Master's Thesis in Theoretical Physics

ROLE OF COULOMB INTERACTIONS IN WEYL SEMIMETALS :
RENORMALISATION AND SYMMETRY BREAKING

written by

Kitinan Pongsangangan

under the supervision of

Dr. Lars Fritz

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Contents

Abstract

The discovery of graphene has drawn a lot of attention to a new phase of condensed matter systems, Dirac/Weyl semimetals. In these materials, electrons have a linear dispersion relation, making them a solid-state analogue of relativistic massless particles. Dirac/Weyl materials in three dimensions are found to be more stable than their 2D counterparts, e.g. graphene. In this research I focus on the role of Coulomb interactions between electrons within these systems. Using renormalisation group and mean filed theory approaches we investigate how the interaction influences the system parameters' flow and breaks the symmetry spontaneously to create a mass gap.

Acknowledgements

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Chapter 1

Introduction

Electronic properties of many crystalline solids are successfully captured in their band structures, the bands of available energy states for electrons in materials, for example, the band structure that is shown in Fig.1.1. The horizontal axis is labelled momentum states of electrons and curves depict possible energies for each of the momenta. Band structures are derived by assuming that electrons are noninteracting and living in a clean crystal. At zero temperature, the states are filled up to the Fermi energy, ϵ_F whose location determines electrical properties of materials. The highest filled band is called valence band and the lowest unfilled band is called conduction band. The Fermi energy of conductors sits within the band while in insulator it is in between bands, there is a gap between the valence and conduction band of an insulator. The density of states at the the Fermi level of an insulator is vanishing, while that of a conductor is not. [1] [2] Semimetal is a state with properties sitting between a conductor and an insulator. While there is no gap between conduction and valence bands, like in a conductor, the density of states at the Fermi level is vanishing like in an insulator. After the discovery of graphene, the most prominent member of the family, in 2004, Weyl/Dirac Semimetallic materials have been gaining tremendous attention due to their intriguing electric properties and the large potential for applications.[3] [4] More generally, Weyl materials are a group of crystalline solids that have so-called Weyl points in their band structure. A Weyl point is a point in momentum space where the conduction and valence bands touch and around which quasi-particle excitations are massless fermions with linear energy dispersion. The electronic behaviours of these materials are effectively captured by the Weyl Hamiltonian,

$$H = v_F \vec{p} \cdot \vec{\sigma}. \tag{1.1}$$

Here, v_F is the speed of the excitations in materials known as the Fermi velocity and $\vec{p}$ is their momentum, while $\vec{\sigma}$ is the vector of Pauli matrices given by

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2}$$

Weyl materials are semimetal because there is no energy gap between the conduction and valence bands and the density of state at the Fermi level vanishes. While many interesting properties are attributed to the semimetallic feature, driving the system to the different phases brings about many more possible novel applications. For instance, by gapping the Weyl node, it is possible to manipulate the valley degree of freedom for information storage and processing. This leads to the concept of valleytronics, in analogy with the conventional spintronics. [5]

In two dimensional materials, the effective Hamiltonian around the Weyl points can be written as

$$H = v_F (p_x \sigma^x + p_y \sigma^y). \tag{1.3}$$

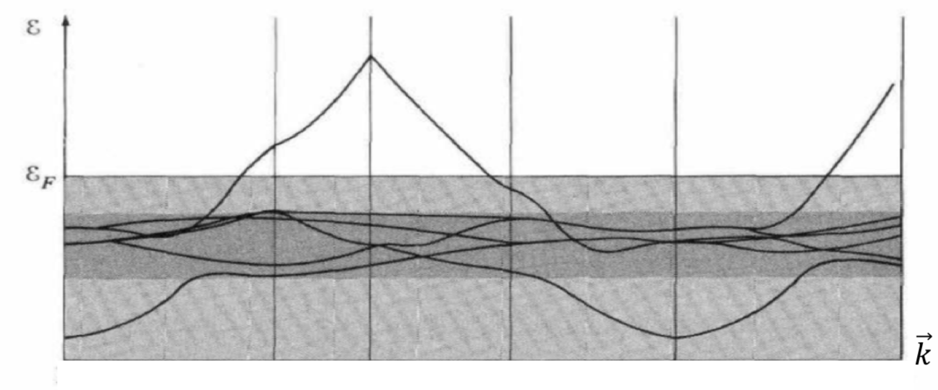

Figure 1.1: Band structure of crystal

The loss of one dimension makes them much more easy to be gapped. By simply adding the mass term , $mv_F^2\sigma^z$, into their Hamiltonian, the Weyl node is removed, a gap is opened in their spectrum, and they are driven from semimetal to the insulating phase.

In contrast, the three-dimension Weyl semimetal is more robust. The effective Hamiltonian is given by

$$H = v_F(p_x\sigma^x + p_y\sigma^y + p_z\sigma^z). \tag{1.4}$$

Since the Hamiltonian consists of all three Pauli matrices, addding $\vec{m}v_F^2\cdot\vec{\sigma}$ into it does not remove the Weyl node to open the mass gap but merely shifts its location in momentum space. Three dimensional Weyl points are protected by topology [5]. In order to gap the three dimension Weyl node, two Weyl cone must be coupled in such a way that effective Hamiltonian of two Weyl cones becomes

$$H = v_F\vec{p}\cdot\vec{\beta}, \tag{1.5}$$

where $\vec{\beta} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$ is composed of the Pauli matrices with the opposite chiralities. The mass term is simply $mv_F^2\beta^0$, where β^0 here is $\begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}$. This mass term might be created in more realistic models with disorder and/or interactions, for example. In this thesis, we aim to study the role of Coulomb interaction in 2D and 3D Weyl systems. It is found that weak Coulomb interaction renormalises parameters of the system while strong coupling is able to remove Weyl points and open mass gaps spontaneously. The effects of electron-electron interaections in 2D Weyl materials are investigated in chapter 2, In chapter 3, the same methods will be applied to discuss interactions in 3D Weyl systems.

Chapter 2

2D Dirac/Weyl Semimetals

In this chapter, we are going to explore the possibility to drive the two-dimensional Weyl semimetal into the insulating phase due to Coulomb interaction among electrons in the system. We first review the electronic properties and the emergence of the Weyl Hamiltonian in a sheet of graphene. Then we look for the field theory of free Weyl electrons and the Coulomb interaction among them. Finally, we implement field theoretical methods including perturbation theory, renormalisation group theory and Schwinger-Dyson Equation to study the phase transition in 2D semimetal.

2.1 Electronic properties of graphene

Graphene is a two dimensional honeycomb lattice of carbon atoms. It is spanned by two lattice generators forming two triangular sublattices A and B as shown in Figure 2.1. When arranged in the hexagonal lattice of a graphene sheet, most of the electrons are tightly bound to the carbon atom except an electron in p^z orbital. The p^z electrons can hop between nearest neighbour sites, and they are responsible for the electronic properties of graphene. The electronic properties of graphene are efficiently captured in the tight binding model [6] [17],

$$H = -t \sum_{\langle ij\rangle,\alpha} a^\dagger_{i,\alpha} b_{j,\alpha} + h.c. \tag{2.1}$$

where $a_{i,\alpha}$ and $b_{i,\alpha}$ are fermionic operators with spin α for sublattice A and B, respectively. Making use of

$$a^\dagger_i = \frac{1}{N}\sum_k a^\dagger_k e^{-ikx_i} \tag{2.2}$$

$$a_i = \frac{1}{N}\sum_k a_k e^{ikx_i} \tag{2.3}$$

and the equivalent equations for fermionic operators of sublattice B, we can Fourier transform the tight binding Hamiltonian to get

$$H = -\frac{t}{N}\sum_{k,\alpha} \phi_k a^\dagger_{k,\alpha} b_{k,\alpha} + h.c. \tag{2.4}$$

$$= \frac{1}{N}\sum_{k,\alpha} \begin{pmatrix} a^\dagger_{k,\alpha} & b^\dagger_{k,\alpha}\end{pmatrix} \begin{pmatrix} 0 & -t\phi_k \\ -t\phi^*_k & 0 \end{pmatrix} \begin{pmatrix} a_{k,\alpha} \\ b_{k,\alpha}\end{pmatrix} \tag{2.5}$$

$$= \frac{1}{N}\sum_{k,\alpha} \bar{\Psi}_{k,\alpha} \begin{pmatrix} 0 & -t\phi_k \\ -t\phi^*_k & 0 \end{pmatrix} \Psi_{k,\alpha} \tag{2.6}$$

where $\phi_k = \sum_{i=1}^{3} e^{i\vec{k}\cdot\vec{\rho}_i}$, N is the number of lattice site, and $\rho_1 = a\hat{y}$, $\rho_2 = \frac{\sqrt{3}}{2}a\hat{x} - \frac{a}{2}\hat{y}$, and $\rho_3 = -\frac{\sqrt{3}}{2}a\hat{x} - \frac{a}{2}\hat{y}$ are nearest neighbour vectors. Diagonalising the Block Hamiltonian (2.6) we get the spectrum of graphene

$$E(\vec{k}) = \pm\,|t\phi_k| \tag{2.7}$$

which is plotted in Figure 2.2. Graphene in it pristine state has one electron per unit cell, hence the Fermi energy is zero. The Fermi surface in a clean graphene is defined by two nonequivalent points in Brillouin zone, $\vec{K} = \frac{4\pi}{3\sqrt{3}a}\hat{x}$ and $\vec{K'} = -\vec{K}$, where the bands touch. [6] [17] In the low energy regime, the properties of the system are governed by the excitations around the Fermi surface. If we expand around $\vec{K}$

$$\begin{aligned} \phi_{K+p} &= \sum_{i=1}^{3} e^{i(\vec{K}+\vec{p})\cdot\vec{\rho}_i} && (2.8)\\ &\approx \sum_{i=1}^{3} e^{i\vec{K}\cdot\vec{\rho}_i}(1 + i\vec{p}\cdot\vec{\rho}_i) && (2.9)\\ &= \frac{3}{2}a(ip_y - p_x), && (2.10)\end{aligned}$$

the tight binding Hamiltonian in momentum space effectively becomes the Weyl Hamiltonian

$$\begin{aligned} H &= \frac{1}{N}\sum_{p,\alpha} \bar{\Psi}_{K+p,\alpha}\left(\begin{smallmatrix} 0 & -\frac{3}{2}at(ip_y-p_x) \\ \frac{3}{2}at(ip_y+p_x) & 0 \end{smallmatrix}\right)\Psi_{K+p,\alpha} && (2.11)\\ &= \frac{1}{N}\sum_{p,\alpha} \bar{\Psi}_{K+p,\alpha} v_F\left(\begin{smallmatrix} 0 & -ip_y+p_x \\ ip_y+p_x & 0 \end{smallmatrix}\right)\Psi_{K+p,\alpha} && (2.12)\\ &= \frac{1}{N}\sum_{p,\alpha} \bar{\Psi}_{K+p,\alpha} v_F\left(p_y\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right) + p_x\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)\Psi_{K+p,\sigma} && (2.13)\\ &= \frac{1}{N}\sum_{p,\alpha} \bar{\Psi}_{p,\alpha} v_F\vec{p}\cdot\vec{\sigma}\ \Psi_{p,\alpha} && (2.14)\end{aligned}$$

with a linear dispersion relation

$$E(\vec{K} + \vec{p}) = \pm\frac{3}{2}at\sqrt{p_x^2 + p_y^2} = \pm\frac{3}{2}at\,|\vec{p}\,| = \pm v_F\,|\vec{p}\,| \tag{2.15}$$

where $v_F \approx 10^6 m/s$ is called Fermi velocity [6]. This is true also around the point $\vec{K'}$. There are two Weyl points in graphene's band structure, both at $\vec{K}$ and $\vec{K'}$.

Later we are going to apply field theoretical techniques such as perturbation theory, renormalisation group analysis and mean field theory to study the phase transition. In the continuum limit, the Hamiltonian becomes

$$H = \int_{|\vec{p}|=0}^{\Lambda\approx\frac{1}{a}} \frac{d\vec{p}}{(2\pi)^2}\bar{\Psi}(\vec{p})v_F\vec{p}\cdot\vec{\sigma}\ \Psi(\vec{p}) \tag{2.16}$$

For technical convenience we change into path integral language. In term of coherent states, the partition function at zero temperature is given by equation (2.17) [7]

$$Z_0 = \int D\bar{\Psi}D\Psi e^{-S_0[\bar{\Psi},\Psi]} \tag{2.17}$$

where the free action

$$S_0[\bar{\Psi},\Psi] = \int d\tau \int_{|\vec{p}|=0}^{\Lambda\approx\frac{1}{a}} \frac{d\vec{p}}{(2\pi)^2}\bar{\Psi}(\vec{p},\tau)(\frac{d}{d\tau} + v_F\vec{p}\cdot\vec{\sigma})\ \Psi(\vec{p},\tau) \tag{2.18}$$

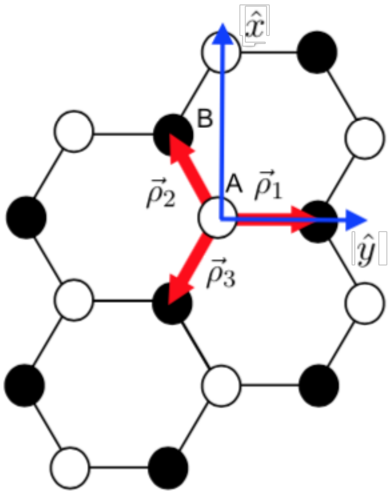

Figure 2.1: Honeycomb lattice of graphene (adapted from [6])

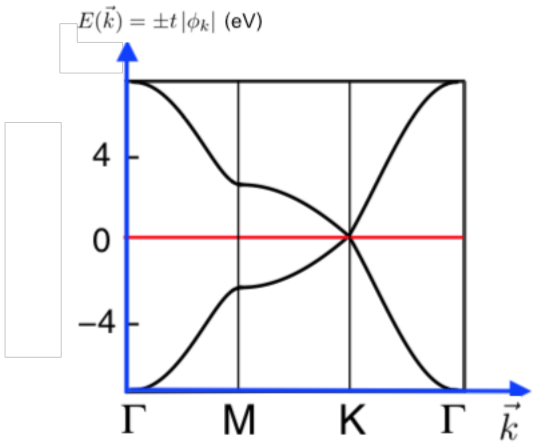

Figure 2.2: Energy spectrum of graphene (adapted from [6])

is written in Euclidean space-time after Wick rotation and results straightforwardly from the above continuum Hamiltonian. After performing Fourier transformations

$$\bar{\Psi}(\vec{p},\tau) = \int \frac{d\omega}{2\pi} \bar{\Psi}(\vec{p},\omega) e^{i\omega\tau} \tag{2.19}$$

$$\Psi(\vec{p},\tau) = \int \frac{d\omega}{2\pi} \Psi(\vec{p},\omega) e^{-i\omega\tau} \tag{2.20}$$

the free action becomes

$$S_0[\bar{\Psi},\Psi] = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{|\vec{p}|=0}^{\Lambda\approx\frac{1}{a}} \frac{d\vec{p}}{(2\pi)^2} \bar{\Psi}(\vec{p},\omega)(-i\omega\mathbb{1} + v_F \vec{p}\cdot\vec{\sigma})\ \Psi(\vec{p},\omega) \tag{2.21}$$

from which the Green function which is the main ingredient for field theoretical analysis can be read off easily and is given by

$$
\begin{aligned}
G_0(\vec{p},\vec{q};\omega,\eta) &= \left\langle \Psi(\vec{p},\omega)\bar{\Psi}(\vec{q},\eta)\right\rangle && (2.22)\\
&= (2\pi)^3\delta(\vec{p}-\vec{q}\,)\delta(\omega-\eta)(-i\omega\mathbb{1}+v\vec{p}\cdot\vec{\sigma})^{-1} && (2.23)\\
&= (2\pi)^3\delta(\vec{p}-\vec{q}\,)\delta(\omega-\eta)\frac{i\omega\mathbb{1}+v\vec{p}\cdot\vec{\sigma}}{\omega^2+v^2\left|\vec{p}\,\right|^2} && (2.24)\\
&= (2\pi)^3\delta(\vec{p}-\vec{q}\,)\delta(\omega-\eta)G_0(\vec{p},\omega) && (2.25)
\end{aligned}
$$

which is true for translationally invariant systems. [8] [11]

2.2 Coulomb interaction

To study the effect of the Coulomb interaction by field theoretical methods, we need to know the Coulomb interaction action. Starting from the electron-electron Coulomb interaction Hamiltonian in the second quantisation formalism it is given by [8]

$$
H_{int} = \frac{1}{2}\int d\vec{r}_1\int d\vec{r}_2\bar{\Psi}(\vec{r}_1)\bar{\Psi}(\vec{r}_2)V_C(\vec{r}_1-\vec{r}_2)\Psi(\vec{r}_2)\Psi(\vec{r}_1). \qquad (2.26)
$$

, Consequently, the action of Coulomb interaction in coordinate space can be written as

$$
S_{int} = \frac{1}{2}\int d\tau\int d\vec{r}_1\int d\vec{r}_2\bar{\Psi}(\vec{r}_1,\tau)\bar{\Psi}(\vec{r}_2,\tau)V_C(\vec{r}_1-\vec{r}_2)\Psi(\vec{r}_2,\tau)\Psi(\vec{r}_1,\tau), \qquad (2.27)
$$

where $V_C(\vec{r}) = \frac{e^2}{4\pi\epsilon|\vec{r}|}$ is Coulomb potential of which the Fourier tranform reads

$$
\begin{aligned}
V_C(\vec{q}) &= \int d\vec{r}\ V_C(\vec{r})e^{i\vec{q}\cdot\vec{r}} && (2.28)\\
&= \frac{e^2}{4\pi\epsilon}\int_0^\infty dr\int_0^{2\pi} rd\theta\frac{1}{r}e^{iqrcos(\theta)}\\
&= \frac{e^2}{4\pi\epsilon}\int_0^\pi d\theta\int_{-\infty}^\infty dre^{iqrcos(\theta)}\\
&= \frac{e^2}{4\pi\epsilon}\int_0^\pi d\theta 2\pi\delta(qcos(\theta))\\
&= \frac{2\pi e^2}{4\pi\epsilon\left|\vec{q}\,\right|}. && (2.29)
\end{aligned}
$$

The positive divergence of the zero mode ($|\vec{q}| = 0$) is compensated by the negative divergence contributions from positive charges background [9], hence Coulomb potential in a momentum space is defined by

$$
V_C(\vec{q}) = \begin{cases} 0 & ; \quad \vec{q} = 0 \\ \frac{2\pi e^2}{4\pi\epsilon|\vec{q}\,|} & ; \quad \text{otherwise.} \end{cases} \qquad (2.30)
$$

Performing the Fourier transformation

$$
\begin{aligned}
\bar{\Psi}(\vec{r},\tau) &= \frac{1}{(2\pi)^2}\int d\vec{p}\ \bar{\Psi}(\vec{p},\omega)e^{-i\vec{p}\cdot\vec{r}+i\omega\tau} && (2.31)\\
\Psi(\vec{r},\tau) &= \frac{1}{(2\pi)^2}\int d\vec{p}\ \Psi(\vec{p},\omega)e^{i\vec{p}\cdot\vec{r}-i\omega\tau} && (2.32)\\
V_C(\vec{r}) &= \frac{1}{(2\pi)^2}\int d\vec{q}\ V_C(\vec{q})e^{i\vec{q}\cdot\vec{r}} && (2.33)
\end{aligned}
$$

on the above interacting action , one gets the Coulomb interaction action in Fourier space written as

$$
\begin{aligned}
S_{int}[\bar{\Psi},\Psi] \;=\; & \frac{1}{2}\frac{e^2}{4\pi\epsilon}\prod_{i=1}^{4}\int_{\omega_i=-\infty}^{\omega_i=\infty}\frac{d\omega_i}{2\pi}\int_{|\vec{p}_i|=0}^{\Lambda}\frac{d\vec{p}_i}{(2\pi)^2}\int_{|\vec{q}|=0}^{\Lambda}\frac{d\vec{q}}{(2\pi)^2}\frac{2\pi}{|\vec{q}\,|} \\
& \bar{\Psi}(\vec{p}_1,\omega_1)\bar{\Psi}(\vec{p}_2,\omega_2)\Psi(\vec{p}_3,\omega_3)\Psi(\vec{p}_4,\omega_4)(2\pi)^2\delta(\vec{p}_1-\vec{p}_4-\vec{q}) \\
& (2\pi)^2\delta(\vec{p}_2-\vec{p}_3+\vec{q})(2\pi)\delta(\omega_1+\omega_2-\omega_3-\omega_4) \qquad (2.34)
\end{aligned}
$$

In the following we will discuss effects of Coulomb interaction in two limits, weak and strong.

