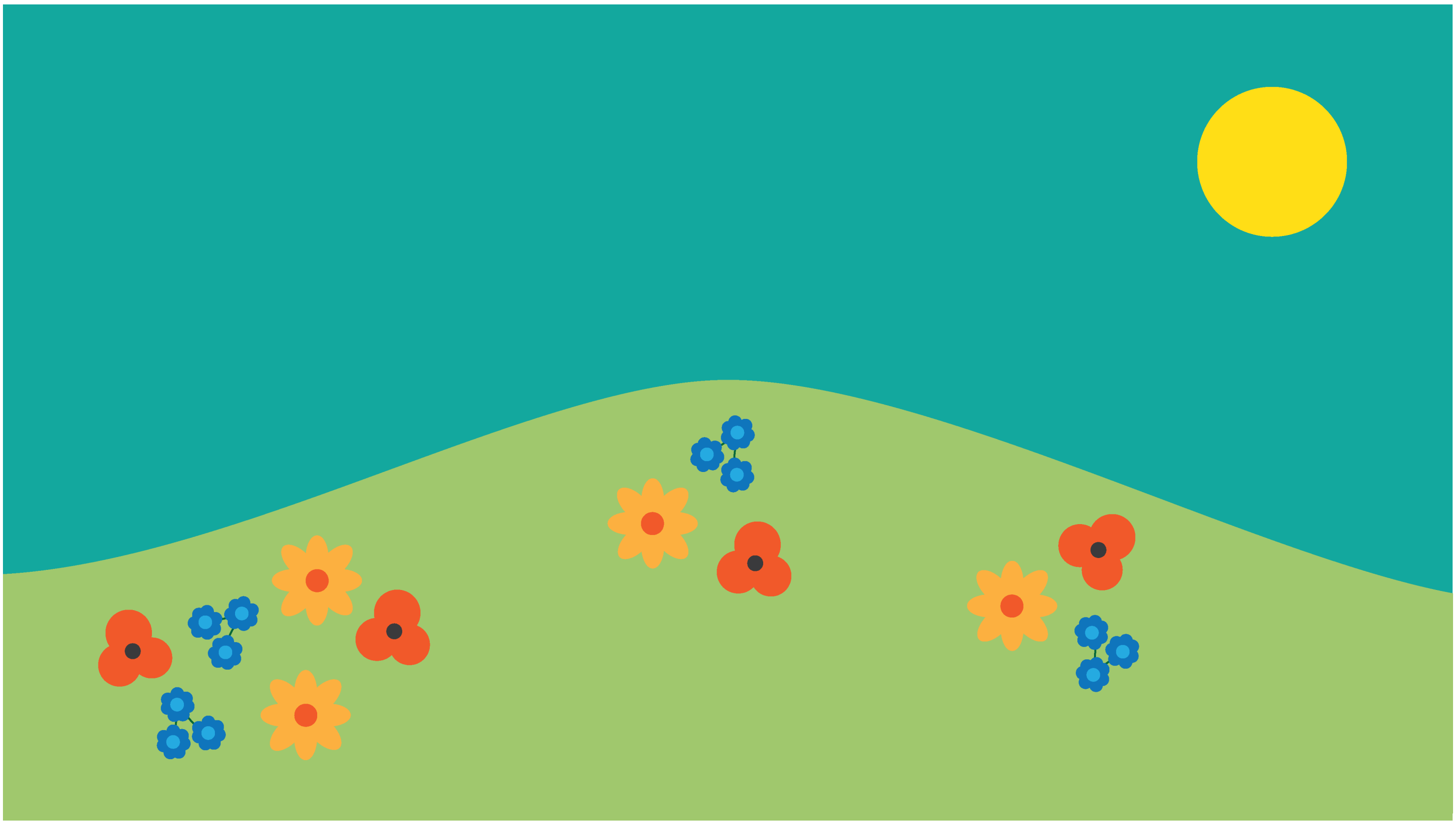
Adiabatic Modes in an Open Universe

Master Thesis Theoretical Physics



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Abstract

Adiabatic modes are solutions in general relativity which are locally indistinguishable from the Friedmann-Lemaître-Robertson-Walker metric after an appropriate transformation. In other words, they are cosmological perturbations which resemble a pure gauge profile. Many adiabatic modes are known for spatially flat universes, providing model-independent solutions and implying soft theorems. In this thesis, we generalize the theory of adiabatic modes to open universes (i.e. universes with negative spatial curvature). The main results are the open-universe versions of Weinberg's tensor adiabatic mode in equation (5.23) and Weinberg's scalar adiabatic modes in equation (6.19). These modes are, however, puzzling. While it appears that for the tensor gauge modes are physical (sub-curvature), it seems that monochromatic scalar modes can never become adiabatic. This could imply, for single-field inflation in an open universe, that inflation does not solely produce adiabatic modes and that Maldacena's consistency condition is violated. Future research to get to the bottom of these issues is suggested.

Acknowledgments

I would like to thank my supervisor Enrico Pajer whose great physical insight has guided me through this very interesting research project. Also, I would like to thank Sadra Jazayeri with whom I have collaborated extensively to write this thesis and who was always willing to help and explain. Lastly, I would like to acknowledge Eva Peet for drawing all original pictures used in this thesis and for mental support.

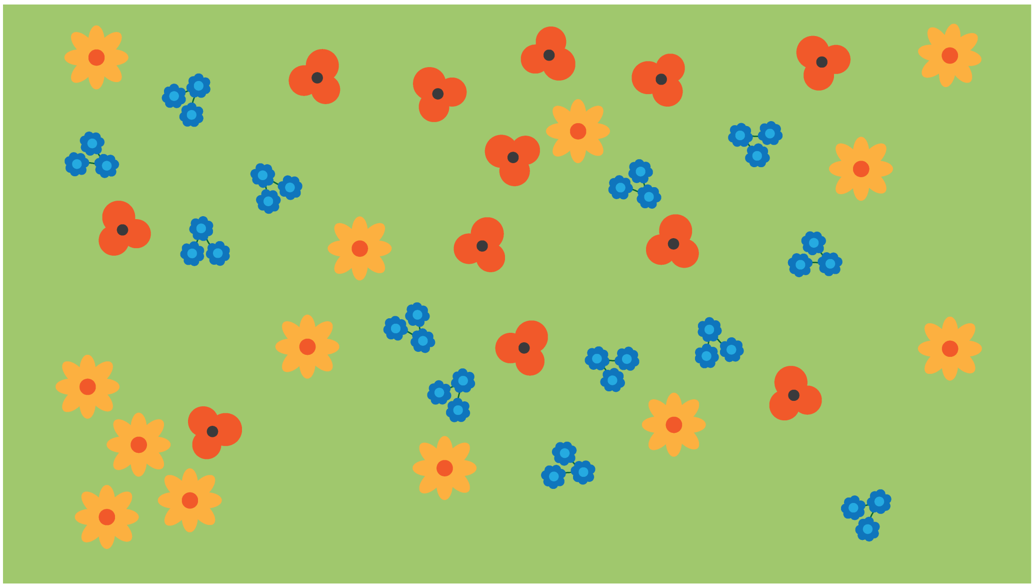
Contents

Introduction

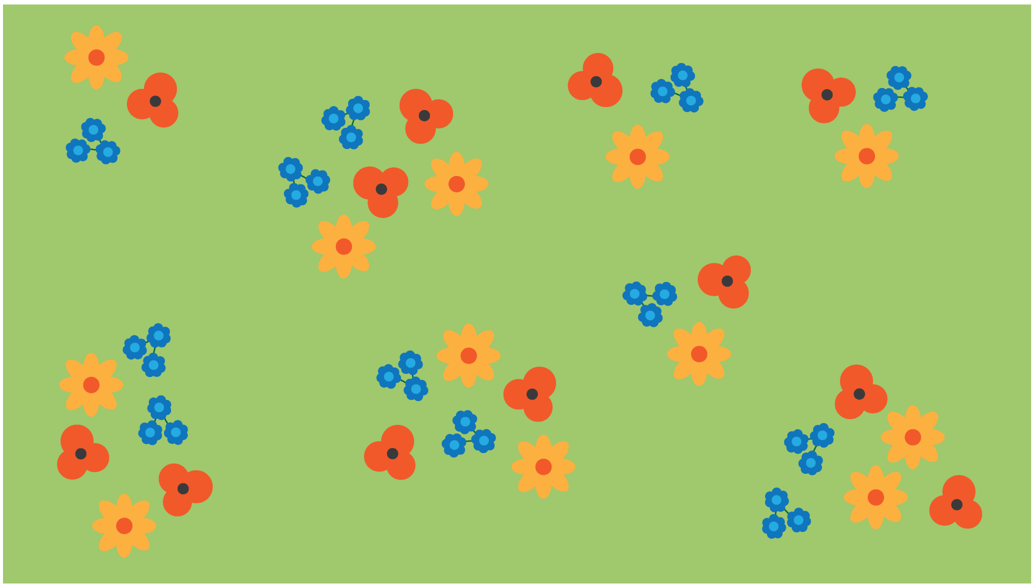
For the moment, we would like you to imagine a quiet farmland meadow. Suppose that, in the meadow, three different types of flowers appear. Without any further information, one might expect these flowers to be distributed throughout the meadow in a random fashion. After all, flower seeds are usually carried by the wind or by animals, both of which have no precognition of where the seeds should end up. The meadow, then, might look like the one in figure 1a, which is one of many random configurations.

One might be surprised to learn if, instead, the different types of flowers are always grouped together. Then, while one part of the meadow might contain more flowers than the other, the different flowers always appear in equal proportions, as illustrated in figure 1b. Instead of thinking the scattering of flower seeds to have proceeded by natural means, it may now seem more likely that all flowers have some common origin. For example, one might hypothesize that the flowers were sown there on purpose, as illustrated by figure 1c. Imagine someone throwing the seeds around the field without too much care, such that the flowers will not be equally divided. Yet as long as she pulls the seeds from a bag which contains a mixture of the three different kinds of flower seeds, the different kinds of flowers will grow everywhere more or less in the same proportions.

The meadow and flowers provide a good analogy to the universe as we observe it. At large scales, the universe looks both homogeneous (meaning that all points are created equal) and isotropic (meaning that there are no special directions). Such a smooth universe can be represented by a field of grass (without flowers), which looks the same everywhere. However, homogeneity and isotropy are not perfect (of course, planet Earth does not look the same as the sun and our galaxy looks very different from the vast empty space dividing it from the next galaxy). Thus, in order to properly describe the universe, we allow for 'perturbations' to the density and velocity of the different contents (baryonic matter, cold dark matter, light and neutrinos) of our cosmos. These perturbations, which are deviations from the average, can then be represented by flowers, having each kind of flower correspond to perturbations of a different inhabitant of the universe. Because of perturbations, the universe is not completely boring, just as the flowers make the meadow a lot more colorful place.



(a) A random distribution of flowers.



(b) A distribution of flowers in which the different types of flowers always appear in equal proportions.

(c) A person sowing flowers randomly from a mixed bag of seeds, resulting in a distribution of flowers in which the different types always appear in equal proportions.

Figure 1: Flowers distributed over a meadow. The meadow can be considered analogous to the homogeneous and isotropic universe, while the flowers can be considered perturbations to this universe of its different content (for example, red flowers might symbolize baryonic matter, yellow flowers might symbolize photons and blue flowers might symbolize cold dark matter). In an adiabatic universe, the flowers always appear in equal proportions, even if for the rest they are distributed randomly. This could be explained by invoking some common origin, such as the flowers having been sown together.

Since each different content of the universe has its own dynamics and way of interacting, one might initially expect the perturbations to be distributed throughout the universe randomly, or at least such that the perturbations of different contents don't have anything to do with each other. This corresponds to the flower seeds being planted through natural processes. Rather, though, the different contents everywhere have the same velocity and the perturbations to the total density always appear in a proportional fashion. Thus, the actual situation of our universe is described better by the picture where the different kinds of flowers are always grouped together. When perturbations behave like this they are called adiabatic, and the study of such perturbations is the topic of this thesis.

Learning about adiabatic perturbations is very interesting not only because they are so dominantly present in the universe, but also for many other reasons. For one thing, they provide model-independent solutions to the cosmological equations governing perturbations. That is to say that, no matter what the contents of the universe are, adiabatic perturbations always behave the same. Since we know what this behavior is, we can reconstruct the perturbations that dominated the universe in the past from the adiabatic perturbations observed today. Thus, they provide a way of looking into the past, even all the way back to the very early universe. Since we do not know much about the contents of the universe at the time, we have in general no idea how perturbations evolve through this period. Yet, since adiabatic perturbations always evolve the same, we can look all the way back to the beginning of this period, when the perturbations were first created.

Another method of probing the very early universe can also be obtained from knowledge about adiabatic perturbations. According to some theories, all perturbations stem from a single substance (a 'scalar field') which dominated the universe during the first fraction of a second. This theory is called single-field inflations, and if the theory holds true, all perturbations in the universe must be adiabatic. This is exactly represented by the sowing scenario described in the analogy: if all contents stem from a common origin, it is small wonder that their perturbations look so similar. In fact, if ever there was found a non-adiabatic perturbation, this would disprove single-field inflation altogether. What is more, properties of adiabatic perturbations can be used to derive certain statistical relations that must hold between perturbations if they were indeed created due to single-field inflation. Observing a violation of those would also rule out the popular theory. Altogether, we can conclude that adiabatic perturbations provide a powerful method of probing the early universe and testing theories about the origin of perturbations.

Still, we have not really discussed what is so special about adiabatic perturbations. What we consider their defining characteristic, is the fact that they *locally look the same as a coordinate transformation.* This sounds a little abstract, but what this means can be seen as follows. Over time, the universe is constantly expanding. Since the amount of matter in the universe remains more or less constant, it is diluted and thus the average density of matter becomes smaller all the time. Consider now one region of the universe where the matter density is constant but slightly smaller than the average. Then, instead of describing this region as having a matter density perturbation, we can describe it as having the average density at some later time. Thus, in this region, the matter density perturbation looks exactly the same as a transformation of the time coordinate. And since all contents of the universe react the same way to such a transformation, in a region of the universe where all the contents are locked together (i.e. they perturbations are adiabatic) all the densities look like the same transformation and thus the whole region of the universe, which is perturbed, looks exactly like an unperturbed region at some later time.

Of course, the adiabatic perturbations do not look exactly the same as a coordinate transformation, since then they would not really be perturbations at all; we would simply have chosen the wrong coordinates to describe the universe, making it seem perturbed. The difference is in the word *locally*: we can only play the trick of the previous paragraph when the perturbations are constant in a region of universe. Outside of this region, the perturbations will be different. For example, in a different region the density might be slightly larger than the average. In that case, we could describe it as a unperturbed region at a time slightly earlier instead of later. Thus, we cannot describe both regions as being unperturbed using a single transformation of the time coordinate. Everywhere, the adiabatic universe looks locally the same as a coordinate transformation, but what this transformation is differs from place to place. Globally, it does not look the same as a coordinate transformation.

The theory of adiabatic perturbations is well-developed for universes which are spatially flat. A property of a spatially flat universe is that the angles of triangles always add up to 180 degrees. While this agrees with everyday experience, we still cannot be entirely sure that we live in such a flat universe. Rather, it may only appear flat because the curvature of our universe is very small. This is reminiscent of how people once believed the earth to be flat. After all, humans are only very small compared to the Earth. Living on its surface, the effects of its curvature are hard to observe. For example, if you draw a triangle on the surface of the earth, it should not add up to 180 degrees. Yet, for any triangle we can draw by hand, the deviation will be negligible. Similarly, the curvature of the universe may become noticeable only at scales which are larger than our observable universe. It is thus worthwhile to account for the possibility of a curved universe.

The goal of this thesis is to extend the theory of adiabatic perturbations to the case of universes with negative curvature (such a universe is called 'open', as it has infinite volume). Instead of the meadow discussed in this introduction, a more apt analogy would be a grassy hill as illustrated in figure 2. The key question will be to find out whether perturbations in our universe can still be described as a local coordinate transformation if there is some negative curvature. That is: do adiabatic perturbations exist in an open universe? And if they do, what do they look like, and how do they evolve in time? The answers might surprise you, as the adiabatic modes that we manage to find have some strange properties.

The plan to this thesis is as follows. In Part I, a review of the already established theory of adiabatic modes will be provided. In order to do so,

Figure 2: When dealing with a curved universe, a hill as shown in this figure is a more fitting analogy than a flat meadow. The goal of this thesis is to find out whether flowers can also be expected to group together on the hills as they did in the meadow.

we first provide a pedagogical description of the homogeneous and isotropic curved universe in Chapter 1. Next, we discuss how perturbations on such a homogeneous and isotropic background can be described in Chapter 2. When then all the preliminaries are in place, Chapter 3 is dedicated to adiabatic modes. In particular, it will be discussed how adiabatic solutions can be found using perturbation theory in a flat universe and what their use is (in a more technical and elaborate way than in this introduction). After that, we are ready to try and generalize these results to the open universe in Part II. While Chapter 4 will focus on how to find adiabatic solutions, Chapter 5 focuses specifically on the simplest possible tensor mode while Chapter 6 focuses on the simplest possible scalar mode.In Chapter 7, conclusions to these results will be drawn and an outlook for possible future research will be sketched. Finally, Part III will contain appendices in which elaborate and/or boring calculations are located to support the rest of the work.

Part I

Review

Chapter 1

Homogeneity and Isotropy

1.1 Hubble's Law

When one looks up at the sky, it appears to be more or less **isotropic**. That is, it looks the same in one direction as in any other direction. This is particularly true if we only look at objects that are far away enough from us, i.e. at galaxies and such rather than at the stars in our own milkyway. While each specific galaxy is visible only at one spot in the sky, and while some areas may contain more galaxies than others, the density of galaxies (when averaged over sufficiently large areas) does not vary much between different directions. What is more, all these galaxies seem to behave more or less the same. Everywhere we look, they recede away from us. The way in which they do is caputerd by the famous Hubble law [17],

$$v = H_0 r. \tag{1.1}$$

Here, v is the magnitude of the velocity of an object relative to us (as measured through redshift), r is the magnitude of its distance from us, and H_0 is the **Hubble constant**. Thus, how fast an object moves away from us is proportional to its distance from us, and does not depend on its direction in the sky.

The isotropy of our sky is perhaps most evident when looking at the cosmic microwave background, which is a faint signal of relic light from the very early universe. It has been observed very accurately by the Planck satellite (which was launched in 2009), which is able to associate a temperature to the radiation from different directions, creating a thermal map of the sky (see figure 1.1) Deviations from the average temperature, which is 2.7 Kelvin, are only of the order of 10^{-5} [1], meaning that the sky once more looks pretty smooth.

One conclusion to draw from this would be that Earth is at the center of the entire universe, from which all matter originates and now moves away (except for us). This seems a bit far fetched though, as in all other aspects Earth appears to be just some random planet that happened to develop life as we know it. The idea that the universe is not tailor made for mankind is expressed by the **Copernican principle**, which states that "humans are not privileged observers" [30, page 66]. The consequence of this would be that the universe is isotropic around *any* point. Whenever we will say the universe is isotropic, this

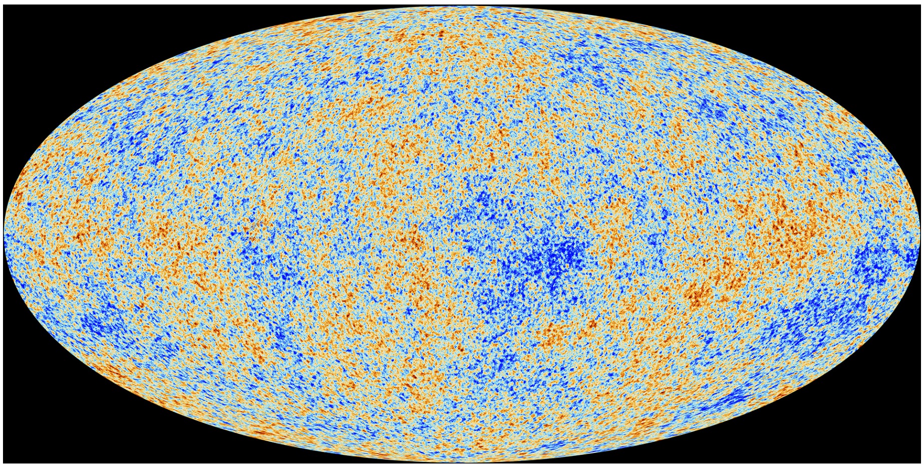


Figure 1.1: An image of the cosmic microwave background (CMB) as observed by the PLANCK satellite. The celestial sphere has been projected onto a plane, where red regions indicate directions where the cosmic microwave background looks slightly hotter and blue regions indicate slightly colder directions. Copyright ESA and the Planck Collaboration. Obtained from **www.esa.int**.

is what we mean.

But if the universe is isotropic around every point, there is another conclusion we can draw: the universe is **homogeneous**. That is, it looks the same as seen from any point. It is interesting to note that while isotropy implies homogeneity, this does not work the other way around. For example, if the whole universe would be filled with a nonzero constant electric field pointing one way, then not all directions are the same to an observer. Yet, because this field is observed in the same way by any observer at any point, this does not stop the universe from being homogeneous.

At first, homogeneity might sound incompatible with Hubble's law, which however has experimental basis. If we perceive all objects in the universe to move away from *us*, they cannot also all move away from some other point at the same time! In fact, Hubble's law gives the only isotropic distribution of velocities which looks the same for any observer (that moves with the flow). This can be seen as follows (which is inspired by [27, pp. 5-6]). Any velocity distribution[1] is specified by some function $\mathbf{v}(\mathbf{r})$, where $\mathbf{r}$ is the position vector of the object that we observe to have vector velocity $\mathbf{v}$. For each component of $\mathbf{v}$, we can write a Taylor expansion

$$v_i(r_1, r_2, r_3) = c_{i0} + c_{ij}r_j + c_{ijk}r_jr_k + ..., \tag{1.2}$$

where r_i are the components of $\mathbf{r}$, the c's are constants, and the dots indicate terms which are the product of three or more components of $\mathbf{r}$. Note that the Einstein summation convention is used, so all repeated indices are summed over implicitly (where latin indices take the values $1, 2, 3$).

[1] Of course, a general velocity distribution can also have time dependence. Since including this does not change the argument or conclusion, this fact is ignored for the moment.

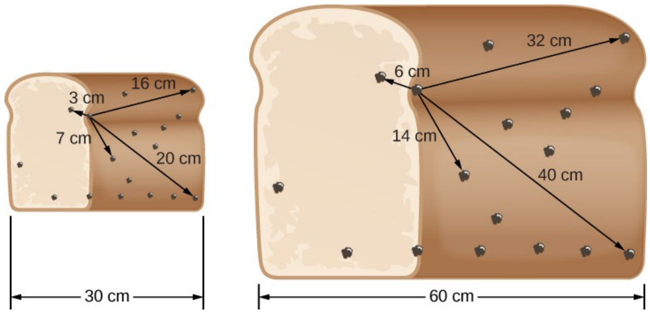


Figure 1.2: A raisin bread rising. As it expands, all raisins move away from each other, in analogy to the isotropic and homogeneous universe. Retrieved from [6].

Now, if the distribution is homogeneous, this distribution must look the same to an observer at any other point. Let's call our position in space for the moment point A. The point B moves away from us with velocity $\mathbf{v}(\mathbf{r}_{AB})$ while the point C moves away form us with velocity $\mathbf{v}(\mathbf{r}_{AC})$. Then, by simple vector addition, point C moves away from point B with velocity $\mathbf{v}(\mathbf{r}_{AC}) - \mathbf{v}(\mathbf{r}_{AB})$. Thus, homogeneity dictates

$$\mathbf{v}(\mathbf{r}_{AC}) - \mathbf{v}(\mathbf{r}_{AB}) = \mathbf{v}(\mathbf{r}_{BC}) = \mathbf{v}(\mathbf{r}_{AC} - \mathbf{r}_{AB}), \tag{1.3}$$

where the last equality once more follows from vector addition. This equation is compatible only with the terms linear in the components of $\mathbf{r}$ in equation (1.2). For example, a term with $r_1 r_2$ is forbidden since in general $r_{AC,1} r_{AC,2} - r_{AB,1} r_{AB,2} \neq (r_{AC,1} - r_{AB,2})(r_{AC,2} - r_{AB,2})$. Thus, we are left with

$$v_i(\mathbf{r}) = H_{ij} r_j, \tag{1.4}$$

where H_{ij} is some matrix. When we also require isotropy, $\mathbf{v}$ must be in the same direction as $\mathbf{r}$ since any angular velocity could be changed direction using a rotation (and is thus incompatible with isotropy). Thus, H_{ij} must be proportional to unity (δ_{ij}) [2]. This leaves us indeed with Hubble's law, $\mathbf{v} \propto \mathbf{r}$.

The situation sketched by Hubble's law can be visualized by the following analogy. Consider baking a loaf of raisin bread. When you add yeast to the dough, the loaf will start rising and its volume will increase. The raisins, which are fixed in the dough, consequently move away from each other. Each raisin moves away from each other raisin in a way such that their instantaneous velocities will exactly follow Hubble's law. The situation is illustrated in figure 1.2.

It now seems very tempting to interpret Hubble's law as the result of expansion of our universe. However, our jugling of velocities has so far been very Newtonian. In order to speak of the expansion of space, we will need to turn to general relativity, which is currently the most important physical theory at large scales. While the intuition developed above will remain valuable, we will

[2]More formally, this can also be shown from equation (1.4) since unity is the only matrix that commutes with all rotation matrices.

see that the interpretation of some concepts will need slight revision. In this section (i.e. Chapter 1), we will develop a general-relativistic description of the homogeneous and isotropic universe. After that, in Chapter 2, we will go beyond the assumption of homogeneity and isotropy to give a more realistic description of our universe.

1.2 Isometries

General relativity describes spacetimes using a metric. Thus, if we want to describe our universe using general relativity, we must find the proper metric to do so. As discussed above, the universe can be considered both homogeneous and isotropic on large enough scales. Conveniently, these symmetries heavily restrict the form the metric can take. In order to find out what form this is exactly, we must first understand what homogeneity and isotropy mean precisely in general relativity.

Although a basic understanding of general relativity will be assumed throughout this thesis, we will here quickly recap the basic notions needed to discuss symmetries and their applications. We will consider a spacetime with D dimensions, of which $d < D$ are spatial (and thus $D - d$ temporal). The geometry of this spacetime is defined by its metric $g_{\mu\nu}(\mathrm{x})$, which enables one to calculate line elements,

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu. \tag{1.5}$$

The greek indices run over all D dimensions of the spacetime. Through a diffeomorphism, general relativity can be expressed in terms of different coordinates:

$$x^\mu \to \tilde{x}^\mu. \tag{1.6}$$

While a geometric quantity like the line element remains invariant under this transformation, the metric does not. Like any tensor, it transforms in a covariant way,

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\mu}{\partial \tilde{x}^\rho}\frac{\partial x^\nu}{\partial \tilde{x}^\sigma} g_{\rho\sigma}(\tilde{x}). \tag{1.7}$$

A derivation of the transformation properties of the metric can be found in Appendix B. The metric is said to be **form-invariant** under a diffeomorphism if

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x). \tag{1.8}$$

A diffeomorphism that leaves the metric form-invariant is called an **isometry**. This is what is usually meant by 'symmetry' in general relativity, as the spacetime looks the same before and after the transformation.

It is often simplest to consider only infinitesimally small diffeomorphisms. Such transformations then take the form

$$x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu(x), \tag{1.9}$$

where $\epsilon^\mu \equiv \epsilon\xi^\mu$ with $\epsilon \ll 1$. This means that we only regard terms that are linear in ϵ^μ. The vector field $\xi^\mu(x)$ is said to generate the diffeomorphism, as every point is moved an infinitesimal amount along the vector field. A finite diffeomorphism can be obtained by repeated application of infinitesimal ones, which amounts to moving points along the integral curves traced out by ξ^μ (see e.g. [10, app. B]. Since every diffeomorphism that is continuously connected to unity (as opposed to e.g. reflections, which are discrete) is generated by such a vector field, limiting ourselves to infinitesimal diffeomorphisms entails no great loss of generality.

It is shown in Appendix B that we then have

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(\tilde{x}) = g_{\mu\nu}(x) - 2\nabla_{(\mu}\epsilon_{\nu)}(x). \tag{1.10}$$

Here, the brackets on the indices indicate symmetrization. More precisely, when used to symmetrize two indices, they are defined by

$$T_{(\mu_1\mu_2)} = \frac{1}{2}\left(T_{\mu_1\mu_2} + T_{\mu_2\mu_1}\right). \tag{1.11}$$

For more details, see Appendix A. Thus, (infinitesimal) isometries are exactly those generated by

$$\nabla_{(\mu}\xi_{\nu)} = 0. \tag{1.12}$$

This is known as **Killing's equation**. Every vector field that satisfies Killing's equation is called a **Killing vector field** (although the 'field' is often omitted for brevity). All isometries which are continuously connected to unity are obtained by moving all points in a spacetime along such a vector field.

In physics, every symmetry is usually associated with a conserved quantity (as follows from Noether's theorem, see e.g. [33, sec. 7.6] or [31, sec. 2.2]). Similarly, each Killing vector satisfies the equation [10, eq. (3.174)]

$$p^\mu \nabla_\mu (\xi_\nu p^\nu) = 0, \tag{1.13}$$

where $p^\mu = m\frac{dx^\mu}{d\tau}$ is the momentum of a test particle with mass m and proper time τ. The equation states that momentum along the Killing vector field is invariant along the geodesic of the test particle, i.e. it is conserved.

Interestingly, a Killing vector field is fully determined by ξ^μ and $\nabla_\mu\xi^\nu$ at a single point. Using Killing's equation, it can be shown that

$$\nabla_\rho\nabla_\sigma\xi_\mu = -R^\lambda_{\rho\sigma\mu}\xi_\lambda., \tag{1.14}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor. This can be used to express any higher-order covariant derivative in terms of the Killing vector and its first-order covariant derivative. Since these can be used to write down the Taylor series for the Killing vector field around point x_0, we have

$$\xi_\mu(x) = A^\rho_\mu(x; x_0)\xi_\rho(x_0) + B^{\rho\sigma}_\mu(x; x_0)\nabla_\rho\xi_\sigma(x_0), \tag{1.15}$$

where the functions A and B are the same for all Killing vectors (they are determined by the geomtery of the spacetime through the Riemann tensor). At the point x_0, there are at most D linearly independent Killing vectors ξ_μ (since μ runs over all D spacetime dimensions, they are vectors in a D-dimenional vector space, of which there can be at most D independent) and $\frac{1}{2}D(D-1)$

linearly independent $\nabla_\mu \xi_\nu$ (since this tensor must be antisymmetric, as follows from Killing's equation). It follows that there are at most $\frac{1}{2}D(D+1)$ linlearly independent Killing vector fields in any spacetime [35, sec. 13.1].

Spacetimes with $\frac{1}{2}D(D+1)$ independent Killing vectors are called **maximally symmetric spaces**. Since there is a great deal to say about these, and since their analysis will significantly help finding the proper metric to describe our own universe, these special spacetimes get their own section.

1.3 Maximally Symmetric Spaces

In order to get some feeling for what it means for a spacetime to be maximally symmetric, let's go back to equation (1.15). When there indeed are $\frac{1}{2}D(D+1)$ linearly independent solutions to Killing's equation, this means we can choose any set of $\frac{1}{2}D(D+1)$ linearly independent vector fields obeying equation (1.15) to use as a basis for all possible Killing vectors [3]. We do this in the simplest way possible by taking a set of D vector fields with nonzero $\xi_\rho(x_0)$ but vanishing $\nabla_\rho \xi_\sigma(x_0)$ and a set of $\frac{1}{2}D(D-1)$ vector fields with vanishing $\xi_\rho(x_0)$ but nonzero $\nabla_\rho \xi_\sigma(x_0)$:

$$\begin{aligned} \xi_\nu^{(\mu)}(x_0) =& \delta^\mu_\nu, \\ \nabla_\rho \xi_\nu^{(\mu)}(x_0) =& 0, \\ \xi_\rho^{(\mu\nu)}(x_0) =& 0, \\ \nabla_\rho \xi_\sigma^{(\mu\nu)}(x_0) =& \delta^\mu_\rho \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma. \end{aligned} \tag{1.16}$$

Here, brackets indicate that the index is a label to differentiate between the different basis vector fields. The two different sets now allow for a nice interpretation.

The first set of basis vectors, $\xi_\nu^{(\mu)}(x)$, span the whole D-dimensional vector space at any point x (the tangent spaces T_x, to be precise). Thus, we can use infinitesimal isometries to move the point x to any point in its immediate neighborhood. This defines our space to be **homogeneous** [35, page 378]. Since the procedure can be repeated at the new point, we can use this to move any point in the space to any other point through isometry (which is an alternate but equivalent definition of homogeneity [10, page 323]). This formalizes the earlier definition of homogeneity. It means that the metric is the same at every point throughout the entire space. While homogeneity is often thought of as translational symmetry, it is discussed below that this can only be done if we generalize what we mean exactly by 'translation'.

[3]This statement can be explained better using linear algebra. There are $\frac{1}{2}D(D+1)$ numbers defining every $\xi_\mu(x)$ obeying equation (1.15). We can consider this a $\frac{1}{2}D(D+1)$ dimensional vector space (closure under addition and scalar multiplication are easily verified). If there are n linearly independent solutions to Killing's equation, these can be represented as n linearly independent vectors in the $\frac{1}{2}D(D+1)$ space (since we know that every Killing vector satisfies equation (1.15)). These span a n-dimensional subspace containing all Killing vectors. When $n = \frac{1}{2}D(D+1)$, this subspace coincides with the whole vector space, and we can thus freely choose any $\frac{1}{2}D(D+1)$ basis vectors since it is impossible to 'accidentally' choose a vector that is not in the n-dimensional subspace. For smaller n, we would have to be more careful and check whether our basis vectors actually solve Killing's equation.

The second set of basis vectors, $\xi_\rho^{(\mu\nu)}(x)$, leaves the point x_0 invariant. Yet, by taking linear combinations, $\nabla_\rho \xi_\sigma^{(\mu\nu)}(x_0)$ is allowed to take any value. A space which has Killing vectors for which this holds is defined to be **isotropic** about the point x_0 [35, page 378]. This formalizes the earlier definition of isotropy. Isotropic space is often thought of as having rotational symmetry. Indeed, the isometries generated by this second set of basis vectors can be thought of as 'rotations', although this might require abandoning the well-known Euclidean notion of the concept. To see in what sense they are rotations, let's consider the action of these specific isometries more closely. A vector at x_0 transforms under a diffeomorphism $x^\mu \to x'^\mu = x^\mu + \epsilon \xi^{(\lambda\tau)\mu}$ as

$$\begin{aligned} V^\mu(x_0) \to V'^\mu(x'_0) &= \frac{\partial x'^\mu}{\partial x^\rho} V^\rho(x_0 + \cancelto{0}{\epsilon \xi^{(\lambda\tau)}(x_0)}) \\ &= \Big((\delta_\rho^\mu + \epsilon \partial_\rho \xi^{(\lambda\tau)\mu}(x_0) \Big) V^\rho(x_0) \\ &= V^\mu(x_0) + \epsilon g^{\mu\sigma} \nabla_\rho \xi_\sigma^{(\lambda\tau)}(x_0) V^\rho(x_0) \\ &= V^\mu(x_0) + \epsilon \Bigg(g^{\mu\tau}(x_0) V^\lambda(x_0) - g^{\mu\lambda}(x_0) V^\tau(x_0) \Bigg), \end{aligned} \tag{1.17}$$

where we have used the general law for vector transformations (see e.g. [10, eq. (2.19)]) and where we have used $\nabla_\rho \xi_\sigma^{(\mu\nu)}(x_0) = \partial_\rho \xi_\sigma^{(\mu\nu)}(x_0)$ since $\xi_\sigma^{(\mu\nu)}(x_0) = 0$.

Now consider a vector $W^\mu(x_0)$ that is orthogonal to the change in $V^\mu(x_0)$. This means

$$W^\tau(x_0) V^\lambda(x_0) = W^\lambda(x_0) V^\tau(x_0). \tag{1.18}$$

Clearly, one solution is $W^\mu(x_0) = V^\mu(x_0)$. However, it can also be shown that this is the unique (nonzero) solution up to scalar multiplication. We can choose our coordinate basis such that $V^\mu(x_0) = \delta_0^\mu$. Then generally, $W^\mu(x_0) = c_0 \delta_0^\mu + c_1 \delta_1^\mu + ...$ for some constants c_n. The above equation for $\lambda = 0$, $\tau = 1$ then implies $c_1 = 0$. The same can be done for any c_n with $0 < n \leq D$. We conclude $W^\mu(x_0) \propto V^\mu(x_0)$. This means that the Killing vectors can be used to move $V^\mu(x_0)$ any direction it does not already point in (if one direction would be excluded, $W^\mu \propto V^\mu$ would not be the only solution). It also follows from the above that

$$V'_\mu(x_0) V'^\mu(x_0) = V_\mu(x_0) V^\mu(x_0), \tag{1.19}$$

i.e. the norm is conserved. It is thus possible to move vectors at x_0 around in a continuous way through isometry, as long as we do not change the norm. This indeed coincides with the basic concept of 'rotation'.

So in what sense do the concepts of translation and rotation need generalization to suit the formal definitions of homogeneity and isotropy above? We know that when there is no curvature ($R_{\mu\nu\rho\sigma} = 0$), all higher-order derivatives of ξ_μ vanish and we can choose coordinates such that covariant derivatives become ordinary derivative. We then have

$$\xi_\mu(x) = \xi_\mu(x_0) + x^\rho \partial_\rho \xi_\mu(x_0). \tag{1.20}$$

Assuming $g_{\mu\nu} = \delta_{\mu\nu}$ (i.e. Euclidean space), it is clear that the $\xi_\nu^{(\mu)}$ generate ordinary translations (moving every point exactly by ϵ under an infinitesimal diffeomorphism) while the $\xi_\rho^{(\mu\nu)}(x)$ generate the ordinary $\frac{1}{2}D(D-1)$ rotations in the space (see e.g. [10, eq. (3.186)]). However, when there are timelike dimensions, some of the rotations become Lorentz boosts. Furthermore, when there is curvature, the functions A and B in equation (1.15) can take some nontrivial form depending on the geometry. Thus, while in a homogeneous curved space we can still use $\xi_\nu^{(\mu)}$ to 'translate' x_0 by an amount ϵ, all other points may at the same time move by some different amount. Similarly, an isotropic space might be defined by an isometry that does not look exactly like rotations as we know them in flat space.

The above construction of basis vectors shows that any maximally symmetric space is both homogeneous and isotropic. The inverse is also easily seen to hold, as homogeneity and isotropy imply the existence of the $\frac{1}{2}D(D+1)$ Killing vectors constructed above. Furthermore, it can be proven that a space that is isotropic about every point is also homogeneous [35, pp. 378-379]. Lastly, it can be seen that in flat space every vector field satisfying equation (1.20) also satisfies Killing's equation (as long as $\partial_\mu \xi_\nu$ is antisymmetric). Since there as $\frac{1}{2}D(D+1)$ such vector fields, it follows that all flat spaces are maximally symmetric.

It can be shown that every maximally symmetric space is a so-called space of constant curvature. This means that the Ricci scalar in constant throughout the entire space and that the Riemann tensor is

$$R_{\lambda\rho\sigma\nu} = K(g_{\sigma\rho}g_{\lambda\nu} - g_{\nu\rho}g_{\lambda\sigma}), \tag{1.21}$$

where K is the **curvature constant**, related to the Ricci scalar by

$$R = -D(D-1)K. \tag{1.22}$$

One of the (for current purposes at least) most important properties of maximally symmetric spaces is that they are **unique**. That is to say, all the maximally symmetric spaces with the same curvature constant and the same metric signature (i.e. the same number of positive and negative eigenvalues), are related to each other through a diffeomorphism. Thus, they are really the same, only represented by different coordinates. An elaborate proof of this can be found in [35, sec. 13.2].

One last theorem that will be of great import in the next section is about a space with maximally symmetric subspaces. A D-dimensional space with a family of M-dimensional subspaces can be described using M coordinates u^i which indicate points within each subspace and $D-M$ coordinates v^a which parametrize the different subspaces in the family. The subspaces are then defined to be maximally symmetric if there are $\frac{1}{2}M(M+1)$ linearly independent Killing vectors of dimension M such that the metric is form-invariant under the infinitesimal diffeomorphism

$$\begin{aligned} u^i &\to u'^i = u^i + \epsilon^i(u,v), \\ v^a &\to v'^a = v^a, \end{aligned} \tag{1.23}$$

with $\epsilon^i = \epsilon\xi^i$ as before. It is proven in [35, sec. 13.5] that it is always possible to pick the u coordinates such that the metric takes the form

$$ds^2 = g_{ab}(v)dv^a dv^b + f(v)\tilde{g}_{ij}(u)du^i du^j, \tag{1.24}$$

where $\tilde{g}_{ij}$ is the metric of an a maximally symmetric space of dimension M.

1.4 Friedmann-Lemaître-Robertson-Walker

In this section, the above discussion homogeneity and isotropy is used to finally write down a metric for our universe.

In the first part of Chapter 1 it was discussed that the sky seems approximately isotropic. In fact, this statement can be seen to be equivalent to the (approximate) presence of the more abstract notion of isotropy developed in Section 1.3. First, it must be noted that every isometry of the metric must be a symmetry of the contents of the universe as well. This is due to the specific way in which the contents of the universe deform spacetime and is explained in the beginning of Section 1.5.

It can be proven that isotropy implies a smooth night sky as follows. If there was some observable that varied over the celestial sphere, we could define a vector at our position ("x_0", in the language of Section 1) pointing towards some specific point on the sphere, like for example in the direction where we see the largest density of stars or the highest temperature in the cosmic microwave background. However, such a vector could be made to change direction using the $\frac{1}{2}d(d-1)$ independent infinitesimal diffeomorphisms which only act on the spatial coordinates and which leave x_0 invariant. If spacetime is isotropic about x_0, these diffeomorphisms are isometries. Yet, they change some observable, which must derive either from the geometry or the contents of the universe (as there is nothing else). An isometry however cannot change these things, and thus the presence of a vector implies they are not isometries after all. This means the presence of vectors implies the absence of isotropy, and hence the presence of isotropy implies the absence of vectors, which means the night sky must look isotropic in the more intuitive sense discussed is the beginning of this chapter.

Conversely, the smoothness of the sky implies isotropy. If all observables at our position would remain invariant under the $\frac{1}{2}d(d-1)$ Euclidean rotations, we could take these to be the spatial part of the diffeomorphisms that leave x_0 invariant (which is obviously the case for rotations). The fact that nothing changes at x_0 implies they can be extended to all other points as isometries using equation (1.15). While this does not provide the required $\frac{1}{2}D(D-1)$ isometries required for isotropy, it does cover the 'spatial part' of the definition.

It is important to be careful here not to draw hasty conclusions. The above discussion really only concerned spatial three-vector and three-dimensional rotations. While one might expect, with the Copernican principle in mind, that the whole universe is maximally symmetric, this does not actually agree with observations. The universe looks invariant under spatial rotations but the same cannot be said for Lorentz boosts. After all, in a boosted frame the Hubble law 1.1 would no longer hold, since objects in one direction will now appear to move faster than objects in another direction. We do thus not know of an isometry that can change timelike vectors, and we do not observe the universe to be a fully isotropic spacetime.

Instead, when we said the universe is isotropic, we meant that it posseses **spatial isotropy**. More precisely, we can choose our coordinates such that constant-time hypersurfaces are isotropic. This defines a preferred time coordinate that is sometimes called **cosmic time** (e.g. in [14]). The fact that we observe the universe to be spatially isotropic reveals that our frame of reference is one of the preferred frames defined by this coordinate (i.e. our four-velocity points along the cosmic time axis)[4]. Invoking the Copernican principle, we conclude that the universe possesses **spatial homogeneity** as well. The universe can thus be described as a family of time-ordered spatial subspaces that are isotropic and homogeneous, parametrized by the cosmic time.

Now, equation (1.24) becomes extremely useful. For $D = 4$ and $M = 3$, this becomes

$$ds^2 = -h(t)dt^2 + a(t)^2\tilde{g}_{ij}(x)dx^i dx^j. \tag{1.25}$$

We have written the function multiplying the subspace metric as $a(t)^2$ because it is customary. It must hold that $h(t) > 0$, and $a(t)^2 g_{ij}$ must have a $+++$ signature, since we we want our metric to have $-+++$ signature (otherwise, we do not obtain Minkowsky space in the local Lorentz frame [10, pp. 73-74], and our theory would in no limit reproduce special relativity.) Here we take $a(t)^2 > 0$, such that $\tilde{g}_{ij}$ must be a Euclidean metric.

