Infinite distances in the moduli space of Calabi-Yau threefolds

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Abstract

In this work we analyze the Swampland Distance Conjecture [1] for the field space of scalar fields, called the complex moduli, that arise in Calabi-Yau threefold compactifications. The Swampland Distance Conjecture states that at any infinite distance point in this moduli space an infinite tower of states will appear, rendering the effective 4D theory unphysical. This is very important when one attempts to realize inflation with a large field distance from string theory, as in essence this renders the complex moduli unable to be identified with the inflaton that causes this type of large field inflation. In this work we elucidate the above motivations for studying the Swampland Distance Conjecture, after which we connect the Swampland Distance Conjecture in physics to the mathematical Wang conjecture [2]. For a moduli space with singularities encoded on subspaces called divisors the Wang conjecture states that the metric on the moduli space diverges if and only if the divisors have monodromies of infinite order. In [3] a connection between infinite order monodromy matrices and infinite towers of states is postulated, bridging the gap between the Wang and Swampland Distance conjectures. Wang himself already showed that an infinite order monodromy is a necessary requirement for an infinite field distance to arise[2]. In this work we study the mechanism behind the Swampland Distance Conjecture, by considering whether having such a monodromy is also a sufficient condition for the divergence of the field distance. We review a proof by Lee [4] of specific cases of this sufficient condition of the Wang conjecture. The cases we revisit and clarify assume the singularity to be located on the intersection of up to two divisors. Finally we present a new result, by giving new criteria for the field distance to diverge, in the case of a singularity on one infinite and one finite divisor in two dimensions.

Contents

Chapter 1

Introduction

Within string theory there is a vast amount of consistent effective theories arising from the equations of motion. These consistent theories are said to lie in the string *landscape*. Because of this very large amount of possible solutions it is not clear whether string theory can make any real predictions at all, as it appears that, with the right compactification scheme, one could construct any theory one wants to find. Then came the claimed BICEP-2 measurement of tensor modes in the CMB spectrum, which would imply a large inflationary field distance. Inflation can be parametrized by a field called the inflaton, which acts as an order parameter for inflation. The difference in value of the inflaton field from one time to the next is referred to as the *field distance*, or *field range*. This lead to string theorists revisiting the following question: "is it possible to engineer transplanckian field ranges from string theory?" That is, is it possible to create a 4D effective theory, from string theory, which can source inflation with a large inflaton field distance. Realizing such inflationary theories proved difficult however, as the inflationary models were plagued by problems. One of these is the *moduli stabilization problem*, where fields related to the geometry of the string landscape, called *moduli*, are too energetic for too long, thereby being relevant well into the early universe and ruining nucleosynthesis. Because of these problems string theorists were hard-pressed to achieve high enough field distances, in a straightforward manner.

The growing pile of string theories that were unable to source large field inflation lead Ooguri and Vafa[1] to consider a simplified question "is it possible to engineer parametrically infinite field distances in string theory?" In order to come to an answer they conjectured the following: any theory that is consistent as a quantum gravitational theory *breaks down* if one lets a field distance traveled in the *space of moduli* (which is just the field space for some specific string theoretical fields) diverge. That is, they conjectured that the divergent field distance leads to the emergence of an infinite tower of massless states that have to be incorporated into the effective theory, rendering the theory invalid. They therefore answered the simplified question, where the field range is taken to infinity, with a no. This would make it impossible to write down UV-complete 4d effective theory in which a moduli field is identified to be the inflaton that sources large field inflation. To use the terminology introduced earlier: any theory which is located in the string theory landscape has this inability to realize transplanckian field distances. Hence their conjecture was named the

swampland distance conjecture (SDC).

As of now the SDC has been expanded into the *refined* swampland distance conjecture (RSDC)[5]. This conjecture specifies the field distance at which the effective theory breaks down to be $\Delta\phi \sim M_{pl}$, the Planck mass.

The (R)SDC is still a conjecture however. More evidence is needed to verify it and also to understand the connections between the field distance and the appearing infinite tower of states. Proving it in full generality is a difficult task: to begin, we therefore restrict ourselves to a particular corner of string theory; those 4d effective theories with a scalar field realized as a complex modulus (i.e. the field describing the complex properties of the string manifold) of a type IIB compactification. Restricting ourselves to these solutions gives us the possibility to use the already existing mathematical formalism of the moduli space.

In this thesis we attempt to motivate the SDC from a mathematical point of view, by studying the theory behind infinite distances in the moduli space. We hope to give strong evidence for the link between a divergent field distance and the emergence of an infinite tower of massive states (hence also evidence for the SDC) through the mathematical *Wang conjecture*. This conjecture pertains to moduli spaces, and relates a divergent field distance to the existence of a monodromy of infinite order.

We begin by developing the knowledge necessary to understand how the measurements of CMB tensor modes could be relevant for string theory. As such we begin with a short review of large field inflation and tensor perturbations, following with a review of the conjectures seeking to limit the field distances realizable from string theory. Having explained the predictive power of tensor modes in the context of string theory, we then, in chapter 3, consider the basics of string theory compactifications. This chapter explains the origin of scalar fields in 4d effective string theories, and introduces the concept of the moduli space. With our understanding of the moduli space, and the relevance of distances therein, we are ready to study the divergence of distances in the moduli space and the Wang conjecture. This is what chapter 4 focuses on.

Chapter 2

Large field inflation and the string swampland

In this section we shall consider the theory of inflation, and the way in which it influences the evolution of the metric. Inflation is postulated as a solution to the horizon and flatness problems[6], by making the *Hubble radius* $(aH)^{-1}$ decrease at early times. This criterion is equivalent to accelerated expansion of the universe, $\ddot{a} > 0$. When considering inflation from a field theoretical point of view we think of a field ϕ, the inflaton, which *drives* inflation. This is explained in a bit more detail in appendix A.

Within inflationary theories we can differentiate between those theories in which the inflaton traverses a subplanckian field distance ($\Delta\phi < M_{pl}$) while sourcing inflation, and those where the inflaton traverses a transplanckian field range ($\Delta\phi > M_{pl}$). The former class of theories is labeled *large field inflation*. This field range is measured by the *tensor to scalar ratio* r and can be observed in CMB fluctuations. It turns out that, for an observable value of r, the field distance has to be of the order of the Planck distance $\Delta\phi \sim M_{pl}$. This bound is called the *Lyth bound* [7]. In this chapter we will provide the motivation for the Lyth bound.

Were tensor perturbations to be discovered in the CMB, the Lyth bound would imply that during inflation the inflaton traverses a transplanckian field range. An inflationary model where this is the case is called *large field inflation*. In this chapter we will restrict ourselves to large field models and mention some points about them. Note that, from a field theoretical point of view, finding such a large field range would be surprising, as this requires the potential to be flat for a long enough time. Inflation however is a UV-sensitive theory, and one would expect Planck-suppressed operators to ruin the flatness of the potential in the UV-limit. One does not allow this to happen though, as such a potential would spoil the inflationary mechanism. This suggests that inflationary models have a shift symmetry to protect the potential from change. This shift symmetry needs to be unbroken also in the UV-limit, which requires inflationary models to have a UV-completion.

Realizing large field inflation from string theory could therefore lead to a measurable prediction for r, as well as UV-complete theories for inflation. This motivates one to ask the question "is it possible to engineer transplanckian field ranges from string theory?" As was

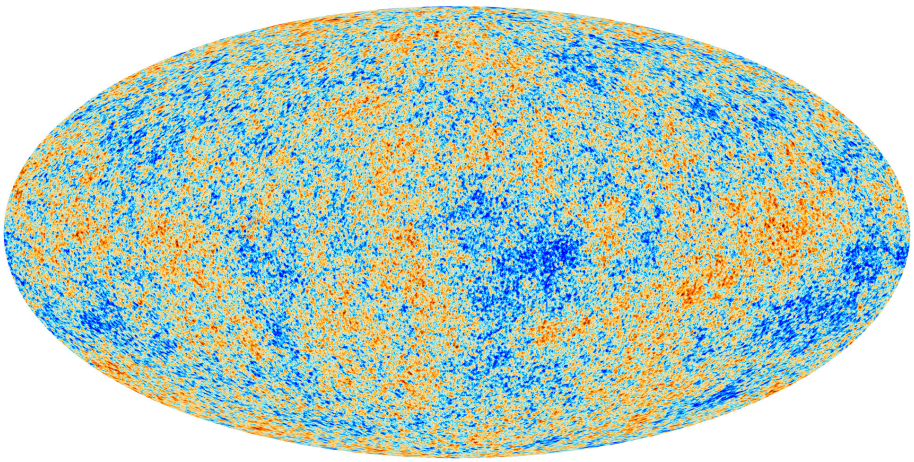

Figure 2.1: Picture showing the anisotropies in the CMB temperature. In the field of cosmology one can model the correlation of fluctuations in the temperature. Source: Planck collaboration.

mentioned in the introduction, this question is difficult to answer, and therefore the (Refined) swampland distance conjectures [1, 5] began by conjecturing the answer to the slightly simpler question, "is it possible to engineer diverging moduli space[1] field ranges from string theory?" to be *no*.

To motivate the conjecture we shall look at a simple example, where the inflaton is identified with the complex structure modulus arising from the string compactification on a circle. One should note that inflation has also been realized from string theory in different, more complicated ways[8, 9], which with the methods presented in this work are beyond our reach to consider.

In this chapter we shall introduce large field inflation and motivate the Lyth bound. Having motivated the question of realizing large field inflation from string theory, we shall then consider the conjectured answer by looking at the (R)SDC.

2.1 Inflation and the Lyth bound

When looking at the temperature distribution in the CMB 2.1 we observe that the universe is not perfectly homogeneous and isotropic. One finds patches of slightly higher, and some of slightly lower temperature. These fluctuations away from a homogeneous solution are small however. Therefore this phenomenon can be explained by *quantum fluctuations* to the classical background evolution of the inflaton $\overline{\phi}(t)$. Because of these fluctuations inflation will end earlier or later in different disconnected patches, leading to different parts of the universe undergoing different types of evolution.

[1]Remember that the moduli space is simply the name for the field space of some specific string theoretical fields, namely those describing the structure of the manifold on which one compactifies.

These same inhomogeneities in the CMB also lead us to consider perturbations to the background metric $\eta_{\mu\nu}$. The various components of the metric fluctuations can then be studied using cosmological perturbation theory.

In this section we will briefly motivate (omitting all calculations) how one can combine the quantum fluctuations to the inflaton with cosmological perturbation theory to derive the *Lyth bound*. This is a relation between the field distance traversed by the inflaton during inflation and the ratio of tensor-to-scalar metric perturbations. We will see that, in order to have an observable tensor-to-scalar ratio, the inflaton field distance needs to be transplanckian.

Having obtained this motivation to study large inflaton field distances, we will briefly review large field inflation.

The review we present in this section will follow [6]. Some notation, the basics of inflation and some important notions of cosmology, like the Hubble radius, have been collected in appendix A. For the motivations behind inflation and a more detailed discussion see [6].

2.1.1 Metric perturbations

The starting point for our discussion is the inflaton Lagrangian as written in (A.23),

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) =: S_{EH} + S_\phi. \tag{2.1}$$

We define the following perturbations around the homogeneous background solutions $\overline{\phi}(t)$, $\overline{g}_{\mu\nu}(t)$ for the inflaton and metric respectively

$$\phi(\mathbf{x},t) = \overline{\phi}(t) + \delta\phi(\mathbf{x},t), \quad g_{\mu\nu}(\mathbf{x},t) = \overline{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(\mathbf{x},t). \tag{2.2}$$

We can then decompose the metric into scalar, vector and tensor modes to find

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu\nu \\ &= -(1+2\Phi)dt^2 + 2aB_i dx^i dt + a^2\left[(1-2\psi)\delta_{ij} + E_{ij}\right]dx^i dx^j, \end{aligned} \tag{2.3}$$

where

$$B_i := \partial_i B - S_i, \quad \partial^i S_i = 0, \tag{2.4}$$

$$E_{ij} := 2\partial_i\partial_j E + 2\partial_{(i}F_{j)}, \quad \partial^i F_i = 0,\ h^i_i = \partial^i h_{ij} = 0. \tag{2.5}$$

We have therefore decomposed the metric $g_{\mu\nu}$ into scalar perturbations E, Ψ, Φ and B, vector perturbations S_i and F_i and a tensor perturbation h_{ij}. The vector perturbations S_i and F_i decay as the universe expands. Also, they are not sourced by inflation[6]. As such we will ignore them in the rest of this discussion.

The relevant parameters to us will therefore be E, Ψ, Φ, B and h_{ij}. The scalar perturbations will be responsible for density fluctuations in the late universe, and the tensor fluctuations as gravitational waves.

Gauge-invariant parameter

Because of the gauge freedom present in general relativity it is necessary, in order to get a physical answer, to define gauge invariant scalars out of the scalars considered above. We note that h_{ij} is already gauge invariant, and therefore does not have to be treated.

The most relevant gauge invariant quantity to this discussion is the comoving curvature perturbation,

$$\mathcal{R} := \Psi - \frac{H}{\overline{\rho} + \overline{p}} \delta q, \tag{2.6}$$

where H is the Hubble parameter, $\overline{\rho}, \overline{p}$ are the background energy density and pressure respectively and δq is related to the scalar part of the 3-momentum density; $T_i^0 = \partial_i \delta q$.

In the epoch of inflation we know the behavior of the stress-energy tensor[6] to be $T_i^0 = -\dot{\overline{\phi}} \partial_i \delta\phi$. Hence

$$\mathcal{R} = \Psi + \frac{H}{\dot{\overline{\phi}}} \delta\phi. \tag{2.7}$$

In the next section we shall consider the Fourier transform of the scalar and tensor perturbations, namely $\mathcal{R}_{\mathbf{k}}$ and $h_{\mathbf{k}}$. We can compare the mode $\mathbf{k}$ of the perturbation to the characteristic wave number of the universe, $(aH)^2$. It can be shown that, when $|\mathbf{k}| = k < aH$, the perturbations *freeze out*, meaning that they remain constant[3].

We know that, by definition, $(aH)^{-1}$ shrinks during inflation. We can therefore talk about modes *exiting* the horizon (as $k < aH$), and *entering* the horizon again (as $k > aH$) at a later time. We will use these ideas later on.

2.1.2 Power spectra of cosmological perturbations

Due to the uncertainty in the initial conditions of the universe we have to work with statistical quantities when doing measurements. The perturbative approach given above could therefore never supply us with an estimate for the value of $\mathcal{R}$. Instead we consider the statistics through its two-point *correlation function* $\langle \mathcal{R}_{\mathbf{x}} \mathcal{R}_{\mathbf{x}'} \rangle$. Given a value of $\mathcal{R}_{\mathbf{x}}$ at $\mathbf{x}$, this function describes the chance of finding a value $\mathcal{R}_{\mathbf{x}'}$ at position $\mathbf{x}'$. Using our assumption that the background of the universe is isotropic and homogeneous we can Fourier transform this correlation function to

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') P_{\mathcal{R}}(k), \qquad \Delta_s^2 := \Delta_{\mathcal{R}}^2 = \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k). \tag{2.8}$$

[2]This characteristic wave number is of course derived from the characteristic length scale $(aH)^{-1}$ as mentioned in appendix A.

[3]Intuitively this is because the wavelength of the mode is larger than that of the causally connected universe; different points on the wave are therefore no longer in causal contact and cannot evolve.

Where P denotes the *power spectrum*. The power spectra of the gauge invariant scalar and tensor modes, $\mathcal{R}$ and h_{ij}, are very important in considering their primordial fluctuations. We will assume the statistics governing the early universe to be *Gaussian*, which means that all the information is absorbed in the two-point function shown above.

We can split the tensor modes up into two polarizations; $h = h^+, h^\times$. The power spectrum for the two polarization modes is then defined to be,

$$\langle h_{\mathbf{k}} h_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') P_h(k), \quad \Delta_h^2 = \frac{k^3}{2\pi^2} P_h(k). \tag{2.9}$$

The presence of the two polarization modes then motivates us to define the full tensor power spectrum as,

$$\Delta_t^2 := 2\Delta_h^2. \tag{2.10}$$

2.1.3 Calculating the power spectra from inflation

We would now like to calculate these two power spectra in the context of inflation, and see what results this leads us to.

Scalar perturbations

Beginning with the action for single-field slow-roll inflation,

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[R - (\nabla\phi)^2 - 2V(\phi) \right], \tag{2.11}$$

with $M_{pl}^{-2} := 1$. After fixing the gauge as in [6] one can expand this action up to second order in the perturbations $\mathcal{R}$ in order to find the following scalar action,

$$S_{(2)} = \frac{1}{2} \int d^4x \frac{\dot{\phi}}{H^2} \left[\dot{\mathcal{R}}^2 - a^{-2} \left(\partial_i \mathcal{R} \right) \right] a^3. \tag{2.12}$$

We will now omit the further calculations, and merely mention the steps taken in [6].

First one uses the Mukhanov variable to re-parameterize the action. This leads to some equations of motion, for which we need boundary conditions in order to find a solution. We therefore quantize the action, the normalization of operators giving us our first boundary condition. The next boundary condition on the modes is given by the fact that, in the far past ($\tau \to -\infty$) the vacuum state of the fluctuations is the Minkowski vacuum.

This makes it possible to compute the power spectrum of $\mathcal{R}$, the moment when the wavelength of the modes k is of the order of the Hubble scale (i.e. at *horizon crossing*, $a_* H_* = k$). We assume that this happens during inflation, such that the slow-roll approximation still holds. The $\mathcal{R}$ power spectrum can then be related to inflation through

$$\Delta_{\mathcal{R}}^2(k) = \frac{H_*^2}{(2\pi)^2} \frac{H_*^2}{\dot{\phi}_*^2}. \tag{2.13}$$

Note that, as the mode no longer evolves, the power spectrum is fixed to this value until horizon re-entering. This makes it very valuable to measure upon re-entering, as it supplies us with knowledge on inflation and the early universe.

Tensor perturbations

Analogously we can also expand the action (2.11) up to second order in the tensor modes h_{ij},

$$S_{(2)} = \frac{M_{pl}^2}{8} \int d\tau dx^3 a^2 \left[(h'_{ij})^2 - (\partial_l h_{ij})^2 \right], \tag{2.14}$$

where h'_{ij} denotes the derivative of h_{ij} with respect to the proper time, and introduction of M_{pl} is done to make h_{ij} dimensionless. Performing a similar, but simpler, procedure leads us to the tensor power spectrum,

$$\Delta_t^2 = 2\Delta_h^2(k) = \frac{2}{\pi^2} \frac{H_*^2}{M_{pl}^2}. \tag{2.15}$$

2.1.4 The Lyth bound

It is useful to express tensor fluctuations as normalized with respect to the scalar fluctuations. This leads one to define the *tensor-to-scalar ratio* r,

$$r := \frac{\Delta_t^2(k)}{\Delta_s^2(k)}. \tag{2.16}$$

Using equations (2.13) and (2.15), we can express this ratio in terms of the evolution of the inflaton as a function of the number of e-folds,

$$r = \frac{8}{M_{pl}^2} \left(\frac{d\phi}{dN} \right)^2. \tag{2.17}$$

Taking the square root and integrating both sides we observe that it is possible to express the inflaton field range $\Delta\phi$ in terms of the tensor-to-scalar ratio $r(N)$,

$$\frac{\Delta\phi}{M_{pl}} = \int_{N_{end}}^{N_{cmb}} dN \sqrt{\frac{r}{8}}. \tag{2.18}$$

Making the final observation that, during inflation, the tensor-to-scalar ratio does not change much as a function of N we can approximate it to be constant, and find the Lyth bound [7],

$$\frac{\Delta\phi}{M_{pl}} \simeq \mathcal{O}(1) \left(\frac{r(N_{cmb})}{0.01} \right)^{1/2}. \tag{2.19}$$

The value that a chosen model has for the scalar to tensor ratio therefore directly influences the field range; for $r > 0.01$ we find that $\Delta\phi > M_{pl}$. An inflationary theory with a transplanckian field range is referred to as *large field inflation*.

The growth of scalar and tensor modes influences the CMB spectrum. As such their values can be observed by analyzing the CMB, as was done by the BICEP-2 measurement mentioned in the introduction. It is through the Lyth bound that observations could lead to a confirmation of large field inflation, and the need for string theory to produce effective theories with transplanckian field ranges.

2.2 Swampland Conjectures

In the previous section and in the introduction we motivated the benefits of considering large field inflationary models, and attempting to find an effective string theory realizing this. It is widely known that the various compactification schemes, parameters and geometries possible within string theory lead to an entire host of string theories: estimated to go up to $\mathcal{O}(10^{1500})$ [10]. This lead to people referring to the "landscape" of viable string theories, which is distinguished from a "swampland" of theories which are proven to be inconsistent in some way.

In this section we will restrict ourselves to those effective theories in which a moduli field[4] is identified with the inflaton driving large field inflation. In other words, we want to study which effective theories the field distance in the moduli space can be transplanckian; $\Delta\phi > M_{pl}$. In this section we will review The swampland distance conjecture (SDC) [1], which declares that any consistent quantum gravitational theory breaks down for such field distances.

The SDC states that letting the moduli field value vary over *any geodesic* of infinite distance leads to a breakdown of the effective theory. Note that the SDC does not mention what the cutoff Δp is up to which the effective theory is still valid. The recently proposed refined swampland distance conjecture [5] improves on the SDC by stating that the allowed field range is of the natural order; the Planck scale.

This chapter therefore acts as the "glue" between the various parts of this work. When building an inflationary model from string theory the question of possible field ranges for string moduli, which is discussed here, arises naturally when one considers the inflaton field range that is required to reach 60 e-folds of inflation. As, for class of theories chosen earlier, the inflaton inhabits the moduli space, the mathematical approach to studying distances in the moduli space taken in chapter 5 is warranted.

In this chapter we will consider the (R)SDC and physical examples thereof. In contrast to chapter 4, where we will use the mathematical Wang conjecture2 to motivate the (R)SDC.

[4]Remember: this is a field describing the geometry of the string space, its field space is referred to as the moduli space

2.2.1 Swampland Distance Conjecture

The Swampland Distance Conjecture (SDC) is a conjecture proposed by Ooguri and Vafa [1]. In [1] it corresponds to conjectures 1 and 2. The essence of these conjectures is as follows.

At any infinite distance singularity in the moduli space an infinite tower of massless states will appear, meaning that for this field value one actually has to incorporate an infinite amount of states into the effective theory. Since this is impossible within a gravitational theory the effective theory becomes unphysical at that point.

When trying to realize large field inflation from a string theory one attempts to approach such singularities, meaning that one then reaches a point where an asymptotically infinite number of states have to be included in the effective field theory. This means that, in order for the theory not to break down, the inflaton field can only slow roll up to that cut-off. This in turn leads to a reduced proper field distance which may be less than 60 e-folds, and is therefore unable to produce inflation.

Note that the SDC does not make a statement on what the cut-off is up to which the effective field theory would still be valid.

Ooguri and Vafa formulated the conjecture as,

Conjecture 1 (Swampland Distance Conjecture) *Let $\mathcal{M}$ be the string moduli space, spanned by the expectation values of the string moduli. Let d be the hermitian metric on $\mathcal{M}$ and let $p_0 \in \mathcal{M}$.*

For any $L \in \mathbb{R}_{>0}$, there exists another point $p_1 \in \mathcal{M}$ such that $d(p_0, p_1) > L$.

Moreover, denote now by p_L any point such that $d(p_0, p_L) > L$. The theory at which the field takes the value p_L has an infinite tower of light particles in comparison to the theory at p_0. These particles have a mass of $e^{-\alpha L}$.

This implies in particular that there always exists a point p_∞ which is at infinite distance from p_0, and at which point the effective theory needs to incorporate an infinite tower of massless states, hence becoming unphysical.

Example of the SDC

The authors of [1] present several examples of effective string theories where the SDC holds. We will consider one of them here.

Consider the compactification of M-theory in 10-dimensions, on a circle. For the corresponding Kaluza-Klein expansion we refer back to equation (3.3),

$$\Phi(X^M) = \sum_{n=0}^{\infty} e^{iny/R} \phi(x^\mu), \qquad M_{KK}^2 = \frac{n^2}{R^2}. \tag{2.20}$$

In this case we note that we have one modulus r which is related to the radius of the circle. The metric on the moduli space is given by

$$g = \left(\frac{dr}{r}\right)^2 . \tag{2.21}$$

As we compactify on a circle we only have the one parameter y, such that there is no path dependence. We find the following distance between a point r and r_0,

$$L(r) = \int_r^{r_0} \frac{dr'}{r'} t = |\log(r/r_0)| . \tag{2.22}$$

Note that the logarithm is a monotonically increasing function, which means that we can always find an r' such that $L(r') > L(r)$, as per the conjecture1.

Defining now $\rho = \lambda^{-1} \log r \propto L(r)$, with $\lambda \in \mathbb{R}_{>0}$ we can express the Kaluza-Klein mass in a different way

$$M_{KK} \sim \frac{n}{r} \sim n e^{-\lambda\rho}, \tag{2.23}$$

which is the dependence conjectured in 1. As we take $r \longrightarrow \infty$ the distance $L(r)$ also diverges, leading to a an infinite tower of massless states, and an unphysical theory.

We can even say that, depending on the chosen mass cutoff of our theory, for $\rho > \rho_c = \lambda^{-1}$ we find an infinite tower of exponentially light states. These all have to be incorporated into the effective field theory, indicating again that the effective theory has to break down. As demonstrated in [11] this parameter λ emerges when writing out the field distance $L(r)$ using our knowledge of the Calabi-Yau structure.

In the attempt to generalize the Wang conjecture to more than one parameter in section 4.2 we will again try to find a $1/r^2$ dependence in the metric, but now for a general case, as such a dependence always leads to a divergent distance for $r \to \infty$.

Refined Swampland Distance Conjecture

In the previous section no mention was made about the actual value of ρ_c at which the theory becomes unphysical. Palti and Kläwer conjecture the critical distance $d_c = \lambda^{-1}$ to always be of natural order[5], i.e. for $\alpha \in \mathbb{R}$ and of order 1, $\lambda = \alpha M_{pl}$. This conjecture was dubbed the *Refined Swampland Distance Conjecture* (RSDC) and would imply that super-Planckian field ranges lead to unphysical theories.

