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Black holes and the phase space of supersymmetric solutions

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Abstract

Supergravity BPS solutions in six dimensions are richer than their four and five dimensional counterparts and the full determination of their phase space remains an open problem. We study here the phase space of supersymmetric solutions of minimal 6D supergravity, which have a stringy microscopic realization in terms of F-theory. We centre our attention on a class of solutions with certain isometries, for which an $Sp(6,\mathbb{R})$ group of endomorphisms was discovered in [1]. This group can be used to generate new backgrounds, and its physical role is still unclear. We study in particular a solution obtained by acting on $AdS_3 \times S^3$ with one of its generators. The transformation changes the geometry noticeably, giving a singularity and a squashed event horizon in the new solution, as well as non-trivial asymptotics and metric signature changes. We show that it is actually a particular limit of a family of solutions with $S^3 \times S^1$ horizon topology and local $AdS_3 \times S^3$ near-horizon geometry. On another note, the singularity happens to be naked for a certain region of the parameter space of the solution. We attempt to give an explanation to this fact in terms of type IIB superstring theory with negative branes.

Contents

1 Introduction **6**

2 Black holes in general relativity **9**
2.1 The Schwarzschild black hole . . . 9
2.2 Einstein-Maxwell theory . . . 11
2.2.1 The Reissner-Nordström black hole . . . 13
2.2.2 The extremal Reissner-Nordström black hole . . . 13
2.3 The Kerr-Newman black hole . . . 14
2.4 Cosmological constant . . . 15
2.4.1 The BTZ black hole . . . 16

3 Supergravity **18**
3.1 Supersymmetry . . . 18
3.2 The Rarita-Schwinger field . . . 19
3.3 Gauging supersymmetry . . . 20
3.4 Solutions . . . 22
3.5 Type IIB supergravity . . . 24
3.6 Embedding in string theory . . . 25
3.6.1 Type IIB superstring theory . . . 27
3.6.2 Negative branes . . . 29

4 6D minimal supergravity **31**
4.1 Description of the theory . . . 31
4.2 Supersymmetric solutions . . . 32
4.3 A precise class of solutions . . . 34
4.3.1 Flat space . . . 35
4.3.2 $AdS_3 \times S^3$ and the BTZ black hole . . . 36
4.3.3 The black string . . . 37
4.4 The symplectic group . . . 39
4.4.1 The entropy conserving subgroup . . . 41
4.5 Reduction to five dimensions . . . 43

5 Spectral flow on $AdS_3 \times S^3$ **45**
5.1 The new solution . . . 45
5.2 The $c = 0$ case . . . 46
5.3 Metric signature . . . 47
5.4 The $c > 0$ case . . . 48
5.4.1 The $r = 0$ surface . . . 48
5.4.2 Near-horizon geometry . . . 48

5.4.3 Asymptotic limit . . . 49
5.5 The $c < 0$ case . . . 51
5.5.1 Negative branes . . . 54

6 Switching on missing charges **57**
6.1 Acting with M_n . . . 57
6.1.1 The $r = 0$ suface . . . 58
6.1.2 Near-horizon geometry . . . 59
6.1.3 Asymptotic limit . . . 60
6.2 Acting with M_j . . . 61

7 Conclusions and outlook **62**

A Mathematical tools **64**

B Spinors **65**

C Three-form G of the solutions **67**

1 Introduction

Solutions of general relativity, particularly black holes, have been extensively studied and understood in four dimensions. The no-hair theorem states that these objects are fully characterized by three classical observables: mass, charge and angular momentum, and moreover they are restricted to have spherical event horizons [2]. Nevertheless, general relativity works for an arbitrary number D of spacetime dimensions, and it turns out that solutions in $D > 4$ are more complex. In five dimensions a black hole with an spherical event horizon, the Myers-Perry black hole, can be constructed [3], but it was found in [4] that solutions with $S^2 \times S^1$ horizon topology, called black rings, are also possible, and that their existence violates the no-hair theorem. The black ring can be thought of as a 4D black hole to which one adds an extra compact direction. One can repeat the procedure to generate black rings in 6D from five dimensional black holes. The result is also called black string and has horizon topology $S^3 \times S^1$. As black holes with spherical horizon topology S^{D-2} exist for any $D \geq 4$ [3], black strings with horizon topology $S^{D-3} \times S^1$ can be obtained for any $D \geq 5$, and these can likewise be uplifted adding extra flat dimensions. In general, the scope of solutions of classical gravity becomes more intricate and is less developed as the number of dimensions increases [5]. The study of higher dimensional solutions is then interesting by itself, but they also receive attention in the context of string theory and the AdS/CFT correspondence. The first because it necessarily lives in more than four dimensions, and the second because it relates the properties of D dimensional black holes with QFTs in $D - 1$ dimensions [6].

Gravity can also be made supersymmetric, meaning that it can be described within a theory whose algebra of transformations contains a number $\mathcal{N}$ of fermionic generators or supercharges together with the Poincaré and internal symmetry generators. This is called supergravity, and necessarily includes a spin-3/2 partner of the graviton, the gravitino. Multiplets of global SUSY containing scalars, vectors and spin-1/2 fermions can be added and coupled to the graviton and gravitino yielding a wide range of possible resulting theories. The fact that consistent interactions for particles with spin $s \geq 5/2$ are not known bounds the number of dimensions and supercharges for which a supergravity theory can be constructed to be $D \leq 11$ and $\mathcal{N} \leq 8$. In supergravity, solutions can be characterized by the number of supercharges they conserve compared to the total supersymmetry of the theory. Solutions that preserve some supersymmetry are called supersymmetric or BPS. In this thesis we will be interested in the supersymmetric solutions of 6D minimal supergravity, which is the six dimensional supergravity with the least possible amount of supercharges and fields. The main reason is that the phase space of BPS solutions in 6D remains less explored than the 4D and 5D ones. In particular, extremal black hole solutions can be made supersymmetric in 4D or 5D [7, 8] and in 5D the black ring can also be made BPS [9]. In six dimensions, a supersymmetric black string can be obtained from the uplift of a 5D black hole or black ring [10] and supersymmetric black tubes with horizon topology $S^2 \times S^1 \times S^1$ do exist as well, but it is not known, for instance, whether the black hole with horizon topology S^4 can be made BPS or if there exist more objects with different horizons.

The regime of validity of general relativity ends at energies around the Plank scale $E_{\text{P}} \sim 10^{19}$

GeV, which translates to distances of $l_{\mathrm{P}} \sim 10^{-35}$ m or time intervals of $t_{\mathrm{P}} \sim 10^{-44}$ s. At these scales the quantum mechanical effects cannot be ignored and one needs a theory of quantum gravity. String theory is among the most popular candidates, and it not only provides quantum gravity but also unifies all the interactions of nature in a unique description. There are five different string theories, all of them living in ten spacetime dimensions, and remarkably their low energy dynamics are described by 10D supergravities[1]. It was discovered during the nineties that these so called superstring theories, together with 11D supergravity, can be effectively seen as different dynamical limits of an eleven dimensional theory called M-theory [11]. In turn, some string vacua can be given a non-perturbative description in terms of a twelve dimensional theory called F-theory [12]. The higher dimensional physics of string theory is related to our 4D experience via the compactification of the extra dimensions, and in fact one of its main successes took place when it was given a microscopic description of 4D and 5D supergravity black holes as compactified configurations of 10D superstring objects called branes, allowing to compute their entropy by counting the microscopic states [13, 14]. It is clear then that the study of supergravity solutions in various dimensions and their relations via compactification or uplifting are of huge interest for string theorists. In the case at hand, 6D minimal supergravity is given in terms of F-theory on an elliptically fibered Calabi-Yau manifold with base $\mathbb{P}^2$ [15].

The research pursued in this thesis is based on the characterization made by Gutowski et al. in [16] of all the supersymmetric solutions of 6D minimal supergravity. In particular, it was discovered there that when the solutions have certain isometries they can be fully determined by six harmonic functions. This fact allowed Crichigno et al. in [1] to discover a six dimensional symplectic group of endomorphisms in the space of solutions, this is, elements of $Sp(6,\mathbb{R})$ transform solutions into solutions. They can take for example flat space into $AdS_3 \times S^3$, or the latter into a black string. It is unclear whether this symmetry has any deeper physical significance or it is just a mathematical curiosity. It might be interesting to study the orbits of each generator of $Sp(6,\mathbb{R})$, to determine if this symmetry can give us information about the structure of the phase space. Additionally, one might be able to obtain new solutions of 6D minimal supergravity acting with these generators on some known backgrounds. Some work in these directions has been done by Flavio Porri in [17] and has served as a starting point for this research. Namely, we have studied one solution obtained there by applying an entropy conserving $Sp(6,\mathbb{R})$ transformation on $AdS_3 \times S^3$. The result has been found to describe a non asymptotically flat spacetime with a curvature singularity. Moreover, the original $AdS_3 \times S^3$ solution has an inert parameter that after the transformation becomes quite relevant. When it is positive, the new solution has an $S^3 \times S^1$ horizon in the limit in which S^1 has zero size. When it is negative there is no horizon, and we try to understand the resulting naked singularity in terms of negative branes in superstring theory following [18]. When the parameter vanishes, the transformation amounts to a simple change of coordinates in $AdS_3 \times S^3$. We show also how a close look at the non-flat asymptotics suggests that the solution might be a superposition of

[1]Recall that string theory becomes relevant at the Plank scale, so for many purposes it is enough to consider its low energy limit, i.e. supergravity.

some object with plane waves radiating to spatial infinity.

The structure of this document is as follows. In section 2 we review the classical black hole solutions of 4D general relativity and its thermodynamics. We also introduce AdS spacetime and the three dimensional BTZ black hole. In section 3 we start by introducing global supersymmetry, and later we gauge it to obtain supergravity. Some simple supergravity theories are reviewed as examples, and the extremal charged black hole is shown to be a BPS solution. We give some more details on 10D type IIB supergravity and an important class of its solutions, p-branes, to continue with superstring theory. We stress the important relation between supergravity p-branes and stringy Dp-branes and introduce the negative version of the latest. Section 4 is dedicated to 6D minimal supergravity. After giving an overview of the theory we reproduce [16] in broad lines to obtain its supersymmetric solutions. The $Sp(6,\mathbb{R})$ group is then introduced, and we finish with a discussion about the dimensional reduction of the theory. The content so far comprises the "literature" part of this thesis. Next, in section 5 we show the results obtained in the study of the transformed $AdS_3 \times S^3$. Section 6 is an extension of the previous one, in which we act with more $Sp(6,\mathbb{R})$ elements on the solution at hand in order to uncover its characteristics. We give some conclusions and suggest future research directions in 7. Finally, appendices A and B provide a brief explanation of the Hodge dual operator, vielbeins and different types of spinors for completeness, and appendix C collects the three form field G expressions for the solutions studied in sections 5 and 6.

Conventions

In this thesis we use natural units for which $c = \hbar = k_B = 1$, but the gravitational constant G_{N} is kept explicit. The signature of spacetime is mostly plus, i.e. $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$. We use the Einstein summation convention: an index appearing as a subscript and as a superscript is summed over all its possible values. The components of a p-form α are given by

$$\alpha = \frac{1}{p!}\alpha_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p},$$

and the volume form of a D dimensional manifold with metric tensor $g_{\mu\nu}$ is

$$\mathrm{vol}_D = \sqrt{|g|}dx^1 \wedge \ldots \wedge dx^D,$$

with $g = \det(g_{\mu\nu})$. The Levi-Civitta symbol is defined

$$\varepsilon_{\mu_1\mu_2\ldots\mu_n} = \begin{cases} +1 & \text{if } (\mu_1, \mu_2, \ldots, \mu_n) \text{ is an even permutation of } (1, 2, \ldots, n) \\ -1 & \text{if } (\mu_1, \mu_2, \ldots, \mu_n) \text{ is an odd permutation of } (1, 2, \ldots, n) \\ 0 & \text{otherwise} \end{cases} .$$

The exterior derivative on forms acts from the left, i.e. for $\alpha = \alpha_\mu dx^\mu$ we have

$$d\alpha = \partial_\nu \alpha_\mu dx^\nu \wedge dx^\mu.$$

2 Black holes in general relativity

This thesis is devoted to study gravity solutions in six spacetime dimensions. However, 4D solutions are more intuitive and have been investigated in detail, as our physical experience takes place in precisely four dimensions. For this reason the rich 6D geometries are usually interpreted in terms of their 4D analogues. This chapter reviews the various types of black holes that one can obtain in 4D general relativity, and by doing this introduces key concepts to be used later. Furthermore, the black hole solution in the non-dynamical 3D gravity is introduced, together with the cosmological constant.

2.1 The Schwarzschild black hole

Our starting point is the Einstein-Hilbert action in four dimensions

$$S = \frac{1}{16\pi G_{\mathrm{N}}} \int d^4x \sqrt{-g} R + S_{\mathrm{m}}, \tag{2.1}$$

that relates the geometry of spacetime with its matter and energy content described by S_{m}. Here g is the determinant of the metric tensor and R the Ricci scalar. One can then define the energy-momentum tensor $T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}}$ and vary the action with respect to the metric in order to obtain the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu}. \tag{2.2}$$

When there is no matter or energy, $S_{\mathrm{m}} = 0$ and thus $T_{\mu\nu} = 0$. In that case the trace of Einstein equations shows that the Ricci scalar vanishes, and (2.2) reduces to

$$R_{\mu\nu} = 0. \tag{2.3}$$

Here we are interested in the Schwarzschild solution of Einstein gravity, which describes the gravitational field of a point-like massive object. It is thus a solution in empty space, and it must be static and spherically symmetric. Actually, it is the most general static, spherically symmetric solution of the vacuum Einstein equations [19]. The metric in Schwarzschild coordinates is

$$ds^2 = -\left(1 - \frac{2G_{\mathrm{N}}M}{r}\right) dt^2 + \left(1 - \frac{2G_{\mathrm{N}}M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{2.4}$$

where $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric on a two-sphere of unit radius, and M is the total mass of the object. A quick look at the metric shows that some of its components blow up at $r = 0$ and $r = r_s \equiv 2G_{\mathrm{N}}M$. The latter is not a physical singularity, but just an artifice of our coordinate choice. This can be seen changing to Krustal-Szekeres or Eddington-Finkelstein coordinates because for them the metric is regular at $r = r_s$. A singularity can be proved to be physical when it shows up in a coordinate independent quantity, i.e. a scalar. This is precisely what happens for $r = 0$

when we look at, for example, the Kretschmann scalar

$$K \equiv R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{12 r_s^6}{r^6}. \tag{2.5}$$

The surface of $r = r_s$ is the event horizon, a boundary inside which events cannot affect the outer region. Any observer within this horizon is unable to travel outside, and will reach the singularity at $r = 0$ in a finite amount of proper time. This can easily be seen from the fact that ∂_t and ∂_r change to spacelike and timelike respectively for $r < r_s$.

Something interesting happens if we take a negative mass, i.e. $M < 0$. In this case $r_s < 0$ so there is no horizon enclosing the singularity: we have a so called naked singularity. There is a broadly accepted hypothesis called cosmic censorship conjecture (CCC) that states that all singularities formed by gravitational collapse (this excludes the Big Bang) must be hidden inside event horizons [20], and thus naked singularities are not physical. Our naked singularity occurs for negative mass, so it is easy to see that it is pathological. In general, spacetimes with naked singularities are linked to these non-physically reasonable phenomena like the violation of some energy condition or the requirement of exotic initial conditions. However, the problem of finding a mathematical proof for the CCC (or instead ruling it out) is still open, partly because a precise formulation of the conjecture has not been found.

A related topic is that of closed timelike curves (CTCs). If a spacetime admits timelike trajectories that can close, it would imply the possibility of an observer to travel backwards in time, with all its problematic implications for causality. For this reason, although CTCs are mathematically possible in general relativity, there is a hypothesis similar to the CCC stating that "the laws of physics do not allow the appearance of closed timelike curves" [21]. It is called chronology protection conjecture (CPC).

Back to black holes, there is a remarkable analogy between their dynamical laws and the laws of thermodynamics. First it was thought to be just an analogy, because classically black holes are not thermodynamical systems, but the study of quantum effects revealed that they actually emit radiation at a temperature [22]

$$T_{\mathrm{BH}} = \frac{\kappa}{2\pi}, \tag{2.6}$$

with κ the surface gravity of the event horizon, which is defined by $\chi^\rho \nabla_\rho \chi^\sigma = \kappa \chi^\sigma$ evaluated on it. In that equation, χ is the Killing vector field for which the event horizon is a Killing horizon. The second law of thermodynamics applied to systems with black holes gives an expression for their entropy, which is

$$S_{\mathrm{BH}} = \frac{A}{4 G_{\mathrm{N}}}, \tag{2.7}$$

where A is the area of the event horizon. This is the Bekenstein-Hawking entropy. In usual statistical mechanics, the entropy is a measure of the different microstates compatible with a given macrostate. What we are studying are macroscopic description of black holes, and the question of what is their microscopic description is one of the most important in theoretical physics nowadays. As we mentioned in the introduction, string theory provides a microscopic picture for black holes in terms of objects called branes. Remarkably, the entropy calculated by counting these brane microstates coincides with the one given in (2.7).

2.2 Einstein-Maxwell theory

An interesting extension of the Schwarzschild solution occurs if we give the massive object a non-zero electric or magnetic charge. These charges fill the entire space with a field that has an associated energy, such that S_{m} does not vanish any more. The proper framework to describe this situation is Einstein-Maxwell theory, whose action is (2.1) with S_{m} given by Maxwell's theory of electromagnetism minimally coupled to gravity:

$$S_{\mathrm{m}} = -\frac{1}{4}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\int F \wedge *F. \tag{2.8}$$

where we have used differential forms notation in the second equality, as well as the Hodge star operator $*$, defined in appendix A. The Maxwell equations written with differential forms are,

$$dF = 0, \qquad d * F = 0, \tag{2.9}$$

If a one-form gauge potential A is used such that

$$F = dA, \tag{2.10}$$

the homogeneous Maxwell equation is automatically satisfied. Gauge freedom implies that A can be chosen up to an exact form, because $A \to A + d\Lambda$ does not change the field strength F. Notice that the equations of motion are invariant under the so called duality transformations, which are

$$\begin{pmatrix} F \\ *F \end{pmatrix} \to G \begin{pmatrix} F \\ *F \end{pmatrix} \qquad \text{with} \qquad G \in GL(2,\mathbb{R})^2. \tag{2.11}$$

The action is not invariant, but transforms as $S_{\mathrm{m}} \to \det(G)S_{\mathrm{m}}$. These transformations mix the electric and magnetic parts of the fields.

We can add a one-form source J to the theory, such that the action becomes

$$S_{\mathrm{m}} = -\frac{1}{2}\int (F \wedge *F + A \wedge *J). \tag{2.12}$$

The Maxwell equations are then

$$dF = 0, \qquad d * F = *J, \tag{2.13}$$

and we can see that duality invariance is spoiled. The electric charge contained inside a closed

[2] Actually, there is a restriction to elements of $Sp(2,\mathbb{R}) \subset GL(2,\mathbb{R})$ in a Lorentzian manifold. The reason is that after a transformation one has F' and $(*F)'$, and consistency requires $*(F') = (*F)'$, which is only possible if the transformation is symplectic.

surface ∂B is obtained integrating the flux through that surface:

$$q = \frac{1}{4\pi} \int_{\partial B} *F. \tag{2.14}$$

Similarly one would expect to get a magnetic charge by doing

$$p = \frac{1}{4\pi} \int_{\partial B} F, \tag{2.15}$$

but application of Stoke's theorem and the homogeneous Maxwell equation in (2.13) yields $p = 0$. This can be bypassed if we subtract a point x_0 from B, obtaining a so called Dirac monopole. In this case Stoke's theorem cannot be applied because ∂B is no longer the boundary of a submanifold, and even though we have $dF = 0$ in $B \setminus \{x_0\}$, (2.15) can yield a non-zero result. What is happening is, in physical terms, that in order to have a non-zero magnetic charge we need to introduce a source of magnetic field in our manifold. Wherever there is a source of this kind $dF = 0$ is not satisfied because our theory does not consider magnetic sources. Hence, in order to avoid the breakdown of the theory we must remove from the manifold the point in which the source is sitting.

Take now a sphere of radius $R > 0$ centred around the monopole. One can then write a local expression for A in an open patch on the surface of this sphere. If we try to extend this patch to cover all of the sphere we will find that it is possible except for a point [23], exactly in the same way that a single coordinate patch cannot map to the full surface. This is a consequence of our space being topologically non-trivial after removing the central point, just like the possibility of covering the sphere with one coordinate patch is spoiled by its non-trivial topology. Note that this situation occurs for every radius $R > 0$, so we actually have a line stretching from the monopole to infinity in which the gauge potential A is not well defined. It is called the Dirac string, and it will be important later in section 4.4.

The Dirac string singularity is just an artefact of the local coordinate representation of the gauge potential, but the actual A is not singular at those points. This implies that the string must not be detectable, which yields interesting consequences. If a charged particle travels around a closed path γ, its wavefunction $\psi(x)$ picks a phase

$$\psi(x) \to \exp\left(i \oint_{\gamma} A\right) \psi(x). \tag{2.16}$$

This phase change could be detected in the interference pattern of particles encircling the Dirac string. If the string must be "invisible" the requirement on this phase change is

$$\oint_{\gamma} A = 2\pi n \qquad \text{with} \qquad n \in \mathbb{Z}, \tag{2.17}$$

when γ wraps around it. Using Stoke's theorem this integral measures the magnetic flux carried by the string, and we see that it must be quantized. This is the Dirac quantization condition, which implies that electric and magnetic charges take discrete values.