It is now possible to redefine the time coordinate to simplify life. We introduce

$$dt'^2 = h(t)dt^2. \tag{1.26}$$

Requiring t' to be a monotonically increasing function of t (and thus keep the proper interpretation as time), this means

$$t'(t) = \int^t dt'' \sqrt{h(t'')}. \tag{1.27}$$

Note that the function $t'(t)$ is invertible because of monotony, and thus we can write $f(t(t'))$ simply as $f(t')$. t' now becomes the new time coordinate and is renamed accordingly (i.e. the prime is dropped). The new metric is

$$ds^2 = -dt^2 + a(t)^2\tilde{g}_{ij}dx^i dx^j \equiv -dt^2 + a(t)^2 d\sigma^2, \tag{1.28}$$

Here, $\tilde{g}_{ij}$ is the metric of a maximally symmetric space with line element is $d\sigma^2$.

The metric describes a family of subspaces with line element $a^2 d\sigma^2$. The space that $d\sigma^2$ describes is the subspace corresponding to $a = 1$ and will be called Σ for convenience. As we consider Σ to be a space on its own, it has its own covariant derivative and Christoffel symbols. We denote these $\bar{\nabla}_i$ and $\bar{\Gamma}^i_{jk}$

[4]In fact, this is not exactly true. Because of the movement of our solar system around the center of the milky way and the movement of the earth around the sun, we are not even at rest relative to this frame. Only after correcting for such movement does the sky look properly isotropic and does Hubble's law hold.

respecively (as opposed to ∇_μ and $\Gamma^\lambda_{\mu\nu}$ for the full spacetime). In Appendix C.3, it is shown that $\bar{\Gamma}^i_{jk} = \Gamma^i_{jk}$, and thus the notation $\bar{\Gamma}^i_{jk}$ is not often used. Now, it remains to find out what Σ looks like. Since it is maximally symmetric, we know it is unique up to specification of the curvature constant K.

Qualitatively, one would expect there to be three different kinds of maximally symmetric spaces: those with positive curvature ($K > 0$), those with negative curvature ($K < 0$), and those with zero curvature ($K = 0$). If an example for each of these can be constructed, we know from uniqueness that we have obtained all spaces that could ever qualify as spatial subspace for our universe. It does not matter how we got by these examples, so here we will simply make an educated guess for each.

The maximally symmetric flat space is of course easy to find (since it was shown in Section 1.3 that every flat space is maximally symmetric): it is just flat Euclidean space with metric

$$d\sigma^2_{K=0} = \delta_{ij} dx^i dx^j \equiv d\mathbf{x}^2 \equiv dr^2 + r^2 d\Omega^2, \tag{1.29}$$

where here x^i are ordinary Cartesian coordinates. The last equality expresses the metric in the usual spherical coordinates, where the line element of the unit two-shere is conveniently written as

$$d\Omega^2 = d\theta^2 + \sin^2(\theta) d\phi^2. \tag{1.30}$$

To construct a space of nonzero constant curvature, it is usefull to apply our intuition about two-dimensional surfaces embedded in a three-dimensional space. Such a surface that is curved in a constant way is the two-sphere, which indeed has $\frac{1}{2} 2 * (2 + 1) = 3$ isometries (which are the $\frac{1}{2} 3 * (3 - 1)$ rotations in three-space, which leave the sphere invariant) and is thus maximally symmetric. It thus seems sensible to consider a three-sphere embedded in four-dimensional Euclidean space as subspace with nonzero curvature.

The strategy is thus to take an embedding space of the form

$$ds^2 = \delta_{ij} dx^i dx^j + dz^2, \tag{1.31}$$

where i runs from one to three. We specify the three-sphere in this space by

$$\delta_{ij} x^i x^j + z^2 = \frac{1}{K}, \tag{1.32}$$

where K is some constant with dimension one over length squared. In Appendix D, we work out what the metric on this sphere exactly looks like. In the coordinates of our choice, we find

$$ds^2 = \frac{1}{(1 + \frac{1}{4} K \mathbf{x}^2)^2} d\mathbf{x}^2, \tag{1.33}$$

where $|\mathbf{x}|$ is bounded by $2/\sqrt{|K|}$ for $K \neq 0$ (see equation (D.15) for more details). Thus, we have written the metric in a way that is conformal to the flat metric (1.29)! Since this thesis will be concerned with generalizing results from flat space to curved space, this is a convenient choice of coordinates. The conformal factor will appear so often in this thesis that we give (the square root of) this function its own symbol,

$$f(K\mathbf{x}^2) \equiv \frac{1}{1 + \frac{1}{4}K\mathbf{x}^2}. \tag{1.34}$$

Note that $K\mathbf{x}^2$ is dimensionless, and thus the same holds for f.

While this metric is manifestly invariant under rotations (i.e. $x^i \to R^i{}_j x^j$ such that $R^i{}_k R^k{}_j = \delta^i{}_j$)), the same cannot be said for translations. In fact, while the space is homogeneous, the corresponding isometry is provided by **quasitranslations** rather than normal translations. These are discussed briefly in Appendix D, and the Killing vectors are constructed explicitly in Appendix C.4. In order to get an idea of what quasitranslations are, we present their Killing vectors here without further discussion:

$$\xi^i = (a - \frac{1}{4}K\mathbf{x}^2)a^i + \frac{1}{2}Ka^k x^k x^i, \tag{1.35}$$

where the vector a^i specifies the quasitranslation (this is the displacement vector of the origin). Since we have explicitly constructed $\frac{1}{2}3(3-1) = 6$ linearly independent Killing vectors (the rotations are also treated as Killing vectors in Appendix C.4), we are certain that we have indeed constructed a maximally symmetric space. It is thus a space of constant curvature as well. In fact, it turns out that the constant K is the curvature constant of the space [35, sec. 13.3] (which is of course why we have chosen this particular way of parametrizing the three-sphere).

Since equation (1.32) only has solutions for positive K, it is clear that, using the sphere, we have constructed the positively curved maximally symmetric spaces. However, the metric (1.33) is in fact much more inclusive then that. It is easily verified that when we put $K = 0$, we find back the metric for flat space (1.29). What is more, there is nothing to prevent us from putting $K < 0$. The Killing vectors found in Appendix C.4 still solve Killing's equation, and thus this also describes a maximally symmetric spacetime. Furthermore, K is still the curvature constant. Thus, we have already found the most general $d\sigma^2$ that can occur in equation (1.28). Note that for $K \leq 0$ we can no longer interpret the space as a sphere. The negatively curved space (which is called Lobachevsky space) can however be embedded in four-dimensional Lorentzian space as a hyperboloid [27, p. 16].

It must be noted that while maximally symmetric spaces have a unique metric, their *global* properties may differ. The flat metric can describe both the simply-connected plane ($\mathcal{R}^3$) or the topologically less trivial three-torus ($S^1 \times S^1 \times S^1$). Thus, the topology of our universe cannot be determined from symmetry alone. If $K > 0$, $d\sigma^2$ can only describe the three-sphere (or the non-orientable space $\mathcal{R}P^3$), suggesting that the universe has finite volume (yet does not have a boundary). Therefore, a universe with $K > 0$ is usually called **closed**. For $K < 0$, the metric could describe a hyperboloid, which has infinite volume. Therefore, a universe with $K < 0$ is usually called **open**. However, it could in principle also describe topologically more complicated spaces that have, in fact, finite volume [10, p. 331].

We then finally arrive at the most general metric that can describe a (spatially) homogeneous and isotropic universe,

$$\begin{aligned} ds^2 &= -dt^2 + a(t)^2 \frac{\delta_{ij} dx^i dx^j}{(1 + \frac{1}{4} K \mathbf{x}^2)^2} \equiv -dt^2 + a^2 f^2 d\mathbf{x}^2 \\ &\equiv -dt^2 + a(t)^2 d\sigma^2 \equiv -dt^2 + a(t)^2 \tilde{g}_{ij} dx^i dx^j . \end{aligned} \tag{1.36}$$

This is the famous **Friedmann-Lemaître-Robertson-Walker metric**, or, as it will be called in the rest of this thesis, simply the **FLRW metric**. The function $a(t)$ is called the **scale factor**, and it is the only degree of freedom the spacetime itself has (although its content can, of course, provide more degrees of freedom). It is a measure of what physical distances correspond to what distances in the spatial subspaces. The coordinates on $d\sigma^2$ are usually called **comoving**. A comoving distance l_{comoving} corresponds, at time t, to a physical distance $l_{\text{physical}}(t) = a(t) l_{\text{comoving}}$.

Thus, in some sense, the degree of freedom that the scale factor represents can be thought of the 'size' of the universe (although the universe need not have a finite size). When $a(t)$ is not constant, we can interpret this as either expansion ($\dot{a} > 0$) or collapsing ($\dot{a} < 0$) of the universe. We already found such behavior was suggested by Hubble's law in the beginning of Section 1. Only we needed general relativity in order to talk about the expansion of space.

It can be easily verified that the FLRW metric results in Hubble's law. If two observers are a comoving distance l_{comoving} apart, their physical velocity relative of each other will be

$$\frac{d}{dt} l_{\text{physical}}(t) = \frac{d}{dt} a(t) l_{\text{comoving}} = H(t) l_{\text{physical}}, \tag{1.37}$$

where

$$H(t) = \frac{\dot{a}(t)}{a(t)} \tag{1.38}$$

is the **Hubble parameter** (which is sometimes also called Hubble constant, since it is constant over space). Identifying this with the Hubble constant in equation (1.1) gives us the original Hubble's law.

It was stated above that cosmic time defines a class of preferred frames of reference. Indeed, the FLRW metric is not invariant under Lorentz boosts. Any observer for whom the constant-time hypersurfaces are isotropic and homogeneous (and who can thus construct an FLRW metric as we have done) is said to be an **fundamental observer** [30, p. 67]. They are on so-called **fundamental trajectories** [35, p. 410], which are defined by x^i =constant. As one would expect, such trajectories are geodesics [35, eq. (14.2.8)]. Galaxies are usually more or less on fundamental trajectories, although they typically have an additional velocity of about 100 km s^{-1} [30, p.67]. The part of the physical velocity that can be attributed to Hubble's law is called the **Hubble flow** [7, p. 7].

The FLRW metric does not offer a unique way of defining the scale factor, the curvature constant and the comoving coordinates. The freedom this leaves is often used to set the scale factor today to unity. The current cosmic time coordinate is denoted t_0, and quantities evaluated at this time are also denoted with a subscript 0, e.g. $a(t_0) = a_0$. We can rescale our quantities as follows,

$$a'(t) = \frac{a(t)}{a_0}, \tag{1.39}$$

$$x'^i = a_0 x^i, \tag{1.40}$$

$$K' = \frac{K}{a_0^2}. \tag{1.41}$$

The FLRW metric does not change form under this transformation, as we now have

$$ds^2 = -dt^2 + a'(t)^2 f(K'\mathbf{x}'^2)^2 d\mathbf{x}'^2. \tag{1.42}$$

This alternative parametrization indeed has the nice property that $a'_0 = 1$, which allows us to interpret comoving lengths as 'lenghts as they are today', making the concept somewhat more tangible. Afterwards, we can simply drop the primes as the new quantities replace the old ones. While we will often keep a_0 explicit for clarity, it should be clear that we can always set $a_0 = 1$ to slightly simplify matters.

The game played above can also be reversed. Some authors like te redefine their coordinates such that they become dimensionless, instead making the scale factor dimensionfull. This is done by defining $k|K| \equiv K$ and letting $\sqrt{|K|}x^i$ be the new coordinate. In that case, the FLRW metric becomes

$$ds^2 = -dt^2 + R(t)^2 \frac{1}{(1 + \frac{1}{4}k\mathbf{x}^2)^2} d\mathbf{x}^2, \tag{1.43}$$

where $R(t) = a(t)/\sqrt{|K|}$ is the dimensionfull scale factor. For a flat universe, $k = 0$, while closed and open universes are described by $k = +1$ and $k = -1$ respectively. Coordinates can then be chosen such that [10, eq. (8.34)]

$$ds^2 = -dt^2 + R(t)^2(d\chi^2 + S_k^2(\chi)d\Omega^2), \tag{1.44}$$

where χ is related to the radial coordinate and

$$S_k(\chi) = \begin{cases} \sin(\chi) & \text{for } k = +1, \\ \chi & \text{for } k = 0, \\ \sinh(\chi) & \text{for } k = -1. \end{cases} \tag{1.45}$$

Such coordinates will not be used in this thesis.

1.5 The Friedmann Equation

We now know that a homogeneous and isotropic universe has only one geometrical degree of freedom. It would be nice to know how it evolves, for then we would be able to actually do cosmology. For this, we will of course need the Einstein field equations. These can be written as [10, eq. (4.45)]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{M_p^2}T_{\mu\nu}, \tag{1.46}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar (i.e. the trace of $R_{\mu\nu}$) and $T_{\mu\nu}$ is the energy-momentum tensor. Furthermore, $M_p = 1/\sqrt{8\pi G}$ is the reduced Planck mass.

We can obtain some information about the tensors appearing in the Einstein equations without doing any calculations by considering the symmetries of the FLRW spacetime. We can change the isometries of the subspaces to isometries of the full spacetime by leaving the time coordinate t invariant and perform the proper diffeomorphism on the spatial coordinates x^i. Since $\tilde{g}_{ij}$ is form-invariant, and since $a(t)$ and g_{00} are trivially so, the full metric $g_{\mu\nu}$ is too (it is also shown explicitly that this way of generalizing isometries works at the level of Killing vectors in Appendix C.4).

It can be verified that a diffeomorphism that leaves the metric form-invariant also leaves derivatives of the metric form-invariant. Define the function

$$f_{\mu\rho\sigma}(x) = \frac{\partial}{\partial x^\mu}g_{\rho\sigma}(x). \tag{1.47}$$

Then, after a diffeomorphism,

$$f'_{\mu\rho\sigma}(x') = \frac{\partial}{\partial x'^\mu}g'_{\rho\sigma}(x'). \tag{1.48}$$

If the diffeomorphism is an isometry, then $g'_{\rho\sigma}(x') = g_{\rho\sigma}(x')$ and thus

$$f'_{\mu\rho\sigma}(x') = \frac{\partial}{\partial x'^\mu}g_{\rho\sigma}(x') = f_{\mu\rho\sigma}(x), \tag{1.49}$$

and thus $f_{\mu\rho\sigma}$ is form-invariant as well. Since Christoffel symbols are built using the metric and its derivatives, and since the Ricci tensor and scalar are built using the Christoffel symbols and the metric, it follows that the Ricci tensor and scalar are themselves form-invariant. Thus, the left-hand side of Einstein field equations (1.46) is form-invariant under isometries. Since the validity of the equations should not be altered by a diffeomorphism, this implies that the energy-momentum tensor is form-invariant under isometries as well. This proves the assumption at the beginning of Section 1.4 that the symmetries of the metric describing the universe are also symmetries of the content in the universe. Here, we will use isometries to draw conclusions about the form of both the metric and the energy-momentum tensor.

One of the isometries we can perform is (non-infinitesimal) rotation, i.e. $x^i \to S^i{}_j x^j$ where $S^i{}_j$ is a matrix in some representation of the group $SO(3)$, which obey $S^k{}_i \delta_{kl} S^l{}_j = \delta_{ij}$ [10, eq. (1.30)]. Since every spatial index on a tensor picks up a matrix $S^i{}_j$ under the diffeomorphism, we can quickly deduce that $R_{0i} = 0$. Any other function would not satisfy $S^k{}_i R_{0k} = R_{0i} \forall S^j{}_i \in SO(3)$, which is required by form-invariance (spatial coordinates are suppressed for the moment). Similarly, requiring $R_{ij} = S^k{}_i S^l{}_j R_{kl}$ reveals $R_{ij} \propto \delta_{ij}$ (by definition of the $SO(3)$ matrices). We can go further: requiring R_{ij} to be invariant under quasitranslations suggests $R_{ij} \propto \tilde{g}_{ij}$, where the proportionality 'constant' is a function of time. In fact, it can be shown generally that any two-tensor in a maximally symmetric space that is form-invariant under all isometries must be proportional to the metric [35, sec. 13.4]. Because of the way the isometries are carried over to the full spacetime, we can effectively consider the spatial parts of tensors to live in the space Σ, requiring the proportionality.

The implication is that we can write

$$R_{\mu\nu}(t, \mathbf{x}) = \begin{pmatrix} r_0(t) & 0 & 0 & 0 \\ 0 & & & \\ 0 & & r_1(t) g_{ij}(\mathbf{x}) & \\ 0 & & & \end{pmatrix}. \tag{1.50}$$

Note that we have used g_{ij} instead of $\tilde{g}_{ij}$, since the two metrics are proportional to each other anyway as long as we allow the propotionality 'constant' to be time dependent. Since $R_{\mu\nu}$ derives from the metric, both r_0 and r_1 must be functions of the scale factor. It can be calculated that [10, eq. (8.45)]

$$r_0(t) = -3\frac{\ddot{a}(t)}{a(t)}, \tag{1.51}$$

$$r_1(t) = \frac{\ddot{a}(t)}{a(t)} + 2\left(\frac{\dot{a}(t)}{a(t)}\right)^2 + 2\frac{K}{a(t)^2}. \tag{1.52}$$

This gives us the trace

$$R = g^{\mu\nu} R_{\mu\nu} = g^{00} r_0 + r_1 g^{ij} g_{ij} = -r_0 + 3r_1. \tag{1.53}$$

The implications for $T_{\mu\nu}$ are, in some sense, even more interesting. Just like we did for $R_{\mu\nu}$, we can write the energy-momentum tensor as

$$T_{\mu\nu}(t, \mathbf{x}) = \begin{pmatrix} \rho(t) & 0 & 0 & 0 \\ 0 & & & \\ 0 & & p(t) g_{ij}(\mathbf{x}) & \\ 0 & & & \end{pmatrix} \tag{1.54}$$

(the reason we chose ρ and p will become clear in a moment). However, this way of writing the tensor is not covariant at all; the equation only holds in the reference frame of a fundamental observer. A way to make it covariant is to invoke the four-velocity

$$u^\mu = \frac{dx^\mu(\tau)}{d\tau}, \tag{1.55}$$

where $d\tau$ is the proper time ($d\tau^2 = -ds^2$) and $x^\mu(\tau)$ is the worldline of an observer. It can be considered to parametrize the frame of reference used. Since the four-velocity is equal to δ^μ_0 for any fundamental observer (since they are at rest w.r.t. the comoving coordinates and thus $x^\mu = \delta^\mu_0$ and $d\tau = dt$), the energy-momentum tensor can then be written as

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \tag{1.56}$$

which is manifestly covariant. But energy-momentum tensors of this form are well-known; it describes a **perfect fluid** with **energy density** ρ, **isotropic pressure** p and fluid velocity u_μ [10, eqs. (1.114) and (8.48)] (we have picked the symbols ρ and p to adhere to the standard nomenclature). Thus, matter in a homogeneous and isotropic universe is necessarily a perfect fluid. What is more, since $u^i = 0$ according to any comoving observer, the fluid is itself comoving (i.e. at rest in comoving coordinates). Note that, while not the case here, a perfect fluid can also have a nonconstant velocity field $u^\mu(x)$.

Now we are finally ready to write down the Einstein equations (for which we can safely use the frame-dependent formulations of $R_{\mu\nu}$ and $T_{\mu\nu}$). It should be clear that the $0i$ equations have no content. The 00 equation becomes

$$\begin{aligned} R_{00} - \frac{1}{2}Rg_{00} &= \frac{1}{M_p^2}T_{00} \\ r_0 + \frac{1}{2}(-r_0 + 3r_1) &= \frac{1}{M_p^2}\rho \\ r_0 + 3r_1 &= \frac{2}{M_p^2}\rho \\ 6\left[\left(\frac{\dot a}{a}\right)^2 + \frac{K}{a^2}\right] &= \frac{2}{M_p^2}\rho \\ \left(\frac{\dot a}{a}\right)^2 &= \frac{1}{3M_p^2}\rho - \frac{K}{a^2}, \end{aligned} \tag{1.57}$$

or, recognizing the Hubble parameter,

$$H^2 = \frac{1}{3M_p^2}\rho - \frac{K}{a^2}. \tag{1.58}$$

There is one other linearly independent Einstein equation. The ij equations are

$$\begin{aligned} R_{ij} - \frac{1}{2}Rg_{ij} &= \frac{1}{M_p^2}T_{ij} \\ \left(r_1 - \frac{1}{2}(-r_0 + 3r_1)\right)g_{ij} &= \frac{1}{M_p^2}pg_{ij} \\ \frac{1}{2}r_0 - \frac{1}{2}r_1 &= \frac{1}{M_p^2}p \\ -2\frac{\ddot a}{a} - H^2 - \frac{K}{a^2} &= \frac{1}{M_p^2}p \\ \frac{\ddot a}{a} &= \frac{1}{2}\left(H^2 + \frac{K}{a^2}\right) - \frac{1}{2M_p^2}p. \end{aligned} \tag{1.59}$$

The expression between large brackets can be eliminated in favor of ρ using the Friedmann equation. This gives

$$\frac{\ddot a}{a} = -\frac{1}{6M_p^2}(\rho + 3p). \tag{1.60}$$

Together, equations 1.58 and (1.60) are called the **Friedmann equations**. The first is the one used most often and is often referred to as ***the*** **Friedmann equation** [10, p. 336]. The second equation is sometimes called the **acceleration**

equation [28, eq. 2.48]. This is the nomenclature that will be used throughout this thesis, since referring to a specific equation is easier this way.

There is an interesting way of rewriting the Friedmann equation. We define the so-called **critical density** by

$$\rho_{\text{crit}} \equiv 3H^2 M_p^2. \tag{1.61}$$

This allows us to write a dimenionless **density parameter**,

$$\Omega \equiv \frac{\rho}{\rho_{\text{crit}}}, \tag{1.62}$$

such that the Friedmann equation becomes

$$\Omega - 1 = \frac{K}{H^2 a^2}. \tag{1.63}$$

This means that when $\Omega = 1$, i.e. when $\rho = \rho_{\text{crit}}$, the universe is spatially flat ($K = 0$). This explains the name 'critical'. When $\Omega > 1$, the universe is closed, while when $\Omega < 1$, the universe is open.

There is one more equation we can find from the above. The energy-momentum tensor is covariantly conserved, meaning

$$\nabla_\nu T^{\mu\nu} = 0. \tag{1.64}$$

The equations for $\mu = i$ yield no information, for it can be calculated that $\nabla_\nu T^{i\nu} = 0$ holds automatically. This is to be expected, as it can be interpreted as a spatial three-vector (which must be zero by isotropy). The equation for $\mu = 0$[5] does yield information though. Before working it out, observe that (due to diagonality of the metric) $\nabla_\nu T^{\mu\nu} = 0$ is equivalent to $\nabla_\nu T^\nu{}_\mu = 0$. The tensor $T^\mu{}_\nu = g^{\mu\rho} T_{\rho\nu}$ looks particularly nice. The 00 component picks up a factor $g^{00} = -1$, while in the ij part $\tilde{g}_{ij}$ is contracted with its inverse. Thus, $T^\nu{}_\mu = \text{diag}(-\rho, p, p, p)$. The covariant divergence of $T^\nu{}_0$ then is

$$\begin{aligned}
\nabla_\nu T^\nu{}_0 &= \partial_\nu T^\nu{}_0 + \Gamma^\nu_{\nu\lambda} T^\lambda{}_0 - \Gamma^\lambda_{\nu 0} T^\nu{}_\lambda \\
&= -\partial_0 \rho - \Gamma^\nu_{\nu 0} \rho + \cancelto{0}{\Gamma^0_{00}} \rho - \Gamma^i_{i0} p \\
&= -\dot{\rho} - \Gamma^i_{i0}(\rho + p) - \cancelto{0}{\Gamma^0_{00}} \rho \\
&= -\dot{\rho} - 3H(\rho + p),
\end{aligned} \tag{1.65}$$

where the results of Appendix C.3 have been used for the Christoffel symbols. Thus, requiring covariant energy conservation means

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{1.66}$$

This is known as the **continuity equation** [28, eq. 2.40] [7, 1.3.98]. In general, ρ and p are not independent variables. They are related by an **equation of**

[5] In some sense, one might think of this as a covariant version of energy conservation. However, the absence of a timelike Killing vector means that energy is in general not conserved in a FLRW universe.

state [10, p. 334], which is specific for the kind of energy being described. The ones we will be dealing with all obey a simple linear relation,

$$\rho = wp, \tag{1.67}$$

where w is a constant.

With this in hand, we can solve the continuity equation so that we know how the matter content of the universe evolves. This, in turn, can be used to solve the Friedmann equation (as will be done in Section 1.6). Thus, the dynamics of the universe (i.e. of $a(t)$) are determined by only three equations: the Friedmann equation, the continuity equation and the equation of state. However, we have four equations, as there is also the acceleration equation. One might wonder if this is even consistent. Rest assured that it is, for these four equations are not independent of one another. Taking a time derivative of the Friedmann equation gives

$$2H\dot{H} = \frac{1}{3M_p^2}\dot{\rho} + 2\frac{K}{a^2}H. \tag{1.68}$$

Since

$$\dot{H} = \partial_t \left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \frac{\dot{a}}{a^2}\dot{a} = \frac{\ddot{a}}{a} - H^2, \tag{1.69}$$

the equation becomes, after dividing by $2H$ and moving terms around a bit,

$$\frac{\ddot{a}}{a} = H^2 + \frac{K}{a^2} + \frac{1}{6M_p^2}\frac{\dot{\rho}}{H}. \tag{1.70}$$

On the right hand side, the first two terms can be rewritten in terms of ρ using the Friedmann equation and the last term can be rewritten in terms of ρ and p using the continuity equation. It is easily verified that one then indeed obtains the acceleration equation, and one can thus do without. We can of course also do without the continuity equation in favor of the acceleration equation, but this is not the approach taken in this thesis. We will however remember the acceleration equation, for it is a convenient expression to use when dealing with second derivatives of the scale factor.

1.6 Solving Friedmann

In this section, we will consider how exactly we can determine the evolution of the homogeneous and isotropic universe using the equations found in Section 1.5. We will start by considering a singly type of energy, and use the continuity equation and equation of state to determine how it evolves with the universe. This in turn can be used to write down an equation for the scale factor alone (and some constants), which fully determines how the universe evolves.

Substituting the equation of state into the continuity equation gives us a first-order differential equation,

$$\dot{\rho} + 3\frac{\dot{a}}{a}(1+w)\rho = 0. \tag{1.71}$$

This can be rewritten as

$$\frac{1}{dt}d\rho = -3(1+w)\rho\frac{da}{a}\frac{1}{dt}. \tag{1.72}$$

The $1/dt$ cancels and we can move the ρ to the left-hand side,

$$\frac{d\rho}{\rho} = -3(1+w)\frac{da}{a}. \tag{1.73}$$

Integrating both sides gives logarithms and an integration constant. Exponentiating and rewriting the constant such that $\rho(t_0) \equiv \rho_0$ gives

$$\rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}. \tag{1.74}$$

Before proceeding, let's consider some typical types of energy that this ρ can describe. Perhaps the simplest example is ordinary, heavy matter. In the limit where the pressure exerted by matter is negligible as compared to its energy density, the matter is called **dust**. It has $w = 0$ and thus

$$\rho_{\text{dust}} \propto \frac{1}{a^3}. \tag{1.75}$$

This has the following easy interpretation. Consider a large comoving cube in the universe with sides of physical length L_0 at t_0 (at which time $a = 1$). The cube contains a certain amount of matter with total mass M. In the dust limit, the (average) energy density of the matter in the cube at t_0 is just $\rho_0 = M/V_0 = ML_0^{-3}$. As the universe evolves, the amount of matter inside the cube will remain the same because it is at rest w.r.t. the comoving coordinates (otherwise, the velocity field would define a vector, which is forbidden by isotropy). Thus, the energy density is $\rho = M/V = ML^{-3}$. Since $L = aL_{comoving}$ with $L_{comoving} = L_0$, this gives $\rho(a) = M(aL_0)^{-3} = \rho_0 a^{-3}$. When the universe expands, dust is simply diluted.

One might be surprised to learn that the same does not hold for other types of energy. Consider, for the moment, **radiation**. While dust has rest mass but no kinetic energy, radiation is the limit where there is only kinetic energy and no rest mass. An obvious example of this is electromagnetic radiation (photons), but relativistic massive particles can often be treated the same way (like e.g. relativistic neutrinos [30, p. 281]). Radiation has $p = \frac{1}{3}\rho$ [14, chap. 2, prob. 14. (a)]. This means

$$\rho_{\text{rad}} \propto \frac{1}{a^4}. \tag{1.76}$$

This may be interpreted as follows. Since the energy of radiation is proportional to one over the wavelength ($E \propto \lambda^{-1}$), and since $\lambda_{\text{physical}} = a\lambda_{\text{comoving}}$, the total energy in a bunch of radiation goes as $E \propto a^{-1}$. Combining this with the dilation effect that also applies to dust, we find $\rho = E/V \propto a^{-4}$.

There is one last type of energy of particular interest. Consider for the moment $w = -1$. This implies $\rho \propto a^0$, i.e. the energy density is constant. Such energy is called **dark energy** [14, sec. 2.4.5]. While the negative pressure and failure to dilute might seem quite queer, it is estimated to make up about 69% of the energy in our observable universe [1, table 4]. Since dark energy does not dilute, it is compelling to think of it as a property of space itself rather then some entity within the space. Indeed, the easiest way to put dark energy into your theory is to invoke **vacuum energy** with $T^{\text{vac}}_{\mu\nu} = -\rho_{\text{vac}} g_{\mu\nu}$, where ρ_{vac} is constant in both time and space. This can also be interpreted in another way. When we write the Einstein equation (1.46), we can move the vacuum energy to the left-hand side to find

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{M_p^2} T_{\mu\nu}, \tag{1.77}$$

where $\Lambda = \frac{1}{M_p^2}\rho_{\text{vac}}$ is the **cosmological constant**. While vacuum energy and cosmological constant are often used interchangeably, the first treats the phenomenon like a property of the matter fields in the universe while the second rather treats it as a geometrical effect [10, sec. 4.5]. While there are other dark-energy candidates (where e.g. the energy density has some sort of time dependence, a class of models often referred to as 'quintessence' [14, p. 47, footnote]), they do not seem as popular [7, p. 21] nor are their basics as easy to explain.

Now the crux is that for some mixture of contents of the universe, we cannot solve the Friedmann equation analytically. We then have

$$\rho = \sum_i \rho_i = \sum_i \rho_{0i} \left(\frac{a}{a_0}\right)^{-3(1+w_i)}. \tag{1.78}$$

However, we can often approximate the universe as containing only one type of energy. In that case, exact solutions are available. First, let's make the approximation a bit more plausible. The different types evolve differently. If the universe (i.e. the scale factor) is small enough, the radiation energy density can be treated as much larger then the others. When the universe is very large, both matter and radiation will have diluted away and only dark energy is significant. Somewhere in between, dust will be the most important contituent of the universe. Thus, this approximation will be realistic at some point in the evolution of the universe, and the whole evolution can be estimated by patching together these different epochs. Much better results however are obtained using numerical methods.

First, we will consider the case $K = 0$, which is the easiest. The Friedmann equation becomes

$$\frac{\dot{a}}{a} = \frac{1}{\sqrt{3}M_p}\sqrt{\rho} = \frac{\rho_0}{\sqrt{3}M_p}\left(\frac{a}{a_0}\right)^{-\frac{3}{2}(1+w)}. \tag{1.79}$$

Writing $\dot{a} = da/dt$ and putting all a and da on one side and all constants and dt on the other gives

$$a^{\frac{1}{2}(1+3w)} da = \frac{\rho_0 a_0^{\frac{3}{2}(1+w)}}{\sqrt{3}M_p} dt. \tag{1.80}$$

This can now be integrated to yield (after multiplying with $\frac{1}{2}(1+3w)$)

$$a^{\frac{3}{2}(1+w)} = A_w a_0^{\frac{3}{2}(1+w)} t + C, \tag{1.81}$$

where C is an integration constant and we have defined

$$A_w \equiv \frac{(1+3w)\rho_0}{2\sqrt{3}M_p}. \tag{1.82}$$

Thus, the solution is

$$a(t) = (A_w a_0^{\frac{3}{2}(1+w)} t + C)^{\frac{2}{3}\frac{1}{1+w}}. \tag{1.83}$$

We have already defined the initial conditions, namely $a(t_0) = a_0$ (and $\rho(t_0) = \rho_0$). Thus, C should be a function of those quantities. To determine what the integration constant looks like, we first put it into a more convenient form. First, we divide a_0 out of the term with the weird power. Next, we'd rather have a function that is linear in $t - t_0$ instead of just t since this term would be conveniently zero at t_0. Since $A_w t_0$ is itself just a constant, this can be pulled out of the integration constant. We thus get

$$a(t) = a_0(A_w(t - t_0) + D)^{\frac{2}{3}\frac{1}{(1+w)}}, \tag{1.84}$$

where $D \equiv a_0^{\frac{2}{3}\frac{1}{(1+w)}} C + A_w t_0$. Since D is a function of C, it is itself an integration constant. The requirement $a(t_0) = a_0$ now trivially reveals that $D = 1$. Thus, our solotion is

$$a(t) = a_0(A_w(t - t_0) + 1)^{\frac{2}{3}\frac{1}{(1+w)}}. \tag{1.85}$$

Before discussing what this solution looks like for different types of energy we change its form one more time. We can define the time $\bar{t}_0 \equiv t_0 - 1/A_w$. Then the term between brakets in the solution becomes $A_w(t - \bar{t}_0)$. We are free to callibrate time however we like, so we can define the zero point of our timeline to be $\bar{t}_0$, i.e. $\bar{t}_0 = 0$. Then,

$$a(t) = \bar{A}_w t^{\frac{2}{3}\frac{1}{(1+w)}} \tag{1.86}$$

with $\bar{A}_w = a_0 A_w^{\frac{2}{3}\frac{1}{(1+w)}}$. This insightfull way of writing things shows clearly that the universe thus evolves according to a simple power law. It shows that in the case of **matter domination** (i.e. when the universe can be approximated as only containing dust)

$$a(t) \propto t^{\frac{2}{3}}, \tag{1.87}$$

while in the case of **radiation domination** (i.e. when the universe can be approximated as only containing radiation)

$$a(t) \propto \sqrt{t}. \tag{1.88}$$

Furthermore, it is usefull to note that the solution for the Hubble constant is always proportional to $1/t$; one can easily calculate

$$H(t) = \frac{2}{3}\frac{1}{1+w}\frac{1}{t}. \tag{1.89}$$

There is one important caveat however. The solution presented above only works for $w \neq -1$! This should be clear from the fact that we are dividing by $1+w$ in the exponent. In order to know what the solution is for $w=-1$, i.e. for a universe dominated by dark energy, we need to go back to equation (1.79). This reveals that the Hubble paramter is, in fact, constant (with value $H = \rho_0/(\sqrt{3}M_p)$ and thus

$$\dot{a} = aH. \tag{1.90}$$

This is solved by

$$a(t) = e^{Ht+C}, \tag{1.91}$$

where C is an integration constant (distinct from the C used while finding the $w \neq -1$ solution). Requiring $a(t_0) = a_0$ quickly reveals

$$a(t) = a_0 e^{H(t-t_0)}. \tag{1.92}$$

Note that there is no time where $a = 0$ to choose as zero point on the timeline. In fact, there is no preferred time at all in this solution.

The above equation is the solution corresponding to **vacuum domination**, where the universe grows exponentially. This has been theorized to have happened during the first 10^{-32} seconds of the universe under the name of **inflation** [8, p. 12]. It can be easily verified from equation (1.53) that (since we assumed $K = 0$) the Ricci scalar is constant for this solution. Indeed, the solution corresponds to a spacetime that is maximally symmetric in $3+1$ dimensions with positive curvature scalar. It is known as **de Sitter space** [35, eq. (13.3.41)] and can be considered to satisfy the 'perfect' Copernican principle where there is no preferred position nor direction in spacetime rather than just in space [10, p. 324].

In this section, we have only considered solutions of the Friedmann equations in a flat universe, i.e. for $K = 0$. We hope that this gives some insight in what kind of different substances fill the universe and how they affect the evolution of the universe. Similar calculations for a curved universe are more difficult and thus will not be included in this thesis (as they have only limited additional value). Results of such calculations can however be found in [26, sec. 5.3].

Chapter 2

Linear Perturbation Theory

In Chapter 1, we have reviewed what our universe must look like and how it evolves assuming spatial homogeneity and isotropy. However, in reality, the universe only displays these assumptions approximately. We can deal with this by allowing the quantities we use to describe the universe to deviate slightly from their background values, which are the values dictated by homogeneity and isotropy. By treating these deviations **perturbatively**, i.e. by neglecting quantities that are the product of too many such deviations, we can simplify their treatment immensely. Symmetry will still have a helpfull role to play, which would not be the case in a generic universe.

In this thesis we will only consider **linear perturbation theory**, i.e. we will neglect any quantity that is the product of two or more perturbations. This is most popular and by far the simplest way of doing cosmic perturbation theory (although higher-order perturbation theory is of course more accurate). This means all perturbations must obey linear equations of motions, which can be derived from Einstein's field equations.

In fact, there are some more very nice properties that make cosmic linear perturbation theory doable. We can classify perturbations based on how they transform under rotations, as is done in Section 2.1. Isotropy of the background dictates that the different modes decouple from ane another. Similarly, homogeneity of the background dictates that different eigenmodes of the Laplacian decouple from one another (in flat space, these are simply the Fourier modes). This decomposition is discussed in Section 2.2.

Finally, in Section 2.3, we see how the invariance under diffeomorphisms of general relativity implies that there is no unique way to split the universe into background and perturbations. This gives rise to the concept of gauge transformations, and the possibility to choose a gauge which suits your needs.

2.1 Scalars, Vectors and Tensors

As we have seen in Section 1, there are two objects needed to describe the universe: the metric to describe its geometry, and the energy-momentum tensor to describe its contents. They are related through the Einstein field equations (1.46). We have seen in Section 1.5 what these objects look like in a universe that is spatially homogeneous and isotropic. We define these to be the background objects. To allow for small deviations from these symmetries, we perturb these tensors:

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x), \tag{2.1}$$
$$T_{\mu\nu}(x) = \bar{T}_{\mu\nu}(x) + \delta T_{\mu\nu}(x). \tag{2.2}$$

Here and everywhere else, overbars denote background quantities (except when used on covariant derivatives and Christoffel symbols).

The results of Section 1.5 reveal that the background metric is just the FLRW metric, $ds^2 = -dt^2 + a^2 f^2 d\mathbf{x}^2$, while the background energy-momentum tensor is that of a perfect fluid, $\bar{T}^{\mu}{}_{\nu} = \mathrm{diag}(\bar{\rho}, \bar{p}, \bar{p}, \bar{p})$. $h_{\mu\nu}$ and $\delta T_{\mu\nu}$ are the perturbations and can in general be any four-dimensional two-tensor. Since we are doing linear perturbation theory, any product of components of these tensors is neglected. This means, for one thing, that the indices on $h_{\mu\nu}$ and $\delta T_{\mu\nu}$ are raised and lowered using only the background metric as opposed to the full metric. Thus, we have for example,

$$h^{\mu}{}_{\nu} = g^{\mu\rho} h_{\rho\nu} = \bar{g}^{\mu\rho} h_{\rho\nu} + \cancelto{0}{h^{\mu\rho} h_{\rho\nu}}. \tag{2.3}$$

Note that this way of defining things implies that $h^{\mu\nu}$ is *not* the inverse of $h_{\mu\nu}$. We do however define $\bar{g}^{\mu\nu}$ to be the inverse of $\bar{g}_{\mu\nu}$. This implies that the inverse of the full metric is

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu}, \tag{2.4}$$

since we then have $g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}{}_{\nu} - h^{\mu}{}_{\nu} + h^{\mu}{}_{\nu} + \mathcal{O}(h^2) = \delta^{\mu}{}_{\nu}$.

As mentioned briefly before, we would like to collect components of the perturbations into objects that obey some distinct transformation law under rotations. This is useful, because we expect objects obeying different observation laws to decouple from one another. An argument for this is as follows. Assume that there is some set, $\mathcal{P}$, of objects with distinct linear transformation rules composed of the components of $h_{\mu\nu}$ and $\delta T_{\mu\nu}$, such that the degrees of freedom of these objects together are exactly all the perturbative degrees of freedom. These objects together then form a parametrization of the perturbations. Furthermore, assume there are subsets T_i of $\mathcal{P}$ for $i \in \{1, 2, ..., n\}$ such that all elements of T_i transform the same under rotations. Since each $p \in \mathcal{P}$ obeys a distinct such rule, the union of all T_i's is exactly $\mathcal{P}$ and their intersection is empty. Lastly, we assume that we can choose the p's such that the most general equation of motion we can write is

$$\sum_{p \in \mathcal{P}} \mathcal{O}_p p = \sum_{p_1 \in T_1} \mathcal{O}_{p_1} p_1 + \sum_{p_2 \in T_2} \mathcal{O}_{p_2} p_2 + ... + \sum_{p_n \in T_n} \mathcal{O}_{p_n} p_n = 0. \tag{2.5}$$

Here, the $\mathcal{O}_p$ are (differential) operators that obey three conditions. First of all, they must be composed from background operators and derivatives only. The second condition is that they must not have (spatial) indices which are contracted with any indices the objects in $\mathcal{P}$ might have, i.e. any components of an object p remain unmixed. Thirdly, spatial derivatives can only occur in the operators in the form of covariant derivatives $\bar{\nabla}_i$ with *contracted* indices. The significance of these conditions is that they ensure us that $\mathcal{O}_p p$ transforms exactly the same way as the object p itself (one might say that the operators are 'covariant'). The second condition ensures that the structure of p is left invariant; the operators act like simple multiplications, except for derivatives. The third condition makes sure the derivatives do not change the transformation law. Then, since the background is isotropic, we know from the first condition that the operators themselves do not change under transformation laws.

Thus, under a transformation, $(\mathcal{O}_p p)' = \mathcal{O}_p p'$. Note that the first condition is satisfied trivially in linear perturbation theory, as putting perturbed quantities in the operators would make the terms negligible anyway. The second and third conditions are the real assumptions, and choosing our parametrization $\mathcal{P}$ such that they must hold for any equation of motion requires some cleverness (as we will see shortly). When the assumptions hold, we can draw the following conclusion. In the equation of motion, the sum over each T_i as a whole transforms under rotations according to the same transformation law that is followed by each $p_i \in T_i$. Since all these laws are distinct, the only way that the sum remains zero after a rotation, is if each sum is zero individually. Since the equation of motion should not depend on the frame of reference that we happen to be in (the action can be expected to be invariant under the full Poincaré group), they must be so indeed. Thus, we really have n equations of motion of the form

$$\sum_{p \in T_i} \mathcal{O}_p p = 0, \tag{2.6}$$

i.e. the p's in different T_i's **decouple**.

Since rotations are a subset of the diffeomorphisms that only leave the time coordinate invariant, the usual suspects for the different kinds of objects are scalars, vectors and tensors[1] under such diffeomorphisms. More precisely, under the diffeomorphism $x^i \to x'^i$, $x^0 \to x^0$, a scalar transforms as

$$S(x) \to S'(x') = S(x), \tag{2.7}$$

a vector transforms as

$$V^i(x) \to \partial_j x'^i V'^j(x') = V^i(x), \tag{2.8}$$

and a tensor transforms as

$$T^{ij}(x) \to \partial_k x'^i \partial_l x'^j T'^{kl}(x') = T^{ij}(x). \tag{2.9}$$

The transformation of the different parts of a tensor (such as $h_{\mu\nu}$) under such a diffeomorphism follow immediately from Appendix B: every spatial index (i.e. latin) picks up a $\partial_j x^i$, while every temporal index (i.e. 0) remains unaltered. From this, we conclude that h_{00} is a scalar, $h_{0i} = h_{i0}$ is a vector, and h_{ij} is a

[1] Of course, scalars and vectors can also be considered tensors (0-tensors and 1-tensors respectively). In the current context we reserve the name tensor (with no further specification) for 2-tensors.

tensor.

However, this decomposition does not obey the second and third conditions in the argument above. For example, we could have an equation of motion of the form $\bar{\nabla}_i h_{00} + h_{0i} = 0$, which indeed mixes the different objects. In order to find a more appropriate parametrization of the perturbations, we have a look at some basic vector calculus. The Helmholtz theorem states that we can (under certain conditions) decompose any vector field uniquely into its divergence and its curl [16, app. B]. In flat space, this means we can write an arbitrary vector as $v_i = w_i + \partial_i \theta$, where w_i carries the curl of the vector field and $\partial_i \theta$ carries the divergence for some scalar θ (the vector w_i is then called 'transverse' [28, eq. 6.8]). This works because the divergence of a curl is always zero, and similarly for the curl of a gradient. The constraint equation $\partial_i w^i = 0$ (i.e. the condition that it is pure curl) means that w_i only describes two degrees of freedom, so together with θ there are still three degrees of freedom in the vector field. Thus, the objects w_i and θ form a valid alternative parametrization of the vector v_i.

