R}$. From this it follows that $1/\sqrt{y} + \sqrt{y} \in (-2, 2)$ and thus from (28) that $\frac{1}{x(t)\sqrt{y(t)}} + x(t)\sqrt{y(t)} \in (-\infty, -2) \cup (2, \infty)$. We conclude that $\frac{1}{x(t)\sqrt{y(t)}} \in \mathbb{R}$ and thus $y(-t + 3w_1/4) = \frac{1}{x(t)^2 y(t)} \in \mathbb{R}$ and $x(-t + 3w_1/4) = x(t) \notin \overline{\mathbb{R}}$.

Finally, assume that $\frac{1}{x(t)^2 y(t)} = y(-t + 3w_1/4) \in \mathbb{R}$ and $x(t) = x(-t + 3w_1/4) \notin \overline{\mathbb{R}}$. Suppose that $x(t)\sqrt{y(t)} \in i\mathbb{R}$. Then it follows from (28) that we must also have $\sqrt{y(t)} \in i\mathbb{R}$. But then $x(t) \in \mathbb{R}$. Thus $x(t)\sqrt{y(t)} \in \mathbb{R}$. Then we must have $\sqrt{y(t)} \notin \mathbb{R}$. Also, combined with (28) we deduce that $1/\sqrt{y(t)} + \sqrt{y(t)} \in \mathbb{R}_{\neq 0}$ and thus $|\sqrt{y}| = 1$ while $\sqrt{y} \neq \pm i$. We thus conclude that $|y(t)| = 1$ and $y(t) \notin \mathbb{R}$. □

Corollary 6.5. *We have*

$$|x(t)| = 1 \Longleftrightarrow t = iq + \left(\frac{n}{2} - \frac{3}{8}\right) w_1, \quad t \in \mathbb{R}, n \in \mathbb{Z}.$$

and

$$|y(t)| = 1 \Longleftrightarrow t = iq + \left(\frac{n}{2} + \frac{1}{4}\right) w_1, \quad t \in \mathbb{R}, n \in \mathbb{Z}.$$

Proof. This follows by combining Proposition 6.4 with Lemma 6.3. Looking at the region $\Pi_\wp$ and the fact that x and y are of order two it follows that there can be at most four points $t \in \Pi_\wp$ such that $x(t) = \pm 1$ (resp. $y(t) = \pm 1$). Combined with the fact that $\{t \in \mathbb{C}\,,\, |x(t)| = 1\}$ and $\{t \in \mathbb{C}\,,\, |y(t)| = 1\}$ are closed in $\mathbb{C}$ the result follows. □

Corollary 6.6. *We have*

$$|x(t)| < 1 \Longleftrightarrow \Re(t)/w_1 \in (5/8, 9/8)(mod\ \mathbb{Z})$$

and

$$|y(t)| < 1 \Longleftrightarrow \Re(t)/w_1 \in (1/4, 3/4)(mod\ \mathbb{Z})$$

Proof. Not that $\{t \in \mathbb{C}\,,\, |x(t)| = 1\}$ and $\{t \in \mathbb{C}\,,\, |y(t)| = 1\}$ divide $\mathbb{C}$ in regions where $|x(t)| > 1$ and $|x(t)| < 1$, resp. $|y(t)| > 1$ and $|y(t)| < 1$. The result then follows from Corollary 6.5, using that $x(0) = 0$ and $y(w_1/2) = 0$ and the fact that $x(t)$ and $y(t)$ are elliptic. □

The following image gives a short insight in the results from this last result.

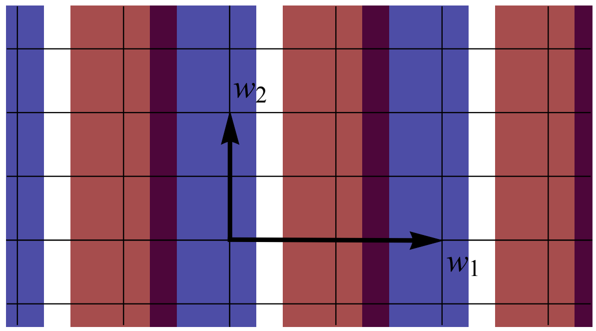


Figure 2: In the above image one can visually see the result of Corollary 6.6. The red stripes are the regions where $|y(t)| < 1$, the blue stripes are the regions where $|x(t)| < 1$ and the purple stripes are the regions where both $|y(t)| < 1$ and $|x(t)| < 1$.