2.2.1 The Reissner-Nordström black hole

The field strength sourced by a point-like q electric and p magnetic charge sitting at $r = 0$ is $F_{tr} = -q/r^2$ and $F_{\theta\phi} = p \sin\theta$ with all other independent components to zero. Deriving the associated energy-momentum tensor and solving the Einstein equations we get the Reissner-Nordström solution, whose line element is (for $Q^2 \equiv q^2 + p^2$)

$$ds^2 = -\left(1 - \frac{2G_\mathrm{N}M}{r} + \frac{G_\mathrm{N}Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2G_\mathrm{N}M}{r} + \frac{G_\mathrm{N}Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2. \tag{2.18}$$

It is, as the Schwarzschild one, a spherically symmetric and static solution, and it also has a singularity at $r = 0$. Other important surfaces are those for which the g_{tt} component of (2.18) vanishes, which happens at radii

$$r_\pm = G_\mathrm{N}M \pm \sqrt{G_\mathrm{N}^2 M^2 - G_\mathrm{N}Q^2}. \tag{2.19}$$

There are three different cases (assuming $M > 0$):

- $G_\mathbf{N}M^2 > Q^2$**:** The two roots $r_\pm$ are real and are called outer and inner horizon, respectively. The outer one is the event horizon, from which light cannot escape, and the inner one is the so called Cauchy horizon. Notice that the norms of ∂_t and ∂_r change sign at both horizons, so in this case the singularity can be avoided by an observer.

- $G_\mathbf{N}M^2 = Q^2$**:** There is only one root at $r = G_\mathrm{N}M$. This is the extremal Reissner-Nordström black hole, to be studied with more detail in the following section.

- $G_\mathbf{N}M^2 < Q^2$**:** The roots are imaginary so there are no horizons. We have a naked singularity, just like for the negative mass Schwarzschild black hole. Then, according to CCC there exists a bound on the charge that a physical black hole can have, which is $Q^2 \leq G_\mathrm{N}M^2$.

2.2.2 The extremal Reissner-Nordström black hole

Let us focus our attention on the extremal case. When $G_\mathrm{N}M^2 = Q^2$ one can rewrite (2.18) as

$$ds^2 = -\left(1 - \frac{G_\mathrm{N}M}{r}\right)^2 dt^2 + \left(1 - \frac{G_\mathrm{N}M}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2. \tag{2.20}$$

With a coordinate change to $\rho = r - G_\mathrm{N}M$ we shift the horizon to $\rho = 0$ and transform this line element into

$$ds^2 = -\left(1 + \frac{G_\mathrm{N}M}{\rho}\right)^{-2} dt^2 + \left(1 + \frac{G_\mathrm{N}M}{\rho}\right)^2 \left(d\rho^2 + \rho^2 d\Omega_2^2\right). \tag{2.21}$$

Now we can analyse the near-horizon geometry taking the limit $\rho \to 0$, which yields

$$ds^2 = -\frac{\rho^2}{G_\mathrm{N}^2 M^2} dt^2 + \frac{G_\mathrm{N}^2 M^2}{\rho^2} d\rho^2 + G_\mathrm{N}^2 M^2 d\Omega_2^2. \tag{2.22}$$

We see that the metric has factorized into a two dimensional space parametrized by t and ρ, which is anti-de Sitter space (defined later in section 2.4), and a two-sphere. We have then $AdS_2 \times S^2$ near-horizon geometry, with the AdS scale and the S^2 radius both being $G_{\rm N}M$, such that the total Ricci curvature vanishes. This metric belongs to the Bertotti-Robinson class of solutions [24, 25] and is maximally symmetric, i.e. it has the same number of independent Killing vector fields as Minkowski spacetime.

On the other hand, it is easy to see that taking $r \to \infty$ in (2.20) yields the Minkowski metric in spherical coordinates. In other words, it is an asymptotically flat geometry. This is actually a common feature of all the 4D black holes described in this chapter. One can then say that the extremal Reissner-Nordström black hole interpolates between two maximally symmetric spacetimes: Minkowski at infinity and Bertotti-Robinson near the horizon.

For a extremal black hole the surface gravity is zero by definition, so their temperature vanishes. Note that, nonetheless, their entropy does not vanish because the area of the horizon is still finite. We conclude then that the third law of thermodynamics does not apply to black holes, at least in its strong formulation.

2.3 The Kerr-Newman black hole

So far we have studied black holes created by massive and charged matter, but what if we also add angular momentum? This question is important when it comes to model astrophysical black holes, because their rotation is often not negligible. The solution that describes a charged rotating black hole is called Kerr-Newman, and in Boyer-Lindquist coordinates its metric is

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 - \frac{2a(r^2+a^2-\Delta)\sin^2\theta}{\Sigma}dtd\phi + \\ &+\frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2, \end{aligned} \tag{2.23}$$

with

$$\begin{aligned} \Sigma &\equiv r^2 + a^2\cos^2\theta, \\ \Delta &\equiv r^2 - 2G_{\rm N}Mr + G_{\rm N}Q^2 + a^2. \end{aligned} \tag{2.24}$$

The total mass, charge and angular momentum are M, Q and $J = aM$ respectively. The gauge field is not necessary in our discussion so we omit it. When $Q = 0$ we have the so called Kerr solution, when $J = 0$ we are back in the Reissner-Nordström case and when $Q = J = 0$ we have of course the Schwarzschild black hole.

This spacetime is no longer static and spherically symmetric. Instead it satisfies two weaker conditions related to the existence of certain Killing vector fields: it is stationary and axisymmetric. The metric tensor degenerates at $\Sigma = 0$ and $\Delta = 0$, the former being a true singularity provided $M \neq 0$. Notice that we have $\Sigma = 0$ only for $\theta = \pi/2$ so we are dealing with a so called ring singularity.

On the other hand, $\Delta = 0$ occurs at

$$r_\pm = G_{\rm N}M \pm \sqrt{G_{\rm N}^2M^2 - G_{\rm N}Q^2 - a^2}, \tag{2.25}$$

We have, like in the Reissner-Nordström black hole, three different cases. When $G_{\rm N}^2M^2 \geq G_{\rm N}Q^2 + a^2$ the above equation has real solutions r_+ and r_-, corresponding to the event and the Cauchy horizons respectively. When the bound is saturated both coincide and we have the extremal case, and when $G_{\rm N}^2M^2 < G_{\rm N}Q^2 + a^2$ the singularity $\Sigma = 0$ is naked. We have then that not only the electromagnetic charge is bounded now, but also the angular momentum.

Notice that in the $G_{\rm N}^2M^2 < G_{\rm N}Q^2 + a^2$ case one can not associate an entropy to the solution because there is no horizon whose area one can measure. If one still insists in substituting $A = 4\pi r_+^2$ in the Bekenstein-Hawking formula, it produces a complex result for the entropy because $r_+ \in \mathbb{C}$ when the bound is not satisfied.

2.4 Cosmological constant

We can further generalise (2.1) if we add a cosmological constant term $-2\Lambda/16\pi G_{\rm N}$ to the lagrangian. It modifies the Einstein field equations yielding

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_{\rm N}T_{\mu\nu}. \tag{2.26}$$

We can see that this term is equivalent to an energy-momentum tensor $T_{\mu\nu} = -g_{\mu\nu}\Lambda/8\pi G_{\rm N}$. For a homogeneous and isotropic perfect fluid the energy-momentum tensor is $T_\mu{}^\nu = \mathrm{diag}(-\rho, p, \ldots, p)$ with ρ the energy density and p the pressure. Hence, the cosmological constant term models the presence of a perfect fluid with equation of state $\rho = -p$ filling the entire space, which acts as a vacuum energy density $\rho = \Lambda/8\pi G_{\rm N}$. As a consequence of the extra term, the vacuum solutions of these equations will now have constant curvature

$$R = \frac{2D}{D-2}\Lambda. \tag{2.27}$$

The maximally symmetric solutions of the vacuum (2.26) equations are de Sitter (dS_D) and anti-de Sitter (AdS_D) spacetimes, for positive and negative Λ respectively.

We are interested in the AdS_D solution, which can be defined by its embedding in $D+1$ flat spacetime with signature $(-,-,+,\ldots,+)$ as the hypersurface satisfying

$$-x_1^2 - x_2^2 + x_3^2 + \ldots + x_{D+1}^2 = \frac{(D-1)(D-2)}{2\Lambda} \equiv -l^2. \tag{2.28}$$

The metric can be written, in global coordinates,

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right)dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1}dr^2 + r^2 d\Omega_{D-2}^2, \tag{2.29}$$

with $t \in [0, 2\pi)$ and $r \in \mathbb{R}^+$. Another parametrization is given by the so called Poincaré coordinates, that do not cover the whole manifold. In terms of these coordinates the line element is

$$ds^2 = \frac{l^2}{z^2}\left(dx_\mu dx^\mu + dz^2\right), \tag{2.30}$$

with $dx_\mu dx^\mu$ a metric on $\mathbb{R}^{1,D-2}$.

2.4.1 The BTZ black hole

General relativity in three (2+1) dimensions has no local dynamics, because a graviton has zero propagating degrees of freedom for $D = 3$. Nevertheless, some interesting solutions can be found in this theory if one adds a negative cosmological constant, namely the Bañados-Teitelboim-Zanelli (BTZ) black hole that we will describe here. This black hole shares many characteristics with its four dimensional analogues, but its construction and causal structure are very different [26]. In addition, it is asymptotically AdS_3 instead of flat Minkowski.

The anti-de Sitter spacetime in three dimensions has $SO(2,2)$ as its isometry group. In terms of the coordinates used in (2.28) we take the Killing vector

$$\xi = \frac{r_+}{l}\left(x_3\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_3}\right) - \frac{r_-}{l}\left(x_4\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_4}\right), \tag{2.31}$$

with r_+ and r_- irrelevant constants. The BTZ black hole is constructed by the identification of points under a discrete subgroup of $SO(2,2)$ generated by ξ. In other words, we take the quotient of AdS_3 under the identification

$$x \sim e^{2\pi n\xi}x \qquad \text{with} \qquad n \in \mathbb{Z}. \tag{2.32}$$

As the transformation is an isometry, the quotient spacetime obtained is still a solution of Einstein equations with negative constant curvature, and in fact it is locally AdS_3.

The fact that we are dealing with a black hole comes as follows. In order to avoid CTCs it is necessary and sufficient to require the Killing vector ξ to be spacelike [27]. Hence, in order to make the above identification physically reasonable we need to cut out from the spacetime the regions in which $\xi \cdot \xi \leq 0$. The resulting space is geodesically incomplete, because there are geodesics from the spacelike to the timelike regions of ξ. We have then that the surface $\xi \cdot \xi = 0$ is a singularity in the quotient space. It is not a curvature singularity, as the previous ones reviewed here, but a singularity in the causal structure.

One can write a line element for the BTZ black hole:

$$ds^2 = -N(r)^2 dt^2 + N(r)^{-2}dr^2 + r^2\left[d\varphi + \tilde{N}(r)dt\right]^2, \tag{2.33}$$

with

$$N(r)^2 = -G_\mathrm{N}M + \frac{r^2}{l^2} + \frac{G_\mathrm{N}^2 J^2}{4r^2}, \qquad \tilde{N}(r) = -\frac{G_\mathrm{N}J}{2r^2}. \tag{2.34}$$

In these coordinates the singularity is in $r = 0$, and ξ corresponds to ∂_φ so the discrete identification amounts to take this coordinate to be periodic, i.e. $\varphi \sim \varphi + 2\pi$. Notice that this identification is what makes the black hole, and if it is absent (2.33) just describes a portion of AdS_3. The mass and the angular momentum of the solution are M and J, respectively. The event horizon is the biggest root of $N(r)$.

For the horizon to exist there are two conditions on the charges. If violated, one obtains a naked singularity like in previous cases. These conditions are

$$M > 0, \qquad |J| \leq Ml. \tag{2.35}$$

When the second condition is saturated both horizons coincide yielding the extremal BTZ black hole. The vacuum state, in which the black hole disappears, corresponds to $M \to 0$, which by the above condition implies $J \to 0$ as well. This gives

$$ds^2 = -\frac{r^2}{l^2}dt^2 + \frac{l^2}{r^2}dr^2 + r^2 d\varphi^2. \tag{2.36}$$

From this vacuum state, one can increase M to produce the continuous spectrum of black holes, but lowering M to negative values violates (2.35) producing non physical states. There is an exception for $G_{\mathrm{N}}M = -1$ and $J = 0$, for which the singularity disappears and one obtains the metric (2.29), i.e. AdS_3 spacetime. We have then a continuous black hole spectrum and a discrete state, separated from the vacuum by a mass gap, that corresponds to anti-de Sitter.

It is worth noticing that the BTZ solution can emerge as a final state of collapsing matter, namely conical defects that in (2+1) dimensional gravity are treated as point particles [28]. This is particularly surprising given the absence of local dynamics in the theory and provides one more reason to call this solution a black hole.

3 Supergravity

Supersymmetry (SUSY) provides a unified description of bosons and fermions by adding fermionic generators to the algebras of usual quantum field theories. These carry half integer spin so, when acting on the fields, they transform bosons into fermions and vice versa. There is no experimental evidence for SUSY, but considering a supersymmetric extension of standard model (in which SUSY must be broken to account for the different masses of physical bosons and fermions) is a way of solving the hierarchy problem between the electroweak and Plank scales. Besides, it provides candidates of dark matter particles and achieves the unification of the strong, weak and electromagnetic forces at high energies. The interesting aspects of supersymmetry extend also to the gravitational interaction. In particular, when the superalgebra generators are allowed to vary independently in each point of spacetime, i.e. when SUSY is gauged, one finds that the resulting theory consists of a supersymmetric extension of general relativity: supergravity. A good review of this topic can be found in [29], but for a full treatment check [30].

In this chapter we will give a general description of global supersymmetry before explaining its local version, supergravity. The mathematical treatment of the quantum fields that are present in the standard model is expected to be known, but we will introduce the spin-3/2 field which is essential in supergravity theories. After this, some simple theories will be presented as examples, as well as some of their solutions. Later, we will review type IIB supergravity, to be used later, with some more detail. The main theory with which we will work, namely six dimensional minimal supergravity, is left for next chapter. Finally, we will introduce string theory and we will study how it is connected to supergravity.

3.1 Supersymmetry

We first review global supersymmetry, that will be abbreviated as SUSY. Usual quantum field theories are invariant under Poincaré and internal symmetry transformations. The Poincaré algebra consists of the $D(D+1)/2$ generators $M_{\mu\nu}$ and P_μ, the former corresponding to Lorentz transformations and the latter to translations, and its structure is given by

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\mu\sigma} M_{\nu\rho}, \\ [M_{\rho\sigma}, P_\mu] &= P_\rho \eta_{\sigma\mu} - P_\sigma \eta_{\rho\mu}, \\ [P_\mu P_\nu] &= 0. \end{aligned} \tag{3.1}$$

The internal symmetry transformations, global or local, have generators denoted T_A, and their Lie algebra has structure constants f_{AB}^C such that

$$[T_A, T_B] = f_{AB}^C T_C \tag{3.2}$$

In SUSY, one includes $\mathcal{N}$ spinor supercharges Q^i_α to the algebra under which the theory is invariant. Here α is a spinor index and $i = 1, \ldots, \mathcal{N}$ labels the various distinct supercharges we might add. These generators join the Poincaré and internal symmetry ones forming a so called superalgebra, that in the $\mathcal{N}=1$ case consists of the new relations

$$\begin{aligned} \left\{Q_\alpha, \bar{Q}^\beta\right\} &= -\frac{1}{2}(\gamma_\mu)_\alpha{}^\beta P^\mu, \\ [M_{\mu\nu}, Q_\alpha] &= -\frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta, \\ [P_\mu, Q_\alpha] &= 0, \end{aligned} \tag{3.3}$$

plus those we already had in (3.1) and (3.2). Notice that we have introduced anti-commutators for the fermionic quantities and that $\bar{Q}$ is the Dirac adjoint of Q. An important quantity of the theory is the number of real supercharges $\tilde{\mathcal{Q}}$, which is the number of real components of Q times $\mathcal{N}$.

Taking the trace of the anti-commutator above one obtains $\mathrm{Tr}\left[QQ^\dagger + Q^\dagger Q\right] = 2P^0$. The left hand side of this expression is always positive, so the energy P^0 of any state of the SUSY theory must be positive. The supercharges, and thus the parameters of SUSY transformations, are spinors so they transform bosons into fermions and vice versa. We can then see that a SUSY theory will contain both bosonic and fermionic states, and due to the third (anti-)commutator in (3.3) those states related by a transformation under Q will have the same mass. In fact, for a superalgebra of the form given above, the numbers of bosonic and fermionic degrees of freedom coincide [31].

When $\mathcal{N} > 1$ we have extended supersymmetry. In the so called minimal extension the different supercharges anti-commute and satisfy $\mathcal{N}$ copies of the relations (3.3). When they do not commute one needs to add some objects called central charges, as we will see later in an example.

The field content of supersymmetric theories is organized in supermultiplets, commonly abbreviated as multiplets. A multiplet is a representation of the superalgebra, so it consists of a set of bosonic and fermionic fields that transform among themselves under supersymmetry. There are several types, classified by the maximum spin s_m of the fields included. The gravity (or supergravity) multiplets are those with $s_\mathrm{m} = 2$, the vector or gauge multiplets have $s_\mathrm{m} = 1$ and the chiral and hypermultiplets $s_\mathrm{m} = 1/2$. Obviously, no multiplet with $s_\mathrm{m} = 0$ is possible.

3.2 The Rarita-Schwinger field

Supergravity is the theory of local supersymmetry. This means that, as we will see in section 3.3, the fermionic SUSY transformation parameters are gauged and thus have an associated gauge field. This field has necessarily spin $s = 3/2$ and two indices, one spacetime and one spinor, such that we will denote it by $\Psi_{\mu\alpha}(x)$. In this section we will review the theory of a free spin-3/2 field, also known as vector-spinor or Rarita-Schwinger field. In the context of supergravity it is called gravitino because it is the superpartner of the graviton, i.e. they transform into each other under supersymmetry.

We are then concerned with a free gauge field in Minkowski spacetime, that has a gauge transformation

$$\Psi_{\mu\alpha}(x) \quad \rightarrow \quad \Psi_{\mu\alpha}(x) + \partial_\mu \epsilon_\alpha(x). \tag{3.4}$$

The Rarita-Schwinger action for such a field is [30, ch. 5]

$$S = -\int d^D x \bar{\Psi}_\mu \left(\gamma^{\mu\rho\nu}\partial_\rho - m\gamma^{\mu\nu}\right)\Psi_\nu, \tag{3.5}$$

where m is the mass and spinor indices are omitted. In supergravity the gravitino is massless and interacts with the rest of the fields. However, it is useful to study this limit because the gravitino kinetic term of supergravity actions is written as (3.5) with $m = 0$ and minimally coupled to gravity.

The Euler-Lagrange equation derived from (3.5) is

$$\left(\gamma^{\mu\rho\nu}\partial_\rho - m\gamma^{\mu\nu}\right)\Psi_\nu = 0. \tag{3.6}$$

In the massless case there are $(D-1)\cdot 2^{[D/2]}$ independent equations of motion ($[x]$ stands for the integer part of x), that determine the $D \cdot 2^{[D/2]}$ components of $\Psi_{\mu\alpha}(x)$ up to gauge transformations. Here we have considered $\Psi_\mu(x)$ and $\epsilon(x)$ to be Dirac spinors with $2^{[D/2]}$ complex components, but we will see that in supergravity the type of spinor is different and these numbers change.

3.3 Gauging supersymmetry

As we have seen in section 3.1, the parameters of SUSY transformations are constant spinors, that we will call ϵ_α. If we gauge this symmetry we have spacetime dependent parameters $\epsilon_\alpha(x)$ instead, and as a consequence of the superalgebra relations (3.3), the Poincaré transformations must also be gauged. These local Poincaré transformations are diffeomorphisms, so the theory includes gravity. This is precisely how supergravity works.

A supergravity theory will of course contain a gravity multiplet, formed by the graviton, $\mathcal{N}$ gravitini $\Psi_{\mu\alpha}(x)$ and additional fields depending on the specific theory. The graviton is often described in terms of the vielbein fields $e^a_\mu(x)$, that satisfy $g_{\mu\nu}(x) = e^a_\mu(x)e^b_\nu(x)\eta_{ab}$ (see appendix A). Apart from the gravity one, other multiplets (chiral, vector, tensor ...) of the superalgebra can be added. They are often denoted matter multiplets. When for a given D and $\mathcal{N}$, the theory contains only the gravity multiplet, it is called minimal.

D	Spinor	# of components
4	M	4
5	S	8
6	SW	8
7	S	16
8	M	16
9	M	16
10	MW	16
11	M	32

Table 1: Fundamental spinors and its number of components in terms of the spacetime dimension D.