We now see that we are indeed not able to contract derivatives with the index on w_i (or rather, such a term would not contribute to the equations of motion anyway). Furthermore, the only way we can make θ transform like a vector is by putting ∂_i in front of it. In that case, we could write an equation like $\partial_i \theta + w_i = 0$, but this would not be an equation of motion; while one vector is pure curl, the other is pure divergence, implying that both are curlless and divergenceless, making it a constraint equation rather than an equation of motion. Furthermore, the only background quantity we have is the spatial part of the metric, but since it is proportional to unity, contracting it with w_i would be not different from scalar multiplication (it does not mix indices). Lastly, as we here allow for the possibility of spatial curvature, we will decompose vectors as $v_i = w_i + \bar{\nabla}_i \theta$. Here, the vectors will be considered to live on Σ. This implies $\theta = \bar{\nabla}^{-2} \bar{\nabla}_i v^i$, where $\bar{\nabla}^{-2}$ is the inverse Laplacian operator on the subspaces. $\bar{\nabla}^{-2} A(x)$ has a unique solution (defined by $\bar{\nabla}^2 \bar{\nabla}^{-2} A = A$) as long as either the subspaces are compact or the object A vanishes fast enough at infinity [21, p. 9]. This will be assumed throughout the thesis. Note that, since covariant derivatives act as ordinary derivatives on scalars, the decomposition still looks like $v_i = w_i + \partial_i \theta$, only the objects w_i and θ are defined differently (such that $\bar{\nabla}_i w^i = 0$ instead of $\partial_i w^i = 0$, which *is* different).

We can perform a similar trick with tensors. Basically, we want to start with a symmetric tensor t_{ij} (as both the metric and the energy-momentum tensor are symmetric), and then extract vectors and scalars until we are left with a tensor t_*^{ij} that obeys the conditions above. This tensor should obey $\nabla_j t^{ij} = 0$, similarly to the transverse vectors. However, we can also do a contraction with a background quantity that is not just equivalent to scalar multiplication; we can take the trace, $\tilde{g}_{ij} t^{ij}$. To fulfill the conditions, this must yield zero. Such a tensor is called transverse-traceless. In order to get this tensor, we need to extract one vector (which is more or less $\nabla_j t^{ij}$), which in turn can be decomposed into a scalar and a transverse vector. Furthermore, we must extract one scalar for the trace. For more details of the decomposition, see [21, eq. 1.3].

In the end, we can decompose the metric perturbations such that the full metric becomes

$$
\begin{aligned}
ds^2 =& -(1+E)dt^2 + 2a(\bar{\nabla}_i F + G_i)dtdx^i \\
&+ a^2\left[(1+A)\tilde{g}_{ij} + \bar{\nabla}_i\bar{\nabla}_j B + 2\bar{\nabla}_{(i}C^V_{j)} + D_{ij}\right]dx^i dx^j,
\end{aligned}
\tag{2.10}
$$

where

$$\bar{\nabla}_i G^i = \bar{\nabla}_i C^{V\,i} = \bar{\nabla}_i D^{ij} = \tilde{g}_{ij}D^{ij} = 0. \tag{2.11}$$

Or, in other words,

$$h_{00} = -E, \tag{2.12}$$

$$h_{i0} = a(\bar{\nabla}_i F + G_i), \tag{2.13}$$

$$h_{ij} = a^2\left[A\tilde{g}_{ij} + \bar{\nabla}_i\bar{\nabla}_j B + \bar{\nabla}_i C^V_j + \bar{\nabla}_j C^V_i + D_{ij}\right]. \tag{2.14}$$

The letters used to indicate the different variables are chosen as to agree with [29]. The decomposition of perturbations into scalars, transverse vectors and transverse-traceless tensors is called **scalar-vector-tensor decomposition**.

Now we can, of course, do the same for the energy-momentum tensor. However, we are often dealing with perfect fluids. While the energy-momentum tensor in a homogeneous and isotropic universe necessarily describes a perfect fluid, not every perfect fluid has to be homogeneous and isotropic. A perfect fluid is defined by the fact that we can always find a reference frame in which the fluid appears locally isotropic. That is, $T_{\mu\nu}(x_0)$ is invariant under rotations, implying $T^{\mu}{}_{\nu} = \mathrm{diag}(\rho, p, p, p)$. This can indeed be made covariant by invoking a four-velocity field which has $u^\mu(x_0) = \delta^\mu{}_0$. This can be considered the velocity field of the fluid itself, as in a frame with this velocity the energy-momentum tensor is diagonal, and thus there are no fluxes of energy and momentum. However, the four-velocity required to put the energy-momentum tensor in this form does not need to be the same everywhere; u^μ can change from place to place. The same holds for ρ and p. Thus, the most general perfect fluid is described by

$$T_{\mu\nu}(t,\mathbf{x}) = \Big(\rho(t,\mathbf{x}) + p(t,\mathbf{x})\Big)u_\mu(t,\mathbf{x})u_\nu(t,\mathbf{x}) + p(t,\mathbf{x})g_{\mu\nu}(t,\mathbf{x}), \tag{2.15}$$

where $u^\mu u_\mu = -1$. Only when requiring homogeneity and isotropy (around every point), as is done in Section 1.5, do we find that all $\mathbf{x}$-dependence (except for that of the metric) must vanish and that the spatial vector $u_i = 0$, giving the background energy-momentum tensor.

So, how do we perturb the energy-momentum tensor of a perfect fluid? It is convenient to remain in the perfect-fluid form and thus perturb the objects u^μ, ρ and p (and, of course, $g_{\mu\nu}$ is already perturbed). We write

$$u_\mu(t,\mathbf{x}) = \bar{u}_\mu + \delta u_\mu(t,\mathbf{x}), \tag{2.16}$$

$$\rho(t,\mathbf{x}) = \bar{\rho}(t) + \delta\rho(t,\mathbf{x}), \tag{2.17}$$

$$p(t,\mathbf{x}) = \bar{p}(t) + \delta p(t,\mathbf{x}). \tag{2.18}$$

Here, we have $\bar{u}^\mu = \delta^\mu{}_0$ (in a comoving frame of reference). $\bar{\rho}(t)$ and $\bar{p}$ are the spatial averages of the energy density and isotropic pressure respectively, and their evolution is governed by the Friedmann equations (and the equation of state). There is one last thing to say about the perturbed fluid velocity. A velocity must always obey $u^\mu u_\mu = -1$, i.e. it must be a timelike unit vector. Since we have, in linear perturbation theory,

$$\begin{aligned} u^\mu u_\mu &= (\bar{u}_\mu + \delta u_\mu)(\bar{u}_\nu + \delta u_\nu) \\ &= \bar{g}^{\mu\nu}\bar{u}_\mu\bar{u}_\nu + h^{\mu\nu}\bar{u}_\mu\bar{u}_\nu + 2\bar{g}^{\mu\nu}\bar{u}_\mu\delta u_\nu \\ &= -1 + h_{00} + 2\delta u_0, \end{aligned} \tag{2.19}$$

we have $\delta u_0 = h_{00}/2$. Furthermore, since $\bar{u}_i = 0$, we have $u_i = \delta u_i$. Therefore, we are in principle free to drop the δ, but in order to remind ourselves of the perturbative nature of the quantity we choose to retain it. Lastly, we can decompose it into a transverse vector and a scalar, $u_i = \bar{\nabla}_i \delta u^S + \delta u_i^V$.

Plugging the above into the equation for perfect fluids, retaining only the linear-order perturbations, and separating the background energy-momentum tensor, we find

$$\delta T_{00} = \delta\rho - \bar{\rho} h_{00}, \tag{2.20}$$

$$\delta T_{0i} = -(\bar{\rho} + \bar{p})(\bar{\nabla}_i \delta u^S + u_i^V) + \bar{p} h_{0i}, \tag{2.21}$$

$$\delta T_{ij} = \delta p \bar{g}_{ij} + \bar{p} h_{ij}. \tag{2.22}$$

We can compare this to the decomposition of the metric perturbations. We see that we indeed have one scalar degree of freedom for the 00 part and we have one scalar and one vector degree of freedom for the $0i$ part. However, for the ij part, we only have one scalar degree of freedom, while the metric has one additional scalar, one additional transverse vector, and one additional transverse-traceless tensor here. To correct for this, and write down the most general decomposition of the energy-momentum tensor, we add these degrees of freedom as follows:

$$\delta T_{ij} = \bar{p} h_{ij} + a^2 \left[\delta p \tilde{g}_{ij} + \bar{\nabla}_i \bar{\nabla}_j \pi^S + 2\bar{\nabla}_{(i}\pi^V_{j)} + \pi^T_{ij} \right], \tag{2.23}$$

where $\bar{\nabla}_i \pi^{Vi} = \bar{\nabla}_j \pi^{Tij} = \pi^{Ti}{}_i = 0$.

The π's make it impossible to go to a frame in which the energy-momentum tensor is locally isotropic. That is why we call them the anisotropic inertia. Their values are properties of the specific fluid that is described, and they will vanish for a perfect fluid. Since we will assume the universe is filled by a perfect fluid throughout this thesis, they will not be considered henceforth. Lastly, while this will not be further considered, we note that a more realistic model of the universe would contain more than one type of energy. In that case, we can perturb the energy-momentum tensor for each fluid as done above, and take the total energy-momentum tensor to be the sum. This, then, completes the scalar-vector-tensor decomposition of the perturbations.

2.2 Eigenfunctions of the Laplacian

In the previous section, we have used the isotropy of the background to find a parametrization of the cosmic perturbations in which different sectors decouple.

We can do something similar using the homogeneity of the background. The reader is probably familiar with Fourier analysis, where an arbitrary function can be expanded in terms of trigonometric functions. These functions of the form $\exp(i\mathbf{k}\cdot\mathbf{x})$ are exactly eigenfunctions of the flat-space Laplacian operator, $\partial_i\partial^i \equiv \partial^2$. When applying this procedure to perturbations on a flat background, modes with a different wave number $\mathbf{k}$ decouple from one another. In curved space, this is generalized as one might expect; instead of expanding in terms of eigenfunction of the flat-space Laplacian, the eigenfunctions of the curved-space Laplacian on Σ (i.e. $\bar{\nabla}^2$) should be used.

The line of thought behind how to make good use of the homogeneity of space is very similar to that behind the use of isotropy in Section 2.1. We assume there is a set of objects $\mathcal{P}$ which we can use to parametrize the cosmic perturbations such that each object has a well-defined linear transformation rule under quasitranslations. Furthermore, we assume $\mathcal{P}$ can be divided into subsets T_i such that all $p \in T_i$ transform the same way under quasitranslations. Next, we assume the most general equation of motion takes the form

$$\sum_{p\in\mathcal{P}} \mathcal{O}_p p = \sum_i \sum_{p\in T_i} \mathcal{O}_p p = 0, \tag{2.24}$$

where the operators are such that $O_p p$ transforms the same as p. Since the equation of motion itself should be invariant under quasitranslations (since the Einstein-Hilbert action describing general relativity is), it follows that

$$\sum_{p\in T_i} \mathcal{O}_p p = 0 \tag{2.25}$$

for every T_i.

It now remains to find a parametrization $\mathcal{P}$ for which the above argument holds. The operators $\mathcal{O}_p$ can be composed only of background quantities and covariant derivatives. We know that the background quantities obey homogeneity, so they are invariant under quasitranslations. If $\mathcal{O}_p$ contains no derivatives, we can expand our perturbations into any complete set of functions we like and use this as $\mathcal{P}$ to fulfill the above requirements (as we are just multiplying functions by constants), e.g. we could use a Taylor series or a Fourier series. When we do include covariant derivatives though, the same only holds if the functions are eigenfunctions; that is, if $\bar{\nabla}_i p = c_i p$ for some vector c_i.

Now it so happens that, since the Einstein equations are second-order differential equations, the equations of motion can be formulated such that spatial derivatives only occur through the Laplacian [23, intro of sec. 3]. Thus, if we can expand our perturbations into a complete basis of such eigenfunctions, the different terms in the expansion will indeed decouple. Such a basis does in fact exist. Scalars can be expanded in terms of the functions Y_k, vectors in ${Y_k}^i$, and tensors in ${Y_k}^{ij}$ such that

$$\begin{aligned}
\bar{\nabla}^2 Y_k &= -k^2 Y_k, && (2.26)\\
\bar{\nabla}^2 {Y_k}^i &= -k^2 {Y_k}^i, && (2.27)\\
\bar{\nabla}_i {Y_k}^i &= 0, && (2.28)\\
\bar{\nabla}^2 {Y_k}^{ij} &= -k^2 {Y_k}^{ij}, && (2.29)\\
{Y_k}^{ij} &= {Y_k}^{ji}, && (2.30)\\
{Y_k}^i_{\ i} &= 0, && (2.31)\\
\bar{\nabla}_j {Y_k}^j_{\ i} &= 0. && (2.32)
\end{aligned}$$

We will not here write an explicit form of these harmonic functions nor of the expansions, as we will have no need for them. It should be noted though that for every k, there can be multiple eigenfunctions (that are all required in the expansion). While these different parts of the expansion corresponding to the same k decouple also from each other, the equations governing them are the same. As an example we mention that the functions Y_k are the well-known spherical harmonics, which can be found in [23, sec. 3.1]. Note that the coordinate r used in that paper is related to the coordinates used in this thesis by

$$\sinh r = \frac{\sqrt{\mathbf{x}^2}}{1 - \frac{1}{4}\mathbf{x}^2}. \tag{2.33}$$

In an open universe ($K < 0$), k is continuous. In order to expand any function (perturbation), we only need to use so-called **sub-curvature modes**, which are the modes for which $k^2 > |K|$ (their name is based on the fact that these modes vary significantly within the curvature radius $1/\sqrt{|K|}$)[2]. For a closed universe ($K > 0$), since its volume in finite, a discrete set of eigenfunctions is used. k can then take the values $(l{+}4)|K|$ where l is an integer (including 0) [21, p. 10].

A big advantage of this decoupling, is that our equations of motion become ordinary differential equations instead of partial differential equations. All spatial derivatives are replaced by the factor k^2, and only temporal derivatives remain. In such an equation, the eigenfunctions (which are generally nonzero) can be 'divided out', so that the equations of motions contain only the coefficients of the expansion, their characteristic k, background quantities and temporal derivatives. For example, one term in the expansion of the tensor D_{ij} can be $D(k, ...)Y_{k,...ij}$, where the dots denote possible other variables needed in the expansion (in order to expand in all functions with eigenvalue $-k^2$). We can do the same for the one other tensor we have, π^T_{ij} (this is the only other perturbation that can enter the equation, on grounds of Section 2.1). In the equation of motion, we can then divide out the $Y_{k,...ij}$ and suppress the argument of the coefficients to find [21, chap. 2, eq. 4.15]

$$\ddot{D} + 2H\dot{D} + \frac{1}{a^2}\left(k^2 + 2K\right) D = \frac{1}{M_p^2}\bar{p}\pi^T. \tag{2.34}$$

[2]Peculiarly, if we want to expand a general Gaussian field in terms of Laplacian eigenfunctions, we also need super-curvature modes, for which $0 < k^2 < |K|$, although these modes are not linearly independent from the sub-curvature modes. For a detailed discussion, see [23].

Sometimes, when we want to make clear that we are dealing with coefficients of the expansion into Laplacian eigenfunctions, or want to make the eigenvalue explicit, we include a subscript k, i.e. we would write the above equation as

$$\ddot{D}_k + 2H\dot{D}_k + \frac{1}{a^2}\left(k^2 + 2K\right)D_k = \frac{1}{M_p^2}\bar{p}\pi_k^T. \tag{2.35}$$

2.3 Gauges

In order to perform perturbation theory, we need to divide our universe into background and perturbations. However, since general relativity has diffeomorphism invariance, there is no unique way to do this. This can be seen as follows. Under a general diffeomorphism, the metric transforms in some way,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = g_{\mu\nu}(x) + \Delta g_{\mu\nu}(x). \tag{2.36}$$

This is nothing but a change of coordinates, which offers an alternate but equivalent description of a physical situation. Now let's say that we have some specific background metric in mind, and would like to describe the metric in both coordinate systems in terms of this background and deviations from it. Thus, we write $g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x)$ and $\tilde{g}_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \tilde{h}_{\mu\nu}(x)$. Since we define the background metric to be the same in both coordinate systems, the perturbations have to account for all the change,

$$\tilde{h}_{\mu\nu}(x) = h_{\mu\nu}(x) + \Delta g_{\mu\nu}(x) \equiv h_{\mu\nu}(x) + \Delta h_{\mu\nu}(x). \tag{2.37}$$

Since we are free to choose what coordinates to use, the perturbations corresponding to a certain state of the universe are not uniquely defined.

It may seem a bit confusing that, all of the sudden, our metric is not uniquely defined because of diffeomorphism invariance. After all, this invariance is a property of general relativity itself, and should thus also be present when dealing with a non-perturbed universe. The difference is that, when we do not allow for perturbations, we have a condition that uniquely[3] defines our coordinates: we have a specific metric $\bar{g}_{\mu\nu}$ in mind, and we require $g_{\mu\nu} = \bar{g}_{\mu\nu}$. While we could perform a diffeomorphism to obtain another description of the situation that is equally valid, the condition would be violated. In some sense, we thus have a preferred set of coordinates (for the FLRW universe, these are the comoving coordinates in which isotropy and homogeneity are manifest).

However, when we do allow for perturbations, this condition becomes somewhat less rigid. Instead of completely fixing the metric, the requirement now is $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where the perturbations can be anything, as long as they are first order in some perturbative parameter ϵ. We are now able to perform any diffeomorphism for which $\Delta g_{\mu\nu}$ is of the same order as $h_{\mu\nu}$ without violating our condition. This exactly holds for the infinitesimal diffeomorphisms already encountered in Section 1.2, i.e.

$$x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu, \tag{2.38}$$

[3] Up to isometries, of course.

where $\epsilon^\mu = \epsilon\xi^\mu$ for some vector field ξ^μ. Thus, ϵ^μ is of the same order as the perturbations (so that products of ϵ^μ and $h_{\mu\nu}$ or $\delta T_{\mu\nu}$ vanish in linear perturbation theory). Such a diffeomorphism (where the background is left invariant) is called a **gauge transformation** (see [10, sec. 7.1] for a more rigorous treatment). It is shown in Appendix B that, under it, the metric perturbations transform as

$$h_{\mu\nu}(x) \to \tilde{h}_{\mu\nu}(x) = h_{\mu\nu}(x) - 2\nabla_{(\mu}\epsilon_{\nu)}. \tag{2.39}$$

Of course, there are similar transformation rules for $\delta T_{\mu\nu}$. These are just the 'normal' covariant transformation rules for tensors [10, eq. (2.30)].

A **gauge condition** is a condition that we impose to limit our freedom to perform (gauge) transformations, and thus limit the set of coordinate systems that we are allowed to use. In fact, one can think of the requirements $g_{\mu\nu} = \bar{g}_{\mu\nu}$ and $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ encountered above as gauge conditions, although the word (in cosmology) usually refers to conditions on $h_{\mu\nu}$ and $\delta T_{\mu\nu}$. 'Choosing a gauge' (i.e. defining a specific gauge condition) usually allows one to set certain perturbations to zero and simplify problems. We will here define some popular gauges, although we will refrain from showing they are valid (i.e. any set of perturbations can be made to satisfy them by applying gauge transformations). It should be noted though that, when we do a gauge transformation, we can scalar-vector-tensor decompose the diffeomorphism parameter ϵ_μ into two scalars (ϵ_0 and ϵ^S) and one vector (ϵ_i^V) (where $\epsilon_i = \bar{\nabla}_i\epsilon^S + \epsilon_i^V$). Thus, we expect we can 'fix' (i.e. eliminate the degrees of freedom of) two scalars and one vector. In fact, we can choose whether we set $C_i = 0$ or $G_i = 0$. In this thesis, we will always use $C_i = 0$. The gauge conditions below are then taken as conditions on the scalar degrees of freeom only. For more details, the reader is encouraged to read [34, sec. 5.3].

Newtonian gauge. This gauge is defined by the conditions

$$B = F = 0. \tag{2.40}$$

It is then customary to rename $E = 2\Phi$, $A = -2\Psi$ and $D_{ij} = \gamma_{ij}$. Because Φ and Ψ (which must be equal for physical solutions, see Appendix E) get the interpretation of Newtonian potentials in the weak-field limit [10, sec. 7.3], this gauge is often the most intuitive to use. Newtonian gauge will be used predominantly in this thesis.

Comoving gauge. The gauge condition is

$$B = \delta u^S = 0. \tag{2.41}$$

For scalar perturbations, the condition $\delta u^S = 0$ implies that $\delta u_i = 0$. That means an observer using these coordinates is moving with the fluid (i.e. it is at rest in this frame), hence the name. Use of this gauge is often (e.g. in [29] and [24]) in combination with the so-called ADM notation, such that we write $A = 2\mathcal{R}_c$, $E = 2N_1$, $D_{ij} = \gamma_{ij}$ (like in Newtonian gauge) and $\bar{\nabla}_i F + G_i = \frac{1}{a}N_i = \frac{1}{a}(\bar{\nabla}_i\phi + N_i^V)$. Apart from Newtonian gauge, comoving gauge is the only gauge that will be considered in this thesis.

Synchronous gauge. This gauge is defined by

$$E = F = 0. \tag{2.42}$$

Because $E = 0$, $g_{00} = -1$ everywhere. This implies that for all comoving observers (i.e. $dx^i = 0$) time runs equally fast, allowing for the synchronization of clocks (and explaining the name).

It should be noted that, instead of imposing a gauge condition, one can also make gauge-invariant combinations of perturbations and use these to parametrize the perturbations. Of course, the same reduction in degrees of freedom should occur (since now all degrees of freedom used to describe the system are actually physical, i.e. the 'gauge' degrees of freedom represented by ϵ_μ are eliminated). Thus, we can write the metric perturbations as two scalars, one vector and one tensor in this formalism. In this thesis, we will only consider the gauge-fixing approach (as one will hopefully understand after reading Chapter 3, this is required in order to find adiabatic modes). For a formulation in terms of gauge-invariant variables, one can read e.g. [21].

Chapter 3

Adiabatic Modes and Soft Theorems

In this chapter, we leave the somewhat more basic cosmological theory behind us and delve deeper into the specific phenomenon which is studied in part II. That is, in this chapter, we will study adiabatic modes and their implications. As will be discussed, an adiabatic mode is a solution of cosmological perturbation theory that can be obtained through symmetry considerations. More specifically, by exploiting the fact that gauge-fixing conditions (as presented in Section 2.3) do often not entirely eliminate the freedom to make gauge transformations. Since adiabatic modes provide solutions that do not depend on the theory governing the contents of the universe, they are extremely useful. What is more, they can be used to derive so-called soft theorems, which make predictions about cosmic correlation functions.

First, in Section 3.1, we present the derivation and use of the adiabatic modes found first (i.e. Weinberg's scalar adiabatic modes). We observe that this approach can be generalized to obtain an infinite number of adiabatic modes, and sketch how this could be done in Section 3.2. This is really a preview of part II of this thesis, where the derivation will be done on a curved FLRW background. In Section 3.3, we discuss the physical relevance of adiabatic modes and why it is useful to study them. One of these reasons is their relation to soft theorems, which are interesting enough to warrant their own section. They will be discussed last, in Section 3.4.

3.1 Weinberg's Theorem

The theory of adiabatic modes was first developed by Weinberg in 2003 [36] (while further explanation can be found in his book [34, sec. 5.4]), assuming the universe is spatially flat ($K = 0$) and using Newtonian gauge. He showed that there are two modes (i.e. solutions to the linearized Einstein equations) for which the gauge-invariant[1] quantity

$$\mathcal{R} = \frac{1}{2}A + H\delta u^S, \tag{3.1}$$

which is called the "curvature perturbation on comoving hypersurfaces" [2], is conserved. In Newtonian gauge it is equal to

$$\mathcal{R}|_{\text{Newtonian gauge}} = -\Psi + H\delta u^S. \tag{3.2}$$

One of these modes has $\mathcal{R} \neq 0$ and, in Newtonian gauge,

$$\Phi_k(t) = \Psi_k(t) = \mathcal{R}_k \left[-1 + \frac{H(t)}{a(t)} \int^t a(t')dt'\right], \tag{3.3}$$

$$\frac{\delta\rho_k}{\dot{\rho}} = \frac{\delta p_k}{\dot{p}} = -\delta u_k^S = -\frac{\mathcal{R}_k}{a(t)} \int^t a(t')dt' \tag{3.4}$$

(while the vector and tensor perturbations are arbitrary, since they decoupled, and where $\int^t$ means that the lower bound of the integral is arbitrary). The other mode has $\mathcal{R} = 0$ and

$$\Phi_k(t) = \Psi_k(t) = \mathcal{C}_k \frac{H(t)}{a(t)}, \tag{3.5}$$

$$\frac{\delta\rho_k}{\dot{\rho}} = \frac{\delta p_k}{\dot{p}} = -\delta u_k^S = -\frac{\mathcal{C}_k}{a(t)}, \tag{3.6}$$

for some constant $\mathcal{C}_k$. Here, the subscript k is the magnitude of the wave number of the mode (see Section 2.2, where the interpretation as wave number arises since we are assuming flat space). What I will refer to as **Weinberg's theorem** is the statement that these modes always solve the linearized Einstein equations in the regime $k/a \ll H$ (i.e. for sufficiently long physical wavelengths, $\lambda_{\text{phys}} = a\lambda_{\text{comoving}} = a/k$), whatever the content of the universe.

So how did Weinberg prove his theorem? This is where, for the purposes of this thesis, things get really interesting. Weinberg used a very clever trick. While it is true that the Newtonian gauge condition limits our possibility to make gauge transformations, there is in fact still some residual gauge freedom. While **small gauge transformations**, i.e. gauge transformations for which ϵ^μ vanishes at spatial infinity, violate the Newtonian gauge condition, we are still able to make **large gauge transformations**, i.e. those for which $\lim_{\mathbf{x}^2\to\infty} \epsilon^\mu(\mathbf{x}) \neq 0$ [29, p. 5].

Now, what Weinberg did, is to start with an unperturbed universe. Here, all the (linearized) Einstein equations are trivially satisfied. Next, he performs a large gauge transformation of the following form:

$$\epsilon_0(\mathbf{x}, t) = \frac{-\mathcal{D}}{a(t)} \int_T^t a(t')dt', \tag{3.7}$$

$$\epsilon^i(\mathbf{x}, t) = -\mathcal{D}x^i. \tag{3.8}$$

[1]It should be noted that $\mathcal{R}$ is gauge invariant only in linear perturbation theory. At higher orders, the transformations of A and δu don't necessarily cancel each other out anymore.

[2]It is called as it is because it reduces to $A/2$ in comoving gauge (see 2.3). This is the reason we usually rename $A/2$ as $\mathcal{R}_c$ in comoving gauge; it just means $\mathcal{R}|_{\text{comoving gauge}}$.

The result of this transformation is that we obtain exactly the $\mathcal{R} \neq 0$ mode with $\mathcal{R} = \mathcal{D}$ from Weinberg's theorem. Since general relativity is diffeomorphism invariant, we are certain the Einstein equations are still satisfied. Since the transformations are (because of linear perturbation theory) linear in ϵ^μ, and since the difference between two solutions is also a solution, we can also perform a diffeomorphism which is the difference between two of the transformations above but with different T. This will give $\epsilon^i = 0$ and $\epsilon_0 = -\mathcal{C}/a$, where $\mathcal{C} = \mathcal{D}\int_{T'}^{T} a(t')dt'$. This results in $\mathcal{R} = 0$, while the other perturbations are exactly the second mode in Weinberg's theorem. Thus, the $\mathcal{R} = 0$ mode is found using the same trick. Note, however, that since the diffeomorphisms are constant in space, the modes we create with it have $k = 0$. Extending this argument to all $k/a \ll H$ and thus finishing the proof of the theorem proceeds below.

Both modes from Weinberg's theorem satisfy

$$\frac{\delta\rho}{\dot{\bar{\rho}}} = \frac{\delta p}{\dot{\bar{p}}} = -\delta u^S. \tag{3.9}$$

What is more, if the universe is filled with multiple fluids (each with their own equation of state), these values are all equal

$$\frac{\delta\rho_\alpha}{\dot{\bar{\rho}}_\alpha} = \frac{\delta p_\alpha}{\dot{\bar{p}}_\alpha} = -\delta u^S_\alpha = \frac{\delta\rho_\beta}{\dot{\bar{\rho}}_\beta} = \frac{\delta p_\beta}{\dot{\bar{p}}_\beta} = -\delta u^S_\beta. \tag{3.10}$$

Since the gauge-invariant quantity

$$\Gamma_\alpha = \frac{\delta p_\alpha}{\bar{p}_\alpha} - \frac{\dot{\bar{p}}_\alpha}{\bar{p}_\alpha}\frac{\delta\rho_\alpha}{\dot{\bar{\rho}}_\alpha} \tag{3.11}$$

is thought of as the amplitude of entropy perturbations [21, eq. 3.38], and since it is zero for any perturbation satisfying the above, they are considered not to carry any entropy. Borrowing the term from thermodynamics, such modes are therefore called **adiabatic modes**. This origin of the name, however, is of small import to us. It will be shown in Section 4.1 that any perturbation created using a diffeomorphism obeys

$$\frac{\delta\rho}{\dot{\bar{\rho}}} = \frac{\delta p}{\dot{\bar{p}}} = \epsilon_0 \tag{3.12}$$

(which also holds for each fluid individually in the case of a multiple-component universe) and is thus adiabatic.

Conversely, for any physical adiabatic mode, we can perform the diffeomorphism

$$\epsilon_0 = -\frac{\delta\rho(x_0)}{\dot{\bar{\rho}}} = -\frac{\delta p(x_0)}{\dot{\bar{p}}} \tag{3.13}$$

to erase all the energy density and pressure perturbations at x_0. In fact, as will become clear in Section 3.2, we can also remove the gradient at x_0. We can however not remove the full adiabatic mode, since a physical mode must go to zero at spatial infinity while ϵ_0 must be large in order to preserve the gauge. Yet, as long as we only look *locally* (i.e. close to x_0), any adiabatic mode looks the same as if it were the the results of some gauge transformation. And that is exactly how we would encourage you to think about adiabatic modes: as a mode that locally looks the same as a coordinate transformation.

Back to the modes found by Weinberg, which we will subsequently refer to as **Weinberg's first (scalar) adiabatic mode** ($\mathcal{R} \neq 0$) and **Weinberg's second (scalar) adiabatic mode** ($\mathcal{R} = 0$). There is still one problem. Since the modes have been obtained using a gauge transformation, they can just as easily be gauged away. They are really nothing but a gauge artifact. It is also clear from the fact that the modes are constant in space that they cannot be physical. They could never be normalized, and have $k = 0$, which can straightforwardly interpreted as zero momentum (since we are dealing with flat space). So what are we so excited about? It is the fact that we can extend the gauge modes to some small but nonzero momentum $k \to 0$ (or, equivalently, we can let the gauge modes go to zero 'near infinity'). They then become physical modes which (locally) resemble gauge modes, only then can we really call them adiabatic modes. In cosmology, $k \to 0$ in practice means $\frac{k}{a} \ll H$, as is explained in Section 3.3.

This does, however, require some sort of continuity condition. Namely, our solution of the $k = 0$ Einstein equations should be the $k \to 0$ limit of some $k \neq 0$ solution. It is shown in [36, pp. 7-8] that when the coefficients in the Einstein equations are continuous around $k = 0$, then the same will hold for the solutions. It is also made plausible that this will generally be the case, as k usually enters the equations only in simple ways. We then only have to worry about the possibility that some equations 'vanish', i.e. that they are trivially satisfied for $k = 0$ because of an overall spatial derivative. We see that there are indeed two such equations (which are written down in Appendix E without any assumption on the curvature constant and treated more carefully for an open universe in Section 6.2).

The most relevant physicality condition is often equation (E.8), which can be rewritten in flat space as

$$k_i k_j (\Phi - \Psi) = 0 \tag{3.14}$$

(where $i \neq j$), suggesting $\Phi = \Psi$ for any $k \neq 0$. This is however not necessarily the case for gauge modes, which still satisfy the equation trivially because $k = 0$. The equality $\Phi = \Psi$ is then not enforced, and this beomces is a constraint condition that determines which adiabatic modes can be extended to finite momentum and thus be made physical. Weinberg, of course, chose his gauge transformation exactly such that this condition is satisfied. The idea of extending a pure gauge mode (resulting from a large gauge transformation) to a physical adiabatic mode when the physicality conditions are solved is illustrated by figure 3.1.

Note that we have not proven Weinberg's theorem explicitly here. The main part that is missing is a derivation of how the different perturbations in Newtonian gauge transform under a large gauge transformation. Such a proof is omitted since Weinberg's theorem can be seen to hold as a corollary to the more general work done in part II of this thesis, where the assumption $K = 0$ is traded for $K < 0$. While the remaining gauge freedom in Newtonian gauge, and

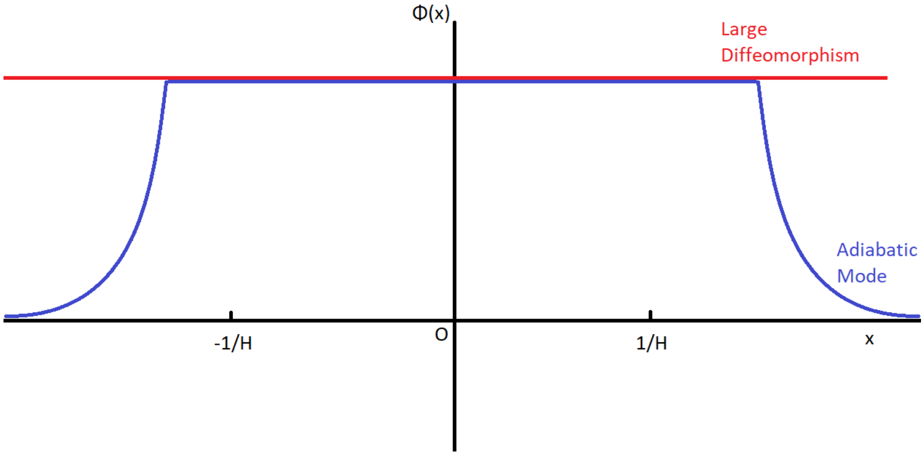


Figure 3.1: While modes obtained through a large diffeomorphism are unphysical, a physical adiabatic mode can be obtained by imposing some appropriate fall-off behavior far away (i.e. outside of the Hubble radius, which is more or less the observable universe). While the gauge mode automatically solves all linearized Einstein equations, the same can only be said for the adiabatic mode if it solves all physicality conditions. $\Phi(x)$ is the Newtonian potential, yet for a physical mode all perturbations must fall off at infinity.

the corresponding transformation rules, are explored in Chapter 4, the existence of Weinberg's adiabatic scalar modes is shown explicitly in Section 6.3 by taking the $K \to 0$ limit.

3.2 Infinitely Many Adiabatic Modes

In Section 3.1 we have discussed how Weinberg found model-independent solutions to the linearized Einstein equations by making clever use of the residual gauge freedom in Newtonian gauge. However, one can imagine there are many more large gauge transformations that can be made physical. Indeed, there exist infinitely many adiabatic modes. This section will discuss the most important of them and the systematic procedure by which they can be found (which is the foundation of Chapter 4).

First, though, let's have a brief discussion on the interpretation of Weinberg's first adiabatic mode. What we really did to create it from the background is to perform a rescaling of the spatial coordinates,

$$x^i \to \lambda x^i \tag{3.15}$$

(where $\lambda = 1+\mathcal{D}$). This must be supplemented with a temporal shift in order to remain in Newtonian gauge. This tells us that Weinberg's first adiabatic mode locally looks the same as a coordinate rescaling. As discussed in Section 3.1, a physical adiabatic mode has some small but nonzero k, but locally such a mode cannot be distinguished from a constant ($k = 0$) mode. After all, the distances over which such a mode varies will be very large. If the physical wavelength is much longer then the Hubble length, it will certainly be impossible to tell the difference. This has some interesting implications for physics in the presence of Weinberg's first adiabatic mode: processes would occur the same if there was no adiabatic mode but as seen through rescaled coordinates. This is exactly the basis for soft theorems, as will be discussed in Section 3.4.

After Weinberg's discovery, people found other adiabatic modes. That is, they found other physical modes that locally look the same as some coordinate transformation. Most predominantly, there is the gradient scalar mode, in which the scalar perturbations are not (locally) constant but linear in x^i [29, eq. 2.37] [12, sec. 3.1] [18, sec. 4] (making the statement 'adiabatic modes look locally like a change of coordinates' even stronger, as discussed in Section 3.1). Such an adiabatic mode is obtained from a so-called special conformal transformation,

$$\epsilon_i = 2b^j x_j x_i - \mathbf{x}^2 b_i \tag{3.16}$$

with constant vector b_i. In order for the mode to be extensible to a physical solution, it must be accompanied with a time-dependent translation. The whole diffeomorphism then becomes [19, eq. 2.28]

$$\epsilon_i = 2b^j x_j x_i - \mathbf{x}^2 b_i - 2b_i \int^t \frac{dt'}{H(t')} \tag{3.17}$$

and $\epsilon_0 = 0$. Thus, space with this adiabatic mode locally looks the same as space without the adiabatic mode, after changing coordinates according to the diffeomorphism above.

Adiabatic modes are not limited to the scalar sector of perturbation theory. Another adiabatic mode attributable to Weinberg is a tensor mode, which we will henceforth refer to as **Weinberg's tensor mode**. It is obtained in a fashion similar to Weinberg's first scalar adiabatic mode, starting with the diffeomorphism

$$\epsilon_i = \omega_{ij} x^j. \tag{3.18}$$

If the matrix ω_{ij} is proportional to unity (δ_{ij}), this is a regular rescaling. When combined with the proper temporal translation (ϵ_0) it can be made physical and it results in Weinberg's first adiabatic mode. When it is antisymmetric ($\omega_{ij} = -\omega_{ji}$), this corresponds to an infinitesimal rotation. Since this is an isometry (see apppendix C.4), no objects transform and no adiabatic mode is obtained. However, a matrix that has no antisymmetric part and no part that is proportional to unity ($\omega_{ij} = \omega_{<ij>} \equiv \omega_{(ij)} - \frac{1}{3}\omega_{kk}\delta_{ij}$, see Appendix A) creates a tensor mode with [34, pp. 249-250]

$$\gamma_{ij} = -2\omega_{<ij>} \tag{3.19}$$

(while all other perturbations do not transform). The transformation that this mode locally looks like can be interpreted as a 'time-independent anisotropic rescaling' [29, eq. (2.34)]. Interestingly, there are no vanishing Einstein equations for tensors, and thus this mode is automatically extensible to a physical mode.

The above discussion, and the form of equation (3.18), already seem to suggest a straightforward way in which more adiabatic modes can be found. Why don't we write the diffeomorphism parameter as a general polynomial,

$$\epsilon_i = c_i + \omega_{ik} x^k + \frac{1}{2!}\sigma_{ikl} x^k x^l + ... \quad , \tag{3.20}$$

and then try to figure out what coefficients we should choose to obtain an adiabatic mode? This is exactly the strategy adopted by Pajer and Jazayeri in [29]. They first wrote this general Taylor expansion of ϵ_i up to third order in x^i and then examined what the constraints on the coefficients are. These constraints arise from the requirements that the gauge condition should not be violated and that the resulting modes should be extensible to the physical domain. They found the coefficients must have nontrivial time dependence and the specific form ϵ_0 should take (which can also be Taylor expanded). This way, they have reproduced all known adiabatic modes (the most important of which were discussed above), and they have discovered some new adiabatic modes as well. Among these are vector modes (which decay in an expanding universe but grow in a contracting universe) and mixed modes (where the diffeomorphism excites not just the scalar, vector or tensor sector of the perturbations but a combination of them). In principle, this approach could be used to find an infinite amount of adiabatic modes, as one could include an infinite amount of terms in the Taylor expansion of ϵ_i.

The paper by Pajer and Jazayeri [29], however, assumes the universe to be spatially flat ($K = 0$), and the same assumption is behind all the adiabatic modes presented in this chapter. The goal of this thesis is to reproduce their analysis in the case of nonzero curvature, and especially to find whether Weinberg's adiabatic modes also exist when dropping the flatness assumption. Part II is dedicated to this, and the results might surprise the reader. First, however, we discuss some additional motivation for the search for adiabatic modes.

3.3 Physical Relevance of Adiabatic Modes

In the previous sections, we have defined what adiabatic modes are, we have shown how they can be obtained and we have given some examples. Still, the whole discussion so far may seem like a rather academic endeavor with little real-world application. In this section we will argue that the study of adiabatic modes is, in fact, essential to obtain a good understanding of the universe we all live in. For one thing, adiabatic modes are related to soft theorems. These are relations between cosmic correlation functions. Since there is a lot to say about these, they get a section of their own. We discuss them in Section 3.4 and focus here on their further use in explaining cosmic observations.

One of the most interesting properties of adiabatic modes is that they are **model-independent solutions** of the linearized Einstein equations. That is to say, in the limit $k \to 0$, the linearized Einstein equations are always solved by them, whatever the contents of the universe may be. Since we are not sure about what's been going on in the universe at all times (in the early universe, the average energy of particles was much higher than any we have ever probed), that is very nice. For any adiabatic mode we observe, we know its time dependence, and thus we can trace it back in time up to the moment it became adiabatic. This way, we can partly reconstruct what went on in the early universe.

This is especially interesting in the context of inflation [8]. As mentioned briefly before in Section 1.6, this is a hypothesized period of accelerated expansion right after the big bang. Such expansion must be caused by a type of energy with equation of state $p < -\frac{1}{3}\rho$. Many theories trying to explain where this energy came from (and where it went, as we don't see it anymore) impose the existence of one or more fundamental scalar fields. The energy is then contained in the potential of these fields. At the end of inflation all this energy must be converted into more conventional types of energy, otherwise inflation will never end. This is called reheating, and it is quite unknown how this took place, as knowing anything about it would require information about how the scalar fields couple to other fields. And after that, who knows what other mysterious events might have occurred. It is thus, in general, hard to test inflationary theory using current-day observations. Adiabatic modes give us some hope that we might.

The good news is that, if inflation produces adiabatic modes, this allows us to trace back many of the perturbations observed today. A perturbation can become adiabatic only if $k \to 0$, but in practice this means $k/a \ll H$. During inflation, H is constant, and a increases exponentially. Thus, by the and of inflation, modes of a wide range of comoving wavenumbers k will be 'outside' of the Hubble radius. By the **Hubble radius**, we mean $1/H$. Since we have set the speed of light $c = 1$, this can be rewritten (more manifestly as a distance) as c/H. Remembering Hubble's law (1.1), we see that this is exactly the distance at which comoving objects move away from us at the speed of light[3]. Thus, light beyond this distance cannot reach us, and therefore the Hubble radius defines roughly the observable universe[4]. Since $\lambda_{\text{physical}} = a\lambda_{\text{comoving}} = a/k$, the condition $k/a \ll H$ is equivalent to $\lambda_{\text{physical}} \gg \frac{1}{H}$ (which we mean by 'outside' of the Hubble radius).

While Weinberg's theorem was proven for $k \to 0$, this can be considered to be equivalent to $k/a \ll H$ because the observable universe is the largest length scale accessible to us. For if $\lambda_{\text{physical}}$ is much larger than the Hubble radius, it will more or less be constant over the whole observable universe. Since everything outside of the observable universe is inaccessible to us anyway, we are unable to tell at what scale the mode will start to vary significantly. In other words, if $\lambda_{\text{physical}} \gg 1/H$, all we can tell is that the wavelength is very large, but we cannot tell *how* large. Thus, for our purposes, it is as large as a wavelength can get, and it is operationally equivalent to $k \to 0$. Adiabatic modes look locally like a change of coordinates, but for $k/a \ll H$ the mode is more or less constant within the observable universe and thus it will even globally be indistinguishable from a change of coordinates. These ideas are illustrated by figure 3.2.

[3]Note that this is no 'real' velocity. The cause is not movement of the objects itself, but rather the fact that the space separating you from the objects is increasing due to the expansion of the universe. They are at rest with respect to the comoving coordinates and their velocity is thus timelike, as it should be.

[4]Light can, of course, reach us if the expansion slows down later on.