7 Visualizing η^* and ξ^*

In this section we will try to get a better visualization of the results from the previous section. This section will be less formal and is included to get a better understanding of the behavior of the automorphisms discussed and also help to get a better intuitive feeling why some of the results were found. It is therefore possible to skip this section, without missing any crucial results.

First, since we are working with elliptic functions, we reduce our interests to the fundamental parallelogram $\Pi_\wp$. This can be done explicitly by taking the quotient $\mathbb{C}/\Lambda$. This parallelogram can then be made into a torus by identifying the opposite sides and we can then embed it in $\mathbb{R}^3$. We will then look at the result from the automorphisms η and ξ on this torus. By doing so one can actually show that the resulting maps become $(x, y, z) \mapsto (x, -y, -z)$ and $(x, y, z) \mapsto (-y, -x, -z)$. These clearly correspond with regular rotations in $\mathbb{R}^3$. In particular, we have the following image:

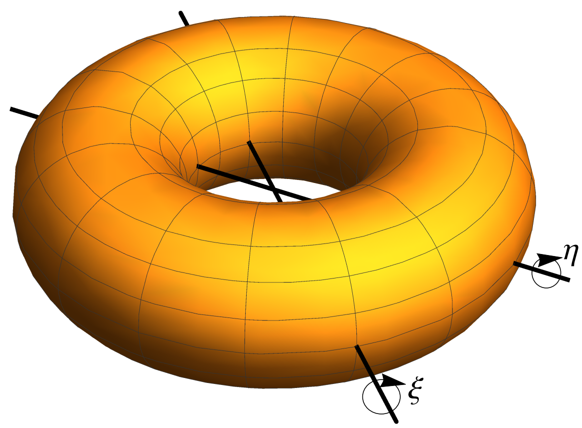


Figure 3: By creating a torus from $\mathbb{C}/\Lambda$ and looking at the resulting maps of η and ξ, we have a more intuitive look at the nature of these automorphisms. For instance that the group they generate is isomorphic to D_4.

From this it also becomes a lot more intuitive that the generated group is isomorphic to D_4. We can go even further by marking the lines where $x(t), y(t) \in \mathbb{R}$ and the lines $|x(t)| = 1$ and $|y(t)| = 1$. This results in the following image:

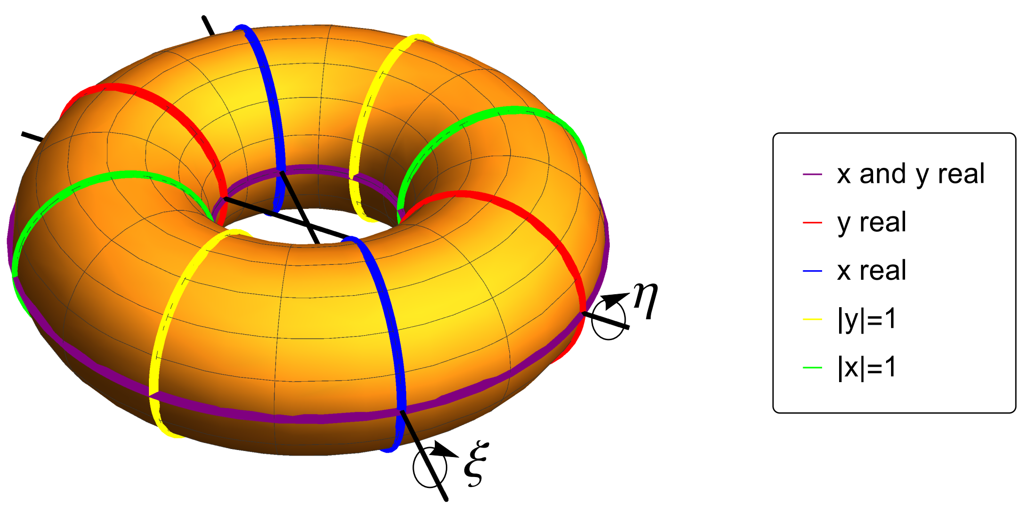


Figure 4: By marking the results from the previous section we get a better insight how specific points are mapped to others by η and ξ.