The structure of the multiplets of a theory highly depends on $\mathcal{N}$. Looking at the possible massless multiplets one can see that supergravity is only possible for $\mathcal{N} \leq 8$, because higher $\mathcal{N}$ requires particles with spin $s \geq 5/2$, for which consistent interacting theories are not known. This gives another constraint: the maximum dimension for a supergravity theory is $D = 11$ [32].

So far, we have not specified which kind of spinor is used for supersymmetry transformations. The rule is to choose the most fundamental spinor, i.e. the one with the fewest independent components. Table 1 shows what is the fundamental spinor for every interesting spacetime dimension D: a Majorana (M), symplectic Majorana (S), Majorana-Weyl (MW) or symplectic Majorana-Weyl (SW) spinor. All these kinds of spinors are explained with more detail in appendix B. It can be seen in the table that in $D = 6$ and $D = 10$ the (symplectic) Majorana and Weyl conditions are compatible, i.e. the chiral components of a (symplectic) Majorana spinor are also (symplectic) Majorana, so the most elementary spinors are those that satisfy both. For this reason the supergravity theories in these dimensions are frequently not denoted by the number $\mathcal{N}$, but by (m, n), where m and n are the numbers of right-chiral and left-chiral pairs of supercharges, respectively.

Example: $4D$ $\mathcal{N} = 1$ supergravity

All these concepts are better understood with a simple example, and for that reason here we introduce $\mathcal{N} = 1$ supergravity in four dimensions. It is the most basic supergravity theory in $4D$, and when no matter multiplets are added it only contains the graviton and one gravitino. Their transformation rules are [30, ch. 9]

$$
\begin{aligned}
\delta e_\mu^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\Psi_\mu, \\
\delta\Psi_\mu &= D_\mu\epsilon \equiv \partial_\mu\epsilon + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\epsilon,
\end{aligned}
\tag{3.7}
$$

with $\omega_{\mu ab}$ the spin connection[3]. From table 1 we know that ϵ and each Ψ_μ are Majorana spinors, and they have four components so the theory has $\tilde{\mathcal{Q}} = 4$ real supercharges.

The action, invariant under the above transformations (and also local Poincaré transformations), is

$$
S = \frac{1}{16\pi G_{\rm N}}\int d^4x eR(e,\omega) - \frac{1}{16\pi G_{\rm N}}\int d^4x e\bar{\Psi}_\mu\gamma^{\mu\rho\nu}\nabla_\rho\Psi_\nu, \tag{3.8}
$$

with

$$
\nabla_\mu\Psi_\nu \equiv \partial_\mu\Psi_\nu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab}\Psi_\nu - \Gamma^\rho_{\mu\nu}\Psi_\rho. \tag{3.9}
$$

The first term is the Einstein-Hilbert action (2.1) written in terms of the vielbein fields and the spin connection ω, and the second is just the massless Rarita-Schwinger field (3.5) minimally coupled to gravity and appropriately rescaled.

[3] In order to work with spinors in curved backgrounds one needs to define curved space gamma matrices. They are defined in terms of the usual gamma matrices in Minkowski by means of the vielbein: $\gamma^\mu(x) \equiv e^\mu_a(x)\gamma^a$. Next, one needs to define a covariant derivative for the spinor fields. This is done by lifting the Levi-Civitta connection on the tangent bundle to the Clifford bundle, obtaining the spin connection. In differential form notation it is

$$
\omega = ede^{-1} + e\Gamma e^{-1},
$$

and the covariant derivative of, for example, a vector field with Minkowski index a is

$$
D_\mu V^a \equiv \partial_\mu V^a + \omega_\mu{}^a{}_b V^b.
$$

3.4 Solutions

The most important solutions of a supergravity theory are the background solutions, also called vacua. They are obtained from the classical equations of motion, and then quantum mechanical perturbations can be studied on them. These backgrounds are usually taken to have vanishing fermion fields for simplicity, so they are characterized by the values of the bosonic ones.

It is interesting to ask whether a solution is invariant under a subset of the supersymmetries of the theory. A field configuration Φ_0 (a solution) is said to preserve some supersymmetry if there is a non-vanishing choice of the fermionic transformation parameter $\epsilon(x)$ that leaves it invariant:

$$\delta_\epsilon \Phi|_{\Phi_0} = 0. \tag{3.10}$$

The parameter $\epsilon(x)$ is the fermionic analogue of a Killing vector, so it is called Killing spinor and (3.10) is called Killing spinor equation. In general, for a background, the solution to this equation is a set of $\tilde{\mathcal{Q}}'$ linearly independent spinors. It is then said that the solution preserves a fraction $\tilde{\mathcal{Q}}'/\tilde{\mathcal{Q}}$ of the supersymmetry. Notice that this residual supersymmetry is a global subset of the original local supersymmetry.

Equation (3.10) can be used to construct supersymmetric backgrounds, imposing it as a condition on the bosonic fields. This approach is usually more favourable than trying to solve the equations of motion directly, as one faces first order instead of second order differential equations. One still has to check that the solutions obtained satisfy the EOMs, but happily for some theories this is already guaranteed by the equation (3.10).

Example: $4D$ $\mathcal{N}=1$ **supergravity**

In order to illustrate the study of classical solutions we consider again the four dimensional $\mathcal{N}=1$ supergravity theory. The simplest solution is Minkoski spacetime, for which the field configuration Φ_0 is

$$\begin{aligned} g_{\mu\nu}|_0 &= \eta_{\mu\nu}, \\ \Psi_\mu|_0 &= 0. \end{aligned} \tag{3.11}$$

The transformation rules (3.7) evaluated in this background values are

$$\begin{aligned} \delta e^a_\mu &= 0, \\ \delta\Psi_\mu &= \partial_\mu \epsilon, \end{aligned} \tag{3.12}$$

because the spin connection vanishes for Minkowski spacetime. Now, in order to find the residual supersymmetry we must impose these variations to be zero. For the first one this is already the case, and from the second we get the Killing spinor equation $\partial_\mu\epsilon = 0$, which is solved by four constant linearly independent Majorana spinors. Hence, Minkowski spacetime preserves all the supersymmetry of the theory.

Another example: $4D$ $\mathcal{N}=2$ supergravity

In this example, we consider again a supergravity theory in four dimensions, but with two supercharges instead of one. Their anti-commutator is

$$\{Q^A_\alpha, Q^B_\beta\} = -\frac{1}{2} P_{L\alpha\beta} \mathcal{Z}^{AB}, \tag{3.13}$$

where $A, B = 1, 2$ label the supercharges, P_L is the left chiral projector and $\mathcal{Z}^{AB}$ are the components of an antisymmetric matrix. These $\mathcal{Z}^{AB}$ are called central charges because they commute with all other operators of the superalgebra. For $\mathcal{N}=2$ we can write $\mathcal{Z}^{AB} = \varepsilon^{AB}\mathcal{Z}^{12} \equiv \varepsilon^{AB}\mathcal{Z}$. In the minimal extension case we have $\mathcal{Z}=0$. Some manipulation of (3.13) together with (3.3) and a unitarity requirement lead to the BPS (Bogomol'nyi-Prasad-Sommerfield) bound [33, ch. 25]

$$M \geq |\mathcal{Z}|\,. \tag{3.14}$$

Solutions that saturate (3.14) are called BPS, and in fact supersymmetric solutions are always BPS. In fact, those that preserve all of the supersymmetry are called full-BPS, those that preserve $1/2$ of it are called half-BPS and so on ... One can then see that central charges are necessary in order to have massive supersymmetric solutions.

Extended supergravity theories contain $\mathcal{N}(\mathcal{N}-1)/2$ vector fields that are $U(1)$ gauge bosons. Then, the supergravity multiplet of the $4D$ $\mathcal{N}=2$ theory contains the graviton, two gravitini and the gauge boson, called graviphoton. We can see then that the bosonic content is the same as for the Einstein-Maxwell theory described in section 2.2, if we identify the graviphoton here with the electromagnetic photon there. The transformations of the gauge group mentioned above are the central charges. Then, following the identification with Einstein-Maxwell theory, we can write our central charge in terms of the electric and magnetic charges getting [7]

$$\mathcal{Z} = \frac{q+ip}{\sqrt{G_\mathrm{N}}} \qquad \rightarrow \qquad |\mathcal{Z}|^2 = \frac{q^2+p^2}{G_\mathrm{N}} \equiv \frac{Q^2}{G_\mathrm{N}}. \tag{3.15}$$

Now we can see that the extremal condition $G_\mathrm{N}M^2 = Q^2$ of the Reissner-Nordström black hole implies that it saturates the BPS bound (3.14). In other words, the metric given by (2.20), its associated Maxwell gauge field and two gravitini set to zero are a BPS solution of $4D$ $\mathcal{N}=2$ supergravity. Many black holes can be made BPS in supergravity, for which they need to be extremal. The converse is not necessarily true.

We can wonder now what is the residual supersymmetry of the extremal Reissner-Nordström solution. The graviton and graviphoton variations vanish because they are proportional to the gravitini, so we only have the gravitini variation

$$\delta\Psi_{\mu A} = \left(\partial_\mu + \frac{1}{4}\gamma^{ab}\omega_{\mu ab}\right)\epsilon_A - \frac{\sqrt{\pi G_\mathrm{N}}}{2} F_{ab}\gamma^{ab}\gamma_\mu \varepsilon_{AB}\epsilon^B, \tag{3.16}$$

where the up or down position of the $A, B = 1, 2$ indices denote the left and right chiral projections of the spinors, respectively. Imposing this variation to be zero we get a Killing spinor equation that can be solved in terms of four independent spinors [30, ch. 22]. The theory has eight real supercharges so the extremal Reissner-Nordström solution is half-BPS.

The number of residual supersymmetries is doubled in the limits studied in section 2.2.2, i.e. Minkowski space and Bertotti-Robinson geometries. These two are then full-BPS solutions of $4D$ $\mathcal{N} = 2$ supergravity, and we can say that the extremal Reissner-Nordström black hole interpolates between two maximally supersymmetric vacua of the theory. This is one of the reasons why the extremal Reissner-Nordström black hole is considered a supersymmetric soliton. Solitons are stationary, regular, stable and finite energy solutions in QFTs that tipically interpolate between vacua, and all these properties are satisfied by the extremal Reissner-Nordström black hole.

3.5 Type IIB supergravity

As we saw in section 3.3, in 10 dimensions the supercharges Q are chiral spinors. There are three types of supergravity theories in 10D depending on the number and chirality of these supercharges:

- **Type I:** One chiral supercharge.
- **Type IIA:** Two supercharges of opposite chirality.
- **Type IIB:** Two supercharges of the same chirality.

The names of these different algebras will be suggestive for those who are familiarized with string theory. In fact, these supergravity theories are the low energy limit of the string theories of the same name.

In type IIB we have then two supercharges satisfying $Q^A = P_L Q^A$ with $A = 1, 2$, and their anti-commutator is

$$\{Q^A_\alpha, Q^B_\beta\} = -\frac{1}{2}\delta^{AB}(\gamma^a)_{\alpha\beta}P_a. \tag{3.17}$$

These supercharges are Majorana-Weyl spinors, which have 16 real components in 10D. The supergravity multiplet is formed by the graviton, two gravitini of the same chirality, a four-form with self-dual field strength, two two-forms, two spinors of the same chirality and two scalars [34].

The bosonic matter content of some supergravity theories includes p-forms, as we have just seen. Just like the one-form potential A of Maxwell theory couples to particles, the natural coupling of a p-form is with objects whose world-volume extends in p dimensions. The so called electric couplings have the form

$$\int_{M_p} A^{(p)}, \tag{3.18}$$

where M_p is the world-volume of the object and $A^{(p)}$ is the form. By analogy to the Maxwell case one can take the Hodge dual of the field strength of $A^{(p)}$ to obtain the magnetic coupling. The extended objects are called strings when their world-volume is two dimensional and branes when it has more dimensions. Hence, supergravity is naturally a theory of strings and branes, and we expect to understand it in the frame of string theory as we said above. In fact, it has been proved that one can work classically with IIB supergravity, but when quantum corrections are studied one needs to consider the string theory in order to avoid nonrenormalizable divergences [35].

We now focus on a particular family of solutions of type II supergravity called p-branes, representing objects that extend in p spatial dimensions. When these objects are taken to be extremal, they are charged with respect to the bosonic fields of the theory, so using equation (3.18) we can know their dimensionality. In the IIB case, the two-forms will have a 1-brane as electric source, and a 5-brane as magnetic source and the four-form will have 3-branes as both electric and magnetic sources[4]. An extremal p-brane solution is, in string frame[5],

$$ds^2 = H_p(\vec{x})^{-1/2}\left(-dy_0^2 + \sum_{m=1}^{p} dy_m^2\right) + H_p(\vec{x})^{1/2}\sum_{n=1}^{D-p-1} dx_n^2, \tag{3.19}$$

$$A^{(p+1)} = \frac{H_p(\vec{x})^{-1} - 1}{g_s} dy^0 \wedge \ldots \wedge dy^p, \tag{3.20}$$

$$e^{-2\phi} = g_s^{-2} H_p(\vec{x})^{(p-3)/2}, \tag{3.21}$$

where H_p is a harmonic function given by

$$H_p(r) = 1 + \left(\frac{r_p}{r}\right)^{7-p} \quad \text{with} \quad r = |\vec{x}| \quad \text{and} \quad r_p^{7-p} = g_s N \alpha'^{(7-p)/2}(4\pi)^{(5-p)/2}\Gamma\left(\frac{7-p}{2}\right). \tag{3.22}$$

$A^{(p+1)}$ is the $(p+1)$-form coupled to the brane and ϕ is one of the scalars of the theory, called dilaton in the string theory setting. Extremal p-brane solutions are important because they are half-BPS solitons and they can be associated with a very important object in string theory, Dp-branes.

3.6 Embedding in string theory

String theory claims that the fundamental objects of nature are not point-like particles, but strings that spatially extend in one dimension. The dynamics of a string in a background with metric $G_{\mu\nu}(X)$ are given by the non-linear sigma-model action [36]

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}h^{ab}G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu, \tag{3.23}$$

where σ^0 and σ^1 are coordinates on the world-sheet Σ, h_{ab} is its metric and α' is the only independent dimensionful parameter of string theory, which is directly related to the length of the strings $l_s = \sqrt{\alpha'}$.

[4] One can also associate brane sources to the scalars, which are zero-forms. These, called (-1)-brane and 7-brane have exotic properties and are important in string theory, but their study is out of the scope of this thesis.

[5] The Einstein frame is the field parametrization in which the standard Einstein-Hilbert action (2.1) is written. However, in string theory one often uses a different parametrization called string frame, in which some power of e^ϕ (with ϕ one of the scalars of the theory called dilaton) multiplies the action. One can go from one frame to the other using

$$g_{\mu\nu}^{(E)} = e^{-4\frac{\phi - \langle\phi\rangle}{D-2}} g_{\mu\nu}^{(s)},$$

where $\langle\phi\rangle$ denotes the v.e.v. of the dilaton.

Usually this is set to be the Planck length, as string theory is a quantum theory of gravity and one expects to find quantum gravity effects at this scale. This done, the theory is free of adjustable parameters. X^μ are D scalar fields in two dimensions that describe the string embedding in the curved D dimensional background.

After quantizing, one finds that the spectrum is given by the oscillation states of the string. The excited states of the string spectrum will have masses given by the inverse of l_s, i.e. in the Planck scale, so they will not be accessible at usual energies. We will then focus our attention in the massless sector of the theory, but first we must specify the boundary conditions in the spatial direction σ^1 of the world-sheet, for which we have several options. When periodic boundary conditions are chosen, one is describing closed strings. The massless states given by closed strings are the graviton $G_{\mu\nu}$, a two-form $B_{\mu\nu}$ and a scalar ϕ called dilaton, whose vacuum expectation value fixes the string coupling parameter g_s. A question arises at this point: if the graviton is produced by the closed string dynamically, is it consistent to introduce a curved background $G_{\mu\nu}(X)$ in the sigma model action? The answer is yes, because one can see that the background in (3.23) is actually a coherent state of gravitons. One can generalize this to include a coupling to the two-form $B_{\mu\nu}$ by adding in the action a term

$$S_B = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \varepsilon^{ab} B_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu. \tag{3.24}$$

Other possible boundary conditions are Neumann and Dirichlet, which describe open strings and are defined as

$$\partial_1 X^\mu|_{\partial\Sigma} = 0, \qquad\qquad \partial_0 X^\mu|_{\partial\Sigma} = 0, \tag{3.25}$$

respectively. Notice that the Dirichlet condition implies that the endpoints of the string are fixed in the X^μ direction, so if we want to satisfy conservation of momentum at these points we must have some dynamical object there. These are the Dp-branes, objects to which the endpoints of the string are attached.

Generically one has Neumann boundary conditions along the time component X^0 and p spatial components, and Dirichlet boundary conditions along the $D-p-1$ remaining directions. We have then a Dp-brane defined by $X^\alpha = X_0^\alpha$ with $\alpha = p+1, \ldots, D-1$, which extends in p spatial dimensions and thus has a $p+1$ dimensional world-volume. The massless states obtained from open strings attached to the brane are a p dimensional gauge boson A_μ with $\mu = 0, \ldots, p$ and $D-p-1$ scalars Φ^α that describe the oscillations of the brane in the transverse directions. These fields live on the world-volume of the brane, which we will parametrize using coordinates ξ^a with $a = 0, \ldots, p$, so the scalar fields $\Phi^\alpha(\xi^a)$ are the embedding fields of the brane in analogy to the $X^\mu(\sigma^a)$ fields for the string. We conclude then that string theory is not only a theory of strings, but also of more extended objects called branes.

Consider the spectrum of strings in a configuration with N parallel Dp-branes. We can label the endpoints of the open strings with numbers $i, j = 1, \ldots, N$ called Chan-Paton factors, associated to the brane in which the endpoint lays. As a consequence, we will have N^2 sectors labelled by the two Chan-Paton factors of the string i, j (we are dealing here with oriented strings, for unoriented the number would be $N(N+1)/2$. The N sectors i, i correspond to strings starting and ending in the same brane, and we know from the previous paragraph that their spectrum contains a massless gauge boson A_μ. When the branes sit at different positions we will have then a $U(1)^N$ symmetry. However, if the N branes coincide we will have N^2 copies of the spectrum of a single brane, and we can arrange the massless fields in a matrix of Chan-Paton factors

$$(A_\mu)_i^{\ j}, \qquad\qquad (\Phi^\alpha)_i^{\ j}. \tag{3.26}$$

One can see that the $(A_\mu)_i^{\ j}$ form now a $U(N)$ gauge connection [37, ch.7], so by putting the N branes together we have enhanced the gauge symmetry from $U(1)^N$ to $U(N)$. In addition, the scalars $(\Phi^\alpha)_i^{\ j}$ transform in the adjoint representation of the group.

So far we have considered only D bosonic fields $X^\mu(\sigma^a)$ in the world-sheet, but the resulting theory, called bosonic string theory, does not properly describe nature. In order to get a realistic theory one needs to extend it adding D world-sheet fermions $\psi^\mu(\sigma^a)$ in the action:

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \sqrt{-h}\eta_{\mu\nu} \left(h^{ab}\partial_a X^\mu \partial_b X^\nu + i2\bar{\psi}^\mu \gamma^a \partial_a \psi^\nu \right), \tag{3.27}$$

where we have considered now a flat background, and γ^a are gamma matrices in the world-sheet. The result is called superstring theory and lives in $D = 10$. Choosing different types of strings leads to the various theories: type I for open strings and type II for closed ones. Among type II the supersymmetry requirements still allow to distinguish between IIA, for which the fermionic ground states are not chiral, and IIB, for which they are.

3.6.1 Type IIB superstring theory

The bosonic massless spectrum of type IIB superstring theory consists of the graviton $G_{\mu\nu}$, a two-form $B_{\mu\nu}$, a dilaton ϕ and three gauge fields $A^{(0)}$, $A^{(2)}$ and $A^{(4)}$ that are zero-, two- and four-forms respectively. The latter has a self-dual field strength $F^{(5)} = *F^{(5)}$. As for the massless fermions, we have two spin-3/2 particles, the gravitini, and two spin-1/2 ones called dilatini, and as we said all of them have the same chirality. It can be noticed that the field content is the same as for the type IIB supergravity, in consistency with our claim that it is the low energy limit of type IIB superstring theory.

At this point we can ask ourselves what the string theory looks like around those backgrounds we found for the type IIB supergravity: extremal p-branes. Polchinski showed in [38] that the string description of a p-brane is given by a $p+1$ surface around which the spectrum of fluctuations of the theory can be obtained quantizing open strings whose endpoints are attached to it. In other words, the string theory description of p-branes are Dp-branes[6]. One can actually check that the tension (energy per unit volume) and charges of the supergravity p-branes and superstring Dp-branes coincide, and that the backreaction of the Dp-branes on a flat background produces the metric of

[6]There are some exceptions to this claim like the NS5-brane, which does not have a good stringy description. In any case, these exceptions can be ignored for our purposes.

the p-brane solution (3.19). It is worth noticing that we are dealing here with a non-perturbative state of string theory, because these Dp-branes are solutions that cannot be described as oscillatory states of strings. In fact, the brane tension decreases (except for the 1-brane case) in the strong coupling limit $g_s > 1$. This implies that these objects become lighter than the strings and effectively dominate the low energy physics in this regime.