The authors of [5] have checked and confirmed their conjecture for several manifolds. They consider for example a case with two singular divisors, one finite and one infinite. Their next step consists of analyzing the field distance one can traverse from in approaching these singular divisors. This of course requires you to find the metric; in chapter 5 we will show

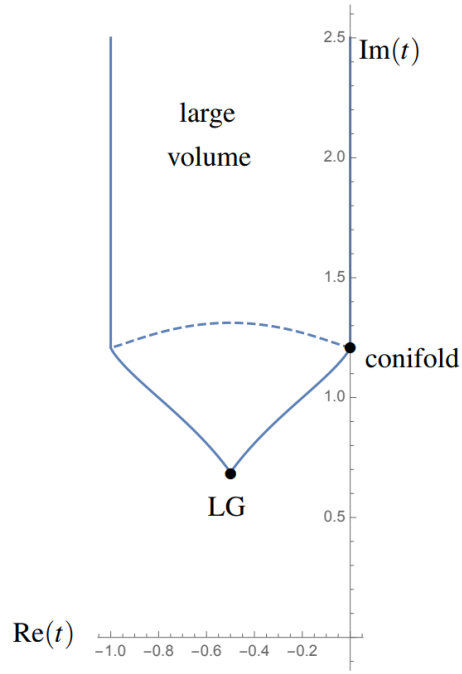

Figure 2.2: Picture showing the Kähler moduli space for a specific example. In the picture we can distinguish between three points; the Landau-Ginzburg (LG), large volume and conifold points. The dotted line represents the radius of convergence for the two expansions of the metric. Source: [10], figure 1.

how one can expand the metric around a singular point using the corresponding monodromy matrix. Here however we will merely assume this to be possible.

It follows that the presence of two singular divisors leads to two different expansions for the metric, each with their own cut-off and valid only in an area around the singularity. If we want to move from a *small volume* area near a finite singular divisor, to a *big volume* area[5] near an infinite singular divisor we therefore have to cross some domain wall, where the expression for the metric changes. This is represented in figure 2.2.1, where the domain wall is given by the dotted line. The patching together of the metric expansions to get a global expansion of the metric is still an open problem. In [5] the authors therefore approach the problem by calculating the maximum field distance one can travel from the small volume limit to the large volume limit, after which they consider the path from the large volume towards the infinite distance singular divisor.

In the notation of the previous part the critical distance for a point in the large volume region would be λ_c^{-1}. However, as explained, if one wants to reach a point at infinity from the small volume region one first has to transition from the small volume to the large volume region. We will denote the maximum distance one can travel in field space to complete this transition by θ_0. The critical distance for a point in the small volume region is then $\theta := \theta_0 + \lambda_c^{-1}$.

Now the small volume region posits a way to falsify the RSDC; if $\theta_0 > M_{pl}$, then the

[5]The small volume area is finite, hence small.

parameter $\alpha := \theta/M_{pl} > \mathcal{O}(1)$, contrary to what the RSDC predicts. In [11] more possible moduli spaces were considered, where for all cases it was shown that $\theta_0 < 1$ and $\lambda_c^{-1} < 1$, verifying the RSDC so far.

Chapter 3

Deriving the effective theory: IIB compactification

Within string theory one starts by analyzing the superstring action, which depending on the chosen type of GSO-projection leaves one with an effective 10-dimensional theory, containing several fields describing the dynamics of our string modes. Using these modes this chapter aims to build the effective 4-dimensional string vacuum from the type IIB theory.

When the alleged discovery of tensor modes seemed to have confirmed large field inflation via the Lyth bound, string theorists attempted to realize such an effective 4D theory, and thereby explain large field inflation from string theory. In doing so some of the effective fields arising in the compactification procedure have to be identified with the inflaton driving inflation. In this chapter we go over a relatively simply compactification, that of the type IIB superstring, to show the emergence of a popular candidate for the inflaton: the scalars describing the complex structure of the manifold, called the *complex moduli*. Remember that the Swampland Distance Conjecture 1 would tell us that at any infinite distance point in the space spanned by the complex moduli, i.e. the *moduli space*, an infinite tower of massless states would appear. In other words, a diverging field distance in the moduli space leads to a breakdown of the 4D effective theory. The Swampland Distance Conjecture would therefore tell us that, in a consistent theory of quantum gravity, the complex moduli are actually not able to source large field inflation.

In this chapter we will also mention the properties of the moduli space, details of which are left to the appendix C. These properties can be very well described using algebraic geometrical tools that are extensively used in chapters 4 and 5.

There are two important points in building the vacuum for an effective theory. The first is that there is a discrepancy in dimension; we know that in general relativity we model our world via a 4-dimensional Minkowski space $\mathcal{M}_4$. A realistic physical theory should therefore be 4-dimensional rather than 10-dimensional. The simplest possibility would be to consider that the 10-dimensional manifold has a product form $\mathcal{M}_{d_c} = \mathcal{M}_d \times K_6$, where K_6 is a compact manifold which we have not yet detected in experiments. Note that for this to be true the characteristic length scale of K_6 should be much smaller than that measured by

particle accelerators; this enables us to see a low energy effective theory which is located on $\mathcal{M}_4$.

The second point is that we would like the 4 dimensional effective theory to be supersymmetric[12]. This greatly simplifies some properties in string theory and could solve for example the hierarchy problem.

Next we can wonder how to realize the above two points. The discrepancy in dimensions leads us to the concept of *compactification*; a way to reduce the number of dimensions from $d_c = 10$ to $d = 4$ by considering the low energy effective theory. We do this by splitting our fields into a part which depends on the *internal manifold* K_6, and a part which depends on the *external manifold* $\mathcal{M}_4$. This idea is well illustrated by considering the metric. On a product manifold the background metric, which is also a 10-dimensional field, splits in a useful way

$$G_{MN}(x,y) = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{mn}(y) \end{pmatrix}. \tag{3.1}$$

The indices we use here and in most of the remainder of this work are capital latin letters for the full 10-dimensional theory, small latin letters for the internal manifold and greek letters for the external manifolds. A point in the 10 dimensional theory is denoted $X^m = x^\mu, y^m$ with $x^\mu \in M_4$ and $y^m \in K_6$. Note that this choice for the metric also allows us to split the laplacian as $\Delta_{d_c} = \Delta_d + \Delta_D$.

The properties of the manifold K_6 will determine the physics we observe in the 4-dimensional theory. This includes the conservation of supersymmetry as we compactify from 10 to 4 dimensions: this is only possible if the internal manifold K_6 admits a *Ricci-flat metric* [12]. As such we will choose K_6 to be a *Calabi-Yau* manifold CY_3. A Calabi-Yau n-fold is a $2n$-dimensional complex, Kähler manifold which admits a Ricci-flat metric, the details to this are worked out in appendix B and C. The Kähler structure enables us to properly reduce the 10-dimensional fields dependent on X^m to 4-dimensional fields dependent on x^μ.

The rest of this chapter is dedicated to writing down the 10 dimensional IIB superstring action and using the properties of the Calabi-Yau threefold to reduce the number of dimensions to 4 through compactification.

3.1 Kaluza-Klein reduction

We will first illustrate the idea of compactification in a simpler example. Consider a scalar field $\phi(X^M)$ in d_c dimensions. We will take the internal manifold to be the one dimensional circle S^1, denote $X^{d_c} = y$ and call it the "internal coordinate". The remaining $d = (d_c - D) = (d_c - 1)$ coordinates will be the "external coordinates", denoted by x^μ. We will now use the geometry of the circle, specifically the periodicity, to expand the field ϕ into a part that lives on the internal manifold and a part on the external manifold.

The periodicity of the circle is realized as $\Phi(y) = \Phi(y + R)$, with R the radius of the circle. For such a field we can perform a Fourier transformation to find,

$$\Phi(X^M) = \sum_{n=0}^{\infty} e^{iny/R} \phi(x^\mu). \tag{3.2}$$

The quantities $\exp(iny/R)$ are solutions to the Laplace equation, and hence can be said to have squared mass n^2/R^2,

$$\Delta e^{in\theta/R} = \frac{\partial^2}{\partial y^2} e^{iny/R} = -\frac{n^2}{R^2} e^{iny/R}. \tag{3.3}$$

If we consider the case where R is very small we see that the first mode already has a very high mass. Keeping our energy cutoff small, it follows then that we can only take the $n = 0$ mode of our expansion into account when writing down the effective theory. This process is called *Kaluza-Klein reduction*; reduction because we went from a d_c to a $d = d_c - D$ dimensional theory.

Our effective theory hence only consists of the massless d-dimensional modes (in the above example this is ϕ). The problem in realizing the d dimensional effective theorey therefore becomes one of finding all the massless modes. We will illustrate what this means in the context of our internal and external space with another example, taking again $d_c = d + D$.

Consider the p-form field $B^{(p)}$ with a field strength $H^{(p+1)}$. The d_c-dimensional action for $B^{(p)}$ will be proportional to,

$$S = \int_{\mathcal{M}_{d_c}} H^{(p+1)} \wedge *H^{(p+1)}. \tag{3.4}$$

If we fix our gauge condition by taking $d^* B^{(p)} = 0$ then this has as equation of motion,

$$\Delta_{d_c} B^{(p)} = 0, \tag{3.5}$$

where the Laplacian Δ is defined in B, and the subscript d_c denotes that it is the d_c dimensional Laplacian on the full space. Now assume that we can neatly split the form $B^{(p)}(X^M)$ in a part dependent on the *internal* coordinates and a part dependent on the *external* coordinates:

$$B^{(p)}(X^M) = \sum_k A^{(k)}(x^\mu) \wedge \tilde{A}^{(k-p)}(y^m).$$

Using also that the laplacian splits over the internal and external space, $\Delta_{d_c} = \Delta_d + \Delta_D$ we find that

$$\Delta_{d_c} B^{(p)}(X^M) = \sum_k (\Delta_d A^{(k)}(x^\mu)) \wedge \tilde{A}^{(k-p)}(y^m) + \sum_k A^{(k)}(x^\mu) \wedge (\Delta_D \tilde{A}^{(k-p)}(y^m)) = 0,$$

and so,

$$\Delta_d A^{(k)}(x^\mu) = 0 \Longleftrightarrow \Delta_D \tilde{A}^{(k-p)}(y^m) = 0. \tag{3.6}$$

Remember that we were interested in finding massless modes for our d-dimensional theory. The above relation tells us that in order to find all of these massless modes it is also sufficient to find the zero modes of the internal laplacian, i.e. Δ_d.

A form ω for which $\Delta_D \omega = 0$ is called a *harmonic form*. Finding these harmonic forms is a problem that can be tackled using the theory of cohomologyB, in which the structure of the internal manifold plays a very important role.

3.1.1 The moduli space

The metric, being another field in our theory, can also be expanded in its zero modes. We can decompose the metric itself through its indices as $g_{MN} \to g_{\mu\nu} \oplus g_{\mu m} \oplus g_{mn}$. Each of these components has a different structure with respect to the internal manifolds; $g_{\mu\nu}$ for example is a scalar on the internal manifold, which means that it only has one zero mode in its expansion C.

In turn g_{mn} corresponds to 2-form in D dimensions. The metric g_{mn} on the internal manifold K_D of course determines its structure; we would therefore like to analyze the behavior of these modes further by considering perturbations from the background value: $g_{mn} = \mathring{g}_{mn} + h_{mn}$. These perturbations h_{mn} are called the (metric) *moduli*.

Remember that we chose $K_6 = CY_3$, where the CY_3 space was defined to be Ricci-flat. The moduli therefore need to conserve this property, meaning that $R(\mathring{g}_{mn} + h_{mn}) = 0$.

What this means is that there are many consistent string vacua possible which have the same topology, but a slightly different structure. An example of this is the radius R that we considered in the compactification on a circle (3.3). This was a free parameter (up to the point where we needed R small such that the energy scale of massive modes would become sufficiently large) of our space, signifying the different circle sizes that could have been used in the compactification process.

Using the equations characterizing the moduli (for example $R(\mathring{g}_{mn} + h_{mn}) = 0$) we can define the *moduli space*, which consists of all possible values for the moduli. The space of all complex structure moduli is also a Kähler manifold, as explained in C.

3.1.2 Further with compactification

Consider a theory with $\Phi(X^M)$ coupled to gravity and electromagnetism. After having written our field $\Phi(X^M)$ only in terms of the field $\phi(x^\mu)$ we would like to expand the entire theory in these modes, such that we are left with only a $(d-1)$ dimensional theory. To achieve this we need to do a massless reduction of the metric tensor (i.e. write it out in massless modes). Consider the following ansatz for the metric,

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} & V_\mu \\ V_\nu & \phi(x^\mu) \end{pmatrix}. \tag{3.7}$$

We note that none of the fields ϕ (which is a Kaluza-Klein field), V_μ or the external metric g depend on the coordinate y. Next it is possible to expand the full metric g_{MN} in its indices again. The d dimensional metric then becomes the $(d-1)$ dimensional metric when one considers only the external indices. When the indices are mixed between internal and external we get a vector structure V_μ, and when the indices are purely internal we get a scalar ϕ. We can therefore reduce the Ricci scalar R to a purely $(d-1)$ dimensional expression[13],

$$R = R + F_{\mu\nu}F^{\mu\nu} + \partial_\mu\phi\partial^\mu\phi.$$

This illustrates the general strategy when performing a compactification,

- Write down the 10 dimensional fields *(g_{MN}).*
- expand the 10 dimensional fields, keeping only massless modes *($g_{\mu\nu}, \phi, V_\mu$).*

This is the step that will truly lead to a reduction in dimension, as for specific internal spaces (in the Kaluza-Klein example it was a circle, in the type IIB example it will be a Calabi-Yau manifold) one finds that the reduced, massless modes do not depend on the internal coordinates.

- reduce the quantities dependent on the fields *(in this example, R)* by expanding them as well.

When considering the action one has to still perform the integral over the internal dimensions,

- integrate out the internal dimensions.

The result is a 4-dimensional theory of supergravity.

3.2 Compactification of the type IIB theory

We are now ready to consider the compactification of the type IIB theory. In doing so we will follow [13, 14]. The type IIB supergravity theory consists of a gravitational part; the Ricci scalar $\hat{R}$ and the dilaton $\hat{\phi}$, and a matter part. The NS-NS spectrum of the matter part is made up of the dilaton and a 2-form $\hat{B}_2$. The R-R spectrum consists of a 0-form $\hat{a}$, a 2-form $\hat{C}_2$ and a 4-form $\hat{A}_4$.

Note that the fields enter the action through their field strengths, which we shall introduce momentarily. In working out the action we shall make use of the Hodge duality outlined in appendix B. For the 4-form $\hat{A}_4$ this poses a problem, as $*\hat{F}_5 = \hat{F}_5$ (i.e. $\hat{F}_5$ is self dual). This means that we have an extra constraint, losing us half the degrees of freedom, which has to be manually imposed on the action. The nature of differential forms however tells us that $*\hat{F}_5 = \hat{F}_5$ putting this condition into the action means that the kinetic term for $\hat{A}_4$ vanishes, as $\hat{F}_5^2 = 0$.

All of these fields can be written within the following 10-dimensional supergravity action,

$$\begin{aligned} S_{IIB} = &\int e^{-2\hat{\phi}} \left(-\frac{1}{2}\hat{R} * 1 + 2d\hat{\phi} \wedge *\hat{\phi} - \frac{1}{4} d\hat{B}_2 \wedge *\hat{B}_2 \right) \\ &- \frac{1}{2} \int \left(d\hat{a} \wedge *d\hat{a} + \hat{F}_3 \wedge *\hat{F}_3 + \frac{1}{2}\hat{F}_5 \wedge *\hat{F}_5 \right) - \frac{1}{2} \int \hat{A}_4 \wedge d\hat{B}_2 \wedge d\hat{C}_2, \end{aligned} \tag{3.8}$$

where $\hat{a}$ is the scalar axion coming from the RR-sector. Note that in this notation we omitted writing down the metric $\sqrt{-g}$ which multiplies all terms in the integral, at all times. The fields strengths are given by,

$$\hat{H}_3 = d\hat{B}_2, \quad \hat{F}_3 = d\hat{C}_2 - \hat{a}\hat{H}_3, \quad \hat{F}_5 = d\hat{A}_4 - \hat{H}_3 \wedge \hat{C}_2. \tag{3.9}$$

We will begin by reducing the gravitational part of the action.

3.2.1 Reducing S_{grav}

We can recognize the gravitational part of equation (3.8), which is

$$S_{grav} = \int \quad e^{-2\hat{\phi}} \quad \left(-\frac{1}{2}\hat{R} * 1 + 2d\hat{\phi} \wedge *\hat{\phi} \right), \tag{3.10}$$

We first note that the metric splits over our internal and external spaces as

$$\int_{\mathcal{M}_{10}} \sqrt{-g} = \int_{\mathcal{M}_4} d^4x \sqrt{-g_4} \int_{CY_3} d^6x \sqrt{g_6}. \tag{3.11}$$

We then complexify the 6 real coordinates as,

$$\xi^1 = \frac{y^1 + iy^2}{\sqrt{2}}; \quad \xi^2 = \frac{y^3 + iy^4}{\sqrt{2}}; \quad \xi^3 = \frac{y^5 + iy^6}{\sqrt{2}}. \tag{3.12}$$

In order to write down the volume form in these coordinates we define the 6-dimensional Levi-Civita tensor with mixed indices as,

$$\epsilon^{\alpha_1\alpha_2\alpha_3\overline{\beta_1\beta_2\beta_3}} = -i\epsilon^{\alpha_1\alpha_2\alpha_3}\epsilon^{\overline{\beta_1\beta_2\beta_3}}, \tag{3.13}$$

with the added condition that $\epsilon^{123\overline{123}} = 1$. Note that we can write the complex volume form as

$$d\xi^{\alpha_1} \wedge ... \wedge d\xi^{\alpha_n} \wedge d\overline{\xi}^{\overline{\beta}_1} ... \wedge d\overline{\xi}^{\overline{\beta}_n} = i\epsilon^{\alpha_1...\alpha_n\overline{\beta_1}...\overline{\beta_n}} d^{2n}\xi. \tag{3.14}$$

Having done this we begin to expand Ricci scalar $\hat{R}$ in its (complex) indices,

$$\begin{aligned} \hat{R} &= R_{MN} g^{MN} \\ &= R_{MPN}{}^{P} g^{MN} \\ &= R + g^{\mu\nu} R_{\mu\alpha\nu}{}^{\alpha} + g^{\alpha\beta} \left(R_{\alpha\mu\beta}{}^{\mu} + R_{\alpha\gamma\beta}{}^{\gamma} + R_{\alpha\overline{\gamma}\beta}{}^{\overline{\gamma}} \right) + g^{\alpha\overline{\beta}} \left(R_{\alpha\mu\overline{\beta}}{}^{\mu} + R_{\alpha\gamma\overline{\beta}}{}^{\mu\gamma} + R_{\alpha\overline{\gamma}\overline{\beta}}{}^{\overline{\gamma}} \right) + \text{ c.c.}, \end{aligned} \tag{3.15}$$

where R is the 4-dimensional Ricci scalar. We expanded the coordinates $M \to \{m, \mu\} \to \{\alpha, \overline{\alpha}, \mu\}$, where the last step represents splitting the internal coordinates m into a real and complex part as done with the ξ above.

We then proceed to expand the internal metric up to second order in the moduli fields, as outlined in B. We define $g = \mathring{g} + \delta g$, with $\mathring{g}$ the original background metric and δg the perturbation leading to the moduli. Considering first the purely holomorphic metric perturbations,

$$g_{\alpha\beta} = \overline{z}^a (\overline{b}_a)_{\alpha\beta} \tag{3.16}$$

$$g^{\alpha\beta} = -z^a (b_a)_{\overline{\alpha}\overline{\beta}} \mathring{g}^{\alpha\overline{\alpha}} \mathring{g}^{\beta\overline{\beta}}, \tag{3.17}$$

with b the complex structure moduli as defined in appendix C. We now introduce the following notation to simplify our expressions a bit,

$$(\omega_i g) = (\omega_i)_{\alpha\overline{\beta}} \mathring{g}^{\alpha\overline{\beta}} \tag{3.18}$$

$$\omega_i \omega_j = (\omega_i)_{\alpha\overline{\alpha}} (\omega_i)_{\beta\overline{\beta}} \mathring{g}^{\alpha\overline{\beta}} \mathring{g}^{\beta\overline{\alpha}} \tag{3.19}$$

$$b_a \overline{b}_b = (b_a)_{\overline{\alpha}\overline{\beta}} (\overline{b}_b)_{\alpha\beta} \mathring{g}^{\alpha\overline{\beta}} \mathring{g}^{\beta\overline{\alpha}}. \tag{3.20}$$

For the mixed perturbations we get,

$$g_{\alpha\overline{\alpha}} = \mathring{g}_{\alpha\overline{\alpha}} - i v^i (\omega_i)_{\alpha\overline{\alpha}} \tag{3.21}$$

$$g^{\alpha\overline{\alpha}} = \mathring{g}^{\alpha\overline{\alpha}} + i v^i (\omega_i)_{\beta\overline{\beta}} \mathring{g}^{\alpha\overline{\beta}} \mathring{g}^{\beta\overline{\alpha}}. \tag{3.22}$$

Note that, as shown in appendix B, the Calabi-Yau structure implies that certain parts of the Levi-Civita tensor vanish. We can use the expansions of the metric to find the only non-zero components, which are

$$\Gamma_{\mu\alpha}{}^{\beta} = \frac{1}{2}(\omega_i)_{\gamma\overline{\gamma}} (\omega_j)_{\alpha\overline{\beta}} \mathring{g}^{\gamma\overline{\beta}} \mathring{g}^{\beta\overline{\gamma}} v^i \partial_\mu v^j - \frac{i}{2}(\omega_i)_{\alpha\overline{\beta}} \mathring{g}^{\beta\overline{\beta}} \partial_\mu v^i - \frac{1}{2}(b_a)_{\overline{\beta}\overline{\gamma}} (\overline{b}_b)_{\alpha\gamma} \mathring{g}^{\gamma\overline{\beta}} \mathring{g}^{\beta\overline{\gamma}} z^a \partial_\mu \overline{z}^b \tag{3.23}$$

$$\Gamma_{\mu\alpha}{}^{\overline{\beta}} = \frac{1}{2}(\overline{b}_a)_{\alpha\beta} \mathring{g}^{\beta\overline{\beta}} \partial_\mu v^i \overline{z}^a \tag{3.24}$$

$$\Gamma_{\alpha\beta}{}^{\mu} = -\frac{1}{2}(\overline{b}_a)_{\alpha\beta} \partial^\mu \overline{z}^a \tag{3.25}$$

$$\Gamma_{\alpha\overline{\beta}}{}^{\mu} = \frac{i}{2}(\omega_i)_{\alpha\overline{\beta}} \partial^\mu v^i. \tag{3.26}$$

We then put these Christoffel symbols into equation (3.15) to find for each of the terms

$$
\begin{aligned}
g^{\mu\nu}R_{\mu\alpha\nu}{}^{\alpha} &= \frac{1}{2}\left[(\omega_i g)(\omega_j g) - \frac{1}{2}\omega_i\omega_j\right]\partial_\mu v^i\partial^\mu v^j + \mathcal{O}(3)4b_a\bar{b}_b\partial_\mu z^a\partial^\mu\bar{z}^b + \mathcal{O}(3) && (3.27)\\
g^{\alpha\beta}R_{\alpha\mu\beta}{}^{\mu} &= \frac{1}{2}b_a\bar{b}_b\partial_\mu z^a\partial^\mu\bar{z}^b + \mathcal{O}(3) && (3.28)\\
g^{\alpha\bar{\beta}}R_{\alpha\mu\bar{\beta}}{}^{\mu} &= \frac{1}{2}\left[(\omega_i g)(\omega_j g) - \frac{1}{2}\omega_i\omega_j\right]\partial_\mu v^i\partial^\mu v^j - \frac{1}{4}b_a\bar{b}_b\partial_\mu z^a\partial^\mu\bar{z}^b + \mathcal{O}(3) && (3.29)\\
g^{\alpha\bar{\beta}}R_{\alpha\gamma\bar{\beta}}{}^{\gamma} &= -\frac{1}{4}\left[(\omega_i g)(\omega_j g) - \omega_i\omega_j\right]\partial_\mu v^i\partial^\mu v^j + \mathcal{O}(3) && (3.30)\\
g^{\alpha\bar{\beta}}R_{\alpha\bar{\gamma}\bar{\beta}}{}^{\bar{\gamma}} &= -\frac{1}{4}(\omega_i g)(\omega_j g)\partial_\mu v^i\partial^\mu v^j - \frac{1}{4}b_a\bar{b}_b\partial_\mu z^a\partial^\mu\bar{z}^b + \mathcal{O}(3). && (3.31)\\
& && (3.32)
\end{aligned}
$$

There are two terms in the expansion (3.15) we have not considered yet,

$$g^{\alpha\beta}\left(R_{\alpha\gamma\beta}{}^{\gamma} + R_{\alpha\bar{\gamma}\beta}{}^{\bar{\gamma}}\right) = g^{\alpha\beta}R_{\alpha m\beta}{}^{m} = g^{\alpha\beta}R_{\alpha\beta}. \tag{3.33}$$

It was mentioned before that the background of the internal Ricci scalar is zero. Because of the index structure we observe in addition that in the expression of the Riemann tensors there will be no derivative term with spacetime indices, which means that all derivate terms vanish. The only terms surviving are the terms quadratic in Γ, which are of second order in the moduli fields. Noting that $g^{\alpha\beta}$ is of first order in the moduli fields we find,

$$g^{\alpha\beta}\left(R_{\alpha\gamma\beta}{}^{\gamma} + R_{\alpha\bar{\gamma}\beta}{}^{\bar{\gamma}}\right) = \mathcal{O}(3). \tag{3.34}$$

We can now combine all of the terms above to find the correct expansion of the Ricci scalar up to second order in moduli fields,

$$S_{EH} = \int d^{10}\hat{x}\sqrt{-\hat{g}}\hat{R} = \int d^4x\sqrt{-g_4}\left(\mathcal{K}R + V_{ij}\partial_\mu v^i\partial^\mu v^j + Z_{ab}\partial_\mu z^a\partial^\mu\bar{z}^b\right), \tag{3.35}$$

where we defined,

$$
\begin{aligned}
\mathcal{K} &= \int_{CY_3}\sqrt{g_6} && (3.36)\\
V_{ij} &= \int_{CY_3}\sqrt{g_6}\left((\omega_i g)(\omega_j g) - \frac{1}{2}(\omega_i\omega_j)\right) && (3.37)\\
Z_{ab} &= \int_{CY_3}\sqrt{g_6}\left(-(\omega_i g)(\omega_j g) + (\omega_i\omega_j)\right). && (3.38)
\end{aligned}
$$

This can be related to the properties of the Calabi-Yau manifold as outlined in appendix C.