Finally we can look at the regions where $|x(t)| < 1$ and $|y(t)| < 1$. By keeping track of $(0, \infty)$ and $(\infty, 0)$ this results in the following top-view of these regions on the torus.

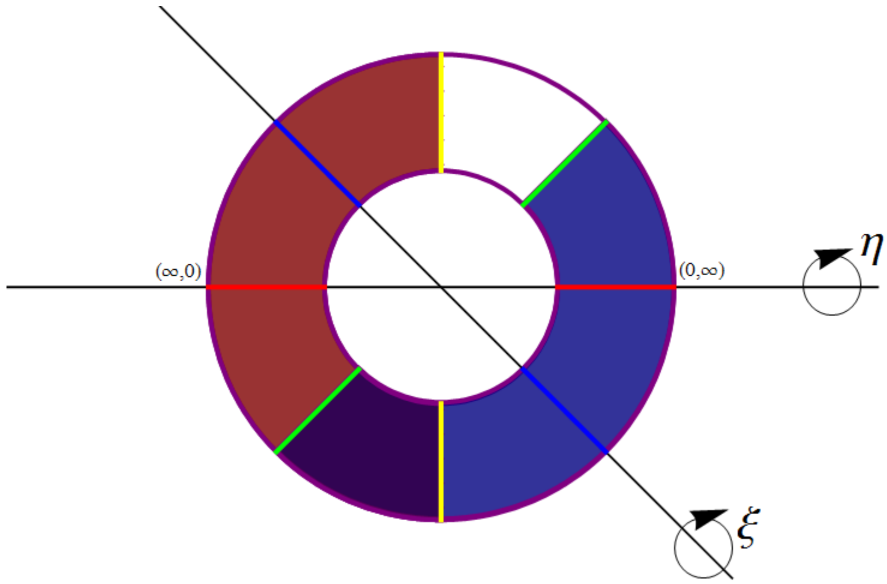


Figure 5: This image shows the regions where $|y(t)| < 1$ and $|x(t)| < 1$ on torus, using the same colors as figure 2. The other interesting points are also marked using the same colors as figure 4

8 The functions r_y and r_x

We begin this section by defining the following regions

Definition Let $\Delta_x \subset \mathbb{C}$ denote the region where $\Re(t) \in (5w_1/8, 9w_1/8)$ and $\Delta_y \subset \mathbb{C}$ denote the region where $\Re(t) \in (w_1/4, 3w_1/4)$

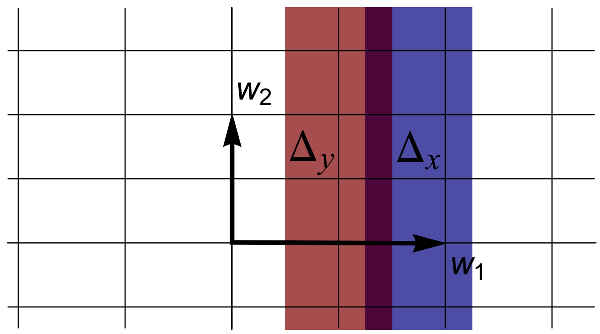


Figure 6: A visualization of Δ_x and Δ_y. Note that they are both one of the stripes for $|x(t)| < 1$ and $|y(t)| < 1$ from figure 2.