2.3 Renormalisation group analysis

2.3.1 First order pertubative approximation

In this section, we study the effects of Coulomb interaction in 2D Weyl semimetal by the renormalisation group analysis. We start with lowest order perturbation theory, the simplest and crudest approximation, of the interaction. The better approximation will be explored in the next subsection. The object we are looking at in the renormalisation group analysis, is the partition function of the system. At zero temperature, the low-energy effective partition function of graphene with electron-electron Coulomb interaction is given by

$$
Z = \int D\bar{\Psi}D\Psi e^{-S_0[\bar{\Psi},\Psi]-S_{int}[\bar{\Psi},\Psi]} \qquad (2.35)
$$

where

$$
S_0[\bar{\Psi},\Psi] = \int dt\int_{|\vec{p}|=0}^{\Lambda\approx\frac{1}{a}}\frac{d\vec{p}}{(2\pi)^2}\bar{\Psi}(\vec{p},t)(\frac{d}{dt}+v_F\vec{p}\cdot\vec{\tau})\;\Psi(\vec{p},t) \qquad (2.36)
$$

$$
\begin{aligned}
S_{int}[\bar{\Psi},\Psi] \;=\; & \frac{1}{2}\frac{e^2}{4\pi\epsilon}\prod_{i=1}^{4}\int_{\omega_i=-\infty}^{\omega_i=\infty}\frac{d\omega_i}{2\pi}\int_{|\vec{p}_i|=0}^{\Lambda}\frac{d\vec{p}_i}{(2\pi)^2}\int_{|\vec{q}|=0}^{\Lambda}\frac{d\vec{q}}{(2\pi)^2}\frac{2\pi}{|\vec{q}\,|} \\
& \bar{\Psi}(\vec{p}_1,\omega_1)\bar{\Psi}(\vec{p}_2,\omega_2)\Psi(\vec{p}_3,\omega_3)\Psi(\vec{p}_4,\omega_4)(2\pi)^2\delta(\vec{p}_1-\vec{p}_4-\vec{q}) \\
& (2\pi)^2\delta(\vec{p}_2-\vec{p}_3+\vec{q})(2\pi)\delta(\omega_1+\omega_2-\omega_3-\omega_4) \qquad (2.37)
\end{aligned}
$$

The Renormalisation group analysis consists of the following three steps [8]

1. Coarse-Graining and integrating out of high energy modes: The coarse-graining of the partition function is to separate the fermionic fields of momenta greater than new cut-off $\frac{\Lambda}{b}$ where $b > 1$ from the lesser ones, i.e.,

$$
\bar{\Psi}(\vec{p},\omega) = \begin{cases} \bar{\Psi}_>(\vec{p},\omega) & ; \quad \frac{\Lambda}{b} < |\vec{p}| < \Lambda \\ \bar{\Psi}_<(\vec{p},\omega) & ; \quad 0 < |\vec{p}| < \frac{\Lambda}{b} \end{cases} \qquad (2.38)
$$

and

$$
\Psi(\vec{p},\omega) = \begin{cases} \Psi_>(\vec{p},\omega) & ; \quad \frac{\Lambda}{b} < |\vec{p}| < \Lambda \\ \Psi_<(\vec{p},\omega) & ; \quad 0 < |\vec{p}| < \frac{\Lambda}{b} \end{cases} \qquad (2.39)
$$

The greater $(\bar{\Psi}_>(\vec{p},\omega),\Psi_>(\vec{p},\omega))$ and lesser fermionic fields $(\bar{\Psi}_<(\vec{p},\omega),\Psi_<(\vec{p},\omega))$ in the free part of the action are decoupled, i.e.,

$$
S_0[\bar{\Psi},\Psi] = S_0[\bar{\Psi}_>,\Psi_>] + S_0[\bar{\Psi}_<,\Psi_<] \qquad (2.40)
$$

where

$$S_0[\bar{\Psi}_>, \Psi_>] = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{|\vec{p}|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\vec{p}}{(2\pi)^2} \bar{\Psi}_>(\vec{p},\omega)(-i\omega + v_F \vec{p}\cdot\vec{\tau})\ \Psi_>(\vec{p},\omega) \tag{2.41}$$

$$S_0[\bar{\Psi}_<, \Psi_<] = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{|\vec{p}|=0}^{\frac{\Lambda}{b}} \frac{d\vec{p}}{(2\pi)^2} \bar{\Psi}_<(\vec{p},\omega)(-i\omega + v_F \vec{p}\cdot\vec{\tau})\ \Psi_<(\vec{p},\omega) \tag{2.42}$$

,whereas, in the interacting part, they are not, i.e.,

$$\begin{aligned} S_{int}[\bar{\Psi}_<, \Psi_<, \bar{\Psi}_>, \Psi_>] \quad = \quad & \frac{1}{2}\frac{e^2}{4\pi\epsilon} \prod_{i=1}^{4} \int_{\omega_i=-\infty}^{\omega_i=\infty} \frac{d\omega_i}{2\pi} \left(\int_{|\vec{q}|=0}^{\Lambda} \frac{d\vec{q}}{(2\pi)^2} \frac{2\pi}{|\vec{q}\,|} \right) \\ & \prod_{i=1}^{2} \left(\int_{|\vec{p}_i|=0}^{\frac{\Lambda}{b}} \frac{d\vec{p}_i}{(2\pi)^2} \bar{\Psi}_<(\vec{p}_i,\omega_i) + \int_{|\vec{p}_i|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\vec{p}_i}{(2\pi)^2} \bar{\Psi}_>(\vec{p}_i,\omega_i) \right) \\ & \prod_{i=3}^{4} \left(\int_{|\vec{p}_i|=0}^{\frac{\Lambda}{b}} \frac{d\vec{p}_i}{(2\pi)^2} \Psi_<(\vec{p}_i,\omega_i) + \int_{|\vec{p}_i|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\vec{p}_i}{(2\pi)^2} \Psi_>(\vec{p}_i,\omega_i) \right) \\ & (2\pi)^2\delta(\vec{p}_1 - \vec{p}_4 - \vec{q})(2\pi)^2\delta(\vec{p}_2 - \vec{p}_3 + \vec{q}) \\ & (2\pi)\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \end{aligned} \tag{2.43}$$

The partition function then becomes

$$\begin{aligned} Z \quad &= \quad \int D\bar{\Psi}_< D\Psi_< e^{-S_0[\bar{\Psi}_<,\Psi_<]} \int D\bar{\Psi}_> D\Psi_> e^{-S_0[\bar{\Psi},\Psi]} e^{-S_{int}[\bar{\Psi}_<,\Psi_<,\bar{\Psi}_>,\Psi_>]} \\ &= \quad \int D\bar{\Psi}_< D\Psi_< e^{-S_0[\bar{\Psi}_<,\Psi_<]} Z_{0>} \left\langle e^{-S_{int}[\bar{\Psi}_<,\Psi_<,\bar{\Psi},_>\Psi_>]} \right\rangle_> \\ &= \quad \int D\bar{\Psi}_< D\Psi_< e^{-S_0[\bar{\Psi}_<,\Psi_<]} e^{\left\langle ln(e^{-S_{int}[\bar{\Psi}_<,\Psi_<,\bar{\Psi}_>,\Psi_>]}) \right\rangle_>} \\ &= \quad \int D\bar{\Psi}_< D\Psi_< e^{-S_0[\bar{\Psi}_<,\Psi_<]} e^{-\langle S_{int}\rangle_> + \frac{1}{2}(\left\langle S_{int}^2 \right\rangle_> - \langle S_{int}\rangle_>^2) + O(S_{int}^3)} \end{aligned} \tag{2.44}$$

Consider the first order perturbation of the partition function in the equation (3.21). Expand equation (3.20) and apply Wick's theorem to find the correlation function with respect to the greater degrees of freedom, the only non-vanishing contributions is given by equation (3.22)

$$\left\langle S_{int}[\bar{\Psi}_<, \Psi_<, \bar{\Psi}_>, \Psi_>] \right\rangle_> = \langle S_{int}\rangle_{>,1} + \langle S_{int}\rangle_{>,2} \tag{2.45}$$

where

$$\begin{aligned} \langle S_{int}\rangle_{>,1} \quad = \quad & \frac{1}{2}\frac{e^2}{4\pi\epsilon} \prod_{i=1}^{4} \int_{\omega_i=-\infty}^{\omega_i=\infty} \frac{d\omega_i}{2\pi} \int_{|\vec{p}_i|=0}^{\frac{\Lambda}{b}} \frac{d\vec{p}_i}{(2\pi)^2} \int_{|\vec{q}|=0}^{\Lambda} \frac{d\vec{q}}{(2\pi)^2} \frac{2\pi}{|\vec{q}\,|} \bar{\Psi}_<(\vec{p}_1,\omega_1) \\ & \bar{\Psi}_<(\vec{p}_2,\omega_2)\Psi_<(\vec{p}_3,\omega_3)\Psi_<(\vec{p}_4,\omega_4)(2\pi)^2\delta(\vec{p}_1 - \vec{p}_4 - \vec{q}) \\ & (2\pi)^2\delta(\vec{p}_2 - \vec{p}_3 + \vec{q})(2\pi)\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \end{aligned} \tag{2.46}$$

and

$$\begin{aligned} \langle S_{int}\rangle_{>,2} \quad = \quad & 2\frac{1}{2}\frac{e^2}{4\pi\epsilon} \prod_{i=1}^{4} \int_{\omega_i=-\infty}^{\omega_i=\infty} \frac{d\omega_i}{2\pi} \int_{|\vec{p}_i|=0}^{\frac{\Lambda}{b}} \frac{d\vec{p}_i}{(2\pi)^2} \int_{|\vec{q}|=0}^{\Lambda} \frac{d\vec{q}}{(2\pi)^2} \frac{2\pi}{|\vec{q}\,|} \bar{\Psi}_<(\vec{p}_2,\omega_2) \\ & (2\pi)^3\delta(\vec{p}_3 - \vec{p}_1)\delta(\omega_3 - \omega_1)G_{0>}(\vec{p}_3,\omega_3)\Psi_<(\vec{p}_4,\omega_4)(2\pi)^2\delta(\vec{p}_1 - \vec{p}_4 - \vec{q}) \\ & (2\pi)^2\delta(\vec{p}_2 - \vec{p}_3 + \vec{q})(2\pi)\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \end{aligned} \tag{2.47}$$

2. Rescaling: The free action in equation (3.19) and $\langle S_{int}\rangle_{>,1}$ are similar to the original free action of equation (3.1) and Coulomb interaction action, except that the upper cutoff has decreased to $\frac{\Lambda}{b}$. Restoring the cutoff to its original value by rescaling

$$\begin{cases} \vec{p}\,' &= b\vec{p} \\ \omega' &= b\omega \\ \bar{\Psi}'(\vec{p}\,',\omega') &= b^{-2}\bar{\Psi}_<(\vec{p},\omega) \\ \Psi'(\vec{p}\,',\omega') &= b^{-2}\Psi_<(\vec{p},\omega) \end{cases} \tag{2.48}$$

and dropping all primes, the transformed action then becomes identical to the original one.

3. Renormalisation:
Now, consider $\langle S_{int}\rangle_{>,2}$;

$$\begin{aligned}
\langle S_{int}\rangle_{>,2} &= 2\frac{1}{2}\frac{e^2}{4\pi\epsilon}\prod_{i=1}^{4}\int_{\omega_i=-\infty}^{\omega_i=\infty}\frac{d\omega_i}{2\pi}\int_{|\vec{p}_i|=0}^{\frac{\Lambda}{b}}\frac{d\vec{p}_i}{(2\pi)^2}\int_{|\vec{q}|=0}^{\frac{\Lambda}{b}}\frac{d\vec{q}}{(2\pi)^2}\frac{2\pi}{|\vec{q}\,|}\bar{\Psi}_<(\vec{p}_2,\omega_2) \\
&\quad (2\pi)^3\delta(\vec{p}_3-\vec{p}_1)\delta(\omega_3-\omega_1)G_{0>}(\vec{p}_3,\omega_3)\Psi_<(\vec{p}_4,\omega_4)(2\pi)^2\delta(\vec{p}_1-\vec{p}_4-\vec{q}) \\
&\quad (2\pi)^2\delta(\vec{p}_2-\vec{p}_3+\vec{q})(2\pi)\delta(\omega_1+\omega_2-\omega_3-\omega_4) \qquad (2.49) \\
&= \frac{e^2}{4\pi\epsilon}\prod_{i=1}^{4}\int_{\omega_i=-\infty}^{\omega_i=\infty}\frac{d\omega_i}{2\pi}\int_{|\vec{p}_i|=0}^{\frac{\Lambda}{b}}\frac{d\vec{p}_i}{(2\pi)^2}\int_{|\vec{q}|=0}^{\frac{\Lambda}{b}}\frac{d\vec{q}}{(2\pi)^2}\frac{2\pi}{|\vec{q}\,|}\bar{\Psi}_<(\vec{p}_2,\omega_2) \\
&\quad (2\pi)^3\delta(\vec{p}_3-\vec{p}_1)\delta(\omega_3-\omega_1)\frac{i\omega_3+v\vec{p}_3\cdot\vec{\sigma}}{\omega_3^2+v^2\,|\vec{p}_3\,|^2}\Psi_<(\vec{p}_4,\omega_4)(2\pi)^2\delta(\vec{p}_1-\vec{p}_4-\vec{q}) \\
&\quad (2\pi)^2\delta(\vec{p}_2-\vec{p}_3+\vec{q})(2\pi)\delta(\omega_1+\omega_2-\omega_3-\omega_4) \qquad (2.50) \\
&= \frac{e^2}{4\pi\epsilon}\int_{\omega_3=-\infty}^{\omega_3=\infty}\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{d\omega_3}{2\pi}\frac{d\vec{p}_3}{(2\pi)^2}\int_{\omega_4=-\infty}^{\omega_4=\infty}\int_{|\vec{p}_4|=0}^{\frac{\Lambda}{b}}\frac{d\omega_4}{2\pi}\frac{d\vec{p}_4}{(2\pi)^2}\frac{2\pi}{|\vec{p}_3-\vec{p}_4\,|}\bar{\Psi}_<(\vec{p}_4,\omega_4) \\
&\quad \frac{i\omega_3+v\vec{p}_3\cdot\vec{\sigma}}{\omega_3^2+v^2\,|\vec{p}_3\,|^2}\Psi_<(\vec{p}_4,\omega_4) \qquad (2.51) \\
&= \frac{2\pi e^2}{4\pi\epsilon}\int_{\omega_4=-\infty}^{\omega_4=\infty}\int_{|\vec{p}_4|=0}^{\frac{\Lambda}{b}}\frac{d\omega_4}{2\pi}\frac{d\vec{p}_4}{(2\pi)^2}\bar{\Psi}_<(\vec{p}_4,\omega_4)\int_{\omega_3=-\infty}^{\omega_3=\infty}\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{d\omega_3}{2\pi}\frac{d\vec{p}_3}{(2\pi)^2}\frac{1}{|\vec{p}_3-\vec{p}_4\,|} \\
&\quad \frac{i\omega_3+v\vec{p}_3\cdot\vec{\sigma}}{\omega_3^2+v^2\,|\vec{p}_3\,|^2}\Psi_<(\vec{p}_4,\omega_4) \qquad (2.52) \\
&= \frac{2\pi e^2}{4\pi\epsilon}\int_{\omega_4=-\infty}^{\omega_4=\infty}\int_{|\vec{p}_4|=0}^{\frac{\Lambda}{b}}\frac{d\omega_4}{2\pi}\frac{d\vec{p}_4}{(2\pi)^2}\bar{\Psi}_<(\vec{p}_4,\omega_4)\Sigma^{(1)}(\vec{p}_4,\omega_4)\Psi_<(\vec{p}_4,\omega_4) \qquad (2.53)
\end{aligned}$$

where $\Sigma^{(1)}(\vec{p}_4,\omega_4)$ is defined by

$$\Sigma^{(1)}(\vec{p}_4,\omega_4) = \int_{\omega_3=-\infty}^{\omega_3=\infty}\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{d\omega_3}{2\pi}\frac{d\vec{p}_3}{(2\pi)^2}\frac{1}{|\vec{p}_3-\vec{p}_4\,|}\frac{i\omega_3+v\vec{p}_3\cdot\vec{\sigma}}{\omega_3^2+v^2\,|\vec{p}_3\,|^2} \tag{2.54}$$

Since

$$\int_{\omega=-\infty}^{\omega=\infty}\frac{\omega}{\omega^2+v^2\,|\vec{p}_3\,|^2} = 0, \tag{2.55}$$

we get

$$\Sigma^{(1)}(\vec{p}_4,\omega_4) = \int_{\omega_3=-\infty}^{\omega_3=\infty}\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{d\omega_3}{2\pi}\frac{d\vec{p}_3}{(2\pi)^2}\frac{1}{|\vec{p}_3-\vec{p}_4\,|}\frac{v_F\vec{p}_3\cdot\vec{\sigma}}{\omega_3^2+v^2\,|\vec{p}_3\,|^2} \tag{2.56}$$

The form of the integrand in (3.31) suggests that $\Sigma^{(1)}(\vec{p}_4,\omega_4)$ can be written as $\Sigma^{(1)}(\vec{p}_4,\omega_4) =$

$\Sigma_v \vec{p}_4 \cdot \vec{\sigma}$, and

$$\Sigma_v = \frac{1}{2|\vec{p}_4|^2} \mathrm{Tr}(\vec{p}_4 \cdot \vec{\sigma} \Sigma^{(1)}(\vec{p}_4, \omega_4)) \tag{2.57}$$

$$= \int_{\omega_3=-\infty}^{\omega_3=\infty} \int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\omega_3}{2\pi} \frac{d\vec{p}_3}{(2\pi)^2} \frac{1}{2|\vec{p}_4|^2} \frac{1}{|\vec{p}_3 - \vec{p}_4\,|} \frac{v_F \mathrm{Tr}((\vec{p}_4 \cdot \vec{\sigma})(\vec{p}_3 \cdot \vec{\sigma}))}{\omega_3^2 + v^2 |\vec{p}_3\,|^2} \tag{2.58}$$

$$= \int_{\omega_3=-\infty}^{\omega_3=\infty} \int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\omega_3}{2\pi} \frac{d\vec{p}_3}{(2\pi)^2} \frac{1}{2|\vec{p}_4|^2} \frac{1}{|\vec{p}_3 - \vec{p}_4|} \frac{v_F \mathrm{Tr}((\vec{p}_3 \cdot \vec{p}_4)\mathbb{1} + i(\vec{p}_3 \times \vec{p}_4) \cdot \vec{\sigma})}{\omega_3^2 + v^2 |\vec{p}_3\,|^2} \tag{2.59}$$

$$= \int_{\omega_3=-\infty}^{\omega_3=\infty} \int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\omega_3}{2\pi} \frac{d\vec{p}_3}{(2\pi)^2} \frac{1}{2|\vec{p}_4|^2} \frac{1}{|\vec{p}_3 - \vec{p}_4|} \frac{2 v_F \vec{p}_3 \cdot \vec{p}_4}{\omega_3^2 + v^2 |\vec{p}_3\,|^2} \tag{2.60}$$

$$= \int_{\theta=0}^{\pi} d\theta \int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda} \frac{|\vec{p}_3|\, dp_3}{(2\pi)^3} \frac{v_F |\vec{p}_3| |\vec{p}_4| \cos(\theta)}{|\vec{p}_4|^2 |\vec{p}_3 - \vec{p}_4|} \int_{\omega_3=-\infty}^{\omega_3=\infty} d\omega_3 \frac{1}{\omega_3^2 + v^2 |\vec{p}_3\,|^2} \tag{2.61}$$