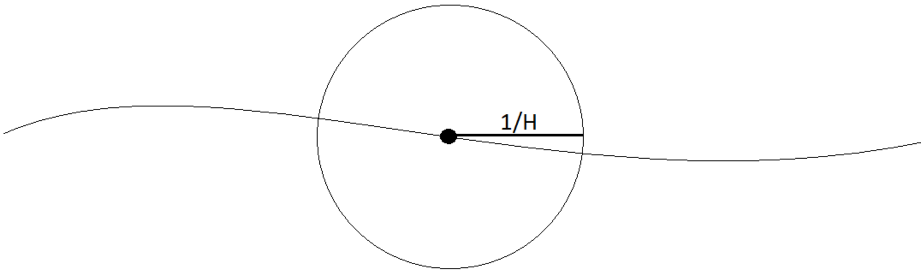


Figure 3.2: Modes for which $\frac{k}{a} \ll H$ have a physical wavelength which is much larger than the Hubble radius. The consequence is that the mode becomes more or less constant throughout the observable universe. Thus, in the $\frac{k}{a} \ll H$ domain, we cannot distinguish between larger and shorter wavelengths and thus the condition is operationally equivalent to $k \to 0$. Furthermore, and adiabatic mode that is constant throughout the observable universe is, for our purposes, not only locally but also globally indistinguishable from a change of coordinates.

Once a mode is made adiabatic during inflation, it will remain so as long as it remains outside of the Hubble radius. After all, whatever the configuration of the universe, it will remain a solution. It will thus remain unaltered. After inflation ends, we assume the universe will be filled with matter and radiation. As discussed in Section 1.6, for flat space, H will be proportional to $1/t$, while the scale factor is proportional to $t^{\frac{1}{3}}$ for radiation domination and $t^{\frac{2}{3}}$ for matter domination. In both cases, H will decay faster than $\frac{1}{a}$ (and the same will hold for a mixture of radiation and matter, or for fluids with $0 < p < \rho/3$, since in both cases one can expect $t^{\frac{1}{3}} \lesssim a \lesssim t^{\frac{2}{3}}$ and $H \sim \frac{1}{t}$). Thus, it will only be a matter of time until $k/a \ll H$. While large modes can still be adiabatic today, smaller modes have reentered the Hubble radius at some earlier time. Yet, this will generally be well after reheating, during a part of cosmological history that we know much more about (after electron positron annihilation was complete and we assume the universe to be filled only with cold dark matter, baryonic plasma, photons and neutrinos). Thus, it is possible to evolve them in time up to the moment we observe them, and hence we can also trace them back in time. A pedagogical, analytical treatment of this evolution can be found in [34, chap. 6]. For more accurate results, however, so-called Boltzmann codes are used to solve the relevant (Boltzmann) equations numerically, such as e.g. CLASS [9] and CAMB [22]. As discussed in Section 3.1, Weinberg's adiabatic modes are characterized by their $\mathcal{R}$. We can reconstruct the $\mathcal{R}$s with which the adiabatic modes were created, and the statistical properties of the different $\mathcal{R}$s at different wave numbers can tell us something about the primordial universe.

We however still haven't answered the question whether adiabatic modes are actually produced. While we know they are just one of many solutions in the long-wavelength regime after electron positron annihilation, they might be picked out during some period before that. This can be checked observationally. Using the methods described above, it can be calculated what the cosmic microwave background would look like if all modes were adiabatic. Such

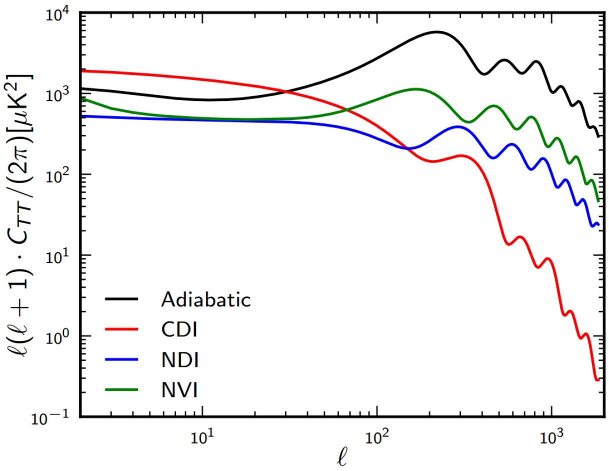


Figure 3.3: Cosmic microwave background anisotropy power spectra that would be caused by adiabatic initial conditions and by the cold dark matter/baryon density (CDI), neutrino density (NDI) and neutrino velocity (CVI) isocurvature modes. Obtained from [2, fig. 21].

an analysis of the data obtained by the Planck satellite, which surveyed the sky in the microwave bandwidth, was performed in [2]. They define adiabatic and non-adiabatic modes (which they refer to as isocurvature modes) at a late enough time such that the universe is considered to only contain cold dark matter, baryons, photons and neutrinos. For adiabatic modes all components are 'tied together'. Because they derive from a diffeomorphism (see Section 4.1 for a derivation), the components (labeled by α) all satisfy

$$\frac{\delta\rho_\alpha}{\dot{\bar{\rho}}_\alpha} = \frac{\delta p_\alpha}{\dot{\bar{p}}_\alpha} = -\delta u^S_\alpha = \frac{\delta\rho}{\dot{\bar{\rho}}} = \frac{\delta p}{\dot{\bar{p}}} = -\delta u^S. \tag{3.21}$$

Deviation from this is a sign of isocurvature. They identify four such possible non-decaying isocurvature modes: deviations of the density of baryons, cold dark matter and neutrinos, and of the neutrino velocity. In practice, the cold dark matter and baryon isocurvature modes are indistinguishable and together abbreviated as CDI. The neutrino density and neutrino velocity isocurvature modes are abbreviated as NDI and NVI respectively.

The authors of [2] have performed a statistical analysis of the anisotropies in the cosmic microwave spectrum. For more details on such an analysis, see e.g. [14, sec. 8.5]. In Figure 3.3, the different power spectrum profiles that the adiabatic and isocurvature modes would create are shown. In the observed spectrum, the amount by which the different modes contribute can be expressed in the amount of 'power' they bring in. The quantity $\alpha_{\mathcal{RR}}$ is the fraction of power due purely to adiabatic modes[5]. If it is one, the primordial modes probed

[5]More exactly, this is the power due to the correlation between adiabatic modes, which explains why the subscript contains two $\mathcal{R}$s. Alternatively, power can be brought in by correlation between isocurvature modes ($\alpha_{\mathcal{II}}$) of by correlation between adiabatic and isocurvature modes ($\alpha_{\mathcal{RI}}$).

by the survey were completely adiabatic.

Now here is the interesting thing: in a more recent paper with new data from the Planck satellite, $\alpha_{\mathcal{RR}}$ has been determined to be unity up to a few percent (at 95% confidence level) [4, Table 16]. Thus, *the primordial modes were almost exclusively adiabatic*. If any isocurvature was created, there was not a lot of it. This is a very significant observation. While all the different fluids in the universe could, in principle, go their own way, their fluctuations are very much in sync with one another. If you observe (at large scales) a lot of baryonic matter in one place, there will be relatively many photons and neutrinos as well. They are all 'locked together'. One might therefore think that, even if all these different fluids look different today, they share a common primordial origin.

Such a line of thought seems to suggest an inflationary scenario with only one scalar field. After all, if all perturbations stem from fluctuations of a single degree of freedom, they will automatically be tied together. More rigorously, it is shown in [34, eqs. 10.1.22, 10.1.23] that the scalar sector in a single-field inflation scenario only has two independent solutions. Since Weinberg's theorem tells us that, in the $k/a \ll H$ regime, his two adiabatic modes must be solutions, we can conclude *all* solutions are adiabatic for $k \to 0$[6] . Thus, has the Planck satellite provided evidence for single-field inflation? No, this would be too rash a conclusion. First of all, the data does not fully exclude the existence of primordial isocurvature modes. Furthermore, there is at least one other scenario in which adiabatic modes could have been created primordially. If the universe went through a phase of local equilibrium, and if there were no conserved quantities (such as electric charge) at the time, any non-adiabatic modes would become adiabatic [38]. Thus, even if all primordial modes are adiabatic, this does not exclude multi-field inflation because such a phase might have occurred shortly after reheating. Yet, if the existence of even a little isocurvature is verified, this rules out single-field inflation altogether. Thus, primordial adiabaticity is an important test of inflation.

The reader might be confused by the counting argument presented above for single-field inflation. It is true that Weinberg's adiabatic modes must always be solutions in the $k \to 0$ limit, but doesn't the same hold for the other adiabatic modes? After all, we claimed there to be infinitely many of them. The secret is in how modes behave exactly (and become adiabatic) when we take $k \to 0$. Consider a mode with comoving wavenumber $\mathbf{k}$ and amplitude $A(t)$. Since we are dealing with flat space for now, this is simply a Fourier mode that decouples from all other Fourier modes (see Section 2.2). The spatial dependence of such a mode is

$$A(t)e^{i\mathbf{k}\cdot\mathbf{x}_{\text{comoving}}} = A(t) + iA(t)\mathbf{k}\cdot\mathbf{x}_{\text{comoving}} - A(t)(\mathbf{k}\cdot\mathbf{x}_{\text{comoving}})^2 + \dots \,. \tag{3.22}$$

[6]Skeptical readers might point out that, even during inflation, the fields that we observe today (such as the electromagnetic field and matter fields) were around, even if they did not contain a lot of energy, which must be coupled to the inflation field one way or another (since, at the end of inflation during reheating, the energy must be transferred from the inflation field to other fields). Thus, the counting argument might not be entirely valid. However, it is proven in [37] that as long as the energy density in these other fields is small at the time, single-field inflation will still only produce adiabatic modes.

Note that this expansion can be done for any cosmological perturbation, so we do not specify which quantity it is here. We write $k = |\mathbf{k}|$ and

$$x = \frac{|\mathbf{x}_{\text{comoving}}|}{a} \frac{\mathbf{x}_{\text{comoving}}}{|\mathbf{x}_{\text{comoving}}|} \cdot \frac{\mathbf{k}}{k} \equiv |\mathbf{x}_{\text{physical}}| \hat{x}_{\text{physical}} \cdot \hat{k}, \tag{3.23}$$

i.e. x is the the physical distance along the direction of $\mathbf{k}$. Since the Hubble radius approximates the observable universe, and since we cannot do physics outside of it, we can take $0 < x < \frac{1}{H}$ as the range. Then, the above expansion becomes

$$A(t) + iA(t)\frac{k}{a}x - A(t)\left(\frac{k}{a}x\right)^2 + ... \; , \tag{3.24}$$

where the dots indicate higher powers of $\frac{k}{a}x$.

It is then clear that in the limit $\frac{k}{a} \ll H$, the mode becomes constant in space. This is exactly the form of Weinberg's adiabatic modes, and thus these modes provide two solutions for $A(t)$ in this limit. Yet, shouldn't the other adiabatic modes also provide solutions for $A(t)$? After all, they derive from large diffeomorphism and thus cannot have $k \neq 0$. The point is that the other adiabatic modes all have higher powers in x. For example, the gradient scalar mode presented in Section 3.2 is linear in x. So, if there is a limit such that the above physical mode only has the term linear in x, it can be approximated using this adiabatic mode and the time dependence of the adiabatic mode would be a solution of $A(t)$ in this limit.

However, such a limit does not exist, as the constant term is always there (the linear term only becomes dominant for $\frac{k}{a} > H$, but then the higher-order terms are even more important). Similar arguments hold for all other adiabatic modes. None of them are eigenfunctions of the Laplacian (in Fourier space, they are represented by derivatives of the Dirac delta function, while the Weinberg adiabatic mode is the Dirac delta function itself). Note that this does not mean that these adiabatic modes are unphysical. Consider a scalar mode that is linear throughout the observable universe and quickly goes to zero outside of it: for our purposes, it looks the same as the gradient scalar mode, yet it is physical and can be described as a linear combination of Fourier modes. Weinberg's adiabatic modes are the only adiabatic modes of definite k and thus are the only ones that provide model-independent solutions for modes of $k \to 0$.

3.4 Soft Theorems

In Section 3.3 we have discussed some of the physical implications and relevance of adiabatic modes. There is however one more important reason to study adiabatic modes: they are intimately connected to the subject of soft theorems and consistency conditions, which provide yet another method of testing single-field inflation. Before discussing these though, we need to quickly introduce the concept of correlation functions.

As discussed in Section 3.3, the adiabatic modes generated during the primordial universe are all characterized by their $\mathcal{R}$, which is constant in time for both Weinberg's first and second adiabatic modes. Each mode of wavenumber **k** which was outside of the Hubble radius during the generation of adiabaticity has been adiabatic with fixed $\mathcal{R}$ until it left the Hubble radius again. Since, for the wavelengths that we study, we can evolve what we observe today back in time until this moment, we can reconstruct the primordial $\mathcal{R}_{\mathbf{k}}$ for each **k**. These are the prime observables that we can use to probe the primordial universe.

However, since it is usually assumed that the fundamental theory of nature is fully Poincaré invariant, there is no way of predicting what such perturbations would look like. The dominant theory of the origin of perturbations is inflation. The scalar fields driving inflation have a quantum nature, and because of the accelerating expansion the quantum fluctuations of the field(s) are blowed up to macroscopic proportions [8, Lecture 2]. But rather than predicting what these perturbations look like, inflation can only tell us something about the statistics of perturbations.

What this means is that our (primordial) universe is only one of many that could have been produced by inflation. The thing characterizing inflation is then not the universe that it produced, but rather the distribution of possible universes from which our universe was pulled. This might seem worrisome. After all, we only have one single universe, and a distribution can never be probed significantly by witnessing one event. But there is hope of testing inflationary theory yet. Because of causality, it is reasonable to assume that the quantum fluctuations in parts of the universe sufficiently far away from each other have nothing to do with each other. That is, they are uncorrelated.

This allows us to think of the situation as follows. If we divide the universe into regions of volume V, where V is large enough such that the inflationary quantum fluctuations in any region can be expected to be uncorrelated with those in any other region, then we can effectively think of each of these regions as a separate 'universe', i.e. a separate realization of the inflationary distribution. Thus, by probing several such regions, we can learn something about this distribution and thus test theories of inflation. This idea is made more exact by the **ergodic theorem**, which states that the ensemble average of (products of) perturbations (i.e. the average over the primordial distribution) is the same as the spatial average in the limit that the average is taken over an infinite volume [34, app. D].

This allows us to define the expectation value of a perturbation $\mathcal{O}$ by $\langle \mathcal{O}(\mathbf{x}) \rangle$, which can be interpreted both as the ensemble average for $\mathcal{O}$ at point **x** (i.e. the average of the perturbation at this specific point in all different possible universes) and the quantity $\mathcal{O}(\mathbf{x})$ in our universe averaged over **x**[7]. Since we will be assuming **statistical homogeneity and isotropy** throughout, which means that expectation values are invariant under both translations and rotations, the **x** can often remain implicit. Statistical homogeneity and isotropy follow from the assumption that the background is homogeneous and isotropic and that the underlying theory in Poincaré invariant [28, sec. 8.1], since the distributions

[7]Note that they are only the same in the limit where the average is taken over an infinite volume. In universes with finite volume (such as a closed universe), the equality can never be made exact.

can be considered to characterize the theory[8].

We will be dealing only with the perturbation $\mathcal{R}$. Since perturbations are deviations from an average, we can certainly expect

$$\langle \mathcal{R} \rangle = 0, \tag{3.25}$$

which is probably understood most easily by thinking about spatial averages. Information about the distribution can be obtained by taking the expectation value of products of perturbations. Such expectation values are called **correlation functions**, as they indicate the correlation between random variables. Prime example is the two-point correlation function

$$\xi_{\mathcal{R}}(r) \equiv \langle \mathcal{R}(\mathbf{x})\mathcal{R}(\mathbf{x}+\mathbf{r}) \rangle, \tag{3.26}$$

where the dependence is only on $r \equiv |\mathbf{r}|$ because of statistical homogeneity and isotropy. A Gaussian distribution is fully characterized by this quantity. The expectation value of any odd number of $\mathcal{R}$ then vanishes, while any even correlation function factorizes into two-point functions (which is similar to 'Wick contracting' in quantum field theory, see e.g. [31, sec. 4.3]). In inflationary scenarios, these correlation functions (which have lost their quantum nature nowadays [7, sec. 6.1]) derive from quantum field theoretical scattering amplitudes [8, sec. 12.2].

Usually, the Fourier transforms of correlation functions are considered. We define this as

$$\mathcal{R}_{\mathbf{k}} = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \mathcal{R}(\mathbf{x}), \tag{3.27}$$

and thus we can calculate the Fourier version of the two-point function

$$\begin{aligned} \langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle &= \int d^3x d^3r e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}'\cdot(\mathbf{x}+\mathbf{r})} \langle \mathcal{R}(\mathbf{x})\mathcal{R}(\mathbf{x}+\mathbf{r}) \rangle \\ &= \int d^3x e^{-i\mathbf{x}\cdot(\mathbf{k}+\mathbf{k}')} \int d^3r e^{-i\mathbf{k}\cdot\mathbf{r}} \xi_{\mathcal{R}}(r) \\ &= (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k}') P_{\mathcal{R}}(k). \end{aligned} \tag{3.28}$$

Here, we introduced the **power spectrum** defined by

$$P_{\mathcal{R}}(k) = \int d^3r e^{-i\mathbf{k}\cdot\mathbf{x}} \xi_{\mathcal{R}}(r), \tag{3.29}$$

i.e. it is the Fourier transform of the correlation function. Be aware that the precise definition of the power spectrum depends on the convention used for the factors $(2\pi)^3$ in the Fourier transformation, as is nicely clarified in [8, Ap. A6]. Note the Dirac delta function $\delta^{(3)}(\mathbf{k}+\mathbf{k}')$, which can be interpreted as momentum conservation. In fact, it can be shown that any Fourier space n-point function has an overall momentum conserving delta function [28, sec. 8.1]. This allows for the notation

$$\langle \mathcal{R}_{\mathbf{k}_1} \mathcal{R}_{\mathbf{k}_2} ... \mathcal{R}_{\mathbf{k}_n} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + ... + \mathbf{k}_n) \langle \mathcal{R}_{\mathbf{k}_1} \mathcal{R}_{\mathbf{k}_2} ... \mathcal{R}_{\mathbf{k}_n} \rangle', \tag{3.30}$$

[8]In fact, statistical homogeneity is required to prove the ergodic theorem, and was thus already assumed implicitly.

according to which $P_{\mathcal{R}} = \langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle'$.

Now that we have introduced correlation functions, we are ready to introduce soft theorems. The first soft theorem was introduced by Maldacena as a consistency condition for his calculation of the three-point function in single-field inflation [24]. It relates the three-point function in the **squeezed limit** to the two-point function. Momentum conservation implies that, for the two-point function, one of the momenta in a correlation function can never be much smaller or larger than the other. This is different for the three-point function. When one of the wavenumbers is much smaller than the others, this is called the squeezed limit. Modes (or, in slightly different context, particles) with small momenta often called 'soft', and therefore the relation of a correlation function in the squeezed limit to other correlation functions is often called a **soft theorem**.

The consistency condition is derived as follows. The two-point function generated by single-field inflation can be calculated by working in comoving gauge, such that the inflation field is unperturbed and $\mathcal{R}$ is the only scalar degree of freedom in the theory. The action is expanded to second order for $\mathcal{R}$ and subsequently the field is quantized. This allows the calculation of the two-point function during inflation (in this approach, the so-called slow-roll approximation is used, which means that the Hubble parameter varies only slowly). Now, the logic is that as soon as the physical momentum becomes larger than the Hubble length, the modes become adiabatic and thus constant. They will evolve no longer. Thus, the primordial power spectrum is approximately given by the power spectrum at the moment when the wavelength was of the same length as the Hubble radius. The calculation is done in [24, sec. 2] by Maldacena and pedagogically presented in [8, sec. 12.2], giving the result

$$P_{\mathcal{R}}(k) = \frac{H^2}{2k^3} \frac{H^2}{\dot{\phi}^2} \Big|_{\frac{k}{a}=H} , \tag{3.31}$$

where ϕ is the unperturbed inflation field.

Now, when a third mode also enters the correlation function with momentum much smaller than that of the other two modes, it will have become adiabatic (i.e. become larger than the Hubble radius) at a time much earlier than the other two. It will no longer evolve and will only have the effect of a 'background wave' on the correlation between the modes of larger momentum. Since the adiabatic mode can be treated in this limit as equivalent to a coordinate transformation, the result will be a coordinate transformation on the two-point function. In comoving gauge, the Weinberg adiabatic modes are nothing but a spatial rescaling ($\epsilon^0 = 0$, as will be made clear in Section 4.3). This alters the momentum and through this, the time at which their wavelengths are equal to the Hubble radius (that is, in the power spectrum, the condition $k/a = H$ is altered). This logic leads to the consistency condition [19, eq. (1.1)]

$$\lim_{q \to 0} \langle \mathcal{R}_{\mathbf{q}} \mathcal{R}_{\mathbf{k}_1} \mathcal{R}_{\mathbf{k}_2} \rangle' = -(n_s - 1) P_{\mathcal{R}}(q) P_{\mathcal{R}}(k_1), \tag{3.32}$$

where $q = |\mathbf{q}|$. $n_s - 1$ is the spectral tilt defined by $P_{\mathcal{R}}(k) \propto k^{3+(n_s-1)}$ [28, eq. (8.19)]. When $n_s = 1$, the power spectrum is said to be scale invariant: it is then invariant under spatial rescaling. Indeed, since the effect of the short mode $\mathcal{R}_{\mathbf{q}}$ is to rescale the correlation of the other two, it is expected not to correlate in the case of scale invariance. This is nicely demonstrated by the above soft theorem.

The result (3.32) was written down first by Maldacena [24, eq. 4.7] and subsequently shown to hold beyond the slow-roll approximation by Creminelli and Zaldarriaga [13]. In [11] the 'background-wave argument' put forward by Creminelli and Zaldarriaga is further formalized, and the result is shown explicitly to hold using the effective field theory of inflation in the slow-roll approximation. In the background-wave argument, it is assumed that the long mode is constant over space (i.e. it is a Weinberg adiabatic mode). Corrections at first order in $q/|\mathbf{k}_1|$ to this were found by Creminelli, Noreña and Simonović [12, eq. (54)] by accounting for the gradient of the mode. After all, while the constant background wave is equivalent to a spatial rescaling, the constant gradient part of the wave is equivalent to a special conformal transformation, as discussed in Section 3.2. In the language of equation (3.24), the $A(t)$ dictated by Weinberg's theorem in the $k \to 0$ limit remains unaltered when the term linear in k is not thrown away, and thus the whole background wave argument runs the same (only now the background wave is equivalent to a rescaling and a special conformal transformation). In the same paper, the consistency condition is generalized to a soft theorem relating any N-point function to a $N-1$-point function.

Although the background wave picture provides an intuitive way of thinking about soft theorems and it has been the origin of the first consistency condition, soft theorems have now been derived using other strategies (which are often easier to generalize). For example, Hinterbichler, Hui and Khoury have found consistency relations to constrain the q^n behaviour of $N+1$-point correlation functions (containing both tensor modes and scalar modes) in terms of N-point functions. The result was obtained by thinking of adiabatic modes as a nonlinearly realized symmetry and finding the corresponding Ward identities [19][9]. While the $n = 0$ relation reduces to the Maldacena result, the $n = 1$ relation describes the correction found by Creminelli, Noreña and Simonović. Another approach was taken by Assassi, Baumann and Green, who derived Maldacena's consistency condition by inserting a quantum-mechanical complete set of states, providing a handle on how the condition would be violated by extra inflationary degrees of freedom and making the analogy with soft-photon physics explicit [5]. As a last example, the Maldacena consistency condition has been derived through a 'wave functional of the universe' approach by Pimentel [32].

Soft theorems are not the topic of this thesis, and therefore no detailed derivation will be provided here. We do discuss them however since they emphasize the importance of adiabaticity. For only if long-wavelength modes are adiabatic during inflation can we know for certain that the modes 'freeze out' at some point, allowing us to treat $\lim_{q\to 0} \mathcal{R}_{\mathbf{q}}$ as a background wave. What's more, the adiabatic nature subsequently allows us to 'remove' the background

[9] in this sense, $\mathcal{R}$ can be thought of as a Goldstone boson, although there are some subtleties in generalizing this concept from particle physics. For a discussion, see the introduction of [29].

wave with a coordinate transformation. An important question in this thesis will be whether the same procedure can be applied in an open ($K < 0$) universe: is there some limit in which physical modes become adiabatic, such that they freeze out (or at least, such that their temporal behavior is known), and allowing us to treat them as a coordinate transformation? The search for curved-space consistency conditions is an important motivation for the study of adiabatic modes in the open universe.

Part II

Finding Adiabatic Modes in Curved Space

In Part II of this thesis, we will search for adiabatic modes in an open universe, which is a universe where there is nonzero spatial curvature with curvature constant $K < 0$. While adiabatic modes have been examined extensively, to the best of my knowledge all research so far has assumed the universe to be spatially flat. It is certainly true that the universe is not *very* curved, yet it has not been excluded that it is. The curvature density parameter

$$\Omega_K = \frac{-K}{H_0^2 a_0^2} \tag{3.33}$$

has been determined to satisfy $|\Omega_K| < 0.5\%$ [3, Table 5] (where the sum of all density parameters is close to one). This implies that

$$1/\sqrt{|K|} \gtrsim 141/H_0, \tag{3.34}$$

i.e. the curvature radius (which is the distance over which effects of curvature become relevant) is at least fourteen times the Hubble radius (which is more or less the observable universe). While this might seem to imply that curvature effects are generally hard to detect within our observable universe, it certainly does not imply that such effects vanish completely. Often the parameter $K\mathbf{x}^2$ pops up. This quantity can grow up to K/H_0^2 within our observable universe, and thus suggests that measurable quantities can get order 10^{-3} corrections. Such quantities may certainly become relevant future high-precision measurements, and thus considering the effects of nonzero curvature is certainly warranted. What's more, knowledge of adiabatic modes in curved universes may be relevant in a flat universe too: processes which occur on the background of a curvature perturbation can be dealt with as if occurring in a curved universe, an idea which is captured by double-soft theorems [25].

In Chapter 4, the general theory required for finding adiabatic modes in an open universe is developed. In particular, the Newtonian gauge is defined and it is examined what subclass of gauge transformations do not violate the gauge condition. In Chapter 5, we generalize Weinberg's tensor mode by working perturbatively in $K\mathbf{x}^2$, finding that the obtained gauge mode is already physical. In Chapter 6, we generalize Weinberg's scalar adiabatic modes. It turns out that monochromatic scalar modes never become adiabatic.

Chapter 4

Preserving the Gauge

This chapter is the essential first step towards finding adiabatic modes on a curved FLRW background. It is discussed in Chapter 1 what it means for space to be curved, while in Chapter 3 it is discussed what adiabatic modes are. An extensive and systematic treatment of adiabatic modes has been performed by Pajer and Jazayeri [29], yet like all who came before them they assumed the universe is spatially flat. While no deviation from flatness has been found yet in our universe, it is interesting to examine whether adiabatic modes also exist when the curvature is small. Finding this out is the goal of part II of this thesis.

Mimicking Pajer and Jazayeri, we first examine how exactly the different perturbations defined in Chapter 2 transform when performing an infinitesimal diffeomorphism This is done in Section 4.1. Next, we will consider what conditions a gauge transformation must meet in order to preserve the gauge used. This will be done for Newtonian gauge in Section 4.2 and for comoving gauge in Section 4.3. In Section 4.5, the gauge transformation parameter is Taylor expanded, and the gauge preservation is translated into conditions on the expansion coefficients using an integration constant technique.

4.1 Transformation Rules

In this section, we examine how the perturbations in our universe transform under a general diffeomorphism. This is necessary in order to determine whether a diffeomorphism violates any gauge conditions. We consider both perturbations to the metric and the energy-momentum tensor.

Transformation of Metric Perturbations

It is described in Appendix B how the perturbations to a general metric transform under infinitesimal diffeomorphisms. We can use this to find out how the perturbations in our universe transform. This metric we use to describe our universe is the perturbed (curved) FLRW metric,

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x), \tag{4.1}$$

where $\bar{g}_{\mu\nu}$ is the background FLRW metric given by

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2 f^2 d\mathbf{x}^2. \tag{4.2}$$

The function f is defined as

$$f(K\mathbf{x}^2) = \frac{1}{1+\frac{1}{4}K\mathbf{x}^2}. \tag{4.3}$$

This metric is motivated and described in detail in Section 1.4, and properties of this metric (and of the function f) are given in Appendix C.

We perform the infinitesimal diffeomorphism

$$x^i \to \tilde{x}^i = x^i + \epsilon^i. \tag{4.4}$$

We will calculate the transformation of perturbations both as a function of ϵ^i (index upstairs) and for ϵ_i (index downstairs). While it is somewhat customary to work with the indices downstairs, as is done e.g. in [29], it turns out to be more convenient to put them upstairs in the case of nonzero curvature ($K \neq 0$). The downstairs expressions are retained for the sake of easy comparison.

Using formula (B.7), we can find the transformation rules for our metric perturbations. For the time-time component, we have

$$\Delta h_{00} = -\epsilon^\lambda \partial_\lambda(-1) - 2(-1)\delta_{\lambda 0}\partial_0\epsilon^\lambda = 2\dot{\epsilon}^0 = -2\dot{\epsilon}_0. \tag{4.5}$$

For the vectorial part of the metric (time-space), we find

$$\begin{aligned}\Delta h_{0i} =& -\epsilon^\lambda\partial_\lambda(0) - (-1)\delta_{\lambda 0}\partial_i\epsilon^\lambda - a^2 f^2 \delta_{\lambda i}\partial_0\epsilon^\lambda \\ =& \partial_i\epsilon^0 - a^2 f^2 \dot{\epsilon}^i.\end{aligned} \tag{4.6}$$

When we put the indices downstairs this becomes

$$\Delta h_{0i} = -a^2 f^2 \partial_0(a^{-2}f^{-2}\epsilon_i) + \partial_i(-\epsilon_0) = -\dot{\epsilon}_i - \partial_i\epsilon_0 + 2H\epsilon_i. \tag{4.7}$$

The tensorial part of the metric (space-space) is the most complicated one, being

$$\begin{aligned}\Delta h_{ij} =& -\epsilon^\lambda\partial_\lambda(\bar{g}_{ij}) - \bar{g}_{ki}\partial_j\epsilon^k - \bar{g}_{kj}\partial_i\epsilon^k \\ =& -\epsilon^0\partial_0\bar{g}_{ij} - \epsilon^k\partial_k\bar{g}_{ij} - 2\bar{g}_{k(i}\partial_{j)}\epsilon^k \\ =& -2H\epsilon^0\bar{g}_{ij} + Kfx^k\epsilon^k\bar{g}_{ij} - 2\bar{g}_{k(i}\partial_{j)}\epsilon^k.\end{aligned} \tag{4.8}$$

Using $\partial_j\epsilon^k = \partial_j\bar{g}^{kl}\epsilon_l = \bar{g}^{kl}\partial_j\epsilon_l - Kx^j f\bar{g}_{kl}\epsilon_l$, it is found that the expression for lowered indices is

$$\Delta h_{ij} = 2H\epsilon_0\bar{g}_{ij} + Kx^k\epsilon_k f\delta_{ij} - 2\partial_{(i}\epsilon_{j)} - 2Kfx^{(i}\epsilon_{j)}. \tag{4.9}$$

Comparing these results to the flat-space results [29, eqs. (2.10 - 2.12)], we see that curvature does not change Δh_{00} or Δh_{0i} in terms of ϵ with lower indices. For Δh_{ij} however, we get two terms which are exclusive to $K \neq 0$. But when we put $K = 0$, it correctly reproduces to the flat-space result,

$$\Delta h_{ij}|_{K=0} = 2a^2 H\delta_{ij}\epsilon_0 - 2\partial_{(i}\epsilon_{j)}. \tag{4.10}$$

Note that we could use these results to determine how the different metric perturbations defined in Section 2.1 transform (by scalar-vector-tensor decomposing the right hand sides of the above equations). It is however easier to gauge fix first, since this means there are less perturbations we need to deal with. Since inverting the Laplace operator is quite a bit harder in curved space than in flat space, this truly is worthwhile.

Transformations of Energy-Momentum Perturbations

The transformation laws for the matter fields are found more easily than those of the metric. From the equation for the energy-momentum tensor (2.15), it is manifest that ρ and p are scalars while u^μ is a vector. The transformation rules or scalars and vectors are described in Appendix B. Thus, the transformations are straight forward.

Under a diffeomorphism,

$$\rho(x) \to \tilde{\rho}(\tilde{x}) = \tilde{\rho}(x+\epsilon) = \rho(x), \tag{4.11}$$

and thus

$$\begin{aligned}\tilde{\rho}(x) =&\rho(x-\epsilon)\\ =&\rho(x) - \epsilon^\mu \partial_\mu \rho(x)\\ =&\rho(x) - \epsilon^0 \dot{\bar{\rho}}(t) + \mathcal{O}(\epsilon\delta\rho).\end{aligned} \tag{4.12}$$

Since we define the background value $\bar{\rho}$ to remain unchanged under the diffeomorphism, and since the same arguments hold for p, we thus find the transformation rules

$$\frac{\Delta\delta\rho}{\dot{\bar{\rho}}} = \frac{\Delta\delta p}{\dot{\bar{\rho}}} = -\epsilon^0 = \epsilon_0 \tag{4.13}$$

(which reveals that any modes obtained this way are indeed 'adiabatic' in the thermodynamic sense, see Section 3.1).

The transformation rule for the vector u^μ is

$$\begin{aligned}\tilde{u}_\mu =&(\delta^\rho_\mu - \partial_\mu\epsilon^\rho)(1-\epsilon^\lambda\partial_\lambda)u_\rho\\ =&u_\mu - u_\rho\partial_\mu\epsilon^\rho - \epsilon^\lambda\partial_\lambda u_\mu + \mathcal{O}(\epsilon^2).\end{aligned} \tag{4.14}$$

Since $u_i = \delta u_i$ is already of linear order in perturbation theory, and since $u_0 = -1 + h_{00}/2$, this reduces (up to linear order) to

$$\tilde{u}_\mu = u_\mu + \partial_\mu \epsilon^0. \tag{4.15}$$

Thus,

$$\partial_i \Delta\delta u^S + \Delta u_i^V = \partial_i\epsilon^0, \tag{4.16}$$

which (since one term is pure divergence and the other is pure curl) must give

$$\partial_i\Delta\delta u^S =\partial_i\epsilon^0, \tag{4.17}$$

$$\Delta u_i^V =0. \tag{4.18}$$

It is now very tempting to conclude $\Delta \delta u^S = \epsilon^0$. However, since we allow for large diffeomorphisms (which do not vanish at infinity), this is not necessarily true. Yet, in this thesis, we will for simplicity assume the universe is filled by some scalar field ϕ with no internal symmetries. We are then required to impose the constraint equation [29, eq. 4.2]

$$\partial_i \delta u^S = -\partial_i \frac{\delta\phi}{\dot{\phi}}. \tag{4.19}$$

If we want our adiabatic mode to be physical, then both δu^S and $\delta\phi$ vanish at infinity. Thus, this equation implies

$$\delta u^S = -\frac{\delta\phi}{\dot{\phi}}. \tag{4.20}$$

Since ϕ is a scalar, it transforms the same under diffeomorphisms as the objects in equation (4.13). Thus, we can safely conclude

$$\Delta \delta u^S = \epsilon^0 = -\epsilon_0. \tag{4.21}$$

Things change, however, when the scalar field is endowed with shift symmetry. We can then combine a diffeomorphism with a transformation $\phi \to \phi + c$, where c is some constant. This increases the freedom of δu^S. In fact, in flat space, this gives rise to an adiabatic mode that is absent for a generic non-shift symmetric scalar field. Studying this mode in curved space may provide a topic for future research. For more discussion on the implications of shift symmetry for adiabatic modes see [29, sec. 4.1] and [15].

4.2 Newtonian Gauge

In this section, we fix the gauge to Newtonian gauge. Then, using the transformation rule (4.9), we can check what gauge transformations leave the Newtonian gauge condition intact. These are exactly the gauge transformations that we can use to find adiabatic modes (in Newtonian gauge). The result will be a condition on the diffeomorphism parameter ϵ^μ, which we will try to solve only in Section 4.5.

The Newtonian gauge condition has been defined already in Section 2.3. For clarity, we once more define it to be

$$F = B = C_i = 0, \tag{4.22}$$

where it is customary to rename E as 2Φ and A as -2Ψ, which are now interpreted as Newtonian potentials, and D_{ij} as γ_{ij}, which contains the graviton degrees of freedom. The metric thus is

$$ds^2 = -(1+2\Phi)dt^2 + 2aG_i dt dx^i + a^2\left[(1-2\Psi)\tilde{g}_{ij} + \gamma_{ij}\right]dx^i dx^j. \tag{4.23}$$

We now want to check whether there are diffeomorphisms that keep us within Newtonian gauge, i.e. such that we can still write the metric as above.

The transformations are

$$\Delta h_{00} = 2\dot{\epsilon}^0 = -2\Delta\Phi, \tag{4.24}$$

$$\Delta h_{0i} = \partial_i \epsilon^0 - a^2 f^2 \dot{\epsilon}^i = a\Delta G_i, \tag{4.25}$$

$$\begin{aligned}\Delta h_{ij} =& -2H\epsilon^0 \bar{g}_{ij} + Kfx^k\epsilon^k \bar{g}_{ij} - 2\bar{g}_{k(i}\partial_{j)}\epsilon^k \\ =& -2\Delta\Psi\bar{g}_{ij} + a^2\Delta\gamma_{ij}\end{aligned} \tag{4.26}$$

For this transformation to keep the metric in Newtonian gauge, we must be able to solve these equations for the Δ objects such that they have the same properties as the original objects in equation (4.23) (i.e. such that $\mathcal{O}' = \mathcal{O} + \Delta\mathcal{O}$ inherits these properties). That is, $\Delta\Phi$ and $\Delta\Psi$ must be scalars, ΔG_i must be a transverse vector and $\Delta\gamma_{ij}$ must be a transverse traceless tensor. For simplicity, we will henceforth drop the Δ in front of the objects (which can be interpreted as the metric prior to the transformation being unperturbed, but this does not matter).

Thus, the equations we will be dealing with are

$$-2\Phi = 2\dot{\epsilon}^0, \tag{4.27}$$

$$aG_i = \partial_i\epsilon^0 - a^2 f^2 \dot{\epsilon}^i, \tag{4.28}$$

$$-2\Psi\bar{g}_{ij} + a^2\gamma_{ij} = -2H\epsilon^0\bar{g}_{ij} + Kfx^k\epsilon^k\bar{g}_{ij} - 2\bar{g}_{k(i}\partial_{j)}\epsilon^k. \tag{4.29}$$

Note that we choose to work with ϵ^i instead of ϵ_i. The main reason for this is the importance of equation (4.29), which becomes messy when lower indices are used.

Obviously, the (00) equation (4.27) gives $\Phi = -\dot{\epsilon}^0$. Since this is a scalar for all diffeomorphisms, the equation puts no conditions on gauge-preserving diffeomorphisms. The other equations are more involved.

We can solve for the $(0i)$ equation (4.28) to find

$$G_i = \frac{1}{a}\partial_i\epsilon^0 - af^2\dot{\epsilon}^i. \tag{4.30}$$

The condition on ϵ we can find from this equation is $\nabla_i G^i = 0$. This means $\bar{g}^{ij}\nabla_i G_j = a^{-2}f^{-2}\delta^{ij}\nabla_i G_j = 0$, and thus the condition is equivalent to $\nabla_i G_i = 0$. This gives

$$\nabla_i G_i = \partial_i G_i - \Gamma^k_{ii} G_k \tag{4.31}$$

$$= \left(\partial_i - \Gamma^i_{kk}\right) G_i \tag{4.32}$$

$$= \left(\partial_i - \frac{1}{2}Kfx^i\right) G_i = 0. \tag{4.33}$$

Here, we have used equation (C.26). Using the identity (C.17), this can be rewritten as

$$\partial_i\left(fG_i\right) = 0 \tag{4.34}$$

or

$$\partial_i\left(f\partial_i\epsilon^0 - a^2f^3\dot{\epsilon}^i\right) = 0. \tag{4.35}$$

This shows that, for gauge preserving diffeomorphisms, ϵ^0 is not fully independent of ϵ^i. Our strategy when finding adiabatic modes will be to choose an ϵ^i, and accompany it by a suitable ϵ^0 such that this condition holds. However, ϵ^0 is not fixed entirely by this condition (for large gauge transformations). In flat space, the relation becomes

$$\nabla^2\epsilon^0 = a^2\partial_i\dot{\epsilon}^i \qquad\qquad \text{for } K=0, \tag{4.36}$$

or

$$\nabla^2\epsilon_0 = -a^2\partial_i\left(a^{-2}\dot{\epsilon}_i\right) = 2H\partial_i\epsilon_i - \partial_i\dot{\epsilon}_i \qquad\qquad \text{for } K=0, \tag{4.37}$$

which agrees with existing flat-space results in [29, eq. 2.17].

The right-hand side of equation (4.29) contains γ_{ij}, which is traceless. This allows us to easily solve for the scalar Ψ by taking the trace of the equation,

$$\begin{aligned}\bar{g}^{ij}h_{ij} = -6\Psi &= -6H\epsilon^0 + 3Kfx^k\epsilon^k - 2\partial_k\epsilon^k\\ \rightarrow -2\Psi\bar{g}_{ij} &= -2H\epsilon^0\bar{g}_{ij} + Kfx^k\epsilon^k\bar{g}_{ij} - \frac{2}{3}\bar{g}_{ij}\partial_k\epsilon^k.\end{aligned} \tag{4.38}$$

We can use this to solve for the tensor mode as

$$\begin{aligned}a^2\gamma_{ij} = h_{ij} + 2\Psi\bar{g}_{ij} &= \frac{2}{3}\bar{g}_{ij}\partial_k\epsilon^k - 2\bar{g}_{k(i}\partial_{j)}\epsilon^k\\ \rightarrow\gamma_{ij} &= f^2\left(\frac{2}{3}\delta_{ij}\partial_k - \delta_{ki}\partial_j - \delta_{kj}\partial_i\right)\epsilon^k.\end{aligned} \tag{4.39}$$

We will consider γ_{ij} to be a tensor on the space Σ with metric $\tilde{g}_{ij}$, and thus

$$\begin{aligned}\gamma^i{}_j &= \tilde{g}^{ik}\gamma_{kj} = \frac{1}{f^2}\gamma_{ij}\\ &= \frac{2}{3}\delta_{ij}\partial_k\epsilon^k - \partial_i\epsilon^j - \partial_j\epsilon^i,\end{aligned} \tag{4.40}$$

which is a somewhat more convenient expression.

We see that γ_{ij} is indeed traceless (by construction). But to remain in Newtonian gauge, it must also also be transverse. That is, $\nabla_i\gamma_{ij} = \nabla_i\gamma^i{}_j = 0$. That is,

$$\begin{aligned}\nabla_i\gamma^i{}_j =&\partial_i\gamma^i{}_j + \Gamma^i_{ik}\gamma^k{}_j - \Gamma^k_{ij}\gamma^i{}_k\\ =&\partial_i\gamma^i{}_j - \frac{3}{2}Kfx^k\gamma_{kj} + \frac{1}{2}Kf\left(x^i\delta_{jk} + x^j\delta_{ik} - x^k\delta_{ij}\right)f^2\gamma_{ik}\\ =&\partial_i\gamma^i{}_j - \frac{3}{2}Kf^3x^i\gamma^i{}_j + \frac{1}{2}Kf^3x^j\cancelto{0}{\gamma_{ii}} + Kf\cancelto{0}{x^{[i}\delta_{k]j}\gamma_{ik}}\\ =&\left(\partial_i - \frac{3}{2}Kfx^i\right)\gamma^i{}_j = 0,\end{aligned} \tag{4.41}$$

where we have used both the tracelessness and the symmetry of γ_{ij}. Furthermore, we have used equation (C.27) for the trace over the Christoffel symbol. Lastly, we can use equation (C.17) to write this condition on ϵ_i in its final, beautiful form

$$\partial_i\left(f^3\gamma^i{}_j\right) = 0. \tag{4.42}$$

For future reference, we here summarize the different transformation rules for the Newtonian gauge:

$$\begin{aligned}
\Phi &= -\dot{\epsilon}^0, \\
\Psi &= H\epsilon^0 + \left(\frac{1}{3}\partial_k - \frac{1}{2}Kfx^k\right)\epsilon^k = H\epsilon^0 - \frac{1}{3f^3}\partial_k\left(f^3\epsilon^k\right), \\
G_i &= \frac{1}{a}\partial_i\epsilon^0 - af^2\dot{\epsilon}^i, \\
\gamma^i{}_j &= \frac{2}{3}\delta_{ij}\partial_k\epsilon^k - \partial_i\epsilon^j - \partial_j\epsilon^i, \\
-\frac{\delta\rho}{\dot{\bar{\rho}}} &= -\frac{\delta p}{\dot{\bar{p}}} = \delta u^S = \epsilon^0, \\
u_i^V &= 0.
\end{aligned} \tag{4.43}$$

4.3 Comoving Gauge

In this section, we go to comoving gauge and examine what the conditions are for diffeomorphisms to remain in comoving gauge. An important difference with Newtonian gauge is that we need not only look at the transformation rules of the metric, but also of the energy-momentum tensor. We will find that the most important result found in Newtonian gauge (in Section 4.2) is shared by comoving gauge.