From Lemma 6.6 it follows that $|x(t)| < 1$ on Δ_x and $|y(t)| < 1$ on Δ_y. Next, we define the following important functions on these regions

$$r_x(t) = zQ(x(t), 0; z), \quad t \in \Delta_x$$
$$r_y(t) = z(1 + y(t))Q(0, y(t); z), \quad t \in \Delta_y.$$

Notice that we know from Proposition 3.1 that these functions are well-defined on these regions. Note also that $r_x(t + w_2) = r_x(t)$ and $r_y(t + w_2) = r_y(t)$. Furthermore, from (2) we can conclude that

$$r_x(t) + r_y(t) - zQ(0, 0) - x(t)y(t) = 0, \quad t \in \Delta_x \cap \Delta_y. \tag{29}$$

Here we use that $K(x(t), y(t)) = 0$. This allows us to continue the functions meromorphically on $\Delta_x \cup \Delta_y$ by

$$r_x(t) = -r_y(t) + zQ(0, 0; z) + x(t)y(t), \quad t \in \Delta_y$$
$$r_y(t) = -r_x(t) + zQ(0, 0; z) + x(t)y(t), \quad t \in \Delta_x.$$

Proposition 8.1. *We have the following identity*

$$r_x(t + 3w_1/4) - r_x(t) = f_x(t), \quad \Re(t) \in (w_1/4, 3w_1/8) \tag{30}$$

where

$$f_x(t) = y(t)[x(-t) - x(t)]. \tag{31}$$

Proof. Let's take a point $\Re(t_0) \in (w_1/4, 3w_1/8)$. Then $t_0 \in \Delta_y$, $-t_0 + w_1 \in \Delta_x \cap \Delta_y$ and $t_0 + 3w_1/4 \in \Delta_x$. Using the definitions of r_x and r_y, we can then deduce that

$$\begin{aligned}
r_x(t_0 + 3w_1/4)) &= zQ(x(\xi^*(-t_0)), 0; z) = zQ(x(-t_0), 0; z) \\
&= zQ(x(-t_0 + w_1), 0; z) = r_x(-t_0 + w_1) \\
&= -r_y(-t_0 + w_1) + zQ(0, 0; z) + x(-t_0 + w_1)y(-t_0 + w_1) \\
&= -r_y(-t_0 + w_1) + zQ(0, 0; z) + x(-t_0)y(t_0) \\
r_x(t_0) &= -r_y(t_0) + zQ(0, 0; z) + x(t_0)y(t_0) \\
&= -z(1 + y(t_0))Q(0, y(t_0); z) + zQ(0, 0; z) + x(t_0)y(t_0) \\
&= -z(1 + y(-t_0))Q(0, y(-t_0); z) + zQ(0, 0; z) + x(t_0)y(t_0) \\
&= -z(1 + y(-t_0 + w_1))Q(0, y(-t_0 + w_1); z) + zQ(0, 0; z) + x(t_0)y(t_0) \\
&= -r_y(-t_0 + w_1) + zQ(0, 0; z) + x(t_0)y(t_0).
\end{aligned}$$

And thus

$$r_x(t_0 + 3w_1/4) - r_x(t_0) = y(t_0)[x(-t_0) - x(t_0)] = f_x(t_0). \tag{32}$$

Which concludes the proof. □

Using (32), we can continue r_x meromorphically on the whole of $\mathbb{C}$. We will now show that this continuation is actually elliptic.

Proposition 8.2. *The continuation of r_x is elliptic with periods $3w_1$ and w_2.*

Proof. From the continuation of $r_x(t)$ it follows that the functional equation of (32) holds for all $t \in \mathbb{C}$. From this we can show that

$$\begin{aligned}
r_x^*(t + 3w_1) &= r_x^*(t + 9w_1/4) + f_x(t + 9w_1/4) \\
&= r_x^*(t + 6w_1/4) + f_x(t + 6w_1/4) + f_x(t + 9w_1/4) \\
&= r_x^*(t + 3w_1/4) + f_x(t + 3w_1/4) + f_x(t + 6w_1/4) + f_x(t + 9w_1/4) \\
&= r_x(t) + f_x(t) + f_x(t + 3w_1/4) + f_x(t + 6w_1/4) + f_x(t + 9w_1/4) \\
&= r_x(t) + \sum_{n=0}^{3} f_x(t + 3nw_1/4).
\end{aligned}$$