Making use of the formula

$$\int_{\omega=-\infty}^{\omega=\infty} d\omega \frac{1}{\omega^2 + v^2 |\vec{p}\,|^2} = \frac{\pi}{v_F |\vec{p}|}, \tag{2.62}$$

we obtain

$$\Sigma_v = \int_{\theta=0}^{\pi} d\theta \int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda} \frac{|\vec{p}_3|\, dp_3}{(2\pi)^3} \frac{v_F |\vec{p}_3| |\vec{p}_4| \cos(\theta)}{|\vec{p}_4|^2 |\vec{p}_3 - \vec{p}_4|} \frac{\pi}{v_F |\vec{p}_3|} \tag{2.63}$$

$$= \int_{\theta=0}^{\pi} d\theta \int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda} \frac{|\vec{p}_3|\, dp_3}{(2\pi)^3} \frac{\pi \cos(\theta)}{|\vec{p}_4| \sqrt{|\vec{p}_3|^2 + |\vec{p}_4|^2 - 2|\vec{p}_3||\vec{p}_4| cos(\theta)}} \tag{2.64}$$

$$= \int_{\theta=0}^{\pi} d\theta \int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda} \frac{|\vec{p}_3|\, dp_3}{(2\pi)^3} \frac{\pi \cos(\theta)}{|\vec{p}_4| |\vec{p}_3| \sqrt{1 + \frac{|\vec{p}_4|^2}{|\vec{p}_3|^2} - 2\frac{|\vec{p}_4|}{|\vec{p}_3|} cos(\theta)}} \tag{2.65}$$

$$= \int_{\theta=0}^{\pi} d\theta \int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda} \frac{dp_3}{(2\pi)^3} \frac{\pi \cos(\theta)}{|\vec{p}_4|} \left(1 + \frac{|\vec{p}_4|}{|\vec{p}_3|} cos(\theta) + O(\frac{|\vec{p}_4|^2}{|\vec{p}_3|^2})\right) \tag{2.66}$$

$$= \int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda} \frac{dp_3}{(2\pi)^2} \frac{\pi^2}{2} \frac{1}{|\vec{p}_3|} + O(\frac{|\vec{p}_4|^2}{|\vec{p}_3|^2}) \tag{2.67}$$

$$\approx \frac{1}{16\pi} \left(\ln(\Lambda) - \ln(\frac{\Lambda}{b})\right). \tag{2.68}$$

We can summarise this in

$$\langle S_{int} \rangle_{>,2} = \frac{1}{16\pi} \left(\ln(\Lambda) - \ln(\frac{\Lambda}{b})\right) \frac{2\pi e^2}{4\pi\epsilon} \int_{\omega_4=-\infty}^{\omega_4=\infty} \int_{|\vec{p}_4|=0}^{\frac{\Lambda}{b}} \frac{d\omega_4}{2\pi} \frac{d\vec{p}_4}{(2\pi)^2} \bar{\Psi}_<(\vec{p}_4, \omega_4)(\vec{p}_4 \cdot \vec{\sigma}) \Psi_<(\vec{p}_4, \omega_4). \tag{2.69}$$

We define the dimensionless interaction parameter $\alpha = \frac{e^2}{4\pi\epsilon v_F}$ which measures the strength of Coulomb interaction relative to the kinetic energy of electrons near Weyl points, including the spin degree of freedom and dropping all indices we get

$$\langle S_{int} \rangle_{>,2} = \int_{\omega=-\infty}^{\omega=\infty} \int_{|\vec{p}|=0}^{\frac{\Lambda}{b}} \frac{d\omega}{2\pi} \frac{d\vec{p}}{(2\pi)^2} \bar{\Psi}_<(\vec{p}, \omega) \left(\frac{\alpha v_F}{4} \left(\ln(\Lambda) - \ln(\frac{\Lambda}{b})\right) \vec{p} \cdot \vec{\sigma}\right) \Psi_<(\vec{p}, \omega). \tag{2.70}$$

We then substitute Eq. (3.46) into the partition function, Eq. (3.21), and find that, up to first order perturbation theory, the Fermi velocity in the free action is renormalised, i.e.,

$$S_0 = \int_{\omega=-\infty}^{\omega=\infty} \int_{|\vec{p}|=0}^{\frac{\Lambda}{b}} \frac{d\omega}{2\pi} \frac{d\vec{p}}{(2\pi)^2} \bar{\Psi}_<(\vec{p}, \omega) \left[\left(v_F + \frac{\alpha v_F}{4} \left(\ln(\Lambda) - \ln(\frac{\Lambda}{b})\right)\right) \vec{p} \cdot \vec{\sigma}\right] \Psi_<(\vec{p}, \omega) \tag{2.71}$$

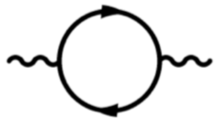

Figure 2.3: Diagrammatic representation of the polarisation function

We define a scale dependent Fermi velocity, $v_F(\Lambda)$, which from Eq. (3.47) obeys [12] [16]

$$v_F\left(\frac{\Lambda}{b}\right) = v_F(\Lambda) + \frac{\alpha v_F(\Lambda)}{4}\left(\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right) \tag{2.72}$$

$$\frac{v_F\left(\frac{\Lambda}{b}\right) - v_F(\Lambda)}{\left(\ln\left(\frac{\Lambda}{b}\right) - \ln(\Lambda)\right)} = -\frac{\alpha v_F(\Lambda)}{4} \tag{2.73}$$

$$\frac{dv_F(\Lambda)}{d\ln(\Lambda)} = -\frac{\alpha v_F(\Lambda)}{4} = -\frac{e^2}{32\pi\epsilon}. \tag{2.74}$$

Additionally, we find the absence of charge renormalisation [6]

$$\frac{de^2}{d\ln(\Lambda)} = 0 \tag{2.75}$$

which leads to an interaction renormalisation

$$\begin{aligned} v_F\frac{d\alpha}{d\ln(\Lambda)} + \alpha\frac{dv_F(\Lambda)}{d\ln(\Lambda)} &= 0 \\ v_F(\Lambda)\frac{d\alpha}{d\ln(\Lambda)} &= -\alpha\frac{dv_F(\Lambda)}{d\ln(\Lambda)} \\ \frac{d\alpha}{d\ln(\Lambda)} &= \frac{\alpha^2}{4}. \end{aligned} \tag{2.76}$$

The beta function of the interaction parameter is positive , $\beta_\alpha = \alpha^2/4$, and vanishes when $\alpha = 0$. In low energy limit, the interaction parameter flows to zero. The interaction is suppressed and the quasiparticle excitations are still free and massless. [6]

2.3.2 Random phase approximation

Usually, the first order perturbation theory is not a good approximation. The quantum effect from loop diagrams always plays a significant role so we will consider loop corrections in this subsection. However, there are infinitely many possible loop diagrams that can be written down, the problem then becomes intractable. In order to proceed, we will assume that only a particular type of diagram contributes to the interaction. Here, we look at the large number of fermion flavours limit where the problem can be simplified. In this limit, only diagrams as shown in Figure 2.4 that have the polarisation function as a building block dominate the other diagrams. This approximation is known as the Random Phase approximation (RPA). There are infinitely many but manageable diagrams.

1. Polarisation function

To study the correction from RPA approximation, we first need to calculate the polarisation function. In 2D, the polarisation function is given by [18]

$$\begin{aligned} \Pi(\omega, \vec{k}) &= N\int \frac{d\nu}{2\pi}\frac{d\vec{q}}{(2\pi)^2}\mathrm{Tr}\left(G_0(\vec{k}+\vec{q}, \nu+\omega)G_0(\vec{q}, \nu)\right) \\ &= N\int \frac{d\nu}{2\pi}\frac{d\vec{q}}{(2\pi)^2}\mathrm{Tr}\left(\frac{i(\nu+\omega) + v_F(\vec{k}+\vec{q})\cdot\vec{\sigma}}{(\nu+\omega)^2 + v_F^2|\vec{k}+\vec{q}\,|^2}\frac{i\nu + v_F\vec{q}\cdot\vec{\sigma}}{\nu^2 + v_F^2\,|\vec{q}\,|^2}\right) \end{aligned} \tag{2.77}$$

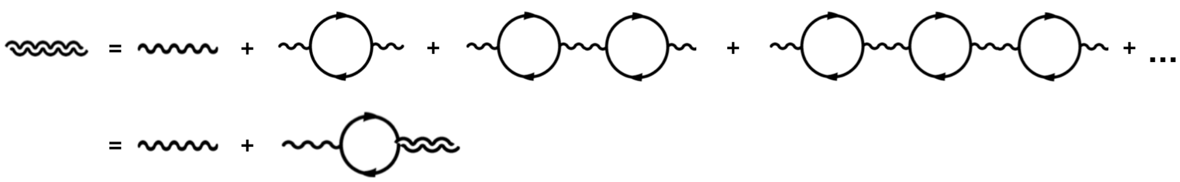

Figure 2.4: Diagrammatic representation of the RPA potential (adapted from [6])

where N is the number of fermion favours. In graphene, for instance, there are 2 spin degrees of freedom for each $\vec{K}$ and $\vec{K}'$ valleys, so in total $N = 4$. [17]
To proceed, consider that

$$\begin{aligned}\Big[(i(\nu+\omega)+v_F(\vec{k}+\vec{q})\cdot\vec{\sigma})(i\nu+v_F\vec{q}\cdot\vec{\sigma})\Big] &= -\nu(\nu+\omega)+iv_F(\nu+\omega)\vec{q}\cdot\vec{\sigma}+v_F(\vec{k}+\vec{q})\cdot\vec{\sigma}\\ &\quad +v_F^2(\vec{k}+\vec{q})\cdot\vec{q}+i((\vec{k}+\vec{q})\times\vec{q})\cdot\vec{\sigma}.\end{aligned}$$

Taking the trace, we get

$$\begin{aligned}\mathrm{Tr}\left(\Big[(i(\nu+\omega)+v_F(\vec{k}+\vec{q})\cdot\vec{\sigma})(i\nu+v_F\vec{q}\cdot\vec{\sigma})\Big]\right) &= -2\nu(\nu+\omega)+2v_F^2(\vec{k}+\vec{q})\cdot\vec{q}\\ &= -2\left\{\nu(\nu+\omega)-v_F^2(\vec{k}+\vec{q})\cdot\vec{q}\right\}. \quad (2.78)\end{aligned}$$

The polarisation function then becomes

$$\Pi(\omega,\vec{k}) = -2N\int\frac{d\nu}{2\pi}\frac{d\vec{q}}{(2\pi)^2}\frac{\nu(\nu+\omega)-v_F^2(\vec{k}+\vec{q})\cdot\vec{q}}{((\nu+\omega)^2+v_F^2|\vec{k}+\vec{q}\,|^2)(\nu^2+v_F^2\,|\vec{q}\,|^2)}. \quad (2.79)$$

Making use of the Feynman trick

$$\frac{1}{AB} = \int_{x=0}^{x=1}dx\frac{1}{(xA+(1-x)B)^2} \quad , \quad (2.80)$$

here $A = (\nu+\omega)^2+v_F^2|\vec{k}+\vec{q}\,|^2$ and $B = \nu^2+v_F^2\,|\vec{q}\,|^2$,
we get

$$\begin{aligned}\Pi(\omega,\vec{k}) &= -2N\int\frac{d\nu}{2\pi}\frac{d\vec{q}}{(2\pi)^2}\int_{x=0}^{x=1}dx\frac{\nu(\nu+\omega)-v_F^2(\vec{k}+\vec{q})\cdot\vec{q}}{(x((\nu+\omega)^2+v_F^2|\vec{k}+\vec{q}\,|^2)+(1-x)(\nu^2+v_F^2\,|\vec{q}\,|^2))^2}\\ &= -2N\int\frac{d\nu}{2\pi}\frac{d\vec{q}}{(2\pi)^2}\int_{x=0}^{x=1}dx\frac{\nu(\nu+\omega)-v_F^2(\vec{k}+\vec{q})\cdot\vec{q}}{\Big[(x\omega+\nu)^2+v_F^2(x\vec{k}+\vec{q})^2+x(1-x)(\omega^2+v_F^2|\vec{k}|^2)\Big]^2}\quad .\end{aligned}$$

Changing variables $\nu\to\nu-x\omega$ and $\vec{q}\to\vec{q}-x\vec{k}$ leads to

$$\begin{aligned}\Pi(\omega,\vec{k}) &= -2N\int\frac{d\nu}{2\pi}\frac{d\vec{q}}{(2\pi)^2}\int_{x=0}^{x=1}dx\frac{(\nu-x\omega)(\nu+\omega-x\omega)-v_F(\vec{k}+\vec{q}-x\vec{k})\cdot(\vec{q}-x\vec{k})}{\Big[\nu^2+v_F^2|\vec{q}|^2+x(1-x)(\omega^2+v_F^2|\vec{k}|^2)\Big]^2}\\ &= = -2N\int\frac{d\nu}{2\pi}\frac{d\vec{q}}{(2\pi)^2}\int_{x=0}^{x=1}dx\frac{\nu^2-x\omega^2+x^2\omega^2+v_F^2x|\vec{k}|^2-v_F^2|\vec{q}|^2-x^2v_F^2|\vec{k}|^2}{\Big[\nu^2+v_F^2|\vec{q}|^2+x(1-x)(\omega^2+v_F^2|\vec{k}|^2)\Big]^2}\end{aligned}$$

and furthermore using

$$\int_{-\infty}^{\infty}dx\frac{x^2}{(Ax^2+B)^2} = \frac{\pi}{2}\frac{1}{\sqrt{A^3B}} \quad ,$$

$$\int_{-\infty}^{\infty} dx \frac{1}{(Ax^2+B)^2} = \frac{\pi}{2}\frac{1}{\sqrt{AB^3}} \quad ,$$
$$\int_0^{\infty} qdq \frac{1}{(Aq^2+B)^{3/2}} = \frac{1}{A\sqrt{B}}$$

we get

$$\begin{aligned}
\Pi(\omega,\vec{k}) &= \frac{-N}{4\pi^2}\int d\vec{q}\int_{x=0}^{x=1} dx \frac{v_F^2|\vec{k}|^2 x(1-x)}{(v_F^2|\vec{q}|^2 + x(1-x)(\omega^2+v_F^2|\vec{k}|^2))^{3/2}} \\
&= \frac{-N}{4\pi^2}\int_{x=0}^{x=1} dx \int_{q=0}^{\infty} qdq \int_{\theta=0}^{2\pi} d\theta \frac{v_F^2|\vec{k}|^2 x(1-x)}{(v_F^2|\vec{q}|^2 + x(1-x)(\omega^2+v_F^2|\vec{k}|^2))^{3/2}} \\
&= \frac{-N}{2\pi}\frac{|\vec{k}|^2}{\sqrt{\omega^2+v_F^2|\vec{k}|^2}}\int_{x=0}^{x=1} dx\sqrt{x(1-x)} \\
&= \frac{-N}{16}\frac{|\vec{k}|^2}{\sqrt{\omega^2+v_F^2|\vec{k}|^2}}. \qquad (2.81)
\end{aligned}$$

Note that the polarisation function is linearly proportional to N, meaning it dominates in the large fermion flavour limit. [10] [11]

2. Random phase approximation analysis

Summing all diagrams in Figure 2.4, Coulomb interaction ($v_F = 1$) is modified and given by [8]

$$\begin{aligned}
V^{RPA}(\vec{p},\eta) &= \frac{1}{V_C^{-1}(\vec{p}) - \Pi(\vec{p},\eta)} \\
&= \frac{2\pi e^2}{4\pi\epsilon|\vec{p}| + \frac{2\pi e^2 N|\vec{p}|^2}{16\sqrt{\eta^2+|\vec{p}|^2}}} \quad . \qquad (2.82)
\end{aligned}$$

Note that $V^{RPA}(\vec{p},\eta)$ is an even function in η and all components of $\vec{p} = (p_x, p_y)$.
Using the RPA potential to calculate the self energy and by virtue of renormalisation group analysis, the limit of integral is from Λ/b to Λ

$$\Sigma^{RPA}(\vec{k},\omega) = -\int_{|\eta|,|\vec{p}|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\eta}{2\pi}\frac{d\vec{p}}{(2\pi)^2} G_0(\vec{k}+\vec{p},\omega+\eta)V^{RPA}(\vec{p},\eta). \qquad (2.83)$$

Since $\vec{k}$ and ω are small compare to $\vec{p}$ and η, we can Taylor expand $G_0(\vec{k}+\vec{p},\omega+\eta)$ up to the first order in $\vec{k}$ and ω

$$\begin{aligned}
G_0(\vec{k}+\vec{p},\omega+\eta) &= G_0(\vec{p},\eta) + \omega\frac{\partial G_0(\vec{p},\eta)}{\partial\eta} + \sum_{i\in x,y} k_i \frac{\partial G_0(\vec{p},\eta)}{\partial p_i} \\
&= G_0(\vec{p},\eta) + \omega\left(\frac{i}{\eta^2+|\vec{p}|^2} - \frac{2i\eta^2}{(\eta^2+|\vec{p}|^2)^2} - \frac{2\eta\vec{p}\cdot\vec{\sigma}}{(\eta^2+|\vec{p}|^2)^2}\right) \\
&\quad + \sum_{i\in x,y} k_i\left(\frac{-2ip_i\eta}{(\eta^2+|\vec{p}|^2)^2} + \sum_{j\in x,y}\frac{\sigma_j\delta_{ij}}{\eta^2+|\vec{p}|^2} - \sum_{j\in x,y}\frac{2\sigma_j p_j p_i}{(\eta^2+|\vec{p}|^2)^2}\right) \qquad (2.84)
\end{aligned}$$

Since $V^{RPA}(\vec{p},\eta)$ is even, all terms in Eq.(3.69) that are odd in η, p_x, and p_y give no contributions to the integral of the RPA self-energy. The RPA self energy the becomes

$$\begin{aligned}
\Sigma^{RPA}(\vec{k},\omega) &= -\int_{|\eta|,|\vec{p}|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\eta}{2\pi}\frac{d\vec{p}}{(2\pi)^2}\Bigg[\omega\left(\frac{i}{\eta^2+|\vec{p}|^2}-\frac{2i\eta^2}{(\eta^2+|\vec{p}|^2)^2}\right) \\
&+\sum_{i\in x,y} k_i\left(\sum_{j\in x,y}\frac{\sigma_j\delta_{ij}}{\eta^2+|\vec{p}|^2}-\sum_{j\in x,y}\frac{2\sigma_j p_j p_i}{(\eta^2+|\vec{p}|^2)^2}\right)\Bigg]V^{RPA}(\vec{p},\eta) \\
&= -\int_{|\eta|,|\vec{p}|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\eta}{2\pi}\frac{d\vec{p}}{(2\pi)^2}\Bigg[\omega\left(\frac{i}{\eta^2+|\vec{p}|^2}-\frac{2i\eta^2}{(\eta^2+|\vec{p}|^2)^2}\right) \\
&+\Bigg(\sum_{i\in x,y}\frac{\sigma_i k_i}{\eta^2+|\vec{p}|^2}-\frac{2k_x\sigma_x p_x p_x}{(\eta^2+|\vec{p}|^2)^2}-\frac{2k_x\sigma_y p_y p_x}{(\eta^2+|\vec{p}|^2)^2}-\frac{2k_y\sigma_x p_x p_y}{(\eta^2+|\vec{p}|^2)^2} \\
&-\frac{2k_y\sigma_y p_y p_y}{(\eta^2+|\vec{p}|^2)^2}\Bigg)\Bigg]V^{RPA}(\vec{p},\eta) \\
&= -\int_{|\eta|,|\vec{p}|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\eta}{2\pi}\frac{d\vec{p}}{(2\pi)^2}\Bigg[\omega\left(\frac{i}{\eta^2+|\vec{p}|^2}-\frac{2i\eta^2}{(\eta^2+|\vec{p}|^2)^2}\right) \\
&+\left(\sum_{i\in x,y}\frac{\sigma_i k_i}{\eta^2+|\vec{p}|^2}-\frac{2k_x\sigma_x p_x^2}{(\eta^2+|\vec{p}|^2)^2}-\frac{2k_y\sigma_y p_y^2}{(\eta^2+|\vec{p}|^2)^2}\right)\Bigg]V^{RPA}(\vec{p},\eta) \\
&= -\int_{|\eta|,|\vec{p}|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\eta}{2\pi}\frac{d\vec{p}}{(2\pi)^2}\Bigg[i\omega\left(\frac{\eta^2+|\vec{p}|^2-2\eta^2}{(\eta^2+|\vec{p}|^2)^2}\right) \\
&+\left(\frac{\sigma_x k_x(\eta^2+|\vec{p}|^2)+\sigma_y k_y(\eta^2+|\vec{p}|^2)-2k_x\sigma_x p_x^2-2k_y\sigma_y p_y^2}{(\eta^2+|\vec{p}|^2)^2}\right)\Bigg]V^{RPA}(\vec{p},\eta) \\
&= -\int_{|\eta|,|\vec{p}|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\eta}{2\pi}\frac{d\vec{p}}{(2\pi)^2}\Bigg[i\omega\left(\frac{|\vec{p}|^2-\eta^2}{(\eta^2+|\vec{p}|^2)^2}\right) \\
&+\left(\frac{\sigma_x k_x\eta^2+\sigma_y k_y\eta^2+\sigma_x k_x p_y^2+\sigma_y k_y p_x^2-k_x\sigma_x p_x^2-k_y\sigma_y p_y^2}{(\eta^2+|\vec{p}|^2)^2}\right)\Bigg]V^{RPA}(\vec{p},\eta)
\end{aligned}$$