We have seen in section 3.5 that p-branes source the form fields of the supergravity theory. Now these fields are $B_{\mu\nu}$, $A^{(4)}$, $A^{(2)}$ and $A^{(0)}$, and certain Dp-branes will carry their charges in the string theory description. This is just another signal that string theory must necessarily include branes, as they are charged under certain fields that the strings alone can describe but not source. Due to their special properties, the objects charged under the Kalb-Ramond field $B_{\mu\nu}$ are called fundamental string or F-string and NS5-brane. The fields and branes of type IIB string theory are summarized in table 2.

Field	Electrically coupled to	Magnetically coupled to
$B_{\mu\nu}$	F-string	NS5-brane
$A^{(4)}$	D3-brane	D3-brane
$A^{(2)}$	D1-brane	D5-brane
$A^{(0)}$	D(-1)-brane	D7-brane

Table 2: Form fields of type IIB superstring theory and the objects they couple to.

The fact that the branes are BPS solutions implies a certain relation between their tension and charges, that causes a cancellation of forces when various branes are set on a background. This allows one to write stable solutions containing an arbitrary number of branes because the gravitational attraction and electrical repulsion exactly compensate each other. Recall that the BPS condition is satisfied (or not) at the level of the supersymmetry algebra, so BPS states do not cease to be BPS under changes of parameters of the theory (like α' or g_s) or quantum corrections. This property is essential to assure the existence of the non-perturbative brane solution beyond the low energy limit [39, ch. 6].

The stringy description of an extremal p-brane is actually more elaborate. The solution (3.19)-(3.22) has N units of charge under the $A^{(p+1)}$ gauge form, meanwhile a Dp-brane has one [7]. The supergravity p-brane is then understood as the field generated by a stack of N Dp-branes located at the same position, and it is also called black brane solution. In a situation with N Dp-branes the calculation of a typical perturbative string diagram includes a trace over the Chan-Paton factors, so together with the string coupling g_s we will have a factor N and the effective expansion parameter will be g_sN. This implies that the perturbation theory is valid for $g_sN < 1$. On the other hand, the supergravity approximation is valid for low energies or, in other words, when the curvature of the geometry is small compared to l_s and string effects do not arise. The curvature is (inversely) related to r_p and from (3.22) we have

$$r_p^{7-p} \sim g_s N \alpha'^{(7-p)/2} = g_s N l_s^{7-p}, \tag{3.28}$$

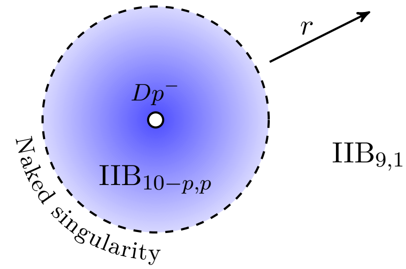

Figure 1: Negative brane Dp^- in $r = 0$ surrounded by a naked singularity at some finite radius. Inside it, the string theory has non-Lorentzian signature $\{10-p, p\}$. The coordinate r measures the distance to the brane in the transverse directions.

so the supergravity solution holds when $g_s N \gg 1$. We see then that the stringy and supergravity descriptions are complementary, which lays the basis for AdS/CFT correspondence [6].

3.6.2 Negative branes

For future convenience, we introduce here negative branes. They are defined as the objects that cancel the effect of usual Dp-branes, meaning that two coinciding branes, one ordinary and one negative, are equivalent to a flat vacuum field configuration. In order to get this, the supergravity description of a stack of N_+ usual branes and N_- negative branes is given by the usual solution (3.19)-(3.22) under the substitution $N \to N_+ - N_-$. In other words, Chan-Paton factors of negative branes carry a minus sign.

We had that a stack of N Dp-branes realizes a $U(N)$ gauge symmetry in the target space, but what is the effect of negative branes? It was shown in [40] that for N_+ usual and N_- negative branes the gauge symmetry group is actually a supergroup $U(N_+|N_-)$ because the negative sign of the Chan-Paton factors change the statistics of the massless string states, making fermions appear.

We focus now on the backreaction of negative branes. We can see in 3.22 that when $N_- > N_+$ there exists a radius r_s for which $H_p = 0$, which causes a naked curvature singularity. This makes sense because the tension of a negative brane is negative (recall that the Schwarzschild metric describes a naked singularity when its mass is negative). The picture is then a stack of branes sitting in $r = 0$ surrounded by a naked singularity at $r = r_s$. Moreover, Dijkgraaf et al. showed in [18] that in the region inside the naked singularity, i.e. $0 < r < r_s$, the metric signature of spacetime is different from the usual $\{9, 1\}$ (nine positive and one negative eigenvalues). This is visible with a heuristical argument that goes as follows. In the bubble $0 < r < r_s$ the harmonic function H_p is negative so in order to study the solution (3.19)-(3.22) in this region one must analytically continue it. This yields

$$ds^2 = i^{-1}\bar{H}_p(\vec{x})^{-1/2}\left(-dy_0^2 + \sum_{m=1}^{p} dy_m^2\right) + i\bar{H}_p(\vec{x})^{1/2}\sum_{n=1}^{D-p-1} dx_n^2, \tag{3.29}$$

$$A^{(p+1)} = \frac{1-\bar{H}_p(\vec{x})^{-1}}{g_s} dy^0 \wedge \ldots \wedge dy^p, \tag{3.30}$$

$$e^{-2\phi} = i^{p-3} g_s^{-2} \bar{H}_p(\vec{x})^{(p-3)/2}, \tag{3.31}$$

with $\bar{H} = -H$. Now a field redefinition of the metric allows to eliminate the imaginary units, getting

$$ds^2 = -\bar{H}_p(\vec{x})^{-1/2}\left(-dy_0^2 + \sum_{m=1}^{p} dy_m^2\right) + \bar{H}_p(\vec{x})^{1/2}\sum_{n=1}^{D-p-1} dx_n^2, \tag{3.32}$$

and we see that the signature of the brane world-volume directions has been flipped, yielding a spacetime signature $\{10-p, p\}$. The conclusion, more rigorously proved in [18], is that the negative branes live in a string theory with non-Lorentzian signature contained inside a bubble, and the boundary of the bubble is a naked singularity beyond which the usual string theory is recovered. A schematic picture is shown in figure 1.

4 6D minimal supergravity

During most of the present thesis we will work with solutions of six dimensional minimal supergravity, also called 6D $(1,0)$ supergravity. In this chapter we will study its field content, main characteristics and the general form of its supersymmetric solutions. We will also present some of them, because they are the starting point of the forthcoming research part. After parametrizing the phase space of a certain class of supersymmetric solutions, we will introduce the $Sp(6,\mathbb{R})$ group of endomorphisms on it. This group is interesting because its action on the solutions is clear in mathematical terms, but its physical significance has not been explored yet. Finally, we talk about the reduction of the theory to five dimensions.

4.1 Description of the theory

The theory we are about to study is minimal so it only contains a supergravity multiplet. It consists of the graviton, a symplectic Majorana-Weyl gravitino Ψ_μ^A and a two-form $B^+_{\mu\nu}$ for which the $+$ index denotes that its field strength $G \equiv dB$ is self-dual [41]. Our symplectic Majorana spinors form representations of the group $Sp(1) = Sp(2,\mathbb{R}) \cap U(2)$ so we need a couple of them. This is the reason for the label $A = 1,2$ in the gravitino, that will be usually omitted. A Majorana-Weyl spinor has $2^{[D/2]-1}$ real components, and we just saw that in our case two of them are necessary because of the symplectic condition. The theory has then $\tilde{\mathcal{Q}} = 8$ real supersymmetries.

The self-duality condition of G cannot be obtained as an equation of motion from any action, unless we add an auxiliary tensor multiplet [42]. The condition is necessary to match the fermionic and bosonic degrees of freedom, so it cannot be relaxed. As we want to stay in the minimal theory adding matter multiplets is not an option, so we will just treat the self-duality of G as an extra condition on our fields. We have then that the equations of motion are

$$G = *G, \tag{4.1}$$

$$dG = 0, \tag{4.2}$$

$$R_{\mu\nu} = G_{\mu\rho\sigma} G_\nu{}^{\rho\sigma}. \tag{4.3}$$

From the first and last equations we can obtain that the Ricci scalar vanishes for any solution of this theory. To show this, recall from equation (A.1) that

$$G \wedge *G = \frac{1}{6} G_{\mu\nu\rho} G^{\mu\nu\rho} \ \mathrm{vol}_6, \tag{4.4}$$

and taking the trace of (4.3) gives $R_\mu{}^\mu = G_{\mu\nu\rho} G^{\mu\nu\rho}$ so

$$G \wedge *G = \frac{1}{6} R_\mu{}^\mu \ \mathrm{vol}_6 = \frac{R}{6} \ \mathrm{vol}_6. \tag{4.5}$$

But due to the self-duality condition on G we have $G \wedge *G = G \wedge G = 0$ and so $R = 0$. We can then consider (4.3) to be the Einstein equations.

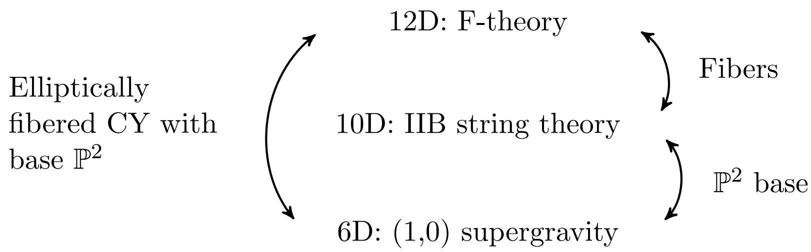

Figure 2: F-theory compactified on a two-torus yields type IIB string theory. Also, when compactified in an elliptically fibered Calabi-Yau manifold with base the complex projective plane, one gets minimal 6D supergravity.

How does 6D minimal supergravity fit in the stringy description of nature? In order to answer this question we need to briefly introduce F-theory. The starting point is the fact that type IIB superstring theory has an $SL(2,\mathbb{Z})$ symmetry called S-duality. This symmetry can be geometrized and made explicit by compactifying a twelve dimensional theory, F-theory, on an elliptical fiber over a base B (due to supersymmetry requirements, the fiber and the base must form a three dimensional complex manifold of a specific class called Calabi-Yau (CY)). This provides a non-perturbative picture of type IIB with $D7$-branes, whose backreaction on the metric and the dilaton is more substantial than for other branes [12]. The backreaction is geometrically taken into account by the fibration, and the loci where the fiber degenerates describe the presence of the brane. When F-theory is compactified on the CY threefold, the result is a 6D theory with (1,0) supersymmetry. The matter content in six dimensions is determined by the Hodge numbers of the particular CY manifold chosen. These numbers contain information about its topological properties and are denoted $h^{p,q}(X)$ with X the manifold. One obtains $h^{1,1}(B)-1$ tensor and $h^{1,1}(X)-h^{1,1}(B)-1$ vector multiplets in the six dimensional theory for a CY manifold X [15]. Although the $D7$-brane sits in singular points of the fiber, we want to keep the total CY manifold non-singular. Possible singularities of the CY manifold in the points where the fiber is singular can be fixed by a procedure called "blowing-up", but it generates gauge fields in the 6D theory that in our minimal supergravity case are undesired. We must look then for spaces that do not become singular when the fiber does, and this in practice means $h^{1,1}(X)=h^{1,1}(B)+1$ so vector multiplets will be absent. Choosing the complex projective plane $\mathbb{P}^2$ as a base satisfies this, and in addition has $h^{1,1}(B)=1$ so tensor multiplets are also absent. It gives then minimal 6D supergravity [43]. Figure 2 shows schematically the relations described.

4.2 Supersymmetric solutions

In section 3.4 it was mentioned that the Killing spinor equation can be used as a starting point to construct supersymmetric solutions of a theory. This method is more systematic than looking for ansatz of a solution of the equations of motion, but is difficult to apply in complicated theories. In contrast, it has been particularly successful for simple supergravities, for which a general form of all supersymmetric solutions has been found. It is the case of some $D=4$ and $D=5$ theories [44, 45], and luckily also of minimal 6D supergravity, whose supersymmetric solutions were described by

Gutowski et al. in [16]. We will follow here their work, although the notation and conventions might not fully coincide.

The Killing spinors are those that yield a vanishing gravitino variation, i.e.

$$\nabla_\mu \epsilon^A + \frac{1}{4} G_{\mu\rho\sigma} \gamma^{\rho\sigma} \epsilon^A = 0, \tag{4.6}$$

where $A = 1, 2$ is the symplectic $Sp(1)$ index. The strategy followed by Gutowski et al. is to write Killing spinor bi-linears. Then, Fierz identities and (4.6) are used to impose some algebraic and differential equations on them, and these equations are enough to determine the local form of the solutions. In our case the bi-linears are

$$\begin{aligned} V_\mu \varepsilon^{AB} &= \bar{\epsilon}^A \gamma_\mu \epsilon^B, \\ \Omega^{AB}_{\mu\nu\rho} &= \bar{\epsilon}^A \gamma_{\mu\nu\rho} \epsilon^B. \end{aligned} \tag{4.7}$$

A Fierz identity implies $V_\mu V^\mu = 0$, and we can then introduce a vielbein e^-, e^+, e^m with $e^- \equiv V$ by writing

$$ds^2 = -2e^- e^+ + \delta_{mn} e^m e^n. \tag{4.8}$$

Besides, the spinor Killing equation makes V a Killing vector field. The Fierz identity also implies

$$\gamma^- \epsilon = 0. \tag{4.9}$$

In the end, all the algebraic and differential relations make the Killing spinor equation simplify to $\partial_\mu \epsilon = 0$, so any constant spinor satisfying (4.9) is a solution. Due to this condition we have that supersymmetric solutions must preserve either none, one half or all the supersymmetry. Other fractions of residual supersymmetry are not allowed. Actually, it is shown in [16] that the only maximally supersymmetric spaces of this theory are three: $\mathbb{R}^{1,5}$, $AdS_3 \times S^3$ and a particular six dimensional Cahen-Wallach space CW_6.

The fact that $V = \partial_v$ is a null Killing vector field allows one to introduce local coordinates v, u and x^m, and partially solve for the vielbein. In these coordinates the metric (4.8) has to be

$$ds^2 = -2H^{-1}(du + \beta)\left[dv + \omega - \frac{F}{2}(du + \beta)\right] + H ds^2_{HK_4}, \tag{4.10}$$

for some v-independent functions H and F and one-forms β and ω. The line element $ds^2_{HK_4}$ corresponds to the four dimensional base space $\mathcal{B}$ in which β and ω live, and is an almost hyper-Kähler manifold[7]. Similarly, one can write the three-form G in terms of these functions and forms. Then, the equations of motion (4.2) and (4.3) impose constraints on them.

[7] A hyper-Kähler manifold is a complex manifold of dimension $4n$ (with $n \in \mathbb{N}$) which admits three complex structures that transform under an $SU(2)$ symmetry. When the structures are almost complex, we have an almost hyper-Kähler manifold. The most general hyper-Kähler fourfold with a Killing vector field that preserves the three complex structures is a Gibbons-Hawking manifold.

4.3 A precise class of solutions

In this thesis we are interested in a particular class of the described supersymmetric solutions. Namely, we will be looking at backgrounds for which ∂_u is a Killing vector field. In this case all the functions and forms are u-independent, $\mathcal{B}$ becomes hyper-Kähler and $d\beta$ is self-dual on $\mathcal{B}$.

Apart from this, we also take $\mathcal{B}$ to be a Gibbons-Hawking (GH) space (see footnote 7). GH spaces consist on a $U(1)$ fibration over $\mathbb{R}^3$:

$$ds^2_{GH} = V_1^{-1}\left(d\psi + \chi\right)^2 + V_1 ds^2_{\mathbb{R}^3}, \tag{4.11}$$

where V_1 is a harmonic function, χ is a one-form satisfying $*_3 d\chi = dV_1$ and $\psi \in [0, 4\pi)$. Both depend only on the $\mathbb{R}^3$ coordinates because the vector field in the fiber direction ∂_ψ is Killing in $\mathcal{B}$. Consequently, the subscript in the Hodge star operator denotes that it is taken in the $\mathbb{R}^3$ base. If we further assume that this ∂_ψ isometry is extended to the full six dimensional spacetime, the complete solution is determined by five additional harmonic functions $V_2, \ldots, V_6$ on $\mathbb{R}^3$.

To sum up, we are considering supersymmetric solutions with two extra symmetries generated by ∂_u and ∂_ψ, the second being the $U(1)$ isometry of a GH base space. Their general form is (now including the self-dual three-form)

$$ds^2 = -2H^{-1}\left(du + \beta\right)\left[dv + \omega - \frac{F}{2}\left(du + \beta\right)\right] + HV_1^{-1}\left(d\psi + \chi\right)^2 + HV_1 ds^2_{\mathbb{R}^3}, \tag{4.12}$$

$$G = -\frac{1}{2} *_4 dH - \frac{1}{2}H^{-1}\left(du + \beta\right) \wedge \left[(d\omega)^- + \frac{F}{2}d\beta\right] + \frac{1}{2}\left(dv + \omega\right) \wedge d\left[H^{-1}\left(du + \beta\right)\right]. \tag{4.13}$$

This set of solutions can be fully specified by six harmonic functions on $\mathbb{R}^3$ that we arrange into a vector $\mathbb{V} = (V_1, \ldots, V_6)$. These functions determine the solution according to

$$\begin{aligned}
\beta &= \frac{V_2}{V_1}(d\psi + \chi) + \tilde{\beta}, \\
\omega &= \left(V_4 + \frac{V_6 V_3 + V_2 V_5}{V_1} + \frac{V_2 V_3^2}{V_1^2}\right)(d\psi + \chi) + \tilde{\omega}, \\
F &= 2V_5 + \frac{V_3^2}{V_1}, \\
H &= V_6 + \frac{V_2 V_3}{V_1},
\end{aligned} \tag{4.14}$$

with

$$*_3\, d\chi = dV_1, \qquad *_3 d\tilde{\beta} = -dV_2, \qquad *_3 d\tilde{\omega} = \langle \mathbb{V}, d\mathbb{V} \rangle\,. \tag{4.15}$$

In (4.15) we have used the symplectic norm on $\mathbb{R}^6$, defined by

$$\langle \mathbb{A}, \mathbb{B} \rangle \equiv \mathbb{A}^{\mathrm{T}}\Omega\mathbb{B} \qquad \text{with} \qquad \Omega = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}. \tag{4.16}$$

We also write, for future convenience, the above solution in terms of the vielbein (4.8)

$$\begin{aligned}
e^- &= H^{-1}\left(du+\beta\right),\\
e^+ &= dv+\omega-\frac{HF}{2}e^-,\\
e^2 &= \sqrt{\frac{H}{V_1}}\left(d\psi+m\cos\theta d\phi\right),\\
e^3 &= \sqrt{HV_1}dr,\\
e^4 &= \sqrt{HV_1}rd\theta,\\
e^5 &= \sqrt{HV_1}r\sin\theta d\phi.
\end{aligned} \tag{4.17}$$

For most of the solutions studied here the harmonic functions will be written in the form $a+b/r$ with r the radial coordinate of the GH base. We will use at some points the notation of [1], in which $\mathbb{V}=\Gamma_\infty+\Gamma/r$ with

$$\Gamma_\infty=\left(m_\infty,q_\infty,p_\infty,j_\infty,\frac{n_\infty}{2},\mu_\infty\right),\qquad \Gamma=\left(m,q,p,j,\frac{n}{2},\mu\right). \tag{4.18}$$

Although we will not encounter them, it is possible to superpose harmonic functions to write multipole solutions, despite they satisfy non-linear equations. This is used to write bound states of various black holes and/or other objects. In these cases the harmonic functions have poles in different points $\vec{x}_a$ of $\mathbb{R}^3$ yielding

$$\mathbb{V}=\Gamma_\infty+\sum_a\frac{\Gamma_a}{|\vec{x}-\vec{x}_a|}. \tag{4.19}$$

4.3.1 Flat space

It is not difficult to construct flat $\mathbb{R}^{1,5}$ spacetime by direct inspection of (4.12). We notice that we want $\beta=\omega=F=0$ and $\chi=\cos\theta d\phi$, from which one easily obtains

$$\mathbb{V}_{\text{flat}}=\begin{pmatrix}1/r\\0\\0\\0\\0\\1\end{pmatrix}. \tag{4.20}$$

As one would expect, this gives $G=0$ and $ds^2=-2dudv+4d\rho^2+4\rho^2d\Omega_3^2=-2dudv+ds^2_{\mathbb{R}^4}$. The reason why ψ was taken to be $\psi\in[0,4\pi)$ was precisely to have the right ranges for the angles in $d\Omega_3^2$ such that it describes a three-sphere. One can alternatively take $V_1=1$ instead of $V_1=1/r$, in which case $\chi=0$ and the metric obtained has the form $ds^2=-2dudv+d\psi^2+ds^2_{\mathbb{R}^3}$. This is locally $\mathbb{R}^{1,5}$ but not globally, as the coordinate ψ was taken to be compact.