So now our goal is to show that $\sum_{n=0}^{3} f_x(t+3nw_1/4) = 0$. From the definition of f_x and Lemma 6.2 we get

$$\begin{aligned} f_x(t) = y(t)[x(-t) - x(t)] = \frac{1}{x(t)} - x(t)y(t) \\ f_x(t+3w_1/4) &= \frac{1}{x(t+3w_1/4)} - x(t+3w_1/4)y(t+3w_1/4) \\ &= x(t)y(t) - x(t) \\ f_x(t+6w_1/4) &= x(t) - \frac{1}{x(t)y(t)} \\ f_x(t+9w_1/4) &= \frac{1}{x(t)y(t)} - \frac{1}{x(t)}. \end{aligned}$$

From which it follows that $\sum_{n=0}^{3} f_x(t+3nw_1/4) = 0$. So we conclude that $r_x(t+3w_1) = r_x(t)$. Since r_x is w_2 periodic on Δ_x, its continuation will be also. Since we continued r_x meromorphically we thus conclude that it is indeed elliptic with periods $3w_1$ and w_2. □

9 The algebraicity of $Q(x, y; z)$

As a start, we need information about the locations of the poles of r_x in $\Pi_{\wp_{3,1}}$.

Lemma 9.1. *Let $P_{r_x} \subset \Pi_{r_x}$ denote the set of poles of r_x in Π_{r_x}. Then $P_{r_x} \subset$ $\{0, w_1/4, ..., 10w_1/4, 11w_1/4\}$*

Proof. To proof this lemma, we look at the location of its poles of r_x in the domain of its original definition and at the location of the poles of f_x. From the definition of r_x and f_x and by using (32) it follows that r_x can only have a pole whenever $x(t)$ and/or $y(t)$ has a pole or a zero. From Lemma 6.1 we know the only possible locations and the result then follows. □

Secondly, we show the following property of $\wp$

Lemma 9.2. *For any $t_0 \in \{0, w_1/4, w_1/2, 3w_1/4\}$ we have that $\wp(t_0)$ rational in z.*

Proof. This follows immediately from Lemma 4.4 and the values of ψ found in the proof of Theorem 4.2. □

Next, we will need the following theorem. Here $\Lambda(z)$ corresponds to the lattice associated to $g_2(z)$ and $g_3(z)$.

Theorem 9.3. *Given a lattice $\Lambda(z)$. Let f be an elliptic function with this period lattice with poles at $a \in A$. Assume that the coefficients of its Laurent expansion around 0 are all rational in z. Let K be the field generated by $\mathbb{C}(z)$ and $\wp(a)$ for $a \in A$. Then $f \in K(\wp, \wp')$.*

Proof. We follow the proof of Theorem 2.6 of [6].
We divide the proof into three cases of increasing generality.

1. If f is an even function with poles contained in $\Lambda(z)$, then f can be written as a polynomial in $\wp$ with coefficients in $K(z)$ in the following way. Use the series expansion $f(t) = a_{-2n}t^{-2n} + ...$ and $\wp(t) = t^{-2} + ..$ to see that $f - a_{-2n}\wp^n$ is an elliptic function with strictly smaller order than f. Repeat this process until the order is zero. Since all $a_n \in \mathbb{C}(z)$ by assumption, the result follows.

2. If f is an even function with arbitrary poles, then f can be written as a rational function of $\wp$ with coefficients in $K(z)$ in the following way. For each pole $t_j \notin \Lambda(z)$, consider the map $t \mapsto f(t)\left(\wp(t) - \wp(t_j)\right)^{N_j}$, where N_j is an integer large enough to remove the pole at t_j. Doing this for each pole $t_j \notin \Lambda(z)$ leads to an elliptic function whose poles are contained in $\Lambda(z)$, namely
$$f(t)\prod_j \left(\wp(t) - \wp(t_j)\right)^{N_j}. \tag{33}$$
By applying part (1) to this new elliptic function and dividing by the product gives the desired result.