Since p_x and p_y are equivalent in the integral, one can simplify to

$$\Sigma^{RPA}(\vec{k},\omega) = -\int_{|\eta|,|\vec{p}|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\eta}{2\pi}\frac{d\vec{p}}{(2\pi)^2}\left[i\omega\left(\frac{|\vec{p}|^2-\eta^2}{(\eta^2+|\vec{p}|^2)^2}\right)+\vec{\sigma}\cdot\vec{k}\left(\frac{\eta^2}{(\eta^2+|\vec{p}|^2)^2}\right)\right]V^{RPA}(\vec{p},\eta)$$

Introducing three-vectors $\mathbf{q}=(q_0,\vec{q})=(\eta,\vec{p})$, we have in spherical coordinate $q_0=|\mathbf{q}|\cos\theta$ and $|\vec{p}\,|=|\mathbf{q}|\sin\theta$, and obtain

$$\Sigma^{RPA}(\vec{k},\omega) = -\int_{|\mathbf{q}|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\mathbf{q}}{(2\pi)^3}\left[\frac{-i\omega}{|\mathbf{q}|^2}\left(\cos^2\theta-\sin^2\theta\right)+\frac{\vec{\sigma}\cdot\vec{k}}{|\mathbf{q}|^2}(\cos^2\theta)\right]V^{RPA}(\vec{p},\eta) \tag{2.85}$$

where

$$V^{RPA}(\vec{p},\eta) = \frac{2\pi e^2}{4\pi\epsilon|\vec{p}|+\frac{2\pi e^2 N|\vec{p}|^2}{16\sqrt{\eta^2+|\vec{p}|^2}}} = \frac{16\lambda/N}{|\mathbf{q}|\sin\theta(1+\lambda\sin\theta)} \quad ;\lambda=\frac{Ne^2}{32\pi\epsilon}. \tag{2.86}$$

This yields,

$$\begin{aligned}\Sigma^{RPA}(\vec{k},\omega) &= -\int_{|\mathbf{q}|=\frac{\Lambda}{b}}^{\Lambda}\frac{d\mathbf{q}}{(2\pi)^3}\left[\frac{-i\omega}{|\mathbf{q}|^2}\left(\cos^2\theta-\sin^2\theta\right)+\frac{\vec{\sigma}\cdot\vec{k}}{|\mathbf{q}|^2}(\cos^2\theta)\right]\frac{16\lambda/N}{|\mathbf{q}|\sin\theta(1+\lambda\sin\theta)}\\ &= -\int_{|\mathbf{q}|=\frac{\Lambda}{b}}^{\Lambda}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{|\mathbf{q}|^2 d|\mathbf{q}|\sin\theta d\theta d\phi}{(2\pi)^3}\left[\frac{-i\omega}{|\mathbf{q}|^2}\left(\cos^2\theta-\sin^2\theta\right)+\frac{\vec{\sigma}\cdot\vec{k}}{|\mathbf{q}|^2}(\cos^2\theta)\right]\\ &\quad \frac{16\lambda/N}{|\mathbf{q}|\sin\theta(1+\lambda\sin\theta)}\\ &= -\frac{8}{N\pi^2}\left(\int_{|\mathbf{q}|=\frac{\Lambda}{b}}^{\Lambda}\frac{d|\mathbf{q}|}{|\mathbf{q}|}\right)\frac{\lambda}{2}\int_{\theta=0}^{\pi}d\theta\left[-i\omega\left(\frac{\cos^2\theta-\sin^2\theta}{(1+\lambda\sin\theta)}\right)+\vec{\sigma}\cdot\vec{k}\left(\frac{\cos^2\theta}{(1+\lambda\sin\theta)}\right)\right].\end{aligned}$$

Making use of the integral formulas [11]

$$F_{0,\lambda}=\frac{\lambda}{2}\int_{\theta=0}^{\pi}d\theta\frac{\cos^2\theta-\sin^2\theta}{(1+\lambda\sin\theta)}=\begin{cases}-\frac{2-\lambda^2}{\lambda\sqrt{1-\lambda^2}}\cos^{-1}\lambda-2+\frac{\pi}{\lambda} & ;\ \lambda<1\\ \frac{\lambda^2-2}{\lambda\sqrt{\lambda^2-1}}\ln(\lambda+\sqrt{\lambda^2-1})-2+\frac{\pi}{\lambda} & ;\ \lambda>1\end{cases} \tag{2.87}$$

$$F_{1.\lambda}=\frac{\lambda}{2}\int_{\theta=0}^{\pi}d\theta\frac{\cos^2\theta}{(1+\lambda\sin\theta)}=\begin{cases}-\frac{\sqrt{1-\lambda^2}}{\lambda}\cos^{-1}\lambda-1+\frac{\pi}{2\lambda} & ;\ \lambda<1\\ \frac{\sqrt{\lambda^2-1}}{\lambda}\ln(\lambda+\sqrt{\lambda^2-1})-1+\frac{\pi}{2\lambda} & ;\ \lambda>1\end{cases} \tag{2.88}$$

and performing the integral in $|\mathbf{q}|$, we get

$$\Sigma^{RPA}(\vec{k},\omega)=-\frac{8}{N\pi^2}\left[\ln(\Lambda)-\ln\left(\frac{\Lambda}{b}\right)\right]\left[-i\omega F_{0,\lambda}+\vec{\sigma}\cdot\vec{k}F_{1,\lambda}\right].$$

Inserting back v_F leads to

$$\Sigma^{RPA}(\vec{k},\omega)=-\frac{8}{N\pi^2}\left[\ln(\Lambda)-\ln\left(\frac{\Lambda}{b}\right)\right]\left[-i\omega F_{0,\lambda}+v_F\vec{\sigma}\cdot\vec{k}F_{1,\lambda}\right]. \tag{2.89}$$

The effect of the self-energy on the Green's function is encoded in Dyson's equation which can be written as

$$G^{-1}(\vec{k},\omega)=G_0^{-1}(\vec{k},\omega)-\Sigma^{RPA}(\vec{k},\omega).$$

Assume that, generically, interacting Green's function $G(\vec{k},\omega)$ and free Green's function, $G_0(\vec{k},\omega)$ are given by

$$\begin{aligned}G^{-1}(\vec{k},\omega) &= Z^{-1}(\Lambda/b)[-i\omega+Z(\Lambda/b)v_F(\Lambda/b)\vec{\sigma}\cdot\vec{k}\ ]\\ G_0^{-1}(\vec{k},\omega) &= Z^{-1}(\Lambda)[-i\omega+Z(\Lambda)v_F(\Lambda)\vec{\sigma}\cdot\vec{k}\ ],\end{aligned}$$

respectively. Z is called quasi-particle residue, and $Z(\Lambda \approx 1/a) = 1$, so

$$\begin{aligned}
\frac{-i\omega + Z(\Lambda/b)v_F(\Lambda/b)\vec{\sigma}\cdot\vec{k}}{Z(\Lambda/b)} &= -i\omega + v_F(\Lambda)\vec{\sigma}\cdot\vec{k} + \frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right] \\
&\quad \times \left[-i\omega F_{0,\lambda} + v_F(\Lambda)\vec{\sigma}\cdot\vec{k}F_{1,\lambda}\right] \\
&= -i\omega\left(1 + F_{0,\lambda}\frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right]\right) \\
&\quad +\vec{\sigma}\cdot\vec{k}v_F(\Lambda)\left(1 + F_{1,\lambda}\frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right]\right) \\
&= \left(1 + F_{0,\lambda}\frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right]\right)\Bigg\{ - i\omega + \vec{\sigma}\cdot\vec{k}v_F(\Lambda)\times \\
&\quad \left(\frac{1 + F_{1,\lambda}\frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right]}{1 + F_{0,\lambda}\frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right]}\right)\Bigg\} \\
&\approx \left(1 + F_{0,\lambda}\frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right]\right)\Bigg\{ - i\omega + \vec{\sigma}\cdot\vec{k}v_F(\Lambda)\times \\
&\quad \left(1 + F_{1,\lambda}\frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right]\right)\left(1 - F_{0,\lambda}\frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right]\right)\Bigg\} \\
&\approx \left(1 + F_{0,\lambda}\frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right]\right)\Bigg\{ - i\omega + \vec{\sigma}\cdot\vec{k}v_F(\Lambda)\times \\
&\quad \left(1 + F_{1,\lambda}\frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right] - F_{0,\lambda}\frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right]\right)\Bigg\}.
\end{aligned}$$

Inserting back $Z(\Lambda)$, the left right hand side of the above equation becomes

$$\begin{aligned}
\text{R.H.S} &= \left(\frac{1 + F_{0,\lambda}\frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right]}{Z(\Lambda)}\right)\Bigg\{ - i\omega + \vec{\sigma}\cdot\vec{k}v_F(\Lambda)\left(\frac{Z(\Lambda)}{1 + F_{0,\lambda}\frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right]}\right) \\
&\quad \times\left(1 + F_{1,\lambda}\frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right] - F_{0,\lambda}\frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right]\right)\Bigg\}. \qquad (2.90)
\end{aligned}$$

Comparing both side of the above equation, we get

$$v_F(\Lambda/b) = v_F(\Lambda)\left\{1 + (F_{1,\lambda} - F_{0,\lambda})\frac{8}{N\pi^2}\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right]\right\}, \qquad (2.91)$$

$$\frac{v_F(\Lambda/b) - v_F(\Lambda)}{\left[\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right]} = v_F(\Lambda)(F_{1,\lambda} - F_{0,\lambda})\frac{8}{N\pi^2},$$

$$\frac{dv_F(\Lambda)}{d\ln(\Lambda)} = -\frac{8}{N\pi^2}(F_{1,\lambda} - F_{0,\lambda})v_F(\Lambda).$$

$F_{1,\lambda} - F_{0,\lambda}$ is positive for all λ, hence the beta function of Fermi velocity is negative , $\beta_v = -\frac{8}{N\pi^2}(F_{1,\lambda} - F_{0,\lambda})v_F(\Lambda)$. In the low energy limit, the Fermi velocity increases toward the speed of light and the interaction parameter which is inversely proportional to Fermi velocity flows to weak coupling. The interaction is suppressed and the quasiparticle excitations are still free and massless . [11] [6]

2.4 Strong coupling

From the renormalisation group analysis, we find a significant result of the low energy behaviour of interacting electrons in 2D Weyl/Dirac materials. If the number of species of electrons is large or the interaction between electrons is very weak, the interaction between low-energy electrons is effectively suppressed and they behave as free and massless quasiparticles. What happen to the system with small number of electron flavours and strong coupling? Will the interaction in this regime be strong enough to generate a gap spontaneously? This is the topic of this section.

The possibility of spontaneous gap generation due to Coulomb interaction in 2D Dirac/Weyl materials, especially graphene, has been analysed extensively.[21][22][23][24][25][26][27] In this section, We will study this problem by finding the non-trivial solution of the so-called Schwinger-Dyson equation of the following action:

$$\begin{aligned} S[\bar{\Psi},\Psi,A_0] &= \int dtd\vec{x}\Big[\bar{\Psi}(\vec{x},t)(\frac{d}{dt}-iv_F\vec{\sigma}\cdot\vec{\bigtriangledown})\Psi(\vec{x},t)-A_0(\vec{x},z=0,t)\bar{\Psi}(\vec{x},t)\Psi(\vec{x},t)\Big] \\ &+\frac{\pi\epsilon}{e^2}\int dtd^2xdz\vec{\bigtriangledown}A_0(\vec{x},z,t)\cdot\vec{\bigtriangledown}A_0(\vec{x},z,t)+\frac{\partial A_0(\vec{x},z,t)}{\partial z}\frac{\partial A_0(\vec{x},z,t)}{\partial z}. \end{aligned}$$

Our original action can be recovered by integrating out A_0 and the two dimensional free propagator of A_0 field in Fourier space is indeed the Coulomb potential in momentum space

$$D(\vec{p},\omega)=\langle A_0(-\vec{p},-\omega)A_0(\vec{p},\omega)\rangle=\frac{2\pi e^2}{4\pi\epsilon|\vec{p}|}. \tag{2.92}$$

Schwinger-Dyson equation is a self-consistent equation relating the interacting Green function to the interaction of the system. The derivation of Schwinger-Dyson equation for the above action is given in Appendix A.

2.4.1 Excitonic mass generation

We are looking for the possibility of spontaneous mass generation due to interaction, so we assume that there is a mass term in the interacting Green function

$$G^{-1}(\vec{a},\alpha)=-i\alpha+\vec{\sigma}\cdot\vec{a}+\Delta(|\vec{a}|)\sigma^z. \tag{2.93}$$

For simplicity, we assume that mass Δ only depends on the magnitude of the momentum and all parameter renormalisations are neglected. The A_0 field propagator is given by the Random phase approximation in the on-shell dynamical approximation, namely

$$D(\vec{p},\omega)=V^{RPA}(\vec{p},\omega=|\vec{p}|)=\frac{2\pi e^2}{\left(4\pi\epsilon+\frac{\sqrt{2}\pi e^2N}{16}\right)|\vec{p}|}. \tag{2.94}$$

Plugging these propagators into the Schwinger-Dyson with the approximation that the vertex function γ is unity equation we get

$$-i\alpha+\vec{\sigma}\cdot\vec{a}+\Delta(|\vec{a}|)\sigma^z=-i\alpha+\vec{\sigma}\cdot\vec{a}+\int\frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi}\frac{2\pi e^2}{\left(4\pi\epsilon+\frac{\sqrt{2}\pi e^2N}{16}\right)|\vec{p}+\vec{a}|}\frac{i\omega+\vec{\sigma}\cdot\vec{p}+\Delta(|\vec{p}|)\sigma^z}{\omega^2+|\vec{p}|^2+\Delta(|\vec{p}|)^2}. \tag{2.95}$$

The coefficient of σ^z gives the self-consistent mass equation [6]

$$\Delta(|\vec{a}|)=\int\frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi}\frac{2\pi e^2}{\left(4\pi\epsilon+\frac{\sqrt{2}\pi e^2N}{16}\right)|\vec{p}+\vec{a}|}\frac{\Delta(|\vec{p}|)}{\omega^2+|\vec{p}|^2+\Delta(|\vec{p}|)^2} \tag{2.96}$$

that we are going to analyse. Integrating out ω to get

$$\Delta(|\vec{a}|) = \int \frac{d\vec{p}}{(2\pi)^2}\frac{1}{2}\frac{2\pi e^2}{\left(4\pi\epsilon + \frac{\sqrt{2}\pi e^2 N}{16}\right)|\vec{p}+\vec{a}|}\frac{\Delta(|\vec{p}|)}{\sqrt{|\vec{p}|^2+\Delta(|\vec{p}|)^2}}. \tag{2.97}$$

Assuming that the angular dependence of $|\vec{p}+\vec{a}|$ is weak, and the only role of Δ in the $\sqrt{|\vec{p}|^2+\Delta^2}$ is to introduce the infrared cut-off, Δ_0, to the integral (the latter assumption is called bifurcation approximation). $\Delta(|\vec{a}| \leq \Delta_0) = \Delta_0$ and for $|\vec{a}| > \Delta_0$ we get

$$\begin{aligned}
\Delta(|\vec{a}|) &= \int_{\Delta_0}^{\Lambda} \frac{d\vec{p}}{(2\pi)^2}\frac{1}{2}\frac{2\pi e^2}{\left(4\pi\epsilon + \frac{\sqrt{2}\pi e^2 N}{16}\right)}\left(\frac{\theta(|\vec{p}|-|\vec{a}|)}{|\vec{p}|}+\frac{\theta(|\vec{a}|-|\vec{p}|)}{|\vec{a}|}\right)\frac{\Delta(|\vec{p}|)}{|\vec{p}|} \\
&= \int_{|\vec{a}|}^{\Lambda} \frac{d\vec{p}}{(2\pi)^2}\frac{1}{2}\frac{2\pi e^2}{\left(4\pi\epsilon + \frac{\sqrt{2}\pi e^2 N}{16}\right)}\frac{1}{|\vec{p}|}\frac{\Delta(|\vec{p}|)}{|\vec{p}|} + \int_{\Delta_0}^{|\vec{a}|} \frac{d\vec{p}}{(2\pi)^2}\frac{1}{2}\frac{2\pi e^2}{\left(4\pi\epsilon + \frac{2\pi e^2 N}{16}\right)|}\frac{1}{|\vec{a}|}\frac{\Delta(|\vec{p}|)}{|\vec{p}|} \\
&= \int_{|\vec{a}|}^{\Lambda} d|\vec{p}|\lambda\frac{\Delta(|\vec{p}|)}{|\vec{p}|} + \int_{\Delta_0}^{|\vec{a}|} d|\vec{p}|\lambda\frac{\Delta(|\vec{p}|)}{|\vec{a}|},
\end{aligned} \tag{2.98}$$

where $\lambda = \frac{1}{4\pi}\frac{2\pi e^2}{\left(4\pi\epsilon+\frac{\sqrt{2}\pi e^2 N}{16}\right)}$. The integral equation (2.98) is equivalent to the following differential equation

$$a^2\frac{d^2\Delta(a)}{da^2} + 2a\frac{d\Delta(a)}{da} + \lambda\Delta(a) = 0 \tag{2.99}$$

with boundary conditions:

$$a^2\frac{d\Delta(a)}{da}\bigg|_{a=\Delta_0} = 0, \tag{2.100}$$

$$\left(a\frac{d\Delta(a)}{da} + \Delta(a)\right)\bigg|_{a=\Lambda} = 0. \tag{2.101}$$

Here, we set $|\vec{a}| = a$. To show that, we rearrage Eq. (2.99) to be

$$\frac{d}{da}\left(a^2\frac{d\Delta(a)}{da}\right) = -\lambda\Delta(a). \tag{2.102}$$

Then, performing integration to get

$$\left(a^2\frac{d\Delta(a)}{da}\right) - \left(a^2\frac{d\Delta(a)}{da}\right)\bigg|_{a=\Delta_0} = -\int_{p=\Delta_0}^{a} dp\lambda\Delta(p) \tag{2.103}$$

Setting the second term on the left hand side of the equation to be the first boundary condition, and the remaining term is

$$a\frac{d\Delta(a)}{da} = -\frac{1}{a}\int_{p=\Delta_0}^{a} dp\lambda\Delta(p). \tag{2.104}$$

Rearraging Eq.(2.99) in another way, namely

$$a\frac{d^2\Delta(a)}{da^2} + \frac{d\Delta(a)}{da} = -\frac{d\Delta(a)}{da} - \frac{\lambda}{a}\Delta(a), \tag{2.105}$$

$$\frac{d}{da}\left(a\frac{d\Delta(a)}{da}\right) = -\frac{d\Delta(a)}{da} - \frac{\lambda}{a}\Delta(a), \tag{2.106}$$