4.3.2 $AdS_3 \times S^3$ and the BTZ black hole

As stated before, another maximally symmetric solution of the theory is $AdS_3 \times S^3$. It can be obtained with the harmonic functions [17]

$$\mathbb{V}_{AdS_3\times S^3} = \begin{pmatrix} 1/r \\ 0 \\ 0 \\ 0 \\ c \\ \mu/r \end{pmatrix}. \tag{4.21}$$

The self-dual three-form is

$$G = -\frac{1}{2\mu} dv \wedge du \wedge dr - \frac{\mu \sin\theta}{2} d\psi \wedge d\theta \wedge d\phi, \tag{4.22}$$

and the metric is

$$ds^2 = -\frac{2r}{\mu} du(dv - cdu) + \frac{\mu}{r^2} dr^2 + 4\mu d\Omega_3^2. \tag{4.23}$$

Notice that the parameter c is absent from the three-form, and it can be eliminated from the metric by a change of coordinates $v \to \tilde{v} = v - cu$. It is then an inert parameter that does not change the solution. To show explicitly that (4.23) describes $AdS_3 \times S^3$ we use a new coordinate z defined by $r = 4\mu^2/z^2$, and we get

$$ds^2 = \frac{4\mu}{z^2}\left(-2dud\tilde{v} + dz^2\right) + 4\mu d\Omega_3^2. \tag{4.24}$$

Comparing with (2.30), we see that the first part of the metric is AdS_3, so we have in total the direct product $AdS_3 \times S^3$. Notice that the radii squared of the three-sphere and of anti-de Sitter are both 4μ. This fact makes the two curvatures cancel each other for the total Ricci scalar[8], yielding $R = 0$ as we expect for a solution of this theory.

We can now "switch on" another parameter in (4.21) and get

$$\mathbb{V}_{AdS_3\times S^3} = \begin{pmatrix} 1/r \\ 0 \\ 0 \\ 0 \\ c + n/r \\ \mu/r \end{pmatrix}. \tag{4.25}$$

[8] Recall that the scalar curvature of the direct product of two manifolds $M = M_1 \times M_2$ is the sum of the individual curvatures, i.e. $R = R_1 + R_2$.

The three-form is again (4.22) but the metric has now an extra term:

$$ds^2 = -\frac{2r}{\mu}dud\tilde{v} + \frac{2n}{\mu}du^2 + \frac{\mu}{r^2}dr^2 + 4\mu d\Omega_3^2. \tag{4.26}$$

It is useful now to make the local change of coordinates

$$r = \frac{\rho^2 - 4n}{4}, \qquad u = \frac{t - 2\sqrt{\mu}\varphi}{\sqrt{2}}, \qquad v = \frac{t + 2\sqrt{\mu}\varphi}{\sqrt{2}}, \tag{4.27}$$

which yields

$$ds^2 = -N(\rho)^2dt^2 + \frac{d\rho^2}{N(\rho)^2} + \rho^2\left(d\varphi - \frac{2n}{\rho^2\sqrt{\mu}}dt\right)^2 + 4\mu d\Omega_3^2 \qquad \text{with} \qquad N(\rho)^2 \equiv \frac{\left(\rho^2-4n\right)^2}{4\mu\rho^2}. \tag{4.28}$$

If we now take φ to be periodic ($\varphi \sim \varphi + 2\pi$) this metric corresponds to the extremal BTZ black hole times S^3. This can be seen comparing to (2.33), which also allows to identify the radius, mass and angular momentum of the BTZ part as

$$l^2 = 4\mu, \qquad G_{\mathrm{N}}M = 2\frac{n}{\mu}, \qquad G_{\mathrm{N}}J = \frac{4n}{\sqrt{\mu}}. \tag{4.29}$$

It is interesting to take $n = 0$ now. Doing so we get $M = J = 0$ and thus we are in the vacuum BTZ state (2.36) (times S^3). There is an apparent contradiction, because we have studied above the $n = 0$ case (4.21) and it yielded $AdS_3 \times S^3$. The difference is in the identification $\varphi \sim \varphi + 2\pi$: as discussed in section 2.4.1 when this identification is absent (4.26) just describes some patch of $AdS_3 \times S^3$.

4.3.3 The black string

The construction of black string solutions was already mentioned in the introduction. Given a black hole solution in D dimensions with horizon topology M, it is possible to construct a solution in $D+1$ dimensions by adding a spatial direction. The resulting object will have an extended horizon $M \times \mathbb{R}$ and receives the name of black string [5]. One can also wrap this direction in a circle to render a $M \times S^1$ horizon. In principle this would cause the gravitational collapse of the object, but in five or more dimensions it is possible to set a non-vanishing angular momentum that compensates it. Here we will write a circular black string solution of 6D minimal supergravity, u being the direction of the string. The six dimensional black string naturally reduces to the five dimensional black hole, so we expect it to have a horizon topology $S^3 \times S^1$. As a supergravity solution, it is an extremal object and its near horizon geometry is $AdS_3 \times S^3$.

The first step to build the black string solution is to impose the right asymptotics, in our case $\mathbb{R}^{1,4} \times S^1$ with the S^1 factor corresponding to the periodic coordinate $u \sim u + L$ in which the string extends. In terms of the harmonic functions, the asymptotics are controlled by Γ_∞, so the asymptotic condition plus the bubble equations (that we will see in next section) amount to some restrictions on the form of this vector, found in [1]. The most general form of $\mathbb{V}$ is

$$\mathbb{V}_{\text{bs}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{m}\left(p + qn_\infty/2\right) \\ n_\infty/2 \\ 1 \end{pmatrix} + \begin{pmatrix} m \\ q \\ p \\ j \\ n/2 \\ \mu \end{pmatrix} \frac{1}{r}, \tag{4.30}$$

with $m \in \mathbb{Z}^+$ to ensure the correct asymptotic limit. It is convenient now to introduce the quantities

$$\tilde{Q} \equiv 4\sqrt{2}(m\mu + qp), \qquad Q \equiv 4(mn + p^2), \qquad J \equiv 8\left(qp^2 + mp\mu + \frac{mqn}{2} + m^2 j\right), \tag{4.31}$$

whose physical meaning will become clear in section 4.5. The line element for this solution is

$$\begin{aligned} ds^2 = &-2\left(1 + \frac{\tilde{Q}}{4\sqrt{2}mr}\right)^{-1} du'\left[dv + \frac{J}{8m^2 r}\left(d\psi + m\cos\theta d\phi\right) - \frac{du'}{2}\left(n_\infty + \frac{Q}{4mr}\right)\right] + \\ &+ \left(1 + \frac{\tilde{Q}}{4\sqrt{2}mr}\right)\left[\frac{r}{m}\left(d\psi + m\cos\theta d\phi\right)^2 + m\frac{dr^2}{r} + mrd\Omega_2^2\right], \end{aligned} \tag{4.32}$$

with $u' \equiv u + \frac{q}{m}\psi$[9]. The singularity is at $r = -\tilde{Q}/4\sqrt{2}$ as one can see in the curvature scalar $R^{\mu\nu}R_{\mu\nu}$, and the horizon is at $r = 0$, where the metric degenerates. In order to check that the horizon topology corresponds to a black string we take $r = 0$ in the metric, getting

$$ds^2\big|_{r=0} = \frac{\sqrt{2}Q}{\tilde{Q}}du'^2 - \frac{\sqrt{2}J}{\tilde{Q}}\left(d\psi' + \cos\theta d\phi\right)du' + \frac{\tilde{Q}}{4\sqrt{2}}\left(d\psi'^2 + 2\cos\theta d\psi' d\phi + d\theta^2 + d\phi^2\right), \tag{4.33}$$

where $\psi' \equiv \psi/m$. With the local change of coordinates $\psi' \to \psi'' \equiv \psi' - 4Ju'/\tilde{Q}^2$ we see that this describes $S^1 \times S^3/\mathbb{Z}_m$:

$$ds^2\big|_{r=0} = \frac{2\sqrt{2}}{\tilde{Q}}\left(\frac{Q}{2} - \frac{J^2}{\tilde{Q}^2}\right)du'^2 + \frac{\tilde{Q}}{\sqrt{2}}d\Omega_3^2. \tag{4.34}$$

The fact that the S^3 factor is an orbifold can be solved changing the periodicity of ψ, but we stay in the general case here. The area of this horizon can be obtained integrating the volume form of (4.33), and allows to calculate the Bekenstein-Hawking entropy using (2.7):

$$S = \frac{1}{4G_{\text{N}}}\int_{r=0} \frac{\sqrt{\tilde{Q}^2 Q - 2J^2}}{8\sqrt{2}} \sin\theta^2 d^4x = \frac{\pi^2 L'}{2G_{\text{N}}}\sqrt{\frac{\tilde{Q}^2 Q}{2m^2} - \frac{J^2}{m^2}} = 2\pi\sqrt{\frac{\tilde{Q}^2 Q}{2m^2} - \frac{J^2}{m^2}}, \tag{4.35}$$

with L' the period of u' and $G_{\text{N}} = \pi L'/4$ following the conventions of [1]. The near-horizon geometry can be obtained taking the $r \to 0$ limit in (4.32), or directly in the harmonic functions (4.30). In

[9]Both u and ψ are periodic, so we need to impose $4\pi q/mL \in \mathbb{Z}$ for the change of coordinates to be well defined.

this solution both procedures are equivalent, but it is not the case in general as we shall see in the forthcoming sections. The result is, with the same coordinate redefinitions as above,

$$ds^2\big|_{r\to 0} = -\frac{8\sqrt{2}mr}{\tilde{Q}}dvdu' + \frac{2\sqrt{2}}{\tilde{Q}}\left(\frac{Q}{2} - \frac{J^2}{\tilde{Q}^2}\right)du'^2 + \frac{\tilde{Q}}{4\sqrt{2}}\left(\frac{dr^2}{r^2} + 4d\Omega_3^2\right). \tag{4.36}$$

This is just (4.26) so we have that the near-horizon geometry is (ignoring the orbifold issue) locally $AdS_3 \times S^3$, as one expects for a black string.

4.4 The symplectic group

In general, solutions for 6D minimal supergravity can contain Dirac-Misner string singularities [46]. These are analogous to Dirac string singularities studied in section 2.2 but this time for vector potentials that are part of the metric, namely $\tilde{\omega}$. These strings are potentially dangerous physical singularities so they must be avoided. Besides, their lack is a necessary condition for the absence of closed timelike curves in our geometry. As explained in section 2.2, the strings are a set of points in which the vector potential is not well defined, so to make sure they are not in our solution we require $\tilde{\omega}$ to be globally defined, i.e.

$$d^2\tilde{\omega} = 0. \tag{4.37}$$

We write this in terms of the harmonic functions by taking the Hodge dual and the exterior derivative of the third equation in (4.15), which yields

$$d\left(*_3 \langle \mathbb{V}, d\mathbb{V}\rangle\right) = 0. \tag{4.38}$$

Using (4.19) and after some calculation we get

$$d\left(*_3 \langle \mathbb{V}, d\mathbb{V}\rangle\right) = 2\sum_b \left[\frac{\langle \Gamma_\infty, \Gamma_b\rangle}{r_b^3} + \sum_a \frac{\langle \Gamma_a, \Gamma_b\rangle}{r_a r_b^3}\right] \mathrm{vol}_3, \tag{4.39}$$

where we have used $r_a \equiv |\vec{x} - \vec{x}_a|$. The right hand side of (4.39) must vanish for all $\vec{x}$, and in particular for $r_b \to 0$ we get a set of conditions on the relative distances of the centres, called bubble equations:

$$\langle \Gamma_\infty, \Gamma_b\rangle + \sum_a \frac{\langle \Gamma_a, \Gamma_b\rangle}{r_{ab}} = 0, \tag{4.40}$$

where $r_{ab} \equiv |\vec{x}_a - \vec{x}_b|$. Summing this over b we get

$$\sum_b \langle \Gamma_\infty, \Gamma_b\rangle = 0, \tag{4.41}$$

because the symplectic product is antisymmetric. Note that for our case of interest, in which there is only one centre, (4.40) and (4.41) are equivalent.

Following the rationale of [1] we realise that because a sum of harmonic functions is harmonic, performing a $GL(6,\mathbb{R})$ transformation on one solution $\mathbb{V}$ gives another solution. In addition, if we want to consider transformations that preserve the regularity of a solution we must ensure that the bubble equations stay invariant. For this reason the transformations that send a regular solution to another regular solution are those that preserve the symplectic norm $\langle\cdot,\cdot\rangle$, i.e. $Sp(6,\mathbb{R})$. $Sp(6,\mathbb{R}) \subset SL(6,\mathbb{R}) \subset GL(6,\mathbb{R})$ is the real six dimensional symplectic group, defined as the set of 6×6 real matrices S such that

$$S^{\mathrm{T}}\Omega S = \Omega, \tag{4.42}$$

with Ω the matrix defined in equation (4.16)[10]. The group has dimension 21.

As an example, consider the $Sp(6,\mathbb{R})$ transformation

$$M_n \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & n & 0 & 1 & 0 & 0 \\ n & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.43}$$

and notice that it transforms the harmonic functions (4.21) into (4.25). If we consider them to describe extremal BTZ$\times S^3$ solutions via the identification explained, we are transforming the vacuum state into one with non-vanishing mass and angular momentum. By further applying the transformation M_n we will be adding $2n/\mu$ units of mass and $4n/\sqrt{\mu}$ units of angular momentum each time.

The action of this $Sp(6,\mathbb{R})$ group on solutions is the main research interest of this thesis, and will be examined via some concrete examples. As mentioned in the introduction, we are interested in how this group acts in physical terms. The $Sp(6,\mathbb{R})$ endomorphisms are also interesting because they might allow to reach unexplored corners of the space of solutions, helping us to find new interesting vacua of minimal 6D supergravity. We will present in what follows a subset of these transformations that later will be applied on some known solutions.

[10] More precisely, in the definition of the symplectic group the matrix Ω can be any fixed invertible real skew-symmetric matrix satisfying $\Omega^2 = -I$. Actually, we use this looser definition when introducing symplectic Majorana spinors in appendix B.

4.4.1 The entropy conserving subgroup

There is a particularly interesting set of transformations in $Sp(6,\mathbb{R})$: those that leave invariant the entropy (4.35). We start to present this set by writing the gauge transformations

$$M_{\rm g}=\begin{pmatrix}1&0&0&0&0&0\\ g_2&1&0&0&0&0\\ 2g_1&0&1&0&0&0\\ 2g_1^2g_2&2g_1^2&2g_1g_2&1&-g_2&-2g_1\\ -2g_1^2&0&-2g_1&0&1&0\\ -2g_1g_2&-2g_1&-g_2&0&0&1\end{pmatrix}. \tag{4.44}$$

These transformations form a two dimensional subgroup of $Sp(6,\mathbb{R})$ whose effect is just a change of coordinates. This can be seen from the way they act on the functions (4.14): V_1, H, F and ω stay invariant and the change in β can be absorbed by a redefinition of u. They are called "gauge" for this reason. Moreover, the transformations generated by g_1 alone do not change the solutions at all. The so called spectral flow transformations also leave the entropy invariant. They are the transpose of the gauge ones, i.e.

$$M_{\rm sf}=\begin{pmatrix}1&\gamma_2&2\gamma_1&2\gamma_1^2\gamma_2&-2\gamma_1^2&-2\gamma_1\gamma_2\\ 0&1&0&2\gamma_1^2&0&-2\gamma_1\\ 0&0&1&2\gamma_1\gamma_2&-2\gamma_1&-\gamma_2\\ 0&0&0&1&0&0\\ 0&0&0&-\gamma_2&1&0\\ 0&0&0&-2\gamma_1&0&1\end{pmatrix}. \tag{4.45}$$

These transformations are studied in [47], specially their non-trivial effects on four and five dimensional solutions. There is another subgroup of $Sp(6,\mathbb{R})$ that just rescales the harmonic functions, and leaves the entropy invariant as well. It is given by

$$M_{\rm r}=\begin{pmatrix}\beta_2&0&0&0&0&0\\ 0&\beta_1^2\beta_2&0&0&0&0\\ 0&0&\beta_1^{-1}&0&0&0\\ 0&0&0&\beta_2^{-1}&0&0\\ 0&0&0&0&\beta_1^{-2}\beta_2^{-1}&0\\ 0&0&0&0&0&\beta_1\end{pmatrix}. \tag{4.46}$$

These entropy conserving transformations can be obtained from a set of six generators t_i as follows

$$M_{\rm g}=\exp{(2g_1t_1+g_2t_2)},\qquad M_{\rm sf}=\exp{(2\gamma_1t_3+\gamma_2t_4)},\qquad M_{\rm r}=\exp{[\ln{(\beta_1)}t_5+\ln{(\beta_1\beta_2)}t_6]}. \tag{4.47}$$

The particular form of the generators is not very illuminating so we will not write them here, but they satisfy an algebra with non-vanishing brackets

$$\begin{aligned}
[t_1, t_3] &= -t_6, & [t_2, t_5] &= -2t_2, \\
[t_1, t_6] &= t_1, & [t_2, t_4] &= t_5, \\
[t_3, t_6] &= -t_3, & [t_4, t_5] &= 2t_4.
\end{aligned} \tag{4.48}$$

We can see that there are two separate subalgebras and they correspond to two copies of the algebra $\mathfrak{sl}(2, \mathbb{R})$, i.e.

$$\begin{aligned}
[x, y] &= 2y, \\
[x, z] &= -2z, \\
[y, z] &= x,
\end{aligned} \tag{4.49}$$

by making the identifications $\{x, y, z\} = \left\{2t_6, \sqrt{2}t_3, \sqrt{2}t_1\right\}$ and $\{x, y, z\} = \{t_5, t_2, t_4\}$. Hence, the algebra of the entropy conserving set of transformations of $Sp(6, \mathbb{R})$ is $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$. The exponentiation of the algebra to the entropy conserving elements is not surjective, though. This can be seen from the fact that (4.47) contains logarithms of β_1 and β_2, such that those elements for which $\beta_1 < 0$ and/or $\beta_2 < 0$ cannot be written in that form.

The factorization of the algebra is telling us something about the structure of the entropy conserving transformations. We pick those transformations generated by the subalgebra $\{t_2, t_4, t_5\}$, i.e. gauge and spectral flow with $g_1 = \gamma_1 = 0$ and rescaling with $\beta_2 = \beta_1^{-1}$. A general product of them has the form

$$M_{\text{cc}} = \begin{pmatrix}
d & b & 0 & 0 & 0 & 0 \\
c & a & 0 & 0 & 0 & 0 \\
0 & 0 & d & 0 & 0 & -b \\
0 & 0 & 0 & a & -c & 0 \\
0 & 0 & 0 & -b & d & 0 \\
0 & 0 & -c & 0 & 0 & a
\end{pmatrix}, \tag{4.50}$$

with $a = \beta_1 + g_2\gamma_2$, $b = \gamma_2\beta_1^{-1}$, $c = g_2$ and $d = \beta_1^{-1}$. The element M_{cc} will obviously belong to $Sl(2, \mathbb{R})$, and in fact amounts to a special linear local change of coordinates [17]

$$\begin{pmatrix} \psi \\ u \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi' \\ u' \end{pmatrix}. \tag{4.51}$$

Note that if the u direction is taken to be compact one is changing coordinates on a torus and thus has quantization conditions on the parameters a, b, c, d for the transformation to be well defined.

Among the rest of transformations, we already mentioned that those generated by g_1 have no effect on the solutions. M_{r} with $\beta_2 = \beta_1$ is easily seen to be a rescaling of the coordinates u and ψ and of the whole lagrangian. We only have left the spectral flow transformations generated by γ_1, which in fact are the only ones able to change the solution in a significant way, as we will see later.

4.5 Reduction to five dimensions

One of the reasons why it is worth to study six dimensional supergravity despite its apparent lack of correspondence with physical reality is the possibility to reduce its solutions to five or four dimensional supergravity solutions. Here we will study how this reduction works for minimal 6D supergravity and we will apply it to the case of the black string, that lays a spinning black hole in five dimensions. As we will justify later, the circle reduction of minimal 6D supergravity generates minimal $\mathcal{N} = 2$, 5D supergravity coupled to a vector multiplet. One can then, if desired, truncate the theory to minimal 5D supergravity by consistently setting to zero the vector multiplet, so all solutions of minimal 5D supergravity arise by dimensional reduction from a subset of minimal 6D supergravity solutions. When reducing it to 4D one finds minimal $\mathcal{N} = 2$ supergravity coupled to three vector multiplets [16].

To start with, we need a spacelike Killing vector field in which we can perform the Kaluza-Klein reduction. There are two clear candidates in the class of solutions that we have studied: ∂_u and ∂_ψ. The ansatz for the reduction is [1]

$$ds^2 = e^{2\varphi} (du + A)^2 + e^{-2\varphi/3} d\tilde{s}^2, \tag{4.52}$$

$$G = \tilde{G} + \frac{1}{2} dA' \wedge (du + A), \tag{4.53}$$

where we have reduced in the u direction in the interest of the later example, but it is generic and can be used to reduce in ψ by just substituting $u \to \psi$. The metric and three-form ds^2 and G are the 6D minimal supergravity ones. The five dimensional metric and three-form are $d\tilde{s}^2$ and $\tilde{G}$, and as a result of the Kaluza-Klein reduction one obtains two extra vectors A and A' and one scalar φ in five dimensions. The self-duality of G allows to write $\tilde{G}$ in terms of A', so the former is not an independent field of the 5D theory. Our 6D theory has $\tilde{\mathcal{Q}} = 8$ real supercharges and this number is conserved in the reduction because so it is the number of gravitini. From table 1 we see that two symplectic Majorana spinors (recall that they always come in even number) assemble eight real components in 5D, so we have an $\mathcal{N} = 2$ theory. The bosonic content of the gravity multiplet of $\mathcal{N} = 2$ 5D supergravity is the metric and a graviphoton, that can be identified with A. This leaves apart the vector A' and the scalar φ, that in five dimensions can be accommodated in a vector multiplet [30, ch. 12] confirming, at least at the level of the bosonic fields, that the theory obtained is minimal $\mathcal{N} = 2$, 5D supergravity coupled to a vector multiplet.