3. If f is an arbitrary elliptic function, then it can be written as $R(\wp) + \wp' S(\wp)$, where R and S are rational functions with coefficients in $K(z)$ in the following way. In general, any function $\mathbb{C} \to \mathbb{C}$ can be decomposed

into its even and odd parts as

$$\begin{aligned} f(t) &= f_{even}(t) + f_{odd}(t) \\ &= 1/2(f(t) + f(-t)) + 1/2(f(t) - f(-t)). \end{aligned}$$

So, we have $f_{even} = R(\wp)$ by part (2). Since f_{odd} and $\wp'$ are both odd functions, their quotient is even and therefore we have $f_{odd} = \wp' S(\wp)$. □

Corollary 9.4. *Let f be as in Theorem 10.2. If moreover all $\wp(a)$ are rational in z, then f is algebraic in z and $\wp$.*

Proof. From the previous theorem it holds that $f \in K(\wp, \wp')$. Combining (5) and the fact that we have $g_2, g_3 \in \mathbb{C}(z)$ we can also deduce that $\wp'$ is algebraic in $\wp$ and z. We conclude that f is the solution to a polynomial where all coefficients are algebraic in z and $\wp$. □

Now we are ready to show that $Q(x, y; z)$ is algebraic.

Theorem 9.5. *The generating function $Q(x, y; z)$ of Gessel walks is algebraic.*

Proof. First we note that clearly $r_x(t)$ is a zero of the following polynomial

$$(X - r_x(t))(X - r_x(t + w_1))(X - r_x(t + 2w_1)) = 0 \tag{34}$$

Notice that the coefficients are elliptic with period w_1 and w_2. Also from Lemma 9.1 it follows that the location of their poles are a subset of $\{0, w_1/4, w_1/2, 3w_1/4\}$. Before we can deduce the final result however, we will present the following lemma without proof. The statement of this lemma seems somewhat intuitive correct, but turns out to be far from trivial.

Lemma 9.6. *All coefficients of the Laurent expansion around* 0 *of the coefficients of* (34) *are rational in z.*

Now we can combine the above with Corollary 9.4 and Lemma 9.2 to conclude that the coefficients of (34) are actually algebraic in z and $\wp$. Finally, from the definition of ψ it follows that $\wp$ is algebraic in z and $x(t)$. Thus r_x is a solution to a polynomial where the coefficients are algebraic in z and $x(t)$. We can thus conclude that $Q(x, 0; z)$ is algebraic in x and z.

The reasoning in these last parts can also be used to conclude that $Q(0, y; z)$ is algebraic in y and z. From the functional equation (2) it then follows that $Q(x, y; z)$ is algebraic in x, y and z. $\square$

10 Appendix

Here we will state some useful properties of the elliptic functions. First we will state some common definitions. These are mostly inspired by [1] and [7].

Definition Let $w_1, w_2 \in \mathbb{C}$ such that $w_1/w_2 \notin \mathbb{R}$. We define the lattice $\Lambda(w_1, w_2)$ generated by w_1 and w_2 as

$$\Lambda(w_1, w_2) = \{mw_1 + nw_2 \,|\, m, n \in \mathbb{Z}\} \tag{35}$$

and we define the fundamental region $\Pi(w_1, w_2)$ as

$$\Pi(w_1, w_2) = \{\alpha w_1 + \beta w_2 \,|\, 0 \leq \alpha, \beta < 1\}. \tag{36}$$

If $w_1, w_2 \in \mathbb{C}$ are clear from context we will usually simply denote $\Lambda \equiv \Lambda(w_1, w_2)$ and $\Pi \equiv \Pi(w_1, w_2)$.

Definition A real lattice is a lattice Λ such that

$$\overline{\Lambda} = \{\overline{w} \,|\, w \in \Lambda\} = \Lambda. \tag{37}$$

Definition A lattice Λ is called real rectangular if $\Lambda = \Lambda(w_1, w_2)$, $w_1 \in \mathbb{R}$ and $w_2 \in i\mathbb{R}$ and real rhombic if $\Lambda = \Lambda(w_1, w_2)$, $\overline{w_1} = w_2$.

Definition A function is called doubly periodic if it has two periods w_1 and w_2 such that $w_1/w_2 \notin \mathbb{R}$. A function is called elliptic if it is doubly periodic and meromorphic.