Weyl rescaling to the Einsteins frame

Having obtained the expansion of $\hat{R}$ in terms of 4 dimensional fields the next step is to rewrite the expanded equation (3.10). We do this in order to get the correct sign in front of the

kinetic term for the dilaton in equation (3.10); this is called transforming into the *Einstein frame*. we perform a Weyl rescaling as outlined in equation (A.45). By using equations (A.48) and (A.51) we find

$$S_{grav} = \int_{10} -\frac{1}{2}\hat{R} * 1 - \frac{1}{4} d\phi \wedge *\phi, \tag{3.39}$$

the action in the Einstein frame, where the dilaton kinetic term has a negative sign.

Compactification

As the dilaton is a scalar on the Calabi-Yau the difficult part in compactifying lies with the Ricci scalar. We have done this work earlier and can plug in equation (3.35) to find

$$S = \int_4 -\frac{1}{2}\mathcal{K}R * 1 - \mathcal{K}\frac{1}{4} d\phi \wedge *\phi - \frac{1}{2} V_{ij} dv^i \wedge *dv^j - \frac{1}{2} Z_{ab} dz^a \wedge *d\overline{z}^b. \tag{3.40}$$

Note that the Ricci scalar is multiplied by the volume of the Calabi-Yau manifold. We would like to scale this away.

Weyl rescaling the normalization of R

We again perform a Weyl transformation as in (A.45), taking $\Omega = \mathcal{K}^{1/2}$. This gives us,

$$S = \int -\frac{1}{2} R * 1 - \frac{3}{4} d\log\mathcal{K} \wedge *d\log\mathcal{K} - \frac{1}{4} d\phi \wedge *\phi - \frac{1}{2\mathcal{K}} V_{ij} dv^i \wedge *dv^j - \frac{1}{2\mathcal{K}} Z_{ab} dz^a \wedge *d\overline{z}^b \tag{3.41}$$

Note now that we have introduced a new scalar term $d\log\mathcal{K} \wedge *d\log\mathcal{K}$. This term can in fact be absorbed within the v_i, as we will do in the next and final step.

Rotating the v^i

In order to eliminate the scalar term depending only on the Calabi-Yau volume we rotate the v^i as follows,

$$v^i = e^{-\frac{1}{2}\phi} \tilde{v}^i. \tag{3.42}$$

All the fundamental forms in the action are independent of the v^i. We therefore only have to consider the effect that such a rotation has on the metric and composite terms. Remember that the v^i are the complex structure moduli, and that they are defined through the expansion (C.53). By considering this expansion we can quickly see that,

$$K_{ijk} = \tilde{K}_{ijk} \tag{3.43}$$

$$K_{ij} = e^{-\frac{1}{2}\phi} \tilde{K}_{ij} \tag{3.44}$$

$$K_i = e^{-\phi} \tilde{K}_i \tag{3.45}$$

$$\mathcal{K} = e^{-\frac{3}{2}\phi} \tilde{\mathcal{K}}. \tag{3.46}$$

From expression (C.56) we can then immediately deduce that,

$$g_{ij} = e^{-\frac{1}{2}\phi}\tilde{g}_{ij}. \tag{3.47}$$

Using equation (C.58) we can write

$$V_{ij} = -K_{ij} - 2\mathcal{K}g_{ij} = 2\mathcal{K}g_{ij} - \frac{1}{4\mathcal{K}}K_iK_j, \tag{3.48}$$

and so

$$V_{ij} = e^{-\frac{1}{2}\phi}\tilde{V}_{ij}. \tag{3.49}$$

The terms that have to be rotated are the $d\log\mathcal{K}$ term, and the kinetic term for the v^i. The rotation gives us,

$$\frac{1}{2\mathcal{K}}V_{ij}\partial v^i\partial v^j = \tilde{g}_{ij}\partial\tilde{v}^i\partial\tilde{v}^j - \frac{15}{16}\partial\phi\partial\phi - \frac{1}{2}\partial\log\tilde{\mathcal{K}}\partial\log\tilde{\mathcal{K}} + \frac{5}{4}\partial\log\tilde{\mathcal{K}}\partial\phi \tag{3.50}$$

$$\frac{3}{4}d\log\mathcal{K}\wedge *d\log\mathcal{K} = -\frac{3}{4}\partial\log\tilde{\mathcal{K}}\partial\log\tilde{\mathcal{K}} + \frac{9}{4}\partial\phi\partial\log\tilde{\mathcal{K}} - \frac{27}{16}\partial\phi\partial\phi. \tag{3.51}$$

Where we used that

$$\partial\mathcal{K} = \frac{1}{2}\partial v^i K_i. \tag{3.52}$$

Putting our expressions together we find the following gravitational action,

$$S_{grav} = \int -\frac{1}{2}R * 1 - d\phi\wedge *d\phi - g_{ij}dv^i\wedge *dv^j - g_{ab}dz^a\wedge *d\bar{z}^b. \tag{3.53}$$

In this expression we defined the 4-dimensional dilaton to be

$$\phi = \phi - \frac{1}{2}\log\tilde{\mathcal{K}}. \tag{3.54}$$

We also dropped the tildes and defined the metric on the complex moduli space to be,

$$g_{ab} = \frac{1}{2\mathcal{K}}Z_{ab}. \tag{3.55}$$

3.2.2 Reducing the IIB matter action

The first step we take in reducing the type IIB action is to expand the fields in terms of a basis of harmonic forms as explained in the appendix B,

$$\hat{B}_2 = B_2 + b^i\wedge\omega_i, \quad i = 1, ..., h^{1,1} \tag{3.56}$$

$$\hat{C}_2 = C_2 + c^i\wedge\omega_i, \tag{3.57}$$

$$\hat{A}_4 = D_2^i\wedge\omega_i + \rho_i\tilde{\omega}^i + V^A\wedge\alpha_A - U_A\wedge\beta^A, \quad A = 0, 1, ..., h^{2,1}, \tag{3.58}$$

the basis elements of the various cohomology spaces are,

$$\omega_i \in H^{1,1}(CY_3), \quad \tilde{\omega}^i \in H^{2,2}(CY_3), \quad \alpha_A, \beta^A \in H^3(CY_3). \tag{3.59}$$

The "coefficients" of the harmonic forms are purely space time dependent, and given by the 2-forms D_2^i, the 1-forms V^A and U_A and the scalars b^i, c^i and ρ_i. Using that a form being harmonic implies that it is closed by (B.10) we can write the field strengths as,

$$\hat{H}_3 = H_3 + db^i \wedge \omega_i, \tag{3.60a}$$

$$d\hat{C}_2 = dC_2 + dc^i \wedge \omega_i, \tag{3.60b}$$

$$d\hat{A}_4 = dD_2^i \wedge \omega_i + d\rho_i \wedge \tilde{\omega}^i + F^A \wedge \alpha_A - G_A \wedge \beta^A, \tag{3.60c}$$

$$\hat{F}_3 = dC_2 + dc^i \wedge \omega_i - a(H_3 + db^i \wedge \omega_i), \tag{3.60d}$$

$$\hat{F}_5 = \left(D_2^i - db^i \wedge C_2 - c^i H_3\right) \wedge \omega_i + F^A \wedge \alpha_A - G_A \wedge \beta^A + d\rho_i \wedge \tilde{\omega}^i - c^i db^j \wedge \omega_i \wedge \omega_j \tag{3.60e}$$

where we took $F^A = dV^A, G_A = dU_A$. Note that $\hat{a} = a$, as it already was a scalar on the Calabi-Yau manifold. We can then plug in these expansions to find the following expressions for the integrals of CY_3,

$$-\frac{1}{4}\int_{CY_3} \hat{H}_3 \wedge *\hat{H}_3 = -\frac{\mathcal{K}}{4} H_3 \wedge *H_3 - \mathcal{K} g_{ij} db^i \wedge *db^j, \tag{3.61a}$$

$$-\frac{1}{2}\int_{CY_3} d\hat{a} \wedge *\hat{a} = -\frac{\mathcal{K}}{2} da \wedge *da, \tag{3.61b}$$

$$\begin{aligned} -\frac{1}{2}\int_{CY_3} d\hat{F}_3 \wedge *\hat{F}_3 = &-\frac{\mathcal{K}}{2} da \left(dC_2 - lH_3\right) \wedge * \left(dC_2 - lH_3\right) \\ &- 2\mathcal{K} g_{ij} \left(dc^i - l db^i\right) \wedge * \left(dc^j - l db^j\right), \end{aligned} \tag{3.61c}$$

$$\begin{aligned} -\frac{1}{4}\int_{CY_3} d\hat{F}_5 \wedge *\hat{F}_5 = &\frac{1}{4}\mathrm{Im}\left(\mathcal{M}^{-1}\right)\left(\tilde{G} - \mathcal{M}\tilde{F}\right) \wedge * \left(\tilde{G} - \overline{\mathcal{M}}\tilde{F}\right) \\ &- \mathcal{K} g_{ij} d\tilde{D}_2^i \wedge *d\tilde{D}_2^i - \frac{1}{16\mathcal{K}} g^{ij} d\tilde{\rho}_i \wedge *d\tilde{\rho}_j, \end{aligned} \tag{3.61d}$$

$$-\frac{1}{2}\int_{CY_3} \hat{A}_4 \wedge \hat{H}_3 \wedge d\hat{C}_2 = -\frac{1}{2}\mathcal{K}_{ijk} D_2^i \wedge db^j \wedge dc^k - \frac{1}{2}\rho_i \left(dB_2 \wedge dc^i + db^i \wedge dC_2\right) \tag{3.61e}$$

where we defined

$$d\tilde{D}_2^i = dD_2^i - db^i \wedge C_2 - c^i dB_2 \tag{3.62}$$

$$d\tilde{\rho}_i = d\rho_i - \mathcal{K}_{ikl} c^k db^l, \tag{3.63}$$

and the matrix $\mathcal{M}$ is the matrix depending on the moduli as defined in (C.47). Now the self duality condition on the 5-form field strength tensor $\hat{F}_5 = *\hat{F}_5$ tells us that only half the

degrees of freedom in $\hat{A}_4$ are physical. This means that the fields on which $\hat{A}_4$ depends come in dual pairs. We shall therefore use Lagrange multipliers to pick ρ_i and V^A as independent fields, which have D_2^i and U_A as their dual respectively.

We will use two different ways to impose the self duality condition of the 5-form to eliminate D_2^i and G_A in terms of the other fields, as presented in [13]. First we write out the duality condition $*\hat{F}_5 = \hat{F}_5$ to find the equivalent conditions on the fields making up F_5

$$\begin{aligned} d\tilde{D}_2^i &= g^{ij}\frac{1}{4\mathcal{K}} * d\tilde{\rho}_j && (3.64)\\ *G &= \mathrm{Re}\mathcal{M} * F - \mathrm{Im}\mathcal{M}F && (3.65)\\ G &= \mathrm{Re}\mathcal{M}F + \mathrm{Im}\mathcal{M} * F, && (3.66) \end{aligned}$$

where in the derivation we made us of the relations between the bases of cohomology classes as outlined in appendix C. We will begin by dualising D_2^i.

Our goal is to write down a Lagrangian with the same physical content (i.e. equations of motion) *after* imposing equation (3.64). In order to do this we write down the most general Lagrangian independent D_2^i, and match its equations of motion to that of the original Lagrangian (3.8). This gives us the correct values for the general coefficients we have to introduce in the general D_2^i independent Lagrangian. Looking also at the expansion of the various terms in equation (3.61) we find that the most general post-dualisation Lagrangian has the form,

$$\begin{aligned} \mathcal{L}_{dualised} = k_1 g^{ij}\left(d\rho_i - \mathcal{K}_{ikl}c^k db^l\right) \wedge *\left(d\rho_j - \mathcal{K}_{jpq}c^p db^q\right) + k_3 d\rho_i \\ \wedge \left(c^i dB_2 + db^i \wedge C_2\right) + k_4 \mathcal{K}_{ijk}c^i c^j dB_2 \wedge db^k. \end{aligned} \tag{3.67}$$

So, by matching the equations of motion arising from the 10-dimensional Lagrangian (3.8) to the equations of motion from (3.67) we find the following values for our parameters [13],

$$k_1 = -\frac{1}{8\mathcal{K}}, \quad k_3 = -1, \quad , k_4 = -\frac{1}{2}, \tag{3.68}$$

giving us the dualized Lagrangian.

We now apply a different procedure to dualize G. The aim is to introduce a term to the 4-dimensional Lagrangian,

$$L_F = \frac{1}{4}\mathrm{Im}\mathcal{M}^{-1}(G - \mathcal{M}F) \wedge *(G - \overline{\mathcal{M}}F) \tag{3.69}$$

thereby changing the equations of motion for G to (3.65). We can then impose these equations of motion in order to eliminate G_A in terms of F^A.

The term we add is a total derivative,

$$\frac{1}{2}F^A \wedge G_A, \tag{3.70}$$

which indeed leads to the correct equation of motion. After eliminating G_A we get the Lagrangian,

$$L_F = \frac{1}{2}\mathrm{Im}\mathcal{M}_{AB}F^A \wedge *F^B + \frac{1}{2}\mathrm{M}_{AB}F^A \wedge F^B. \tag{3.71}$$

As was explained earlier while compactifying the gravitational part of the action 3.2.1, we now Weyl rescale our terms with a factor $\mathcal{K}^{\frac{1}{2}}$. This is followed by a rotation of the Kähler class moduli v^i into $v^i = e^{-\frac{1}{2}\hat{\phi}}\tilde{v}^i$, as done in 3.2.1, to eliminate a scalar term. This leads to the following action,

$$\begin{aligned} S^4_{IIB} = \int & -\frac{1}{2} - g_{ab}dz^a \wedge *d\bar{z}^b - g_{ij}dt^i \wedge *d\bar{t}^j - d\phi \wedge *d\phi - \frac{1}{4}e^{-4\phi}dB_2 \wedge *dB_2 \\ & - \frac{1}{2}e^{-2\phi}\mathcal{K}(dC_2 - ldB_2 \wedge *(dC_2 - ldB_2) - \frac{1}{2}\mathcal{K}e^{2\phi}dl \wedge *dl - 2\mathcal{K}e^{2\phi}g_{ij}(dc^i - ldb^i) \wedge \\ & *(dc^j - ldb^j) - \frac{e^{2\phi}}{8\mathcal{K}}(g^{ij})^{-1}\left(d\rho_i - K_{ikl}c^k db^l\right) \wedge *\left(d\rho_j - \mathcal{K}_{jmn}c^m db^n\right) + \left(db^i \wedge C_2 + c^d B_2\right) \\ & \wedge \left(d\rho_i - \mathcal{K}_{ijk}c^j db^k\right) + \frac{1}{2}\mathcal{K}_{ijk}c^i c^j dB_2 \wedge db^k + \frac{1}{2}\mathrm{Re}\mathcal{M}_{AB}F^A \wedge F^B + \frac{1}{2}\mathrm{Im}\mathcal{M}_{AB}F^A \wedge *F^B. \end{aligned} \tag{3.72}$$

Now, in order to get a simpler Lagrangian, it remains us to dualize the last two 2-forms in the expression, namely C_2 and B_2 with scalar duals h_1 and h_2 respectively. To achieve this we proceed in a similar fashion as with the dualization of G_A. We add a total derivative to the Lagrangian, given by

$$dC_2 \wedge dh_1. \tag{3.73}$$

The C_2 dependent part of the Lagrangian then becomes

$$L_{C_2} = -\frac{1}{2}e^{-2\phi}\mathcal{K}\left(dC_2 - ldB_2\right) \wedge *\left(dC_2 - ldB_2\right) - b^i dC_2 \wedge d\rho_i + dC_2 \wedge dh_1. \tag{3.74}$$

By calculating the equation of motion for C_2 and using it to express C_2 in terms of its dual h_1 we find,

$$\mathcal{L}_{h_1} = -\frac{1}{2\mathcal{K}}e^{2\phi}(dh_1 b^i d\rho_i) \wedge *(dh_1 - b^j d\rho_j) + ldB_2 \wedge (dh_1 - b^i d\rho_i). \tag{3.75}$$

We can apply exactly the same procedure to B_2 in order to dualize it to h_2, which leads us to the final action for type IIB supergravity on a Calabi-Yau manifold,

$$\begin{aligned} S^{(4)}_{IIB} = \int & -\frac{1}{2}R * 1 - g_{ab}dz^a \wedge *d\bar{z}^b - g_{ij}dt^i \wedge *d\bar{t}^j - d\phi \wedge *d\phi - \frac{e^{2\phi}}{8\mathcal{K}}(g^{ij})^{-1}\left(d\rho_i - \mathcal{K}_{ikl}c^k db^l\right) \\ & \wedge *(d\rho_j - \mathcal{K}_{jmn}c^m db^n) - 2\mathcal{K}e^{2\phi}g_{ij}(dc^i - ldb^i) \wedge *(dc^j - ldb^j) \\ & - \frac{1}{2}\mathcal{K}e^{2\phi}dl \wedge *dl - \frac{1}{2\mathcal{K}}e^{2\phi}\left(dh_1 - b^i d\rho_i\right) \wedge *(dh_1 - b^j d\rho_j) \\ & - e^{4\phi}D\tilde{h} \wedge *D\tilde{h} + \frac{1}{2}\mathrm{Re}\mathcal{M}_{AB}F^A \wedge F^B + \frac{1}{2}\mathrm{Im}\mathcal{M}_{AB}F^A \wedge *F^B, \end{aligned} \tag{3.76}$$

where we defined

$$D\tilde{h} = dh_2 + ldh_1 + (c^i - lb^i)d\rho_i - \frac{1}{2}\mathcal{K}_{ijk}c^i c^j db^k. \tag{3.77}$$

Chapter 4

Distances in the moduli space

In chapter 2 we saw that the Swampland Distance Conjecture (SDC) implies that there might be a mechanism by which an infinite distance in the moduli space leads to an infinite tower of states. This would motivate why attempts at sourcing inflation with moduli fields lead to a breakdown of the theory.

In this chapter we will further consider the link between divergent field distances and infinite towers of massless states. Remember that in chapter 3 we introduced the moduli space of IIB string compactifications; this is the setting to keep in mind in the following discussion on moduli spaces.

The situation in chapter 3 is slightly changed here however, as we now place singularities in the moduli space. These singularities lie at a finite or infinite distance, and induce a monodromy upon circling them. It was recently proposed in [3] that, if such a monodromy has infinite order, then it leads to an infinite tower of states: the authors have shown this for the moduli space of type IIB with D3 branes.

This proposal leads to a promising avenue for studying the mechanism behind the SDC through a mathematical framework. If one can further understand the relation between divergent field distances and monodromies, then through the proposed link between monodromies and infinite towers of states this would provide very strong evidence for the SDC.

This chapter is an attempt to further study the mechanics of divergent field distances in the moduli space; in particular we study the *Wang conjecture*. It states that, in the moduli space, one has an infinite field distance if and only if there exists a monodromy of infinite order.

Wang himself already proved one direction of his conjecture, that an infinite field distance implies a monodromy matrix of infinite order. When also taking into account the proposed connection between infinite order monodromy matrices and infinite towers of states, we note that in their paper[3] the authors have demonstrated an example of the proposed mechanism by which the SDC comes about.

Having seen evidence for the fact that the existence of a diverging field distance implies an

infinite tower of states, we could wonder whether the opposite is also true. This is in fact proposed in [3]. Specifically, the authors motivate that integrating out an infinite tower of states would lead to corrections to the field distance, resulting in a divergent distance in the field space. Such a mechanism would imply that the SDC, and the Wang conjecture, indeed go both ways.

In this chapter we study the implication from right to left; does having an infinite order monodromy matrix imply the existence of an infinite distance. Thereby we get closer to understanding the workings of the SDC. We consider the simpler cases where a singularity is located on one or two divisors, and we review a proof by Lee[4]. In his work the author manages to prove the Wang conjecture for these cases, as long as the paths considered are constant in the x direction. We then present a new result, giving criteria for when the case with one finite and one infinite divisor can be solved on a general path.

The mathematics underlying this discussion can be found in chapter 5. The mathematical theory provides an important background, but is not essential to understand the current chapter.

Motivation: infinite towers of states and monodromies

For completion we will briefly sketch the outline of the postulated link between infinite towers of states and infinite monodromies, as given in [3]. In this work the authors consider type IIB string theory compactifications with D3-branes wrapping 3-cycles. These 3-cycles are purely on the internal manifold, therefore intuitively compactifying the (3+1)D object that is a D3-brane would leave us with a $(0+1)$-dimensional object, without spatial dimensions but with one time dimension: in other words, a particle.

The mass of this scalar particle is proportional to the inverse of the string coupling, and to the volume of the cycle. Shrinking the volume of the cycle to 0 leads one to a vanishing mass, and a singularity in the scalar particle field space. Note that we have now only found a single massless state, which we have yet to relate to monodromies. This will be done by noting that the D3-branes are charged under gauge bosons. Then, by using the $\mathcal{N} = 2$ supersymmetry relation we can relate the scalar particle to a gauge boson. The mass of the gauge boson is $M \propto q \cdot \Pi e^{K/2}$, with q the charge, Π the period map and K the Kähler potential. We know that the period map interacts with the monodromy matrix relating to the mass of the gauge boson. The monodromy matrix is a mathematical concept used to describe singularities, and applying it should not change a physical object like the mass of a particle. Nonetheless we see that Π is changed by N, and therefore M is changed as well. We can fix this by stating that it is not the mass of the particle that should remain the same, but rather the entire spectrum of masses that we have in our theory. The mass/charge combination that the original particle had, before applying the monodromy matrix to move to a different "representation" of the theory, is then taken over by another particle. We would therefore need an extra particle per order of the monodromy matrix, as each change requires a different particle to take the place of the old one. An infinite order monodromy matrix would therefore imply an infinite tower of states, the superpartners of which are

massless.

The setup

We consider an n-dimensional moduli space B which parametrizes a family of Calabi-Yau manifolds,

$$\phi : X \to B. \tag{4.1}$$

For $s \in B$ the corresponding manifold is given by $X_s := \phi^{-1}(s)$.

The singularities of the moduli space are located on *divisors*[1]; such a divisor can be located at finite or infinite distance. Let r be the number of singular divisors in our moduli space. As the points not on the singular divisor will not be relevant to our discussion we can choose $n = r$.

Talking about distances in field space needs us to define some sort of metric. We shall therefore focus on the complex structure moduli space of the type IIB superstring compactification[2]. In this moduli space we can use the special geometry outlined in appendix C. First we define,

$$\tilde{Q}(\Omega, \overline{\Omega}) := -i^D \int_{Y_D} \Omega \wedge \overline{\Omega} \tag{4.2}$$

and name $\tilde{Q}$ the *polarization*[3]. Then, for a family of Calabi-Yau threefolds, the metric measuring the field distance on the moduli space (called the *Weil-Petersson metric*) can be expressed in terms of the unique $(3,0)$-form as,

$$\omega_{WP} = \frac{\sqrt{-1}}{2} \partial\overline{\partial} \log\left(\tilde{Q}(\Omega, \overline{\Omega})\right). \tag{4.3}$$

This result was already used in chapter 3, and is mentioned in section 5.5.3.

Monodromy matrices and filtrations

Having introduced our distance measure we will now look at some other concepts needed in formulating the Wang conjecture. Consider for a moment the Hodge decomposition as in appendix B, and its filtration as given in the appendix B. Both these structures are defined nicely for a moduli space *without* singularities. The moment we add singular divisors we need to consider so called *limiting filtrations*

$$F^p_\infty = \lim_{z \to 0} e^{-\frac{1}{2}\pi i \log(z) N} F^p,$$

[1]It suffices to think of them as $(n-1)$ complex dimensional submanifolds of the moduli space. See 5.2.

[2]Mirror symmetry tells us that this space is dual to the type IIA Kähler moduli space, so in fact both can be considered.

[3]The polarization is similar to the metric, but now takes into account a family of spaces, as one sees in the context of Hodge decomposition. See B.

where intuitively the singularities have been divided out. Here N denotes the monodromy matrix associated to the singular divisor.

Remember that a_0 is an element of the limiting filtration space F_n^∞ [3]. A divisor corresponding to a monodromy matrix N for which $Na_0 \neq 0$ will be referred to as an *infinite divisor*.

As explained further in the beginning of section 5.5 the monodromy operators map elements of the filtration space to a lower filtration; $N(F_\infty^n) \subset F_\infty^{n-1}$. For a Calabi-Yau manifold the Hodge diamond structure therefore ensures that

$$\text{There exists a } d \in \mathbb{N} \text{ such that } N^{d+1}a_0 = 0, \tag{4.4}$$

and as we have a Calabi-Yau threefold $d \leq 3$. This statement further limits the order of the monodromy matrices.

The Wang conjecture

We can now consider the Wang conjecture[2], which states that the singular divisors encode important information about divergences of the metric,

Conjecture 2 (Wang conjecture) *Let X/B be a family of n dimensional Calabi-Yau manifolds, the moduli space of which is smooth outside a divisor*[4] *$\cup_i E_i$. Then X_s, with $s \in \cup_i E_i$, has finite Weil-Petersson distance if $N_i F_\infty^n = 0$ for all i with $s \in E_i$.*

Some comments are in order. Note that the other direction has already been proven by Wang [2], and is therefore no longer a conjecture but a theorem. When we say that the manifold X_s has finite Weil-Petersson distance, we mean that there exists a point $p \in B$ which maximizes the field distance as measured from s. When we write $N_i F_\infty^n = 0$ we mean that N_i applied to an element of F_∞^n gives 0; note that $a_0 \in F_\infty^n$, hence we can also write this condition as $N_i a_0 = 0$.

In this chapter We will review a proof of simpler cases of the contraposition of conjecture 2. That is for some cases we will show that if s is located on the intersection of any number of infinite divisors, then any path length towards the singularity diverges. We therefore assume that $N_i a_0 \neq 0$. Note that combining this with equation (4.4) means that for any monodromy N_i with order d_i we have $0 < d_i \leq 3$.