Now one can reduce the black string solution of section 4.3.3 using the above ansatz. The result is, as we mentioned, a five dimensional black hole with event horizon topology S^3. The general form of the five dimensional solution is not very illuminating, but the relevant thing is that it gives a physical interpretation for the charges (4.31). Namely they are the integrals

$$
\begin{aligned}
J &= \frac{1}{4\pi^2}\int_{\mathcal{H}} *_5 dK, \\
\tilde{Q} &= -\frac{\sqrt{2}}{8\pi^2}\int_{\mathcal{H}} *_5 dA', \\
Q &= -\frac{1}{8\pi^2}\int_{\mathcal{H}} *_5 dA,
\end{aligned}
\tag{4.54}
$$

with K the Killing vector field ∂_ψ and $\mathcal{H}$ the event horizon. For asymptotically flat spacetimes, the result does not change if one integrates in a three-sphere at infinity. We have then that J is the angular momentum of the five dimensional black hole, and $\tilde{Q}$ and Q its electric charges under the $U(1)$ fields A' and A respectively. These charges also give the mass of the black hole via the BPS condition [1]

$$
M = \frac{1}{4}\left(\sqrt{2}\tilde{Q} + Q\right). \tag{4.55}
$$

5 Spectral flow on $AdS_3 \times S^3$

In this section we will phrase the results of the study of a new solution of six dimensional minimal supergravity obtained via a transformation of $AdS_3 \times S^3$ under the symplectic group introduced previously. This has been the main research task along the thesis project, and provides an example of the potential of the $Sp(6,\mathbb{R})$ group to explore unknown regions in the phase space of solutions.

The solution found is not easy to describe and has not been fully understood. In particular, we will not get a good description of what the geometry is far from the origin or near the horizon. This fact makes it difficult to characterize the solution via Komar conserved quantities of its Killing vector fields, for example. However, from the information of a pseudo-horizon, we will conjecture that we are dealing with a particular case of a bigger family of solutions. This will be confirmed later in section 6. We will also find a case which describes a naked singularity and closed timelike curves, endangering the conjectures that protect causality and determinism. We will fail at finding sings that this is a not physical solution, but we shall make a tentative dynamical explanation of the naked singularity in terms of the underlying string theory in section 5.5.1. We add that the solution here encountered was reduced to five dimensions in both u and ψ, but we were not able to interpret the results in terms of known 5D solutions and they did not provide any valuable information. For this reason we will not present them here.

5.1 The new solution

Our starting point are the harmonic functions (4.21) corresponding to $AdS_3 \times S^3$. We act on them with a spectral flow transformation (4.45) with $\gamma_1 \equiv \gamma$ and $\gamma_2 = 0$, that will be denoted M_γ (we saw that it is the only non-trivial entropy conserving transformation). The result is

$$\mathbb{V} = M_\gamma \cdot \mathbb{V}_{AdS_3\times S^3} = \begin{pmatrix} -2c\gamma^2 \\ 0 \\ -2c\gamma \\ 0 \\ c \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2\mu\gamma \\ 0 \\ 0 \\ 0 \\ \mu \end{pmatrix} \frac{1}{r}. \tag{5.1}$$

The transformation M_γ is part of the entropy conserving subset, so the new solution will have, just like $AdS_3 \times S^3$, zero entropy. This is consistent with the fact that the charges (4.31) for this solution are

$$\tilde{Q} = 4\sqrt{2}\mu, \qquad Q = 0, \qquad J = 0. \tag{5.2}$$

The metric given by the above harmonic functions is

$$\begin{aligned}
ds^2 = &-\frac{2r}{\mu(1+\alpha r)}dv\left[(1-\alpha r)du - 2\mu\gamma d\psi - 2\mu\gamma\alpha r\cos\theta d\phi\right]+\\
&+\frac{2cr}{\mu(1+\alpha r)}du\left(du+4\mu\gamma\cos\theta d\phi\right)+\frac{\mu}{1+\alpha r}d\psi\left(d\psi+2\cos\theta d\phi\right)+\\
&+\mu(1+\alpha r)\frac{dr^2}{r^2}+\mu(1+\alpha r)d\theta^2+\mu\frac{1+4\alpha r}{1+\alpha r}\cos^2\theta d\phi^2+\mu(1+\alpha r)\sin^2\theta d\phi^2,
\end{aligned} \tag{5.3}$$

where we have defined $\alpha \equiv 2c\gamma^2$ for convenience. The Ricci scalar of this solution is of course $R = 0$, so if we want information about possible curvature singularities we have to look, like in the Schwarzschild case, to other scalars. The Kretschmann scalar is

$$K = \frac{3 - 12\alpha r + 10\alpha^2 r^2 - 60\alpha^3 r^3 + 11\alpha^4 r^4}{2\mu^2(1+\alpha r)^6}, \tag{5.4}$$

so there is a curvature singularity in

$$r = -\frac{1}{\alpha}. \tag{5.5}$$

The expression for the three-form G of this solution can be found in appendix C, and is also singular at this point. The Gibbons-Hawking base is fully determined by the first harmonic function, which in this case is $V_1 = 1/r - \alpha$. When getting close to the pole in $r = 0$ we have $V_1 \to 1/r$ that, as shown in section 4.3.1, gives flat $\mathbb{R}^4$.

In section 4.3.2 we noticed that the parameter c is inert in the $AdS_3 \times S^3$ solution and can be given any value. The first striking feature of our new solution is that it changes drastically depending on this parameter. We can yet see this in the fact that (5.5) is c dependent (for example there is not singularity for $c = 0$). This is only one aspect of the c dependence so in what follows we will divide the analysis in three cases: zero, positive and negative c.

5.2 The $c = 0$ case

We start by analysing the case in which c vanishes, which turns out to be the simplest. Notice that taking $c = 0$ in (5.1) gives the same result as transforming $AdS_3 \times S^3$ with M_{cc}. More precisely, take (4.21) with $c = 0$ and apply a transformation (4.50) with $a = d = 1$, $b = 0$ and $c = -2\mu\gamma$. The result is

$$\mathbb{V}(c=0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2\mu\gamma & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2\mu\gamma & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2\mu\gamma & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1/r \\ 0 \\ 0 \\ 0 \\ 0 \\ \mu/r \end{pmatrix} = \begin{pmatrix} 1/r \\ -2\mu\gamma/r \\ 0 \\ 0 \\ 0 \\ \mu/r \end{pmatrix}, \tag{5.6}$$

i.e. (5.1) with $c = 0$.

From the discussion about M_{cc} in section 4.4.1 we know that the above transformation is equivalent to an innocuous change of coordinates $u \to u' = u + 2\mu\gamma\psi$ on $AdS_3 \times S^3$. We conclude then that our new solution, when evaluated in $c = 0$, is just the original $AdS_3 \times S^3$.

5.3 Metric signature

The six dimensional solutions we are treating are supposed to have signature $\{5, 1\}$, and $\{4, 0\}$ in the GH base. However, when the H and V_1 functions change sign in certain ways at some points of the manifold, the signature of the metric can become different. It is the case for our solution. The specific signature changes and the way we will deal with them are studied in this section. Later in section 5.5.1, we will see that these signature changes can have an explanation in terms of the underlying type-IIB superstring theory.

Let us start with the GH base metric (4.11). One can calculate the general form of the eigenvalues and see that they have signature $\{4, 0\}$ for $V_1 > 1$ and $\{0, 4\}$ for $V_1 < 0$. In our case $V_1 = 1/r - \alpha$ so we have[11]

- $c > 0$ case:
$$\begin{cases} \{0,4\} & \text{for} \quad r<0 \quad \text{and} \quad r>\frac{1}{\alpha} \\ \{4,0\} & \text{for} \quad 0<r<\frac{1}{\alpha} \end{cases}, \tag{5.7}$$

- $c < 0$ case:
$$\begin{cases} \{4,0\} & \text{for} \quad r<\frac{1}{\alpha} \quad \text{and} \quad r>0 \\ \{0,4\} & \text{for} \quad \frac{1}{\alpha}<r<0 \end{cases}. \tag{5.8}$$

The signature of the base space is important because it affects the original derivation of the three-form G. Looking at (4.13) we can see that the Hodge dual operator restricted to the base space (denoted $*_4$) enters the solution in two points: explicitly in the first term and implicitly in $(d\omega)^-$. If the signature of the base changes to $\{0, 4\}$, this Hodge operator must pick up a minus sign.

We repeat now the analysis with the full metric (5.3). In this case the eigenvalues are not solvable analytically with standard tools so numerical methods have been used. The result is that the metric has opposite signature at both sides of the singularity, i.e.

- $c > 0$ case:
$$\begin{cases} \{1,5\} & \text{for} \quad r<-\frac{1}{\alpha} \\ \{5,1\} & \text{for} \quad r>-\frac{1}{\alpha} \end{cases}, \tag{5.9}$$

- $c < 0$ case:
$$\begin{cases} \{5,1\} & \text{for} \quad r<-\frac{1}{\alpha} \\ \{1,5\} & \text{for} \quad r>-\frac{1}{\alpha} \end{cases}. \tag{5.10}$$

As our convention is to work with $(-1, 1, \ldots, 1)$ metrics, we will multiply (5.3) by an overall minus sign in those cases and regions in which the signature is reversed, to get the canonical one $\{5, 1\}$.

[11] Here we need to consider separately the cases in which α is positive or negative. This is controlled by the sign of c because $\alpha = 2c\gamma^2$.

Although we have detailed the signatures for every value of r, in practice we are interested only in some regions of the solution, just like we did not consider $r < 0$ for the Scharzschild black hole. In fact this is the key point: we will study the region on the "right hand side" of the singularity, i.e. $r > -1/\alpha$.

5.4 The $c > 0$ case

When c is positive the singularity $r = -1/\alpha < 0$ sits in a negative value of the radius. The region of interest includes then another relevant point of the metric (5.3), $r = 0$, in which it degenerates. We try to explore now what is going on at this point.

5.4.1 The $r = 0$ surface

By plugging $r = 0$ in the metric (5.3) we can see that all the v and u components vanish, so the $r = 0$ surface extends in the three dimensions parametrized by the coordinates ψ, θ and ϕ. The metric restricted to this surface is

$$ds^2\big|_{r=0} = \mu \left(d\psi^2 + d\theta^2 + d\phi^2 + 2\cos\theta d\psi d\phi\right) = 4\mu d\Omega_3^2. \tag{5.11}$$

So the constant time slices of $r = 0$ are three-spheres of squared radius 4μ. We see then that this surface extends in three spacial dimensions plus the temporal one, giving in total co-dimension two. Hence, $r = 0$ is not a hypersurface and this implies that it cannot be a Killing horizon in the strict sense of the term.

If we are to consider this surface as an event horizon, we must conclude that it is shrunk to zero size in one direction because event horizons in 6D extend in four spatial dimensions. This implies that the volume of our horizon $r = 0$ is zero, and is consistent with the fact that the entropy vanishes for this solution. All this suggests that we are dealing with an object for which some quantum number has been taken to zero, contracting the size of the horizon to a point in one of its directions. If that is the case, we would be able to obtain a more general solution with extended horizon and non-vanishing entropy by means of an $Sp(6, \mathbb{R})$ transformation that "switches on" the missing charge. In section 6 we will precisely do this.

5.4.2 Near-horizon geometry

We are going to study now the $r \to 0$ limit of our solution. The title of this section is an abuse of terminology, as we have just shown that $r = 0$ is not exactly a horizon, but it helps to compare our solution with previous and future cases.

Taking the $r \to 0$ limit in the line element (5.3) gives

$$\begin{aligned} ds^2\big|_{r\to 0} &\to -\frac{2r}{\mu}du(dv - cdu) + \frac{\mu}{r^2}dr^2 + 4\mu d\Omega_3^2 + 4\gamma r dv d\psi + 8\gamma c r \cos\theta du d\phi = \\ &= ds^2_{AdS_3\times S^3} + 4\gamma r dv d\psi + 8\gamma c r \cos\theta du d\phi, \end{aligned} \tag{5.12}$$

i.e. the metric on $AdS_3 \times S^3$ plus two crossed terms between the anti-de Sitter and spherical parts. This spacetime is not a solution of minimal 6D supergravity, as one can see from the fact that $R \neq 0$.

In general we want these asymptotic geometries to be also solutions of our theory so the previous result is not satisfying. If we want to make sure that our near-horizon limit does not lead us out of the phase space of solutions we can take it directly in the harmonic functions $\mathbb{V}$, taking care not to spoil the bubble equation. In this case, taking the $r \to 0$ limit in (5.1) gives

$$\mathbb{V} \to \begin{pmatrix} 0 \\ 0 \\ -2c\gamma \\ 0 \\ c \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2\mu\gamma \\ 0 \\ 0 \\ 0 \\ \mu \end{pmatrix} \frac{1}{r}. \tag{5.13}$$

Of course, the geometry given by these harmonic functions is a solution of the theory, but the problem is now that it is not much simpler than the full metric (5.3) and we cannot tell which space does it correspond to. We know though that it is not $AdS_3 \times S^3$, because its Kretschmann scalar is not constant. One might try to take the $r \to 0$ limit in this metric to simplify it further, but the result is (5.12) so we are again out of the phase space of solutions.

One can see then that taking these limits is not an easy task. Firstly because there are various possibilities: one can do it in the harmonic functions that characterise the solutions or in the solutions themselves, secondly because these different ways do not yield the same result in general and finally because we can end up with a geometry that is not accepted by our theory.

The conclusion is that we have not found a good near-horizon geometry for our solution. In [16] it was proved that any supersymmetric solution of 6D minimal supergravity with a compact horizon has near-horizon geometry $\mathbb{R}^{1,1} \times T^4$, $\mathbb{R}^{1,1} \times K3$ or some identification of $AdS_3 \times S^3$, so we might be tempted to use that result in our case. However, it was assumed in their proof that the event horizon is a Killing horizon of v, which in our case is not exactly true. In section 6 we will avoid that problem by setting a charge to be different from zero and the result will be, among other things, a well-defined near-horizon geometry.

5.4.3 Asymptotic limit

During the last lines we have realised that the application of a spectral flow transformation on $AdS_3 \times S^3$ results in a complicated geometry. One of its most important troubles is the fact that it does not have a simple asymptotic limit, as we will show in this section. But what do we mean by simple asymptotic limit? In general the gravitational solutions studied by physicists consist on an empty background geometry over which some kind of matter and energy is placed in a certain region, bending the surrounding spacetime. Then, far away from the region in which matter and energy are, one shall recover the background geometry, which is a solution of the vacuum Einstein equations. For Schwarzschild and Reissner-Nordström black holes the asymptotic geometry is flat

Minkowski, for the BTZ black hole it is anti-de Sitter, etc.

In our case of study we started with $AdS_3 \times S^3$, which is a vacuum solution[12], and then acted on it with a spectral flow transformation. We could expect then to find $AdS_3 \times S^3$ when taking the $r \to \infty$ limit in the new solution, i.e. when going far away from the pole at $r = 0$, but it is not the case. The key here is that the spectral flow transformation, regardless how small the parameter γ is, changes the matter content at big r. This can be seen in appendix C, in which G is expanded in powers of γ to show that the first order term is linear in r.

One could still argue that the r dependence of G can cancel when calculating the energy-momentum tensor (which, from (4.3) we know it is $T_{\mu\nu} = G_{\mu\rho\sigma} G_\nu{}^{\rho\sigma}/8\pi G_{\rm N}$), but it is not the case and the influence in the geometry takes place. This can be seen by expanding, this time the metric (5.3), in powers of γ:

$$\begin{aligned} ds^2 &= -\frac{2r}{\mu} du(dv - cdu) + \frac{\mu}{r^2} dr^2 + 4\mu d\Omega_3^2 + 4\gamma r dv d\psi + 8\gamma c r \cos\theta du d\phi + \mathcal{O}(\gamma^2) = \\ &= ds^2_{AdS_3 \times S^3} + \gamma r \left(4 dv d\psi + 8c \cos\theta du d\phi\right) + \mathcal{O}(\gamma^2). \end{aligned} \tag{5.14}$$

The zero order is of course the $AdS_3 \times S^3$ metric we started from, and notice that the linear term in γ is also linear in r, showing that the asymptotic region is heavily changed by the transformation M_γ. Apart from this, we see that the transformation is mixing the anti-de Sitter and three-sphere parts (coordinates $\{v, u, r\}$ and $\{\psi, \theta, \phi\}$ respectively) already at first order, such that the geometry is no longer a direct product. This coincides with Bena et al., who state in [47] that the spectral flow transformation mixes the coordinates u and ψ and when the GH base asymptotes to $\mathbb{R}^4$, as it is the case for $AdS_3 \times S^3$, the circle ψ in the base becomes infinitely large and the spectral flow changes the asymptotics.

Now that we know that the asymptotic geometry is not a simple one, we try to find it explicitly. Taking the limit $r \to \infty$ in the metric drives us out of the space of solutions, just like in the near-horizon case. Hence, we take it in the harmonic functions, getting

$$\mathbb{V} \to \begin{pmatrix} -2c\gamma^2 \\ 0 \\ -2c\gamma \\ 0 \\ c \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2\mu\gamma \\ 0 \\ 0 \\ 0 \\ \mu \end{pmatrix} \frac{1}{r}. \tag{5.15}$$

[12] One might think that it is not a vacuum solution because it has a non-zero three form (4.22), and then its Einstein equations are not of the form $R_{\mu\nu} = 0$. However, the three-form contains in this case the information about the cosmological constant, so it acts in the Einstein equations as the term $\Lambda g_{\mu\nu}$ in (2.26).

They generate the solution

$$ds^2 = \frac{2r}{\mu}dvdu + \frac{2}{c\gamma}dvd\psi + 4\gamma r\cos\theta dvd\phi + \frac{\mu}{\alpha r}d\psi^2 + \frac{\mu\alpha}{r}dr^2 + \mu\alpha r\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{5.16}$$

$$G = \frac{1}{2\mu}dv\wedge du\wedge dr + \gamma r\sin\theta dv\wedge d\theta\wedge d\phi - \gamma\cos\theta dv\wedge dr\wedge d\phi + \frac{\mu}{2}\sin\theta d\psi\wedge d\theta\wedge d\phi. \tag{5.17}$$

The charges (4.31) are all zero for this solution, so the entropy vanishes as well. We can confirm that it is not just $AdS_3\times S^3$ in some convoluted coordinate system because its Kretschmann scalar is not constant and shows a curvature singularity:

$$K = \frac{11}{2\mu^2\alpha^2 r^2}. \tag{5.18}$$

It is useful to explore the behaviour of the energy-momentum tensor at $r\to\infty$ with a bit more detail. The energy-momentum tensor of our solution (5.1) tends to a constant in the asymptotic limit $r\to\infty$. This fact is, as one would expect, also true for the $T_{\mu\nu}$ of (5.15). The story is different for other objects: the energy-momentum tensor of a black string (4.30) tends to zero and that of $AdS_3\times S^3$ becomes linear in r. However, an asymptotically constant $T_{\mu\nu}$ is a characteristic of pp-wave spacetimes, which suggests that our solution could be describing a bound state of some object (associated to the singularity) with pp-waves radiating to infinity.

Pp-wave spacetimes, abbreviation for plane-fronted gravitational waves with parallel rays, are solutions of general relativity that model light-like radiation of any kind. This class of solutions includes for example gravitational waves, electromagnetic radiation and their combinations. Mathematically, pp-waves are spacetimes that admit a covariantly constant null vector field [48, ch. 24]. One easily notices that our asymptotic spacetime (5.16) contains the two null vector fields ∂_v and ∂_u, but they are not covariantly constant:

$$\nabla_\alpha\left(\partial_v\right)^\beta = \Gamma^\beta_{\alpha v} \neq 0, \tag{5.19}$$

$$\nabla_\alpha\left(\partial_u\right)^\beta = \Gamma^\beta_{\alpha u} \neq 0. \tag{5.20}$$

Attempts to obtain other null covariantly constant vector fields in this spacetime have failed, so we can neither confirm nor deny the suspicion that we have pp-waves asymptotically.