Definition Given a lattice Λ, we define the Weierstrass elliptic function $\wp$ as

$$\wp(t) = \frac{1}{t^2} + \sum_{\substack{w \in \Lambda \\ w \neq 0}} \frac{1}{(t-w)^2} - \frac{1}{w^2} \tag{38}$$

and for $n \geq 3$ the Eisenstein series of order n as

$$G_n = \sum_{\substack{w \in \Lambda \\ w \neq 0}} \frac{1}{w^n}. \tag{39}$$

Definition We denote by e_1, e_2, e_3 the values of $\wp$ at the half-periods,

$$e_1 = \wp(w_1/2), \quad e_2 = \wp(w_2/2), \quad e_3 = \wp((w_1 + w_2)/2). \tag{40}$$

Definition A meromorphic function $f(t)$ is called real if $\overline{f(t)} = f(\bar{t}), \quad z \in \mathbb{C}$ (here we interpret $\overline{\infty}$ as ∞).

We refer to [1] for proofs about the convergence of $\wp$ and G_n. Next we state some important theorems we will need

Theorem 10.1 ([1],1.8)**.** *The number of zeros of an elliptic function in any period parallelogram is equal to the number of poles, each counted with multiplicity.*

Theorem 10.2 ([1],1.10)**.** *The function $\wp$ defined as above has periods w_1 and w_2. It is analytic except for a double pole at each period $w \in \Lambda$. Moreover $\wp(t)$ is an even function of z.*

Theorem 10.3 ([1],1.12)**.** *The function $\wp$ satisfies the nonlinear differential equation*

$$[\wp'(t)]^2 = 4\wp^3(t) - g_2\wp(t) - g_3 \tag{41}$$

where

$$g_2 = 60G_4, \quad g_3 = 140G_6. \tag{42}$$

Theorem 10.4 ([7],3.16.2)**.** *The following conditions are equivalent*

$$(i) \quad g_2, g_3 \in \mathbb{R};$$

$$(ii) \quad \wp \textit{ is a real function};$$

$$(iii) \quad \Lambda \textit{ is a real lattice.}$$

Theorem 10.5 ([7],3.16.4)**.** *A lattice Λ is real if and only if it is real rectangular or real rhombic.*

Theorem 10.6 ([1],2.9)**.** *Given two complex numbers a_2 and a_3 such that $a_2^3 - 27a_3^2 \neq 0$. Then there exist complex numbers w_1 and w_2 with $w_1/w_2 \notin \mathbb{R}$ such that*

$$g_2(w_1, w_2) = a_2, \quad g_3(w_1, w_2) = a_3. \tag{43}$$

This last theorem is especially import, since it tells us that any elliptic curve $y = 4x^3 - a_2x - a_3$ with $a_2^3 - 27a_3^2 \neq 0$ can be associated by a lattice Λ (this is often called the Uniformization Theorem for Elliptic Curves). We have now all the tools available to prove the following deep result

Theorem 10.7. *Let* $g_2, g_3 \in \mathbb{R}$ *and* $g_2^3 - 27g_3^2 > 0$*. Then the associated lattice* Λ *is real rectangular and*

$$\wp(t) \in \mathbb{R} \cup \{\infty\} \iff z = iq + \frac{n}{2}w_1 \vee z = q + \frac{n}{2}w_2, \quad q \in \mathbb{R}, n \in \mathbb{Z}. \tag{44}$$

Furthermore, $\wp$ *is strictly decreasing on* $]0, 1/2w_1[$ *and strictly increasing on* $]1/2w_1, w_1[$.

Proof. From Theorem 10.4 we have that Λ is a real lattice and that $\wp$ is a real function. From Theorem 10.5 that it is either real rectangular or rhombic. From $g_2^3 - 27g_3^2 > 0$ it follows that the roots of $4y^3 - g_2y - g_3 = 0$ are all real and distinct. If Λ were rhombic we have $e_1 = \wp(w_1/2) = \wp(1/2\overline{w_2}) = \overline{\wp(w_2/2)} = \overline{e_2} = e_2$. So Λ must be real rectangular.
Now, first suppose that $q \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then we have

$$\begin{aligned}\overline{\wp(iq + n/2w_1)} &= \wp(\overline{iq + n/2w_1}) = \wp(-iq + n/2w_1) \\ &= \wp(-iq + n/2w_1 + nw_1) = \wp(-iq - n/2w_1) = \wp(iq + n/2w_1).\end{aligned} \tag{45}$$