Outline of the proof

The rest of the current chapter will be devoted to using the mathematical formalism in chapter 5 to expand Ω. In doing so we find a series in $N_i a_0$, and, knowing what we know about the possible order, we find that the series truncates. From there we estimate a lower

[4] This divisor is a *simple normal crossing* divisor, this basically means that the divisors act nicely as they intersect.

bound for the integral defining the field distance. We will show that, for specific cases, this lower bound diverges.

In this introduction we have already mentioned two very relevant parameters: the number of intersecting divisors that our point s is contained in, and the dimension r of the moduli space. Considering different numbers of intersecting divisors changes the number of coordinates along which one approaches the singularity. One can visualize this as having a plane in three dimensions [5]; in order to reach the plane one only has to change the coordinate perpendicular to the plane. When considering a singularity placed on two intersecting, non-parallel divisors however, one has to change *both* perpendicular coordinates in order to reach the singularity located on the intersection. In the case with one infinite divisor we will see that the metric depends non-trivially on only one coordinate. Increasing the number of dimensions still complicates the story, as a higher dimension increases the number of possible paths towards the singularity, thereby making it difficult to control the various terms popping up in the estimation.

Our case distinction will be based on these two parameters: the number of divisors that create the intersection on which the singularity is located, and the number of dimensions.

- Case I: 1 infinite divisor in an r dimensional moduli space,
- Case II: 1 infinite and 1 finite divisor in a 2 dimensional moduli space,
- Case III: 1 infinite and 1 finite divisor in an r dimensional moduli space,
- Case IV: 2 infinite divisors in a 2 dimensional moduli space,
- Case V: 2 infinite divisors in an r dimensional moduli space.

Case I can be proven in full generality, however we will not be able to prove cases III-IV generally, and we will not prove V at all. For case II we will give new results, in the form of criteria for when the case could be proven generally. The results of the review and further simplifications done are summarized in section 4.8.

4.1 Case I

We now consider the case where our singularity is located only on one infinite, singular divisor $E \subset B$, and therefore one infinite order monodromy matrix N, in an r dimensional moduli space.

Case 1 (One parameter Wang conjecture) *If $s \in E_1$ with E_1 an infinite divisor, and $s \notin E_j$ for all $j \neq 1$, then X_s has infinite Weil-Petersson distance.*

[5] Note that this example uses real dimensions, whereas in the moduli space we are working with complex codimension 1. For the divisor to take the form of a plane this would require us to consider a 4-dimensional real space, which remains tricky.

Note that even though we have only one singular divisor, the moduli space itself can still be parametrized by many moduli. It does however mean that we can reach the singularity from any point by only varying one of our moduli.

In the one singular divisor scenario Wang showed that the Weil-Petersson metric diverges[2]. We will present the proof here, following [4]. Let us begin by considering the expression for the Weil-Petersson metric 5.16. Remember that our goal is to show that it diverges as we approach the singularity (located on the divisor) at $z = (0, 0)$.

Consider any p-form Ω, which we can expand in periods Π^i as $\Omega(z) = \Pi_i(z)\gamma^i := \Pi^T(z) \cdot \vec{\gamma}$ where the γ^i are complex $(3, 0)$-forms. We can then define the *inner product matrix* $\eta_{IJ} = \int_{Y_D} \gamma_I \wedge \gamma_J$ to get

$$Q(\Omega, \overline{\Omega}) = -i^D \Pi^T(z)\eta\overline{\Pi}. \tag{4.5}$$

Using the nilpotent orbit theorem 2 we can expand the period map $\Pi(z)$ as

$$\Pi(z) = \exp(tN)A(z, z'), \tag{4.6}$$

where we take z' to be the coordinates outside of the divisor E, z the coordinate on E and $A \in \mathbb{C}$. The new variable t is defined to be $t = \log(z)/(2\pi i)$. So the singularity is now located at $t \to \infty$. We now note a property of the monodromy operators N_i, and η, namely that [3]

$$N_i^T \eta = -\eta N_i. \tag{4.7}$$

We can use this property to write out the Weil-Petersson metric,

$$\begin{aligned} Q(\Omega, \overline{\Omega}) &= -i^D \Pi^T \eta \overline{\Pi} \\ &= -i^D A^T \eta e^{-tN} e^{\overline{tN}} \overline{A} \\ &= -i^D A^T \eta e^{(t-\bar{t})N} \overline{A} \\ &= -i^D A^T \eta e^{-2i\mathrm{Im}(t)N} \overline{A}. \end{aligned} \tag{4.8}$$

Next we consider the limit we are taking. Since we consider the metric around the point $z = (0, 0)$ we can expand the function $A(z, z')$ around 0 in z. Keeping only the 0 order terms in z we find,

$$Q(\Omega, \overline{\Omega}) = -i^D a_0(z')\eta e^{-2i\mathrm{Im}(t)N} \overline{a_0}(z') + \mathbf{H}. \tag{4.9}$$

Here we used $\mathbf{H}$ to denote the class of functions that decay exponentially with $\mathrm{Im}(t) \to \infty$, and also whose partial derivatives decay exponentially in this limit. We now proceed to put in the expansion for the exponent that depends on $\mathrm{Im}(t)$, giving

$$Q(\Omega, \overline{\Omega}) = -i^D a_0(z')\eta \sum_{k=0}^{\infty} (-2i\mathrm{Im}(t)N)^k \overline{a_0}(z') + \mathbf{H}. \tag{4.10}$$

We will now make use of the fact that, for d as in 4.4, we have $0 < d \leq 3$. This will make the power series truncate. For simplicity we use the following notation,

$$S_j(a,b) := a^T \eta N^j b. \tag{4.11}$$

Note that if N is of order d then $S_{d+1}(a_0, \overline{a_0}) = 0$.

We will now prove the conjecture 2 for the case with only one parameter. This was first done by Wang [2], and we follow the proof given by Lee [4].

We start by taking $\mathrm{Im}(t) = y_1$ and renaming the first term in the potential (4.10);

$$p(y_1) := -i^D a_0(z')\eta \sum_{k=0}^{\infty} (-2iy_1 N)^k \overline{a_0}(z') \tag{4.12}$$

Here p is a polynomial in the parameter on the divisor, i.e. y_1. The polynomial p has a guaranteed finite order d because of the nilpotency of the monodromy matrix. Note that $p(y_1)\mathbf{H} \to 0$ for $y_1 \to \infty$, as $\mathbf{H}$ decays exponentially in y_1.

Note from appendix B that the polarization Q is restricted to have $Q(\omega, \overline{\omega}) > 0$. If we write $p(y_1) := \sum_{l=0}^{d} s_l(z_2, ..., z_r) y_1^l$ then for all z_i near 0 we get that $y_1 \to \infty$. Now in this limit the term $s_d y_1^d$ dominates $p(y_1)$. The polarization condition B therefore tells us that $s_d(z_2, ..., z_r) > 0$ for all z_i near 0. So p will remain a polynomial of order d even near 0.

Remember now the expression for the metric two form(5.16), which we will expand in forms to get

$$-\frac{\sqrt{-1}}{2} \sum_{i,j} \partial_i \partial_j \log \tilde{Q}(\Omega, \overline{\Omega}) dz_i \wedge \overline{z}_j. \tag{4.13}$$

We will prove theorem 1 by considering the terms $i = j = 1$, $i = 1, j \neq 1$ and $i \neq 1, j \neq 1$ in the above sum separately, and showing that their sum has to diverge as $y_1 \to \infty$. We begin with the term $i = j = 1$.

Putting in $\tilde{Q} = p(y_1) + \mathbf{H}$ and taking the derivatives (the derivative $\partial_{z_1}\partial_{\overline{z_1}} = \partial_{x_1}^2 + \partial_{y_1}^2$, so we compute the double x_1 and y_1 derivatives separately to come to the full result). For the x_1 derivative we find,

$$\begin{aligned} -\frac{\sqrt{-1}}{2}\partial_{x_1}^2 \log \tilde{Q} &= -\frac{\sqrt{-1}}{2}\partial_{x_1}^2 \log(p(y_1) + \mathbf{H}) \\ &= \frac{\sqrt{-1}}{2}\partial_{x_1} \frac{\mathbf{H}}{p(y_1) + \mathbf{H}} \\ &= -\frac{\sqrt{-1}}{2} \frac{\mathbf{H}}{(p(y_1) + \mathbf{H})^2}, \end{aligned} \tag{4.14}$$

where we kept denoting any exponentially divergent term simply as $\mathbf{H}$. Next, for the double y_1 derivative we find,

$$-\frac{\sqrt{-1}}{2}\partial_{y_1}^2 \log(p(y_1) + \mathbf{H}) = \frac{\sqrt{-1}}{2}\partial_{y_1} \frac{\mathbf{H} + p'(y_1)}{p(y_1) + \mathbf{H}} = \frac{\sqrt{-1}}{2} \frac{\mathbf{H} + p''(y_1)p(y_1) - (p'(y_1))^2}{(p(y_1) + \mathbf{H})^2}.$$

Now considering the limit where $y_1 \to \infty$ we neglect the lower order terms and terms proportional to $\mathbf{H}$, and take $p(y_1) \approx s_d y_1^d$ to finally get,

$$-\frac{\sqrt{-1}}{2}\partial_{y_1}^2 \log \tilde{Q} \approx \frac{d(d-1)s_d^2 y_1^{d-2} y_1^d - d^2 s_d^2 y_1^{2d-2}}{s_d^2 y_1^2 d} = \frac{\sqrt{-1}}{2}\frac{d}{y_1^2}.$$

Putting these two results together we have the following expression for the $i = j = 1$ term,

$$-\frac{\sqrt{-1}}{2}\partial_{z_1}\partial_{\overline{z_1}} \log \tilde{Q} \approx \frac{\sqrt{-1}}{2}\frac{d}{y_1^2}.$$

We then consider the term with $i = 1, j \neq 1$, again splitting up the derivative into a derivative over x and y, this time also getting mixing terms.

$$\begin{aligned} -\frac{\sqrt{-1}}{2}\partial_{x_1}\partial_{x_j} \log(p(y_1) + \mathbf{H}) &= -\frac{\sqrt{-1}}{2}\partial_{x_1}\frac{\mathbf{H}}{p(y_1) + \mathbf{H}} \\ &\sim -\frac{\sqrt{-1}}{2}\frac{\mathbf{H}}{(p(y_1) + \mathbf{H})^2}\frac{\partial y_1}{\partial x_j} \\ &= \mathbf{H}, \end{aligned} \tag{4.15}$$

where the last two steps were taken using that we denote anything decaying exponentially, also when multiplied by $p(y_1)$, with $\mathbf{H}$. Now for the y derivatives,

$$\begin{aligned} &-\frac{\sqrt{-1}}{2}\partial_{y_1}\partial_{y_j} \log(p(y_1) + \mathbf{H}) \\ &= -\frac{\sqrt{-1}}{2}\partial_{y_j}\frac{\mathbf{H} + p'(y_1)}{p(y_1) + \mathbf{H}} \\ &= -\frac{\sqrt{-1}}{2}\frac{-(p(y_1) + \mathbf{H})\partial_{y_j}(\mathbf{H} + p'(y_1)) + (\mathbf{H} + p'(y_1))\partial_{y_j}(\mathbf{H} + p(y_1))}{(p(y_1) + \mathbf{H})^2}. \end{aligned} \tag{4.16}$$

Instead of calculating this quantity explicitly for the limit $y_1 \to \infty$ we will give an argument to motivate the scaling of the term (as per [4]), which is all we are interested in. By putting in the power series expansion for $p(y_1)$ we find that,

$$\begin{aligned} (p(y_1) + \mathbf{H})\partial_{y_j}(\mathbf{H} + p'(y_1)) &= d \cdot y_1^{2d-1} s_d \partial_{y_j} s_d + ... \\ (\mathbf{H} + p'(y_1))\partial_{y_j}(\mathbf{H} + p(y_1)) &= d \cdot y_1^{2d-1} s_d \partial_{y_j} s_d + ... \ . \end{aligned}$$

So the highest order terms in both expansions cancel, leaving us with a numerator which has a maximum order of y_1^{2d-2}. For the total y derivative term we then find,

$$-\frac{\sqrt{-1}}{2}\partial_{y_1}\partial_{y_j} \log(p(y_1) + \mathbf{H}) \sim \frac{C_j}{y_1^2}$$

with C_j a constant in y_1, a real number (possibly zero). All that rests us now is to examine the mixed derivatives. The term with derivatives $\partial_{x_1}\partial_{y_j}$ will scale as $\mathbf{H}$, since just as in

(4.15) we differentiate twice to a variable that $p(y_1)$ is independent of.
The term with derivatives $\partial_{y_1}\partial_{x_j}$ gives,

$$-\frac{\sqrt{-1}}{2}\partial_{y_1}\partial_{x_j}\log(p(y_1)+\mathbf{H}) = -\frac{\sqrt{-1}}{2}\partial_{x_j}\frac{\mathbf{H}+p'(y_1)}{p(y_1)+\mathbf{H}}, \tag{4.17}$$

which is the same case we had in (4.16), hence by analogy we find that

$$-\frac{\sqrt{-1}}{2}\partial_{y_1}\partial_{x_j}\log(p(y_1)+\mathbf{H}) \sim \frac{C_j'}{y_1^2}, \tag{4.18}$$

where C_j' again gives a constant real number. For simplicity we will denote the sum $C_j + C_j'$ as C_j from this point onwards.

This takes care of all terms but the one with $i \neq 1$, $j \neq 1$, luckily this term does not contain any y_1 derivatives, so as we have seen it will scale as $\mathbf{H}$, meaning that g_{ij} decays exponentially as well.

Putting all of this together we find that the metric two-form is given by

$$\begin{aligned} g &\sim -\frac{\sqrt{-1}}{2}\left(\frac{d}{y_1^2}dz_1\otimes\overline{dz_1} + \left(\frac{C_j}{y_1^2}+\mathbf{H}\right)dz_1\otimes\overline{dz_j} + g_{ij}dz_i\otimes\overline{dz_j}\right) \\ &\geq -\frac{\sqrt{-1}}{2}\left(\frac{d}{y_1^2}dz_1\otimes\overline{dz_1} + \frac{C_j}{y_1^2}dz_1\otimes\overline{dz_j} + g_{ij}dz_i\otimes\overline{dz_j}\right). \end{aligned} \tag{4.19}$$

We can now use this expression to show that the Weil-Petersson distance indeed diverges by further estimating a lower bound for the second term of the metric. We will use the following relation for complex numbers a, b:

$$(|a|-|b|)^2 \geq 0 \Rightarrow -|a|^2-|b|^2 \leq -2|ab|.$$

Noting that the one forms dz_i are maps into $\mathbb{C}$ we can use this inequality with $a = \epsilon dz_1$ and $b = C_j dz_j/\epsilon$ to get,

$$-\frac{C_j}{y_1^2}(dz_1\otimes\overline{dz_j}) \geq -\frac{C_j}{y_1^2}|dz_1\otimes\overline{dz_j}| \geq -\left(\frac{\epsilon^2}{y_1^2}dz_1\otimes\overline{dz}_1 + \frac{C_j^2}{\epsilon^2 y_1^2}dz_j\otimes\overline{dz}_j\right)/2. \tag{4.20}$$

which means that we can estimate the metric as follows:

$$g \geq \left(\frac{A}{y_1^2}dz_1\otimes\overline{dz_1} + \frac{C_j^2}{\epsilon^2 y_1^2}dz_j\otimes\overline{dz_j} + g_{ij}dz_i\otimes\overline{dz_j}\right). \tag{4.21}$$

Where we choose ϵ such that $A := d - \epsilon^2(r-1) > 0$. Since the third term decays exponentially in all coordinates it will remain finite as we integrate over a curve. The second term is larger than zero, as it corresponds to the sum over the diagonal elements of our semi-positive definite metric, we have

$$\int_\gamma ds \geq \int_c^\infty \frac{\sqrt{A}}{y_1}dy_1 + \text{ finite terms} = \infty. \tag{4.22}$$

And so indeed the Weil-Petersson distance from any point $c \neq 0$ towards singular point will diverge, and the conjecture 2 has been proven.

4.2 Two divisor setup

We will now consider the situation where the singular point is contained in two divisors; $s \in E_1 \bigcap E_2$, $s \notin E_j$. The conjecture we would like to prove is therefore

Conjecture 3 (Two parameter Wang conjecture) *If $s \in E_1$ and $s \in E_2$ with E_1 or E_2 an infinite divisor, and $s \notin E_j$ for all $j \neq 1, 2$, then $\mathcal{X}_s$ has infinite Weil-Petersson distance.*

The two singular divisor case of the Wang conjecture 2 requires a different, but in essence similar, approach as the previous one parameter case. This is because with the addition of more parameters one introduces path dependence. In this paragraph we will be following Lees attempt at a proof [4].

Here we introduce the notation for any two divisors contained in an r dimensional moduli space. In the next sections, when we consider more specific cases, we shall change the assumption on the number of dimensions.

We begin with the potential function $\tilde{Q}(z)$, which depends on all our r coordinates z_i. We have two divisors E_1, E_2 the coordinates on which we will denote by z_1 and z_2 respectively, which we collectively write as z. The remaining $r-2$ coordinates will be denoted ζ. As we are sending z_1 and z_2 to 0 it makes sense to attempt to expand our potential in these variables. We will, like before, map z to $t = \log(z)/(2\pi i)$.

We will begin by applying the nilpotent orbit theorem 2 to our period map $\Pi(z, \eta)$ to get

$$\Pi(z, \eta) = \exp(t_1 N_1 + t_2 N_2) A(z, \zeta) \to \exp(t_1 N_1 + t_2 N_2) A(t),$$

where we leave the ζ dependence of $A(t)$ implicit. The next step is to consider the expansion of $A(t)$ around $z = 0$,

$$\begin{aligned} A(t) &= \sum_{m,n} a_{mn}(\zeta) z_1^m z_2^n \\ &= a_0(\zeta) + \sum_{m \geq 1, n=0} a_{m,0}(\zeta) z_1^m + \sum_{m=0, n \geq 1} a_{0,n}(\zeta) z_2^n + \sum_{m \geq 1, n \geq 1} a_{m,n}(\zeta) z_1^m z_2^n \\ &\equiv a_0 + f_1(t_1) + f_2(t_2) + h(t), \end{aligned} \tag{4.23}$$

where $h(t)$ denotes all terms of order at least one in both z_1 and z_2 (which means of order one in both $\exp(2\pi i t_1)$ and $\exp(2\pi i t_2)$); hence $h(t)$ decays exponentially for $\mathrm{Im}(t_1)$ or $\mathrm{Im}(t_2)$ approaching infinity. The terms $f_1(t_1)$ and $f_2(t_2)$ are at least first order in $\exp(2\pi i t_1)$ and $\exp(2\pi i t_2)$ respectively, hence f_1 decays exponentially for $\mathrm{Im}(t_1)$ and f_2 for $\mathrm{Im}(t_2)$ approaching infinity.

We will now put these expressions into our Kähler potential $\tilde{Q}(\Omega, \overline{\Omega})$ and, using commutativity of monodromy matrices arising from different divisors [15], find,

$$\begin{aligned} \tilde{Q}(\Omega(z, \zeta), \overline{\Omega}(z, \zeta)) &= \tilde{Q}(e^{t_1 N_1 + t_2 N_2} A(z, \zeta), e^{\overline{t_1} N_1 + \overline{t_2} N_2} \overline{A}(z, \zeta)) \\ &= \tilde{Q}(e^{2i \mathrm{Im}(t_1) N_1} A(z, \zeta), e^{-2i \mathrm{Im}(t_2) N_2} \overline{A}(z, \zeta)). \end{aligned}$$

Labeling $A_1 := e^{2i\mathrm{Im}(t_1)N_1}$ and $A_2 := e^{-2i\mathrm{Im}(t_2)N_2}$ as per [4] and putting in our expansion for $A(z,\zeta)$ gives,

$$\begin{aligned}\tilde{Q}(\Omega(z,\zeta),\overline{\Omega}(z,\zeta)) = \tilde{Q}(A_1a_0, A_2\overline{a_0}) + \tilde{Q}(A_1a_0, A_2\overline{f_1}) + \tilde{Q}(A_1a_0, A_2\overline{f_2}) \\ + \tilde{Q}(A_1f_1, A_2\overline{a_0}) + \tilde{Q}(A_1f_1, A_2\overline{f_1}) + \tilde{Q}(A_1f_1, A_2\overline{f_2}) \\ + \tilde{Q}(A_1f_2, A_2\overline{a_0}) + \tilde{Q}(A_1f_2, A_2\overline{f_1}) + \tilde{Q}(A_1f_2, A_2\overline{f_2}) + \mathbf{H}_{12},\end{aligned} \tag{4.24}$$

where $\mathbf{H}_{12}$ denotes all the terms that decay exponentially for $\mathrm{Im}(t_1)$ or $\mathrm{Im}(t_2)$ approaching infinity. We will refer to each of the terms in equation (4.24) by $\tilde{Q}_{i,j}$, where i,j refer to the lower index of the a_0, f_1, f_2 terms in first and second argument of $\tilde{Q}$ respectively. For example $\tilde{Q}_{1,1}$, corresponds to $\tilde{Q}(A_1f_1, A_2\overline{f_1})$. Our goal is now to study each of these terms and determine their behavior when we take $\mathrm{Im}(t_1)$, $\mathrm{Im}(t_2)$ to infinity, remembering that due to the nilpotency of the monodromy matrices the terms A_1 and A_2 are polynomials of finite order in t_1 and t_2 respectively.

All the terms dependent on f_1 but not f_2 will therefore be polynomials in y_2 with coefficients decaying exponentially when $\mathrm{Im}(t_1)$ diverges. The same holds in the case where $\mathrm{Im}(t_2)$ goes to infinity for all terms that have an f_2 term and not an f_1. Hence we will define $p_2(t_2) := \tilde{Q}_{1,1} + \tilde{Q}_{0,1} + \tilde{Q}_{1,0}$ and $p_1(t_1) = \tilde{Q}_{2,2} + \tilde{Q}_{0,2} + \tilde{Q}_{2,0}$. Both p_1 and p_2 depend on t, but for simplicity we omit the dependence on the parameter that they decay in exponentially. Note that the dependence of p_1 on t_2 and p_2 on t_1 arises from A_1 and A_2. We cannot say anything about the decay properties of $\tilde{Q}_{0,0} =: p(t_1,t_2)$, and will thus simply relabel it. Finally the terms $\tilde{Q}_{1,2}$ and $\tilde{Q}_{2,1}$ decay exponentially in both t_1 and t_2, so we absorb them in $\mathbf{H}_{12}$. This leaves us with

$$\tilde{Q}(\Omega(z,\zeta),\overline{\Omega}(z,\zeta)) = p(t_1,t_2) + p_1(t_1) + p_2(t_2) + \mathbf{H}_{12}. \tag{4.25}$$

Comparing this to the one parameter case $\tilde{Q} = p(y_1) + \mathbf{H}$ we see that we have gained two non-trivial terms due to the introduction of an additional parameter.

We can still say something about the order of these polynomials. In order to do this we introduce d_j again as the order of the nilpotent matrix N_j. Since p_1 is a polynomial in $\mathrm{Im}(t_1)$ it follows that the order of p_1 in $\mathrm{Im}(t_1)$ is less than or equal to d_1. Analogously the order of p_2 in $\mathrm{Im}(t_2)$ is less than or equal to d_2. Finally the order of p in $\mathrm{Im}(t_1)$ and $\mathrm{Im}(t_2)$ is less than or equal to d_1 and d_2 respectively; hence the total order is less than or equal to $d_1 + d_2$.

The next step is to again study the resulting Weil-Petersson metric. However, with the introduction of the two extra terms we can no longer simply compute all derivatives; the expression will become too involved to make sense of the path integration, in a general case. The approach taken by Lee [4] is to calculate the distance for $p(t_1,t_2)$ first, and then view the metric term resulting from

$$f := p_1 + p_2 + \mathbf{H}_{12} \tag{4.26}$$

as a small perturbation on p. This matrix perturbation to the metric will be denoted E. One can calculate it by simply performing the derivatives as worked out in the appendix D.

With the introduction of a second nilpotent matrix we can also no longer be certain of the order of the polynomial p in t_1 and t_2; we can only state an upper bound. Lee [4] attempts to remedy this by doing a *case distinction* of all possible polynomials with an order between 0 and 3, labeling these *candidates of the Weil-Petersson potential*. After characterizing all the possible polynomials one attempts to integrate each of them over a path in order to determine whether the Weil-Petersson distance diverges if the polynomial p would be of that order.

At the end of the proof we shall consider the full Weil-Petersson potential, which means incorporating the perturbation to the candidate potential as well. We will see that this is possible for all cases but case V.

In doing these calculations we shall find it difficult to work with the path dependence; having two divisors will lead to a path in two complex directions. As will be mentioned again later, to simplify our life we can *work on constant angular slices*: $\mathrm{Re}(z_j) = c_j$. In this case our path will only depend on the imaginary parts of the coordinates. We will actually turn out to need it in cases II-V.

The candidate potential

We will assume now that $i, j \in \{1, 2\}$. The first step in doing the case distinction is to check whether the Weil-Petersson candidate actually dominates the potential metric at large values of $\mathrm{Im}(t_1)$ and $\mathrm{Im}(t_2)$. This has been done by Lee [4], and we will repeat some results here. Take M to be given by $M_{ij} = -\partial_i \partial_j \tilde{Q}$. For the case with only two dimensions[6] we find,

$$M_{11} \sim \frac{(\partial_{y_1} p)^2 - p\partial_{y_1}^2 p}{p^2} + e^{-y_1} \text{ (bounded terms)} \tag{4.27}$$

$$M_{22} \sim \frac{(\partial_{y_2} p)^2 - p\partial_{y_2}^2 p}{p^2} + e^{-y_2} \text{ (bounded terms)} \tag{4.28}$$

$$M_{12} \sim \frac{(\partial_{y_1} p)(\partial_{y_2} p) - p\partial_{y_1}\partial_{y_2} p}{p^2} + C_1 e^{-y_2} \frac{y_1^{2D_1-2}}{p^2} + C_2 e^{-y_1} \frac{y_2^{2D_2-2}}{p^2} + C_3 \frac{e^{-y_1-y_2}}{p^2}, \tag{4.29}$$

where D_j is the order of p in y_j. We see that in the limit of large y_1, y_2 the metric is indeed dominated by the polynomial p. The terms in (4.27) proportional to exponents arise from what was in the previous section called the matrix E. Remember p is a polynomial in the nilpotent monodromy matrices. Therefore we have $1 \leq D_j \leq d_j \leq 3$. As on the intersection of two divisors $y_1 N_1 + y_2 N_2$ defines the same monodromy weight filtration for any $y_1, y_2 \neq 0$ [15], we see that the Hodge diamond structure implies that applying any monodromy operator more than three times gives zero (e.g. $N_1 N_2 N_2 N_1 a_0 = 0$). Since p is

[6] In section 4.6 we will remark on the r dimensional case.