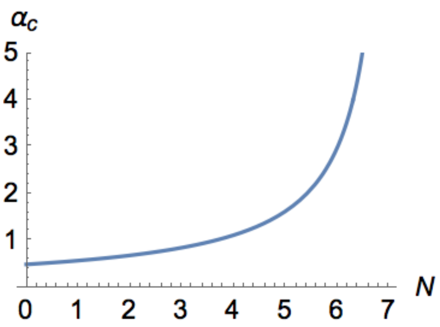

Figure 2.5: phase diagram of 2D Dirac/Weyl materials in the $\alpha - N$ plane

and integrate both side of this equation to get

$$\left(a\frac{d\Delta(a)}{da}\right)\Bigg|_{a=\Lambda} - \left(a\frac{d\Delta(a)}{da}\right) = \Delta(a) - \Delta(a)\Bigg|_{a=\Lambda} - \int_a^\Lambda dp\frac{\lambda}{p}\Delta(p), \tag{2.107}$$

$$\left[\left(a\frac{d\Delta(a)}{da}\right)\Bigg|_{a=\Lambda} + \Delta(a)\Bigg|_{a=\Lambda}\right] - \left(a\frac{d\Delta(a)}{da}\right) = \Delta(a) - \int_a^\Lambda dp\frac{\lambda}{p}\Delta(p). \tag{2.108}$$

The square bracket on the left hand side provides the second boundary condition,

$$a\frac{d}{da}\Delta(a) + \Delta(a) = \int_a^\Lambda dp\frac{\lambda}{p}\Delta(p). \tag{2.109}$$

We arrive at the integral equation (2.98) by subtracting Eq.(2.105) from Eq.(2.109) the non-trivial solution of differential equation (2.99) satisfying boundary conditions Eq.(2.100) and Eq.(2.101) is given by [25] [23]

$$\Delta(a) = \frac{\Delta_0^{3/2}}{\sin(\arctan\sqrt{4\lambda-1})\sqrt{a}}\sin\left(\frac{\sqrt{4\lambda-1}}{2}\ln\left(\frac{a}{\Delta_0}\right) + \arctan\sqrt{4\lambda-1}\right), \tag{2.110}$$

with the restriction that

$$\frac{\sqrt{4\lambda-1}}{2}\ln\left(\frac{\Lambda}{\Delta_0}\right) + 2\arctan\sqrt{4\lambda-1} = \pi. \tag{2.111}$$

Equivalently,

$$\Delta_0 = \Lambda\exp\left[-\frac{2}{\sqrt{4\lambda-1}}\left(\pi - 2\arctan\sqrt{4\lambda-1}\right)\right]. \tag{2.112}$$

This non-trivial real solution exists when λ is at least at critical value $\lambda_c = 1/4$,

$$\lambda = \frac{1}{4\pi}\frac{2\pi e^2}{\left(4\pi\epsilon + \frac{\sqrt{2}\pi e^2 N}{16}\right)} \geq \frac{1}{4}, \tag{2.113}$$

which implies that

$$\alpha \geq \alpha_c = \frac{16}{32 - \sqrt{2}\pi N}. \tag{2.114}$$

There exist a critical flavours of electron, $N_c = \frac{32}{\sqrt{2}\pi}$, at which $\alpha_c \to \infty$. 2D Weyl/Dirac systems with $N > N_c$ are always in their semimetallic state. A phase diagram in the $\alpha - N$ plane is shown in Figure.2.5

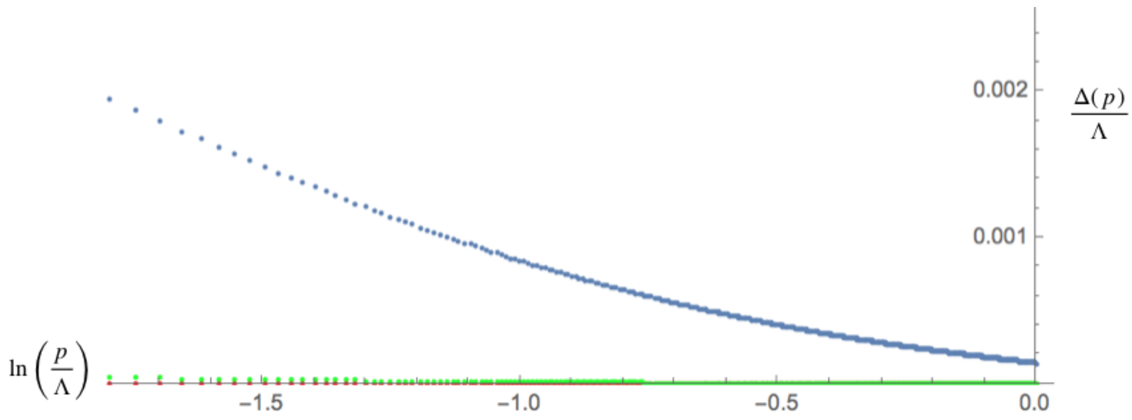

Figure 2.6: Momentum dependence of the mass gap for $\lambda = 0.20$ (red), $\lambda = 0.25$ (green), and $\lambda = 0.30$ (blue)

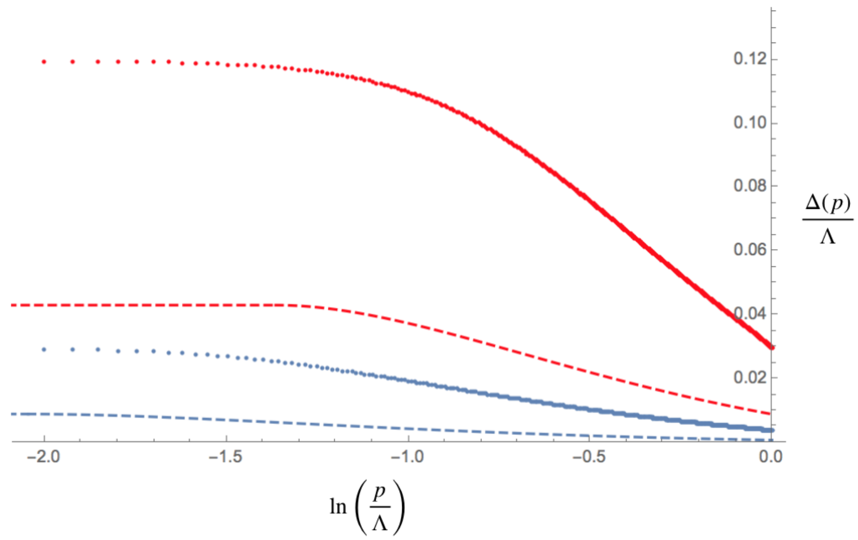

Figure 2.7: Momentum dependence of the mass gap for $\lambda = 0.40$ (blue), $\lambda = 0.50$ (red) from numerical (dotted) and analytical (dashed) calculations

In the case of graphene, for instance, where $N = 4$, $\alpha_c \approx 1.125$. [23] The value of α for a sheet of pristine graphene is 2.16, hence suspended graphene is theoretically an insulator, not a semimetal. [25] Furthermore, the Schwinger-Dyson equation (2.97) is solved numerically without any approximations. The numerical results of the momentum dependent gap for different values of λ are presented in Figure 2.6. We find that the critical value, λ_c, is in agreement with the analytical analysis. However, as depicted in Figure.2.7, the analytical method underestimates the spontaneous mass gap of the electrons. The larger the value of λ, the more the two methods differ.

Chapter 3

3D Weyl Semimetals

In the last chapter, we studied the possibility of a semimetal to insulator phase transition in 2D Dirac/Weyl semimetals induced by Coulomb interaction. Now we will do the analogous calculations for three-dimensional Weyl semimetals.

3.1 Model

Free Weyl electrons in three dimension are described by

$$S_0[\bar{\Psi}, \Psi] = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{|\vec{p}|=0}^{\Lambda \approx \frac{1}{a}} \frac{d\vec{p}}{(2\pi)^3} \bar{\Psi}(\vec{p}, \omega)(-i\omega \mathbb{1} + v_F \vec{p} \cdot \vec{\sigma})\ \Psi(\vec{p}, \omega) \tag{3.1}$$

where $\vec{p} = (p_x, p_y, p_z)$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, Pauli matrices. The Green's function is given by

$$\begin{aligned} G_0(\vec{p}, \vec{q}; \omega, \eta) &= \left\langle \Psi(\vec{p}, \omega) \bar{\Psi}(\vec{q}, \eta) \right\rangle & (3.2) \\ &= (2\pi)^3 \delta(\vec{p} - \vec{q}\,) \delta(\omega - \eta)(-i\omega \mathbb{1} + v\vec{p} \cdot \vec{\sigma})^{-1} & (3.3) \\ &= (2\pi)^3 \delta(\vec{p} - \vec{q}\,) \delta(\omega - \eta) \frac{i\omega \mathbb{1} + v\vec{p} \cdot \vec{\sigma}}{\omega^2 + v^2 \left|\vec{p}\,\right|^2} & (3.4) \\ &= (2\pi)^3 \delta(\vec{p} - \vec{q}\,) \delta(\omega - \eta) G_0(\vec{p}, \omega). & (3.5) \end{aligned}$$

The electron-electron interaction in the second quantisation formalism reads

$$H_{int} = \frac{1}{2} \int d\vec{r}_1 \int d\vec{r}_2 \bar{\Psi}(\vec{r}_1) \bar{\Psi}(\vec{r}_2) V_C(\vec{r}_1 - \vec{r}_2) \Psi(\vec{r}_2) \Psi(\vec{r}_1) \tag{3.6}$$

where

$$V_C(\vec{r}) = \frac{e^2}{4\pi\epsilon\, |\vec{r}|} = \lim_{a \to 0^+} \frac{e^2}{4\pi\epsilon\, |\vec{r}|} e^{-ar}$$

is the Coulomb potential of which the Fourier transform is given by

$$\begin{aligned} V_C(\vec{q}) &= \int d\vec{r}\ V_C(\vec{r}) e^{i\vec{q} \cdot \vec{r}} & (3.7) \\ &= \lim_{a \to 0^+} \frac{e^2}{4\pi\epsilon} \int_0^{\infty} r^2 dr \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \frac{1}{r} e^{iqr\cos\theta - ar} \\ &= \lim_{a \to 0^+} \frac{2\pi e^2}{4\pi\epsilon} \int_0^{\infty} r dr \int_{-1}^{1} d\cos\theta e^{iqr\cos\theta - ar} \\ &= \lim_{a \to 0^+} \frac{2\pi e^2}{4\pi\epsilon i q} \int_0^{\infty} dr \left(e^{(iq-a)r} - e^{(-iq-a)r} \right) \\ &= \lim_{a \to 0^+} \frac{2\pi e^2}{4\pi\epsilon i q} \left(-\frac{1}{iq - a} + \frac{1}{-iq - a} \right) \\ &= \frac{e^2}{\epsilon q^2}. \end{aligned}$$

Assuming that the positive divergence of the zero mode Coulomb interaction is compensated by the negative divergence from the positive's charged background [9], the Coulomb potential in a momentum space reads

$$V_C(\vec{q}) = \begin{cases} 0 & ; \quad \vec{q} = 0 \\ \frac{e^2}{\epsilon|\vec{q}\,|^2} & ; \quad \text{otherwise,} \end{cases} \tag{3.8}$$

and the interacting part of the action in configuration space is given by

$$S_{int} = \frac{1}{2}\int dt \int d\vec{r}_1 \int d\vec{r}_2 \bar{\Psi}(\vec{r}_1, t)\bar{\Psi}(\vec{r}_2, t)V_C(\vec{r}_1 - \vec{r}_2)\Psi(\vec{r}_2, t)\Psi(\vec{r}_1, t). \tag{3.9}$$

Performing the Fourier transformation

$$\bar{\Psi}(\vec{r}, t) = \frac{1}{(2\pi)^3}\int d\vec{p}\ \bar{\Psi}(\vec{p}, \omega)e^{-i\vec{p}\cdot\vec{r}+i\omega t} \tag{3.10}$$

$$\Psi(\vec{r}, t) = \frac{1}{(2\pi)^3}\int d\vec{p}\ \Psi(\vec{p}, \omega)e^{i\vec{p}\cdot\vec{r}-i\omega t} \tag{3.11}$$

$$V_C(\vec{r}) = \frac{1}{(2\pi)^3}\int d\vec{q}\ V_C(\vec{q})e^{i\vec{q}\cdot\vec{r}} \tag{3.12}$$

on the action Eq. (3.9), one obtains the Coulomb interaction action according to

$$\begin{aligned} S_{int}[\bar{\Psi}, \Psi] = {} & \frac{1}{2}\frac{e^2}{\epsilon}\prod_{i=1}^{4}\int_{\omega_i=-\infty}^{\omega_i=\infty}\frac{d\omega_i}{2\pi}\int_{|\vec{p}_i|=0}^{\Lambda}\frac{d\vec{p}_i}{(2\pi)^3}\int_{|\vec{q}|=0}^{\Lambda}\frac{d\vec{q}}{(2\pi)^3}\frac{1}{|\vec{q}\,|^2} \\ & \bar{\Psi}(\vec{p}_1, \omega_1)\bar{\Psi}(\vec{p}_2, \omega_2)\Psi(\vec{p}_3, \omega_3)\Psi(\vec{p}_4, \omega_4)(2\pi)^3\delta(\vec{p}_1 - \vec{p}_4 - \vec{q}) \\ & (2\pi)^3\delta(\vec{p}_2 - \vec{p}_3 + \vec{q})(2\pi)\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4). \end{aligned} \tag{3.13}$$

We repeat the steps of two dimension in Chapter 2 to study the phase diagram of three dimensional Weyl semimetals.

3.2 Renormalisation group analysis

3.2.1 First order pertubative approximation

The object we are looking at, in the renormalisation group analysis, is the partition function of the system. At zero temperature, the partition function of Weyl electrons with Coulomb interaction is given by

$$Z = \int D\bar{\Psi}D\Psi e^{-S_0[\bar{\Psi},\Psi]-S_{int}[\bar{\Psi},\Psi]} \tag{3.14}$$

where $S_0[\bar{\Psi}, \Psi]$ and $S_{int}[\bar{\Psi}, \Psi]$ are given above.

The renormalisation group analysis consists of the three steps [8]

1. Coarse-Graining and integrating out of high energy modes: The First step of coarse-graining of the partition function is to separate the fermionic fields of momenta greater than new cut-off $\frac{\Lambda}{b}$ $(b > 1)$ from the lesser ones, i.e.,

$$\bar{\Psi}(\vec{p}, \omega) = \begin{cases} \bar{\Psi}_>(\vec{p}, \omega) & ; \quad \frac{\Lambda}{b} < |\vec{p}| < \Lambda \\ \bar{\Psi}_<(\vec{p}, \omega) & ; \quad 0 < |\vec{p}| < \frac{\Lambda}{b} \end{cases} \tag{3.15}$$

and

$$\Psi(\vec{p}, \omega) = \begin{cases} \Psi_>(\vec{p}, \omega) & ; \quad \frac{\Lambda}{b} < |\vec{p}| < \Lambda \\ \Psi_<(\vec{p}, \omega) & ; \quad 0 < |\vec{p}| < \frac{\Lambda}{b}. \end{cases} \tag{3.16}$$

The greater $(\bar{\Psi}_>(\vec{p},\omega), \Psi_>(\vec{p},\omega))$ and lesser fermionic fields $(\bar{\Psi}_<(\vec{p},\omega), \Psi_<(\vec{p},\omega))$ in the free part of the action are decoupled, i.e.,

$$S_0[\bar{\Psi},\Psi] = S_0[\bar{\Psi}_>,\Psi_>] + S_0[\bar{\Psi}_<,\Psi_<] \tag{3.17}$$

where

$$S_0[\bar{\Psi}_>,\Psi_>] = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{|\vec{p}|=\frac{\Lambda}{b}}^{\Lambda}\frac{d\vec{p}}{(2\pi)^3}\bar{\Psi}_>(\vec{p},\omega)(-i\omega + v_F\vec{p}\cdot\vec{\tau})\ \Psi_>(\vec{p},\omega) \tag{3.18}$$

$$S_0[\bar{\Psi}_<,\Psi_<] = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{|\vec{p}|=0}^{\frac{\Lambda}{b}}\frac{d\vec{p}}{(2\pi)^3}\bar{\Psi}_<(\vec{p},\omega)(-i\omega + v_F\vec{p}\cdot\vec{\tau})\ \Psi_<(\vec{p},\omega), \tag{3.19}$$

whereas, in the interacting part, they are not, i.e.,

$$\begin{aligned}
S_{int}[\bar{\Psi}_<,\Psi_<,\bar{\Psi}_>,\Psi_>] \;=\; & \frac{1}{2}\frac{e^2}{\epsilon}\prod_{i=1}^{4}\int_{\omega_i=-\infty}^{\omega_i=\infty}\frac{d\omega_i}{2\pi}\left(\int_{|\vec{q}|=0}^{\Lambda}\frac{d\vec{q}}{(2\pi)^3}\frac{1}{|\vec{q}\,|^2}\right)\\
& \prod_{i=1}^{2}\left(\int_{|\vec{p}_i|=0}^{\frac{\Lambda}{b}}\frac{d\vec{p}_i}{(2\pi)^3}\bar{\Psi}_<(\vec{p}_i,\omega_i) + \int_{|\vec{p}_i|=\frac{\Lambda}{b}}^{\Lambda}\frac{d\vec{p}_i}{(2\pi)^3}\bar{\Psi}_>(\vec{p}_i,\omega_i)\right)\\
& \prod_{i=3}^{4}\left(\int_{|\vec{p}_i|=0}^{\frac{\Lambda}{b}}\frac{d\vec{p}_i}{(2\pi)^3}\Psi_<(\vec{p}_i,\omega_i) + \int_{|\vec{p}_i|=\frac{\Lambda}{b}}^{\Lambda}\frac{d\vec{p}_i}{(2\pi)^3}\Psi_>(\vec{p}_i,\omega_i)\right)\\
& (2\pi)^3\delta(\vec{p}_1-\vec{p}_4-\vec{q})(2\pi)^3\delta(\vec{p}_2-\vec{p}_3+\vec{q})\\
& (2\pi)\delta(\omega_1+\omega_2-\omega_3-\omega_4).
\end{aligned} \tag{3.20}$$

The partition function then becomes

$$\begin{aligned}
Z \;&=\; \int D\bar{\Psi}_< D\Psi_< e^{-S_0[\bar{\Psi}_<,\Psi_<]}\int D\bar{\Psi}_> D\Psi_> e^{-S_0[\bar{\Psi}_>,\Psi_>]}e^{-S_{int}[\bar{\Psi}_<,\Psi_<,\bar{\Psi}_>,\Psi_>]}\\
&=\; \int D\bar{\Psi}_< D\Psi_< e^{-S_0[\bar{\Psi}_<,\Psi_<]}Z_{0>}\left\langle e^{-S_{int}[\bar{\Psi}_<,\Psi_<,\bar{\Psi},_>\Psi_>]}\right\rangle_>\\
&=\; \int D\bar{\Psi}_< D\Psi_< e^{-S_0[\bar{\Psi}_<,\Psi_<]}e^{\left\langle \ln(e^{-S_{int}[\bar{\Psi}_<,\Psi_<,\bar{\Psi}_>,\Psi_>]})\right\rangle_>}\\
&=\; \int D\bar{\Psi}_< D\Psi_< e^{-S_0[\bar{\Psi}_<,\Psi_<]}e^{-\langle S_{int}\rangle_> + \frac{1}{2}(\left\langle S_{int}^2\right\rangle_> - \langle S_{int}\rangle_>^2) + O(S_{int}^3)}.
\end{aligned} \tag{3.21}$$

Consider the first order perturbation of the partition function in Eq. (3.21). Expand Eq.(3.20) and apply Wick's theorem to find the correlation function with respect to the greater degrees of freedom. The only non-vanishing contributions is

$$\left\langle S_{int}[\bar{\Psi}_<,\Psi_<,\bar{\Psi}_>,\Psi_>]\right\rangle_> = \langle S_{int}\rangle_{>,1} + \langle S_{int}\rangle_{>,2} \tag{3.22}$$

where

$$\begin{aligned}
\langle S_{int}\rangle_{>,1} \;=\; & \frac{1}{2}\frac{e^2}{\epsilon}\prod_{i=1}^{4}\int_{\omega_i=-\infty}^{\omega_i=\infty}\frac{d\omega_i}{2\pi}\int_{|\vec{p}_i|=0}^{\frac{\Lambda}{b}}\frac{d\vec{p}_i}{(2\pi)^3}\int_{|\vec{q}|=0}^{\Lambda}\frac{d\vec{q}}{(2\pi)^3}\ \frac{1}{|\vec{q}\,|^2}\bar{\Psi}_<(\vec{p}_1,\omega_1)\\
& \bar{\Psi}_<(\vec{p}_2,\omega_2)\Psi_<(\vec{p}_3,\omega_3)\Psi_<(\vec{p}_4,\omega_4)(2\pi)^3\delta(\vec{p}_1-\vec{p}_4-\vec{q})\\
& (2\pi)^3\delta(\vec{p}_2-\vec{p}_3+\vec{q})(2\pi)\delta(\omega_1+\omega_2-\omega_3-\omega_4),
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
\langle S_{int}\rangle_{>,2} \;=\; & 2\frac{1}{2}\frac{e^2}{\epsilon}\prod_{i=1}^{4}\int_{\omega_i=-\infty}^{\omega_i=\infty}\frac{d\omega_i}{2\pi}\int_{|\vec{p}_i|=0}^{\frac{\Lambda}{b}}\frac{d\vec{p}_i}{(2\pi)^3}\int_{|\vec{q}|=0}^{\Lambda}\frac{d\vec{q}}{(2\pi)^3}\ \frac{1}{|\vec{q}\,|^2}\bar{\Psi}_<(\vec{p}_2,\omega_2)\\
& (2\pi)^3\delta(\vec{p}_3-\vec{p}_1)\delta(\omega_3-\omega_1)G_{0>}(\vec{p}_3,\omega_3)\Psi_<(\vec{p}_4,\omega_4)(2\pi)^3\delta(\vec{p}_1-\vec{p}_4-\vec{q})\\
& (2\pi)^3\delta(\vec{p}_2-\vec{p}_3+\vec{q})(2\pi)\delta(\omega_1+\omega_2-\omega_3-\omega_4).
\end{aligned} \tag{3.24}$$

2. Rescaling: $S_0[\bar{\Psi}_<, \Psi_<]$ together with $\langle S_{int}\rangle_{>,1}$ are similar to the original action, except that the upper cutoff has decreased to $\frac{\Lambda}{b}$. Restoring the cutoff to its original value by rescaling

$$\begin{cases} \vec{p}\,' &= b\vec{p} \\ \omega' &= b\omega \\ \bar{\Psi}'(\vec{p}\,',\omega') &= b^{-5/2}\bar{\Psi}_<(\vec{p},\omega) \\ \Psi'(\vec{p}\,',\omega') &= b^{-5/2}\Psi_<(\vec{p},\omega) \end{cases} \tag{3.25}$$

and dropping all primes, the transformed action then becomes identical to the original one.