Following the definition in Section 2.3, we have the comoving-gauge condition

$$B = \delta u = C_i = 0, \tag{4.44}$$

and (using ADM notation) we rename $E = 2N_1$, $\partial_i F + G_i = \frac{1}{a}N_i = \frac{1}{a}(\partial_i\phi + N_i^V)$, $D_{ij} = \gamma_{ij}$ (like in Newtonian gauge) and $A = 2\mathcal{R}_c$. The metric then becomes

$$ds^2 = -(1+2N_1)dt^2 + 2N_i dx^i dt + a^2\left[(1+2\mathcal{R}_c)\tilde{g}_{ij} + \gamma_{ij}\right]. \tag{4.45}$$

Just like in Newtonian gauge, γ_{ij} is transverse-traceless. As opposed to Newtonian gauge though, N_i need not be transverse. Thus, there is one condition less to remain in the gauge.

The transformations of the metric perturbations are

$$\Delta h_{00} = 2\dot{\epsilon}^0 = -2\Delta N_1, \tag{4.46}$$

$$\Delta h_{0i} = \partial_i\epsilon^0 - a^2 f^2\dot{\epsilon}^i = \Delta N_i, \tag{4.47}$$

$$\begin{aligned}
\Delta h_{ij} =& -2H\epsilon^0\bar{g}_{ij} + Kfx^k\epsilon^k\bar{g}_{ij} - 2\bar{g}_{k(i}\partial_{j)}\epsilon^k \\
=& 2a^2\Delta\mathcal{R}_c\bar{g}_{ij} + a^2\Delta\gamma_{ij}.
\end{aligned} \tag{4.48}$$

Because the transformation rules are the same if $h_{\mu\nu}$ is zero before the transformation as when this is would not be the case, we once more drop the Δ's. Solving for them proceeds in a way that is extremely similar to Newtonian gauge. However, there is one simplifying matter. Combining equation (4.21) with the condition $\delta u^S = 0$ reveals that, in order to remain in the comoving gauge (for a generic scalar field), we must have

$$\epsilon^0 = 0. \tag{4.49}$$

Using this, we simply copy (4.43) with the proper adjustments to find

$$
\begin{aligned}
N_1 &= 0, \\
\mathcal{R}_c &= -\left(\frac{1}{3}\partial_k - \frac{1}{2}Kfx^k\right)\epsilon^k = -\frac{1}{3f^3}\partial_k\left(f^3\epsilon^k\right), \\
N_i &= -af^2\dot{\epsilon}^i, \\
\gamma^i{}_j &= \frac{2}{3}\delta_{ij}\partial_k\epsilon^k - \partial_i\epsilon^j - \partial_j\epsilon^i, \\
-\frac{\delta\rho}{\dot{\bar{\rho}}} &= -\frac{\delta p}{\dot{\bar{p}}} = \delta u^S = 0, \\
u_i^V &= 0.
\end{aligned}
\tag{4.50}
$$

The only of this objects on which comoving gauge imposes a special condition is γ_{ij}, which must be transverse. Thus, this condition is

$$
\bar{\nabla}_i\gamma^i{}_j = 0, \tag{4.51}
$$

which is exactly the same condition on ϵ^i as we have found in Newtonian gauge. The only difference between the gauges is the constraint on ϵ^0, which is extremely simple in comoving gauge, making it a convenient gauge. However, since Newtonian gauge is important in the literature (most notably, it is the gauge which Weinberg used to derive his theorem, see Section 3.1), we will focus on that gauge. Conveniently, though, one of the prime equations we need to solve in both cases is the same. This is further examined in Section 4.5.

4.4 Conditions for Adiabatic Modes

In this thesis, we focus on solving the gauge-preservation condition $\bar{\nabla}_i\gamma^i{}_j$, which constrains the spatial gauge parameters, ϵ^i. After all, this condition occurs both in Newtonian gauge and comoving gauge, making it a universally important condition. Solving the constraint is done systematically, using integration constants, in Section 4.5. However, there are more conditions involved with finding adiabatic mode. For one, we also need a proper ϵ^0 to accompany the spatial diffeomorphism. While it is trivially zero in comoving gauge, this is not true in Newtonian gauage, where the condition is more complicated. It must certainly be chosen such that the vector G_i is transverse (i.e. $\bar{\nabla}_iG^i = 0$). However, in this thesis, we will focus on finding tensor and scalar adiabatic modes, and thus we will typically choose ϵ^0 such that $G_i = 0$. This implies $\partial_i\epsilon^0 = a^2f^2\dot{\epsilon}^i$. In Chapter 5, where we look for the simples possible tensor adiabatic mode, this condition is solved by setting $\epsilon^0 = 0$ and $\epsilon^i = \epsilon^i(\mathbf{x})$. In Chapter 6, ϵ^i is allowed time dependence and ϵ^0 becomes less trivial. In any case, solving the condition $G_i = 0$ is not very complicated and does not lend itself for a more general treatment. Therefore, we do not further consider the condition in this chapter.

Other conditions which must be solved by the diffeomorphisms are the so-called physicality conditions (which were discussed for flat universes in Section 3.1). Any mode obtained by a diffeomorphism is sure to solve all the Linearized Einstein equations (because the unperturbed universe does, and general relativity is diffeomorphism covariant). However, such a mode is just a gauge artifact.

We are interested in extending it to a physical mode, which we then call adiabatic. However, we must then check that such a physical mode, in the limit where it closely resembles the gauge mode, also solves the linearized Einstein equations. While most equations can be expected to be continuous with respect to this, things become more complicated when there are derivatives involved. Generally, when the spatial derivative of A is equal to the spatial derivative of B, this implies $A = B$ *if* they both vanish at infinity (such that no integration constants are allowed). While physical modes must vanish at infinity, the same needs not hold for the gauge modes. Thus, only gauge modes for which, in such an equation, $A = B$ holds are extensible to physical adiabatic modes. Therefor, such conditions are called physicality conditions.

The only physicality conditions occur in the scalar sector. Thus, we need not concern ourselves with them when looking for pure tensor modes. Yet, in Chapter 6, where we look for a pure scalar mode, they are important. The physicality conditions are derived for Newtonian gauge from the linearized Einstein equations in Appendix E. The most important of them is

$$\Phi = \Psi, \tag{4.52}$$

making it clear why we tend to think of both as *the* Newtonian potential. The other is

$$\dot{\Phi} + H\Phi = \left(\dot{H} - \frac{K}{a^2}\right)\partial_i \delta u^S. \tag{4.53}$$

However, just as with the condition $G_i = 0$, the conditions are not very complicated and do not warrant a general treatment. Thus, they will not be further considered in this chapter.

It should now be clear what the conditions are that we have to deal with in order to find adiabatic modes. Since the transverseness condition of the tensor is the most complicated of all, and since the resulting framework is rather general (it could be used for finding other tensor or mixed adiabatic modes), we will deal with solving it in this chapter before proceeding to the search for specific adiabatic modes.

4.5 Integration Constants

We have seen in both the Newtonian and the comoving gauge that our diffeomorphisms need to obey the same condition $\bar{\nabla}_i \gamma^i{}_j$ in order to preserve the gauge. In this section, we will examine what the diffeomorphisms look like that solve these using an integration constant strategy.

We have used equation (C.17) to express the gauge-preserving condition on ϵ_i as

$$\partial_i \left(f^3 \gamma^i{}_j\right) = 0, \tag{4.54}$$

where, as a reminder,

$$\gamma^i{}_j = \frac{2}{3}\delta_{ij}\partial_k \epsilon^k - 2\partial_{(i}\epsilon^{j)}. \tag{4.55}$$

We solve this equation by introducing an integration “constant”,

$$f^3\gamma^i{}_j = M_{ij}. \tag{4.56}$$

Here, the “constant”, which is actually a function of x, must satisfy the conditions

$$M_{ij} = M_{ji}, \tag{4.57}$$
$$M_{ii} = \partial_i M_{ij} = 0. \tag{4.58}$$

While the symmetry and tracelessness are inherited from $\gamma^i{}_j$, the transverseness is required in order to solve the gauge-preserving condition.

This can be rewritten as

$$\gamma^i{}_j = f^{-3}M_{ij}, \tag{4.59}$$

or more explicitly,

$$\frac{2}{3}\delta_{ij}\partial_k\epsilon^k - \partial_i\epsilon^j - \partial_j\epsilon^i = (1+\frac{1}{4}K\mathbf{x}^2)^3 M_{ij}. \tag{4.60}$$

Now, part of the freedom we have in choosing our solution parametrized by M_{ij}. Note that in the flat-space case this equation becomes

$$\gamma^i{}_j = \gamma_{ij} = M_{ij} \qquad\qquad \text{for } K=0, \tag{4.61}$$

The equation corresponding to $M_{ij}=0$ is

$$\frac{2}{3}\delta_{ij}\partial_k\epsilon^k - \partial_i\epsilon^j - \partial_j\epsilon^i = 0, \tag{4.62}$$

and this is exactly the conformal Killing equation [19, eq. (2.7)]. This reveals that pure gauge modes that are obtained by conformal transformations (such as, for example, dilations, which corresponds to Weinberg’s first scalar adiabatic mode when accompanied by a time-dependent time translation) exist within Newtonian gauge both when there is and when there isn’t spatial curvature. Whether they can be extended to finite momentum remains to be checked.

A more elaborate analysis of what solutions are allowed can be obtained by solving equation (4.59) order-by-order in x. This analysis is simplified by working up to first order in $K\mathbf{x}^2$, because then we can write

$$\gamma^i{}_j = (1+\frac{1}{4}K\mathbf{x}^2)^3 M_{ij} = (1+\frac{3}{4}K\mathbf{x}^2)M_{ij} + \mathcal{O}\left((K\mathbf{x}^2)^2\right). \tag{4.63}$$

Since $K\mathbf{x}^2$ reaches its maximum within the observable universe at the Hubble radius, $\mathbf{x}^2 = 1/H^2$, where it is at order 10^{-3} or smaller (today), such a perturbative approach is certainly justified as a first analysis.

Effect on First Orders in x

First, to get a feel for what’s going on, we work with $M_{ij} = m^{(0)}_{ij}$ a constant. In this case, $\gamma^i{}_j$ must be quadratic in x (while we would need sixth-order terms when we work non-perturbatively in $K\mathbf{x}^2$). Since $\gamma^i{}_j$ is composed of single spatial derivatives working on ϵ^i, this means we are choosing our diffeomorphism ϵ^i to be at most third order in x. The expansion is defined as follows:

$$\epsilon^i = c_i + \omega_{il}x^l + \frac{1}{2}\sigma_{ikl}x^k x^l + \frac{1}{6}\mu_{ijkl}x^j x^k x^l. \tag{4.64}$$

Here, we choose $\sigma_{ikl} = \sigma_{i(kl)}$ and $\mu_{ijkl} = \mu_{i(jkl)}$, which we can do because the antisymmetric parts do not contribute to ϵ^i anyway. The symbols used as coefficients have been chosen as to match the paper by Pajer and Jazayeri [29]. From the expansion, we find

$$\partial_j \epsilon^i = \omega_{ij} + \sigma_{ijl}x^l + \frac{1}{2}\mu_{ijkl}x^k x^l, \tag{4.65}$$

and thus

$$\begin{aligned}\gamma^i{}_j =& \frac{2}{3}\delta_{ij}\omega_{mm} - 2\omega_{(ij)} + \left(\frac{2}{3}\delta_{ij}\sigma_{mml} - 2\sigma_{(ij)l}\right)x^l \\ &+ \frac{1}{2}\left(\frac{2}{3}\delta_{ij}\mu_{mmkl} - 2\mu_{(ij)kl}\right)x^k x^l.\end{aligned} \tag{4.66}$$

The right-hand side of equation (4.63) is $m^{(0)}_{ij} + \frac{3}{4}Km^{(0)}_{ij}\delta_{kl}x^k x^l$. Thus, equating the polynomial coefficients gives

$$\frac{2}{3}\delta_{ij}\omega_{mm} - 2\omega_{(ij)} = m^{(0)}_{ij}, \tag{4.67}$$

$$\frac{2}{3}\delta_{ij}\sigma_{mml} - 2\sigma_{(ij)l} = 0, \tag{4.68}$$

$$\frac{1}{3}\delta_{ij}\mu_{mmkl} - \mu_{(ij)kl} = \frac{3}{4}Km^{(0)}_{ij}\delta_{kl}. \tag{4.69}$$

The first equation does not constrain the allowed diffeomorphisms. We can choose whatever ω_{ij} we like, and calculate what the $m^{(0)}_{ij}$ is corresponding to it. $m^{(0)}_{ij}$ must be symmetric and traceless, but these conditions are automatically satisfied by the left-hand side of the equation and thus impose nothing on ω_{ij}. The second equation is exactly the equation we would get for σ_{ijk} when we put $M_{ij} = 0$. The third equation can be rewritten by eliminating $m^{(0)}_{ij}$ from the system of equations, giving

$$\frac{2}{3}\delta_{ij}\mu_{mmkl} - 2\mu_{(ij)kl} = K\left(\delta_{ij}\omega_{mm} - 3\omega_{(ij)}\right)\delta_{kl}. \tag{4.70}$$

Note that this condition is weaker than the conditions obtained for $M_{ij} = 0$, as this would require both sides of the equations to be zero. Thus, we have constrained the set of gauge-preserving diffeomorphisms by imposing a relation between the first-order and third-order coefficients of ϵ^i.

Now let's consider what happens when we allow M_{ij} to also have a second-order term,

$$M_{ij} = m^{(0)}_{ij} + \frac{1}{2}m^{(2)}_{ijkl}x^k x^l. \tag{4.71}$$

Here, $m^{(2)}$ is symmetric in the last two indices. Note that this does not force ϵ^i to have a fifth-order term, as $m^{(2)}_{ijkl} = 0$ is still an allowed solution. But to see what it does force on ϵ^i, we return to the equation obtained from equating the second-order coefficients (i.e. equation (4.69)). Now, this equation becomes

$$\frac{1}{3}\delta_{ij}\mu_{mmkl} - \mu_{(ij)kl} = \frac{1}{2}m^{(2)}_{ijkl} + \frac{3}{4}Km^{(0)}_{ij}\delta_{kl}. \tag{4.72}$$

Once more eliminating $m^{(0)}$ and solving for $m^{(2)}$ gives

$$m^{(2)}_{ijkl} = \frac{2}{3}\delta_{ij}\mu_{mmkl} - 2\mu_{(ij)kl} - K\left(\delta_{ij}\omega_{mm} - 3\omega_{(ij)}\right)\delta_{kl}. \tag{4.73}$$

Like before, we can argue that we can pick any μ and ω. The integration constant is then determined by these coefficients, and not the other way around. However, there is an extra condition that $m^{(2)}$ must satisfy that did not play a role before, and thus not all μ and ω are allowed.

While symmetry and tracelessness in the first two indices are still trivially satisfied, the transversality of the integration constant is not. Because $\partial_j M_{ij} = 0$, we have $m^{(2)}_{ikkj} = 0$ (in fact, by symmetry, it is traceless in all two indices except for the last two). This means

$$\begin{aligned} m^{(2)}_{ikkj} &= \frac{2}{3}\delta_{ik}\mu_{mmkj} - \mu_{ikkj} - \mu_{kikj} - K\left(\delta_{ik}\omega_{mm} - \frac{3}{2}\omega_{ik} - \frac{3}{2}\omega_{ki}\right)\delta_{kj} \\ &= \frac{2}{3}\mu_{mmij} - \mu_{ijkk} - \mu_{kkij} - K\left(\delta_{ij}\omega_{mm} - \frac{3}{2}\omega_{ij} - \frac{3}{2}\omega_{ji}\right) \\ &= -\frac{1}{3}\mu_{mmij} - \mu_{ijmm} - K\left(\delta_{ij}\omega_{mm} - 3\omega_{(ij)}\right) = 0, \end{aligned} \tag{4.74}$$

or

$$\mu_{ijkk} + \frac{1}{3}\mu_{kkij} = -K\left(\delta_{ij}\omega_{kk} - 3\omega_{(ij)}\right). \tag{4.75}$$

When we put $K = 0$, this exactly agrees with the most general condition on μ found in flat space [29, eq. (2.27)].

General Order in x

Now that we have seen explicitly what the effect of the integration constant is at first and third order, we are ready for a more general treatment to see it has a very similar effect at every order. We write both ϵ^i and M_{ij} as a general Taylor expansion (in a new way that will turn out to be convenient),

$$\epsilon^i = c^{(0)}_i + \sum_{n=1} \frac{1}{n} c^{(n)}_{ik_1\ldots k_n} x^{k_1}\ldots x^{k_n}, \tag{4.76}$$

$$M_{ij} = m^{(0)}_{ij} + \sum_{n=1} m^{(n)}_{ijk_1\ldots k_n} x^{k_1}\ldots x^{k_n}. \tag{4.77}$$

Then, from equating the different terms in the polynomials in equation (4.63), we find the analogue of equation (4.74),

$$m^{(n)}_{ijk_1\ldots k_n} = \frac{2}{3}\delta_{ij}c^{(n+1)}_{mmk_1\ldots k_n} - 2c^{(n+1)}_{(ij)k_1\ldots k_n} - \frac{3}{4}Km^{(n-2)}_{ij(k_1\ldots k_{n-2}}\delta_{k_{n-1}k_n)}. \tag{4.78}$$

When we enforce the transversality condition, this gives

$$m^{(n)}_{immk_1\ldots k_{n-1}} = -\, c^{(n+1)}_{immk_1\ldots k_{n-2}} - \frac{1}{3}c^{(n+1)}_{mmik_1\ldots k_{n-1}} - \frac{3}{2n}Km^{(n-2)}_{i(k_1\ldots k_{n-1})} = 0, \tag{4.79}$$

where use has been made also of the transversality condition on $m^{(n-2)}$. We can now use this equation, which holds for $n > 1$, to fill in the equation for $m^{(n-2)}$ back into the equation for $m^{(n)}$. In principle, this leads to a regress, as $m^{(n-2)}$ refers to $m^{(n-4)}$ in turn. Thus, only when $m^{(0)}$ and $m^{(1)}$ are known, all the other coefficients can be found by filling in the equation recursively (*ad infinitum*). However, since we work perturbatively, we can neglect the K^2 terms[1]. Thus, no recursion occurs, and we find

$$\begin{aligned} c^{(n+1)}_{immk_1\ldots k_{n-1}} + \frac{1}{3}c^{(n+1)}_{mmik_1\ldots k_{n-1}} = \\ -\frac{3}{2n}K\left(\frac{2}{3}c^{(n-1)}_{mm(k_1\ldots k_{n-2}}\delta_{k_{n-1})i} - c^{(n-1)}_{(k_1|i|k_2\ldots k_{n-1})} - c^{(n-1)}_{ik_1\ldots k_{n-1}}\right). \end{aligned} \tag{4.80}$$

This can be rewritten, using symmetry properties, as

$$\begin{aligned} c^{(n+2)}_{immk_1\ldots k_n} + \frac{1}{3}c^{(n+2)}_{mmik_1\ldots k_n} = \\ -\frac{3}{2(n+1)}K\left(-c^{(n)}_{ik_1\ldots k_n} + \frac{1}{n}\sum_{j=1}^{n}\left[\frac{2}{3}c^{(n)}_{mmk_1\ldots k_{j-1}k_{j+1}\ldots k_n}\delta_{ik_j} - c^{(n)}_{k_jik_1\ldots k_{j-1}k_{j+1}\ldots k_n}\right]\right), \end{aligned} \tag{4.81}$$

which holds for all $n > 0$. While equation (4.81) may not look very nice, it is the final grand result of this analysis. As long as the Taylor expansion coefficients of ϵ^i satisfy this relation, $\nabla^i\gamma_{ij} = 0$ (to first order in $K\mathbf{x}^2$).

In Chapter 5, we we only consider the relation that is imposed between ω and μ. The idea will be to start with ω at order $\mathcal{O}(K^0)$, forcing μ to be at order $\mathcal{O}(K)$. However, one could also take the approach of setting all coefficients up to order n in x^i to zero and having the n^{th} coefficient at order $\mathcal{O}(K^0)$. Then, this would force the order $n+2$ coefficient to be of order $\mathcal{O}(K)$. Thus, even when working linearly in $K\mathbf{x}^2$, the machinery developed in this section allows for finding more tensor modes than is actually done in this thesis.

[1] In any actual perturbation, the indices are contracted with x^i's, and each order in K is effectively $K\mathbf{x}^2$, allowing perturbative treatment.

Chapter 5

Pure Tensor

In this chapter, we use the machinery developed in Chapter 4 (and Section 4.5 in particular) to generalize Weinberg's tensor adiabatic mode to the open universe. Since Weinberg's tensor mode is time-independent (with $\epsilon^0 = 0$), we attempt to find a similarly time-independent mode. The calculation is performed to first order in $K\mathbf{x}^2$ in Section 5.1 and the resulting tensor mode can be found in equation (5.23). Next, a non-trivial check of the result is performed by verifying that the mode solves the Linearized Einstein equations in Section 5.2. Surprisingly, we find that the tensor mode we found is *already physical*, without the need to extend the gauge mode to the physical domain in order to get an adiabatic mode. In Section 5.3, we therefore check whether the time-independent adiabatic tensor more still exists at second order in $K\mathbf{x}^2$. It is verified explicitly that it does, with the resulting tensor mode in equation (5.64). Finally, in the same section, a brief discussion on the (non-)existence of the time-independent tensor mode is offered.

5.1 Generalizing Weinberg's Tensor Mode

In this section, we try to generalize Weinberg's tensor adiabatic mode ($\epsilon^i = \omega_{ij}x^j$, with no time dependence, $\omega_{ij} = \omega_{(ij)}$ and $\omega_{mm} = 0$ [34, sec. 5.4]), which is the simplest pure tensor mode in flat space. Our strategy is to require the only nonzero term at $\mathcal{O}(K^0)$ in the expansion of ϵ^i to be ω_{ij}. To make it satisfy the $\bar{\nabla}_i \gamma^i{}_j = 0$ equation, μ_{ijkl} must be nonzero at linear order in K, such that it satisfies equation (4.75), i.e.

$$\mu_{ijkk} + \frac{1}{3}\mu_{kkij} = -K\left(\delta_{ij}\omega_{kk} - 3\omega_{(ij)}\right). \tag{5.1}$$

Defining $\mu \equiv K\mu'$, we have

$$\epsilon^i = \omega_{il}(t)x^l + \frac{1}{6}K\mu'_{ijkl}(t)x^j x^k x^l, \tag{5.2}$$

where both ω and μ' are of order $\mathcal{O}(K^0)$. Before proceeding further, it will be useful to study the solutions to equation (5.1).

The only tensorial building blocks we have for μ are ω and the 3-space metric, which is proportional to the Kronecker delta. Thus, we will be working with ω and δ, which the equation itself also seems to suggest. Since only the symmetrical part of ω occurs in the equations, it will be convenient to define $\bar{\omega}_{ij} = \omega_{(ij)}$ and use this instead. We thus look for solutions of the form $\bar{\omega}_{ij}\delta_{kl}$, but such a term does by itself not respect the symmetry in the last three indices of μ. Instead, we symmetrize, which can be done in two unique ways, giving us the most general ansatz

$$\mu'_{ijkl} = n_1 \delta_{i(j}\bar{\omega}_{kl)} + n_2 \bar{\omega}_{i(j}\delta_{kl)}. \tag{5.3}$$

Now we calculate the relevant traces,

$$\begin{aligned}
\delta_{m(m}\bar{\omega}_{ij)} =& \frac{5}{3}\bar{\omega}_{ij}, \\
\delta_{i(j}\bar{\omega}_{mm)} =& \frac{1}{3}\delta_{ij}\bar{\omega}_{mm} + \frac{2}{3}\bar{\omega}_{ij}, \\
\bar{\omega}_{m(m}\delta_{ij)} =& \frac{1}{3}\delta_{ij}\bar{\omega}_{mm} + \frac{2}{3}\bar{\omega}_{ij}, \\
\bar{\omega}_{i(j}\delta_{mm)} =& \frac{5}{3}\bar{\omega}_{ij}.
\end{aligned} \tag{5.4}$$

Note that the trace over the first two indices of the first term is equal to the trace over the last two indices of the last term, and vice versa. We can plug this into the right-hand side of equation (5.1), which gives

$$(n_1 + \frac{n_2}{3})\frac{1}{3}\delta_{ij}\bar{\omega}_{mm} + \left((n_1 + \frac{n_2}{3})\frac{2}{3} + (n_2 + \frac{n_1}{3})\frac{5}{3}\right)\bar{\omega}_{ij} = -\delta_{ij}\bar{\omega}_{mm} + 3\bar{\omega}_{ij}, \tag{5.5}$$

or, equating terms,

$$3n_1 + n_2 = -9, \tag{5.6}$$

$$11n_1 + 17n_2 = 27. \tag{5.7}$$

This system of equations is solved by

$$n_1 = -\frac{9}{2}, \tag{5.8}$$

$$n_2 = \frac{9}{2}. \tag{5.9}$$

Now, since we are looking for a pure tensor mode, we want the scalar and vector perturbations to be zero. $\phi = 0$ is obtained by setting $\epsilon^0 = \epsilon^0(\mathbf{x})$ (no time dependence). The easiest way now to put the other perturbations to zero is to choose $\epsilon^0 = 0$. This is how Weinberg obtained his tensor mode in flat space, and thus it is this choice that we consider to be the curved-space equivalent of Weinberg's tensor mode. This choice immediately gives $\epsilon^i = \epsilon^i(\mathbf{x})$ from the requirement $G_i = \frac{2}{a}\partial_i\epsilon^0 - 2af^2\dot{\epsilon}^i = 0$. Now, we are left with the equation

$$\begin{aligned}
\Psi = 0 =& -\frac{1}{2}Kfx^k\epsilon^k + \frac{1}{3}\partial_k\epsilon^k \\
=& -\frac{1}{2}Kx^k\omega_{kl}x^l + \frac{1}{3}\omega_{mm} + \frac{1}{6}\mu_{mmkl}x^kx^l.
\end{aligned} \tag{5.10}$$

At zeroth order, this simply requires that ω is traceless, which is the same as for the flat-space Weinberg mode. At first order, it suggests

$$\mu'_{mmij} = 3\omega_{(ij)}. \tag{5.11}$$

This last equation is incompatible with the solution found for μ' above. But since we now have the condition $\omega_{mm} = 0$, one of the equations found before no longer applies. To be precise, equation (5.6) is replaced by

$$5n_1 + 2n_2 = 9, \tag{5.12}$$

which becomes a new exactly solvable system of equations together with equation (5.7). The system is solved by

$$n_1 = \frac{11}{7}, \tag{5.13}$$

$$n_2 = \frac{4}{7}, \tag{5.14}$$

and thus the generalization of Weinberg's adiabatic mode is obtained by the diffeomorphism

$$\begin{aligned} \epsilon^i =& \omega_{il} x^l + \frac{1}{42} K \left(11 \delta_{i(j} \bar{\omega}_{kl)} + 4 \bar{\omega}_{i(j} \delta_{kl)} \right) x^j x^k x^l, \\ \omega_{kk} =& 0, \qquad \bar{\omega}_{ij} = \omega_{(ij)}, \qquad \omega_{ij} = \text{constant}. \end{aligned} \tag{5.15}$$

So what does the tensor mode γ_{ij} now look like? When we use the notation

$$A_{<ij>} = A_{(ij)} - \frac{1}{3} \delta_{ij} A_{kk} \tag{5.16}$$

for the symmetric traceless part of a tensor, we can write

$$\gamma_{ij} = -2f^2 \partial_{<i} \epsilon^{j>} \tag{5.17}$$

(where $\gamma^i{}_j$, which was used before, equals $-2\partial_{<i}\epsilon^{j>}$). We can easily calculate

$$\begin{aligned} \partial_j \epsilon^i =& \omega_{ij} + \frac{3}{42} K \left(11 \delta_{i(j} \bar{\omega}_{kl)} + 4 \bar{\omega}_{i(j} \delta_{kl)} \right) x^k x^l \\ =& \omega_{ij} + \frac{1}{42} K \left(11 \delta_{ij} \bar{\omega}_{kl} + 22 \delta_{ik} \bar{\omega}_{lj} + 4 \bar{\omega}_{ij} \delta_{kl} + 8 \bar{\omega}_{ik} \delta_{lj} \right) x^k x^l \\ =& \omega_{ij} + \frac{2}{21} \bar{\omega}_{ij} K \mathbf{x}^2 + \frac{11}{42} K \delta_{ij} \bar{\omega}_{kl} x^k x^l + \frac{1}{42} K \left(22 \bar{\omega}_{kj} x^i + 8 \bar{\omega}_{ik} x^j \right) x^k. \end{aligned} \tag{5.18}$$

Since ω is traceless, we have

$$\omega_{<ij>} = \bar{\omega}_{<ij>} = \bar{\omega}_{ij}. \tag{5.19}$$

Furthermore, since the Kronecker delta is 'pure trace', $\delta_{<ij>} = 0$. Thus,

$$\begin{aligned} \partial_{<i} \epsilon^{j>} =& \bar{\omega}_{ij} \left(1 + \frac{2}{21} K \mathbf{x}^2 \right) + \frac{30}{42} K \bar{\omega}_{k<j} x^{i>} x^k \\ =& \bar{\omega}_{ij} \left(1 + \frac{2}{21} K \mathbf{x}^2 \right) + \frac{5}{7} K \bar{\omega}_{k<j} x^{i>} x^k \\ =& \bar{\omega}_{ij} \left(1 + \frac{2}{21} K \mathbf{x}^2 \right) + \frac{5}{14} K \bar{\omega}_{ki} x^j x^k + \frac{5}{14} K \bar{\omega}_{kj} x^i x^k - \frac{5}{21} K \delta_{ij} \bar{\omega}_{kl} x^k x^l. \end{aligned} \tag{5.20}$$

At first order in $K\mathbf{x}^2$,

$$f^2 = \frac{1}{\left(1+\frac{1}{4}K\mathbf{x}^2\right)^2} = \frac{1}{1+\frac{1}{2}K\mathbf{x}^2} = 1 - \frac{1}{2}K\mathbf{x}^2, \tag{5.21}$$

and thus the tensor mode is

$$\begin{aligned}\gamma_{ij} =& \bar{\omega}_{ij}\left(-2+\frac{17}{21}K\mathbf{x}^2\right) - \frac{10}{7}K\bar{\omega}_{k<j}x^{i>}x^k \\ =& \bar{\omega}_{ij}\left(-2+\frac{17}{21}K\mathbf{x}^2\right) - \frac{10}{14}K\bar{\omega}_{ki}x^jx^k - \frac{10}{14}K\bar{\omega}_{kj}x^ix^k + \frac{10}{21}K\delta_{ij}\bar{\omega}_{kl}x^kx^l..\end{aligned} \tag{5.22}$$

We see that only the symmetric part of ω enters the actual mode. We can interpret this as follows. When ω is antisymmetric, the diffeomorphism $x^i \to x^i + \omega_{ik}x^k$ is an infinitesimal rotation. The background space however is invariant under rotations (it is isotropic), and thus we cannot obtain any perturbation this way. Therefore it is better to define the diffeomorphism that generates the tensor mode such that ω is symmetric.

In summary, then, the generalization of Weinberg's tensor adiabatic mode is

$$\begin{aligned}\epsilon^i =& \omega_{il}x^l + \frac{1}{42}K\left(11\delta_{i(j}\omega_{kl)} + 4\omega_{i(j}\delta_{kl)}\right)x^jx^kx^l \\ \gamma_{ij} =& \omega_{ij}\left(-2+\frac{17}{21}K\mathbf{x}^2\right) - \frac{10}{7}K\omega_{k<i}x^{j>}x^k \\ \omega_{ij} =& \omega_{<ij>} = \text{ const.}\end{aligned} \tag{5.23}$$

5.2 Checking: Einstein Equations

To convince ourselves that the pure tensor mode found above is correct, we check whether it solves Einstein's equations (as it should, by diffeomorphism invariance of general relativity). The Einstein equation for γ_{ij} is found in [21, chap. II, eq. (4.15)], equation (4.15), and reads (in the absence of anisotropic stress, which is valid since we have found in Section 4.1 that all adiabatic perturbations are of the perfect-fluid form)

$$\ddot{\gamma}_{ij} + 2H\dot{\gamma}_{ij} + \frac{1}{a^2}\left(-\bar{\nabla}^2 + 2K\right)\gamma_{ij} = 0. \tag{5.24}$$

Since the mode is time-independent, it solves the equation if

$$\bar{\nabla}^2\gamma_{ij} = \tilde{g}^{kl}\bar{\nabla}_k\bar{\nabla}_l\gamma_{ij} = \frac{1}{f^2}\bar{\nabla}_k\bar{\nabla}_k\gamma_{ij} = 2K\gamma_{ij}. \tag{5.25}$$

We are working perturbatively in $K\mathbf{x}^2$, and thus cannot expect the function γ_{ij} that we have found to be exactly the $k^2 = -2K$ Laplacian eigenvalue. After all, it seems very likely that higher-order terms in $K\mathbf{x}^2$ would be presence if we had not thrown them away, since non-perturbative treatment of equation (4.75) would couple the linear term in ϵ^i, which is ω_{ij}, to the fifth-order term in ϵ^i through a factor K^2. Since this full solution would also only be specified by ω_{ij}, it would also be time independent. Thus, this solution should also obey $\bar{\nabla}^2\gamma_{ij} = 2K\gamma_{ij}$. The Laplacian contains a term ∂^2 which would certainly make the $(K\mathbf{x}^2)^2$ term contribute to the $K\mathbf{x}^2$ in $\bar{\nabla}^2\gamma_{ij}$. Thus, neglecting the second-order term means that the first-order term will be 'missing' a contribution, and thus we can only expect the eigenvalue equation to hold at zeroth order in $K\mathbf{x}^2$. Thus, this is the level at which we will validate the Einstein equation, and it is the best we can do. Since one K enters through $K\mathbf{x}^2$ and one K enters as eigenvalue, working at zeroth order in $K\mathbf{x}^2$ is effectively working at first order in K, which is slightly easier to track. Working at this order certainly simplifies life, as it allows us to calculate the right-hand side of equation (5.25) trivially to be

$$2K\gamma_{ij} = -4K\omega_{ij} + \mathcal{O}(K^2). \tag{5.26}$$

The left-hand side is slightly more involved, but fortunately simplifies. Because ω_{ij} is a constant, $\partial_k\gamma_{ij}$ only contains terms of order K. Since the Christoffel symbols (as given in equation (C.19)) are also of order K, all the terms in $\bar{\nabla}_k\gamma_{ij}$ are of order K. Thus, the Christoffel symbols from the first covariant derivative do not contribute at first order and

$$\bar{\nabla}^2\gamma_{ij} = f^{-2}\left(\partial_k\bar{\nabla}_k\gamma_{ij} - \Gamma\cdot\bar{\nabla}\cdot\gamma\right) = \partial_k\bar{\nabla}_k\gamma_{ij} + \mathcal{O}(K^2) \tag{5.27}$$

(where $\Gamma\cdot\bar{\nabla}\cdot\gamma$ is only schematic). Calculating the covariant derivative gives

$$\begin{aligned}\bar{\nabla}_k\gamma_{ij} =& \partial_k\gamma_{ij} - \Gamma^l_{ki}\gamma_{lj} - \Gamma^l_{kj}\gamma_{il}\\ =& \partial_k\gamma_{ij} - \frac{1}{2}Kf\Big(\left(x^l\delta_{ki} - x^i\delta_{lk} - x^k\delta_{li}\right)\gamma_{lj} + (i\leftrightarrow j)\Big)\\ =& \partial_k\gamma_{ij} - \frac{1}{2}K\Big(x^l\delta_{ki}\gamma_{jl} - x^i\gamma_{jk} - x^k\gamma_{ij} + (i\leftrightarrow j)\Big) + \mathcal{O}(K^2)\\ =& \partial_k\gamma_{ij} - K\Big(x^l\delta_{k(i}\gamma_{j)l} - x^{(i}\gamma_{j)k} - x^k\gamma_{ij}\Big).\end{aligned} \tag{5.28}$$

When we now fill in the equation for γ_{ij},

$$\begin{aligned}\bar{\nabla}_k\gamma_{ij} =& K\Big(\frac{34}{21}x^k\omega_{ij} - \frac{10}{7}\omega_{k<i}x^{j>} - \frac{10}{7}x^l\omega_{l<i}\delta_{j>k} + 2x^l\delta_{k(i}\omega_{j)l} - 2x^{(i}\omega_{j)k} - 2x^k\omega_{ij}\Big)\\ =& K\Big(-\frac{8}{21}x^k\omega_{ij} - \frac{24}{7}x^{(i}\omega_{j)k} + \frac{4}{7}x^l\omega_{l(i}\delta_{j)k} + \frac{20}{21}x^l\omega_{lk}\delta_{ij}\Big).\end{aligned} \tag{5.29}$$

This formula allows us to easily check that the tensor mode is indeed transverse, as it should be by construction,

$$\begin{aligned}\bar{\nabla}_k\gamma_{ki} =& K\Big(-\frac{8}{21}x^k\omega_{ki} - \frac{12}{7}x^k\omega_{ik} - \frac{12}{7}x^i\cancelto{0}{\omega_{kk}} + \frac{2}{7}x^l\omega_{li}\delta_{kk} + \frac{2}{7}x^l\omega_{lk}\delta_{ik} + \frac{20}{21}x^l\omega_{lk}\delta_{ki}\Big)\\ =& \frac{1}{21}K\omega_{ik}x^k\Big(-8 - 36 + 18 + 6 + 20\Big) = 0.\end{aligned} \tag{5.30}$$

Now we can finally calculate the Laplacian of the tensor mode to check whether it indeed solves the equation of motion,

$$\begin{aligned}\bar{\nabla}^2\gamma_{ij} =&\partial_k\bar{\nabla}_k\gamma_{ij}\\ =&K\omega_{ij}\left(-\frac{8}{7}-\frac{24}{7}+\frac{4}{7}\right)+\frac{20}{21}\cancelto{0}{\omega_{kk}\delta_{ij}}\\ =&-\frac{28}{7}K\omega_{ij}=-4K\omega_{ij}=2K\gamma_{ij}.\end{aligned} \tag{5.31}$$

Furthermore, as an extra check, it can be easily verified using the metric transformation rules in Section 4.1 directly that for the diffeomorphism under consideration $\Delta h_{00}=\Delta h_{0i}=0$ (since $\epsilon^0=\dot{\epsilon}^i=0$) while $\Delta h_{ij}=a^2\gamma_{ij}$, where γ_{ij} is as calculated above. Since these rules were written down before defining a gauge, and since transverseness is also the gauge condition of comoving gauge on γ_{ij}, we clearly see that the tensor adiabatic mode is valid for both Newtonian gauge and comoving gauge.

Since we have explicitly checked that the tensor mode solves the linearized Einstein equation and does not violate any gauge condition, we can be quite confident that no technical error was made deriving it.

5.3 To Second Order and Beyond

The tensor mode (5.23) that we have found in Section 5.1 is in principle a pure gauge mode. The usual prescription to obtain an adiabatic mode is to find some physical solution which can be made to resemble the gauge mode arbitrarily closely in some limit. However, something strange is going on here. While the mode we have found appears to diverge at infinity because the part that is of order $K\mathbf{x}^2$ does, this seems to be only a perturbative feature. If we work non-perturbatively, any nonzero mode must be an infinite series of consecutive terms of ever higher order in $K\mathbf{x}^2$ in order to satisfy equation (4.63) (which would then couple ω_{ij} to all expansion parameters with an even number of indices). Since we chose $\epsilon^0=0$ and thus made our mode time independent, the linearized Einstein equation (5.24) dictates that the tensor mode must obey

$$\bar{\nabla}^2\gamma_{ij}=2K\gamma_{ij}. \tag{5.32}$$

Thus, the infinite series must be exactly one of the Laplacian eigenfunctions with eigenvalue $-k^2=2K=-2|K|$.

Here is the odd thing: this eigenfunction is a perfectly physical sub-curvature mode (which are the Laplacian eigenfunctions with $k^2>|K|$, and which form a complete basis to expand any mode vanishing at infinity as discussed in Section 2.2). In an analogy to flat space, instead of having a gauge mode at $k=0$, it already has some finite wavelength. This while there usually is quite some fuss with physicality conditions to extend the gauge mode to nonzero k! While one might find it convenient that it is now very easy to identify which specific physical modes are adiabatic, this is actually very suspect. For if a mode is equivalent to a gauge artifact, it should not be considered physical. Yet, the set of physical modes and the set of gauge modes appear to overlap here.

These considerations give rise to the suspicion that at the non-perturbative level, there is no time-independent tensor adiabatic mode. Just as we have thought of the $K\mathbf{x}^2$ term in equation (4.63) as imposing a condition relating the n^{th} coefficient in ϵ^i to the $n+2^{\text{th}}$, we can think of the $K^2\mathbf{x}^4$ term as imposing a relation between the n^{th} and $n+4^{\text{th}}$ terms. Yet, the $K\mathbf{x}^2$ term also relates the $n+2^{\text{th}}$ term to the $n+4^{\text{th}}$ term, effectively relating the n^{th} term to the $n+4^{\text{th}}$ term. Perhaps, then, these conditions are contradictory. This would mean that no gauge-preserving gauge transformation with $\epsilon^0 = 0$ and $\gamma^i{}_j \neq 0$ exists, and the apparent existence of a time-independent tensor mode vanishes beyond perturbative treatment in $K\mathbf{x}^2$.

As a first attempt to see whether this is true, one can see what happens when we keep treating the problem perturbatively but also retain $\mathcal{O}\left((K\mathbf{x}^2)^2\right)$. If inconsistency already appears at this order, we can be certain that the adiabatic nature of the tensor mode is only a perturbative feature. However, it turns out that the time-independent tensor mode also exists at this order. Even though we obtain three equations for two variables, a solution for the mode exists. Apparently, the equation obtained from imposing $\Psi = 0$ together with $\epsilon^0 = 0$ is not independent from the equations obtained from the gauge-preservation condition $\bar{\nabla}_i \gamma^i{}_j = 0$. An explicit construction of the tensor mode at second order follows below, after which we have a brief discussion about how to interpret the result.