Table 4.1: Possible orders of the dominant polynomial [4].

	d_2			
d_1	3,0	3,1	3,2	3,3
	2,0	2,1	2,2	2,3
	1,0	1,1	1,2	1,3
	0,0	0,1	0,2	0,3

the truncation of an expansion in the monodromy operators, the polynomial p can contain no term of total order higher than three.

We can now start to classify the different possible polynomials p by their value for D_1 and D_2. This polynomial will be dominated by the top-right possible terms in the table below 4.1, e.g. taking $D_1 = 2$ and $D_2 = 3$ would lead to the polynomial $p(y_1, y_2) = Ay_1^2 + By_2^3 + Cy_1^2y_2 + Dy_1y_2^2$. This is because for any bottom-left term there exists a top-right term dominating it for large values of y_1 and y_2, for example terms of order $(1, 1)$ are dominated by terms $(2, 1)$.

As such we can distinguish a total of nine different possible polynomials which are worked out in [4].

Using the above outlined approach we will not be able to prove 3 in generality. For the cases II-IV we will need the angular slice assumption, and case V we will not manage at all.

4.3 Case II

Consider the case with one infinite divisor and one finite divisor, in a 2 dimensional moduli space. Note that this case is still different from having only a single divisor, as our path still depends on the extra parameter y_2.

Case 2 (Infinite/finite Wang conjecture in 2 dimensions) *If $s \in E_1 \bigcap E_2$ with E_1 a finite and E_2 an infinite divisor, let*

- *γ a path for which $y_1, y_2 \to \infty$, situated on an angular slice (i.e. $Re(z_j) = c_j$),*

then $\forall\gamma$ as above $\int_\gamma ds = \infty$, and so the Weil-Petersson distance diverges.

we can prove the theorem 5, albeit *on angular slices.* This case is simpler than the two infinite divisor cases, because the period map $\Pi(z)$ no longer depends on t_2 as per the nilpotent orbit theorem 2. Hence in the discussion in section 3 we can take A_2 to be the identity, and the two parameter potential (4.25) $\tilde{Q}$ reduces to $\tilde{Q} = p(y_1) + p_1(y_1) + \mathbf{H}$.

For simplicity we consider the case where our moduli space B has only two parameters; $r = 2$.

The perturbation matrix E depends on the function f (4.26) which now reduces to $f = p_1(y_1) + \mathbf{H}$. Remember that $\mathbf{H} = \mathbf{H}(y_1, y_2)$ is exponentially decaying in both parameters,

and p_1 is exponentially decaying in y_2. Since every term in the numerator of E is proportional to a derivative of $\mathbf{H}$ or p_1 as per appendix D we find that $E_{i\bar{j}}$ is exponentially decaying in y_2.

The dominant polynomial $p = p(y_1)$, hence the only non-zero contribution comes from $\partial_1\partial_{\bar{1}}p$. This case reduces back to the one-parameter case reviewed in subsection 4.1. We can therfeore write $\partial_1\partial_{\bar{1}}p \sim 1/y_1^2$. Note that we did not need to use the setup with Weil-Petersson candidates to reach this conclusion.

This leads us to the following expression for the full metric,

$$g = \begin{pmatrix} \frac{1}{y_1^2} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{y_1^2} & g(y_1)e^{-y_2} \\ g(y_1)e^{-y_2} & E_{22} \end{pmatrix}. \tag{4.30}$$

Note that because we only have one infinite divisor we only approach the singularity along the z_1 direction (or y_1 because of our angular slice assumption), motivating the first term. In the last expression we chose to neglect E_{11} with respect to $1/y_1^2$, as E_{11} decays exponentially. We also wrote $E_{12} = E_{21} = g(y_1)e^{-y_2}$. We took out part of the exponential y_2 dependence, and left the y_1 dependence in the function g which, taken together with the exponential, is bounded from above. The E_{22} term is left as is because its explicit form is not needed in the rest of the derivation.

We then proceed to estimate a lower boundary for the integral over the metric, and therefore the Weil-Petersson distance, and show that this diverges. Remember that the metric is semi-positive definite[16]. We complete the square to find

$$\begin{aligned} g &= \frac{2}{y_1^2}\left(dy_1^2 + 2e^{-y_2}y_1^2 g dy_1 dy_2 + y_1^2 E_{22} dy_2^2\right) \\ &= \frac{2}{y_1^2}\left|dy_1 + e^{-y_2}y_1^2 g dy_2\right|^2 + 2y_1^2\left(\frac{E_{22}}{y_1^2} - e^{-2y_2}g^2\right)dy_2^2, \end{aligned}$$

where we made us of the angular slice assumption to ignore the displacement along the x direction. Note that semi-positive definiteness of the metric implies that all diagonal elements of g have to be ≥ 0, hence $\det g = E_{22}/y_1^2 - \exp(-2y_2)g^2 \geq 0$. This is exactly the second term in the expression above, and so

$$g \;\geq\; \frac{2}{y_1^2}\left|dy_1 + e^{-y_2}y_1^2 g dy_2\right|^2. \tag{4.31}$$

As we take $y_1, y_2 \to \infty$ we can absorb the factor of y_1^2 into g without changing any of the relevant, large y_1 properties of $g(y_1)e^{-y_2}$. In the distance integral we will work with the square root of the metric, and so we will proceed to estimate that from now on:

$$\sqrt{g} \geq \sqrt{2}\frac{|dy_1 + e^{-y_2}g dy_2|}{y_1}. \tag{4.32}$$

We would now like to estimate this expression in such a way that makes the evaluation of the integral simpler. The form of the fraction invites us to attempt to estimate the term by

the derivative of a logarithm. Let $\min(y_1) = c$ and $\max|g\exp(-y_2)| = g_m$. We can reduce to the case where $c > g_m$, as the path taken until y_1 reaches g_m will always be finite, and give a finite contribution. Therefore if we can show that the infinite path after $y_1 = g_m$ diverges we will not have to bother about $y_1 < g_m$. In this case we can rescale y_1 to find,

$$y_1 \leq \frac{(y_1 - g_m)(c+\epsilon)}{c - g_m} \leq \frac{(y_1 - ge^{-y_2})(c+\epsilon)}{c - g_m}, \tag{4.33}$$

where ϵ is a real positive constant we choose such that $c + \epsilon > c - g_m$. As y_1 is a linear function in y_1 that starts at $y_1 = c$, we have to show that the linear function on the right increases faster and starts at a higher value. We therefore check the value at $y_1 = c$, and the first derivative of the first inequality in 4.33:

$$\frac{c - g_m}{c - g_m}(c+\epsilon) = c + \epsilon > c \tag{4.34}$$

$$\frac{c+\epsilon}{c - g_m} > 1 \text{ by our choice for } \epsilon, \tag{4.35}$$

so we have rescaled this linear function to always be greater than y_1. Note that the constant factor is positive. Inverting the found inequality (4.33) we conclude that,

$$\sqrt{g} \geq A\frac{|dy_1 + e^{-y_2}gdy_2|}{y_1 - ge^{-y_2}} \geq A\frac{(dy_1 + e^{-y_2}gdy_2)}{y_1 - ge^{-y_2}}, \tag{4.36}$$

where A denotes the positive constant term we leave out for simplicity.

We are now ready to consider the Weil-Petersson distance, by integrating the metric over a general curve γ.

$$\int_\gamma ds = \int_c^\infty \sqrt{g}dy_1dy_2 \geq A\int_c^\infty \frac{(dy_1 + e^{-y_2}gdy_2)}{y_1 - ge^{-y_2}} \tag{4.37}$$

$$= A\int_c^\infty \left(d\log(y_1 - ge^{-y_2}) - \frac{(\partial_1 gdy_1 + \partial_2 gdy_2)e^{-y_2}}{y_1 - ge^{-y_2}}\right). \tag{4.38}$$

The logarithmic term clearly diverges, so it rests us to show that the term on the right hand side remains finite. This is true because, as g is at most polynomial in y_1, any integral will again be polynomial, and dominated by the exponential decay we have in y_2. Noting that the integral over the logarithm diverges we see that,

$$A\int_c^\infty \left(d\log(y_1 - ge^{-y_2}) - \frac{(\partial_1 gdy_1 + \partial_2 gdy_2)e^{-y_2}}{y_1 - ge^{-y_2}}\right) = \infty. \tag{4.39}$$

Here we dodged the problems of path dependence by estimating the Weil-Petersson distance by an indefinite integral which we could calculate, thereby proving the Wang conjecture for the two divisor case with one finite divisor, and $r = 2$.

4.4 Case III

Note that in Case II we have proven the conjecture using only that the function g is a bounded function. If take r to be any natural number (while keeping the number of infinite singular divisors equal to one) then we can prove

Case 3 (Infinite/finite Wang conjecture in r dimensions) *If $s \in E_1 \bigcap E_2$ with E_1 a finite and E_2 an infinite divisor, and $s \notin E_j$ for all $j \neq 1, 2$, let*

- *γ a path for which $y_1, y_2 \to \infty$, situated on an angular slice (i.e. $Re(z_j) = c_j$),*

then $\forall \gamma$ as above $\int_\gamma ds = \infty$, and so the Weil-Petersson distance diverges.

in the same way. For r dimensions the expression in equation (4.3) changes to [4],

$$g = \left(dy_1 + \sum_{i \neq 1} E_{1i} dy_i \right)^2 - \left(\sum_{i \neq 1} E_{1i} dy_i \right)^2 + \sum_{i,j \geq 2} E_{ij} dy_i \otimes dy_j, \tag{4.40}$$

Where the matrix E is the perturbation matrix. The parameters are again convergent power series in $1/y_1$, and proportional to e^{-y_2}. However now they can also depend on y_j with $j > 2$. These will not make a difference; since they are not on a divisor their range remains finite.

The boundedness and convergence of the matrix elements ensures that the previous derivation remains largely the same, so we omit it here and refer to [4].

4.5 Case IV

In the case with two infinite divisors we can no longer write the simpler expression we had for the metric in (4.30). Instead we need to use the candidate potential formalism that was built up previously. In [4] all cases are treated, here we will only go over two in order to illustrate the procedure. Remember that D_i denotes the order of the dominant polynomial p in y_i; we used this to characterize the different cases in section 4.2. The theorem that will be proven is

Case 4 (Infinite/infinite Wang conjecture in 2 dimensions) *If $s \in E_1 \bigcap E_2$ with E_1, E_2 infinite divisors, let*

- *γ a path for which $y_1, y_2 \to \infty$, situated on an angular slice (i.e. $Re(z_j) = c_j$),*

then $\forall \gamma$ as above $\int_\gamma ds = \infty$, and so the Weil-Petersson distance diverges.

The two cases we will be looking at in detail are $(D_1, D_2) = (1, 1)$ and $(D_1, D_2) = (1, 3)$.

Studying the dominant polynomial

The first step is writing down the most general polynomial. Here we want to preserve as much information from the mixed variation of Hodge structure as we can, as it is this

structure which should imply divergence of the Weil-Petersson distance. We will make use of the polarization of the MVHS to constrain the coefficients of our polynomials. One of the properties of the polarization is that for any $v \in F^p_\infty$ we have

$$\tilde{Q}(v, \overline{v}) > 0. \tag{4.41}$$

Remember that for the polarization we had $\tilde{Q} \sim p$ as per (4.25), where p is the dominant polynomial. By considering different regimes for y_1, y_2 we can determine constraints for the coefficients on p. Let $p = \sum_{i \leq d_1, j \leq d_2} a_{ij} y_1^i y_2^j$.

- In the limit of $y_1, y_2 \to \infty$ it is clear that $\tilde{Q} \sim p \sim a_{d_1,d_2} y_1^{d_1} y_2^{d_2} > 0$, and it follows that $a_{d_1,d_2} \geq 0$.
- In the limit where $y_1 \to 0$, $y_2 \to \infty$ we have $\tilde{Q} \sim p \sim a_{0,d_2} y_2^{d_2} > 0$, and it follows that $a_{0,d_2} \geq 0$.
- By analogy to the case above $a_{d_1,0} \geq 0$.

The constrains we had on the order of our polynomial were that $d_1 + d_2 \leq d$, so it is possible to have terms with a monomial of order $D_1 + D_2 = d$. For these terms we have the following constraint,

- Consider $d_1 \leq D_1$. By taking $y_2 >> |a_{d_1,0}/a_{D_1,D_2}|$ we have that $a_{D_1,D_2} y_1^{D_1} y_2^{D_2} + a_{d_1,0} y_1^{d_1} \sim a_{D_1,D_2} y_1^{D_1} y_2^{D_2}$. Out of all terms in p this would make the monomial the one of highest order in y_1. Hence sending $y_1 \to \infty$ gives $a_{D_1,D_2} \geq 0$.
- Analogous to the case above, for $d_2 \leq D_2$ we can again find $a_{D_1,D_2} \geq 0$.

We can now consider two cases and calculate the dominant polynomial for each of them. Note that the cases we distinguish here are for different orders of the polynomial. This will generate a total of 9 possible metrics for which to estimate the WP candidate.

Case 1 $d_1 = 1$**,** $d_2 = 1$**,** $d = 1$

We have the polynomial $p = Ay_1 + By_2$. By the discussion above we can note that $A, B > 0$. The corresponding metric is as per equation (4.27), given by

$$g_c = \frac{1}{(Ay_1 + By_2)^2} \begin{pmatrix} A^2 & AB \\ AB & B^2 \end{pmatrix}. \tag{4.42}$$

which is a semi-positive definite matrix, and hence correct. It follows that $p = Ay_1 + By_2$ is a WP candidate.

Case 2 $d_1 = 1$**,** $d_2 = 3$**,** $d = 3$

We could have several possible candidates, namely $p = Ay_2^3 + By_2^2 y_1 + Cy_1$, $p = Ay_2^3 + By_2 y_1 + Cy_1$ and $p = Ay_2^3 + Cy_1$, with $A, B, C > 0$ by the discussion above. Note that this case allows multiple candidates, as we generally do not know the nilpotency order of a

product of N_1's and N_2's. Therefore some coefficients may vanish, or they may not: we do not know and have to check everything.

In [4] the author proceeds to calculate the determinant of the various metrics implied by the polynomials we found. Imposing that they be semi-positive definite leads one to the viable candidates, the details are omitted here.

Estimating the candidate Weil-Petersson distance

The next step is to take our candidate Weil-Petersson potentials and calculate the corresponding metrics to find the Weil-Petersson distance itself. We will only be able to do this by estimating a lower bound for the integral in such a way that the lower bound can be written as an indefinite integral, which we will show diverges. We will consider only two possible candidate polynomials, distinguished by the highest order monomials. Note that this case distinction is different from the one done in the previous subsection, as the cases in the previous subsection could lead to more than one candidate polynomial.

Case i $D_1 = 1$**,** $D_2 = 1$**,** $d = 1$

The candidate Weil-Petersson potential is given by $p(y_1, y_2) = Ay_1 + By_2$, with $A, B > 0$. It follows that the metric g_c corresponding to the candidate potential is,

$$g_c = \frac{1}{(Ay_1 + By_2)^2} \begin{pmatrix} A^2 & AB \\ AB & B^2 \end{pmatrix}. \tag{4.43}$$

We therefore find that the candidate Weil-Petersson distance is,

$$\begin{aligned} L_{cWP} &= \int_\gamma \frac{\sqrt{A^2 dy_1^2 + B^2 dy_2^2 + 2AB dy_1 dy_2}}{Ay_1 + By_2} \\ &= \int_\gamma \frac{|A dy_1 + B dy_2|}{Ay_1 + By_2} \\ &\geq \int_\gamma \frac{A dy_1 + B dy_2}{Ay_1 + By_2} \\ &= \log(Ay_1 + By_2)|_c^\infty \\ &= \infty. \end{aligned} \tag{4.44}$$

So the *candidate* of the Weil-Petersson metric diverges for case i.

Case ii $D_1 = 1$**,** $D_2 = 3$**,** $d = 3$

The candidate potential is $p(y_1, y_2) = Ay_2^3 + By_2^2 y_1 + Cy_1$, where $A, B, C > 0$. We note that in the limit as $y_1, y_2 \to \infty$ the Cy_1 term is dominated by $By_2^2 y_1$, so we can safely omit Cy_1

to keep the dominant polynomial $p(y_1, y_2) = y_2^2(Ay_2 + By_1)$, resulting in the metric

$$\begin{aligned} g_c &= \frac{1}{y_2^4(Ay_1 + By_2)^2} \begin{pmatrix} A^2y_2^4 & ABy_2^4 \\ ABy_2^4 & y_2^2(3y_2^2A^2 + 4y_1y_2AB + 2B^2y_1^2) \end{pmatrix}. \\ &= \frac{1}{(Ay_1 + By_2)^2} \left[\underbrace{\begin{pmatrix} A^2 & AB \\ AB & 3A^2 \end{pmatrix}}_{M} + \underbrace{\frac{1}{y_2} \begin{pmatrix} 0 & 0 \\ 0 & 4ABy_1 \end{pmatrix}}_{N} + \underbrace{\frac{1}{y_2^2} \begin{pmatrix} 0 & 0 \\ 0 & 2B^2y_1^2 \end{pmatrix}}_{O} \right]. \end{aligned} \tag{4.45}$$

Now note that the matrices M, N and O are all semi-positive definite, hence have determinant ≥ 0. We then have the property of the determinant that $\det(M + N + O) \geq \det(M) + \det(N) + \det(O)$. Consider now the candidate Weil-Petersson distance for any path $\gamma(s)$,

$$\begin{aligned} L_{cWP} &= \int_{\gamma(s)} \sqrt{\det(g_c)} ds \\ &= \int_{\gamma(s)} \frac{1}{Ay_1 + By_2} \sqrt{\det(M + N + O)} ds \\ &\geq \int_{\gamma(s)} \frac{1}{Ay_1 + By_2} \sqrt{\det(M) + \det(N) + \det(O)} ds \\ &= \int_{\gamma(s)} \frac{1}{Ay_1 + By_2} \sqrt{\det(M)} ds, \end{aligned} \tag{4.46}$$

filling in M from (4.45) we get,

$$\begin{aligned} &= \int_\gamma \frac{\sqrt{A^2dy_1^2 + 9B^2dy_2^2 + 2ABdy_1dy_2}}{Ay_1 + By_2} \\ &\geq \int_\gamma \frac{\sqrt{A^2dy_1^2 + B^2dy_2^2 + 2ABdy_1dy_2}}{Ay_1 + By_2} \\ &\geq \infty, \end{aligned} \tag{4.47}$$

where the last inequality follows from equation (4.44). This means that any path distance has to diverge, also the one used previously which was parametrized by s. So, putting this into equation (4.46), we find

$$L_{cWP} \geq \infty. \tag{4.48}$$

Thereby we have shown that the *candidate* Weil-Petersson metric diverges for case ii.

Note that in [4] all different polynomials are considered, we will omit them here.

Estimating the Weil-Petersson distance

Having shown that the main part of the potential causes a divergence we are ready to consider adding the perturbations. These were analyzed for the two dimensional case in equation 4.27, where one can see that all of the terms added are bounded, with an exponentially decaying factor multiplying them. We can therefore argue as above 4.33 that the terms can only contribute a finite amount before being dominated by the candidate potential. The remaining infinite path will diverge as worked out in the first part of this section.

We have then proven theorem 5.

4.6 Case V

The final case to be considered, and the only one which is not proven, is the case with two infinite divisors in r dimensions.

Case 5 (Infinite/infinite Wang conjecture in r dimensions) *If $s \in E_1 \bigcap E_2$ with E_1, E_2 infinite divisors, and $s \notin E_j$ for all $j \neq 1, 2$, let*

- *γ a path for which $y_1, y_2 \to \infty$, situated on an angular slice (i.e. $Re(z_j) = c_j$),*

then $\forall \gamma$ as above $\int_\gamma ds = \infty$, and so the Weil-Petersson distance diverges.

In this section we will note the reason why the approach outlined in section 4.2 failed for this case. In principle we would like to repeat the generalizing procedure as shown for 4.4, where we had a potential $\tilde{Q} = p(y) + \mathbf{H}$. We noted in our proof in section 4.4 that the generalization proceeds smoothly because the terms added are still *bounded*. This has everything to do with the fact that $\mathbf{H}$ decays exponentially with our one relevant parameter y.

For the case with two infinite parameters in section 4.5 we have however $\tilde{Q} = p(y_1, y_2) + p_1(y_1) + p_2(y_2) + \mathbf{H} := p(y_1, y_2) + f(y_1, y_2)$, where p_1, p_2 is not exponentially decaying in y_1, y_2 respectively, but actually a polynomial. We can consider the components of the metric as in 4.27. In the limit as $y_1, y_2 \to \infty$ we find for e.g. $-\partial_j \partial_1 \log\left(\tilde{Q}\right)$ a term of the form

$$\frac{-p(y_1, y_2)\partial_{y_1}\partial_{x_j} f(y_1, y_2)}{p(y_1, y_2)} \propto \frac{-p(y_1, y_2)\partial_{x_j} a_{1,0}(\zeta)}{p(y_1, y_2)}, \tag{4.49}$$

where $a_{n,m}(\zeta)$ is defined in (4.23), and ζ denotes the coordinates not on either infinite divisor. The first expression is reduced so far from what is given in equation (D) because of the limit we took: x derivatives of f are still exponentially decaying in y_1 and y_2, hence they drop out. Note that in section 4.5 we could also get terms not multiplied by exponentially decaying terms, however as we still had $r = 2$ the expansion coefficients were constants. They could therefore be ignored, but with the introduction of the ζ coordinates this situation is changed.

We do not know enough of the properties of the expansion coefficients $a_{n,m}(\zeta)$ to guarantee boundedness of the integral when considering paths dependent on ζ. The argument as

presented in section 4.2 therefore does not hold, and to finish the proof (even when taking just the candidate potential into account) more information is needed.

4.7 Away from the angular slice: case II

The Wang conjecture does make mention of a specific path over which the distance has to be computed. In this section we attempt to move away from the angular slice, and see what this implies for the allowed forms of the perturbation, and for the field distance that we can derive.

4.7.1 Example: exponential growth in x

We construct an example as shown by Lee [4] for attempting to generalize case II 5. As mentioned in 4.2 and can be seen from appendix D we have only two restrictions on our perturbation matrix E: the entries of E, excluding E_{11} , should decay exponentially in y_i for one i, and E has to be hermitian.

Consider the perturbation matrix E given by

$$E = \frac{1}{y_1^2}\begin{pmatrix} 0 & ie^{-y_2} \\ -ie^{-y_2} & e^{-2y_2} \end{pmatrix}, \tag{4.50}$$

and the candidate matrix,

$$M = \begin{pmatrix} 1/y_1^2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.51}$$

The perturbed metric is now given by $g = M + E$, which leads to a field distance given by

$$L_{WP} = \int_\gamma ds = \int \frac{1}{y_1}\sqrt{|dx_1 + e^{-y_2}dy_2|^2 + |dy_1 - e^{-y_2}dx_2|^2}.$$

Consider the curve $\gamma : t \to (C, t, e^t, t)$, $t > 1$,where the vector represents $(z_1, z_2) = (x_1, y_1, x_2, y_2)$. Denoting derivatives with respect to t with a prime we then have,

$$L_{WP} = \int \frac{1}{y_1}\sqrt{|x_1' + e^{-y_2}y_2'|^2 + |y_1' - e^{-y_2}x_2'|^2}dt = \int_1^\infty \frac{e^{-t}}{t}dt < \infty, \tag{4.52}$$

where we recognize the last term to be the convergent exponential integral. Note that the chosen path is exponential in x_2. This is in stark contrast with the assumption we made in our proof; namely that all paths are situated on angular slices.

This example shows us that when moving away from the angular slice, the restrictions that we can derive for the perturbation matrix E are not enough to prove case IV 4.5 anymore. We need to therefore know more about the possible perturbations to the metric. Remember that the metric perturbations arose from the polarization of our mixed variation of Hodge

structure, which had as argument the period map corresponding to our $(3,0)$-form. We would therefore need to increase our knowledge of the terms arising in the expansion of the $(3,0)$-form due to the nilpotent orbit theorem, and the exact workings of the polarization $\tilde{Q}$ if we want to tighten the restrictions on E. This requires a deeper understanding of the mixed Hodge structure.

4.7.2 Relaxing the angular slice assumption

In this section we present some new results for the case with one infinite and one finite divisor, in two dimensions.

Case 6 (Infinite/finite Wang conjecture in 2 dimensions) *Let $s \in E_1 \bigcap E_2$ with E_1 an infinite divisor and E_2 a finite divisor, parametrized by z_1 and z_2 respectively, and $s \notin E_j$ for all $j \neq 1,2$. Denote by E the hermitian 2×2 perturbation matrix of which all components decay exponentially in y_2, and let γ be any path in the moduli space ending on the singular divisor. If either of the following conditions hold*

- *$Im(E_{21}) = Im(E_{12}) = 0$,*
- *γ is a path which grows at most polynomially in the real direction,*

then $\int_\gamma ds = \infty$

Note that, because of the hermitian condition on E, having real off-diagonals is equivalent to E being symmetric. We will derive the metric *without* assuming to be on an angular slice, and give criteria for when the Weil-Petersson distance diverges independent of the path. These criteria are based on the form of the perturbation matrix E introduced in section 4.2.

Note that the angular slice is an assumption on the *path* taken towards the infinite distance singularity. The expression for the metric, in terms of a main contribution p and a perturbation matrix E derived in section 4.2 are still valid. We therefore begin with

$$g = \begin{pmatrix} 1/y_1^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & g(y_1)e^{-y_2} \\ \overline{g}(y_1,y_2)e^{-y_2} & E_{22} \end{pmatrix}, \tag{4.53}$$

where g is bounded and a polynomial in y_1, as per section 4.2.