3. Renormalisation:
Now, consider $\langle S_{int}\rangle_{>,2}$;

$$\begin{aligned} \langle S_{int}\rangle_{>,2} &= 2\frac{1}{2}\frac{e^2}{\epsilon}\prod_{i=1}^{4}\int_{\omega_i=-\infty}^{\omega_i=\infty}\frac{d\omega_i}{2\pi}\int_{|\vec{p}_i|=0}^{\frac{\Lambda}{b}}\frac{d\vec{p}_i}{(2\pi)^3}\int_{|\vec{q}|=0}^{\frac{\Lambda}{b}}\frac{d\vec{q}}{(2\pi)^3}\frac{1}{|\vec{q}\,|^2}\bar{\Psi}_<(\vec{p}_2,\omega_2) \\ &\quad (2\pi)^3\delta(\vec{p}_3-\vec{p}_1)\delta(\omega_3-\omega_1)G_{0>}(\vec{p}_3,\omega_3)\Psi_<(\vec{p}_4,\omega_4)(2\pi)^3\delta(\vec{p}_1-\vec{p}_4-\vec{q}) \\ &\quad (2\pi)^3\delta(\vec{p}_2-\vec{p}_3+\vec{q})(2\pi)\delta(\omega_1+\omega_2-\omega_3-\omega_4) \qquad (3.26) \\ &= \frac{e^2}{\epsilon}\prod_{i=1}^{4}\int_{\omega_i=-\infty}^{\omega_i=\infty}\frac{d\omega_i}{2\pi}\int_{|\vec{p}_i|=0}^{\frac{\Lambda}{b}}\frac{d\vec{p}_i}{(2\pi)^3}\int_{|\vec{q}|=0}^{\frac{\Lambda}{b}}\frac{d\vec{q}}{(2\pi)^3}\frac{1}{|\vec{q}\,|^2}\bar{\Psi}_<(\vec{p}_2,\omega_2) \\ &\quad (2\pi)^3\delta(\vec{p}_3-\vec{p}_1)\delta(\omega_3-\omega_1)\frac{i\omega_3+v\vec{p}_3\cdot\vec{\sigma}}{\omega_3^2+v^2\,|\vec{p}_3\,|^2}\Psi_<(\vec{p}_4,\omega_4)(2\pi)^3\delta(\vec{p}_1-\vec{p}_4-\vec{q}) \\ &\quad (2\pi)^3\delta(\vec{p}_2-\vec{p}_3+\vec{q})(2\pi)\delta(\omega_1+\omega_2-\omega_3-\omega_4) \qquad (3.27) \\ &= \frac{e^2}{\epsilon}\int_{\omega_3=-\infty}^{\omega_3=\infty}\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{d\omega_3}{2\pi}\frac{d\vec{p}_3}{(2\pi)^3}\int_{\omega_4=-\infty}^{\omega_4=\infty}\int_{|\vec{p}_4|=0}^{\frac{\Lambda}{b}}\frac{d\omega_4}{2\pi}\frac{d\vec{p}_4}{(2\pi)^3}\frac{1}{|\vec{p}_3-\vec{p}_4\,|^2}\bar{\Psi}_<(\vec{p}_4,\omega_4) \\ &\quad \frac{i\omega_3+v\vec{p}_3\cdot\vec{\sigma}}{\omega_3^2+v^2\,|\vec{p}_3\,|^2}\Psi_<(\vec{p}_4,\omega_4) \qquad (3.28) \\ &= \frac{e^2}{\epsilon}\int_{\omega_4=-\infty}^{\omega_4=\infty}\int_{|\vec{p}_4|=0}^{\frac{\Lambda}{b}}\frac{d\omega_4}{2\pi}\frac{d\vec{p}_4}{(2\pi)^3}\bar{\Psi}_<(\vec{p}_4,\omega_4)\Sigma^{(1)}(\vec{p}_4,\omega_4)\Psi_<(\vec{p}_4,\omega_4) \qquad (3.29) \end{aligned}$$

where $\Sigma^{(1)}(\vec{p}_4,\omega_4)$ is defined by

$$\begin{aligned} \Sigma^{(1)}(\vec{p}_4,\omega_4) &= \int_{\omega_3=-\infty}^{\omega_3=\infty}\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{d\omega_3}{2\pi}\frac{d\vec{p}_3}{(2\pi)^3}\frac{1}{|\vec{p}_3-\vec{p}_4\,|^2}\frac{i\omega_3+v\vec{p}_3\cdot\vec{\sigma}}{\omega_3^2+v^2\,|\vec{p}_3\,|^2} \qquad (3.30) \\ &= \int_{\omega_3=-\infty}^{\omega_3=\infty}\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{d\omega_3}{2\pi}\frac{d\vec{p}_3}{(2\pi)^3}\frac{1}{|\vec{p}_3-\vec{p}_4\,|^2}\frac{v_F\vec{p}_3\cdot\vec{\sigma}}{\omega_3^2+v^2\,|\vec{p}_3\,|^2}. \qquad (3.31) \end{aligned}$$

The form of the integrand in (3.31) suggests that $\Sigma^{(1)}(\vec{p}_4,\omega_4)$ can be written as $\Sigma^{(1)}(\vec{p}_4,\omega_4) =$

$\Sigma_v \vec{p}_4 \cdot \vec{\sigma}$, and

$$\begin{aligned}
\Sigma_v &= \frac{1}{2\left|\vec{p}_4\right|^2}\mathrm{Tr}(\vec{p}_4\cdot\vec{\sigma}\Sigma^{(1)}(\vec{p}_4,\omega_4)) && (3.32)\\
&= \int_{\omega_3=-\infty}^{\omega_3=\infty}\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{d\omega_3}{2\pi}\frac{d\vec{p}_3}{(2\pi)^3}\frac{1}{2\left|\vec{p}_4\right|^2}\frac{1}{\left|\vec{p}_3-\vec{p}_4\ \right|^2}\frac{v_F\mathrm{Tr}((\vec{p}_4\cdot\vec{\sigma})(\vec{p}_3\cdot\vec{\sigma}))}{\omega_3^2+v^2\left|\vec{p}_3\ \right|^2} && (3.33)\\
&= \int_{\omega_3=-\infty}^{\omega_3=\infty}\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{d\omega_3}{2\pi}\frac{d\vec{p}_3}{(2\pi)^3}\frac{1}{2\left|\vec{p}_4\right|^2}\frac{1}{\left|\vec{p}_3-\vec{p}_4\right|^2}\frac{v_F\mathrm{Tr}((\vec{p}_3\cdot\vec{p}_4)\mathbb{1}+i(\vec{p}_3\times\vec{p}_4)\cdot\vec{\sigma})}{\omega_3^2+v^2\left|\vec{p}_3\ \right|^2} && (3.34)\\
&= \int_{\omega_3=-\infty}^{\omega_3=\infty}\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{d\omega_3}{2\pi}\frac{d\vec{p}_3}{(2\pi)^3}\frac{1}{2\left|\vec{p}_4\right|^2}\frac{1}{\left|\vec{p}_3-\vec{p}_4\right|^2}\frac{2v_F\vec{p}_3\cdot\vec{p}_4}{\omega_3^2+v^2\left|\vec{p}_3\ \right|^2} && (3.35)\\
&= -\int_{\phi=0}^{\phi=2\pi}d\phi\int_{\theta=0}^{\pi}d\cos\theta\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{\left|\vec{p}_3\right|^2dp_3}{(2\pi)^4}\frac{v_F\left|\vec{p}_3\right|\left|\vec{p}_4\right|\cos(\theta)}{\left|\vec{p}_4\right|^2\left|\vec{p}_3-\vec{p}_4\right|^2} && (3.36)\\
&\quad \int_{\omega_3=-\infty}^{\omega_3=\infty}d\omega_3\frac{1}{\omega_3^2+v^2\left|\vec{p}_3\ \right|^2}.
\end{aligned}$$

Making use of the formula

$$\int_{\omega=-\infty}^{\omega=\infty}d\omega\frac{1}{\omega^2+v^2\left|\vec{p}\ \right|^2}=\frac{\pi}{v_F\left|\vec{p}\right|} \tag{3.37}$$

we obtain

$$\begin{aligned}
\Sigma_v &= -\int_{\phi=0}^{\phi=2\pi}d\phi\int_{\theta=0}^{\pi}d\cos\theta\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{\left|\vec{p}_3\right|^2dp_3}{(2\pi)^4}\frac{v_F\left|\vec{p}_3\right|\left|\vec{p}_4\right|\cos(\theta)}{\left|\vec{p}_4\right|^2\left|\vec{p}_3-\vec{p}_4\right|^2}\frac{\pi}{v_F\left|\vec{p}_3\right|} && (3.38)\\
&= -2\pi\int_{\theta=0}^{\pi}d\cos\theta\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{\left|\vec{p}_3\right|^2dp_3}{(2\pi)^4}\frac{\pi\cos(\theta)}{\left|\vec{p}_4\right|(\left|\vec{p}_3\right|^2+\left|\vec{p}_4\right|^2-2\left|\vec{p}_3\right|\left|\vec{p}_4\right|cos(\theta))} && (3.39)\\
&= -2\pi\int_{\theta=0}^{\pi}d\cos\theta\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{\left|\vec{p}_3\right|^2dp_3}{(2\pi)^4}\frac{\pi\cos(\theta)}{\left|\vec{p}_4\right|\left|\vec{p}_3\right|^2(1+\frac{\left|\vec{p}_4\right|^2}{\left|\vec{p}_3\right|^2}-2\frac{\left|\vec{p}_4\right|}{\left|\vec{p}_3\right|}cos(\theta))} && (3.40)\\
&= -2\pi\int_{\theta=0}^{\pi}d\cos\theta\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{dp_3}{(2\pi)^4}\frac{\pi\cos(\theta)}{\left|\vec{p}_4\right|}\left(1+2\frac{\left|\vec{p}_4\right|}{\left|\vec{p}_3\right|}cos(\theta)+O(\frac{\left|\vec{p}_4\right|^2}{\left|\vec{p}_3\right|^2})\right) && (3.41)\\
&= -4\pi\int_{\theta=0}^{\pi}d\cos\theta\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{dp_3}{(2\pi)^4}\frac{\pi\cos^2\theta}{\left|\vec{p}_3\right|}+O(\frac{\left|\vec{p}_4\right|^2}{\left|\vec{p}_3\right|^2}) && (3.42)\\
&= -4\pi\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{dp_3}{(2\pi)^4}\frac{\pi}{\left|\vec{p}_3\right|}\frac{\cos^3\theta}{3}\Bigg|_{\theta=0}^{\pi}+O(\frac{\left|\vec{p}_4\right|^2}{\left|\vec{p}_3\right|^2}) && (3.43)\\
&= \frac{8\pi}{3}\int_{|\vec{p}_3|=\frac{\Lambda}{b}}^{\Lambda}\frac{dp_3}{(2\pi)^4}\frac{\pi}{\left|\vec{p}_3\right|}+O(\frac{\left|\vec{p}_4\right|^2}{\left|\vec{p}_3\right|^2}) && (3.44)\\
&\approx \frac{1}{6\pi^2}\left(\ln(\Lambda)-\ln(\frac{\Lambda}{b})\right). && (3.45)
\end{aligned}$$

We summarises this in

$$\langle S_{int}\rangle_{>,2}=\frac{1}{6\pi^2}\left(\ln(\Lambda)-\ln(\frac{\Lambda}{b})\right)\frac{e^2}{\epsilon}\int_{\omega_4=-\infty}^{\omega_4=\infty}\int_{|\vec{p}_4|=0}^{\frac{\Lambda}{b}}\frac{d\omega_4}{2\pi}\frac{d\vec{p}_4}{(2\pi)^3}\bar{\Psi}_<(\vec{p}_4,\omega_4)(\vec{p}_4\cdot\vec{\sigma})\Psi_<(\vec{p}_4,\omega_4). \tag{3.46}$$

we then substitute Eq. (3.46) into the partition function Eq. (3.21), and find that Fermi velocity is renormalised, i.e.,

$$S_0 = \int_{\omega=-\infty}^{\omega=\infty} \int_{|\vec{p}|=0}^{\frac{\Lambda}{b}} \frac{d\omega}{2\pi} \frac{d\vec{p}}{(2\pi)^3} \bar{\Psi}_<(\vec{p},\omega) \left[\left(v_F + \frac{e^2}{6\pi^2\epsilon} \left(\ln(\Lambda) - \ln(\frac{\Lambda}{b}) \right) \right) \vec{p}\cdot\vec{\sigma} \right] \Psi_<(\vec{p},\omega). \quad (3.47)$$

A scale dependent Fermi velocity, $v_F(\Lambda)$, in three dimension obeys [14]

$$v_F\left(\frac{\Lambda}{b}\right) = v_F(\Lambda) + \frac{e^2}{6\pi^2\epsilon}\left(\ln(\Lambda) - \ln\left(\frac{\Lambda}{b}\right)\right), \quad (3.48)$$

$$\frac{v_F\left(\frac{\Lambda}{b}\right) - v_F(\Lambda)}{\left(\ln\left(\frac{\Lambda}{b}\right) - \ln(\Lambda)\right)} = -\frac{e^2}{6\pi^2\epsilon}, \quad (3.49)$$

$$\frac{dv_F(\Lambda)}{d\ln(\Lambda)} = -\frac{e^2}{6\pi^2\epsilon} = -\frac{2\alpha}{3\pi}v_F \quad (3.50)$$

In addition to velocity renormalisation, the dielectric constant of 3D Weyl materials is also renormalised.[10] The diagrammatic representation of this contribution is shown in Figure.3.1, and it can be calculated straightforwardly from polarisation function.The polarisation function of N species Weyl fermions is defined by

$$\begin{aligned}\Pi(\omega,\vec{k}) &= N\int \frac{d\nu}{2\pi}\frac{d\vec{q}}{(2\pi)^3}\mathrm{Tr}\left(G_0(\vec{k}+\vec{q},\nu+\omega)G_0(\vec{q},\nu)\right) \\ &= N\int \frac{d\nu}{2\pi}\frac{d\vec{q}}{(2\pi)^3}\mathrm{Tr}\left(\frac{i(\nu+\omega)+v_F(\vec{k}+\vec{q})\cdot\vec{\sigma}}{(\nu+\omega)^2+v_F^2|\vec{k}+\vec{q}\,|^2}\frac{i\nu+v_F\vec{q}\cdot\vec{\sigma}}{\nu^2+v_F^2\,|\vec{q}\,|^2}\right). \quad (3.51)\end{aligned}$$

Using

$$\begin{aligned}\mathrm{Tr}\left(\left[(i(\nu+\omega)+v_F(\vec{k}+\vec{q})\cdot\vec{\sigma})(i\nu+v_F\vec{q}\cdot\vec{\sigma})\right]\right) &= -2\nu(\nu+\omega)+2v_F^2(\vec{k}+\vec{q})\cdot\vec{q} \\ &= -2\left\{\nu(\nu+\omega)-v_F^2(\vec{k}+\vec{q})\cdot\vec{q}\right\}, \quad (3.52)\end{aligned}$$

we obtain

$$\Pi(\omega,\vec{k}) = -2N\int\frac{d\nu}{2\pi}\frac{d\vec{q}}{(2\pi)^3}\frac{\nu(\nu+\omega)-v_F^2(\vec{k}+\vec{q})\cdot\vec{q}}{((\nu+\omega)^2+v_F^2|\vec{k}+\vec{q}\,|^2)(\nu^2+v_F^2\,|\vec{q}\,|^2)}. \quad (3.53)$$

Making use of the Feynman trick

$$\frac{1}{AB} = \int_{x=0}^{x=1} dx\frac{1}{(xA+(1-x)B)^2} \quad , \quad (3.54)$$

here $A = (\nu+\omega)^2 + v_F^2|\vec{k}+\vec{q}\,|^2$ and $B = \nu^2 + v_F^2\,|\vec{q}\,|^2$.
We get

$$\begin{aligned}\Pi(\omega,\vec{k}) &= -2N\int\frac{d\nu}{2\pi}\frac{d\vec{q}}{(2\pi)^3}\int_{x=0}^{x=1}dx\frac{\nu(\nu+\omega)-v_F^2(\vec{k}+\vec{q})\cdot\vec{q}}{(x((\nu+\omega)^2+v_F^2|\vec{k}+\vec{q}\,|^2)+(1-x)(\nu^2+v_F^2\,|\vec{q}\,|^2))^2} \\ &= -2N\int\frac{d\nu}{2\pi}\frac{d\vec{q}}{(2\pi)^3}\int_{x=0}^{x=1}dx\frac{\nu(\nu+\omega)-v_F^2(\vec{k}+\vec{q})\cdot\vec{q}}{\left[(x\omega+\nu)^2+v_F^2(x\vec{k}+\vec{q})^2+x(1-x)(\omega^2+v_F^2|\vec{k}|^2)\right]^2}.\end{aligned}$$

Changing variables $\nu \to \nu - x\omega$ and $\vec{q} \to \vec{q} - x\vec{k}$ leads to

$$\begin{aligned}\Pi(\omega,\vec{k}) &= -2N\int\frac{d\nu}{2\pi}\frac{d\vec{q}}{(2\pi)^3}\int_{x=0}^{x=1}dx\frac{(\nu-x\omega)(\nu+\omega-x\omega)-v_F(\vec{k}+\vec{q}-x\vec{k})\cdot(\vec{q}-x\vec{k})}{\left[\nu^2+v_F^2|\vec{q}|^2+x(1-x)(\omega^2+v_F^2|\vec{k}|^2)\right]^2}\\ &= = -2N\int\frac{d\nu}{2\pi}\frac{d\vec{q}}{(2\pi)^3}\int_{x=0}^{x=1}dx\frac{\nu^2-x\omega^2+x^2\omega^2+v_F^2x|\vec{k}|^2-v_F^2|\vec{q}|^2-x^2v_F^2|\vec{k}|^2}{\left[\nu^2+v_F^2|\vec{q}|^2+x(1-x)(\omega^2+v_F^2|\vec{k}|^2)\right]^2}.\end{aligned}$$