We expand equation (4.63) as

$$\gamma^i{}_j = \left[1 + \frac{3}{4}K\mathbf{x}^2 + \frac{3}{16}(K\mathbf{x}^2)^2 + O\left((K\mathbf{x}^2)^3\right)\right] M_{ij}. \tag{5.33}$$

We can equate terms at same order in x as done before to obtain the equations

$$\gamma^i{}_j{}^{(0)} = M^{(0)}_{ij}, \tag{5.34}$$

$$\gamma^i{}_j{}^{(2)} = M^{(2)}_{ij} + \frac{3}{4}K\mathbf{x}^2 M^{(0)}_{ij}, \tag{5.35}$$

$$\gamma^i{}_j{}^{(4)} = M^{(4)}_{ij} + \frac{3}{4}K\mathbf{x}^2 M^{(2)}_{ij} + \frac{3}{16}(K\mathbf{x}^2)^2 M^{(0)}_{ij}, \tag{5.36}$$

where (n) means n^{th} order in x. The first two equations are exactly the same as those found before, the third one is new. To see its implication it is easiest to take ∂_i of both sides and use the property $\partial_i M_{ij} = 0$,

$$\partial_i \gamma^i{}_j{}^{(4)} = \frac{3}{2}Kx^i M^{(2)}_{ij} + \frac{3}{4}K^2\mathbf{x}^2 x^i M^{(0)}_{ij}. \tag{5.37}$$

The first two equations relating $\gamma^i{}_j$ and M_{ij} can be used to eliminate M_{ij} now entirely in terms of $\gamma^i{}_j$ to find

$$\partial_i \gamma^i{}_j{}^{(4)} = \frac{3}{2}Kx^i \gamma^i{}_j{}^{(2)} - \frac{3}{8}K^2\mathbf{x}^2 x^i \gamma^i{}_j{}^{(0)}. \tag{5.38}$$

Writing

$$\epsilon^i = \omega_{ij}x^j + \frac{1}{3}K\mu'_{ijkl}x^j x^k x^l + \frac{1}{5}K^2\tau'_{ijklmn}x^j x^k x^l x^m x^n \tag{5.39}$$

(where the definition of μ' differs a factor two with the one used previously but is more convenient for current purposes, and where the prime signifies that we have extracted the appropriate power of K), we have

$$\partial_i \epsilon^j = \omega_{ij} K \mu'_{ijkl} x^k x^l K^2 \tau'_{ijklmn} x^k x^l x^m x^n \tag{5.40}$$

and thus

$$\gamma^i{}_j = -2\partial_{<i}\epsilon^{j>} = -2\omega_{<ij>} - 2K\mu'_{<ij>kl} x^k x^l - 2K^2 \tau'_{<ij>klmn} x^k x^l x^m x^n. \tag{5.41}$$

Thus, equation (5.38) becomes

$$-8K^2 \tau'_{<ij>iklm} x^k x^l x^m = -3K^2 \mu'_{<ij>kl} x^i x^k x^l + \frac{3}{4} K^2 \omega_{<i(j>}\delta_{kl)} x^i x^k x^l, \tag{5.42}$$

or

$$\tau'_{<ij>jklm} x^k x^l x^m = \frac{3}{8}\left(\mu'_{<ik>lm} - \frac{1}{4}\omega_{<i(k>}\delta_{lm)}\right) x^k x^l x^m. \tag{5.43}$$

Now, the contraction with the different x's on the right-hand side imposes some additional symmetrization conditions. Here things become rather confusing, since different symmetrization operations generally do not commute and thus one has to be careful the handle them in the right order (starting with the condition imposed last and ending with the one imposed first). Let's start with μ_{iklm}, which is symmetric in the last three indices; this is the condition imposed first and thus to take care of last, and in that sense similar to the other term at the right hand side of the above equation. Due to the contraction with the x's, the part that we need in the equality is

$$\begin{aligned}
\mu'_{<i(k>lm)} &= \frac{1}{3}\left(\mu'_{<ik>lm} + \mu'_{<il>km} + \mu'_{<im>kl}\right) \\
&= \frac{1}{6}\left(\mu'_{iklm} + \mu'_{kilm} + \mu'_{ilkm} + \mu'_{likm} + \mu'_{imkl} + \mu'_{mikl}\right) \\
&- \frac{1}{9}\left(\mu'_{jjlm}\delta_{ik} + \mu'_{jjkm}\delta_{il} + \mu'_{jjkl}\delta_{im}\right) \\
&= \frac{1}{3}\mu'_{iklm} + \frac{1}{6}\left(\mu'_{iklm} + \mu'_{kilm} + \mu'_{likm} + \mu'_{mikl}\right) \\
&- \frac{1}{9}\left(\mu'_{jjlm}\delta_{ik} + \mu'_{jjkm}\delta_{il} + \mu'_{jjkl}\delta_{im}\right) \\
&= \frac{1}{3}\mu'_{iklm} + \frac{2}{3}\mu'_{(iklm)} - \frac{1}{3}\delta_{i(k}\mu'_{|jj|lm)}
\end{aligned} \tag{5.44}$$

Filling in the solution for μ found in Section 5.1, which for our slightly-different definition is

$$\mu'_{ijkl} = \frac{11}{14}\delta_{i(j}\omega_{kl)} + \frac{4}{14}\omega_{i(j}\delta_{kl)}, \tag{5.45}$$

gives (using the identities (5.4))

$$\mu'_{jjkl} = \omega_{kl}\left(\frac{11}{14}\cdot\frac{5}{3} + \frac{4}{14}\cdot\frac{2}{3}\right) = \frac{3}{2}\omega_{kl}. \tag{5.46}$$

Furthermore, $\mu'_{(ijkl)} = \frac{15}{14}\omega_{(ij}\delta_{kl)}$. Putting it all together, we get

$$\begin{aligned}
\mu'_{<i(k>lm)} &= \left(\frac{1}{3}\cdot\frac{11}{14} - \frac{1}{3}\cdot\frac{3}{2}\right)\delta_{i(k}\omega_{lm)} + \frac{1}{3}\cdot\frac{4}{14}\omega_{i(k}\delta_{lm)} + \frac{2}{3}\cdot\frac{15}{14}\omega_{(ik}\delta_{lm)} \\
&= -\frac{5}{21}\delta_{i(k}\omega_{lm)} + \frac{2}{21}\omega_{i(k}\delta_{lm)} + \frac{5}{7}\omega_{(ik}\delta_{lm)} \\
&= \left(\frac{2}{21} + \frac{5}{21}\right)\omega_{i(k}\delta_{lm)} + \left(\frac{5}{7} - \frac{10}{21}\right)\omega_{(ik}\delta_{lm)} \\
&= \frac{1}{3}\omega_{i(k}\delta_{lm)} + \frac{5}{21}\omega_{(ik}\delta_{lm)}
\end{aligned} \tag{5.47}$$

where we we have used the identity $\delta_{i(j}\omega_{kl)} + \omega_{i(j}\delta_{kl)} = 2\omega_{(ij}\delta_{kl)}$.

Mimicking the procedure for μ, the $\omega_{<i(k>}\delta_{lm)}$ after the proper symmetrization due to the contraction in equation (5.43) is

$$\begin{aligned}
\omega_{<i((k>}\delta_{lm))} &= \frac{1}{3}\omega_{i(k}\delta_{lm)} + \frac{2}{3}\omega_{(ik}\delta_{lm)} - \frac{1}{3}\delta_{i(k}\omega_{|j(j|}\delta_{lm))} \\
&= \frac{1}{3}\omega_{i(k}\delta_{lm)} + \frac{2}{3}\omega_{(ik}\delta_{lm)} - \frac{2}{9}\delta_{i(k}\omega_{lm)} \\
&= \frac{5}{9}\omega_{i(k}\delta_{lm)} + \frac{2}{9}\omega_{(ik}\delta_{lm)}.
\end{aligned} \tag{5.48}$$

We are aware that the notation with multiple symmetrization brackets is extremely confusing. One must be careful to keep track of the order in which the symmetrizations must be handled. It is easiest to think about what happens above by defining $\sigma_{ijkl} = \omega_{i(k}\delta_{lm)}$, which is an object that is symmetric in the last three indices, after which the results for μ'_{ijkl} can be used straightforwardly (making it manifest how to handle the 'original' symmetrization). Furthermore, we can rewrite the left-hand side of equation (5.43) as

$$\tau'_{<ij>jklm} = \frac{1}{2}\tau'_{ijjklm} + \frac{1}{2}\tau'_{jijklm} - \frac{1}{3}\tau'_{nnjklm}\delta_{ij} = \frac{1}{2}\tau'_{ijjklm} + \frac{1}{6}\tau'_{jjiklm}. \tag{5.49}$$

We can now combine the results into equation (5.43) to obtain

$$\begin{aligned}
\frac{1}{2}\tau'_{ijjklm} + \frac{1}{6}\tau'_{jjiklm} &= \frac{3}{8}\left[\left(\frac{5}{21} - \frac{1}{4}\cdot\frac{2}{9}\right)\omega_{(ik}\delta_{lm)} + \left(\frac{1}{3} - \frac{1}{4}\cdot\frac{5}{9}\right)\omega_{i(k}\delta_{lm)}\right] \\
&= \frac{1}{8}\left[\left(\frac{5}{7} - \frac{1}{6}\right)\omega_{(ik}\delta_{lm)} + \left(1 - \frac{5}{12}\right)\omega_{i(k}\delta_{lm)}\right] \\
&= \frac{23}{336}\omega_{(ik}\delta_{lm)} + \frac{7}{96}\omega_{i(k}\delta_{lm)}
\end{aligned} \tag{5.50}$$

or

$$\tau'_{ijjklm} = \frac{23}{168}\omega_{(ik}\delta_{lm)} - \frac{1}{3}\tau'_{jjiklm} + \frac{7}{48}\omega_{i(k}\delta_{lm)}. \tag{5.51}$$

The transversality equation for $\gamma^i{}_j$ is not the only equation we impose. After all, we are looking for a time-independent scalar mode. For $\epsilon^0 = 0$, the condition $\Psi = 0$ becomes

$$\frac{1}{3}\partial_k\epsilon^k - \frac{1}{2}Kfx^k\epsilon^k = 0. \tag{5.52}$$

Using $f = 1 - \frac{1}{4}K\mathbf{x}^2 + \mathcal{O}\left((K\mathbf{x}^2)\right)$ and filling in the expansion of ϵ^i gives

$$\begin{aligned}
&\frac{1}{3}\omega_{ii} + K\left[\frac{1}{3}\mu_{iikl} - \frac{1}{2}\omega_{kl}\right]x^k x^l \\
&+ K^2\left[\frac{1}{3}\tau'_{iiklmn} - \frac{1}{6}\mu'_{klmn} + \frac{1}{8}\omega_{kl}\delta_{mn}\right]x^k x^l x^m x^n + \mathcal{O}\left((K\mathbf{x}^2)^3\right) = 0
\end{aligned} \tag{5.53}$$

The equation must vanish order-by-order in x. The first two resulting equations are exactly the same as when doing linear perturbation theory, as expected (forcing $\omega_{kk} = 0$, of which we make (implicit) use often in this section). Together with the fact that the equations up to first order in $K\mathbf{x}^2$ obtained from the transversality condition were also the same as before justifies us using the result for μ as found in Section 5.1; otherwise we have will never be able to solve all equations at the same time. The vanishing of the term that is fourth-order in x implies

$$\tau'_{iiklmn} = \frac{1}{2}\mu'_{(klmn)} - \frac{3}{8}\omega_{(kl}\delta_{mn)}. \tag{5.54}$$

The symmetrization over all indices makes it particularly easy to fill in equation (5.45) and thus we find

$$\tau'_{jjiklm} = \frac{9}{56}\omega_{(ik}\delta_{lm)} \tag{5.55}$$

Combining equations (5.55) and (5.51) (plugging the first into the second) now gives the system of equations

$$\begin{aligned}
\tau'_{ijjklm} &= \frac{1}{12}\omega_{(ik}\delta_{lm)} + \frac{7}{48}\omega_{i(k}\delta_{lm)} = \frac{3}{16}\omega_{i(k}\delta_{lm)} + \frac{1}{24}\delta_{i(k}\omega_{lm)}, \\
\tau'_{jjiklm} &= \frac{9}{56}\omega_{(ik}\delta_{lm)}.
\end{aligned} \tag{5.56}$$

The question, then, is whether this system has a solution. Due to our limited building blocks and because of symmetry, there is only one ansatz we can make for the solution of τ':

$$\tau'_{ijklmn} = n_1\omega_{i(j}\delta_{kl}\delta_{mn)} + n_2\delta_{i(j}\delta_{kl}\omega_{mn)}. \tag{5.57}$$

It then seems the system of equations is overdeterminated. Taking traces can only give linear combinations of $\omega_{i(k}\delta_{lm)}$ and $\omega_{(ik}\delta_{lm)}$ (or, equivalently, $\omega_{i(k}\delta_{lm)}$ and $\delta_{i(k}\omega_{lm)}$; these are just two different bases which span the same vector space, allowing us to use them interchangeably). Matching terms will give three equations for n_1 and n_2: two for the first equation in (5.56) and one for the second (since $\omega_{(ik}\delta_{lm)}$ is the only term that can occur due to symmetry). Yet, the system might be solvable if some of the equations are not independent. To check this, we should calculate some more traces (which is mostly an exercises in combinatorics):

$$\begin{aligned}
\omega_{j(j}\delta_{ik}\delta_{lm)} &= \frac{1}{5}\cancelto{0}{\omega_{jj}}\delta_{(ik}\delta_{lm)} + \frac{4}{5}\omega_{j(i|}\delta_{j|k}\delta_{lm)} \\
&= \frac{4}{5}\omega_{(ik}\delta_{lm)},
\end{aligned} \tag{5.58}$$

$$\begin{aligned}
\delta_{j(j}\delta_{kl}\omega_{lm)} &= \frac{1}{5}\cancelto{3}{\delta_{jj}}\delta_{(ik}\omega_{lm)} + \frac{2}{5}\delta_{j(i}\delta_{|j|k}\omega_{lm)} + \frac{2}{5}\delta_{j(i}\delta_{kl}\omega_{m)j} \\
&= \frac{7}{5}\omega_{(ik}\delta_{lm)},
\end{aligned} \tag{5.59}$$

$$
\begin{aligned}
\omega_{i(j}\delta_{jk}\delta_{lm)} &= \frac{2}{5}\omega_{ij}\delta_{j(k}\delta_{lm)} + \frac{1}{5}\omega_{i(k}\delta_{lm)}\cancelto{3}{\delta_{jj}} + \frac{2}{5}\omega_{i(k}\delta_{l|j}\delta_{j|m)} \\
&= \frac{7}{5}\omega_{i(k}\delta_{lm)},
\end{aligned}
\tag{5.60}
$$

$$
\begin{aligned}
\delta_{i(j}\delta_{jk}\omega_{lm)} = &(\frac{2}{5}\cdot\frac{1}{4}\cdot 2)\delta_{i(k}\delta_{lm)}\cancelto{0}{\omega_{jj}} + (\frac{2}{5}\cdot\frac{1}{4}\cdot 2)\delta_{i(j}\omega_{kl)}\cancelto{3}{\delta_{jj}} \\
&+ (\frac{2}{5}\cdot\frac{1}{2})\delta_{ij}\delta_{j(k}\omega_{lm)} + (\frac{2}{5}\cdot\frac{1}{2})\delta_{ij}\omega_{j(k}\delta_{lm)} + (\frac{2}{5}\cdot\frac{3}{4}\cdot\frac{2}{3})\delta_{i(k}\delta_{l|j}\omega_{j|m)} \\
&= \delta_{i(k}\omega_{lm)} + \frac{1}{5}\omega_{i(k}\delta_{lm)}.
\end{aligned}
\tag{5.61}
$$

Thus, the system of equations (5.56) becomes

$$
\begin{aligned}
\frac{7}{5}n_1 + \frac{1}{5}n_2 &= \frac{3}{16}, \\
n_2 &= \frac{1}{24}, \\
\frac{4}{5}n_1 + \frac{7}{5}n_2 &= \frac{9}{56}
\end{aligned}
\tag{5.62}
$$

This system of equations is solved by

$$
(n_1, n_2) = \left(\frac{43}{336}, \frac{1}{24}\right), \tag{5.63}
$$

and thus we must conclude that the time-independent adiabatic tensor mode also exists at second order in perturbation theory as advertised. More explicitly, the adiabatic mode is given by

$$
\begin{aligned}
\epsilon^i &= \omega_{ik}x^k + \frac{1}{42}K\left(11\delta_{i(j}\omega_{kl)} + 4\omega_{i(j}\delta_{kl)}\right)x^j x^k x^l \\
&+ \frac{1}{1680}K^2\left(43\omega_{i(j}\delta_{kl}\delta_{mn)} + 70\delta_{i(j}\delta_{kl}\omega_{mn)}\right)x^j x^k x^l x^m x^n \\
\gamma_{ij} &= -2\partial_{<i}\epsilon_{k>} \\
\omega_{ij} &= \omega_{<ij>}
\end{aligned}
\tag{5.64}
$$

It seems surprising that the different equations are not independent. What happens is reminiscent of what happened at first order, where $\Psi = 0$ implied $\omega_{jj} = 0$, making one of the equations that followed from $\bar{\nabla}_i\gamma^i{}_j = 0$ vanish. Here, the exact reason why this happens is not as clear, but it evidently does. This makes one wonder whether the same will occur at every order in $K\mathbf{x}^2$, such that the time-independent adiabatic tensor mode exists non-perturbatively. Perhaps this is caused by the common origin of the two equations. Writing things covariantly, we have (see Appendix B and Section 4.2)

$$
-2\nabla_{(i}\epsilon_{j)} = -2\Psi\bar{g}_{ij} + a^2\gamma_{ij} \tag{5.65}
$$

For $\epsilon^0 = 0$, the Christoffel symbols do not have to sum over the 0 component, and since $\bar{\Gamma}^i_{jk} = \Gamma^i_{jk}$, this becomes

$$
-2\bar{\nabla}_{(i}\epsilon_{j)} = -2\Psi\bar{g}_{ij} + a^2\gamma_{ij}. \tag{5.66}
$$

The requirement $\Psi = 0$ is then equivalent to the equation (using $\gamma^k{}_k = 0$)

$$\bar{\nabla}_k \epsilon^k = 0, \tag{5.67}$$

while the requirement $\bar{\nabla}_i \gamma^i{}_j$ is equivalent to

$$\bar{\nabla}_i \gamma^i{}_j = 2\tilde{g}_{ij} \partial^i \Psi - \frac{1}{a^2} \left(\bar{\nabla}^2 \epsilon_j + \bar{\nabla}_j \bar{\nabla}_i \epsilon^i \right) = 0. \tag{5.68}$$

If the condition $\Psi = 0$ is enforced, then it becomes

$$\bar{\nabla}^2 \epsilon_j = 0. \tag{5.69}$$

This shows how (for $\epsilon^0 = 0$) the different conditions mix. Furthermore, it seems likely that the system of equations

$$\begin{aligned} \bar{\nabla}_i \epsilon^i &= 0 \\ \bar{\nabla}^2 \epsilon^i &= 0 \end{aligned} \tag{5.70}$$

always has solutions (with nonzero $\bar{\nabla}_{<i} \epsilon_{j>}$). Attempting to solve them in a general way is however beyond the scope of this thesis.

In conclusion, we are not certain whether the time-independent adiabatic tensor mode exists non-perturbatively. We do know that it exists up to second order in $K\mathbf{x}^2$ (containing corrections up to order $K/(aH)^2$ to the flat-space result within the observable universe), yet there remains the possibility that the calculations become inconsistent at some higher order in $K\mathbf{x}^2$. This does seem unlikely though, as we have observed already at two orders that the three imposed equations are not independent, allowing one to solve for the two parameters which characterize each order. This seems to indicate a deeper relation between the different conditions which might then hold at every order. It is beyond the scope of this thesis to clarify the precise relation or to find the full non-perturbative tensor mode. Another question is whether time-*dependent* adiabatic tensor modes exist. Since the $\bar{\nabla}_i \gamma^i{}_j$ equations (which do not depend on the choice for ϵ^0) seem to imply the $\Psi = 0$ equation for $\epsilon^0 = 0$, this seems to suggest $\epsilon^0 = 0$ is the only valid choice, meaning the tensor mode *must* be time independent. Yet this too requires more careful analysis.

If the time-independent tensor mode indeed exists non-perturbatively, we must accept that adiabatic modes can be sub-curvature. Since these are the 'finite momentum' modes in an open universe, this seems strange and calls for further investigation and interpretation.

Chapter 6

Pure Scalar

In this section, we try to generalize Weinberg's scalar adiabatic mode, which is probably the simplest and most useful adiabatic mode of all (in flat space, at least). We start in Section 6.1 by mimicking Weinberg's method of obtaining his scalar adiabatic modes, as described in Section 3.1. This means we simply rescale our coordinates and check the necessary physicality conditions. Next, we double check our adiabatic solution in Section 6.2 against the linearized Einstein equations. As a last check for consistency, we show that our solution reduces to Weinberg's solution in Section 6.3. Before going into general discussion, we present a concrete example of what the adiabatic mode looks like when the background is dominated by radiation in Section 6.4. Lastly, we observe that there are some difficulties in making the found scalar mode physical. This problem and its possible consequences are discussed in Section 6.5.

6.1 A Simple Rescaling

Weinberg obtained if flat-space scalar adiabatic modes by performing a rescaling of the spatial coordinates. While he allowed for this rescaling to be time dependent, he discovered that it had to be constant [36]. Furthermore, he accompanied it with an appropriate temporal diffeomorphism. We will mimick this procedure in this section for the open universe, for which case we find that the rescaling should not be time independent. Note that the machinery developed in Section 4.5 is not required here as for (isotropic) rescalings we have ${\gamma^i}_j = 0$ (otherwise, it would not be a pure scalar mode), and thus the condition $\bar{\nabla}_i {\gamma^i}_j = 0$ is trivially satisfied.

Time-dependent (isotropic) rescalings are diffeomorphism of the form

$$\epsilon^i = \lambda(t) x^i \tag{6.1}$$

(such that $x^i \to (1+\lambda)x^i$). Then we have

$$\partial_j \epsilon^i = \lambda \delta_{ij}. \tag{6.2}$$

It is then readily verified that

$$\gamma_{ij} = f^2 \left(\frac{2}{3} \delta_{ij} \partial_k \epsilon^k - \partial_i \epsilon^j - \partial_j \epsilon^i \right) = f^2 \lambda \left(\frac{2}{3} \delta_{ij} \delta_{kk} - 2\delta_{ij} \right) = 0. \tag{6.3}$$

Furthermore, we choose ϵ^0 such that the vector perturbations are also zero (and the mode is a pure scalar),

$$G_i = \frac{2}{a}\partial_i\epsilon^0 - 2af^2\dot{\epsilon}^i = \frac{2}{a}\partial_i\epsilon^0 - 2af^2\dot{\lambda}x^i = 0. \tag{6.4}$$

This is solved by

$$\epsilon^0 = -2\frac{a^2}{K}f\dot{\lambda} + \mathcal{D}(t), \tag{6.5}$$

where $\mathcal{D}$ is an integration 'constant'.

In order to make this mode physical, it must satisfy the physicality conditions. These are found in Appendix E (see also Section 4.4) to be

$$\Phi = \Psi \tag{6.6}$$

and

$$\dot{\Psi} + H\Phi = \left(\dot{H} - \frac{K}{a^2}\right)\delta u. \tag{6.7}$$

The relevant perturbations are given by (see equations (4.43))

$$\delta u = \epsilon^0, \tag{6.8}$$

$$\Phi = -\dot{\epsilon}^0, \tag{6.9}$$

$$\Psi = H\epsilon^0 - \frac{1}{2}Kfx^k\epsilon^k + \frac{1}{3}\partial_k\epsilon^k. \tag{6.10}$$

Thus, the second physicality condition becomes

$$\dot{H}\epsilon^0 + H\dot{\epsilon}^0 - \frac{1}{2}Kfx^k\dot{\epsilon}^k + \frac{1}{3}\partial_k\dot{\epsilon}^k - H\dot{\epsilon}^0 = \dot{H}\epsilon^0 - \frac{K}{a^2}\epsilon^0, \tag{6.11}$$

which reduces to

$$\frac{K}{a^2}\epsilon^0 - \frac{1}{2}Kfx^k\dot{\epsilon}^k + \frac{1}{3}\partial_k\dot{\epsilon}^k = 0. \tag{6.12}$$

The ϵ^k bit can be simplified a bit when filled in,

$$\begin{aligned} -\frac{1}{2}Kfx^k\epsilon^k + \frac{1}{3}\partial_k\epsilon^k =& \lambda(1 - \frac{1}{2}K\mathbf{x}^2 f) = \lambda\left(1 - \frac{\frac{1}{2}K\mathbf{x}^2}{1+\frac{1}{4}K\mathbf{x}^2}\right) \\ =& \lambda\frac{1+\frac{1}{4}K\mathbf{x}^2 - \frac{1}{2}K\mathbf{x}^2}{1+\frac{1}{4}K\mathbf{x}^2} = \left(1 - \frac{1}{4}K\mathbf{x}^2\right)f\lambda. \end{aligned} \tag{6.13}$$

Thus, the equation becomes

$$\left(-2 + (1 - \frac{1}{4}K\mathbf{x}^2)\right)f\dot{\lambda} + \frac{K}{a^2}\mathcal{D} = 0. \tag{6.14}$$

This is solved by

$$\mathcal{D} = \frac{a^2}{K}\left(1 + \frac{1}{4}K\mathbf{x}^2\right)f\dot{\lambda} = \frac{a^2\dot{\lambda}}{K}. \tag{6.15}$$

The result of this is that the temporal part of of the diffeomorphism takes the simple form

$$\begin{aligned}\epsilon^0 =& \frac{a^2}{K}\dot{\lambda}(1-2f)\\ =& \frac{a^2}{K}\dot{\lambda}\left(1-\frac{2}{1+\frac{1}{4}K\mathbf{x}^2}\right)\\ =& \frac{a^2}{K}\dot{\lambda}\frac{1+\frac{1}{4}K\mathbf{x}^2-2}{\frac{1}{4}K\mathbf{x}^2}\\ =& -\frac{a^2}{K}\left(1-\frac{1}{4}K\mathbf{x}^2\right)f\dot{\lambda}.\end{aligned} \tag{6.16}$$

We then have

$$\dot{\epsilon}^0 = -\frac{1}{K}\left(1-\frac{1}{4}K\mathbf{x}^2\right)f\partial_t(a^2\dot{\lambda}) = -\frac{a^2}{K}f\left(1-\frac{1}{4}K\mathbf{x}^2\right)(2H\dot{\lambda}+\ddot{\lambda}). \tag{6.17}$$

We can write the first physicality condition as

$$\begin{aligned}\Phi-\Psi =& -\dot{\epsilon}^0 - H\epsilon^0 - \left(-\frac{1}{2}Kfx^k\epsilon^k + \frac{1}{3}\partial_k\epsilon^k\right)\\ =& \frac{a^2}{K}f\left(1-\frac{1}{4}K\mathbf{x}^2\right)\ddot{\lambda} + 3\frac{a^2H}{K}f\left(1-\frac{1}{4}K\mathbf{x}^2\right)\dot{\lambda} - f\left(1-\frac{1}{4}K\mathbf{x}^2\right)\lambda = 0.\end{aligned} \tag{6.18}$$

Simply dividing out the factor $\frac{a^2}{K}f\left(1-\frac{1}{4}K\mathbf{x}^2\right)$ then gives us the final constraint equation on $\lambda(t)$. Since it is a second-order differential equation, we are dealing with two distinct adiabatic modes. This is no surprise, as Weinberg also found two. It is shown in Section 6.3 that the solutions for λ indeed become Weinberg's first and second scalar adiabatic mode for $K\to 0$, even if we cannot solve the differential equation analytically (for arbitrary $a(t)$). Combining the differential equation with the equations for the perturbations (together fully specifying the scalar adiabatic mode) gives

$$\begin{aligned}&\ddot{\lambda}+3H\dot{\lambda}-\frac{K}{a^2}\lambda = 0,\\ &\Phi=\Psi=\frac{1-\frac{1}{4}K\mathbf{x}^2}{1+\frac{1}{4}K\mathbf{x}^2}a^2\left(\frac{1}{a^2}\lambda-\frac{H}{K}\dot{\lambda}\right),\\ &\frac{\delta\rho}{\dot{\rho}}=\frac{\delta p}{\dot{p}}=-\delta u^S=\frac{1}{K}\frac{1-\frac{1}{4}K\mathbf{x}^2}{1+\frac{1}{4}K\mathbf{x}^2}a^2\dot{\lambda},\\ &\mathcal{R}\equiv-\Psi+H\delta u^S=-\frac{1-\frac{1}{4}K\mathbf{x}^2}{1+\frac{1}{4}K\mathbf{x}^2}\lambda,\end{aligned} \tag{6.19}$$

which is the final result of this section.

6.2 Checking: Einstein Equations

In this section, we check whether the two adiabatic scalar modes specified by (6.19) indeed satisfy all the linearized Einstein equations. These equations are given, in Newtonian gauge, by (E.22). If the diffeomorphism that underlies the adiabatic mode has been performed correctly, we can be quite certain that the equations are all satisfied because of diffeomorphism invariance of general relativity (after all, we know that the unperturbed universe solves all of Einstein's field equations). Yet, checking explicitly that the equations are solved for the mode that we found is a way of testing the validity of all steps that came before. After all, if we performed some miscalculation somewhere, this is a nice, independent way of finding out.

We start with the second equation in (E.22). This is a physicality condition which we have used to find the scalar mode in Section (6.1):

$$\dot{\Phi} + H\Phi = \left(\dot{H} - \frac{K}{a^2}\right)\delta u^S. \tag{6.20}$$

It thus should hold trivially. However, we check it explicitly once more here as a consistency check. Also, it will result in a differential equation for ϵ^0 (rather than for λ) which is convenient to use for the checking of the remaining Einstein equations.

For an adiabatic mode, we can plug in (4.43) to turn the equation into a second differential equation for ϵ^0,

$$\ddot{\epsilon}^0 + H\dot{\epsilon}^0 + \left(\dot{H} - \frac{K}{a^2}\right)\epsilon^0 = 0. \tag{6.21}$$

Since we are solving a homogeneous temporal differential equation, we can ignore all spatial dependence of ϵ^0. Since the temporal part of ϵ^0 is contained by an overall factor $a^2\dot{\lambda}$ (see equation (6.16)), and since we would like to write down a differential equation for λ, we calculate the objects

$$\frac{1}{a^2}\frac{d}{dt}(a^2\dot{\lambda}) = \ddot{\lambda} + 2H\dot{\lambda}, \tag{6.22}$$

$$\frac{1}{a^2}\frac{d^2}{dt^2}(a^2\dot{\lambda}) = \dddot{\lambda} + 4H\ddot{\lambda} + (4H^2 + 2\dot{H})\dot{\lambda}. \tag{6.23}$$

We then find the differential equation

$$\dddot{\lambda} + 5H\ddot{\lambda} + \left(6H^2 + 3\dot{H} - \frac{K}{a^2}\right)\dot{\lambda} = 0. \tag{6.24}$$

We now want to check whether it is automatically solved when λ obeys the differential equation in (6.19). First, we take the derivative of this equation, which gives

$$\dddot{\lambda} + 3H\ddot{\lambda} + \left(3\dot{H} - \frac{K}{a^2}\right)\dot{\lambda} + 2H\frac{K}{a^2}\lambda = 0. \tag{6.25}$$

We subtract this equation from (6.24) to find

$$2H\ddot{\lambda} + 6H^2\dot{\lambda} - 2H\frac{K}{a^2}\lambda = 0. \tag{6.26}$$

Since this is just $2H$ times the differential equation in (6.19), it follows trivially that any λ solving this equation also solves equation (6.24).

Next, we check the two remaining Einstein equations (which do not provide a physicality condition since they contain no overall derivative). We take an approach similar to the one above and rewrite both of them in terms of ϵ^0. As a result, we obtain two differential equations for ϵ^0,

$$3H\ddot{\epsilon}^0 + \left(3H^2 + \frac{k^2 - 3K}{a^2}\right)\dot{\epsilon}^0 + \frac{\dot{\rho}}{2M_p^2}\epsilon^0 = 0, \tag{6.27}$$

$$\dddot{\epsilon}^0 + 4H\ddot{\epsilon}^0 + \left(2\dot{H} + 3H^2 - \frac{K}{a^2}\right)\dot{\epsilon}^0 - \frac{\dot{p}}{2M_p^2}\epsilon^0 = 0. \tag{6.28}$$

Since we encounter k^2, which is minus the eigenvalue of the Laplacian of ϵ^0 (and all other nonzero objects in the scalar mode), we must first calculate what it is.

Since the spatial dependence of ϵ^0 factors out, we are really calculating the Laplacian of the function

$$\frac{1 - \frac{1}{4}K\mathbf{x}^2}{1 + \frac{1}{4}K\mathbf{x}^2} = (1 - \frac{1}{4}K\mathbf{x}^2)f. \tag{6.29}$$

This proceeds as follows:

$$\bar{\nabla}^2\epsilon^0 = \tilde{g}^{ij}\bar{\nabla}_i\bar{\nabla}_j\epsilon^0 = f^{-2}\bar{\nabla}_i\partial_i\epsilon^0. \tag{6.30}$$

Neglecting for the moment to write the time-dependent part of ϵ^0, its spatial derivative is

$$\begin{aligned}\partial_i\epsilon^0 &= \partial_i(\frac{1}{4}K\mathbf{x}^2)\left[\frac{-1}{1 + \frac{1}{4}K\mathbf{x}^2} - \frac{1 - \frac{1}{4}K\mathbf{x}^2}{(1 + \frac{1}{4}K\mathbf{x}^2)^2}\right] \\ &= \frac{1}{2}Kf^2x^i\left[\left(-1 - \frac{1}{4}K\mathbf{x}^2\right) - \left(1 - \frac{1}{4}K\mathbf{x}^2\right)\right] \\ &= -Kf^2x^i.\end{aligned} \tag{6.31}$$

We can then calculate, using equations (C.26) and (C.2),

$$\begin{aligned}\bar{\nabla}_i\partial_i\epsilon^0 &= \partial_i\partial_i\epsilon^0 - \Gamma^j_{ii}\partial_j\epsilon^0 \\ &= \partial_i\left(-Kx^if^2\right) - \frac{1}{2}Kfx^j\left(-Kx^jf^2\right) \\ &= -Kf^2\partial_ix^i - Kx^i\partial_if^2 + \frac{1}{2}f^3K62\mathbf{x}^2 \\ &= -3Kf^2 + f^3K^2x^2 + \frac{1}{2}f^3K^2\mathbf{x}^2 \\ &= -3Kf^2\left(1 - \frac{1}{2}fK\mathbf{x}^2\right) \\ &= -3Kf^3\left((1 + \frac{1}{4}K\mathbf{x}^2) - \frac{1}{2}K\mathbf{x}^2\right) \\ &= -3Kf^3\left(1 - \frac{1}{4}K\mathbf{x}^2\right).\end{aligned} \tag{6.32}$$

And thus,

$$\begin{aligned}\bar{\nabla}^2\epsilon^0 &= -3Kf\left(1-\frac{1}{4}K\mathbf{x}^2\right)\\ &= -3K\frac{1-\frac{1}{4}K\mathbf{x}^2}{1+\frac{1}{4}K\mathbf{x}^2}\\ &= -3K\epsilon^0,\end{aligned} \tag{6.33}$$

leading finally to the conclusion that $k^2 = 3K$.

Let's start by checking equation (6.27). Using $k^2 = 3K$, the term with K conveniently cancels. Furthermore, we can use the Friedmann equation (1.58) to write

$$\frac{\bar{\rho}}{3M_p^2} = H^2 + \frac{K}{a^2}. \tag{6.34}$$

This gives

$$\frac{\dot{\bar{\rho}}}{3M_p^2} = 2H\dot{H} - 2H\frac{K}{a^2}. \tag{6.35}$$

Multiplying by $3/2$ gives

$$\frac{\dot{\bar{\rho}}}{2M_p^2} = 3H\left(\dot{H} - \frac{K}{a^2}\right). \tag{6.36}$$

Thus, the equation becomes

$$3H\ddot{\epsilon}^0 + 3H^2\dot{\epsilon}^0 + 3H\left(\dot{H} - \frac{K}{a^2}\right)\epsilon^0 = 0. \tag{6.37}$$

This is simply $3H$ times equation (6.21), which we checked to hold for the scalar mode above. Thus, this Einstein equation is indeed satisfied.

To find out whether equation (6.28) also holds, we first subtract (6.27). Since we just saw that this equation indeed holds for the scalar mode, (6.28) will hold if the resulting equation holds. This resulting equation is

$$\dddot{\epsilon}^0 + H\ddot{\epsilon}^0 + (2\dot{H} - \frac{K}{a^2})\dot{\epsilon}^0 - \frac{\dot{\bar{\rho}} + \dot{\bar{p}}}{2M_p^2} = 0. \tag{6.38}$$

This can be easily checked to equal the derivative of equation (6.21) (which we know holds) if

$$\frac{\dot{\bar{\rho}} + \dot{\bar{p}}}{2M_p^2} = -\ddot{H} - 2H\frac{K}{a^2}. \tag{6.39}$$

Thus, if this is the case, we know this Einstein equation is also solved by the scalar mode. In order to find out, we use the continuity equation (1.66) to write

$$\frac{\bar{\rho} + \bar{p}}{2M_p^2} = \frac{-1}{3H}\frac{\dot{\bar{\rho}}}{2M_p^2}. \tag{6.40}$$

Thus,

$$\begin{aligned}\frac{\dot{\bar{\rho}} + \dot{\bar{p}}}{2M_p^2} &= \frac{\dot{H}}{3H^2}\frac{\dot{\bar{\rho}}}{2M_p^2} - \frac{1}{3H}\frac{\ddot{\bar{\rho}}}{2M_p^2}\\ &= \frac{1}{3H}\left(\frac{\dot{H}}{H}\frac{\dot{\bar{\rho}}}{2M_p^2} - \frac{\ddot{\bar{\rho}}}{2M_p^2}\right).\end{aligned} \tag{6.41}$$

Now we can apply equation (6.36). By calculating its derivative we find

$$\begin{aligned}\frac{\ddot{\bar{\rho}}}{2M_p^2} &= 3\dot{H}^2 + 3H\ddot{H} - 3\dot{H}\frac{K}{a^2} + 6H^2\frac{K}{a^2} \\ &= 3H\left[\frac{\dot{H}^2}{H} + \ddot{H} + \frac{K}{a^2}\left(2H - \frac{\dot{H}}{H}\right)\right].\end{aligned} \tag{6.42}$$

Plugging this result together with (6.36) itself then gives

$$\begin{aligned}\frac{\dot{\bar{\rho}} + \dot{\bar{p}}}{2M_p^2} &= \frac{\dot{H}}{H}\left(\dot{H} - \frac{K}{a^2}\right) - \frac{\dot{H}^2}{H} - \ddot{H} - 2H\frac{K}{a^2} + \frac{\dot{H}}{H}\frac{K}{a^2} \\ &= -\ddot{H} - 2H\frac{K}{a^2}.\end{aligned} \tag{6.43}$$

Thus, this Einstein equation holds as well.

In conclusion, the gauge mode that we have found satisfies all Einstein equations and physicality conditions. Thus, we can conclude that the procedure in Section 6.1 was most probably applied correctly and the gauge mode can safely be extended to the physical domain by giving it some appropriate fall-off at infinity.

6.3 Flat-Space Limit

When we take the flat-space limit $K \to 0$ for the scalar adiabatic mode above, we should regain the scalar modes that have already been found in the flat-space analysis. Because we have assumed $\delta u = \epsilon^0$ (see Section 4.1), the time-dependent scalar mode (which was found by [29] for the special case of shift-symmetric scalar fields) should be absent. We do however correctly obtain both of Weinberg's scalar adiabatic modes.

The chief difficulty with taking the flat-space limit lies in the $\frac{1}{K}$ appearing in the solution for $\Phi = \Psi$ (and the matter fields, see equations (6.19)). One might be surprised by the presence of these terms, as $1/K$ is ill-defined for $K = 0$. This is however not a problem, since these quantities are defined differently for $K = 0$. This is a result of equation (6.4), the solution of which does not reduce to the $K = 0$ solution when taking $K \to 0$:

$$\epsilon_0 = \begin{cases} \frac{1}{K}\frac{1-\frac{1}{4}K\mathbf{x}^2}{1+\frac{1}{4}K\mathbf{x}^2}a^2\dot{\lambda} & K \neq 0 \\ \frac{-\lambda}{a} & K = 0 \end{cases} \tag{6.44}$$

The appearance of $\frac{1}{K}$ can be dealt with by considering the differential equation on λ in equation (6.19). Multiplying the whole equation by a^3, it can be recast in the form

$$\frac{d}{dt}\left(a^3\dot{\lambda}\right) = aK\lambda. \tag{6.45}$$

This reveals to us that

$$\dot{\lambda}(t) = \frac{K}{a(t)^3}\int^t dt' a(t')\lambda(t') + \frac{C}{a(t)^3}, \tag{6.46}$$

where C is an integration constant. When we plug this back into the scalar mode (6.19) we find

$$\Phi = \Psi = \frac{1 - \frac{1}{4}K\mathbf{x}^2}{1 + \frac{1}{4}K\mathbf{x}^2}\left(\lambda - \frac{H}{a}\int^t dt' a(t')\lambda(t') - \frac{CH}{aK}\right), \tag{6.47}$$

where dependence on t has been suppressed for convenience.

Now we assume $\lambda(t) = \lambda_0(t) + \mathcal{O}(K)$ where $\lim_{K\to 0}\lambda = \lambda_0$ [1]. Plugging this into the differential equation for λ and taking $K \to 0$ then gives the solution

$$\lambda_0(t) = c_1 + c_2 \int^t dt' \frac{1}{a(t')^3}. \tag{6.48}$$

We must note now that the introduced integration constant c_2 is not new. When we match

$$\lim_{K\to 0} \dot{\lambda}(t) = \dot{\lambda}_0(t), \tag{6.49}$$

it is revealed that

$$c_2 = \lim_{K\to 0} C. \tag{6.50}$$

Therefore, we will write $C = c_2 + Kc_3 + \mathcal{O}(K^2)$, where both constants are of order $\mathcal{O}(K^0)$. Now, we are able to write down the general solution

$$\begin{aligned}\Phi = \Psi =& c_1\left(1 - \frac{H}{a}\int^t dt' a(t')\right)\\ &+ c_2\left(\int^t dt' \frac{1}{a(t')^3} - \frac{H}{a}\int^t dt' a(t') \int^{t'} dt'' \frac{1}{a(t'')^3}\right)\\ &- \frac{1}{K}\frac{c_2 H}{a} - \frac{c_3 H}{a} + \mathcal{O}(K).\end{aligned} \tag{6.51}$$

We now observe that, in order for the $K \to 0$ limit to exist for the Newtonian potentials, it is required that $c_2 = 0$. This gives us the final solution,

$$\lim_{K\to 0}\Phi = \lim_{K\to 0}\Psi = c_1\left(1 - \frac{H}{a}\int^t dt' a(t')\right) - c_3\frac{H}{a}. \tag{6.52}$$

Now we can also easily find the other scalar perturbations in the flat-space limit,

$$\lim_{K\to 0}\frac{\delta\rho}{\dot{\bar{\rho}}} = \lim_{K\to 0}\frac{\delta p}{\dot{\bar{p}}} = -\lim_{K\to 0}\delta u^S = \frac{c_1}{a}\int^t dt' a(t') + \frac{c_3}{a} \tag{6.53}$$

$$\lim_{K\to 0}\mathcal{R} = -c_1 \tag{6.54}$$

We recognize that the first solution, multiplied by c_1, is the flat-space Weinberg's first scalar adiabatic mode. Indeed, it arises from $\lambda = \text{constant} = -\mathcal{R}$, as we would expect. Similarly, the second solution, multiplied by c_3, is Weinberg's second scalar adiabatic mode. When all constants except for c_3 are zero, we have $\lambda = \epsilon^i = 0$ while $\epsilon_0 = c_3/a$. Note that, even though $\lambda = 0$ in the $K \to 0$

[1] Requiring the convergence of matter fields as $K \to 0$, it can be shown that there are no solutions for λ with poles in K.

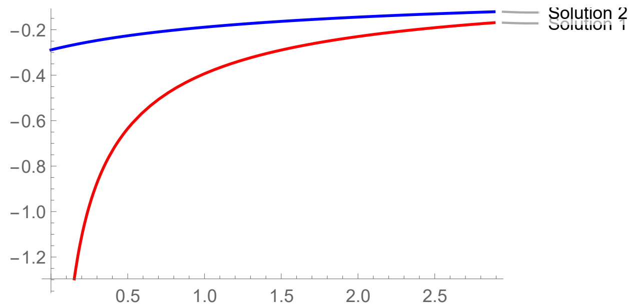


Figure 6.1: Adiabatic scalar mode in radiation dominatione: $\lambda(t)$.

limit, that does not mean that the diffeomorphism generated by c_3 is an empty one. This is because ϵ_0 contains $\frac{1}{K}$, making terms that are linear in K relevant in the $K \to 0$ limit.

Thus, in the limit $K \to 0$, we obtain exactly the same scalar adiabatic modes as for $K = 0$. In combination with the validation of the Einstein equations in Section 6.2, this gives us some confidence that our treatment of the scalar adiabatic mode has been correct.

6.4 Example: Radiation Dominated

It is possible to solve the differential equation for $\lambda(t)$ in (6.19), which governs the time dependence of the scalar adiabatic mode, exactly using the Mathematica software package [20] in some specific cases. In this section we focus on finding the scalar mode for a radiation-dominated universe, i.e. a universe where $p = \frac{1}{3}\rho$. This is for the purpose of providing an explicit example of how the time dependence of the scalar can look and what kind of solutions the differential equation for $\lambda(t)$ has. The results are reported in this section, the Mathematica notebook in which the calculations are actually performed is provided in Appendix F. In this notebook, it is also explicitly verified that the mode solves the linearized scalar Einstein equations in a radiation dominated universe, providing an extra consistency check.