Losing the angular slice assumption does show up when writing out the expression of the metric in terms of a basis of forms, namely, we now need to take $dx \neq 0$. This gives us

$$g = \frac{1}{y_1^2}\left|dz_1^2\right| + E_{22}\left|dz_2^2\right| + g(y_1)e^{-y_2}dz_1 d\overline{z}_2 + \overline{g}(y_1,y_2)e^{-y_2}dz_2 d\overline{z}_1. \tag{4.54}$$

Expanding $dz = dx + idy$ and rewriting leads to

$$\begin{aligned} g = &\frac{1}{y_1^2}\left(\left|dy_1 + e^{-y_2}y_1^2 g(y_1)dy_2\right|^2 + \left|dx_1 + e^{-y_2}y_1^2 g(y_1)dx_2\right|^2\right) \\ &+ y_1^2\left(E_{22}/y_1^2 - e^{-y_2}\left|g\right|^2\right)(dy_2^2 + dx_2^2) - 2e^{-y_2}\mathrm{Im}(g)(dy_1 dx_2 - dy_2 dx_1). \end{aligned} \tag{4.55}$$

Table 4.2: Current status of the Wang conjecture

			Proven?	
Case	# of infinite/finite divisors	# of dimensions	Type of path	Type of perturbation
I	1/0	r	Any	Any
II	1/1	2	Any	Symmetric
			Polyn. growth in x	Any
III	1/1	r	Angular slice	Any
IV	2/0	2	Angular slice	Any
V	2/0	r	No	No

We note that, in the above expression, the final term is the only term with a possibly negative contribution (this is shown in section 4.3 in detail). We have shown in section 4.3 that the first term by itself already leads to a divergence of the field distance: absence of the final term proportional to $\mathrm{Im}(g)$ therefore implies divergence of the field distance, and so the first condition in 6 is proven.

If we limit the possible paths to only those *not* growing exponentially in the direction of x, then we see that the factor e^{-y_2} exponentially suppresses the last term, thereby again ensuring divergence, as the negative term only contributes up to a finite cutoff in y_2. This proves the second condition in 6.

The general case considered here reduces to example 4.7.1 upon taking $E_{22} = e^{-2y_2}$ and $g = i$ (i.e. $\mathrm{Im}(g) = 1$).

4.8 Summary and outlook

In the previous parts we have reviewed the proof of Wang's conjecture 2 for some special cases, completing the steps and clarifying the assumptions made by Lee [4].

The attempted proof was a constructive one, where various mathematical techniques outlined in chapter 5 were used to expand the metric, split it into a main contribution and an exponentially decaying contribution, and use this to calculate the field distance through integration. We showed for various cases that the field distance diverges, the results are summarized in table 4.2.

The other barrier we ran into during the proof is the fact that the expansion coefficients suddenly become relevant in case V, meaning that we can no longer guarantee boundedness of the extra terms (i.e. those appearing in the r dimensional analogue of equation (4.27)) along paths that also run in the ζ direction.

It appears that due to the intricacy with which the metric is produced the resulting terms become highly non-trivial when one tries to expand them in generality. The subsequent integration to find the Weil-Petersson distance is difficult because of the large number of available paths in four real dimensions. Both the intricacy of the metric, and the failure to incorporate the perturbation matrix E to finish the proof of case 4.5 follow from the

mathematical formalism used to calculate the metric. A better understanding of the polarization of mixed variations of Hodge structures could lead us to improved restrictions on the expression for the dominant term and its perturbations. This could also tell us what form the perturbations generally take, as 6 seems to imply that, for the Wang conjecture to hold in generality, we should be able to use variations of Hodge structure to prove that the perturbation matrix is always symmetric, i.e. that $\mathrm{Im}(g) = 0$.

Chapter 5

Mathematical background

This chapter provides an overview of the mathematics used in chapter 4. The setup we consider is the same: we have a family of Calabi-Yau manifolds $\phi : X \longrightarrow B$, where B is the moduli space. In this moduli space some points lie on singular divisors; which are hypersurfaces of one dimension lower than the full space, on which a singularity is located.

We build up the mathematics needed to consider Hodge decomposition in the context of singularities. This chapter continues with the theory presented in appendix B, which was already needed for the compactification in chapter 3.

5.1 Variation of Hodge structure

In this section we closely follow [17]. One can consider a family of different Kähler manifolds M, each manifold specified by several parameters (for example the Kähler and complex structure moduli). Now remember that in string theory one is interested in the space of harmonic forms, which one can relate to the de Rham cohomology, which in turn can be studied through Hodge theory. Note that here we assume the reader to be familiar with all of these concepts, including Hodge decomposition. If that is not the case then it warrants itself to read appendix B (which was also used "behind the scenes" in chapter 3) as most of this chapter leans on these concepts.

A very interesting topic of study is then how Hodge structures change based on the parameters that specify the family of manifolds. This is precisely what this chapter is aimed at; elucidating some of the mathematical theory behind the study of *variations of Hodge structure*.

Let $\phi : X \to B$ be a map between two complex manifolds. We take ϕ to be a proper, holomorphic submersion. In this scenario B will be the moduli space parametrizing a family of Kähler manifolds. Each of these Kähler manifolds is given by the *fibre* $X_b := \phi^{-1}(b)$. By [17] we know that each X_b is a complex submanifold within X, of codimension $dim(B)$.

We now have the fibres $X_b : b \in B$ as the manifolds of which we want to study the Hodge structure.

We now use the formalism of filtrations, as outlined in appendix B, to find a map between parameters corresponding to the family of manifolds and the Hodge structure. First we shall find a map relating the fibres, i.e. a map between two manifolds X_{b_1} and X_{b_2}, for $b_1, b_2 \in B$. This map will then be used to relate a parameter $t \in B$ to a filtration, and therefore a Hodge structure.

Assume B to be contractible, and X to be C^∞-trivial[1], over B. We can then find a diffeomorphism $g_t : X = X_{t_0} \to X_t$ [17]. The pull-back of which induces an isomorphism

$$g_t^* : H^k(X_t, \mathbb{C}) \to H^k(X_{t_0}, \mathbb{C}) \tag{5.1}$$

between the de Rham cohomologies of the two manifolds X_t and X_{t_0} in our family. Now comes the step where we can relate the moduli space B to realizations of Hodge structure. This is done via the following map,

$$\mathcal{P}^p : B \to G(f^p, H^k(X, \mathbb{C})); \qquad \mathcal{P}^p(t) = g_t^*(F^p(X_t)), \tag{5.2}$$

where F^p is the Hodge filtration as defined in appendix B. In the above equation $G(f^p, H^k(X, \mathbb{C}))$ is a *Grassmannian* space. It is the set of all f^p dimensional subsets of the space $H^k(X, \mathbb{C})$; in other words it is the set of filtrations of the Hodge structure. So $\mathcal{P}^p$ relates t, the parameter for our family of manifolds, to the Hodge filtration of some basic Hodge filtration corresponding to t_0. It therefore tells us how the Hodge filtration varies as a function of t.

Based on this discussion we can now present the relevant properties of a variation of Hodge structures as it is used in this work. Note that we omit a large part of the more general definition given in [17], however for our purposes this is enough.

Let B be a connected complex manifold. A variation of Hodge structure (VHS) of weight k over B is a local system[2] of vector spaces over $\mathbb{Z}$, and the corresponding holomorphic vector bundle $\mathbb{V}$[3]. This vector bundle has an associated filtration,

$$... \subset \mathbb{F}^p \subset \mathbb{F}^{p-1} \subset ... \tag{5.3}$$

by holomorphic sub bundles $\mathbb{F}^p$ satisfying $\mathbb{V} = \mathbb{F}^p \bigoplus \overline{\mathbb{F}^{k-p+1}}$, and the property that the flat connection on $\mathbb{V}$ can also be applied to the sections of $\mathbb{F}^p$, the space of which is denoted by $\mathcal{F}^p$.

In our case the holomorphic vector bundle is the collection of de Rham cohomologies of the different Calabi-Yau manifolds, which are distinguished by the values of their parameters which lie in B, and the $\mathbb{F}^p$ are the Hodge filtrations. By placing this cohomology into a VHS we get a framework for studying the changes in the Hodge structure as parametrized by B.

[1]This means that X as a bundle allows a smooth global frame.

[2]Intuitively this just means a "collection of".

[3]This bundle is "corresponding" in the sense that it contains all of the aforementioned vector spaces as fibres.

We can also consider the changes in the Hodge numbers $h^{p,q}(X_t)$ as a function of t. It is not unconceivable that different manifolds within the same fibre have different Hodge numbers. This turns out to not be the case,

$$h^{p,q}(X_t) = h^{p,q}(X_{t_0}), \quad \forall t \in B. \tag{5.4}$$

For the full proof see [17].

5.2 Divisors

The section above introduces variations of *pure* Hodge structure, i.e. the considered spaces do not contain any singularities. We will now introduce infinite distance singularities to our moduli space, by considering them to be located on *divisors*. Given a manifold M, a divisor is a subspace N of M of *codimension* one,

$$\text{codim}(N) = \dim(M) - \dim(N) = 1. \tag{5.5}$$

These are complex dimensions; so a divisor of a complex one dimensional space is simply a point. Being codimension one locks down the direction of rotations, and one can then consider paths that move around the divisor (i.e. the point), leading to the concept of monodromies and monodromy operators. Each singularity has one monodromy operator related to it, and this operator encodes information about the singularity.

In this work we will consider all of our singularities to lie on such divisors[4].We then consider paths moving towards these divisor-singularities. In an n-dimensional complex space divisors are necessarily $(n-1)$ dimensional, and can overlap an arbitrary amount of times. Hence when moving towards a singularity there could be any number of relevant monodromy operators, depending on which divisors overlap.

5.3 Monodromy

Now that we can consider the singularities within our moduli space, we can start to learn more about the behavior around the singularities. The class of transformations that move a path around a point is given by the group of *monodromy* transformations denoted by Γ. We will now proceed to define this.

Let B be a moduli space corresponding to a family of manifolds as before. We can consider the group containing all transformations of a point $b_0 \in B$ which leave the point b_0 invariant, and call it $\pi_1(B, b_0)$[5].By considering curves on B one can then define a representation of

[4]It was proven in [18] that one can resolve any other type of singularity to a normal crossing divisor, i.e. a divisor which somehow acts nicely when intersecting other divisors.

[5]In the theory of homotopy $\pi_1(B, b_0)$ is called the *fundamental group* of B at the point b_0. It is defined as the group of all loops at b_0 (with composition of two loops meaning traveling first over one, then over the other) modulo *homotopy* (i.e. in the fundamental group any two loops which can be continuously deformed into each other have the same representative under homotopy).

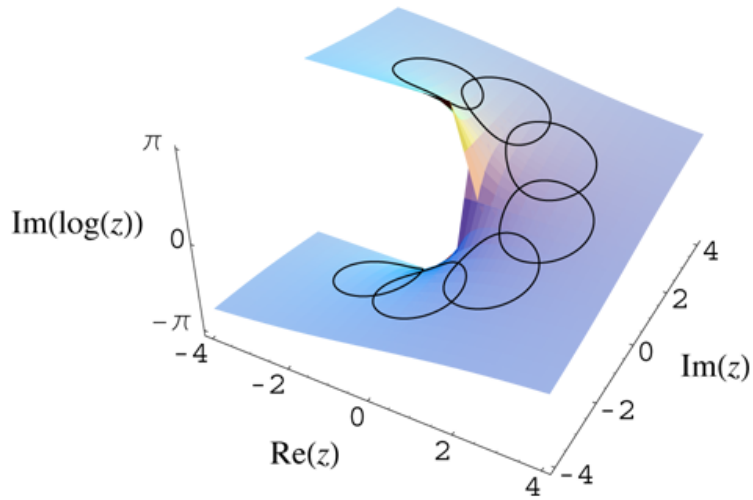

Figure 5.1: When one attempts to define the logarithm for complex numbers one finds different answers along different paths. This induces an infinite order monodromy, where the outcome of the complex logarithm is changed by a term $2\pi i$ for each rotation around the origin. Source: [19]

$\pi_1(B, b_0)$,

$$\rho : \pi_1(B, b_0) \to GL(\mathbb{V}_{b_0}), \tag{5.6}$$

where $\mathbb{V}_{b_0}$ is the fibre of the family of Hodge structures parametrized by the moduli space B. We then name ρ the *monodromy representation* and

$$\Gamma := \rho(\pi_1(B, b_0)) \subset GL(\mathbb{V}_{b_0}, \mathbb{Z}), \tag{5.7}$$

the monodromy group. Intuitively any path which begins at a point $z \in B$ and ends at the same point $z \in B$ is like the identity; applying this path to the argument of a function of z *intuitively* should not change the function value.

This is true in a lot of cases, but not when singularities are introduced into the complex space B. When one analytically continues a function $F(z)$ then moving the argument z on a loop around the singularity can change the function value. This creates a *covering space* B' of B, which intuitively consists of layers above and below B, where one moves between the layers by going around the singularity; in the same sense as using a spiral staircase.

We note that two monodromy matrices N_1, N_2 corresponding to different divisors E_1, E_2 commute [15].

In this work we keep the discussion general, and do not mention how to derive the monodromy matrices N; they depend on the type of singularity however. One can view [3] for more information.

5.4 Period map

Consider now the conceptual space D of all Hodge structures of a certain weight k, dimension n and with certain Hodge numbers; $D(\mathbb{V}, Q, k, h^{p,q})$. Here $\mathbb{V}$ signifies the collection of vector spaces defining the Hodge structure, k is the weight of the Hodge structure, Q its polarization (see appendix B) and $h^{p,q}$ the Hodge numbers. This appears as a very abstract notion, and it is. As the Hodge numbers remain constant under variations in the parameters per (5.4) we can specify all parameters that D depends on. The space D will then reduce to a collection of cohomology vector spaces, which, as complex vector spaces, can be identified with$\mathbb{C}^n$. In other words D reduces to the space of complex moduli characterizing a specific Hodge structure, where n denotes the dimension of the parameter space B.

We can now consider the following map,

$$\Phi : B \to D/\Gamma, \tag{5.8}$$

called the *period map*. The period map relates a family of Kähler manifolds (as specified by B) to a family of Hodge structures. In this definition we do not want to consider the covering spaces of the Hodge structures implied by monodromy, hence we take the quotient of all Hodge structures with the monodromy group. This quotient under the group action then "glues together" all the points which are related to each other by moving around singularities, as explained in the context of monodromies 5.3.

The name *period map* comes from the fact that a Hodge filtration F^p is also referred to as a "period". The period map therefore connects a parameter B corresponding to a type of Kähler manifold to a period, or Hodge filtration.

Within a physical context we find period maps whenever we perform a Poincaré duality. Consider for example the holomorphic $(3,0)$-form $\Omega(z)$. Through Poincaré duality we can expand $\Omega(z) = \Pi_i(z)\gamma^i$, where in this context we refer to the $\Pi_i(z) \in \mathbb{C}$ as the *periods*. The map $\Pi = (\Pi_1, ..., \Pi_n)$ is therefore a *period map*, in the sense that it is a map into coefficients of Ω as $(3,0)$-form in the cohomology space. Noting that this $(3,0)$-form, being an element of F_3, generates F_2 and F_1 leads us to consider Π as a map into the collection of cohomology vector spaces, and therefore to be a period map in the mathematical sense described above.

As we now use the space D this period map is defined in a more general way than the map $\mathcal{P}^p$ 5.2 constructed in the previous section, however their function is similar, in the sense that they both relate a parameter to a Hodge structure.

5.5 Asymptotic behaviour of Hodge structures

We have now come to the last part of this preliminary, where we introduce the structure that will be used in the rest of the thesis. This structure is called the *mixed Hodge structure*; it is the asymptotic version[6] of a variation of Hodge structure. There are several theorems,

[6]In the sense that the moduli space contains singularities.

vital for the next chapter, regarding period maps of mixed Hodge structures that will be introduced in this section.

For our purposes mixed Hodge structures are variations of Hodge structure over spaces with singularities. This changes the Hodge structure To account for this one uses the following spaces (due to Schmid [20]),

$$F^p_\infty = \lim_{z\to 0} e^{-\frac{1}{2}\pi i \log(z) N} F^p, \tag{5.9}$$

called the limiting space of F^p, where N is the monodromy operator. In the context of cohomologies one can move from one filtration to the other by applying a derivative. Now, for the limiting filtration given above, we note that the properties of the exponent give us that applying N is similar to applying the derivative. This motivates the fact that $N(F^p_\infty) \subset F^{p-1}_\infty$ [3].

The above procedure effectively filters out the singularities. However now the F^p_∞ spaces and corresponding $H^{p,q}_\infty$ no longer define a VHS. We therefore need to consider a different object, which takes into account both the Hodge structure without infinities, as the information on the singularities.

5.5.1 Monodromy theorem

It was mentioned in section 5.2 that the monodromy matrices N_i encode the properties of the singularities. In order to use them in the construction of a mixed Hodge structure we first need to mention an extra property.

For this purpose we consider now $c_1, ..., c_r$, the generators of the fundamental group $\pi_1((\Delta^*)^r)$. We denote $\gamma_j = \rho(c_j)$ the generator of the monodromy group $\rho(\pi_1((\Delta^*)^r))$. We can state our first monodromy related theorem, the proof of which is due to Landman [21] and Borel (cf. [20]):

Theorem 1 (The monodromy theorem) *For all monodromy transformations γ_j, $j = 1, .., r$, there exist integers ν_j such that $(\gamma_j^{\nu_j} - \mathrm{id})$ is nilpotent. This property is called quasi-unipotence of the γ_j. The index of nilpotency of $(\gamma_j^{\nu_j} - \mathrm{id})$ is at most $k+1$.*

In the above theorem k is the weight of the Hodge structure.

The nilpotency property of monodromy matrices allows us to use them in the definition of a new filtration, the *weight filtration*, which together with the Hodge filtration will make up our mixed variation of Hodge structure.

5.5.2 Monodromy weight filtration

Using the monodromies We can define the weight filtration made up of spaces $W_i(N_i)$ [22]. It is defined in the following way, following [17].

Let $N : B \longrightarrow B$ be a nilpotent matrix of order k, such that $N^{k+1} = 0$. Let $m \in \mathcal{Z}$, $m \leq k$. For $A \subset B$, we can define A_m to be a *Jordan block* of weight m, if A_m has an m-dimensional basis given by $f_0, f_1, ..., f_m$, and for each f_i we have $N(f_i) = f_{i+1}$, with $f_{m+1} = 0$. We redefine the basis f to a basis e,

$$e_{m-2j} := f_j. \tag{5.10}$$

Now let $U_m := \bigoplus_n A_n | n = m$ be the direct sum of all Jordan blocks of B with weight m. We can then decompose V as the sum,

$$B = \bigoplus_{m=0}^{k} U_m. \tag{5.11}$$

We can also decompose U_m further, into subspaces of U_m which are spanned by the basis vectors e^n_{m-2j} as n runs over all Jordan blocks of weight m. This gives,

$$U_m = \bigoplus_{j=0}^{m} U_{m,m-2j}. \tag{5.12}$$

Finally this leads us to the last space,

$$E_l = E_l(N) = \bigoplus_{m=0}^{k} U_m, l. \tag{5.13}$$

It is these spaces that will make up our weight filtration.

For the decomposition (5.13) we have per [17]

1. $N(E_l) \subset E_{l-2}$
2. For $l \geq 0$, $N^l : E_l \longrightarrow E_{-l}$ is an isomorphism.

For the first statement we used that $N(f_j) = f_{j+1}$ for f_j the basis of a Jordan block.

Now via this decomposition we can define the weight filtration. Let N be a nilpotent matrix with index k. We need the matrix N to be rational; $N \in \mathfrak{gl}_{\mathbb{Q}}$ [17]. Then for a vector space B there exists a unique, increasing filtration $W = W(N)$,

$$0 \subset W_{-k} \subset W_{-k+1} \subset ... \subset W_{k-1} \subset W_k = B, \tag{5.14}$$

with

1. $N(W_l) \subset W_{l-2}$
2. For $l \geq 0$, $N^l : \mathrm{Gr}^W_l \longrightarrow \mathrm{Gr}^W_{-l}$, with $\mathrm{Gr}^W_l := W_l/W_{l-1}$, is an isomorphism.

This filtration is called the *monodromy weight filtration*. Note that we can apply the above to the cohomology spaces, thereby getting a Hodge monodromy weight filtration to contrast the Hodge filtration.

5.5.3 Mixed Hodge structure

If we combine the two structures, i.e. the limiting Hodge filtration carrying information on the Hodge structure (5.9) and the weight filtration with information on the singularities, then we can define what is called a *mixed of Hodge structure*, which encompasses all the relevant information.

Following [17] we take V a vector space over $\mathbb{C}$. A mixed Hodge Structure (MHS) on V consists of a pair of filtrations of V; (W, F), where W is increasing and F is decreasing, such that F induces a Hodge structure of weight k on $\mathrm{Gr}_k^W := W_k/W_{k-1}$ for each k.

In the case of the de Rham cohomologies and monodromies on the moduli space, the filtration W is of course the weight filtration as in (5.5.2) and F is the Hodge filtration as in appendix B.

In the context of an MHS it is not possible anymore to work with the polarization as defined in appendixB, as the Hodge structure is now given in terms of Gr_k, which also depends on the weight filtration. In other words, there is now much more structure that the polarization needs to account for. Following [15] as written in [17], we therefore define the *polarized mixed Hodge structure* (PMHS) as below.

A polarized MHS of weight $k \in \mathbb{Z}$ on a complex vector space V is composed of an MHS (W,F) on V as above, a morphism $N \in \mathfrak{g} \bigcap \}\updownarrow_{\mathbb{Q}}$ (where $\mathfrak{g}$ is the Lie algebra of the monodromy group) and a nondegenerate, rational bilinear form Q where

1. $N^{k+1} = 0$,
2. $W_l = W(N)_{l-k}$,
3. $Q(F^a, F^{k-a+1}) = 0$,
4. the Hodge structure of weight $k + l$ that F induces on

$$\ker\left(N^{l+1} : \mathrm{Gr}_{k+l}^W \to \mathrm{Gr}_{k-l-2}\right) \tag{5.15}$$

   is polarized by $Q(\cdot, N^l \cdot)$.

The metric on the moduli space

On a polarized MHS we can use the polarization to define what is called the *Weil-Petersson* metric on the moduli space. In the specific context of the Calabi-Yau manifold Todorov and Tian[23][24] showed that the Weil-Petersson metric can be written in terms of the unique $(n, 0)$-form. It can even be rewritten significantly to[23],

$$\omega_{WP} = \frac{\sqrt{-1}}{2} \partial\overline{\partial} \log \tilde{Q}(\Omega, \overline{\Omega}) \tag{5.16}$$

Griffiths then showed [16] that the Weil-Petersson metric is semi-positive definite, actually making it into a well-defined metric on the moduli space.

5.5.4 Period maps

Remember the concept of a period map introduced in section 5.4. We will now study the effect that the presence of singularities has on the period maps. Consider a family of Kähler manifolds $\phi : X \to B$ as before. In the case where B has one or more singularities at a point we can consider an open disk of radius L around the singular point. Here r denotes the number of overlapping divisors, i.e. the number of singularities, within the considered disk. With the singularity removed we find

$$B \supset U = (\Delta^*)^r \times \Delta^{n-r} \tag{5.17}$$

where $\Delta^* := z \in C : 0 < |z| < L$ and $\Delta := z \in C : 0 \leq |z| < L$. We can then proceed to study period maps in the context of the polarized mixed Hodge structure that is defined by the cohomology of the Calabi-Yau space over this open $U \subset B$. This lets us define period maps,

$$\Phi : (\Delta^*)^r \times \Delta^{n-r} \to D/\Gamma. \tag{5.18}$$

Noting that the universal cover of Δ^* is given by $H = z \in \mathbb{C} : \mathrm{im}(z) > 0$ we can also lift the period maps to the universal cover,

$$\tilde{\Phi} : H^r \times \Delta^{n-r} \to D. \tag{5.19}$$

Note that in the lifting process we chose to consider all the copies of U under the monodromy operations, hence our period map is now lifted to a map with image D.

Using the monodromy of our space we can say a lot about the lifted period map of our mixed Hodge structure. This is done in the context of *Nilpotent orbits*.

5.5.5 Nilpotent orbits

From this point on forward we make some simplifying assumptions as per [17], which will not reduce our level of generality. We will take the number of overlapping divisors $r = n$, and assume that the monodromy transformations γ_j are quasi-unipotent (as explained in (1)) with $\nu_j = 1$. This means that they are actually *unipotent*, by definition.

Consider $\mathfrak{g}$ the Lie algebra of our monodromy group. We write,

$$\gamma_j = e^{N_j}; \qquad j = 1, .., r, \tag{5.20}$$

here the N_j are *nilpotent* (not unipotent) elements of order $k + 1$ in $\mathfrak{g}$. For a lifted period map $\tilde{\Phi}$ we then find,

$$\tilde{\Phi}(z_1, ..., z_j + 1, ..., z_r) = e^{N_j} \cdot \tilde{\Phi}(z_1, ..., z_j, ..., z_r). \tag{5.21}$$

This means that the monodromy operator γ_j rotates the j'th argument of the period map once around the singularity. Consider now $\check{D}$ to be the space of all Hodge filtrations (also

referred to as the dual of the classifying space D). We can then define some variations of the mapping Φ which will be useful in stating the *Nilpotent Orbit Theorem*. We can define

$$\Psi(z_1, ..., z_r) := \exp\left(-\sum_{j=1}^{r} z_j N_j\right) \cdot \tilde{\Phi}(z_1, ..., z_j + 1, ..., z_r), \tag{5.22}$$

which is in fact the lifting of a different holomorphic map $\psi : (\Delta^*)^r \to D$;

$$\psi(t_1, ..., t_r) = \Psi(\frac{\log t_1}{2\pi i}, ..., \frac{\log t_r}{2\pi i}). \tag{5.23}$$

We can now state the following theorem as per [20] :

Theorem 2 (Nilpotent Orbit Theorem) *Let $\Phi : (\Delta^*)^r \times \Delta^{n-r} \to D$ a period map, and $N_1, ..., N_r$ the monodromy logarithms*[7] *and $\psi : (\Delta^*)^r \times \Delta^{n-r} \to D$ as defined in* (5.23)*. Then*

1. *The map ψ extends holomorphically to $(\Delta^*)^r \times \Delta^{n-r}$*
2. *For each $w \in \Delta^{n-r}$, the map $\theta : \mathbb{C}^r \times \Delta^{n-r} \to \check{D}$ given by*

$$\theta(z, w) = \exp\left(\sum_{j=1}^{r} z_j N_j\right) \cdot \psi(0, w)$$

   is a nilpotent orbit[8]*. Also if there exists a compact $C \subset \Delta^{n-r}$ with $w \in C$ then there exists a constant $\alpha \geq 0$ such that for $Imz_j > \alpha$ for all $1 \leq j \leq n$ we have $\theta(z, w) \in D$.*

3. *For any G-invariant distance d on D there exist positive constants β, K, such that, for $Im(z_j) > \alpha$ for all $1 \leq j \leq n$ we have*

$$d(\Phi(z, w), \theta(z, w)) \leq K \sum_{j=1}^{r} (Im(z_j))^{\beta} \exp\left(-2\pi Im(z_j)\right).$$

Furthermore, the constants α, β, K only depend on the choice of d and the weight and Hodge number used to defining D. The constants may be chosen uniformly for w in a compact subset $C \subset \Delta^{n-r}$.