Furthermore, we use

$$\int_{-\infty}^{\infty}dx\frac{x^2}{(Ax^2+B)^2} = \frac{\pi}{2}\frac{1}{\sqrt{A^3B}} \quad ,$$
$$\int_{-\infty}^{\infty}dx\frac{1}{(Ax^2+B)^2} = \frac{\pi}{2}\frac{1}{\sqrt{AB^3}} \quad ,$$

and obtain

$$\Pi(\omega,\vec{k}) = -N\int\frac{d\vec{q}}{(2\pi)^3}\int_{x=0}^{x=1}dx\frac{v_F^2|\vec{k}|^2x(1-x)}{(v_F^2|\vec{q}|^2+x(1-x)(\omega^2+v_F^2|\vec{k}|^2))^{3/2}}. \tag{3.55}$$

We want to use

$$\int\frac{d\vec{q}}{(2\pi)^d}\frac{1}{(q^2+B)^n} = \frac{1}{4\pi^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}\left(\frac{1}{B}\right)^{n-d/2},$$

to integrate $d\vec{q}$. However, this formula is not well-defined for our case, $d=3$ and $n=3/2$. To proceed further, we calculate the integral for $n=3/2$ and $d=3-\delta$ and expand the result around $\delta=0$ using

$$\Gamma(3/2-d/2) = \Gamma(\delta/2) = \frac{2}{\delta}-\gamma+O(\delta), \tag{3.56}$$

and

$$A^{\delta/2} = 1+\frac{\delta}{2}\ln A+O(\delta^2), \tag{3.57}$$

where $\gamma\approx 0.5772$ known as Euler-Mascheroni constant. We obtain

$$\begin{aligned}\Pi(\omega,\vec{k}) &= -\frac{N|\vec{k}|^2}{4v_F\pi^2}\int_{x=0}^{x=1}dx\left\{\frac{2}{\delta}-\gamma+\ln\left(\frac{v_F^2}{x(1-x)(\omega^2+v_F^2|\vec{k}|^2)}\right)+O(\delta)\right\}x(1-x)\\ &= -\frac{N|\vec{k}|^2}{4v_F\pi^2}\int_{x=0}^{x=1}dx\left\{\frac{2}{\delta}-\gamma+\ln v_F^2-\ln x(1-x)-\ln(\omega^2+v_F^2|\vec{k}|^2)+O(\delta)\right\}x(1-x)\\ &= \frac{N|\vec{k}|^2}{4v_F\pi^2}\left(\int_{x=0}^{x=1}dx\left\{\ln(\omega^2+v_F^2|\vec{k}|^2)\right\}x(1-x)\right)-\frac{N|\vec{k}|^2}{4v_F\pi^2}C\\ &= \frac{N|\vec{k}|^2}{24v_F\pi^2}\ln(\omega^2+v_F^2|\vec{k}|^2)-\frac{N|\vec{k}|^2}{4v_F\pi^2}C\\ &= \frac{N|\vec{k}|^2}{24v_F\pi^2}\left(\ln(\omega^2+v_F^2|\vec{k}|^2)-6C\right)\end{aligned}$$

where

$$C = \int_{x=0}^{x=1}dx\left(\frac{2}{\delta}-\gamma+\ln v_F^2-\ln x(1-x)+O(\delta)\right)x(1-x) \tag{3.58}$$

which diverges when $\delta=0$. We introduce momentum cut-off, Λ, such that

$$6C = \lim_{\Lambda\to\infty}\ln(v_F^2\Lambda^2). \tag{3.59}$$

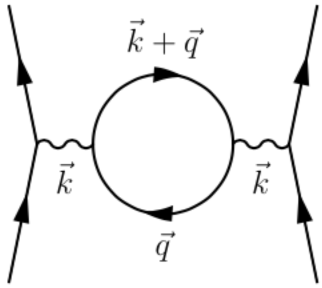

Figure 3.1: Feynman diagram contributing to dielectric constant renormalisation

The polarisation function $\Pi(\omega, \vec{k})$ of 3D Weyl semimetals diverges logarithmically,

$$\Pi_\Lambda(\omega, \vec{k}) = \frac{N|\vec{k}|^2}{24 v_F \pi^2} \ln \frac{\omega^2 + v_F^2|\vec{k}|^2}{v_F^2 \Lambda^2}. \tag{3.60}$$

Setting $v_F = 1$, we get

$$\Pi_\Lambda(\omega, \vec{k}) = \frac{N|\vec{k}|^2}{24 \pi^2} \ln \frac{\omega^2 + |\vec{k}|^2}{\Lambda^2}. \tag{3.61}$$

The dielectric constant renormalisation is given by

$$\begin{aligned} A(\omega, \vec{k}) &= -N\left(-\frac{e^2}{\epsilon|\vec{k}|^2}\right)^2 \int_{-\infty}^{\infty} d\nu \int_{|\vec{q}|=\Lambda/b}^{\Lambda} d\vec{q}\, \mathrm{Tr}[G_0(\vec{k}+\vec{q}, \nu+\omega) G_0(\vec{q}, \nu)] \qquad (3.62) \\ &= -\left(-\frac{e^2}{\epsilon|\vec{k}|^2}\right)^2 \left(\Pi_\Lambda(\omega, \vec{k}) - \Pi_{\Lambda/b}(\omega, \vec{k})\right) \\ &= \left(-\frac{e^2}{\epsilon|\vec{k}|^2}\right)^2 \frac{N|\vec{k}|^2}{12 v_F \pi^2} (\ln(\Lambda) - \ln(\Lambda/b)) \\ &= -\frac{1}{2}\frac{e^2}{\epsilon|\vec{k}|^2}\left\{\frac{2\alpha N}{3\pi}(\ln(\Lambda) - \ln(\Lambda/b))\right\} \end{aligned}$$

from which we can conclude that a scale dependent dielectric constant, $\epsilon(\Lambda)$, obeys [10]

$$\frac{1}{\epsilon\left(\frac{\Lambda}{b}\right)} = \frac{1}{\epsilon(\Lambda)}\left(1 - \frac{2\alpha N}{3\pi}(\ln(\Lambda) - \ln(\Lambda/b))\right) \tag{3.63}$$

$$\frac{d}{d\ln\Lambda}\frac{1}{\epsilon} = \frac{2\alpha N}{3\pi\epsilon} \tag{3.64}$$

$$\frac{d\epsilon}{d\ln\Lambda} = -\frac{2\alpha N}{3\pi}\epsilon. \tag{3.65}$$

As a result, the interaction parameter gets renormalised and obeys [14]

$$\begin{aligned} \frac{d\alpha}{d\ln\Lambda} &= \frac{d}{d\ln\Lambda}\left(\frac{e^2}{4\pi\epsilon v_F}\right) \qquad (3.66) \\ &= \frac{e^2}{4\pi}\left(\frac{1}{v_F}\frac{d}{d\ln\Lambda}\frac{1}{\epsilon} + \frac{1}{\epsilon}\frac{d}{d\ln\Lambda}\frac{1}{v_F}\right) \\ &= \frac{e^2}{4\pi}\left(\frac{2\alpha N}{3\pi\epsilon v_F} + \frac{2\alpha}{3\pi\epsilon v_F}\right) \\ &= \frac{2\alpha^2}{3\pi}(N+1) \end{aligned}$$

We find that the beta function of the interaction parameter of 3D Weyl semimetals is also positive and vanishes when $\alpha = 0$, so the interaction parameter flows to zero in the low energy limit. The interaction is suppressed and the quasiparticle excitations are free and massless .

3.3 Random phase approximation analysis

In the large number of fermion flavours limit, the higher order perturbation theory is dominated by RPA loop diagrams. Coulomb interaction is modified and given by

$$\begin{aligned} V^{RPA}(\vec{p},\eta) &= \frac{1}{V_C^{-1}(\vec{p}) - \Pi(\vec{p},\eta)} \\ &= \frac{1}{\epsilon|\vec{p}|^2/e^2 - \frac{N|\vec{p}|^2}{24\pi^2}\ln\frac{\eta^2+|\vec{p}|^2}{\Lambda^2}} \quad . \end{aligned} \tag{3.67}$$

$V^{RPA}(\vec{p},\eta)$ is an even function in η and all components of $\vec{p} = (p_x, p_y, p_z)$.
The RPA self-energy is given by

$$\Sigma^{RPA}(\vec{k},\omega) = -\int_{|\eta|,|\vec{p}|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\eta}{2\pi}\frac{d\vec{p}}{(2\pi)^2} G_0(\vec{k}+\vec{p},\omega+\eta)V^{RPA}(\vec{p},\eta). \tag{3.68}$$

Since $\vec{k}$ and ω is small compare to $\vec{p}$ and η, we can Taylor expand $G_0(\vec{k}+\vec{p},\omega+\eta)$ up to the first order in $\vec{k}$ and ω

$$\begin{aligned} G_0(\vec{k}+\vec{p},\omega+\eta) &= G_0(\vec{p},\eta) + \omega\frac{\partial G_0(\vec{p},\eta)}{\partial\eta} + \sum_{i\in x,y} k_i \frac{\partial G_0(\vec{p},\eta)}{\partial p_i} \\ &= G_0(\vec{p},\eta) + \omega\left(\frac{i}{\eta^2+|\vec{p}|^2} - \frac{2i\eta^2}{(\eta^2+|\vec{p}|^2)^2} - \frac{2\eta\vec{p}\cdot\vec{\sigma}}{(\eta^2+|\vec{p}|^2)^2}\right) \\ &\quad + \sum_{i\in x,y} k_i \left(\frac{-2ip_i\eta}{(\eta^2+|\vec{p}|^2)^2} + \sum_{j\in x,y}\frac{\sigma_j\delta_{ij}}{\eta^2+|\vec{p}|^2} - \sum_{j\in x,y}\frac{2\sigma_j p_j p_i}{(\eta^2+|\vec{p}|^2)^2}\right) \end{aligned} \tag{3.69}$$

Because of the even function $V^{RPA}(\vec{p},\eta)$, all terms in $G_0(\vec{k}+\vec{p},\omega+\eta)$ that are odd in η, p_x, p_y, and p_z are zero. Moreover, p_x, p_y, and p_z are equivalent in the integral. This can be used to simplify the RPA self energy to

$$\Sigma^{RPA}(\vec{k},\omega) = -\int_{|\eta|,|\vec{p}|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\eta}{2\pi}\frac{d\vec{p}}{(2\pi)^3}\left[i\omega\left(\frac{|\vec{p}|^2-\eta^2}{(\eta^2+|\vec{p}|^2)^2}\right) + \vec{\sigma}\cdot\vec{k}\left(\frac{\eta^2+\frac{1}{3}|\vec{p}|^2}{(\eta^2+|\vec{p}|^2)^2}\right)\right]V^{RPA}(\vec{p},\eta).$$

Introducing three-vectors $\mathbf{q} = (q_0,\vec{q}) = (\eta,\vec{p})$, we have in spherical coordinate $q_0 = |\mathbf{q}|\cos\theta$ and $|\vec{p}\,| = |\mathbf{q}|\sin\theta$, and obtain

$$\Sigma^{RPA}(\vec{k},\omega) = -\int_{|\mathbf{q}|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\mathbf{q}}{(2\pi)^4}\left[\frac{-i\omega}{|\mathbf{q}|^2}\left(\cos^2\theta - \sin^2\theta\right) + \frac{\vec{\sigma}\cdot\vec{k}}{|\mathbf{q}|^2}(\cos^2\theta + \frac{1}{3}\sin^2\theta)\right]V^{RPA}(\vec{p},\eta). \tag{3.70}$$

To proceed further, we expand the RPA potential

$$V^{RPA}(\vec{p},\eta) = \frac{e^2/\epsilon}{\left(1 - \frac{e^2N}{12\epsilon\pi^2}\ln\frac{\mathbf{q}}{\Lambda}\right)|\vec{p}|^2} \approx \frac{e^2/\epsilon}{\left(1 - \frac{e^2N}{12\epsilon\pi^2}\left\{\frac{\mathbf{q}}{\Lambda}-1\right\}\right)|\vec{p}|^2} \approx \frac{e^2/\epsilon}{\left(1 + \frac{e^2N}{12\epsilon\pi^2}\right)|\vec{p}|^2}. \tag{3.71}$$

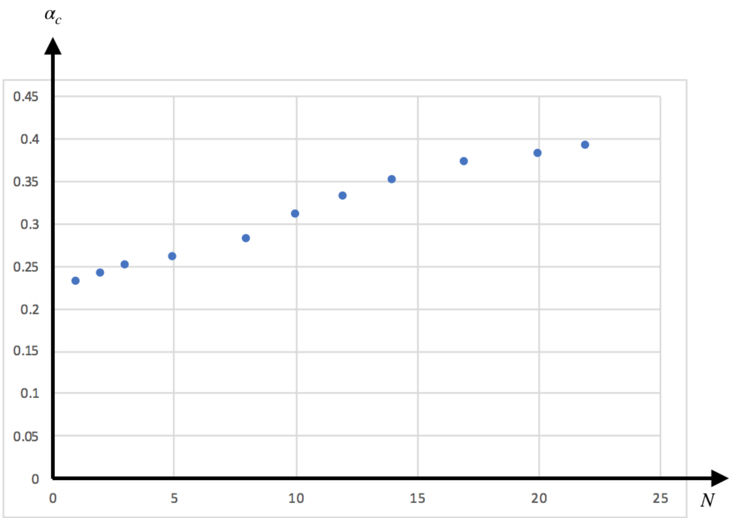

Figure 3.2: phase diagram of 3D Weyl semimetals in the $\alpha - N$ plane

This yields

$$\begin{aligned}\Sigma^{RPA}(\vec{k},\omega) &= -\int_{|\mathbf{q}|=\frac{\Lambda}{b}}^{\Lambda} \frac{d\mathbf{q}}{(2\pi)^4}\left[\frac{-i\omega}{|\mathbf{q}|^2}\left(\cos^2\theta-\sin^2\theta\right)+\frac{\vec{\sigma}\cdot\vec{k}}{|\mathbf{q}|^2}(\cos^2\theta+\frac{1}{3}\sin^2\theta)\right]\frac{e^2/\epsilon}{\left(1+\frac{e^2N}{12\epsilon\pi^2}\right)|\mathbf{q}|^2\sin^2\theta} \\ &= -\frac{1}{6\pi^2}\frac{e^2/\epsilon}{\left(1+\frac{e^2N}{12\epsilon\pi^2}\right)}\left[\ln(\Lambda)-\ln\left(\frac{\Lambda}{b}\right)\right]\vec{\sigma}\cdot\vec{k}. \end{aligned} \tag{3.72}$$

Inserting back v_F leads to

$$\Sigma^{RPA}(\vec{k},\omega) = -\frac{1}{6\pi^2}\frac{e^2/\epsilon}{\left(1+\frac{e^2N}{12\epsilon\pi^2}\right)}\left[\ln(\Lambda)-\ln\left(\frac{\Lambda}{b}\right)\right]v_F\vec{\sigma}\cdot\vec{k}, \tag{3.73}$$

from which we find that a scale dependent Fermi velocity obeys

$$\frac{dv_F(\Lambda)}{d\ln(\Lambda)} = -\frac{1}{6\pi^2}\frac{e^2/\epsilon}{\left(1+\frac{e^2N}{12\epsilon\pi^2}\right)}v_F(\Lambda).$$

The beta function of the Fermi velocity is negative. In the low energy limit, the Fermi velocity increases towards the speed of light and the interaction parameter flows towards weak coupling. The interaction is suppressed and the quasiparticle excitations are still free and massless .

3.4 Strong coupling

From the renormalisation group analysis, we find that effects of Coulomb interaction on 3D Weyl semimetals are exactly the same as on their 2D counterparts. It is reasonable to look for the semimetal-insulator phase transition in 3D. We are looking for the possibility of spontaneous mass generation in three dimensional Weyl semimetal due to electron-electron interaction. Different from 2D materials, two valleys of Weyl semimetals must be coupled to generate mass, so we assume that the interacting Green function can be written as

$$G^{-1}(\vec{a},\alpha) = -i\alpha\mathbb{1}_{4\times4}+\vec{\beta}\cdot\vec{a}+\Delta(|\vec{a}|)\beta^0, \tag{3.74}$$

where

$$\beta^0 = \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix} \qquad \vec{\beta} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} \tag{3.75}$$

We assume that the mass Δ only depends on the magnitude of the momentum and all parameter renormalisations are neglected. The A_0 field propagator is given by the on-shell random phase approximated potential, namely

$$D(\vec{p},\omega) = V^{RPA}(\vec{p},\omega = |\vec{p}|) = \frac{e^2/\epsilon}{\left(1 - \frac{e^2 N}{12\epsilon\pi^2}\ln\frac{\sqrt{2}|\vec{p}|}{\Lambda}\right)|\vec{p}|^2}. \tag{3.76}$$

Plugging these propagators into the Schwinger-Dyson equation with the approximation that the vertex function γ is unity equation we get

$$\begin{aligned} -i\alpha\mathbb{1}_{4\times 4} + \vec{\beta}\cdot\vec{a} + \Delta(|\vec{a}|)\beta^0 &= -i\alpha\mathbb{1}_{4\times 4} + \vec{\beta}\cdot\vec{a} \\ &+ \int \frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi}\frac{e^2/\epsilon}{\left(1 - \frac{e^2 N}{12\epsilon\pi^2}\ln\frac{\sqrt{2}|\vec{p}+\vec{a}|}{\Lambda}\right)|\vec{p}+\vec{a}|^2}\frac{i\omega + \vec{\beta}\cdot\vec{p} + \Delta(|\vec{p}|)\beta^0}{\omega^2 + |\vec{p}|^2 + \Delta(|\vec{p}|)^2}. \end{aligned}$$

The coefficient of β^0 gives the self-consistent mass equation

$$\Delta(|\vec{a}|) = \int \frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi}\frac{e^2/\epsilon}{\left(1 - \frac{e^2 N}{12\epsilon\pi^2}\ln\frac{\sqrt{2}|\vec{p}+\vec{a}|}{\Lambda}\right)|\vec{p}+\vec{a}|^2}\frac{\Delta(|\vec{p}|)}{\omega^2 + |\vec{p}|^2 + \Delta(|\vec{p}|)^2}. \tag{3.77}$$

Integrating out ω to get

$$\begin{aligned} \Delta(|\vec{a}|) &= \int \frac{d\vec{p}}{(2\pi)^2}\frac{1}{2}\frac{e^2/\epsilon}{\left(1 - \frac{e^2 N}{12\epsilon\pi^2}\ln\frac{\sqrt{2}|\vec{p}+\vec{a}|}{\Lambda}\right)|\vec{p}+\vec{a}|^2}\frac{\Delta(|\vec{p}|)}{\sqrt{|\vec{p}|^2 + \Delta(|\vec{p}|)^2}} \\ &= \int |\vec{p}|^2 d|\vec{p}| \int_0^{\pi}\sin\theta d\theta \int_0^{2\pi} d\phi \frac{e^2/8\pi^2\epsilon}{\left(1 - \frac{e^2 N}{12\epsilon\pi^2}\ln\frac{\sqrt{2}\sqrt{|\vec{p}|^2+|\vec{a}|^2+2|\vec{p}||\vec{a}|\cos\theta}}{\Lambda}\right)(|\vec{p}|^2 + |\vec{a}|^2 + 2|\vec{p}||\vec{a}|\cos\theta)} \\ &\times \frac{\Delta(|\vec{p}|)}{\sqrt{|\vec{p}|^2 + \Delta(|\vec{p}|)^2}} \end{aligned} \tag{3.78}$$

Using

$$\begin{aligned} &\int_0^{\pi} \frac{\sin\theta}{\left[|\vec{p}|^2 + |\vec{a}|^2 + 2|\vec{p}||\vec{a}|\cos\theta\right]\left[1 - z\ln(\sqrt{2}\sqrt{|\vec{p}|^2 + |\vec{a}|^2 + 2|\vec{p}||\vec{a}|\cos\theta})\right]} d\theta \\ &= \frac{1}{z|\vec{a}||\vec{p}|}\left\{\ln\left[-2 + z\ln\left[2(|\vec{a}| - |\vec{p}|)^2\right]\right] - \ln\left[-2 + z\ln\left[2(|\vec{a}| + |\vec{p}|)^2\right]\right]\right\}, \end{aligned} \tag{3.79}$$

we obtain

$$\begin{aligned} \Delta(|\vec{a}|) &= \int_0^1 d|\vec{p}|\frac{3\pi}{N}\frac{|\vec{p}|}{|\vec{a}|}\left\{\ln\left[-2 + \frac{\alpha N}{3\pi}\ln\left[2(|\vec{a}| - |\vec{p}|)^2\right]\right] - \ln\left[-2 + \frac{\alpha N}{3\pi}\ln\left[2(|\vec{a}| + |\vec{p}|)^2\right]\right]\right\} \\ &\times \frac{\Delta(|\vec{p}|)}{\sqrt{|\vec{p}|^2 + \Delta(|\vec{p}|)^2}} \end{aligned} \tag{3.80}$$

which is solved numerically. A phase diagram in the $\alpha - N$ plane is shown in Figure.3.2, we find that the critical value of N_c is 23 above which semimetal-insulator phase transition does not happen.