We find two solutions for $\lambda(t)$. The general solution is a superposition of the two, giving

$$\lambda(t) = \frac{C_1}{\sqrt{t\left(-3KM_pt + 2\sqrt{3\rho_0}\right)}} + \frac{C_2 \arctan\left(\sqrt{\frac{-3KM_p}{t}}\left(-3KM_pt + 2\sqrt{3\rho_0}\right)\right)}{\sqrt{-3KM_pt\left(-3KM_pt + 2\sqrt{3\rho_0}\right)}}, \tag{6.55}$$

where ρ_0 is the unperturbed energy density in the universe at t_0 (which is defined by $a(t_0) = 1$ and can be taken to be 'today') and C_1 and C_2 are dimensionless integration constants. $\Phi = \Psi$ is then obtained from (6.19). The expression becomes a bit involved so we will not write it here. A (qualitative) plot of the mode however is provided by figure 6.2.

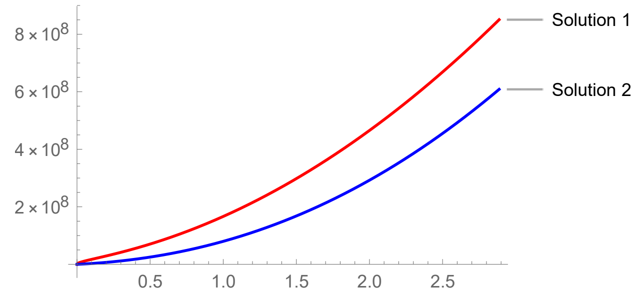


Figure 6.2: Adiabatic scalar mode in radiation domination: $\Phi(t)$.

6.5 Scalar Modes are Unphysical

We have seen in Section 6.2 that the generalized versions of Weinberg's first and second adiabatic modes satisfy

$$k^2 = 3K = -3|K|, \tag{6.56}$$

where k^2 is minus the Laplacian eigenvalue of all the scalar perturbations. This eigenvalue corresponds to an unphysical mode, which should be clear from the fact that the corresponding eigenfunction

$$\frac{1 - \frac{1}{4}K\mathbf{x}^2}{1 + \frac{1}{4}K\mathbf{x}^2} \tag{6.57}$$

diverges at infinity (remember that, in the coordinates we use, 'infinity' is at $\mathbf{x}^2 = -4/K$, see equation (D.15) and the discussion preceding it). Of course, this does not immediately pose a problem. The scalar mode is just a large gauge transformation, and these are usually unphysical. Since the mode satisfies all physicality conditions, they can be extended to physical adiabatic modes by giving them an appropriate falloff behavior outside of the observable universe (as illustrated by figure 3.1).

There is however a problem with this scalar mode that is not present in the flat-space case (i.e. for Weinberg's scalar adiabatic modes). It is discussed and shown in [23] that in an open universe, only sub-curvature modes are needed to expand any physical perturbation (as was also mentioned in Section 2.2). These modes vary significantly within the curvature scale $1/\sqrt{|K|}$ and have $k^2 > |K|$. In contrast, super-curvature modes are more or less constant all through the curvature scale and fall off outside of it. These are functions for which $0 < k^2 < |K|$, and while they are not linearly independent from the sub-curvature modes, they are needed to generate the most general Gaussian random field. For $k^2 = 0$, the eigenfunction is constant throughout space (just like for flat space). Yet, eigenfunctions with $k^2 < 0$ are a different story: these are modes which diverge at infinity, just like a Laplacian eigenfunction in flat space $\exp(i\mathbf{k} \cdot \mathbf{x})$ increases exponentially for imaginary $\mathbf{k}$.

The distinction between sub-curvature and super-curvature modes, which is completely absent in a flat universe, is a little confusing. We have discussed in

Section 2.2 that the different Laplacian eigenfunction components of perturbations decouple. Yet this does not mean that super-curvature modes decouple from the sub-curvature modes. After all, we can expand each super-curvature mode in terms of sub-curvature modes. Thus, to describe any perturbation, we only need to specify the time-dependence of every eigenfunction coefficient with $k^2 > |K|$. Therefore, we will think of a 'monochromatic' perturbation as a perturbation that is in a pure sub-curvature mode. A pure super-curvature mode is then not monochromatic in this sense, since it composed of different sub-curvature modes.

Thus, the largest monochromatic perturbation that we could ever observe in an open universe has $k^2 \to |K|^+$. This corresponds to how, in flat space, the largest monochromatic mode has $k^2 \to 0^+$. In flat space, a $k^2 = 0$ mode corresponds to a pure gauge mode and thus these largest modes can be treated as adiabatic modes: the closer we take k^2 to zero, the better the mode is approximated by a coordinate transformation. Yet, in flat space, the Weinberg scalar adiabatic mode analogue does not have $k^2 = |K|$ and can thus not be approximated by such a monochromatic mode. Since the scalar mode sits at $k^2 = -3|K|$, there is a gap between even the largest monochromatic mode and the scalar mode. Thus, there exists no limit in which monochromatic physical modes can be approximated with arbitrary accuracy by the gauge mode described in this chapter.

This does, of course, not mean that the gauge mode cannot be approximated by some physical mode. That is, there are still adiabatic modes. These can be created trivially by having the perturbation behave spatially exactly the same as the gauge mode up to some coordinate distance $|\mathbf{x}|_{\max}$ from the origin, and having it go to zero beyond there [2]. The approximation will be especially good if $|\mathbf{x}|_{\max} \gg 1/H$[3](using $a = a_0 = 1$), such that deviations will only appear beyond the observable universe. Within the observable universe, the time dependence of the perturbation as seen within the Hubble radius will then be approximately solved by the one dictated in this chapter (equation (6.19)) (although this time dependence does not need to be the only solution of the linearized Einstein equations for a perturbation with this spatial profile). A perturbation with this specific spatial dependence within the Hubble radius and this time dependence is then the adiabatic mode.

Yet, the fact that such a mode will not be monochromatic makes it not as useful as its flat-space analogue. After all, it are monochromatic modes which decouple and which can thus be 'isolated' easily from others. A monochromatic mode on a perturbed background (i.e. when other Laplacian eigenfunctions are also 'excited') will evolve the same as a monochromatic mode on an unperturbed

[2]This might cause some of the derivatives of the mode to be discontinuous. If one considers this to be problematic, there are smoother ways of making the gauge mode physical, like multiplying it by a factor $\exp(-(x/x_{max})^2)$.

[3]More accurately, $|\mathbf{x}|_{\max} \gg l_H$ where l_H is the Hubble radius in our specific coordinates. This distance can be found by equating comoving distance from the origin to the physical Hubble radius, $\int_0^{l_H} a_0 d\sigma = 1/H$. Using $1/\sqrt{|K|} \gtrsim 141/H$ (which is valid nowadays), it is found that the effect of curvature in the Hubble radius is so small that $l_H \approx 1/H$, and thus we use this value as not to overcomplicate the main text.

background. Since adiabatic modes are shown to be solutions by performing a diffeomorphism starting with an unperturbed universe, only monochromatic adiabatic modes are certain to be a solution under any circumstances. The fact that the adiabatic scalar mode seems to locally satisfy $k^2 = 3|K|$ may mean that it also decouples 'locally' (since any physics within the Hubble radius will not be aware of what is going on far beyond at $x_{\max}$). It is however still unclear what kind of conclusions we can draw.

The question of the status of the scalar adiabatic mode is especially interesting with respect to single-field inflation. We have explained in Section 3.1 that, in a flat universe, inflation can only produce adiabatic modes because all long modes become adiabatic. Due to the non-monochromatic nature of scalar adiabatic modes in an open universe it is unclear whether similar arguments can be made. Perhaps it is possible to expand perturbations in terms of the spatial profiles of the adiabatic modes (with different $x_{\max}$) and the 'remaining' sub-curvature modes (since the adiabatic modes are just a linear combination of sub-curvature modes which are the 'basis vectors', one can imagine doing a 'change of basis'). If, then, there is indeed some form of local decoupling, it might be argued that the coefficients corresponding to adiabatic modes in the expansion have the same time dependence as the modes found in this chapter (since single-field inflation only allows for two solutions).

Something else which one needs to consider are soft theorems. While Maldacena's consistency condition [24], as presented in Section 3.4, depends on the adiabatic nature of $k \to 0$ modes, we cannot do something similar in an open universe. After all, we cannot take the limit $k^2 \to -3|K|$ for physical modes. This implies that in an open universe there are corrections. The consistency conditions carries three length scales: λ_{long} is the wavelength of the $k \to 0$ mode, λ_{short} is the wavelength of the other modes, and $a/\sqrt{K}$ is the (physical) curvature scale. While the Maldacena consistency condition is only valid up to corrections of order $\lambda_{\text{short}}/\lambda_{\text{long}}$ (or the same quantity squared, if one also includes the corrections due to the constant gradient (see Section 3.4) [12, eq. (54)], open-universe corrections can be either of order $\sqrt{K}\lambda_{\text{long}}/a$ or $\sqrt{K}\lambda_{\text{short}}/a$.

Since λ_{long} should be at least the Hubble radius to be considered adiabatic, and since the curvature length is at least $\sim 10^3$ times as long, $\sqrt{K}\lambda_{\text{long}}/a$ must be smaller or equal to 10^{-3}. While these corrections might be of the same order as $\lambda_{\text{short}}/\lambda_{\text{long}}$ (for sufficiently large λ_{short}), corrections of order $\sqrt{K}\lambda_{\text{short}}/a$ will be much smaller than those of order $\lambda_{\text{short}}/\lambda_{\text{long}}$ already present. It remains for future research to find what kind of corrections appear. If the result would be that corrections are of order $\sqrt{K}\lambda_{\text{long}}/a$ only, violation of Maldacena's consistency condition might indicate that the universe has nonzero curvature (or it could mean that no single-field inflation has occurred). Otherwise, we cannot expect deviation due to curvature to be detectable.

One might think that there is perhaps another scalar adiabatic mode with a more convenient Laplacian eigenvalue. However, it can be shown that every scalar adiabatic mode has the same eigenvalue $k^2 = -3|K|$. The argument is as follows. Consider the physicality condition (4.53). Using equation (6.36), we can rewrite $3H$ times this as

$$3H\dot{\Psi} + 3H^2\Psi = \frac{\dot{\bar{\rho}}}{2M_p^2}\delta u^S. \tag{6.58}$$

When we add this equation to the first linearized Einstein equation in (E.22), we get

$$\frac{1}{a^2}\left(3K - k^2\right)\Psi = \frac{\dot{\bar{\rho}}}{2M_p^2}\left(\frac{\delta\rho}{\dot{\bar{\rho}}} + \delta u^S\right). \tag{6.59}$$

For an adiabatic mode, $\delta\rho/\dot{\bar{\rho}} = -\epsilon^0$ while $\delta u^S = -\epsilon^0$ (see equations (4.43)) and thus, the right-hand side equals zero. The conclusion can then only be that $k^2 = 3K = -3|K|$.

For now, we must conclude that it is far from certain whether the adiabatic modes we have found have any physical implications. It is, in fact, quite interesting if they don't; for if single-field inflation does not generate an adiabatic universe, any observation of non-adiabaticity might indicate that we do not live in a flat universe, and the Maldacena consistency condition might be violated. Of course, it could also indicate that single-field inflation never happened. A proper analysis of the consequences of curvature for single-field inflation might provide some qualitative predictions for the adiabaticity, which could be compared to the predictions by non-single-field scenarios, perhaps providing a handhold to differentiate them.

Chapter 7

Conclusion

In this thesis, we have investigated the existence of adiabatic modes in an open universe (i.e. a universe with curvature scalar $K < 0$). To this end, we have first reviewed cosmological perturbation theory and the theory of adiabatic modes in a flat universe in Part I. In Part II, we started by examining the general conditions which a coordinate transformation must fulfill in order to provide an adiabatic mode. This machinery has been used to find a tensor adiabatic mode and a scalar adiabatic mode, which are the generalizations of Weinberg's tensor adiabatic mode and Weinberg's scalar adiabatic modes [34, sec. 5.4] respectively. These are the main results of this thesis, and can be found in equations (5.23) and (6.19).

These adiabatic modes are found to differ from their flat-space counterparts in a few significant ways. For instance, both of them are not constant in space. This has the significant implication that their eigenvalue for the Laplacian operator $-k^2$ is nonzero. Calculating these eigenvalues gives $k^2 = 2|K|$ for the tensor and $k^2 = -3|K|$ for the scalar. This implies that the tensor mode is a physical sub-curvature mode (defined by $k^2 > |K|$) while the scalar mode cannot be obtained as (the limit of) a physical monochromatic mode. Since consistency conditions are usually relations between correlation functions of monochromatic perturbations, in which one of the perturbations in the larger correlation function is an adiabatic mode, we can conclude the following:

- In an open universe, there are soft theorems in which a tensor mode becomes 'soft', which is then to mean $k^2 = 2|K|$.
- In an open universe, no soft theorems in which a scalar becomes soft exists.
- In a flat universe, double-soft theorems (in which one scalar is very long and acts as an effective spatial curvature and one intermediately long mode is adiabatic against this background [25]) only exist when the intermediately long mode is a tensor mode.

There are also consequences for the generation of adiabaticity by single-field inflation. In a flat universe all the monochromatic perturbations for which the physical wavelength is longer than the Hubble radius at the end of inflation must be adiabatic. Yet, such $k^2 \to 0$ modes are no longer adiabatic in an open universe (which holds both in the scalar and the tensor sector). It is thus unclear to what extent single-field inflation would result in an adiabatic universe. While tensor modes will in general become non-adiabatic when their 'wavelength' becomes *too* long, there is no k^2 at all for which scalar perturbations are guaranteed to be adiabatic.

The obtained results are intriguing and call for further investigation. Some of the open questions that require answering are:

- Does the time-independent adiabatic tensor mode also exist non-perturbatively in $K\mathbf{x}^2$? While the results in Section 5.3 seem to suggest so, it has only been verified explicitly up to second order in $K\mathbf{x}^2$. There are reasons to expect that the gauge-preservation condition $\bar{\nabla}_i \gamma^i{}_j$ and the $\Psi = 0$ condition for time-independent modes are not independent. Understanding this relation might provide the insight necessary to construct the tensor non-perturbatively or to prove that it always exists. Alternatively, an explicit construction for the n^{th} order part of the tensor (mimicking the procedure in Chapter 5 in a way that does not depend on the order of the parameters considered) might settle the matter. Until then, the existence of the adiabatic time-independent tensor mode beyond second order in $K\mathbf{x}^2$ should be considered tentative.

- Do time-dependent tensor adiabatic modes exist? Such a mode would perhaps have a more 'convenient' Laplacian eigenvalue which is manifestly unphysical (see next bullet point). Yet, if the $\Psi = 0$ equation for time-independent tensor modes is indeed implied by the gauge-preservation condition, this might make $\epsilon^0 = 0$ the only consistent choice, which would mean that any pure tensor mode is time independent.

- If the time-independent tensor mode exists, how should we interpret the fact that it is sub-curvature (i.e. 'finite momentum')? The sub-curvature nature of the adiabatic mode is a puzzling feature as physical modes should not result from gauge transformations. We might have to change our notion of 'large gauge transformations' in the case of nonzero curvature. Alternatively, we might have to reconsider whether pure sub-curvature modes are 'physical'; in flat space, only superpositions of monochromatic waves vanish at infinity and can thus truly be considered physical. Yet, the construction of sub-curvature modes in [23, sec. 3] seems to suggest that they fall off at infinity (unlike their flat-space counterparts), and thus it seems not very likely that we can consider the tensor gauge mode to be unphysical.

- What are the soft theorems for tensors in an open universe? Usually, in soft theorems, the limit $k^2 \to 0$ is taken. Yet, for a tensor mode in an open universe, this seems to be the wrong limit: it only becomes adiabatic when $k^2 = 2|K|$. Calculation of soft theorems and checking to what extent they differ from the flat-space ones would be interesting.

- What are the corrections to scalar soft theorems (and in particular, Maldacena's consistency condition) in an open universe? One could try to calculate this explicitly by examining the deviation from adiabaticity of

$k^2 \to 0$ modes within the Hubble radius. Similarly, corrections to double-soft theorems could be calculated.

- What are the consequences of these results for single-field inflation? To what extent can we expect single-field inflation to produce adiabatic modes in an open universe? Finding a way to expand an arbitrary perturbation in terms of adiabatic modes might help doing such an analysis. If not even single-field inflation generates purely adiabatic modes in an open universe, we cannot expect the open universe to look perfectly adiabatic.

Quantitative results with respect to the last three questions would be extremely interesting, as they are about measurable effects. Potentially, they could provide a method for probing the curvature of the universe. A problem is that such effects would generally be hard to discern from effects that might be caused by the non-existence of single-field inflation. Both single-field inflation in an open universe and multi-field inflation in a flat universe may generate non-adiabaticity, and similarly both scenarios could lead to a violation of established consistency conditions. Both deviation from adiabaticity and violation of consistency conditions would still be very significant observations though, as it would disprove at least one popular theory about the universe.

As a final note, we would like to mention that the analysis in this thesis could probably be extended to closed universes ($K > 0$) without too much trouble. One should however take into account that the topology of the universe would change drastically, imposing periodicity conditions on all solutions. The eigenvalues of the Laplacian would become discretized, potentially making it hard to take any limit in which a physical solution becomes similar to a gauge mode. This might mean that adiabatic modes do not exist, or it would at least complicate their formulation. For the rest, the search for gauge modes should certainly proceed along the same lines as presented in Part II of this thesis.

Part III

Appendices

Appendix A

Conventions

Einstein summation is assumed throughout this thesis (i.e. any repeated index is summed over). While Greek indices run over all spacetime dimensions, i.e. $0, 1, 2, 3$ (where 0 corresponds to the temporal direction), latin indices only run over spatial dimensions, i.e. $1, 2, 3$. A $(-+++)$ metric signature is used (i.e. a timelike interval is characterized by negative ds^2, where ds is the interval as measured by the metric).

In this thesis, we are dealing with a background metric (the FLRW metric, see Section 1.4) of the form

$$ds^2 = -dt^2 + a(t)^2 d\sigma^2, \tag{A.1}$$

where ds^2 is the metric of the universe and $d\sigma^2$ is the metric on the space Σ (which has three spacelike dimensions). We use the convention of considering any quantity which has latin indices (which run over $1, 2, 3$ and which will appear when doing perturbation theory, see Section 2.1) to live in Σ rather than on the spatial part of the whole space (which is '$a^2\Sigma$'). This means that the indices are raised and lowered with the metric of Σ, which is denoted $d\sigma^2 = \tilde{g}_{ij} dx^i dx^j$. Because we are dealing with two separate spaces, we need to differentiate between the covariant derivatives and Christoffel symbols of the two. We use the convention that

$$\nabla_\mu,\ \Gamma^\mu_{\nu\rho} \tag{A.2}$$

correspond to the full space (i.e. the universe) while

$$\bar{\nabla}_\mu,\ \bar{\Gamma}^i_{jk} \tag{A.3}$$

correspond to Σ. Because it turns out in Appendix C.3 that $\bar{\Gamma}^i_{jk} = \Gamma^i_{jk}$, we often use the second rather than the first for simplicity even when dealing with Σ. The same does not hold for the covariant derivatives.

Symmetrization throughout this thesis is defined by

$$T_{(\mu_1 \ldots \mu_n)} = \frac{1}{n!} \sum_{(k_1, \ldots, k_n) \in P_n} T_{\mu_{k_1} \ldots \mu_{k_n}}, \tag{A.4}$$

where P_n is the set of all $n!$ permutations of the numbers 1 through n (e.g. $P_2 = \{(1,2),(2,1)\}$). In our definition, it does not matter whether indices are upstairs or downstairs, e.g.

$$\partial_{(i}\epsilon^{j)} = \frac{1}{2!}\left(\partial_i\epsilon^j + \partial_j\epsilon^i\right). \tag{A.5}$$

Furthermore, when indices are enclosed by two $|$'s, this means that they must not be included in the symmetrization. For example,

$$T_{(\mu_1|\mu_2|\mu_3)} = \frac{1}{2!}\left(T_{\mu_1\mu_2\mu_3} + T_{\mu_3\mu_2\mu_3}\right). \tag{A.6}$$

The symmetric-traceless part of a two-tensor is defined by

$$T_{<\mu\nu>} = T_{(\mu\nu)} - \frac{1}{3}\delta_{\mu\nu}T_{\lambda\lambda}, \tag{A.7}$$

such that $T_{<\mu\nu>} = T_{<\nu\mu>}$ and $T_{<\mu\mu>} = 0$.

Throughout the thesis, natural units are used, i.e. units such that $c = \hbar = 1$. Only the Planck mass M_p is retained, and thus every dimensional quantity is a multiple of some power of M_p (at least, for the quantities considered in this thesis, where e.g. electric charge is not considered; otherwise, one might also put the electron charge to -1). Thus, dimensional analysis can be used to check equations for consistency.

Appendix B

Transformation of the Metric

In this appendix, the basic transformation rules of the metric under infinitesimal diffeomorphisms. This is used subsequently to derive how metric perturbations transform under a gauge transformation (where the 'background metric' is defined to remain invariant while the infinitesimal perturbations, of the same order as the diffeomorphism, do transform).

The metric, i.e. the line element, is a property of the space it describes and thus does not change when we choose new coordinates. Thus, when we perform the infinitesimal diffeomorphism

$$x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu(x), \tag{B.1}$$

the metric tensor transforms as $g_{\mu\nu}(x) \to \tilde{g}(\tilde{x})_{\mu\nu}$ where

$$\begin{aligned} ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu =& \tilde{g}_{\mu\nu}(\tilde{x})d\tilde{x}^\mu d\tilde{x}^\nu \\ =& \, g_{\rho\sigma}(x)\frac{dx^\rho}{d\tilde{x}^\mu}\frac{dx^\sigma}{d\tilde{x}^\nu}d\tilde{x}^\mu d\tilde{x}^\nu \\ =& g_{\rho\sigma}(\tilde{x}-\epsilon)\frac{d\tilde{x}^\rho - d\epsilon^\rho}{d\tilde{x}^\mu}\frac{d\tilde{x}^\sigma - d\epsilon^\sigma}{d\tilde{x}^\nu}d\tilde{x}^\mu d\tilde{x}^\nu \end{aligned} \tag{B.2}$$

Comparing these lines, using the usual notation $\partial_\mu = \frac{d}{dx^\mu}$, renaming $\tilde{x}$ to x and Taylor expanding to first order in ϵ gives

$$\begin{aligned} \tilde{g}_{\mu\nu}(x) =& (\delta^\rho_\mu - \partial_\mu\epsilon^\rho)(\delta^\sigma_\nu - \partial_\nu\epsilon^\sigma)(1-\epsilon^\lambda\partial_\lambda)g_{\rho\sigma}(x) \\ =& g_{\mu\nu}(x) - g_{\mu\lambda}\partial_n u\epsilon^\lambda - g_{\nu\lambda}\partial_\mu\epsilon^\lambda - \epsilon^\lambda\partial_\lambda g_{\mu\nu} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{B.3}$$

Note that this result can be rewritten as

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) - 2\nabla_{(\mu}\epsilon_{\nu)}, \tag{B.4}$$

where $2\nabla_{(\mu}\epsilon_{\nu)} = \mathcal{L}_\epsilon g_{\mu\nu}$ is the Lie derivative of the metric, as in accordance with e.g. [10, eq. B.21]. This can be seen by computing

$$\begin{aligned}
2\nabla_{(\mu}\epsilon_{\nu)} &= 2\partial_{(\mu}\epsilon_{\nu)} - 2\Gamma^{\lambda}_{(\mu\nu)}\epsilon_{\lambda} \\
&= 2\partial_{(\mu}\epsilon_{\nu)} - g^{\lambda\rho}\left(\partial_{\mu}g_{\rho\nu} + \partial_{\nu}g_{\rho\mu)} - \partial_{\rho}g_{\mu\nu}\right)g^{\lambda\rho}\epsilon_{\lambda} \\
&= \partial_{\mu}[g_{\lambda\nu}\epsilon^{\lambda}] - \epsilon^{\lambda}[\partial_{\mu}g_{\lambda\nu}]\partial_{\nu}[g_{\lambda\mu}\epsilon^{\lambda}] - \epsilon^{\lambda}[\partial_{\nu}g_{\lambda\mu}] + \epsilon^{\lambda}\partial_{\lambda}g_{\mu\nu} \\
&= g_{\lambda\nu}\partial_{\mu}\epsilon^{\lambda} + g_{\lambda\mu}\partial_{\nu}\epsilon^{\lambda} + \epsilon^{\lambda}\partial_{\lambda}g_{\mu\nu},
\end{aligned} \tag{B.5}$$

where the definition of the covariant derivative and Christoffel symbols have been used and the product rule for differentiation.

We now consider a perturbed metric, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the background metric and $h_{\mu\nu}$ is an infinitesimal perturbation to it. As discussed in Section 2.3, requiring the non-infinitesimal part of the metric to be of a specific form (i.e. to be $\bar{g}_{\mu\nu}$) defines the coordinates up to infinitesimal coordinate transformations. Only when ϵ_{μ} and $h_{\mu\nu}$ are of the same order, can we do a coordinate transformation where the non-infinitesimal background metric remains the same. Then, all of the change in the metric can be 'blamed' on the perturbations. Plugging the decomposition of the metric into equation (B.3), and remembering that we should not keep $\mathcal{O}(\epsilon^2)$, $\mathcal{O}(h^2)$ and $\mathcal{O}(\epsilon h)$), we get

$$\begin{aligned}
\bar{g}_{\mu\nu} + \tilde{h}_{\mu\nu} =& (\delta^{\rho}_{\mu} - \partial_{\mu}\epsilon^{\rho})(\delta^{\sigma}_{\nu} - \partial_{\nu}\epsilon^{\sigma})(1 - \epsilon^{\lambda}\partial_{\lambda})\left(\bar{g}_{\rho\sigma} + h_{\rho\sigma}\right) \\
=& \bar{g}_{\mu\nu} + h_{\mu\nu} - \epsilon^{\lambda}\partial_{\lambda}\bar{g}_{\mu\nu} - 2g_{\lambda(\mu}\partial_{\nu)}\epsilon^{\lambda}.
\end{aligned} \tag{B.6}$$

When we write $\tilde{h}_{\mu\nu} = h_{\mu\nu} + \Delta h_{\mu\nu}$, our final transformation rule for metric perturbations is

$$\Delta h_{\mu\nu} = -\epsilon^{\lambda}\partial_{\lambda}\bar{g}_{\mu\nu} - \bar{g}_{\lambda\mu}\partial_{\nu}\epsilon^{\lambda} - \bar{g}_{\lambda\nu}\partial_{\mu}\epsilon^{\lambda}. \tag{B.7}$$

Appendix C

Properties of the Background Metric

As derived in Section 1.4, the unique homogeneous and isotropic background metric is the FLRW metric

$$ds^2 \equiv g_{\mu\nu}(t,x)dx^\mu dx^\nu = -dt^2 + a(t)^2 d\sigma^2, \tag{C.1}$$

where

$$d\sigma^2 \equiv \tilde{g}_{ij}(K\mathbf{x}^2)dx^i dx^j = f(K\mathbf{x}^2)^2 \delta_{ij} dx^i dx^j \tag{C.2}$$

defines the spatial subspaces which foliate the universe's spacetime. The space on which we can consider $d\sigma^2$ to be the metric is called Σ, which has curvature constant K. The function f is defined by

$$f(K\mathbf{x}^2) = \frac{1}{1 + \frac{1}{4}K\mathbf{x}^2}. \tag{C.3}$$

Note that the space is bounded by $|\mathbf{x}| < \frac{2}{\sqrt{-K}}$.

Since the space of interest in this thesis is often Σ instead of the full spacetime specified by ds^2, one should differentiate between objects defined both. While it should be clear from the definitions of tensors in which of the two spaces they live, different symbols are used for the covariant derivatives and their Christoffel symbols. The covariant derivative on the full FLRW spacetime is here defined by

$$\begin{aligned}\nabla_\sigma T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} =& \partial_\sigma T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} \\ &+ \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\ldots\mu_k}_{\nu_1\ldots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} + \ldots \\ &- \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2\ldots\mu_k}_{\lambda\nu_2\ldots\nu_l} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\lambda\ldots\nu_l} - \ldots ,\end{aligned} \tag{C.4}$$

where $T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l}$ is a four-dimensional $k+l$-tensor (transforming covariantly under diffeomorphisms $x^\mu \to x'^\mu$) and $\Gamma^\rho_{\mu\nu}$ are the Christoffel symbols of the FLRW space. The covariant derivative on the subspaces is defined similarly by

$$\begin{aligned}\bar{\nabla}_s T^{i_1 i_2\ldots i_k}_{j_1 j_2\ldots j_l} =& \partial_s T^{i_1 i_2\ldots i_k}_{j_1 j_2\ldots j_l} \\ &+ \bar{\Gamma}^{i_1}_{sp} T^{p i_2\ldots i_k}_{j_1\ldots j_l} + \bar{\Gamma}^{i_2}_{sp} T^{i_1 p\ldots i_k}_{j_1 j_2\ldots j_l} + \ldots \\ &- \bar{\Gamma}^{p}_{s j_1} T^{i_1 i_2\ldots i_k}_{p j_2\ldots j_l} - \bar{\Gamma}^{p}_{s j_2} T^{i_1 i_2\ldots i_k}_{j_1 p\ldots j_l} - \ldots ,\end{aligned} \tag{C.5}$$

where now $T^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_l}$ is a three-dimensional $k+l$-tensor (transforming covariantly under diffeomorphisms $x^i \to x'^i$) and $\bar{\Gamma}^k_{ij}$ are the Christoffel symbols of the subspaces. The Christoffel symbols are defined and calculated in Appendix C.3. For a discussion of covariant derivatives and a derivation of the form of the Christoffel symbols, see e.g. [10, sec. 3.2].

In this appendix, some of the properties of this metric are examined. The main purpose of this is to reduce the number of calculations in the main body of the text. Also, since these properties are of import to most parts of this thesis, it is convenient to have them collected here.

C.1 Raising and Lowering Indices

In this appendix it is derived how the indices on tensors are lowered and raised, both for tensors living in the full FLRW spacetime or in the spatial subspaces.

The full metric can be written as

$$ds^2 = -dt^2 + a^2 f^2 d\mathbf{x}^2. \tag{C.6}$$

Thus, the indices of tensors living in the full spacetime are raised and lowered quite easily,

$$\begin{aligned} T_0 =& g_{0\mu} T^\mu = -T^0, && \text{(C.7)} \\ T_i =& g_{i\mu} T^\mu = a^2 f^2 T^i. && \text{(C.8)} \end{aligned}$$

A tensor defined on the space Σ with metric

$$d\sigma^2 = f^2 d\mathbf{x}^2 \tag{C.9}$$

is raised and lowered by

$$S_i = \tilde{g}_{ij} S^j = f^2 S^i \tag{C.10}$$

Note that the action would be the same on any other index that both the tensors S and T might have. For simplicity, only one index is used here.

C.2 Derivatives of f and Metric

Since derivatives of the function $f(K\mathbf{x}^2)$ will appear all to often in this thesis, they are calculated in this section for easy reference. First, it is convenient to define

$$f'(K\mathbf{x}^2) \equiv \frac{df(K\mathbf{x}^2)}{d(K\mathbf{x}^2)} = -\frac{1}{4} f(K\mathbf{x}^2)^2. \tag{C.11}$$

This gives

$$\begin{aligned} \partial_i f =& f' \partial_i (K\mathbf{x}^2) = K f' \delta_{jk} \partial_i (x^j x^k) \\ =& K f' \delta_{jk} (x^j \delta^k_i + x^k \delta^j_i) = 2K f' x^i \\ =& -\frac{1}{2} K f^2 x^i. \end{aligned} \tag{C.12}$$

From this result we find easily that

$$\partial_i f^2 = 2f\partial_i f = -Kf^3 x^i. \tag{C.13}$$

This in turn can be used to calculate the spatial derivatives of the (full) metric,

$$\partial_i g_{jk} = \partial_i f^2 (a^2\delta_{jk}) = -Kf^3 x^i a^2 \delta_{jk} = -Kx^i f g_{jk}. \tag{C.14}$$

Since a^2 factors out above, the same holds for the metric of the spatial subspaces,

$$\partial_i \tilde{g}_{jk} = -Kx^i f \tilde{g}_{jk}. \tag{C.15}$$

Temporal derivatives are found in a similar fashion,

$$\partial_t g_{ij} = 2a\dot{a}f^2\delta_{ij} = 2Hg_{ij}. \tag{C.16}$$

A last useful derivative identity is the following:

$$\frac{1}{f^n}\partial_i\left(f^n\mathcal{O}\right) = \left(\partial_i - \frac{n}{2}Kfx^i\right)\mathcal{O}. \tag{C.17}$$

Here, $\mathcal{O}$ can be anything. Its validity is easily proven using the product rule and $\partial_i f^n = nf^{n-1}\partial_i f = -\frac{n}{2}Kf^{n+1}x^i$.

C.3 Christoffel Symbols

In this appendix, the Christoffel symbols of the full metric ($\Gamma^\rho_{\mu\nu}$) and of the subspaces ($\bar{\Gamma}^k_{ij}$) are calculated for easy reference. Christoffel symbols are defined by [10, eq. 3.27]

$$\Gamma^\rho_{\mu\nu} = g^{\rho\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right). \tag{C.18}$$

Note that these are always symmetric in the two lower indices.

Let's start with the Christoffel symbols of the FLRW spacetime. Those with only spatial indices yield

$$\begin{aligned}
\Gamma^k_{ij} =& \frac{1}{2}g^{k\lambda}\left(\partial_i g_{j\lambda} + \partial_j g_{i\lambda} - \partial_\lambda g_{ij}\right)\\
=& -\frac{1}{2}Kfg^{k\lambda}\left(x^i g_{j\lambda} + x^j g_{i\lambda} - x^\lambda g_{ij}\right)\\
=& -\frac{1}{2}Kf\left(x^i\delta^k{}_j + x^j\delta^k{}_i - x^l(a^{-2}f^{-2}\delta^{kl})(a^2f^2\delta_{ij}\right)\\
=& \frac{1}{2}Kf\left(x^k\delta_{ij} - x^i\delta_{jk} - x^j\delta_{ik}\right)
\end{aligned} \tag{C.19}$$

(where use has been made of $g_{\mu\rho}g^{\rho\nu} = \delta^\rho_\mu$). Furthermore, we can calculate

$$\begin{aligned}
\Gamma^0_{ij} &= \frac{1}{2}g^{0\lambda}\left(\partial_i g_{j\lambda} + \partial_j g_{i\lambda} - \partial_\lambda g_{ij}\right)\\
&= -\frac{1}{2}\left(\partial_i \cancelto{0}{g_{0j}} + \partial_j \cancelto{0}{g_{0i}} - \delta_{ij}\partial_0(a^2f^2)\right)\\
&= \dot{a}af^2\delta_{ij} = Hg_{ij}.
\end{aligned} \tag{C.20}$$

The other symbols with two spatial indices are

$$\begin{aligned}\Gamma^i_{0j} &= \frac{1}{2} g^{i\lambda} \left(\partial_0 g_{\lambda j} + \partial_j g_{\lambda 0} - \partial_\lambda \cancelto{0}{g_{0j}} \right) \\ &= \frac{1}{2} \frac{1}{a^2} \frac{1}{f^2} \left(\partial_0 g_{ij} + \partial_j \cancelto{0}{g_{i0}} \right) \\ &= \frac{1}{2} \frac{1}{a^2} \frac{1}{f^2} (2\dot{a} a f^2) \delta_{ij} \\ &= H \delta_{ij}.\end{aligned} \tag{C.21}$$

Christoffel symbols with a single spatial index should be zero, since these indices transform when we perform a rotation on the space. Since the space itself is isotropic, it must be invariant and thus can only be zero. For completeness, this is shown explicitly here,

$$\begin{aligned}\Gamma^0_{0i} &= \frac{1}{2} g^{0\lambda} \left(\partial_0 g_{i\lambda} + \partial_i g_{0\lambda} - \partial_\lambda \cancelto{0}{g_{0i}} \right) \\ &= -\frac{1}{2} \left(\partial_0 \cancelto{0}{g_{i0}} + \partial_i (-1) \right) \\ &= 0,\end{aligned} \tag{C.22}$$

and

$$\begin{aligned}\Gamma^i_{00} &= \frac{1}{2} g^{i\lambda} \left(2\partial_0 g_{0\lambda} - \partial_\lambda g_{00} \right) \\ &= \frac{1}{2} \frac{1}{a^2} \frac{1}{f^2} \left(2\partial_0 \cancelto{0}{g_{0i}} - \partial_i (-1) \right) \\ &= 0.\end{aligned} \tag{C.23}$$

Lastly, the Christoffel symbols without spatial indices are also zero,

$$\begin{aligned}\Gamma^0_{00} &= \frac{1}{2} g^{0\lambda} \left(2\partial_0 g_{0\lambda} - \partial_\lambda g_{00} \right) \\ &= -\frac{1}{2} \partial_0 g_{00} = -\frac{1}{2} \partial_0 (-1) = 0.\end{aligned} \tag{C.24}$$

Calculation of the Christoffel symbols of the metric $\tilde{g}_{ij}$ is now trivial. In equation (C.19), there is not one term where an index of 0 enters and the scale factor that g_{ij} has but $\tilde{g}_{ij}$ does not have cancels out everywhere against the inverse metric. Thus,

$$\bar{\Gamma}^k_{ij} = \Gamma^k_{ij}. \tag{C.25}$$

Since the spatial Christoffel symbols on $d\sigma^2$ are the same as those on ds^2, the symbol $\bar{\Gamma}$ will not be used anywhere in this thesis except for in this appendix. After all, accidentally using the wrong one will not result in an error. Furthermore, it should usually be clear on what space the calculations are done since the vectors used live in only one of the spaces and since each space has its own symbol for the covariant derivative.

Often, we want to calculate the covariant divergence of spatial vectors (i.e. living in the space Σ), i.e. $\bar{\nabla}_i V^i = \tilde{g}^{ij} \bar{\nabla}_i V_j = a^{-2} f^{-2} \delta^{ij} \bar{\nabla}_i V_j$. This involves either $\bar{\Gamma}^k_{ii}$ or $\bar{\Gamma}^i_{ik}$ (depending on whether the index of the vector is upstairs and downstairs), and thus it will be useful to calculate these here in advance:

$$
\begin{aligned}
\bar{\Gamma}^k_{ii} =& \frac{1}{2} K f \left(x^k \delta_{ii} - 2 x^i \delta_{ik} \right) \\
=& \frac{1}{2} K f (3x^k - 2x^k) \\
=& \frac{1}{2} K f x^k .
\end{aligned}
\tag{C.26}
$$

$$
\begin{aligned}
\bar{\Gamma}^i_{ik} =& \frac{1}{2} K f \left(x^i \delta_{ik} - x^i \delta_{ik} - x^k \delta_{ii} \right) \\
=& \frac{1}{2} K f (-3x^k) \\
=& -\frac{3}{2} K f x^k .
\end{aligned}
\tag{C.27}
$$

C.4 Killing Vectors

In this section, Killing's equation is solved for the spatial metric $d\sigma^2$. The equation is

$$\bar{\nabla}_{(i} \xi_{j)} = 0. \tag{C.28}$$

For every Killing vector ξ_i that solves this equation, the spatial metric will have an infinitesimal isometry of the form

$$x^i \to x^i + \epsilon^i \equiv x^i + \epsilon \xi^i . \tag{C.29}$$

It follows that the full metric then has the infinitesimal isometry

$$
\begin{aligned}
t &\to t, \\
x^i &\to x^i + \epsilon^i \equiv x^i .
\end{aligned}
\tag{C.30}
$$

The four-dimensional Killing vector corresponding to this isometry is

$$\tilde{\xi}^\mu = (0, \xi^i). \tag{C.31}$$

It can also be seen directly from Killing's equation that a Killing vector in the subspaces is also a Killing vector in the full space if carried over this way. Using (see Appendix C.1)

$$\tilde{\xi}_\mu = (0, a^2 \xi_\mu), \tag{C.32}$$

we find

$$
\begin{aligned}
\nabla_{(\mu} \tilde{\xi}_{\nu)} &= \partial_{(\mu} \tilde{\xi}_{\nu)} - \Gamma^\lambda_{\mu\nu} \xi_\lambda \\
&= \partial_{(\mu} \tilde{\xi}_{\nu)} - \Gamma^k_{\mu\nu} \xi_k .
\end{aligned}
\tag{C.33}
$$

For $\mu = i$, $\nu = j$, i.e. only spatial indices, we find $\nabla_{(i} \tilde{\xi}_{j)} = \bar{\nabla}_{(i} \xi_{j)} = 0$ since $\tilde{\xi}_i = \xi_i$ and $\Gamma^k_{ij} = \bar{\Gamma}^k_{ij}$ (see Appendix C.3). For $\mu = 0$, $\nu = i$ we find

$$
\begin{aligned}
\nabla_{(0} \tilde{\xi}_{i)} &= \frac{1}{2} \partial_0 \tilde{\xi}_j + \partial_i \cancelto{0}{\tilde{\xi}_0} - \Gamma^j_{0i} \tilde{\xi}_j \\
&= \frac{1}{2} \partial_0 (a^2 \xi_i) - H a^2 \xi_i = 0.
\end{aligned}
\tag{C.34}
$$

Furthermore, $\nabla_0 \tilde{\xi}_0 = 0$ trivially since all the components of $\tilde{\xi}$ and Christoffel symbols involved are zero individually. Thus, $\nabla_{(\mu}\tilde{\xi}_{\nu)} = 0$, i.e. $\tilde{\xi}_\mu$ is a Killing vector if ξ_i is. This makes looking for Killing vectors of the subspaces extra worthwhile.

Filling in the Christoffel symbols (and multiplying by two), Killing's equation becomes

$$\partial_i \xi_j + \partial_j \xi_i + Kf\left(x^i \xi_j + x^j \xi_i - \delta_{ij} x^k \xi_k\right) = 0. \tag{C.35}$$

As discussed in Section 1.2, this space should be both isotropic and homogeneous. Isotropy motivates guessing Killing vectors that correspond to rotations in flat Euclidean space (see e.g. [10, pp. 138-139]),

$$\xi^i = \omega^i_j x^j, \qquad \omega^i_j = -\omega^j_i. \tag{C.36}$$

It is easily verified that this indeed works. We have $\xi_i = f^2 \omega^i_j x^j$ and thus $\partial_i \xi_j = f^2(\omega^j_i - K x^i f \omega^j_k x^k)$. This gives $2\partial_{(i}\xi_{j)} = -Kfx^i\xi_j - Kfx^j\xi_i$, which exactly cancels against the next two terms in Killing's equation. The last term is $\delta_{ij} f^2 \omega^k_j x^k x^j$ and vanishes because an antisymmetric tensor is contracted with a symmetric one. Thus, Killing's equation is satisfied and there is indeed symmetry under spatial rotations. Note that there are three independent components in an antisymmetric two-tensor of dimension three, and thus there are three of these rotations (as there should be, of course).

Since the subspaces are supposed to be maximally symmetric, they should be homogeneous. Thus, we expect there to be a Killing vector that is translation-like. These are dubbed quasitranslations. In particular, when $K \to 0$, the subspaces become flat and these quasitranslations should reduce to regular translations,

$$\xi^i_{K=0} = a^i \tag{C.37}$$

for some constant vector a^i. Thus, each quasitranslation should be a function of a^i. There cannot be more than three linearly independent quasi-translations (nor any other Killing vectors), since there are already three rotations and a maximally symmetric space of dimension three has only six isometries (see Section 1.2). Since there are three numbers in a^i, there cannot be any other numbers specifying a quasitranslation. Suppressing dependence on x^i and K, this means $\xi^i = \xi^i(a)$

What is more, the sum of two Killing vectors must itself be a Killing vector. Since the sum of two quasitranslations can never be a rotation (as is clear at $K = 0$) it must be another quasitranslation. Thus,

$$\xi^i(a) + \xi^i(b) = \xi^i(c(a,b)). \tag{C.38}$$

At $K = 0$, this equation implies $c^i = a^i + b^i$. At finite K, there can in principle be other terms present, like $Ka^k b_k(a^i + b^i)$. Let's for the moment assume this is not the case, such that we have $\xi^i(a) + \xi^i(b) = \xi^(a + b)$. This is only an educated guess. It will be proven correct once we use it to find explicit Killing vectors.