5.5 The $c<0$ case

For the negative c case, the singularity occurs for a positive radius $r=-1/\alpha>0$. As the horizon surface was in $r=0$, there is no horizon enclosing the singular point from the point of view of an observer in $r>-1/\alpha$. We are then dealing with a naked singularity. In addition, recall from section 5.3 that the metric signature is $\{1,5\}$ in this case, so in order to recover the $\{5,1\}$ convention we will consider (5.3) multiplied by an extra minus sign throughout this section. Given this, one can notice that $g_{\phi\phi}$ becomes negative for certain values of r. The orbits of ∂_ϕ must be closed because ϕ is a compact coordinate, so this implies the presence of closed timelike curves. Namely, we have

$g_{\phi\phi} < 0$ for $r \in (-1/\alpha, r')$ with

$$\alpha r' = -1 - 2\cot^2\theta - \sqrt{4\cot^4\theta + 3\cot^2\theta}. \tag{5.21}$$

This was also true for $c > 0$, but in that case the region with CTCs laid between the singularity and the horizon $r = 0$ such that it was hiding them. When closed timelike curves are hidden by an event horizon they are not considered problematic, because an observer would not be able to detect the causal violation. In the present case however, we have no horizon so we must be concerned about the CTCs.

As it was explained in section 2.1, naked singularities are accompanied by some non-physical matter content that explains them and keeps the cosmic censorship conjecture safe, and the same can be said for CTCs and the chronology protection conjecture. In this case both undesirable objects are present, so we would like to find some physical indication that the solution is pathological.

In the Schwarzschild black hole, when one takes a negative mass the spacetime describes a naked singularity. By analogy, it would be comforting to calculate the Komar mass of our solution and find a negative value. For this we need an asymptotically timelike Killing vector field, in our case ∂_u, that must be properly normalized:

$$\xi \equiv \sqrt{\mu}\gamma\partial_u \qquad \text{giving} \qquad \lim_{r\to\infty} |\xi|^2 = \lim_{r\to\infty} \frac{-\alpha r}{1+\alpha r} = -1. \tag{5.22}$$

Then, the Komar mass is [49]

$$Q_\xi \equiv \frac{1}{8\pi G_\mathrm{N}} \int_{\partial V_\infty} *d\xi, \tag{5.23}$$

where ∂V_∞ is a closed spatial surface at infinity, which must be four dimensional in our case as $*d\xi$ is a four-form. For instance, a Komar integral for any of the four dimensional black holes reviewed in section 2 would be performed over an S^2 centred in the singularity and with infinite radius. The problem comes precisely at this point, because we do not know what the topology of spatial infinity is in our spacetime. The solution obtained previously for the asymptotic limit, i.e. the metric (5.16), is still valid in the $c < 0$ case (with the -1 prescription because of the signature, that is also $\{1, 5\}$) and it does not give many clues about what spatial surface we have as a "boundary" of our spacetime. In sum, we have a manifestation of the fact that the definition of the Komar mass is fully satisfactory only for asymptotically flat spacetimes, as pointed out by Wald in [19]. It is then very difficult (probably impossible) to give a well defined notion of total mass for our spacetime.

An alternative way to prove that we are dealing with odd matter, if that is the case, is to find a violation of some energy condition. The simplest case is the null energy condition (NEC), which states that every future-pointing null vector field l must satisfy [50]

$$T_{\mu\nu} l^\mu l^\nu \geq 0. \tag{5.24}$$

Of course, fully checking this condition implies finding first all possible null vector fields l, i.e.

solving $g_{\mu\nu}l^\mu l^\nu = 0$. We can do this using the vielbein (4.17), such that

$$g_{\mu\nu}l^\mu l^\nu = g_{\mu\nu}e^\mu_a e^\nu_b l^a l^b = \eta_{ab}l^a l^b = 0 \qquad \rightarrow \qquad -2l^+l^- + (l^2)^2 + (l^3)^2 + (l^4)^2 + (l^5)^2 = 0. \quad (5.25)$$

We divide the solutions in two classes:

1. $l^m = 0 \;\forall\; m = 2, \dots, 5$, in which case there are two possibilities:
   (a) $l^- = 0$ and l^+ arbitrary or
   (b) $l^+ = 0$ and l^- arbitrary.
2. $l^m \neq 0$ for some $m = 2, \dots, 5$, in which case we have

$$l^+ = \frac{(l^2)^2 + \dots + (l^5)^2}{2l^-}. \quad (5.26)$$

The first class is tractable but the second consists of vector fields with five independent parameters, and checking (5.30) for them is impossible in practice (notice that we also must change to spacetime coordinates). We go then for the case 1a first. The vector corresponding to l^+ is $l^\mu = e^\mu_+ = \eta_{+a}(e^a)^\mu = -(e^-)^\mu = -\delta^\mu_v$, i.e. $-\partial_v$. The NEC for this vector field is then $T_{vv} \geq 0$, and it is satisfied by our solution and its asymptotic limit because they have

$$T_{vv} = \frac{1}{8\pi G_\mathrm{N}}\frac{4\gamma^2\alpha^2 r^4}{\mu^2(1+\alpha r)^4} > 0, \qquad T_{vv} = \frac{1}{8\pi G_\mathrm{N}}\frac{1}{\mu^2 c^2\gamma^2} > 0, \quad (5.27)$$

respectively. The other possible check is for 1b, for which we have that the vector associated to l^- is $l^\mu = -(e^+)^\mu = -H\left(F\delta^\mu_v/2 + \delta^\mu_u\right)$. This implies that the NEC is in this case

$$\frac{F^2}{4}T_{vv} + T_{uu} + FT_{vu} \geq 0, \quad (5.28)$$

and its is also satisfied for our solution and its asymptotic limit, for which

$$\frac{F^2}{4}T_{vv}+T_{uu}+FT_{vu} = \frac{1}{8\pi G_\mathrm{N}}\frac{\alpha^2 r^2(1-2\alpha r)^2}{\mu^2\gamma^2(1-\alpha r)^2(1+\alpha r)^4} > 0, \qquad \frac{F^2}{4}T_{vv}+T_{uu}+FT_{vu} = 0, \quad (5.29)$$

respectively.

We can now try with other energy conditions, like the weak energy condition (WEC). It stipulates the same as the NEC but with timelike vector fields instead of null. In our solution, two timelike vector fields we can easily think of are ∂_u and ∂_ϕ for $r < r'$, but T_{uu} and $T_{\phi\phi}$ in $r < r'$ are always positive or zero. We have also the strong energy condition (SEC), that requires

$$\left(T_{\mu\nu} - \frac{T}{2}g_{\mu\nu}\right)t^\mu t^\nu \geq 0, \quad (5.30)$$

for every future-pointing timelike vector field t, but notice that in our theory $R = 0$, so $T = 0$ and this condition is equivalent to the WEC.

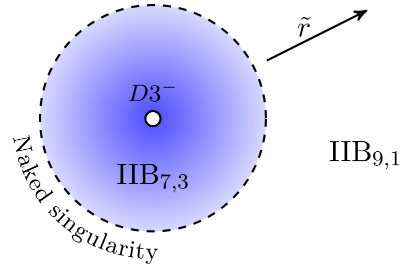

Figure 3: Negative $D3$-brane and the signatures of the type IIB theories at each side of the naked singularity. The radius, which represents the distance to the brane in the transverse directions, is denoted $\tilde{r}$ to avoid confusion with the six dimensional coordinate r.

We conclude that the spectral flow on $AdS_3 \times S^3$ with negative parameter c has a naked singularity and CTCs in a region close to it, and those attempts to prove the presence of abnormal matter have failed. Nonetheless, it cannot be claimed that we have a violation of CCC and CPC because not all the possibilities to prove that our solution is non-physical have been exhausted. The reason is that the problem of checking all potential violations of the energy conditions is quite hard, as we have seen.

5.5.1 Negative branes

We have not been able to explain the naked singularity and signature changes in the metric via the presence of exotic matter. Nevertheless, we saw in section 3.6.2, following the work in [18], that these phenomena can have a microscopic explanation in terms of negative branes in the underlying string theory. Namely, we stated that negative branes dynamically change the signature of spacetime in a region around them, a region whose boundary is a naked singularity. In this section we try to find out whether this idea can be applied to our case at hand.

Our attempt consists on placing a negative brane in the type IIB ten dimensional theory in such a way that the compactification in $\mathbb{P}^2$ (recall figure 2) yields a 6D minimal supergravity solution with a naked singularity at $r = -1/\alpha$ and signature $\{5, 1\}$ at each side (taking into account the minus sign prescription). We need to determine which particular negative Dp-brane (that we will denote Dp^- brane) we place in 10D and whether it wraps in the compact dimensions or not.

As we will see now, the numbers work if we consider a $D3^-$ brane. Figure 3 shows what the signature of the ten dimensional theory is in each side of the singularity, according to the analysis of section 3.6.2. We also take the brane to be wrapped around a 2-cycle in the manifold in which we will compactify to 6D, $\mathbb{P}^2$. The situation is summarized in table 3, where we have followed the notation of section 3.6.2 (y for the brane parallel directions and x for the transverse ones). The table shows schematically the signature flip of the brane directions, and indicates which coordinates are to be compactified. We see that $\mathbb{P}^2$ comprises two of the directions of the brane, y_2 and y_3, and two other spatial directions, x_5 and x_6. It is easy to see then that after compactifying these four coordinates, the resulting spacetime has $\{5, 1\}$ signature at each side of the naked singularity interface, as desired.

Now we need to determine what is the other direction in which the brane lays, i.e. which direction does y_1 correspond to in our six dimensional solution. We will call ζ the vector field that generates translations in that direction. Let us assume for simplicity that it is a linear combination of ∂_u and ∂_v, i.e.

$$\zeta = \zeta_1 \partial_u + \zeta_2 \partial_v, \tag{5.31}$$

with ζ_1 and ζ_2 constants. We know that the brane directions change signature at the naked singularity interface, so we need a change of sign of $|\zeta|^2$ at $r = -1/\alpha$. Taking $\zeta_1 = 0$ is not an option because ∂_v is everywhere null, so we can write $\zeta = \partial_u + \zeta_2 \partial_v$ without loss of generality. The norm is

$$|\zeta|^2 = g_{\mu\nu}\zeta^\mu \zeta^\nu = \frac{2r}{\mu\,(1+\alpha r)} \left[c - \zeta_2\,(1-\alpha r)\right], \tag{5.32}$$

in the region $r < -1/\alpha$. When crossing the singularity to $r > -1/\alpha$ we have an overall change of sign because of the minus sign prescription for the metric. Also, we have another sign flip from the $(1 + \alpha r)$ factor in the denominator, so both cancel and the norm sign stays the same. In order to avoid this we can try to cancel the $(1 + \alpha r)$ factor in the denominator by making

$$c - \zeta_2\,(1-\alpha r) \propto 1 + \alpha r. \tag{5.33}$$

This condition fixes our constant to be $\zeta_2 = c/2$. Up to a proportionality factor and assuming that it does not lay in the base, we have uniquely determined ζ to be

$$\zeta = \partial_u + \frac{c}{2}\partial_v \qquad \text{giving} \qquad |\zeta|^2 = \begin{cases} \frac{cr}{\mu} & \text{for} \quad r < -\frac{1}{\alpha} \\ -\frac{cr}{\mu} & \text{for} \quad r > -\frac{1}{\alpha} \end{cases} . \tag{5.34}$$

To sum up, we can give a tentative explanation to the naked singularity and the signature changes in our $c < 0$ solution via an uplift to type IIB superstring theory with a negative $D3$-brane wrapped around a two-cycle in the compact manifold. Such a ten dimensional configuration reproduces qualitatively the main characteristics of our six dimensional solution. Of course, this correspondence would be properly proved by writing a negative $D3$-brane solution in IIB and compactifying it to get our 6D metric and G form. Compactification on a four-torus is easy to do in practice, but when

	y_0	y_1	y_2	y_3	x_1	x_2	x_3	x_4	x_5	x_6
Outside	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
Inside	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$
Compactification	$\circ$	$\circ$	$\bullet$	$\bullet$	$\circ$	$\circ$	$\circ$	$\circ$	$\bullet$	$\bullet$

Table 3: Schematic representation of the spacetime directions and their signature in the $D3^-$ brane configuration. The brane world-volume directions are called y and the transverse ones are called x. Their timelike ($-$ sign) or spacelike ($+$ sign) character is indicated for the regions inside and outside the naked singularity. The last row denotes which directions are compactified when going to the 6D theory ($\bullet$) and which are not ($\circ$).

doing it for type IIB one ends up in (2,2) 6D supergravity (32 real supercharges) instead of (1,0) supergravity (8 real supercharges) [30, ch. 12]. We know from section 4.1 that the right procedure to obtain 6D minimal (1,0) supegravity is to compactify F-theory on an elliptically fibered Calabi-Yau manifold with base $\mathbb{P}^2$, but how to do this explicitly is not known. We conclude that there might be a connection between negative branes in type IIB and naked singularities and signature changes of six dimensional solutions, but some work is still required to properly establish or discard it.

6 Switching on missing charges

In the previous pages we realised that acting with a spectral flow transformation on $AdS_3 \times S^3$ gives, when c is positive, a spacetime with a squashed horizon and, hence, no entropy. We suggested that we could be dealing with a limit of a more general solution in which some charge has been taken to zero, shrinking to a point one of the directions in which the horizon extends. In this section, as announced, we will use the $Sp(6,\mathbb{R})$ group of transformations to turn on some charges and obtain the more general solution.

6.1 Acting with M_n

Recall from sections 4.3.2 and 4.4 that the transformation M_n switches on one of the parameters in the harmonic functions. In that case it takes $AdS_3 \times S^3$ into $AdS_3 \times S^3$, or when the proper discrete identification is made, a vacuum extremal BTZ$\times S^3$ into a non-vacuum one. The strategy here will be to act with M_n on our solution (5.1). This gives

$$M_n \cdot M_\gamma \cdot \mathbb{V}_{AdS_3\times S^3} = \begin{pmatrix} -2c\gamma^2 \\ 0 \\ -2c\gamma \\ 0 \\ c - 2c\gamma^2 n \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2\mu\gamma \\ 0 \\ -2\mu\gamma n \\ n \\ \mu \end{pmatrix} \frac{1}{r}. \tag{6.1}$$

Notice that M_n and M_γ do not commute, so this result is not the same as giving charge to $AdS_3 \times S^3$ with M_n and then applying the spectral flow transformation. The harmonic functions (6.1) generate

$$\begin{aligned} ds^2 = & -\frac{2r}{\mu(1+\alpha r)} dv \left[(1-\alpha r)du - 2\mu\gamma d\psi - 2\mu\gamma\alpha r \cos\theta d\phi\right] + \\ & + \frac{2\left[cr + n(1-\alpha r)^2\right]}{\mu(1+\alpha r)} du \left(du + 4\mu\gamma \cos\theta d\phi\right) + \frac{\mu(1-8n\gamma^2)}{1+\alpha r} d\psi \left(d\psi + 2\cos\theta d\phi\right) + \\ & + \mu(1+\alpha r)\frac{dr^2}{r^2} + \mu(1+\alpha r)d\theta^2 + \mu\frac{1+4\alpha r + 8n\gamma^2\alpha r(\alpha r - 2)}{1+\alpha r} \cos^2\theta d\phi^2 + \mu(1+\alpha r)\sin^2\theta d\phi^2, \end{aligned} \tag{6.2}$$

and a three-form G written in appendix C. There is still a singularity in $r = -1/\alpha$ for this metric, as one can see in the Kretschmann scalar (that we will omit here due to its complicated form) and other scalars like

$$R^{\mu\nu}R_{\mu\nu} = \frac{3\left[1 + \alpha r(\alpha r - 2)(1 - 8n\gamma^2)\right]^2}{2\mu^2(1+\alpha r)^6}. \tag{6.3}$$

The transformation M_n neither does change the signature of the solution and the points in which it flips. We still have the same pattern analysed in section 5.3.

The charges (4.31) are now

$$\tilde{Q} = 4\sqrt{2}\mu, \qquad Q = 8n, \qquad J = -32\mu\gamma n, \tag{6.4}$$

so we have non-vanishing entropy

$$S = 16\pi\mu\sqrt{2n(1 - 8n\gamma^2)}. \tag{6.5}$$

Notice that for this entropy to be well defined n must take values in $n \in \left[0, 1/8\gamma^2\right]$. This is consistent with the fact that, for $n > 1/8\gamma^2$, ∂_ψ becomes timelike, so we have CTCs.

In the following sections we partially repeat the analysis of the spectral flow on $AdS_3 \times S^3$ solution, looking for the differences. We will centre our attention in the $c > 0$ case because it is the one in which the discussion about missing charges raised (it is the only case for which there is a horizon).

6.1.1 The $r = 0$ suface

We turn now our attention to the surface of $r = 0$, and start by reading off its geometry:

$$ds^2\big|_{r=0} = \frac{2n}{\mu}du^2 + \mu\left(1 - 8n\gamma^2\right)d\psi^2 + \mu d\theta^2 + \mu d\phi^2 + 2\cos\theta d\phi\left[4n\gamma du + \mu\left(1 - 8n\gamma^2\right)d\psi\right]. \tag{6.6}$$

We notice that now it extends in four spatial dimensions, as one would expect for a regular event horizon in 6D. We can also say then that, taking into account the time component, $r = 0$ has co-dimension one and thus it is a hypersurface.

The $r = 0$ suface has normal vector n with components $n_\mu = \nabla_\mu r = \partial_\mu r = \delta^r_\mu$. Covariantly it is then

$$n = g^{rr}\partial_r = \frac{r^2}{\mu(1 + \alpha r)}\partial_r, \tag{6.7}$$

and we can see that its norm $|n|^2 = r^2/\mu(1 + \alpha r)$ vanishes in the $r = 0$ surface and conclude that it is a null hypersurface. Besides, the Killing vector field ∂_v is everywhere null so we can say that $r = 0$ is a Killing horizon of $\chi \equiv \partial_v$. We have

$$\nabla_\rho\chi^\sigma = \Gamma^\sigma_{\rho v} \qquad \rightarrow \qquad \chi^\rho\nabla_\rho\chi^\sigma = \Gamma^\sigma_{vv} = 0 \quad \forall\,\sigma, \tag{6.8}$$

so the surface gravity is $\kappa = 0$ for this Killing horizon. Recall that a zero surface gravity, and thus zero temperature, was a characteristic of extremal black holes, and then it is what one expects for supersymmetric solutions.

To render the horizon compact we consider the coordinate u to be periodic, just like for the black string. Back to the metric (6.6) and going to coordinates

$$\begin{aligned} \xi &\equiv -8n\gamma(1-8n\gamma^2)u + 16\mu n\gamma^2(1-8n\gamma^2)\psi, \\ \eta &\equiv \frac{4n\gamma}{\mu}u + (1-8n\gamma^2)\psi, \end{aligned} \tag{6.9}$$

we get

$$ds^2\big|_{r=0} = \frac{d\xi^2}{32\mu n\gamma^2(1-8n\gamma^2)} + \mu\left(d\eta^2 + d\theta^2 + d\phi^2 + 2\cos\theta d\eta d\phi\right) = \frac{d\xi^2}{32\mu n\gamma^2(1-8n\gamma^2)} + 4\mu d\Omega_3^2. \tag{6.10}$$

This implies that the horizon has $S^1 \times S^3$ topology, as in the black string case, and confirms our suspect that the solution studied in section 5.4 describes a limit in which the S^1 factor has zero length. We can calculate the area of this horizon integrating the determinant of (6.6)

$$A = \int_{r=0} \mu\sqrt{2n(1-8n\gamma^2)}\sin\theta du d\psi d\theta d\phi = 16\pi^2\mu L\sqrt{2n(1-8n\gamma^2)}, \tag{6.11}$$

with L the period of u. Notice that when the charge n is taken to zero the horizon shrinks in size, as we had predicted. As a consistency check, we substitute this area in the Bekenstein-Hawking entropy formula to get the same result as in (6.5):

$$S = \frac{A}{4G_{\rm N}} = 16\pi\mu\sqrt{2n(1-8n\gamma^2)}. \tag{6.12}$$

This horizon not only hides a singularity, but also closed timelike curves. For $c > 0$ and $n \in \left[0, 1/8\gamma^2\right]$ we have negative g_{uu} in a region $\alpha r \in (-1, \alpha r')$ with

$$\alpha r' = \frac{4n\gamma^2 - 1 + \sqrt{1-8n\gamma^2}}{4n\gamma^2}, \tag{6.13}$$

and as we took u to be periodic, this implies the presence of CTCs. Thanks to the fact that $r' < 0$ always for positive c and physically reasonable n, they lay behind the horizon and the cosmic censorship conjecture holds.

6.1.2 Near-horizon geometry

Next we study the geometry near the $S^1 \times S^3$ horizon we just described. Taking the limit $r \to 0$ in the metric (6.2) yields

$$\begin{aligned} ds^2 \to &-\frac{2r}{\mu}dvdu + 4\gamma r dv d\psi + \frac{2n}{\mu}du^2 + \mu(1-8n\gamma^2)d\psi^2 + \\ &+ 2\cos\theta d\phi\left[4n\gamma du + \mu(1-8n\gamma^2)d\psi\right] + \frac{\mu}{r^2}dr^2 + \mu d\theta^2 + \mu d\phi^2, \end{aligned} \tag{6.14}$$

that under the change of coordinates (6.9) becomes

$$ds^2 \to -\frac{r}{4\mu n\gamma(1-8n\gamma^2)}dvd\xi + \frac{d\xi^2}{32\mu n\gamma^2(1-8n\gamma^2)} + \frac{\mu}{r^2}dr^2 + 4\mu d\Omega_3^2. \tag{6.15}$$

This is locally $AdS_3 \times S^3$ as one can check by comparing with (4.23). Notice however that we have taken u, and thus ξ, to be periodic so we have a discrete identification of $AdS_3 \times S^3$ as the near-horizon geometry of this solution. Recall from the analysis of the previous solution that this is one of the three possible near-horizon spaces in 6D minimal supergravity, according to [16]. In contrast with the previous chapter, here the limit $r \to 0$ in the fields does result in a valid solution and it is not necessary to take it directly in the harmonic functions (which, in turn, provides a complicated result with few valuable information). Notice that (6.14) is equivalent to the full metric (6.2) with $c = 0$, so in this case the vanishing c case is again $AdS_3 \times S^3$, at least locally.