This theorem tells us that any period map Φ can asymptotically be approached by the map θ, which is properly defined because the first argument of the map ψ can be holomorphically extended to include the origin. Hence, because N_j is nilpotent, and the expansion will truncate, we can actually express every period map near a singularity as a polynomial in the z_j.

Note that the expression given by θ only depends on the r monodromy matrices $N_j, j \in \{1, ..., r\}$, that cross at the singularity. The nilpotent orbit is independent of the non-singular directions.

[7] In the sense that $N_j = \log \gamma_j$.

[8] That is, the fact that θ can be expanded as such, in terms of nilpotent matrices, is the definition of being a nilpotent orbit.

Chapter 6

Conclusion

In this work we studied field distances in the moduli space of type IIB compactifications on a Calabi-Yau manifold. This study was motivated by the Swampland Distance Conjecture [1], a conjecture stating that at any infinite distance singularity in the moduli space an infinite tower of massless states will appear, rendering the effective theory unphysical. We aimed to enlighten the workings behind it. As such the Wang conjecture [2] was studied, which states that an infinite order monodromy will lead to an infinite field distance. It has already been proven by Wang [2] that, in the presence of a singular divisor, a point at infinity has an infinite order monodromy associated to it. In other words, Wang showed that an infinite order monodromy is a necessary requirement for a divergent field distance. In this work we studied whether an infinite order monodromy is also sufficient. Or, in the context of the Swampland Distance Conjecture, whether an infinite tower of massless states is also a sufficient condition for having a divergent field distance[1].

In order to study this we began by revisiting and clarifying a proof due to Lee [4]. Following his lead we used the special geometry of the moduli space as worked out in chapter 3 to express the field distance in terms of the unique $(3,0)$-form $\Omega(z)$. We then applied the nilpotent orbit theorem 2 to write $\Omega(z)$ as an exponent in the monodromy matrix N, after which we could make use of the nilpotent properties of N to let the corresponding Taylor series truncate, leaving a polynomial to be studied further.

This set us up to study some simpler cases of the Wang conjecture, characterized by the number of infinite/finite divisors and the number of dimensions of the moduli space. We follow the proof by Lee and expand on some of the arguments made. As described in section 4.2 the presented proof hinges on the analyzing the metric in terms of a dominant term, and an exponentially decaying perturbative term. The following results [4] are shown,

- In the presence of one infinite divisor the Wang conjecture holds for arbitrary dimensions (theorem 1).
- In the presence of one infinite divisor and one finite divisor the Wang conjecture holds

[1] It is important to note again that, while the link between infinite towers of states and infinite monodromies has been motivated [3], it has not been proven.

only for paths situated on angular slices, in arbitrary dimensions (theorem 3).

- In the presence of two infinite divisors the Wang conjecture holds only for for paths situated on angular slices, in a two dimensional moduli space (theorem 5).

After having analyzed the work by Lee [4] we proceed to consider the angular slice assumption for the case with one infinite divisor and one finite divisor in two dimensions. In section 4.7 we show an example due to Lee that, for a general perturbation, a path growing exponentially in the real direction can offset the growth in the imaginary direction, leading to a finite field distance.

We then use the same formalism to derive a new result, namely a criterion based on the perturbation for the path divergence

- In the presence of one infinite divisor and one finite divisor in a two dimensional moduli space the Wang conjecture holds for a perturbation matrix with *real* off diagonal elements, for a general path (theorem 3).

In addition we show that

- for one infinite divisor and one finite divisor in two dimensions, for a general perturbation matrix, it is not necessary to be on the angular slice: demanding at most polynomial growth in the real direction is sufficient to show the Wang conjecture. (theorem 3)

The relevance of these results was shown to come from the Lyth bound in inflation, which would relate the existence of tensor modes in the CMB to large field inflation. The discovery of such tensor modes would imply that when realizing inflation from string theory this bound has to be obeyed, demanding a minimum value to be traversed in the inflaton field space. We then used the type IIB string theory compactification to motivate the emergence of complex moduli scalars from string theory, these scalars could be candidates for the inflaton. Finally, the many failed attempts at realizing inflation through complex moduli lead us to consider the Swampland Distance Conjecture, bringing us to a full circle.

For the cases considered the Wang conjecture now motivates the relation between infinite monodromies and infinite distances. It is however painfully clear that an important ingredient is still missing, namely the relation between infinite monodromies and an infinite tower of states. Without this, the results presented here cannot be related to a physical theory. In this work we have thus only managed to scratch the surface by showing how and when the field distance diverges for up to two dimensions. In the future we would like to use these results to constrain the possible field distance in saxion, and even axion inflation.

Linking to actual physics comes with an extra caveat, namely specifying the cut-off at which the effective theory breaks down. In this work, no mention was made of the cut-off that the Wang conjecture could imply for a physical theory: the discussion in this work has only considered field distances for singularities at infinity. The reason for this is that the cut-off for a theory depends on at which mass the new states need to be taken into account. As we have also ignored the link between infinite monodromies and states, the cut-off of the theory was especially beyond the scope of this work.

A lot of progress can also still be made in relaxing the angular slice assumption in higher dimensions, in generalizing to more dimensions and more divisors and of course in proving the most important part in linking this work to actual inflationary theories: proving the connection between infinite monodromies and infinite towers of states.

While this review is far from exhaustive on the topic of string cosmology, we hope nonetheless that this work provides a clear overview of the study of infinite distances in the moduli space, and that the results shown convey how promising the usage of variations of Hodge structure is to study the inner workings of the Swampland Distance Conjecture.

Acknowledgements

I would like to take this small portion of my thesis to thank my fellow students for studying with me every day for the past year, and for the many fruitful (and not so fruitful) discussions. There were many times when I felt confused or frustrated and talking to you always made me feel better; either because you shared my feelings, or because you could alleviate them.

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Appendix A

Basics of cosmology and inflation

Here we will present some equations and definitions necessary in the study of inflation in the context of field theory. In our discussion of inflation we will use natural units

$$c = \hbar := 1. \tag{A.1}$$

The reduced Planck mass is used

$$M_{pl} = (8\pi G)^{-\frac{1}{2}} . \tag{A.2}$$

The flat background metric for our Minkowski space is,

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.3}$$

where throughout this text μ, ν are used to denote spacetime indices (note that the greek indices α, β also see different uses within the context of string compactification).

FLRW spacetime

In cosmology it is assumed that the universe is homogeneous and isotropic. In the large scale limit this leads to the Friedmann-Lemaître-Robertson-Walker (FLRW) metric,

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right). \tag{A.4}$$

The scale factor $a(t)$ describes the expansion of the universe. The parameter k corresponds to the curvature of the universe, $k = -1, 0, 1$. This expression of the FLRW metric (A.4) uses *comoving coordinates*, which means that during the expansion of the universe the location of an object in space has fixed r, θ, ϕ as long as there are no external forces acting on it.

The Hubble parameter

The Hubble parameter

$$H = \frac{\dot{a}}{a} \tag{A.5}$$

characterizes the expansion rate. It is positive for an expanding universe and negative for a collapsing universe. One uses it as the main scale of the FLRW universe, in the sense that H^{-1} is the characteristic time scale of a homogeneous universe, and in natural units also the characteristic length scale called the *Hubble radius*.

If we want to define the *comoving Hubble radius* we need to divide out the scale factor, giving us $(aH)^{-1}$.

Composition of the universe

The constituents of the universe collectively determine the energy density and pressure. We therefore denote by ρ and p the sum of all parts making up the entire universe,

$$\rho := \sum_i \rho_i, \quad p := \sum_i p_i. \tag{A.6}$$

We can define the *critical energy density* ρ_c through,

$$\frac{1}{\rho_c}\left(\sum_i \rho_i + \rho_k\right) = 1, \tag{A.7}$$

where ρ_k is the energy density associated to curvature, evaluated today. From the Friedmann equation one can find that $\rho_c = 3H_0^2$ with H_0 the current value for the Hubble parameter.

We can then define the ratio of the energy density at the current time and the critical energy density

$$\Omega_i := \frac{\rho_i}{\rho_c} \tag{A.8}$$

In these terms we can rewrite the Friedmann equation (A.14),

$$\left(\frac{H}{H_0}\right)^2 = \sum_i \Omega_i a^{-3(1+w_i)} + \Omega_k a^{-2} \tag{A.9}$$

with $\Omega_k = -k/a_0^2 H_0^2$ the critical density for curvature.

We can combine the continuity equation (A.15) and the Friedmann equation (A.14) to find the *acceleration* equation

$$\dot{H} + H^2 = -\frac{1}{6}(\rho + 3p). \tag{A.10}$$

Evaluating this equation at $t = t_0$, i.e. today, we find

$$\frac{1}{a_0 H_0^2} \frac{d^2 a}{dt^2}\bigg|_{t=t_0} = -\frac{1}{2} \sum_i \Omega_i (1 + 3w_i), \tag{A.11}$$

the equation which determines whether accelerated expansions takes place today.

Einstein equations and Dynamics

The dynamics of the FLRW-universe are found through the Einstein equations

$$G_{\mu\nu} = 8\pi G T\mu\nu. \tag{A.12}$$

We can assume the universe to be a perfect fluid, with stress energy tensor

$$T^{\mu}{}_{\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}, \tag{A.13}$$

where ρ and p define the proper energy density and prsure of the fluid in rest frame. This stress energy tensor greatly simplifies the Einstein equations, leading to the *Friedmann equation*,

$$H^2 = \frac{1}{3}\rho - \frac{k}{a^2} \tag{A.14}$$

which can be supplemented with the continuity equation,

$$\frac{d\rho}{dt} + 3H(\rho + p) = 0. \tag{A.15}$$

We can define the *equation of state* parameter,

$$w := \frac{p}{\rho} \tag{A.16}$$

which relates the pressure and density to each other in terms of a parameter w, which is known for some specific cases: for a universe with only matter, radiation or a cosmological constant we have $w = 0$, $w = -1/3$ and $w = -1$ respectively. The equations (A.14), (A.15) and (A.16) can be combined to calculate the scale factor,

$$a(t) \propto \begin{cases} t^{2/(3(1+w))} & w \neq -1 \\ e^{Ht} & w = -1, \end{cases} \tag{A.17}$$

where H is taken to be the Hubble constant.

The main goal of inflation is to have a *decreasing Hubble radius* at very early times. This solves for example the horizon and flatness problems. In order to achieve this we need to

have an epoch of very rapid expansion, and this expansion is referred to as "inflation." As fundamental definition of inflation we therefore take

$$\frac{d}{dt}\left(\frac{1}{\dot{a}}\right) = \frac{d}{dt}\left(\frac{1}{aH}\right) < 0. \tag{A.18}$$

It is however possible to relate the above definition to the condition of *accelerated expansion*,

$$\frac{d}{dt}\left(\frac{1}{aH}\right) = -\frac{\ddot{a}}{(aH)^2}. \tag{A.19}$$

From the above equation we can clearly see that for the Hubble radius to be shrinking we need $\ddot{a} > 0$. The acceleration equation (A.10) gives us a way in which to relate this second derivative of a to the first derivative $\dot{a}$,

$$\frac{\ddot{a}}{a} = H^2(1-\epsilon), \quad \text{with } \epsilon := -\frac{\dot{H}}{H^2}. \tag{A.20}$$

In the above expression we have introduced a new parameter ϵ called the *slow roll parameter*, which is useful to consider in the context of inflation. The requirement of accelerated expansion directly translates to the condition

$$\epsilon = -\frac{\dot{H}}{H^2} = -\frac{d\log H}{dN} < 1, \tag{A.21}$$

with $dN = Hdt = d\log a$ measures the number of *e-folds*, i.e. the number of times the scale factor $a(t)$ has increased by a factor of e.

The components of the universe also predict whether or not expansion takes place. This is done through the equation of state parameter w. By combining equations (A.10) and (A.14) we find the relation

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{1}{6}\left(\rho + 3p\right). \tag{A.22}$$

Inflation in field theory

We would now like to realize the above mentioned behaviour of the scale factor from a field theory perspective. The simplest way of doing this is to link the scale factor to some scalar field ϕ called the *inflaton*. We can then write down a Lagrangian for ϕ, whose equations of motion we can relate to $a(t)$ (we will note later how $a(t)$ and ϕ are related). The inflaton has as a role to parametrize the evolution of the "inflationary energy". We will not yet think about the physical origin of this field.

In this section we proceed to write down the action for the inflaton, which we vary to obtain the stress energy tensor. Assuming again the FLRW-metric (A.4) and a homogeneous and isotropic universe we know the stress energy tensor to be that of a perfect fluid, with pressure and density specified by the inflaton ϕ. Via the Friedmann equations (A.14), relating $a(t)$ to the properties of this perfect fluid, we can then link the inflaton ϕ to the scale factor $a(t)$.

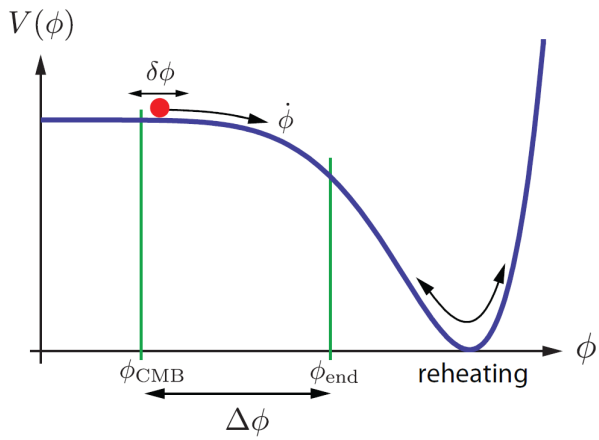

Figure A.1: An example of the inflationary potential. Note how inflation takes place in between ϕ_{CMB} and ϕ_{end}, after which the inflaton rolls down, ending inflation. Source: [6], figure 10.

The action for the inflaton is given by

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) =: S_{EH} + S_\phi. \tag{A.23}$$

The energy momentum tensor corresponding to ϕ is

$$T_{\mu\nu}(\phi) := -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\phi\partial_\sigma\phi + V(\phi)\right) \tag{A.24}$$

and the equation of motion is

$$\frac{\delta S_\phi}{\delta\phi} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) + v_{,\phi}, \tag{A.25}$$

where we take $V_{,\phi} := \partial V/\partial\phi$. For an FLRW-metric (A.4) in a homogeneous and isotropic universe this stress energy tensor assumes the form of a perfect fluid with[6]

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{A.26}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - v(\phi). \tag{A.27}$$

In order to connect ϕ to inflation we calculate the state parameter,

$$w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 - v(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{A.28}$$

Depending on the chosen potential $V(\phi)$ it is therefore possible to achieve $w < -1/3$, leading to accelerated expansion and therefore inflation. For the inflaton in an FLRW-geometry we find the following Friedmann and continuity equations,

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi}, \qquad H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \tag{A.29}$$

The first equation is akin to a harmonic oscillator with a friction term $3H\dot{\phi}$.

We can now express the slow roll parameter ϵ in terms of ϕ,

$$\epsilon = \frac{3}{2}(w_\phi + 1) = \frac{1}{2}\frac{\dot{\phi}^2}{H^2}. \tag{A.30}$$

Remember that accelerated expansion occurs if $\epsilon < 1$. The extreme case is given by $\epsilon \longrightarrow 0$, which happens in the *de Sitter* limit, $p_\phi \longrightarrow -\rho_\phi$. In this case we have that

$$\dot{\phi}^2 \ll V(\phi). \tag{A.31}$$

We will now derive some useful conditions for inflation in terms of so-called slow roll parameters. Remember that one of the reasons for adopting the theory of inflation was the horizon problem. In order for this problem to be solved we need the scale factor to change by a certain amount; otherwise points at a large distance now would still not have been in causal contact early on [6]. It turns out that we need $a(t)$ to increase by about 40-60 e-foldings to achieve this. The inflaton therefore needs to source inflation for a long enough time, since as the value for $\dot{\phi}$ changes the value for ϵ changes too, causing the inflationary period to end.

The constraint that inflation has to last long enough is expressed in terms of $\ddot{\phi}$,

$$\left|\ddot{\phi}\right| \ll \left|3H\dot{\phi}\right|, \ \ |V_{,\phi}|. \tag{A.32}$$

This constraint can then be recast in terms of a second slow-roll parameter η,

$$\eta = -\frac{\ddot{\phi}}{H\phi} = \epsilon - \frac{1}{2\epsilon}\frac{d\epsilon}{dN}. \tag{A.33}$$

The *slow roll* conditions are then $\epsilon, |\eta| < 1$.

We can rewrite both of these conditions also in terms of the inflationary potential. In order to do this we introduce the *potential slow-roll parameters* ϵ_v and η_v

$$\epsilon_v(\phi) \ := \ \frac{M_{pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2, \tag{A.34}$$

$$\eta_v(\phi) \ := \ M_{pl}^2\frac{V_{,\phi\phi}}{V}, \tag{A.35}$$

which under the slow-roll approximation are related to the *Hubble slow roll parameters* ϵ and η through

$$\epsilon \approx \epsilon_v, \quad \eta \approx \eta_v - \epsilon_v. \tag{A.36}$$

Using the above relations we can now relate the potential $V(\phi)$ to H and the inflaton ϕ in the slow-roll approximation

$$H^2 \approx \frac{1}{3}V(\phi) \tag{A.37}$$

$$\dot{\phi} \approx -\frac{V_{,\phi}}{3H}, \tag{A.38}$$

in this regime we find

$$a(t) \sim e^{Ht}. \tag{A.39}$$

Now as mentioned before inflation has to end at some time. We can express this moment in terms of the slow roll conditions, as inflation ends when these conditions are no longer valid

$$\epsilon(\phi_{end}) = 1, \quad \epsilon_v(\phi_{end}) \approx 1. \tag{A.40}$$

The value of the inflaton field ϕ thus changes with a value $\Delta\phi = \phi_{end} - \phi_{start}$. Using the quantity $N(\phi) := \log(a_{end}/a)$ we can calculate the number of e-folds, which in the slow-roll approximation can be expressed as[6]

$$N(\phi) = \int_{\phi_{end}}^{\phi} \frac{d\phi}{\sqrt{2\epsilon}} \approx \int_{\phi_{end}}^{\phi} \frac{d\phi}{\sqrt{2\epsilon_v}}, \tag{A.41}$$

where we made us of equation (A.37). As mentioned before the total number of e-folds between the creation of CMB fluctuations and the end of inflation, $N(\phi_{CMB})$ has to exceed approximately 60 for the horizon and flatness problems to be solved,

$$N_{tot} := \log \frac{a_{end}}{a_{start}} \gtrsim 60. \tag{A.42}$$

Gravity and Weyl rescaling

We have the Ricci scalar and tensor given by

$$R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho} \tag{A.43}$$
$$R = g^{\mu\nu} R_{\mu\nu}. \tag{A.44}$$

A Weyl rescaling with parameter Ω is given by,

$$g_{\mu\nu} = \Omega^{-2} \tilde{g}_{\mu\nu} \tag{A.45}$$
$$g^{\mu\nu} = \Omega^{2} \tilde{g}^{\mu\nu} \tag{A.46}$$
$$\sqrt{-g} = \Omega^{-d} \sqrt{-\tilde{g}} \tag{A.47}$$

which affects the Ricci scalar as,

$$\int dx^d \sqrt{-g} \Omega^{d-2} R = \int dx^4 \sqrt{-\hat{g}'} \left(R + (d-1)(d-2) \left(\frac{\partial \omega}{\Omega} \right) \right). \tag{A.48}$$

We can also consider the effect Weyl rescaling has on a metric independent p-form, which we will denote by F_p. Such a term will arise in our action as,

$$F_p \wedge *F_p = \frac{1}{\sqrt{-g} p^2 (d-p)!} F_{m_1 \ldots m_p} F_{n_1 \ldots n_p} \epsilon^{m_1 m_2}{}_{m_3 \ldots m_{10}} dy^{m_1} \ldots dy^{m_1 0}. \tag{A.49}$$

We note that because of the $(d-p)$ low indices on the Levi-Civita symbol

$$\epsilon^{m_1 m_2}{}_{m_3 \ldots m_{10}} = \underbrace{g_{m_3 n_3} \cdots g_{m_{10} n_{10}}}_{d-p} \epsilon^{m_1 m_2 n_3 \ldots n_{10}} \tag{A.50}$$

which, combined with equation (A.45) tells us how the kinetic term for a p-form F_p transforms:

$$F_p \wedge *F_p = \Omega^{2p-d} \tilde{F}_p \wedge *\tilde{F}_p. \tag{A.51}$$

Appendix B

Basics of differential geometry

We will assume notions like the differential p-form, the exterior derivative and the tangent bundle to be known. One can read more about these concepts in for example[25].

In this appendix we will recall some notions of differential geometry, like the Hodge star, the inner product of p-forms complex manifolds, Kähler manifolds and Hodge decomposition. The information in this appendix was largely drawn from [26].

The Hodge star

Let M a smooth m-dimensional manifold, $\epsilon_{\mu_1\mu_2\ldots\mu_m}$ the Levi-Civita symbol and $\omega \in \Omega^r(M)$. In the presence of a metric/volume form g we can define the *Hodge star* operator to be[27],

$$*\omega = \frac{\sqrt{g}}{r!\,(m-r)!}\omega_{\mu_1\mu_2\ldots\mu_r}\epsilon^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_{r+1}\ldots\nu_m}dx^{\nu_{r+1}} \wedge \ldots \wedge dx^{\nu_r}. \tag{B.1}$$

The Hodge star is an isomorphism $* : \Omega^r(M) \to \Omega^{m-r}(M)$; this means that any r-form can be equally well described by an $(m-r)$-form. For a Riemannian manifold we also have,

$$**\omega = (-1)^{r(m-r)}\omega. \tag{B.2}$$

Inner product of forms

We can consider two forms $\omega, \eta \in \Omega^r(M)$. Then $\omega \wedge *\eta \in \Omega^m(M)$, and hence the integral of $\omega \wedge *\eta$ over M is well defined. We can then define the inner product as,

$$(\omega, \eta) = \int_M \omega \wedge *\eta \tag{B.3}$$

$$= \frac{1}{r!}\int_M \omega_{\mu_1\mu_2\ldots\mu_r}\eta^{\mu_1\mu_2\ldots\mu_r}\sqrt{g}dx^1\ldots dx^m. \tag{B.4}$$

Note that $(\omega, \eta) = (\eta, \omega)$ as

$$\int_M \omega \wedge *\eta = \int_M \eta \wedge *\omega \tag{B.5}$$

The Hodge star and the adjoint exterior derivative

Note that the exterior derivative is a map $d : \Omega^p(M) \to \Omega^{p+1}(M)$. Using the Hodge star we can define the *adjoint* exterior derivative $d^* : \Omega^{p+1}(M) \to \Omega^p(M)$,

$$d^* = (-1)^{mr+m+1} * d * . \tag{B.6}$$

Consequently we have the alternative definition of the adjoint exterior derivative,

$$(d\omega, \eta) = (\omega, d^*\eta). \tag{B.7}$$

The Laplacian

In the context of differential geometry we use the adjoint of the exterior derivative d^* to define the Laplace-Beltrami operator,

$$\Delta = dd^* + d^*d. \tag{B.8}$$

Now let ω a p-form. We note that using (B.7) we can show

$$\begin{aligned}(\Delta\omega, \omega) &= (dd^*\omega, \omega) + (d^*d\omega, \omega) \\ &= |d^*\omega|^2 + |d\omega|^2 ,\end{aligned} \tag{B.9}$$

hence

$$\Delta\omega = 0 \Longleftrightarrow d^*\omega = 0, d\omega = 0, \tag{B.10}$$

proving that a form is harmonic if and only if it is closed and co-closed.

Cycles and homology

Homology classifies manifolds by their *cycles*. Each cycle of M is a closed submanifold which cannot be continuously deformed into another cycle. Consider for example the circle around the hole of the torus and the circle one can make around the actual tube making up the torus. Because one encompasses a hole and the other does not, these cannot be deformed into each other. A torus therefore has 2 cycles. One can capture the information of the number of cycles of a manifold in the *homology class*.

Let Z_p be the set of cycles, and B_p the set of cycles which are themselves boundaries of other cycles. We define the p-th homology group of M to be

$$H_p = Z_p/B_p. \tag{B.11}$$

The elements of H_p are equivalence classes of cycles under the operation which adds the boundary of a submanifold to the cycle.

Forms and cohomology

We can also consider the space of *closed* p-forms $Z^p = \omega_p | d\omega_p = 0$ and the set of *exact* p-forms $B^p = d\omega_{p-1}$. In words the closed forms are all forms which are mapped to zero by the exterior derivative, and the exact p-forms are all those forms which can be written as the exterior derivative acting on a $p-1$-form. Since the exterior derivative is nilpotent with order 2 we have that $B^p \subset Z^p$. The p-th *De Rham cohomology* is then defined to be

$$H^p = Z^p / B^p. \tag{B.12}$$

There is a clear analogy with the homology class (hence the name), where the cycles correspond to closed forms. The elements of H^p are equivalence classes of closed forms up to exact forms, i.e. $\omega_p \simeq \omega_p + d\eta_{p-1}$, which are called cohomology classes and are denoted by $[\omega_p]$. In this example ω_p would be the *representative* of the equivalence class.