This condition means that $\xi^i(a)$ must be linear in a^i, since both zeroth-order terms and higher-order terms will not obey it (not that this is very similar to the argument made to obtain equation (1.4)). Furthermore, ξ^i must have one free index. The objects we can use to construct $\xi^i(a)$ with indices are a^i, x^i and the metric $\tilde{g}_{ij}$. However, since the metric is proportional to δ_{ij}, contracting one of its indices will give us nothing new. Thus, we can have a term with a^i and a term with x^i. The second term must still be linear in a^i though. The only way to contract the index is $a^i x^i$. Furthermore, both terms can be multiplied by functions that depend on K and x^i. Thus, our guess is that the quasitranslations are of the form

$$\xi^i(a, K, x) = f(K\mathbf{x}^2)^{-2} F(K\mathbf{x}^2) a^i + K f(K\mathbf{x}^2)^{-2} G(K\mathbf{x}^2) a^k x^k x^i. \tag{C.39}$$

The factors f^{-2} have been extracted from the functions F and G for computational convenience (since they disappear when lowering the index, as is required for Killing's equation). Note that the factor K in front of G is required to obtain equation (C.37) (just like this requires $F_{K=0} = 1$). Since K and x^i are the only dimensionfull quantities to be used in F and G, and since $K\mathbf{x}^2$ is the only dimensionless quantity that can be composed of these, F and G can only be functions of this particular combination (otherwise they might not be dimensionless and the dimensions of the Killing vector would be off).

Writing $\frac{\partial F(K\mathbf{x}^2)}{\partial K\mathbf{x}^2} = F'$, and similarly for G, we get

$$\partial_i \xi_j = 2KF' x^i a^j + 2K^2 G' a^k x^k x^i x^j + KG a^i x^j + KG a^k x^k \delta_{ij} \tag{C.40}$$

and

$$K f x^i \xi_j = K f F x^i a^j + K^2 f G a^k x^k x^i x^j. \tag{C.41}$$

Thus, Killing's equation becomes

$$\begin{aligned} &2Kx^{(i}a^{j)}\Big[2F' + G + fF\Big] \\ &+2K^2 a^k x^k x^i x^j \Big[2G' + fG\Big] \\ &+K a^k x^k \delta_{ij}\Big[2G - K\mathbf{x}^2 fG - fF\Big] = 0. \end{aligned} \tag{C.42}$$

This can only hold if all the terms between square brackets are zero individually. Using the differentiation properties of f in Appendix C.2, we immediately see that the vanishing of the second term implies $G = gf^2$, where g is an integration constant. The vanishing of the third term then gives

$$F = (\frac{2}{f} - K\mathbf{x}^2) = (2[1 + \frac{1}{4}K\mathbf{x}^2] - K\mathbf{x}^2)G = 2g(1 - \frac{1}{4}K\mathbf{x}^2)f^2. \tag{C.43}$$

It is easily checked that indeed the first term also vanishes for this solution. Lastly, requiring $F_{K=0} = 1$ implies $g = \frac{1}{2}$.

Thus, the Killing vectors generating quasitranslations are

$$\xi^i(a, K\mathbf{x}^2) = (1 - \frac{1}{4}K\mathbf{x}^2)a^i + \frac{1}{2}K a^k x^k x^i. \tag{C.44}$$

Note that, in the end, it does not matter what guesses have been made to arrive at this equation. The only thing that matters is that it solves Killing's equation, and that there are three of them. Thus, all the Killing vectors have been found. Since non of them are linearly dependent, this proves that the space described by $d\sigma^2$ is indeed maximally symmetric.

Appendix D

Constant Curvature Space

In this appendix, we derive the metric of a maximally symmetric three-dimensional Euclidean space. Our strategy is to start with a four-dimensional flat Euclidean space, and hope that the metric of a three-sphere embedded in this sphere is indeed the maximally symmetric space we are looking for. Because of the uniqueness of maximally symmetric spaces, making an educated guess is fine, as long as we check that the resultant space is indeed of the maximally symmetric form. While the sphere is a maximally symmetric space with constant curvature, it can be analytically continued to flat space and negatively curved space.

We consider an Euclidean embedding space with metric

$$ds^2 = \delta_{ij} dx^i x^j + dz^2, \tag{D.1}$$

where i runs form one to three. A three-sphere in this space is specified by the equation

$$\delta_{ij} x^i x^j + z^2 = \frac{1}{K}. \tag{D.2}$$

This constant might seem like a weird way to parametrize a sphere (why not replace $1/K$ by a^2 instead, where a is a constant with the interpretation of the radius of the circle?), but this way of doing things will turn out to be convenient. Differentiating the formula defining the sphere gives us

$$dz = -\frac{\delta_{ij} x^i dx^j}{z}. \tag{D.3}$$

We can use this to eliminate dz^2 from equation D.1 so that we are left with a metric on the sphere itself, with coordinates x^i (in two dimensions, these rectilinear coordinates can be thought of as describing the sphere by projecting all points unto the equitorial plane). We can consider these three-vectors, and write them $\mathbf{x}$. This also enables the notation of dot products, $\delta_{ij} x^i y^j = \mathbf{x} \cdot \mathbf{y}$. Note that, since equation (D.2) has two roots for z, each point $\mathbf{x}$ denotes two distinct points on the sphere (in two dimensions, $\mathbf{x} = \mathbf{0}$ corresponds to both the north and south poles).

We can now write the metric on the three-sphere as

$$ds^2 = d\mathbf{x}^2 + \frac{(\mathbf{x}\cdot d\mathbf{x})^2}{\frac{1}{K} - \mathbf{x}^2}, \tag{D.4}$$

where z has been eliminated using equation (D.2). This equation also reveals the condition $|\mathbf{x}| \leq 1/\sqrt{K}$. Thus, the coordinates we use describe exactly a ball (i.e. a three-disc) (similarly, the equatorial plane in two dimensions is a 2-disc, i.e. a circle and its interior). This suggests the use of spherical coordinates,

$$\begin{aligned} x^1 &= r\cos\phi\sin\theta, \\ x^2 &= r\sin\phi\sin\theta, \\ x^3 &= r\cos\theta. \end{aligned} \tag{D.5}$$

This implies $\mathbf{x}^2 = r^2$, and differentiating this equation gives us $\mathbf{x}\cdot d\mathbf{x} = rdr$. Furthermore, $d\mathbf{x}^2$ is simply the three-dimensional Euclidean line element as in equation (1.29), $dr^2 + r^2 d\Omega^2$. Thus, in these coordinates the metric becomes

$$ds^2 = dr^2\left(1 + \frac{r^2}{\frac{1}{K} - r^2}\right) + r^2 d\Omega^2. \tag{D.6}$$

This can be rewritten in a more convenient way using

$$1 + \frac{r^2}{\frac{1}{K} - r^2} = 1 + \frac{Kr^2}{1 - Kr^2} = \frac{1 - Kr^2 + Kr^2}{1 - Kr^2} = \frac{1}{1 - Kr^2}, \tag{D.7}$$

giving

$$ds^2 = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2. \tag{D.8}$$

While this is a beautifull way to write the metric, we will apply one last coordinate transformation. We define a new radial coordinate, r', by

$$r = \frac{r'}{1 + \frac{1}{4}Kr'^2}. \tag{D.9}$$

From this we get

$$\begin{aligned} dr &= \frac{dr'}{1 + \frac{1}{4}Kr'^2} - \frac{\frac{1}{2}Kr'^2 dr'}{(1 + \frac{1}{4}Kr'^2)^2} = dr' \frac{1 + \frac{1}{4}Kr'^2 - \frac{1}{2}Kr'^2}{(1 + \frac{1}{4}Kr'^2)^2} \\ &= dr' \frac{1 - \frac{1}{4}Kr'^2}{(1 + \frac{1}{4}Kr'^2)^2}. \end{aligned} \tag{D.10}$$

This implies

$$\begin{aligned} \frac{dr^2}{1 - Kr^2} &= dr'^2 \frac{(1 - \frac{1}{4}Kr'^2)^2}{(\frac{1}{4}Kr'^2)^4} \frac{1}{1 - K\frac{r'^2}{(1+\frac{1}{4}Kr'^2)^2}} \\ &= \frac{dr'^2}{(1 + \frac{1}{4}Kr'^2)^2} \frac{(1 - \frac{1}{4}Kr'^2)^2}{(1 + \frac{1}{4}Kr'^2)^2 - Kr'^2} \quad 1 \end{aligned} \tag{D.11}$$

We now replace r in the metric by r' and drop the prime for convenience. This gives us

$$ds^2 = \frac{1}{(1 + \frac{1}{4}Kr^2)}(dr^2 + r^2 d\Omega^2). \tag{D.12}$$

What is more, we can now easily return to rectilinear coordinates (which are often more convenient to use), giving

$$ds^2 = \frac{1}{(1 + \frac{1}{4}K\mathbf{x}^2)^2} d\mathbf{x}^2. \tag{D.13}$$

It still remains to check whether this educated guess is indeed a maximally symmetric space. It follows immediately from the invariance under four-dimensional rotations of both equation (D.1) and equation (D.2), of which there are $\frac{1}{2}4(4-1) = 6$. Since the space only needs $\frac{1}{2}3(3-1) = 6$ continuous symmetries to be maximally symmetric, this immediately follows. It is however useful to know what these six isometries actually look like in term of the coordinates used. While the metric is manifestly invariant under three-dimensional rotations (this is especially clear from equations (D.4) and (D.13) since dot products are invariant under rotations), it does not seem invariant under translations. It however is homogeneous. As discussed in Section 1.3, the 'translations' in a curved space may look a bit different then $x^i \to x^i + a^i$. So what does it look like in this space?

One way to derive the explicit symmetries is to start from the four-dimensional rotations of the embedding space. While rotations around the z-axis are simply three-dimensional rotations of $\mathbf{x}$, the rotations that do not leave the z-coordinate invariant are the **quasitranslations** that make the three-dimensional space homogeneous. This strategy is applied by [35, eq. (13.3.17)]. However, the coordinates used here are different and it is not entirely clear what the infinitesimal transformations look like. Instead we solve the Killing equation directly. It is shown in Appendix C.4 that the Killing vectors that generate quasitranslations are

$$\xi^i(a, K\mathbf{x}^2) = (1 - \frac{1}{4}K\mathbf{x}^2)a^i + \frac{1}{2}Ka^k x^k x^i, \tag{D.14}$$

where a^i is the vector by which the origin in displaced. Note that this approach is the ultimate proof that our space of constant curvature is indeed maximally symmetric, as we have six linearly independent vector fields that solve Killing's equation (the Killing vectors for rotations are also found in Appendix C.4).

It can be shown that the curvature constant of the space we have found is exactly K [35, sec. 13.3], which is of course the reason why we parametrized the sphere as we did in equation D.2. Since this equation only has solutions for positive K, it follows that we have found the spaces of constant curvature for which the curvature is positive. However, the metric we have found describes a space in its own right that does no longer depend on its embedding. Therefore, we are able to analytically continue the metric D.13 to different values of K. We see immediately that for $K = 0$, we obtain the flat metric (1.29) as would be expected on ground of this argument. Indeed, the explicit construction of the Killing vectors in Appendix C.4 does in no way depend on the sign of K.

Thus, whatever the sign of K, the space we have found has six linearly independent Killing vectors and is thus maximally symmetric. We conclude that metric (D.13) is the unique metric (up to coordinate transformations) which describes a maximally symmetric three-dimensional Euclidean space of curvature constant K for each (real) value of K.

Note that, in the coordinates we started our considerations with, we had the condition $\mathbf{x}^2 < \frac{1}{K}$, since we were describing sphere. For $K \leq 0$, we can drop this condition, and our coordinates can run all the way to infinity. Curiously, the coordinates we have adopted now are bounded by $2/\sqrt{|K|}$ for both positive and negative values of K. For $K > 0$, equation (D.9) with $r = \frac{1}{\sqrt{K}}$ is solved by $r' = 2/\sqrt{K}$. For $K < 0$, r in the same equation becomes infinite $r' = 2/\sqrt{-K}$. When $K \to 0$, the limits move to infinity. Indeed, for $K = 0$, transformation (D.9) is not a transformation at all. Thus, we conclude

$$|\mathbf{x}| \begin{cases} < \frac{2}{\sqrt{|K|}} & \text{for } K < 0 \\ \leq \frac{2}{\sqrt{|K|}} & \text{for } K > 0 \\ < \infty & \text{for } K = 0. \end{cases} \tag{D.15}$$

Appendix E

Linearized Einstein Equations for Scalars

In this appendix, we write up the linearized Einstein equations for the scalar sector of linear cosmological perturbation theory. Rather then deriving them ourselves, we follow the cosmological review written by Mukhanov, Feldman and Brandenberger [26].

The equations are written in terms of gauge-invariant variables in [26, eq. (5.14-5.16)]. This can be translated to the variables defined in Section 2.1 by (using [26, eqs. (2.9), (2.12), (2.14), (3.13), (5.11) and (5.12)]

$$\mathcal{O}' \equiv \frac{d}{d\eta}\mathcal{O} = a\frac{d}{dt}\mathcal{O} = a\dot{\mathcal{O}}, \tag{E.1}$$

$$\mathcal{H} \equiv \frac{a}{a'} = aH, \tag{E.2}$$

$$\Psi = \frac{1}{2}A + aH(F - \frac{1}{2}a\dot{B}), \tag{E.3}$$

$$\Phi = \frac{1}{2}E - a(\dot{F} + HF) + \frac{1}{2}a^2(\ddot{B} + 2H\dot{B}), \tag{E.4}$$

$$\delta\epsilon^{(gi)} = \delta\rho + a\dot{\bar{\rho}}(\frac{1}{2}a\dot{B} - F), \tag{E.5}$$

$$\delta p^{(gi)} = \delta p + a\dot{\bar{p}}(\frac{1}{2}a\dot{B} - F), \tag{E.6}$$

$$\delta u_i^{(gi)} = \partial_i \left(-\delta u^S + aF - \frac{1}{2}a^2\dot{B} \right) \tag{E.7}$$

(furthermore, they use $\mathcal{O}_0$ where we use $\bar{\mathcal{O}}$ and $D \equiv \Phi - \Psi$).

The space-space Einstein equations [26, eq. (5.16)] has most terms proportional to the Kronecker delta, $\delta^i{}_j$. Only one term is not diagonal. Thus, requiring the equation to hold at $i \neq j$, reveals

$$\gamma^i{}_k \bar{\nabla}_k \bar{\nabla}_j (\Phi - \Psi) = 0 \qquad \text{for } i \neq j. \tag{E.8}$$

Using equation C.19, and writing $D = \Phi - \Psi$, this becomes

$$
\begin{aligned}
a^{-2} f^{-2} \bar{\nabla}_i \partial_j D &= 0 \\
(\partial_i \partial_j - \Gamma^k_{ij} \partial_k) D &= 0 \\
\left(\partial_i \partial_j + \frac{1}{2} K f (x^k \delta_{ij} - x^i \delta_{jk} - x^j \delta_{ik}) \partial_k \right) D &= 0 \\
\left(\partial_i \partial_j + \frac{1}{2} K f (\cancelto{0}{\delta_{ij}} x^k \partial_k - x^i \partial_j - x^j \partial_i) \right) D &= 0 \\
\left(\partial_i \partial_j - \frac{1}{2} K f \left(x^i \partial_j + x^j \partial_i \right) \right) D &= 0 \\
\left(\partial_{(i} - K f x^{(i} \right) \partial_{j)} D = 0 &
\end{aligned}
\tag{E.9}
$$

(where the cancelation occurs since $i \neq j$). Using equation (C.17), this can be rewritten as

$$\partial_{(i} \left(f^2 \partial_{j)} D \right) = 0, \tag{E.10}$$

or

$$\partial_i \left(f^2 \partial_j D) \right) = -\partial_j \left(f^2 \partial_i D \right). \tag{E.11}$$

Let us write the vector field $f^2 \partial_j D$ as $g_i(t, \mathbf{x})$. Since we are dealing with a scalar perturbation, the vector index cannot come from some vector perturbation. Thus, the index must come from a background vector. Since our background is isotropic, there is no such vector and the index must come from x^i (or, equivalently, from the derivative of some spatial function). Thus, we can write $g_i(t, \mathbf{x}) = g(t, \mathbf{x}) x^i$ from some scalar function g. But, since this scalar function does not carry an index, all the xs in its spatial dependence must be contracted. Again, the only object this can be done with, is another x^i. Thus, $g_i(t, \mathbf{x}) = g(t, \mathbf{x}^2) x^i$. Now, we can calculate

$$\partial_i g_j = (\partial_i g) x^j + g \partial_i x^j = \frac{dg}{d\mathbf{x}^2} x^i x^j + g \delta_{ij}. \tag{E.12}$$

But this quantity is automatically symmetric! Thus, $\partial_{(i} g_{j)} = 0$ implies $\partial_i g_j = 0$.

We thus find the equation

$$\partial_i \left(f^2 \partial_j D) \right) = 0. \tag{E.13}$$

This can be solved using integration constants (although a little different from the strategy in Section 4.5, where only the divergence had to vanish),

$$f^2 \partial_i D = C_i, \tag{E.14}$$

where $\partial_i C_j = 0$. This implies

$$\partial_i D = f^{-2} C_i = \left(1 + \frac{1}{2} K \mathbf{x}^2 + \frac{1}{16} (K \mathbf{x}^2)^2 \right) C_i. \tag{E.15}$$

The only solution for this is of the form

$$D = a + b x^i + c \mathbf{x}^2 x^i + d \mathbf{x}^4 x^i. \tag{E.16}$$

Now, in order for Φ and Ψ to be physical, they must go to zero at spatial infinity. Furthermore, since they are perturbations, their averages should be zero. The only way to achieve this through the above form is

$$D = 0. \tag{E.17}$$

Thus, we obtain a **physicality condition**, a constraint equation which any physical solution to the linearized Einstein equations must obey:

$$\Phi = \Psi. \tag{E.18}$$

Using this identification, the scalar Einstein equations become [26, eqs (5.17 - 5.19)]

$$\begin{aligned} -3H\dot{\Phi} + \left(\frac{3K - k^2}{a^2} - 3H^2\right)\Phi &= \frac{\delta\epsilon^{(gi)}}{2M_p^2}, \\ \partial_i\left(\dot{\Phi} + H\Phi\right) &= \left(\frac{K}{a^2} - \dot{H}\right)\delta u_i^{(gi)}, \\ \ddot{\Phi} + 4H\dot{\Phi} + \left(2\dot{H} + 3H^2 - \frac{K}{a^2}\right)\Phi &= \frac{1}{2M_p^2}\delta p^{(gi)} \end{aligned} \tag{E.19}$$

,where I have used $\bar{\rho} + \bar{p} = 2M_p(K/a^2 - \dot{H})$, which follows from the continuity equation and the Friedmann equation, and $\bar{\nabla}^2 \to -k^2$ since the different eigenmodes decouple anyway. Also, it should be noted that to derive these equations, one should be careful with the differentiating with respect to comoving time. In particular,

$$\mathcal{H}' = a\frac{d}{dt}(aH) = a^2\dot{H} + a^2H^2 \tag{E.20}$$

and

$$\Phi'' = a\frac{d}{dt}(a\dot{\Phi}) = a^2\ddot{\Phi} + a^2H\dot{\Phi}. \tag{E.21}$$

We see now that the second equation provides another physicality condition, since it implies $\dot{\Phi} + H\Phi = (K/a^2 - \dot{H})\delta u^S$ for physical solution, while this does not need to be the case for solutions that do not vanish at infinity.

Since we will mostly be dealing with Newtonian gauge in this thesis, we write the linearized scalar Einstein equations here in Newtonian gauge as well. These are easily obtained from (E.19) by filling in the Newtonian gauge condition (4.22)[1], which gives

$$\begin{aligned} -3H\dot{\Phi} + \left(\frac{3K - k^2}{a^2} - 3H^2\right)\Phi &= \frac{\delta\rho}{2M_p^2}, \\ \partial_i\left(\dot{\Phi} + H\Phi\right) &= \left(\dot{H} - \frac{K}{a^2}\right)\partial_i\delta u^S, \\ \ddot{\Phi} + 3H\dot{\Phi} + \left(2\dot{H} + 3H^2 - \frac{K}{a^2}\right)\Phi &= \frac{1}{2M_p^2}\delta p \end{aligned} \tag{E.22}$$

Note that the second of these equations then also becomes a physicality condition, since $\partial_i A = \partial_i B$ implies $A = B$ when A and B fall off at infinity.

[1] One might find it peculiar that $\delta u_i^{(gi)} = -\partial_i \delta u^S$ (rather than being positive). This has to do with the fact that Mukhanov, Feldman and Brandenberger use a $(+ - - -)$ metric convention. Since u^μ is the 'fundamental' object that is independent of convention, our u_μ is minus their u^μ.

Appendix F

Mathematica Notebook

In Section 6.4, an example of what the found scalar gauge mode looks like in a radiation-dominated universe is provided. This mode was found by first solving for the scale factor $a(t)$ of the background universe and subsequently solving the differential equation for $\lambda(t)$ in (6.19), which then let's us find the time dependence of the scalar mode. Rather then solving all these equations by hand, the Mathematica software package [20] has been used.

The same Mathematica notebook that has been used to find the scalar mode has also been used to check whether different versions of the linearized scalar Einstein equations (as found from [21] and [26] and manipulated in different ways) are really equivalent (making sure that no mistakes have been made). This also allows for checking explicitly whether the scalar mode solves the Einstein equations. It is found that it indeed does, making us ever more certain that the procedure in Section 6.1 has been performed correctly.

For completeness, the Mathematica notebook in which all these calculations have been performed is supplied in this appendix. It begins on the next page.

Adiabatic Scalar Mode in Curved Universe

This Mathematica notebook is written by Guus Avis as a supplement to his master thesis in theoretical physics at Utrecht University.
It serves two distinct purposes:
1) Finding what the open-universe versions of Weinberg's scalar adiabatic modes look like in the case of radiation domination (in order to provide an explicit example).
2) Checking whether the different versions of the linearized Einstein equations in the scalar sector, as found in the works by Kodama and Sasaki (1984) and Mukhanov, Feldman and Brandenberger (1992), and as further processed by me (Guus) and Sadra Jazayeri, all agree with one another.
Note that the two purposes overlap in the end, as the scalar adiabatic mode found at 1) is checked against the different versions of the Einstein equations of 2).

Indeed, all the versions of the Einstein equations are in the end equivalent (in a radiation-dominated universe), and are solved by the scalar mode.

For readers who are interested in the solutions yet do not wish to delve through this notebook, the solutions for λ(t) (which generates the scalar adiabatic mode) and the Newtonian potential ϕ are provided here.

In[86]:= `λ[t] /. Solution // Simplify`

Out[86]=
$$-\frac{\text{AdiabaticC1}\, M_{pl}}{\sqrt{t\left(-3\,K\,M_{pl}\,t+2\sqrt{3}\sqrt{\rho_0}\right)}}-\frac{\text{AdiabaticC2}\,\text{ArcTanh}\left[\sqrt{3}\sqrt{\frac{K\,M_{pl}\,t}{3\,K\,M_{pl}\,t-2\sqrt{3}\sqrt{\rho_0}}}\right]}{\sqrt{3}\sqrt{K\,M_{pl}\,t\left(3\,K\,M_{pl}\,t-2\sqrt{3}\sqrt{\rho_0}\right)}}$$

In[87]:= `ϕ[t] /. Solution // Simplify`

Out[87]=
$$\left(\frac{2\,\text{AdiabaticC1}\left(K\,M_{pl}\,t\left(3\,K\,M_{pl}\,t-2\sqrt{3}\sqrt{\rho_0}\right)\right)^{3/2}\rho_0}{K\,t}+\text{AdiabaticC2}\,t\left(9\left(-K\,M_{pl}\right)^{7/2}t^3+24\left(-K\,M_{pl}\right)^{3/2}t\,\rho_0+15\sqrt{3}\,K^2\,t^2\sqrt{-K\,M_{pl}^5\,\rho_0}+4\sqrt{3}\sqrt{-K\,M_{pl}\,\rho_0^3}\right)+2\,\text{AdiabaticC2}\left(\sqrt{3}\,K\,M_{pl}\,t-2\sqrt{\rho_0}\right)\sqrt{t\left(-3\,K\,M_{pl}\,t+2\sqrt{3}\sqrt{\rho_0}\right)}\,\rho_0\,\text{ArcTanh}\left[\sqrt{3}\sqrt{\frac{K\,M_{pl}\,t}{3\,K\,M_{pl}\,t-2\sqrt{3}\sqrt{\rho_0}}}\right]\right)\Big/\left(2\left(-K\,M_{pl}\right)^{3/2}t^2\left(3\,K\,M_{pl}\,t-2\sqrt{3}\sqrt{\rho_0}\right)^3\right)$$

Preliminaries

Preamble

In[3]:=
```
Needs["Notation`"];
Symbolize[ParsedBoxWrapper[SubscriptBox["_","_"]]] ;
(*This makes sure we can use subscripted variables as a single symbolic*)

$Assumptions = {cssq>0, K<0, ρ0>0, k≠0 , a[t]>0, t>0, a'[t] ≠ 0, Mpl > 0};
```

Definitions

In[6]:=
```
ClearAll[a, H, K, w, ρ, p, PlotSos, Mp, Mpl]
w = 1/3; (*w = 1/3: radiation dominated. Currently only works
    properly for this value!!! Other (not-yet working) options: w =
  0: matter dominated. w = -1: dark energy dominated.*)
cssq = w; (* cs^2, speed of sound. For adiabatic modes, it is equal to w.*)
p[t_] := w ρ[t]; (*equation of state *)
H[t_] = a'[t]/a[t];
(*Mpl=1; *) (*Working completely unitless.*)

PlotSol = {K → -1, ρ0 → 1, Mpl → 1};
(* Specific numbers to be used for plotting functions. Here
 I have just used easy numbers, not necessarily accurate ones.*)
```

Eliminating Higher-Order Derivatives of a(t)

In[12]:=
```
ClearAll[AccelEq]
a''[t_] =.
AccelEq = H'[t] == -3/2 (1 + w) H[t]^2 - 1/2 (1 + 3 w) K/a[t]^2;
(*This form of the acceleration equation can be derived from
 the Friedmann and continuity equations (has been done by hand).*)
a''[t_] = a''[t] /. Solve[AccelEq, a''[t]][[1]] // Simplify;
(* This makes sure no second derivates will occur. *)
a'''[t_] = D[a''[t], t] // Simplify;
a''''[t_] = D[a'''[t], t] // Simplify;
(*Make sure even higher-order derivatives are also properly dealt with.*)
```

Unset: Assignment on Derivative for a''[t_] not found.

Out[13]= $Failed

Defining Custom Functions

In[18]:=
```
(* The function below makes sure the coefficient of the second
 derivative in a second order differential equation is 1. This
 `normalization' allows for fair comparison between equations.*)

normalize[eqn_, fct_] :=
 Collect[
  DivideSides[

   SubtractSides[eqn, eqn[[2]]]

    , Subtract[

     SubtractSides[eqn, eqn[[2]] ] [[1]] /. {fct[t] → 0, fct'[t] → 0, fct''[t] → 1},
     SubtractSides[eqn, eqn[[2]]] [[1]] /. {fct[t] → 0, fct'[t] → 0, fct''[t] → 0}]]

  , {fct''[t], fct'[t], fct[t]}]
```

Homogeneous and Isotropic Universe

Solving Friedmann

In[19]:= ClearAll[Friedmann, Continuity, FriedSol, FriedC1, FriedC2]

Friedmann = H[t] == $-\sqrt{\frac{\rho[t]}{3\,M_{pl}{}^{2}} - \frac{K}{a[t]^{2}}}$; (*Needed to select H>0 solution. *)

Continuity = ρ'[t] + 3 H[t] (ρ[t] + p[t]) == 0;
(*Friedmann Equation and Continuity Equation
together fully specify evolution of unperturbed universe *)

FriedSol = DSolve[{Friedmann(*/.K→0*), Continuity}, {a, ρ}, {t, 0, ∞}][[2]] /.
C[1] → FriedC1 * $M_{pl}{}^{4}$ /. C[2] → $\frac{\text{FriedC2}}{M_{pl}{}^{2}}$ (*FriedSol contains the
general solution (with abstract integration constants C[1] and C[2],
which are renamed FriedC1 and FriedC2 to avoid ambiguity when other differential
equations are introduced). [[2]] selects the solution with a>0.*)

Out[22]=

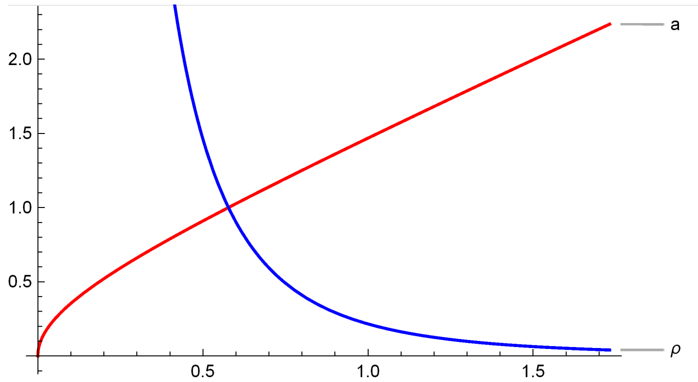
$$\left\{\rho \to \text{Function}\left[\{t\}, \frac{9\left(\text{FriedC1}\, M_{pl}^{4}\right)}{\left(\frac{\text{FriedC1}\, M_{pl}^{4}}{K\, M_{pl}^{2}} - 3\,K\left(t^{2} - \frac{2\sqrt{3}\ t\ \text{FriedC2}}{M_{pl}^{2}} + 3\left(\frac{\text{FriedC2}}{M_{pl}^{2}}\right)^{2}\right)\right)^{2}}\right],\right.$$

$$\left. a \to \text{Function}\left[\{t\}, \sqrt{\left(\frac{\text{FriedC1}\, M_{pl}^{4}}{3\,K\, M_{pl}^{2}} - K\left(t^{2} - \frac{2\sqrt{3}\ t\ \text{FriedC2}}{M_{pl}^{2}} + 3\left(\frac{\text{FriedC2}}{M_{pl}^{2}}\right)^{2}\right)\right)}\right]\right\}$$

Plugging in Integration Constants (only one remaining: ρ_0)

In[23]:=
```
ClearAll[asol, ρsol, FriedC2Rule, FriedC1Rule, ρ, ρ0, tin, t0, FriedSolFull]

tin = t /. Solve[a[t] == 0 /. FriedSol, t][[1]];
(*We only need to consider t>tin, otherwise a(t) becomes imaginary*)
t0 = t /. Solve[a[t] == 1 /. FriedSol, t][[1]];(*This is the time at which a=1*)

FriedC2Rule = Solve[tin == 0, FriedC2][[1]];(*Setting a(0)=0*)

FriedC1Rule = Solve[ρ[t0] == ρ0 /. FriedSol, FriedC1][[1]] /. FriedC2Rule;
(*Setting ρ(a=1)=ρ0*)

FriedC2Rule = FriedC2Rule /. FriedC1Rule;

FriedSolFull = Join[FriedSol /. FriedC2Rule /. FriedC1Rule, FriedC1Rule, FriedC2Rule];
(*Rules needed to fill in the integration
 constants. Combines the above conditions. *)

plota = Plot[a[t] /. FriedSolFull /. PlotSol,
   {t, 0, 3 t0 /. FriedSolFull /. PlotSol}, PlotStyle → Red, PlotLabels → "a"];
plotρ = Plot[ρ[t] /. FriedSolFull /. PlotSol, {t, 0, 3 t0 /. FriedSolFull /. PlotSol},
   PlotStyle → Blue , PlotLabels → "ρ"];
Show[
 plota,
 plotρ]
```

Out[32]=



Switch to Conformal Time (as check)

Finding t(η)

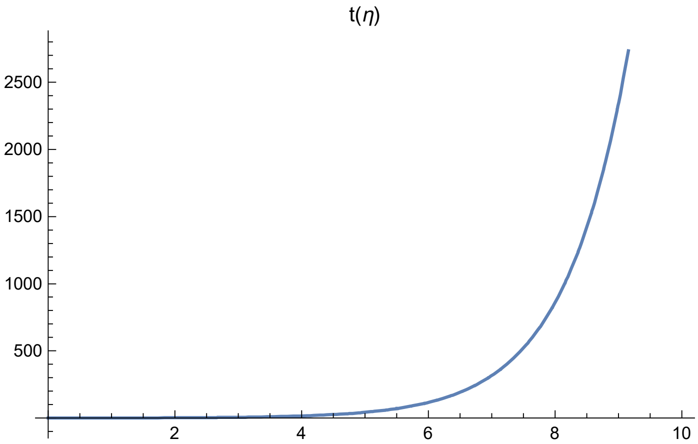
In[33]:=
```
ClearAll[η_sol, t_sol]
```

$$\eta_{sol}[t\_] = \text{Simplify}\left[\int_0^t \frac{dT}{a[T]} \text{ /. FriedSolFull, } t > 0\right];$$

$$t_{sol}[\eta\_] = \text{Simplify}[t \text{ /. Solve}[\eta == \eta_{sol}[t], t][[1]]]$$

```
Plot[t_sol[η] /. PlotSol, {η, 0, 10}, PlotLabel → "t(η)"]
```

Out[35]=
$$-\frac{2\sqrt{\rho_0}\,\text{Sinh}\left[\frac{\sqrt{-K}\,\eta}{2}\right]^2}{\sqrt{3}\,K\,M_{pl}}$$

Out[36]=


Finding a(η)

Comparing to Analytical Solutions

In[39]:=
```
ClearAll[a_an, t_an]
```

$$a_{an}[\eta] = \sqrt{\frac{\rho_0}{-3\,K\,M_{pl}{}^2}}\ \text{Sinh}\left[\eta\sqrt{-K}\right];$$

$$t_{an}[\eta] = \frac{1}{-K}\sqrt{\frac{\rho_0}{3\,M_{pl}{}^2}}\left(\text{Cosh}\left[\eta\sqrt{-K}\right] - 1\right);$$

```
(*These are the analytical solutions for w=1/3, K<0, η(0) = a(0) = 0*)

FullSimplify[a_an[η] == a_sol[η], η > 0]
FullSimplify[t_an[η] == t_sol[η], η > 0]
```

Out[42]= True

Out[43]= True

Deriving Adiabatic Mode Time Dependence

Solving the equation

In[44]:=
```
ClearAll[AdiabaticEq, λ, λSol, AdiabaticC1, AdiabaticC2]
AdiabaticEq = λ''[t] + 3 H[t] λ'[t] - K/a[t]^2 λ[t] == 0;
(*This is the equation we derived for WAM I *)
λSol = DSolve[AdiabaticEq /. FriedSolFull, λ, t][[1]] /. C[1] → -I * AdiabaticC1 * Mpl /.
   C[2] → 1/2 AdiabaticC2 // Simplify
  (* We name the integration constants to avoid confusion,
  and make them dimensionless by taking out appropriate powers of Mpl. *)
```

Out[46]=

$$\left\{\lambda \to \text{Function}\left[\{t\},\; -\frac{i\left(-i\,\text{AdiabaticC1}\,M_{pl}\right)}{\sqrt{t\left(-3\,K\,M_{pl}\,t+2\sqrt{3}\sqrt{\rho_0}\right)}} - \frac{2\,\text{ArcTanh}\left[\sqrt{3}\sqrt{-\frac{K\,M_{pl}\,t}{-3\,K\,M_{pl}\,t+2\sqrt{3}\sqrt{\rho_0}}}\right]\text{AdiabaticC2}}{\sqrt{3}\sqrt{K\,M_{pl}\,t\left(3\,K\,M_{pl}\,t-2\sqrt{3}\sqrt{\rho_0}\right)}\;\;2}\right]\right\}$$

Defining Bardeen Potential

In[47]:=
```
ClearAll[ϕtoλ, ϕSol]
ϕtoλ = {ϕ → Function[t, λ[t] - λ'[t] (a[t] a'[t])/K]};
ϕSol = ϕtoλ /. FriedSolFull /. λSol // Simplify;
```

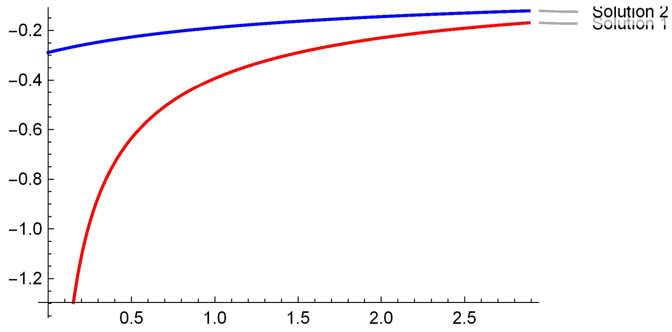
Combining Solutions

In[50]:=
```
ClearAll[Solution]
Solution = Join[FriedSolFull, λSol, ϕSol];
(*Should we fill in some boundary conditions? *)
```

Plotting Solutions

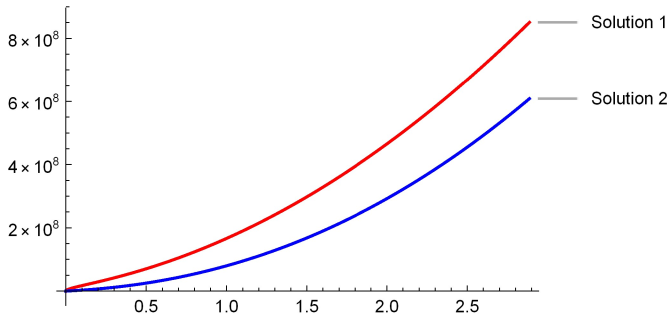
In[52]:=
```
Adiabaticλ1 = Plot[λ[t] /. Solution /. PlotSol /. AdiabaticC1 → 1 /. AdiabaticC2 → 0,
    {t, 0, 5 t0 /. Solution /. PlotSol}, PlotStyle → Red, PlotLabels → "Solution 1"];
Adiabaticλ2 = Plot[λ[t] /. Solution /. PlotSol /. AdiabaticC1 → 0 /. AdiabaticC2 → 1,
    {t, 0, 5 t0 /. Solution /. PlotSol}, PlotStyle → Blue, PlotLabels → "Solution 2"];
Show[Adiabaticλ1, Adiabaticλ2]
```

Out[54]=



In[55]:=
```
Adiabaticϕ1 = Plot[ϕ[t] /. Solution /. PlotSol /. AdiabaticC1 → 1 /. AdiabaticC2 → 0,
    {t, 0, 5 t0 /. Solution /. PlotSol}, PlotStyle → Red, PlotLabels → "Solution 1"];
Adiabaticϕ2 = Plot[ϕ[t] /. Solution /. PlotSol /. AdiabaticC1 → 0 /. AdiabaticC2 → 1,
    {t, 0, 5 t0 /. Solution /. PlotSol}, PlotStyle → Blue, PlotLabels → "Solution 2"];
Show[Adiabaticϕ1, Adiabaticϕ2]
```

Out[57]=



Einstein Equation from Sasaki

Equations in Sasaki

In[58]:=
```
ClearAll[SasakiEq1, SasakiEq2, SasakiEqA, SasakiEqϕ, Btoϕ, A, B]

SasakiEq1 = a[t] D[A[t], t] +
    1/(a[t] H[t]) ((2 - 3 - 3 w) K + 3 (cssq - w) (a[t] H[t])^2) A[t] - cssq k B[t] == 0;
SasakiEq2 = a[t] D[B[t], t] + 2 a[t] H[t] B[t] + k A[t] == 0;
(*These are the two equations in Sasaki,
in terms of gauge-invariant expressions \mathcal A and \mathcal B.*)
```

Combining Equations

In[61]:=
```
SasakiEqB = Collect[SasakiEq1 /. DSolve[SasakiEq2, A, t][[1]], {B[t], B'[t], B''[t]}];
(*Reduce the two first-
 order differential equations above to one second-order for \mathcal B. *)

Btoϕ = B → Function[t, 1/(a[t] H[t]) ϕ[t]]; (*This substitution
 can be made since the identity holds up to spatial dependence. *)

SasakiEqϕ = Collect[SasakiEqB /. Btoϕ, {ϕ[t], ϕ'[t], ϕ''[t]}];
(*This is the final equation for ϕ from Sasaki. *)
```

Checking Adiabatic Mode against Equation

First Solution

In[64]:=
```
SasakiEqϕ /. Solution /. AdiabaticC1 → 1 /. AdiabaticC2 → 0 /. k → √(3 K) // Simplify
```

Out[64]= True

Second Solution

In[65]:=
```
SasakiEqϕ /. Solution /. AdiabaticC1 → 0 /. AdiabaticC2 → 1 /. k → √(3 K) // Simplify
```

Out[65]= True

Einstein Equation from Sadra (Mukhanov)

Setting Up the equation

In[66]:=
```
ClearAll[SadraEqϕ]

SadraEqϕ = Collect[-3 H[t]^2 ϕ[t] - 3 H[t] ϕ'[t] + (3 K/a[t]^2 - k^2/a[t]^2) ϕ[t] ==
    1/cssq (ϕ''[t] + 4 H[t] ϕ'[t] + (3 H[t]^2 + 2 H'[t] - K/a[t]^2) ϕ[t]) //
   Simplify, {ϕ''[t], ϕ'[t], ϕ[t]}];
```

In[68]:=
```
SasakiEqϕ // Simplify
SadraEqϕ // Simplify
GuusEqϕ // Simplify
```

Out[68]= $\left(k^2 - 12\,K\right)\,\phi[t] + 3\,a[t]\,\left(5\,a'[t]\,\phi'[t] + a[t]\,\phi''[t]\right) == 0$

Out[69]= $\left(k^2 - 12\,K\right)\,\phi[t] + 3\,a[t]\,\left(5\,a'[t]\,\phi'[t] + a[t]\,\phi''[t]\right) == 0$

Out[70]= GuusEqϕ

Checking Equivalence to Sasaki

In[71]:=
```
SasakiEqϕ == SadraEqϕ // Simplify
```

Out[71]= True

Checking Adiabatic Mode against Equation

First Solution

In[72]:= SadraEqϕ /. Solution /. AdiabaticC1 → 1 /. AdiabaticC2 → 0 /. k → $\sqrt{3\,K}$ // Simplify

Out[72]= True

Second Solution

In[73]:= SadraEqϕ /. Solution /. AdiabaticC1 → 0 /. AdiabaticC2 → 1 /. k → $\sqrt{3\,K}$ // Simplify

Out[73]= True

Checking Guus' Equations

In[74]:=
```
(*In this section, I check the equations I had
 already derived by hand against the Mathematica results.*)
```

Defining the equations I found by hand from Sasaki

In[75]:=
```
ClearAll[GuusEqϕ, GuusEqB]
```

$$\text{GuusEqB} = B''[t] + \frac{1}{a[t]^2\,H[t]}\left((-1-3\,w)\,K + 3\,(cssq - w + 1)\,(a[t]\,H[t])^2\right) B'[t] + \frac{2}{a[t]^2}\left((-1-3\,w)\,K + 3\,(cssq - w)\,(a[t]\,H[t])^2 + a[t]^2\left(H[t]^2 + H'[t]\right) + \frac{1}{2}\,cssq\,k^2\right) B[t] == 0;$$

$$\text{GuusEq}\phi = \phi''[t] + \left(\frac{-1-3\,w}{a[t]^2\,H[t]}\,K + 3\,H[t]\left(cssq - w + \frac{1}{3}\right) - 2\,\frac{H'[t]}{H[t]}\right)\phi'[t] + \left(\left(\frac{-1-3\,w}{a[t]^2}\,K + 3\,(cssq - w)\,H[t]^2\right)\left(1 - \frac{H'[t]}{H[t]^2}\right) + 2\left(\frac{H'[t]}{H[t]}\right)^2 - \frac{H''[t]}{H[t]} + cssq\left(\frac{k}{a[t]}\right)^2\right)\phi[t] == 0;$$

Defining the equation I got by hand from Sadra

In[78]:=
```
ClearAll[GuusSadraEqϕ]
GuusSadraEqϕ = ϕ''[t] + (4 + 3 cssq) H[t] ϕ'[t] +
```
$$\left(3\ (1+\text{cssq})\ \text{H[t]}^2 + 2\,\text{H'[t]} - \frac{\text{K}}{\text{a[t]}^2}\ (1+3\,\text{cssq}) + \text{cssq}\ \frac{\text{k}^2}{\text{a[t]}^2}\right)\ \phi\text{[t]} == 0;$$

Checking whether they agree with Mathematica

In[80]:=
```
GuusEqB == SasakiEqB // Simplify;
normalize[GuusEqB, B][[1]] == normalize[SasakiEqB, B][[1]] // Simplify
(*For some reason,
this works but the above does not. Some error in Mathematica probably. *)
GuusEqϕ == GuusEqB /. Btoϕ // Simplify
GuusEqϕ == SasakiEqϕ // Simplify
GuusSadraEqϕ == SadraEqϕ // Simplify
GuusEqϕ == SadraEqϕ // Simplify
```

Out[81]= True

Out[82]= True

Out[83]= True

Out[84]= True

Out[85]= True

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