The topology and geometry of the horizon and near-horizon for the present solution are those that characterize a black string in six dimensions. We can conclude then that in the region of small r we are dealing with a black string, but what about the asymptotes?

6.1.3 Asymptotic limit

We have seen that the application of an M_n transformation to $M_\gamma \cdot \mathbb{V}_{AdS_3\times S^3}$ has changed noticeably the horizon, giving it a non-zero area. Next, we study whether the asymptotic geometry of the solution changes as well. Taking the limit $r \to \infty$ in the harmonic functions gives

$$M_n \cdot M_\gamma \cdot \mathbb{V}_{AdS_3\times S^3} \to \begin{pmatrix} -2c\gamma^2 \\ 0 \\ -2c\gamma \\ 0 \\ c - 2c\gamma^2 n \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2\mu\gamma \\ 0 \\ -2\mu\gamma n \\ 0 \\ \mu \end{pmatrix} \frac{1}{r}, \tag{6.16}$$

which generate the solution

$$\begin{aligned} ds^2 = {} & \frac{2r}{\mu}dvdu + \frac{2}{c\gamma}dvd\psi + 4\gamma r\cos\theta dvd\phi + \frac{2n\alpha r}{\mu}du^2 + 8n\gamma\alpha r\cos\theta dud\phi + \\ & + \frac{\mu(1-8n\gamma^2)}{\alpha r}d\psi^2 + \frac{\mu\alpha}{r}dr^2 + \mu\alpha r d\theta^2 + \mu\alpha r d\theta^2(8n\gamma^2\cos^2\theta + \sin^2\theta)d\phi^2, \end{aligned} \tag{6.17}$$

$$G = \frac{1}{2\mu}dv\wedge du\wedge dr - \gamma\cos\theta dv\wedge dr\wedge d\phi + \gamma r\sin\theta dv\wedge d\theta\wedge d\phi + \frac{\mu(1-8n\gamma^2)}{2}\sin\theta d\psi\wedge d\theta\wedge d\phi. \tag{6.18}$$

It does not look like the asymptotes are simpler now. In fact, (6.16) is just (5.15) transformed with M_n, i.e. the operations of taking the $r \to \infty$ limit in the harmonic functions and M_n commute for our solution $M_\gamma \cdot \mathbb{V}_{AdS_3\times S^3}$.

Unlike for the near-horizon, it is not easier to find out what our asymptotic space is after transforming with M_n. Attempts have been made to transform the harmonic functions (6.16) with those elements of the entropy conserving subset of $Sp(6,\mathbb{R})$ that amount to coordinate redefinitions, in order to get a more tractable metric. However, all possible gauge transformations render a more complicated form of the line element. In any case, we can say that this asymptotic solution has an energy-momentum tensor that asymptotes to a constant when $r \to \infty$ so the conclusion made in section 5.4.3 that we could be dealing with pp-waves in the large r limit still holds.

6.2 Acting with M_j

We can further explore the $Sp(6,\mathbb{R})$ group and act with a transformation that switches on a different parameter of the harmonic functions. We will use now

$$M_j \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ j & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.19}$$

which gives

$$M_j \cdot M_\gamma \cdot \mathbb{V}_{AdS_3 \times S^3} = \begin{pmatrix} -2c\gamma^2 \\ 0 \\ -2c\gamma \\ -2c\gamma^2 j \\ c \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2\mu\gamma \\ 0 \\ j \\ 0 \\ \mu \end{pmatrix} \frac{1}{r}. \tag{6.20}$$

This solution has charges $\tilde{Q} = 4\sqrt{2}\mu$, $Q = 0$ and $J = 8j$, which give an imaginary entropy

$$S = 16\pi\sqrt{-j^2}. \tag{6.21}$$

Recall the interpretation of $\tilde{Q}$, Q and J as charges and angular momentum of the five dimensional associated object. Even though we do not know exactly what object are we dealing with, this helps us understanding the imaginary entropy. The situation here is analogous to that of a Kerr-Newman black hole (see section 2.3) with too much angular momentum, for which a naked singularity is developed. Such a configuration does not have a physically well defined entropy, but if one computes it with the classical formula the result is imaginary, like in (6.21). The conclusion is then that M_j has switched on the angular momentum of our original solution without switching on any other charge, resulting in a non-physical object. We add that the seek for a violation of the NEC or the WEC in this solution did not succeed.

7 Conclusions and outlook

We have presented in this thesis a review of black holes in general relativity and supergravity and the intimate relation of the latter with string theory. Later we have focused in a particular class of BPS solutions of six dimensional minimal supergravity and the $Sp(6,\mathbb{R})$ group of transformations acting on them. The research part of this thesis has been centred around a transformation performed on $AdS_3 \times S^3$ with an entropy conserving element of this $Sp(6,\mathbb{R})$ group, called spectral flow transformation. The first conclusion is that the $Sp(6,\mathbb{R})$ group can transform solutions in very non-trivial ways, as we could see from the appearance of curvature singularities and signature changes in the geometry, or from the fact that the new solution is highly dependant on an inert parameter c of the original $AdS_3 \times S^3$.

For $c > 0$ the new geometry has a singularity whose horizon is squeezed to zero size. We showed this by acting with another $Sp(6,\mathbb{R})$ transformation that blew up the horizon to finite size "switching on" some charges of the solution. The topology of this horizon is $S^3 \times S^1$ and the near-horizon geometry is $AdS_3 \times S^3$, coinciding with those of a black string. As for the asymptotic behaviour, we found out that it is also changed by the transformation, and that the resulting asymptotic geometry resembles that of a pp-wave. The information extracted from the new solution does not allow us to conclude whether it is a new solution of the theory *stricto sensu* or just a superposition of a black string over a pp-wave background. This last possibility can motivate future research, for instance trying to find the covariantly constant null vector field that corresponds to the pp-wave.

The $c < 0$ case describes a naked singularity and CTCs close to it. We have attempted to find signs of odd matter in the solution, like a negative Komar mass or violations of energy conditions. The first was not possible to compute given the non-asymptotically flat character of the geometry. As for the energy condition violations, we have failed to found them but a more thorough search could be pursued with more powerful computational resources. We have shown, however, that negative $D3$-branes in type-IIB string theory can qualitatively explain the naked singularity and signatures of our solution. Formalising this relation by explicitly describing the reduction from ten to six dimensions is also a possibility for future research.

In a broader sense, it can be said that the $Sp(6,\mathbb{R})$ group of transformations deserves our attention. At the very least, it is able to take us out of the usual range of BPS solutions and looks like a very promising tool to continue the exploration of the six dimensional minimal supergravity theory. Being optimistic, it could also have some microscopic origin or it could give structure to the phase space of solutions. A systematic approach, in which the 21 generators of the group are studied individually in terms of their effects on the solutions, could be a good path to follow. We want to recall once more the difficulties that we had in our research due to the non asymptotical flatness of our spacetimes. Given this, we add that in a prospective future research about the orbits of $Sp(6,\mathbb{R})$ in the space of solutions, it would be advisable to focus in those with flat or simple asymptotics, at least in the first place. In this sense, it would be very convenient to find which subset of the symplectic group fixes the asymptotics.

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A Mathematical tools

Hodge star operator

For $\Lambda^p(M)$ the space of p-forms in an D dimensional manifold M with metric g, there is an isomorphism between $\Lambda^p(M)$ and $\Lambda^{D-p}(M)$ given by the Hodge star or Hodge dual operator $*$. It can be defined as the operation such that $\forall\ \alpha, \beta \in \Lambda^p(M)$

$$\alpha \wedge *\beta = \frac{1}{p!}\alpha_{\mu_1 \dots \mu_p}\beta^{\mu_1 \dots \mu_p}\ \mathrm{vol}_D, \tag{A.1}$$

with vol_D the volume form of M. The form $*\alpha \in \Lambda^{D-p}(M)$ is called the Hodge dual of α, and its components are [51]

$$(*\alpha)_{\mu_{p+1}\dots\mu_D} = \frac{\sqrt{|g|}}{p!}\alpha^{\mu_1\dots\mu_p}\varepsilon_{\mu_1\dots\mu_p\mu_{p+1}\dots\mu_D} = \frac{\sqrt{|g|}}{p!}\alpha_{\nu_1\dots\nu_p}g^{\nu_1\mu_1}\dots g^{\nu_p\mu_p}\varepsilon_{\mu_1\dots\mu_D} \tag{A.2}$$

with ε the Levi-Civitta symbol and g the determinant of the metric tensor. The Hodge star operation applied twice to $\alpha \in \Lambda^p(M)$ yields

$$* *\alpha = (-1)^{p(D-p)}\mathrm{s}(g)\alpha, \tag{A.3}$$

where $\mathrm{s}(g)$ is the sign of g. We have $s = 1$ for Riemannian and $s = -1$ for Lorentzian manifolds. When $D = 2m$ is even and $(-1)^{m^2}s(g) = 1$ such that $* * \alpha = \alpha$ for m-forms, one can impose on them a self-duality or anti-self-duality condition

$$* \alpha = \pm\alpha. \tag{A.4}$$

Vielbein

The vielbein on a D dimensional manifold M, also called tetrad or frame field, is a set of D vector fields $e_a(x)$ with $a = 0, \dots, D-1$ that form an orthonormal basis of the tangent bundle $T(M)$ at every point $x \in M$, i.e. they locally diagonalize the metric [36, ch. 2]. For a Lorentzian manifold they are one timelike and $D-1$ spacelike vector fields, and physically correspond to a family of ideal observers at spacetime whose world-lines are the integral curves of the timelike one. At each point along these world-lines the spacelike vectors correspond to the axis of a local laboratory frame. We write them

$$e_a(x) = e_a^\mu(x)\partial_\mu, \tag{A.5}$$

and from the above definition

$$\eta_{ab} = g_{\mu\nu}(x)e_a^\mu(x)e_b^\nu(x), \tag{A.6}$$

with η_{ab} the metric of flat D dimensional Minkowski spacetime. Note that, from the fact that the vielbein locally diagonalizes $g_{\mu\nu}(x)$, the determinant of the $D \times D$ matrix $e_\mu^a(x)$ is $e \equiv \det(e_\mu^a) = \sqrt{-g}$.

One can define the dual frame fields raising and lowering indices with the appropriate metric tensor, i.e. $e^a_\mu(x) = \eta^{ab} g_{\mu\nu}(x) e^\nu_b(x)$. Under a Lorentz transformation in the flat local coordinates the fields transform

$$e'^\mu_a(x) = \left(\Lambda^{-1}\right)_a{}^b(x) e^\mu_b(x), \qquad e'^a_\mu(x) = \left(\Lambda^{-1}\right)^a{}_b(x) e^b_\mu(x), \tag{A.7}$$

while under diffeomorphisms on M they transform as contravariant and covariant vectors respectively. All tensorial quantities on the manifold can be expressed in the local frame using the vielbein and its dual:

$$T^{a_1\ldots a_m}{}_{b_1\ldots b_n}(x) = e^{a_1}_{\mu_1}(x)\ldots e^{a_m}_{\mu_m}(x) e^{\nu_1}_{b_1}(x)\ldots e^{\nu_n}_{b_n}(x) T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}(x). \tag{A.8}$$

B Spinors

We briefly introduce here the different types of spinors used in supergravity theories. We refer to [30] for a more detailed treatment of the following content. In order to work with spinors one needs to consider the Clifford algebra associated with the Lorentz group. This algebra is generated by the gamma matrices satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I. \tag{B.1}$$

There is a unique (up to conjugation) irreducible representation of the γ^μ matrices by $2^m \times 2^m$ matrices in spacetime dimension $D = 2m$. For odd dimension $D = 2m+1$, there are two inequivalent irreducible representations by $2^m \times 2^m$ matrices. The dimension of the irreducible representation is then always $2^{[D/2]}$. A basis for the Clifford algebra is given by I, γ^μ and all their independent products, and is denoted

$$\Gamma^A = \{I, \gamma^\mu, \gamma^{\mu_1\mu_2}, \ldots, \gamma^{\mu_1\ldots\mu_D}\}, \tag{B.2}$$

with $\mu_1 < \mu_2 < \ldots < \mu_D$ and $\gamma^{\mu_1\cdots\mu_r} \equiv \gamma^{[\mu_1}\ldots\gamma^{\mu_r]}$, where the antisymmetrization has total weight one and r is the rank of the element. For even D the algebra has dimension 2^D. The highest rank element $\gamma^{\mu_1\cdots\mu_D}$ is usually denoted γ_{D+1} and defines the chiral projectors

$$P_L \equiv \frac{1}{2}\left(I + \gamma_{D+1}\right) \qquad \text{and} \qquad P_R \equiv \frac{1}{2}\left(I - \gamma_{D+1}\right). \tag{B.3}$$

For odd dimension $D = 2m+1$ the basis contains the elements given in (B.2) but only up to rank m. There exists a so called charge conjugation matrix C such that every matrix $C\Gamma^A$ is either symmetric or antisymmetric, depending on the rank r of Γ^A:

$$\left(C\Gamma^A\right)^{\mathrm{T}} = -t_r C\Gamma^A \qquad \text{with} \qquad t_r = \pm 1. \tag{B.4}$$

The numbers t_r depend on the dimension D with periodicity 8, i.e. they are the same for e.g. $D = 4$ and $D = 12$. This symmetry property allows to write the complex conjugate of a gamma matrix as

$$(\gamma^\mu)^* = -t_0 t_1 B\gamma^\mu B^{-1} \qquad \text{with} \qquad B \equiv i t_0 C\gamma^0. \tag{B.5}$$

Majorana spinors

The Majorana conjugate of any spinor ψ is defined

$$\bar{\psi} \equiv \psi^{\mathrm{T}} C, \tag{B.6}$$

such that $\bar{\psi}_1 \psi_2$ is Lorentz invariant. Next, one defines the charge conjugate of a spinor as

$$\psi^c \equiv B^{-1} \psi^*, \tag{B.7}$$

with ψ^* the complex conjugate spinor. This allows to impose a generic reality constraint on any spinor ψ, which is

$$\psi = \psi^c \qquad \rightarrow \qquad \psi^* = B\psi. \tag{B.8}$$

This constraint is compatible with Lorentz symmetry and a spinor ψ satisfying it will have half of the components of a Dirac spinor, i.e. $2^{[D/2]-1}$ complex components. Taking the complex conjugate of the right hand side of (B.8) one finds $\psi = B^* B\psi$, so the reality condition is only consistent if $B^* B = I$, which implies $t_1 = -1$. If one has in addition $t_0 = 1$ the spinors satisfying (B.8) are called Majorana. This only happens for $D = 2, 3, 4$ modulo 8. In these dimensions one can find representations such that $B = I$, the Majorana condition is then $\psi^* = \psi$ and the field is actually real. For $D = 8, 9$ modulo 8, where $t_1 = t_0 = -1$, the spinors satisfying (B.8) are called pseudo-Majorana and a real representation is not available.

Majorana-Weyl spinors

Recall that a Weyl spinor is a massless field with well defined chirality, i.e. $P_L\psi = \psi$ or $P_R\psi = \psi$. For spacetime dimension $D = 2$ modulo 8, the reality condition (B.8) and the Weyl condition are compatible. In other words, each chiral projection of a Majorana spinor ψ satisfies the reality condition:

$$(P_L\psi)^c = P_L\psi, \qquad (P_R\psi)^c = P_R\psi, \tag{B.9}$$

so it is Majorana as well. In these dimensions the most fundamental spinors are taken to be those that satisfy both constraints, called Majorana-Weyl spinors, because they have the least number of independent components: $2^{[D/2]-1}$ real ones.

Symplectic Majorana spinors

When $t_1 = 1$ the reality condition (B.8) is not consistent and cannot be imposed. This occurs for $D = 5, 6, 7$ modulo 8. One can, however, define a different reality condition using an even number of spinors χ^i with $i = 1, \ldots, 2k$ and a $2k \times 2k$ invertible real skew-symmetric matrix Ω:

$$\chi^i = \Omega^i{}_j \left(\chi^j\right)^c \qquad \rightarrow \qquad \left(\chi^i\right)^* = \Omega^i{}_j B \chi^j. \tag{B.10}$$

As now $B^*B = -I$, the matrix Ω must satisfy $\Omega^2 = -I$ in order to get χ^i when complex conjugating the right hand side. There is an internal group acting on the indices i, j, i.e. transforming $\chi^i \to M^i{}_j \chi^j$. It must be consistent with the above definition, so omitting i, j indices we have

$$(M\chi)^* = \Omega M B \chi \qquad \to \qquad \chi^* = (M^*)^{-1}\Omega M B \chi \qquad \to \qquad (M^*)^{-1}\Omega M = \Omega. \tag{B.11}$$

The transformation matrices M belong then to the symplectic unitary group $USp(2k)$ of unitary matrices satisfying $M^{\mathrm{T}}\Omega M = \Omega$ [52, ch. 5]. For that reason this class of spinors are called symplectic Majorana spinors. They have $2^{[D/2]}$ real components, but as they necessarily come in pairs the minimum number of components effectively doubles. In $D = 6$ modulo 8 dimensions it is also possible to combine this reality constraint with chirality, such that the fundamental spinor is the symplectic Majorana-Weyl spinor. They have $2^{[D/2]-1}$ real components so the minimum number in six dimensions is 8.

The information given in this appendix about the fundamental spinor for every dimension and its number of real components is gathered in table 1.

C Three-form G of the solutions

The three-form G for the solution studied in section 5 is written here. As mentioned there, its calculation is different for different regions of the manifold due to the signature changes of the base metric, but the final result is the same. We have

$$\begin{aligned} G = & \frac{\alpha^2 r^2 + 2\alpha r - 1}{2\mu(1+\alpha r)^2} dv \wedge du \wedge dr + \frac{\gamma}{(1+\alpha r)^2} dv \wedge d\psi \wedge dr - \frac{\gamma \alpha r(2+\alpha r)}{(1+\alpha r)^2} \cos\theta dv \wedge dr \wedge d\phi + \\ & + \frac{\gamma \alpha r^2}{1+\alpha r} \sin\theta dv \wedge d\theta \wedge d\phi - \frac{c\gamma}{(1+\alpha r)^2} du \wedge d\psi \wedge dr + \frac{c\gamma}{(1+\alpha r)^2} \cos\theta du \wedge dr \wedge d\phi - \\ & - \frac{\gamma c r}{1+\alpha r} \sin\theta du \wedge d\theta \wedge d\phi - \frac{\mu\alpha}{(1+\alpha r)^2} \cos\theta d\psi \wedge dr \wedge d\phi - \frac{\mu(1-\alpha r)}{2(1+\alpha r)} \sin\theta d\psi \wedge d\theta \wedge d\phi. \end{aligned} \tag{C.1}$$

We can write an expansion of G in powers of γ to study how the transformation M_γ affects it. The result up to first order is

$$\begin{aligned} G = & -\frac{1}{2\mu} dv \wedge du \wedge dr - \frac{\mu}{2} \sin\theta d\psi \wedge d\theta \wedge d\phi + \gamma \left(dv \wedge d\psi \wedge dr - \right. \\ & \left. - c du \wedge d\psi \wedge dr + c\cos\theta du \wedge dr \wedge d\phi - cr \sin\theta du \wedge d\theta \wedge d\phi \right) + \mathcal{O}(\gamma^2). \end{aligned} \tag{C.2}$$

The zero order part is of course the $AdS_3 \times S^3$ three-form (4.22). Notice that one of the first order terms is linear in r, so a small transformation M_γ already changes noticeably the matter content in the asymptotic region.

In section 6.1 we have transformed the previous solution with the $Sp(6, \mathbb{R})$ element M_n. The resulting three-form is

$$
\begin{aligned}
G = \frac{1}{(1+\alpha r)^2}\Big[&\frac{\alpha^2 r^2 + 2\alpha r - 1}{2\mu} dv \wedge du \wedge dr + \gamma dv \wedge d\psi \wedge dr - \gamma\alpha r(2+\alpha r)\cos\theta dv \wedge dr \wedge d\phi + \\
&+ \gamma\alpha r^2(1+\alpha r)\sin\theta dv \wedge d\theta \wedge d\phi + \gamma c(1-8n\gamma^2)\cos\theta du \wedge dr \wedge d\phi - \gamma c(1-8n\gamma^2) du \wedge d\psi \wedge dr - \\
&- \gamma(1+\alpha r)[cr + 2n(1-\alpha r)]\sin\theta du \wedge d\theta \wedge d\phi - \mu\alpha(1-8n\gamma^2)\cos\theta d\psi \wedge dr \wedge d\phi + \\
&+ \frac{\mu}{2}(\alpha^2 r^2 - 1)(1-8n\gamma^2)\sin\theta d\psi \wedge d\theta \wedge d\phi\Big].
\end{aligned}
\tag{C.3}
$$

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