Chapter 4

Conclusion

In this thesis, We have investigated the role of Coulomb interaction in 2D and 3D Dirac/Weyl Semimetals. Dirac/Weyl Semimetals are materials in which electrons have a linear dispersion relation. They are solid-state analogue of relativistic massless Dirac particles. Graphene is the first experimentally observed Weyl semimetal, before their 3D counterparts which are less stable were realised. Coulomb interaction is an instantaneous interaction by photons and live in 3 dimension. Its strength is characterised by the dimensionless coupling parameter $\alpha = e^2/4\pi\epsilon v_F$ which is the ratio of coulomb potential energy to the kinetic energy. It is marginal in the RG sense, meaning it might get stronger or weaker in the low energy limit . Thus we have studied its effects in both weak and strong regimes by perturbation theory and mean field theory, respectively.

First, we investigated the effects of weak interaction. In the low energy sector they are obtained from renormalisation group analysis by integrating out the higher energy modes, and absorb the result into the parameters of the theory. In this cases, they are Fermi velocity, v_F, and dielectric constant, ϵ. We have done the calculation for the first order perturbation theory, and the random phase approximation, where all higher order corrections of a particular type of diagrams namely the poralisation function are included in the Coulomb potential. This approximation is more accurate for the systems with more Weyl fermion flavour because the polarisation function is linearly proportional to the number of fermion flavours , N. In the large N limit, this class of diagram dominates the others. The RPA Coulomb potential depends on both α and N. It is found that, in 2D Weyl semimetals, the first order perturbation theory renormalises the Fermi velocity while the dielectric constant is kept constant. Lowering the energy scale increases v_F and consequently α decrease when energy scale is lowered. The interaction is weaker and weaker and get suppressed in the end. Even though, results from RPA approximation look more complicated, the situation does not change. In low energy limit, the Fermi velocity approaches the speed of light. interactions are suppressed. The quasiparticles are effectively free and massless. For 3D Weyl semimetals, In addition to velocity and coupling parameter renormalisations, the dielectric constant is also renormalised. It gets larger when the energy scale decreases. Since α is inversely proportional to the product of ϵ and v_F which is very large in low energy scale, the interaction is again suppressed.

Next, we looked at the strong coupling effects using mean field theory. We examined the possibility to open the gap when Coulomb interaction is included by self-consistently solving the Schwinger-Dyson equation which is an integral equation relating the Green function to the interaction. We solve this equation both analytically and numerically for 2D. It is found that mass is not constant but momentum dependence. It is spontaneously generated when α is larger than critical value α_c and N is smaller than its critical value reaches N_c. When N is larger than its critical value, the gap cannot be opened no matter how strong interaction is. The critical line divides two different phase of Weyl systems in 2D. Above the line is insulating phase and below is semimetal. Critical values α_c and N_c from both numerical and analytical methods are in agreement while the analytical solution underestimate the value of mass. The gap is underestimated by the analytical method. For the 3D Weyl semimetals, the presence of a logarithmic divergence in the RPA potential render the integral equation too difficult to solve analytically. We only perform numerical calculations and find the same feature as in 2D. the mass can be induced spontaneously in the system with lesser fermion flavours than its critical value when the interaction is strong enough. The mass is momentum dependent and more likely to be created than in 2D.

Spontaneously mass can be induced by Coulomb interaction in both 2D and 3D Weyls semimetals but only at specific value of α and N, it is very unlikely to happen. From our analysis we conclude that both in 2D and 3D Weyl semimetals a mass can spontaneously be generated. The critical value of α_c and N_c are within a realistic range. It has to be noted, however, that so far no experimental evidence, especially in graphene has been found. [20] [21]

Appendix A

Schwinger-Dyson Equation

In this appendix, we are going to derive the Schwinger-Dyson equation. We follow ref.[13]. Given the partition function

$$Z[J,\eta,\bar{\eta}] = \int DA_0 D\Psi D\bar{\Psi} e^{-S[\bar{\Psi},\Psi,A_0]-\int d\vec{x}d\tau\{A_0(\vec{x},t)J(\vec{x},t)+\bar{\Psi}(\vec{x},t)\eta(\vec{x},t)+\bar{\eta}(\vec{x},t)\Psi(\vec{x},t)\}}, \tag{A.1}$$

its the total derivative is vanishing

$$\begin{aligned}
0 &= \int DA_0 D\Psi D\bar{\Psi} \frac{\delta}{\delta\bar{\Psi}(\vec{y},t')} e^{-S[\bar{\Psi},\Psi,A_0]-\int d\vec{x}dt\{A_0(\vec{x},t)J(\vec{x},t)+\bar{\Psi}(\vec{x},t)\eta(\vec{x},t)+\bar{\eta}(\vec{x},t)\Psi(\vec{x},t)\}} \\
&= \int DA_0 D\Psi D\bar{\Psi} \Big(\frac{\delta S}{\delta\bar{\Psi}(\vec{y},t')}[\bar{\Psi},\Psi,A_0] + \eta \Big) e^{-S[\bar{\Psi},\Psi,A_0]-\int d\vec{x}dt\{A_0(\vec{x},t)J(\vec{x},t)+\bar{\Psi}(\vec{x},t)\eta(\vec{x},t)+\bar{\eta}(\vec{x},t)\Psi(\vec{x},t)\}} \\
&= \left[\int d\vec{x}dt\delta(\vec{x}-\vec{y})\delta(t-t') \left(\left(\frac{d}{dt} - iv_F\vec{\sigma}\cdot\vec{\bigtriangledown}_{\vec{x}} \right) \left(-\frac{\delta}{\delta\bar{\eta}(\vec{x},t)} \right) - \frac{\delta}{\delta J(\vec{x},t)} \frac{\delta}{\delta\bar{\eta}(\vec{x},t)} + \eta(\vec{x},t) \right) \right] \\
&\quad Z[J,\eta,\bar{\eta}] \\
&= \left[\left(\frac{d}{dt'} - iv_F\vec{\sigma}\cdot\vec{\bigtriangledown}_{\vec{y}} \right) \left(-\frac{\delta}{\delta\bar{\eta}(\vec{y},t')} \right) - \frac{\delta}{\delta J(\vec{y},t')} \frac{\delta}{\delta\bar{\eta}(\vec{y},t')} + \eta(\vec{y},t') \right] Z[J,\eta,\bar{\eta}] \\
&= \left(\frac{\delta}{\delta\eta(\vec{x},t)} \right) \left\{ \left[\left(\frac{d}{dt'} - iv_F\vec{\sigma}\cdot\vec{\bigtriangledown}_{\vec{y}} \right) \left(-\frac{\delta}{\delta\bar{\eta}(\vec{y},t')} \right) - \frac{\delta}{\delta J(\vec{y},t')} \frac{\delta}{\delta\bar{\eta}(\vec{y},t')} + \eta(\vec{y},t') \right] Z[J,\eta,\bar{\eta}] \right\} \\
&= \left[\left(\frac{d}{dt'} - iv_F\vec{\sigma}\cdot\vec{\bigtriangledown}_{\vec{y}} \right) \left(\frac{\delta}{\delta\eta(\vec{x},t)} \right) \left(-\frac{\delta}{\delta\bar{\eta}(\vec{y},t')} \right) - \frac{\delta}{\delta J(\vec{y},t')} \left(\frac{\delta}{\delta\eta(\vec{x},t)} \right) \frac{\delta}{\delta\bar{\eta}(\vec{y},t')} + \right. \\
&\quad \left. \delta(\vec{x}-\vec{y})\delta(t-t') \right] Z[J,\eta,\bar{\eta}] \\
&= -\left(\frac{d}{dt'} - iv_F\vec{\sigma}\cdot\vec{\bigtriangledown}_{\vec{y}} \right) G(\vec{y}-\vec{x},t'-t;J,\eta,\bar{\eta}) \\
&\quad -\frac{1}{Z[J,\eta,\bar{\eta}]} \frac{\delta}{\delta J(\vec{y},t')} \Big(Z[J,\eta,\bar{\eta}] G(\vec{y}-\vec{x},t'-t;J,\eta,\bar{\eta}) \Big) + \delta(\vec{x}-\vec{y})\delta(t-t') \\
&= -\left(\frac{d}{dt'} - iv_F\vec{\sigma}\cdot\vec{\bigtriangledown}_{\vec{y}} \right) G(\vec{y}-\vec{x},t'-t;J,\eta,\bar{\eta}) - \frac{\delta}{\delta J(\vec{y},t')} G(\vec{y}-\vec{x},t'-t;J,\eta,\bar{\eta}) + \delta(\vec{x}-\vec{y})\delta(t-t').
\end{aligned}$$

Making use of the relation between effective action Γ and the propagator,

$$G^{-1}(x-y) = \frac{\delta^2\Gamma}{\delta\langle\Psi(x)\rangle\,\delta\langle\bar{\Psi}(x)\rangle}, \tag{A.2}$$

to compute

$$\begin{aligned}
\frac{\delta}{\delta J(\vec{y},t')}G(\vec{y}-\vec{x},t'-t) &= \frac{\delta}{\delta J(\vec{y},t')}\left(\frac{\delta^2\Gamma}{\delta\left<\Psi(\vec{y},t')\right>\delta\left<\bar{\Psi}(\vec{x},t)\right>}\right)^{-1} \\
&= \int d\vec{u}dt''\frac{\delta\left<A_0(\vec{u},t'')\right>}{\delta J(\vec{y},t')}\frac{\delta}{\delta\left<A_0(\vec{u},t'')\right>}\left(\frac{\delta^2\Gamma}{\delta\left<\Psi(\vec{y},t')\right>\delta\left<\bar{\Psi}(\vec{x},t)\right>}\right)^{-1} \\
&= \int d\vec{u}d\vec{v}d\vec{w}dt''dt'''dt''''D(\vec{u}-\vec{y},t''-t')G(\vec{y}-\vec{v},t'-t''') \\
&\quad \left(\frac{\delta^3\Gamma}{\delta\left<A_0(\vec{u},t'')\right>\delta\left<\Psi(\vec{v},t''')\right>\delta\left<\bar{\Psi}(\vec{w},t'''')\right>}\right)G(\vec{w}-\vec{x},t''''-t) \\
&= \int d\vec{u}d\vec{v}d\vec{w}dt''dt'''dt''''D(\vec{u}-\vec{y},t''-t')G(\vec{y}-\vec{v},t'-t''') \\
&\quad \gamma(\vec{v},\vec{w},t''',t'''';\vec{u},t'')G(\vec{w}-\vec{x},t''''-t).
\end{aligned}$$

So,

$$\begin{aligned}
0 &= -\left(\frac{d}{dt'}-iv_F\vec{\sigma}\cdot\vec{\bigtriangledown}_{\vec{y}}\right)G(\vec{y}-\vec{x},t'-t)-\int d\vec{u}d\vec{v}d\vec{w}dt''dt'''dt''''D(\vec{u}-\vec{y},t''-t')G(\vec{y}-\vec{v},t'-t''') \\
&\quad \gamma(\vec{v},\vec{w},t''',t'''';\vec{u},t'')G(\vec{w}-\vec{x},t''''-t)+\delta(\vec{x}-\vec{y})\delta(t-t') \\
&= -\left(\frac{d}{dt'}-iv_F\vec{\sigma}\cdot\vec{\bigtriangledown}_{\vec{y}}\right)\int dxdtG(\vec{y}-\vec{x},t'-t)G^{-1}(\vec{x}-\vec{z},t-t''''')- \\
&\quad \int d\vec{x}d\vec{u}d\vec{v}d\vec{w}dtdt''dt'''dt''''D(\vec{u}-\vec{y},t''-t')G(\vec{y}-\vec{v},t'-t''')\gamma(\vec{v},\vec{w},t''',t'''';\vec{u},t'')\times \\
&\quad G(\vec{w}-\vec{x},t''''-t)G^{-1}(\vec{x}-\vec{z},t-t''''')+\int dxdt\delta(\vec{x}-\vec{y})\delta(t-t')G^{-1}(\vec{x}-\vec{z},t-t''''') \\
&= -\left(\frac{d}{dt'}-iv_F\vec{\sigma}\cdot\vec{\bigtriangledown}_{\vec{y}}\right)\delta(\vec{y}-\vec{z})\delta(t'-t''''')+G^{-1}(\vec{y}-\vec{z},t'-t''''')- \\
&\quad \int d\vec{u}d\vec{v}dt''dt'''D(\vec{u}-\vec{y},t''-t')G(\vec{y}-\vec{v},t'-t''')\gamma(\vec{v},\vec{z},t''',t''''';\vec{u},t'').
\end{aligned}$$

Then, performimg Fourier transformation, we get

$$\begin{aligned}
0 &= \int\frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi}\left[\left(\frac{d}{dt'}-iv_F\vec{\sigma}\cdot\vec{\bigtriangledown}_{\vec{y}}\right)-G^{-1}(\vec{p},\omega)\right]e^{i\vec{p}\cdot(\vec{y}-\vec{z})-i\omega(t'-t''''')}+ \\
&\quad \int\frac{d\vec{q}}{(2\pi)^2}\frac{d\phi}{2\pi}\frac{d\vec{m}}{(2\pi)^2}\frac{d\chi}{2\pi}\int d\vec{u}d\vec{v}dt''dt'''D(\vec{q},\phi)e^{-i\vec{q}\cdot(\vec{u}-\vec{y})+i\phi(t''-t')}G(\vec{m},\chi)e^{-i\vec{m}\cdot(\vec{y}-\vec{v})+i\chi(t'-t''')}\times \\
&\quad \int\frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi}\frac{d\vec{r}}{(2\pi)^2}\frac{d\rho}{2\pi}\frac{d\vec{s}}{(2\pi)^2}\frac{d\tau}{2\pi}\gamma(\vec{p},\vec{r},\omega,\rho;\vec{s},\tau)e^{-i\vec{p}\cdot\vec{v}+i\omega t'''}e^{-i\vec{r}\cdot\vec{z}+i\rho t'''''}e^{-i\vec{s}\cdot\vec{u}+i\tau t''} \\
&= \int\frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi}\left[(-i\omega+v_F\vec{\sigma}\cdot\vec{p})-G^{-1}(\vec{p},\omega)\right]e^{i\vec{p}\cdot(\vec{y}-\vec{z})-i\omega(t'-t''''')}+ \\
&\quad \int\frac{d\vec{q}}{(2\pi)^2}\frac{d\phi}{2\pi}\frac{d\vec{m}}{(2\pi)^2}\frac{d\chi}{2\pi}\frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi}\frac{d\vec{r}}{(2\pi)^2}\frac{d\rho}{2\pi}\frac{d\vec{s}}{(2\pi)^2}\frac{d\tau}{2\pi}d\vec{u}d\vec{v}dt''dt'''D(\vec{q},\phi)G(\vec{m},\chi)\gamma(\vec{p},\vec{r},\omega,\rho;\vec{s},\tau)\times \\
&\quad e^{-i\vec{u}\cdot(\vec{q}+\vec{s})+it''(\phi+\tau)}e^{-i\vec{v}\cdot(\vec{p}-\vec{m})+it'''(\omega-\chi)}e^{-i\vec{r}\cdot\vec{z}+i\rho t'''''}e^{-i\vec{y}\cdot(\vec{m}-\vec{q})+it'(\chi-\phi)} \\
&= \int\frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi}\left[(-i\omega+v_F\vec{\sigma}\cdot\vec{p})-G^{-1}(\vec{p},\omega)\right]e^{i\vec{p}\cdot(\vec{y}-\vec{z})-i\omega(t'-t''''')}+ \\
&\quad \int\frac{d\vec{q}}{(2\pi)^2}\frac{d\phi}{2\pi}\frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi}\frac{d\vec{r}}{(2\pi)^2}\frac{d\rho}{2\pi}D(\vec{q},\phi)G(\vec{p},\omega)\gamma(\vec{p},\vec{r},\omega,\rho;-\vec{q},-\phi)\times \\
&\quad e^{-i\vec{r}\cdot\vec{z}+i\rho t'''''}e^{-i\vec{y}\cdot(\vec{p}-\vec{q})+it'(\omega-\phi)}.
\end{aligned}$$

Notice that this equation holds true iff $\vec{r} = \vec{p} - \vec{q}$ and $\rho = \omega - \phi$, we get

$$
\begin{aligned}
0 \;=\; & \int \frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi}\left[(-i\omega + v_F\vec{\sigma}\cdot\vec{p}) - G^{-1}(\vec{p},\omega)\right] e^{i\vec{p}\cdot(\vec{y}-\vec{z})-i\omega(t'-t''''')} + \\
& \int \frac{d\vec{q}}{(2\pi)^2}\frac{d\phi}{2\pi}\frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi} D(\vec{q},\phi)G(\vec{p},\omega)\gamma(\vec{p},\vec{q}-\vec{p},\omega,\phi-\omega;-\vec{q},-\phi) \times \\
& e^{i(\vec{q}-\vec{p})\cdot(\vec{y}-\vec{z})-i(\phi-\omega)(t'-t''''')} \\
= \; & \int d(\vec{y}-\vec{z})d(t'-t''''')\Bigg[\int \frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi}\left[(-i\omega + v_F\vec{\sigma}\cdot\vec{p}) - G^{-1}(\vec{p},\omega)\right] e^{i\vec{p}\cdot(\vec{y}-\vec{z})-i\omega(t'-t''''')} + \\
& \int \frac{d\vec{q}}{(2\pi)^2}\frac{d\phi}{2\pi}\frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi} D(\vec{q},\phi)G(\vec{p},\omega)\gamma(\vec{p},\vec{q}-\vec{p},\omega,\phi-\omega;-\vec{q},-\phi) \times \\
& e^{i(\vec{q}-\vec{p})\cdot(\vec{y}-\vec{z})-i(\phi-\omega)(t'-t''''')}\Bigg] e^{-i\vec{a}\cdot(\vec{y}-\vec{z})+i\alpha(t'-t''''')} \\
= \; & \int d\vec{p}d\omega\left[(-i\omega + v_F\vec{\sigma}\cdot\vec{p}) - G^{-1}(\vec{p},\omega)\right]\delta(\vec{p}-\vec{a})\delta(\omega-\alpha) + \\
& \int \frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi} d\vec{q}d\phi D(\vec{q},\phi)G(\vec{p},\omega)\gamma(\vec{p},\vec{q}-\vec{p},\omega,\phi-\omega;-\vec{q},-\phi) \times \delta(\vec{q}-\vec{p}-\vec{a})\delta(\phi-\omega-\alpha) \\
= \; & (-i\alpha + v_F\vec{\sigma}\cdot\vec{a}) - G^{-1}(\vec{a},\alpha) + \int \frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi} D(\vec{p}+\vec{a},\omega+\alpha)G(\vec{p},\omega)\gamma(\vec{p},\vec{a},\omega,\alpha;-\vec{p}-\vec{a},-\omega-\alpha).
\end{aligned}
$$

This gives the Schwinger-Dyson equation

$$
G^{-1}(\vec{a},\alpha) = (-i\alpha + v_F\vec{\sigma}\cdot\vec{a}) + \int \frac{d\vec{p}}{(2\pi)^2}\frac{d\omega}{2\pi} D(\vec{p}+\vec{a},\omega+\alpha)G(\vec{p},\omega)\gamma(\vec{p},\vec{a},\omega,\alpha;-\vec{p}-\vec{a},-\omega-\alpha). \tag{A.3}
$$

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