Similar we find that

$$\begin{aligned}\overline{\wp(q + n/2w_2)} &= \wp(\overline{q + n/2w_2}) = \wp(q - n/2w_2) \\ &= \wp(q - n/2w_2 + nw_2) = \wp(q + n/2w_2).\end{aligned} \tag{46}$$

Now suppose $\wp(t_0) \in \mathbb{R} \cup \{\infty\}$ for some t_0 not satisfying the right condition of (44). From the periodicity of $\wp$ we can take $t_0 \in \Pi$. Suppose without lose of generality that $t_0 = \alpha w_1 + \beta w_2$ with $0 < \alpha, \beta < 1/2$. Then we have $-t_0 + w_1 + w_2 = (1-\alpha)w_1 + (1-\beta)w_2 \in \Pi$ and $\overline{t_0} + w_2 = \alpha w_1 + (1-\beta)w_2 \in \Pi$. Because $\wp$ is even, real, periodic in w_1 and w_2 and $\wp(t_0) \in \mathbb{R} \cup \{\infty\}$ it follows that

$$\wp(t_0) = \wp(-t_0 + w_1 + w_2) = \wp(\overline{t_0} + w_2). \tag{47}$$

So we have that the elliptic function $\wp(t) - \wp(t_0)$, which has the same periods as $\wp$, has at least three zeros in the period parallelogram Π. But from Theorem

10.2 we know it has only one pole of order two in Π. Combined with Theorem 10.1 this contradicts the existence of at least three zeros in Π. We conclude that such t_0 can not exist.

Finally, we will look at $\wp$ restricted to $]0, w_1[$, where it is a real valued continuous function. We then have $\lim_{t\to 0^+} \wp(t) = +\infty$. Since for the contrary, let's assume that $\lim_{t\to 0^+} \wp(t) = -\infty$. From the equation in Theorem 10.3 we would then have

$$\frac{[\wp'(t)]^2}{\wp(t)} = 4\wp(t)^2 - g_2 - \frac{g_3}{\wp(t)} \tag{48}$$

which leads to a contradiction, since as $t \to 0^+$ the left side goes to $-\infty$, while the right side goes to $+\infty$. From this we can show that $\wp$ is injective on $]0, w_1/2[$ and similarly on $]w_1/2, w_1[$. Suppose we have $t_0, t_1 \in]0, w_1/2[$ with $\wp(t_0) = \wp(t_1)$. Then since $\wp$ is even and periodic it follows that $w_1 - t_0, w_1 - t_1 \in]w_1/2, w_1[$ are also zeros of the elliptic function $\wp(t) - \wp(t_0)$. But this elliptic function can only have at most two roots, so we conclude that $t_0 = t_1$. Since every injective continuous function $f\colon I \to \mathbb{R}$ is strictly monotone and $\lim_{t\to 0+} \wp(t) = +\infty$ we can conclude that $\wp$ is strictly decreasing on $]0, w_1/2[$ and strictly increasing on $]w_1/2, w_1[$. □

Lemma 10.8. *From the conditions of Theorem 10.7 it follows that if $t \in \Pi$ with $\wp(t) \in \mathbb{R}$ and $\wp(t) > e_1$ then $t \in]0, w_1[$ and thus in particular $e_1 > e_2$ and $e_1 > e_3$.*

Proof. Suppose that we have $\wp(t) > e_1$. Then from the intermediate value theorem it follows that there is some $t_0 \in]0, w_1/2[$ and $t_1 \in]w_1/2, w_1[$ such that $\wp(t_0) = \wp(t_1) = \wp(t)$. So, since the elliptic function $\wp(t) - \wp(t_0)$ can have no more than two zeros, these are the only zeros and thus $t \in]0, w_1[$. Since all e_1, e_2, e_3 are distinct combined with definition 10, the last part of the theorem follows immediately from this. □

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