Poincarè duality

The analogy between homology and cohomology classes is realized by the Poincaré duality, which states that given a p-cycle a there exists a closed $(m-p)$-form α which is called the Poincaré dual of a such that for any *closed* p-form ω we have

$$\int_a \omega = \int_M \alpha \wedge \omega. \tag{B.13}$$

The form ω is closed, therefore α is defined up to an exact form.

Harmonic forms

Hodge theory (which is introduced in this section) then enables one to study these cohomology classes as solutions of differential equations, by linking the cohomology class to so called *harmonic forms*,

Definition 1 *A harmonic form is a form $\omega_k \in \Omega^k(M)$ for which*

$$\Delta \omega_k = 0. \tag{B.14}$$

The space of harmonic k-forms is denoted $\mathcal{H}^k_\Delta(M)$.

The Laplacian here is defined in terms of the *exterior derivative*. These harmonic forms are related to the De Rham cohomology through the Hodge theorem, which states that

Theorem 3 *the space of Harmonic k-forms on M and the k-th de Rham cohomology group are isomorphic as vector spaces,*

$$\mathcal{H}^k_\Delta(M) \cong H^k(M). \tag{B.15}$$

This means that each equivalence class of closed forms modulo exact forms (cohomology class) has a harmonic representative through Hodge theory. One can also formulate the Hodge theorem as Hodge decomposition,

Theorem 4 *any differential form ω_k on a closed Riemannian manifold can be written as,*

$$\omega_k = d\alpha_{k-1} + d^*\beta_{k+1} + \gamma_k. \tag{B.16}$$

where d^* is the adjoint operator of d, α_{k-1} is a $k-1$ form on M, β_{k+1} is a $k+1$ form on M and γ_k is a harmonic k-form.

Complex structure

We will now look at the implications a complex structure has for the composition of a manifold and its tangent space. For more details see [26]. Let M be a complex manifold of dimension n, and so a real manifold of dimension $m = 2n$. We note that we can decompose a point $z \in M$ as

$$z^j = x^{2j-1} + ix^{2j}, \quad \overline{z}^j = x^{2j-1} - ix^{2j}, \quad j = 1, ..., n, \tag{B.17}$$

where we use $i = \sqrt{-1}$.

We can decompose the manifold tangent space in this same way to find,

$$T_{\mathbb{C}}(M) = T^{1,0}(M) \bigoplus T^{0,1}(M). \tag{B.18}$$

Intuitively this is the same as splitting a complex number into a real and imaginary part. The space $T^{1,0}(M)$ is spanned by $\{\partial_i\}$ and $T^{0,1}(M)$ is spanned by $\{\bar{\partial}_i\}$. One can perform this same decomposition at the level of the dual space: $T^*_{\mathbb{C}}(M) = T^{*1,0}(M) \bigoplus T^{*0,1}(M)$, spanned by $\{dz^i\}$ and $\{\bar{dz}^i\}$ respectively. Because of this we also find a decomposition of differential k-forms into differential (p, q) forms, with $p + q = k$, defined through

$$A^k = \bigoplus_{p+q=k} A^{p,q}. \tag{B.19}$$

Here we have $A^k = \wedge^k(T^*_{\mathbb{C}}M)$ as the space of k-forms on M and $A^{p,q} = \wedge^p(T^{*1,0}M) \bigoplus \wedge^q(T^{*0,1}M)$ as the space of holomorphic p-forms and anti-holomorphic q-forms respectively. Based on this decomposition we also see that the exterior derivative expands into an exterior derivative on the decomposition fields as $d = \partial + \bar{\partial}$.

The *complex structure* is in essence the way in which a manifold is realized as a $2n$-dimensional real manifold, from an n-dimensional complex manifold. We can define the complex structure through a globally defined, differentiable linear map J that mimics the properties of $i = \sqrt{-1}$

$$J : T(M) \to T(M), \quad v^\mu \mapsto J^\mu_\nu v^\nu \quad J^2 = -1. \tag{B.20}$$

The splitting that we defined in equation (B.18) is therefore a splitting into eigenspaces of J, where the space $T^{1,0}(M)$ has eigenvalue $+i$ and $T^{0,1}(M)$ has eigenvalue $-i$.

Where J has to be *torsionless* (the precise definition of which is irrelevant for this work, see [26]) as well in order to define an actual complex structure.

Kähler structure

A Kähler manifold is a complex manifold, on which a closed differential $(1,1)$-form ω exists, which is compatible with the complex structure J as defined above. This means that the form $\eta(u,v) := \omega(u, Jv)$ is symmetric, positive definite, and closed. In other words we can use J and ω to define a *hermitian metric* on the manifold.

The Kähler manifold therefore has an added *Riemannian structure* with respect to the complex manifold.

This Kähler structure has consequences for the geometric properties of the manifold, like the connection. To see this let g be the metric corresponding to the 2-form ω. From the fact that $d\omega = 0 \Rightarrow \partial\omega = 0, \overline{\partial}\omega = 0$ we get that

$$\partial_i g_{j\bar{k}} = \partial_j g_{i\bar{k}}, \quad \overline{\partial}_i g_{j\bar{k}} = \overline{\partial}_k g_{j\bar{i}}, \tag{B.21}$$

and so the only non-zero components of the Levi-Civita connection are,

$$\Gamma^k_{ij} = g^{k\bar{l}}\partial_i g_{j\bar{l}} \Gamma^{\bar{k}}_{\bar{i}\bar{j}} = g^{l\bar{k}}\overline{\partial}_{\bar{i}} g_{l\bar{j}}. \tag{B.22}$$

The equation (B.21) also implies the existence of a real Kähler potential K for the metric, such that

$$g_{i\bar{j}} = \partial_i \overline{\partial}_j K. \tag{B.23}$$

Hodge decomposition with Kähler structure

For a Kähler manifold (i.e. a manifold with a complex, Riemannian and symplectic structure) one can decompose the k-th de Rham Cohomology $H^k(M)$ according to the complex structure present on the manifold. To do this we apply the derivative we used in defining the complex structure; $\bar{\partial}$.

Definition 2 (Dolbeault cohomology) *The Dolbeault cohomology is defined as*

$$H^{p,q}_{\bar{\partial}}(M) = \frac{Z^{p,q}_{\bar{\partial}}(M)}{\bar{\partial}(A^{p,q-1}_{\bar{\partial}}(M))}. \tag{B.24}$$

Here $Z^{p,q}_{\bar{\partial}}(M)$ denotes the space of (p,q)-forms on M closed under $\bar{\partial}$ and $A^{p,q-1}_{\bar{\partial}}(M)$ is the space of (p,q)-forms on M exact under $\bar{\partial}$.

Note that because of the nature of the complex decomposition we have that $H^{p,q}_{\bar{\partial}}(M) = \overline{H^{q,p}_{\bar{\partial}}}(M)$. Now we present the definition of a *Hodge structure.*

Definition 3 (Hodge structure) *A pure Hodge structure of weight k of an abelian group H is defined to be the decomposition of the complexified version of this group, $H_{\mathbb{C}}$, in a direct*

sum of complex subspaces $H^{p,q}$, with $p+q=k$:

$$H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}, \tag{B.25}$$

$$H^{p,q} = \overline{H^{q,p}}. \tag{B.26}$$

For a Kähler manifold one can decompose the de Rham cohomology into the Dolbeault cohomology spaces, finding such a pure Hodge structure:

$$H^k(M) = \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}. \tag{B.27}$$

Remembering the isomorphism between cohomology spaces and harmonic forms, we would now like to say something about the dimensions of these spaces. We therefore introduce the *Betti* numbers $b^k = \dim(H^k)$, and *Hodge* numbers $h^{p,q} = \dim(H^{p,q})$. From the Dolbeault decomposition the following relation between the two numbers holds,

$$b^k = \sum_{p+q=k} h^{p,q}. \tag{B.28}$$

We can arrange the Hodge numbers of a manifold in a so called "Hodge diamond" as follows. Consider the Hodge diamond of a manifold with complex dimension $dim_{\mathbb{C}}(M) = 3$,

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array} \tag{B.29}$$

for a Calabi-Yau manifold we note that many of these Hodge numbers are not independent, as we will mention in appendix C.

The Hodge decomposition has far-reaching consequences for the expansion of our forms, as we will now show. Let M a 10-dimensional manifold, realized as the cross product of four dimensional Minkowski space $M_{1,3}$ and a Calabi-Yau threefold CY_3. Let $B_p \in \Omega^p(M)$, as in the discussion around equation (3.6). We are interested in finding the harmonic modes of B, as we want an effective theory with massless fields. Consider the expansion,

$$B^{(p)}(X^M) = \sum_k A^{(k)}(x^\mu) \wedge \tilde{A}^{(p-k)}(y^m).$$

By the Hodge decomposition (B.16)we see that we can decompose $\tilde{A}$ into a harmonic part, and a non-harmonic part which we will neglect as it is massive. Now we know by the Dolbeault decomposition above that any harmonic k-form has a b_k dimensional basis of

harmonic k-forms in which we can expand. Denoting the I'th element of this basis by $\tilde{A}^I$ we therefore find,

$$B^{(p)}(X^M) = \sum_{k=0}^{4} \sum_{I=1}^{b^{p-k}} A_k^I(x^\mu) \wedge \tilde{A}^I_{(p-k)}(y^m). \tag{B.30}$$

Note that the forms on the Minkowski space have also received an index I, as the coefficients of $\tilde{A}^I_{(p-k)}(y^m)$ in the expansion can be dependent on x^μ.

From the expansion (B.30) it follows for example that a 2-form will lead to b_2 0-forms, b_1 1-forms and b_0 2-forms, each of them massless, on $M_{1,3}$.

Polarization

Next, we would like to have a way to measure distances within the Hodge structure. This is done, in a way that takes the decomposition into account, by the *polarization*[17].

Let $V = \bigoplus_{p+q=k} V^{p,q}$ be a Hodge structure of weight k. A polarization of the Hodge structure is a real, bilinear form $Q : V \times V \longrightarrow \mathbb{R}$ such that

1. $Q(\omega, \eta) = (-1)^k Q(\eta, \omega)$. In words this means that Q is symmetric for k even and skew-symmetric for k odd.
2. $i^{p-q} Q(\omega, \overline{\omega}) > 0$ for all $0 \neq \omega \in V^{p,q}$
3. Let $\omega_{p,q} \in V^{p,q}$ and $\eta_{p',q'} \in V^{p',q'}$. Then $Q(\omega_{p,q}, \eta_{p',q'}) = 0$ if $p' \neq k - p$.

Note that for a complex Hodge structure the real dimension k will always be even, hence the polarization will always be symmetric. Looking at the properties of the polarization it is apparent that we can use the polarization to define a Hermitian form on V.

Hodge filtrations

It is possible to consider Hodge structures of weight k in a different way. Instead of looking at a direct sum composition of the space $H_\mathbb{C}$ we build it up from the following *Hodge filtration* [20],

$$H_\mathbb{C} = F^0 \supset ... \supset F^{p-1} \supset F^p \supset F^{p+1} ... \supset 0, \tag{B.31}$$

where we have defined

$$F^p = \bigoplus_{i \geq p} H^{i,k-i}, \tag{B.32}$$

and for which the following relation holds,

$$H_\mathbb{C} = F^p \bigoplus \overline{F^{k-p+1}}, \text{ for each } p. \tag{B.33}$$

We call the filtration defined in (B.31) a *decreasing* filtration, as the higher index spaces are contained within the lower index ones. The spaces $H^{i,k-i}$ and $H_{\mathbb{C}}$ are the same ones as used in definition 3. We have $f_p = \sum_{a \geq p} h^{a,k-a}$ the dimension of the filtration spaces.

Through property B.33 we can see that a Hodge filtration is equivalent to a Hodge structure;

$$H^{p,q} = F^p \cap \overline{F^q}, \quad p + q = k \tag{B.34}$$

This defines a bijection between weighted Hodge structures and Hodge filtrations.

Appendix C

The Calabi-Yau manifold

In the introduction it was mentioned that one of the properties we want for our 4-dimensional theory is supersymmetry. This implies the existence of a nowhere-vanishing holomorphic three-form Ω [12]. Assuming that the internal manifold is Kähler one can relate this 3-form to the metric of the internal manifold to find[28],

$$R = i\partial\overline{\partial}\log\sqrt{g} = -i\partial\overline{\partial}\log||\Omega||^2. \tag{C.1}$$

Now since $\log||\Omega||^2$ is globally defined we see that the Ricci scalar is an exact form. As such its equivalence class in the cohomology space, which we will denote by $2\pi i c_1$, is 0. We call c_1 the *first Chern class*, which we can hence conclude vanishes if one wants to preserve supersymmetry.

It is precisely these properties which define a Calabi-Yau manifold: a Kähler manifold with vanishing first Chern class, or equivalently, a Kähler manifold with a unique nowhere vanishing $(3,0)$-form.

On a Calabi-Yau manifold we can make more statements about the Hodge structures introduced in appendix B.

Hodge numbers for CY_3

With the added structure of a Calabi-Yau manifold we can say a lot more about the dimensions of the spaces in our Hodge decomposition. For example the existence of a unique $(3,0)$ form tells us that $h^{3,0} = 1$. Using mirror symmetry, the Hodge-$*$ map and complex conjugation as outlined in [26] we find out that the only independent Hodge numbers are $h^{2,1}$ and $h^{1,1}$; $h^{3,0}$ is fixed to be 1. Any Hodge number which is not related to either of these through one of the mentioned operations vanishes. This leaves us with the following Hodge

diamond,

$$
\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & & & h^{3,3} & & &
\end{array}
=
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\tag{C.2}
$$

where for example $h^{1,1} = h^{2,2}$ through Hodge duality. Hence by equation (B.28) a 2-form on a CY_3 will lead to $b_2 = h^{1,1} + h^{0,2} + h^{2,0} = h^{1,1}$ 0-forms, $b_1 = 0$ 1-forms and $b_0 = h^{0,0} = 1$ 2-forms on $M_{1,3}$.

The moduli space & metric deformations

So far we have studied the different components of the Calabi-Yau manifold, consisting of the complex structure J and the Kähler form that build the Hermitian metric $g_{i\bar{j}}$ (which we can also refer to as the Kähler structure). We have considered both of these to be *fixed* parameters, however one could consider perturbations to the complex structure and the metric that leave the defining properties of the Calabi-Yau manifold invariant. The complex coefficients upon expanding these perturbations are called the *moduli*, and they span a *moduli space* which itself is again a manifold. The properties of this manifold are specified by the geometry of the Calabi-Yau manifold that the perturbations to the structure, and therefore the moduli, have to preserve. This geometry is called *special geometry*.

Through the existence of the moduli it is made clear that a Calabi-Yau manifold with certain Hodge numbers is not unique; in fact, the moduli space specifies a continuously infinite family of Calabi-Yau manifolds.

Metric space deformations

When considering the zero mode expansion of the metric $g_{MN} = g_{\mu\nu} \oplus g_{mn}$ we shall find that its 0-modes are given by the external metric $g_{\mu\nu}$ and a set of massless scalars coming from the internal metric 2-form g_{mn}. The fact that $b_1 = 0$ implies that no 1-forms are formed in the expansion.

Let us consider fluctuations of the internal Calabi-Yau metric g_{mn},

$$g_{mn} \to g_{mn} + \delta g_{mn}. \tag{C.3}$$

We need to impose that the perturbed metric still satisfies the conditions of a Calabi-Yau manifold, one of them being Ricci-flatness,

$$R_{mn}(g_{mn} + \delta g_{mn}) = 0. \tag{C.4}$$

We fix the gauge

$$\nabla^m g_{mn} = \frac{1}{2}\nabla_n g^{mp}\delta g^n_m. \tag{C.5}$$

By expanding the Ricci-flatness constraint to first order in perturbations we get

$$\nabla^k\nabla_k\delta g_{mn} + 2R_m{}^p{}_n{}^q\delta g_{pq} = 0, \tag{C.6}$$

which is the *Lichnerowicz* equation for metric perturbations.

Now, looking at the index structure of the metric perturbations we can see that the pure and mixed components δg_{ij} and $\delta g_{i\bar{j}}$ decouple as solutions to the Lichnerowicz equation. We can therefore look at them each separately.

Consider $\delta g_{i\bar{j}}$. One can show[26] that the Lichnerowicz equation applied to a form with mixed indices is equivalent to $(\Delta\delta g)_{i\bar{j}} = 0$, hence the mixed perturbations are harmonic $(1,1)$-forms. We can therefore expand $\delta g_{i\bar{j}}$ in the basis ω^α with $\alpha \in \{1, ..., h^{1,1}\}$,

$$\delta g_{i\bar{j}} = \sum_{\alpha=1}^{h^{1,1}} \tilde{z}^\alpha \omega^\alpha_{i\bar{j}}, \qquad \tilde{z}^\alpha \in \mathbb{R}. \tag{C.7}$$

Now for $g + \delta g$ to still be a metric we need to pick the Kähler moduli $\tilde{z}^\alpha$ in such a way that the perturbed metric is still positive definite. Positive definiteness of the metric corresponds to the following constraint on the Kähler form

$$\int_{M_r} \underbrace{J \wedge ... \wedge J}_{r-times} > 0, \qquad r = 1, 2, 3. \tag{C.8}$$

In the above expression M_r is any complex r-dimensional submanifold of CY_3.

The subset of metric deformations that satisfy the Lichnerowicz equation and the positive definiteness constraint is called the *Kähler cone*. This name is chosen because for any solution J and number $\lambda \in \mathbb{R}_+$ equation (C.8) is also solved by λJ.

Note that the Kähler form J is real, meaning that also the Kähler moduli are real. In string theory We can complexify the Kähler form by combining it with the real closed $(1,1)$-form $B_{a\bar{b}}$, which, being a $(1,1)$-form like the metric, has $h^{1,1}$ massless modes. We get as the *complexified Kähler form*,

$$\mathcal{J} = B + iJ. \tag{C.9}$$

This form results in $h^{1,1}$ *complex* scalar degrees of freedom in the 4-dimensional effective theory,

$$(\delta B_{i\bar{j}} + i\delta g_{i\bar{j}})dz^i \wedge d\bar{z}^{\bar{j}} = \sum_{\alpha=1}^{h^{1,1}} z^\alpha \omega^\alpha, \tag{C.10}$$

where the $z^\alpha = b^\alpha + i\tilde{t}^\alpha$ denote the complexified Kähler moduli (not to be confused with the complex moduli)

Now let us look at δg_{ij}. Note that the original background metric has no purely holomorphic or anti-holomorphic entries, due to the fact that it is hermitian. We can however consider varying these components to become non-zero. The Lichnerowicz equation applied to the holomorphic perturbation to the metric, δg_{ij}, turns out to be equivalent to [26]

$$\Delta_{\overline{\partial}} \delta g^i = 0, \tag{C.11}$$

with

$$\delta g^i = \delta g^i_{\overline{j}} d\overline{z}^{\overline{j}}, \quad \delta g^i_{\overline{j}} = g^{i\overline{k}} \delta g_{\overline{kj}}. \tag{C.12}$$

In other words, the Lichnerowicz equation implies the existence of a harmonic $(0,1)$-form δg^i contained in the de Rham cohomology group $H^1(M, T_M)$.

For the metric resulting from the perturbation to again be Kähler it is necessary that coordinates exist in which $g + \delta g$ does not have any purely (anti)holomorphic components (as hermiticity of the metric is part of the definition of a Kähler manifold). To arrive at this basis we would have to perform a non-holomorphic coordinate transformation, as that is the only way to change the index type. A non-holomorphic transformation also changes the complex structure however [26], meaning that the new Kähler metric has a different complex structure from the original. The difference between the two is encoded in the δg^i, which we therefore refer to as *deformations of the complex structure*.

In order to expand the complex structure deformations on a basis we are familiar with we can use the unique $(3,0)$-form Ω to define an isomorphism $H^1(M) \to H^{2,1}_{\overline{\partial}}(M)$, $\delta g \mapsto \Omega_{ijk} \delta g^k_{\overline{l}} dz^i \wedge dz^j \wedge \overline{z}^{\overline{l}}$. Note that the image in $H^{2,1}_{\overline{\partial}}$ is again harmonic due to properties of Ω and harmonicity of δg^i.

So we expand the complex structure deformations in a basis of harmonic $(2,1)$-forms $b^\alpha_{ij\overline{l}}$,

$$\Omega_{ijk} \delta^k_{\overline{l}} = \sum_{\alpha=1}^{h^{2,1}} t^\alpha b^\alpha_{ij\overline{l}}, \tag{C.13}$$

where the complex parameters t^α are the complex structure moduli.

We have thus found two sets of moduli: $h^{1,1}$ complexified Kähler moduli z^α and $h^{2,1}$ complex structure moduli t^α, for a total of $2(h^{1,1} + h^{2,1})$ real deformation parameters.

These moduli inhabit the so called *moduli space*, which by itself is again a Kähler manifold that locally has a product structure arising from the combination of the complex structure moduli space $\mathcal{M}_{cs}$ and the complexified Kähler structure moduli space $\mathcal{M}_k$,

$$\mathcal{M} = \mathcal{M}_{cs} \times \mathcal{M}_k. \tag{C.14}$$

In appendix C we will define a metric on this moduli space as per equation (C.55), which encodes the distance between two different realizations of a string vacuum.

The Kähler potential as function of the moduli

In the context of Calabi-Yau manifolds we can say more about the Kähler potential [26]. This goes under the name of *special geometry*. As it turns out we can express the Kähler potential K in terms of the complex moduli t^a, with $a = 1, ..., h^{2,1}$, as follows. Define projective coordinates $Z^I = (Z^0, Z^a)$ through $t^a = Z^a/Z^0$, and then take $\mathcal{F}(Z) = (Z^0)^2 F(t)$ with $F(t)$ a holomorphic function derived from the Lagrangian. This function F is called the *prepotential*. Using these coordinates mathematicians have found that one can write the Kähler potential as,

$$K = \log i(\overline{Z}^I \mathcal{F}_i - Z^I \overline{\mathcal{F}}_I). \tag{C.15}$$

Here we defined $\mathcal{F}_I = \partial F/\partial Z^I$, and $I = 0, ..., h^{2,1}$.

3-forms on a Calabi-Yau

In the Hodge decomposition only $H^{2,1}(CY_3)$ and $H^{0,3}(CY_3)$ are relevant (when considering 3-forms that is). There are two ways to pick a basis for these spaces. One is to take $(\eta_a)_{\alpha\beta\overline{\gamma}}$, with $a \in \{1, ..., h^{2,1}\}$, as basis for the $(2,1)$-forms and $\Omega_{\alpha\beta\gamma}$ as the unique $(3,0)$-form.

One can also choose a basis that works for both spaces; i.e. one in which the $(3,0)$ form can also be expanded [13]. This basis consists of $2(h^{2,1}+1)$ components α_A, β^A with $A \in \{0, ..., h^{2,1}\}$. They are Poincaré dual to a homology basis (A^I, B_I) [26]. The cohomology basis is then orthogonal according to the following relations,

$$\int_{A^J} \alpha_I = \int_{CY_3} \alpha_I \wedge \beta^J = -\int_{B_I} \beta^J = \delta_I^J, \tag{C.16}$$

where all other combinations vanish. We call this is a *symplectic basis*.

When performing a compactification as in chapter 3 we will expand our forms in terms of this basis. Since kinetic terms have the form

$$\mathcal{L}_{kinetic} = \int_{CY_3} F \wedge *F, \tag{C.17}$$

with F a 3-form. Now we can expand $F = A^I \alpha_I + B_I \beta^I$, with A, B coefficients. Upon filling in this expansion it becomes clear that in order to integrate out the 6 internal dimensions we will have to give an expression for the following integrals,

$$\int_{CY_3} \alpha_I \wedge *\beta^J, \quad \int_{CY_3} \alpha_I \wedge *\alpha_J, \quad \int_{CY_3} \beta^I \wedge *\beta^J. \tag{C.18}$$

This appendix is devoted to expressing these integrals in terms of the moduli Z^I and $\mathcal{F}$. We need to use various relations between the $(3,0)$-form Ω and the Kähler potential to do this.

To begin we remark that we can expand the holomorphic $(3,0)$-form Ω in terms of its periods over the cycles A^I and B_J. We shall define first $Z^I = \int_{A^I} \Omega$, $\mathcal{F}_I = \int_{B_I} \Omega$, which leads to

$$\Omega = Z^I \alpha_I - \mathcal{F}_I \beta^I. \tag{C.19}$$

Note now that the A-periods of Ω, i.e. the Z^I, correspond to local projective coordinates on the complex structure moduli space of CY_3. These are the same coordinates as introduced earlier in section C. The same goes for the B-periods; they correspond to the function $\mathcal{F}_I(z)$ defined in C. We can combine the expansion of the 3-form (C.19) and the expression for the Kähler potential (C.15) to find a relation between K and Ω,

$$K = -\log(i\int \Omega \wedge \overline{\Omega}). \tag{C.20}$$

Note now that $*\alpha_A \in H^{2,1} \cong H^{1,2}$. Therefore we can express $*\alpha_A$ in terms of the basis elements α_A and β^B. The same holds for $*\beta^B$. We take

$$*\alpha_A = A_A{}^B \alpha_B + B_{AB}\beta^B, \quad *\beta^A = C^{AB}\alpha_B + D^A{}_B\beta^B. \tag{C.21}$$

Clearly, using equation (C.16), the matrix elements A, B, C and D correspond to the integrals in (C.18)

$$\int \alpha_J \wedge *\alpha_I = B_{IJ} = B_{JI}, \tag{C.22}$$

$$\int \beta^J \wedge *\alpha_I = -A_I{}^J, \tag{C.23}$$

$$\int \alpha_J \wedge *\beta^J = D^I{}_J = -A_J{}^I, \tag{C.24}$$

$$\int \beta^J \wedge *\beta^I = -C^{IJ} = -C^{JI}. \tag{C.25}$$

Our goal now is therefore to express the matrices A, B, C and D in terms of the moduli. In our derivation we follow [29, 30, 31, 32] as per [13]. We note that for π any $(2,1)$-form we have

$$*\Omega = -i\Omega \tag{C.26}$$

$$*\pi = i\pi. \tag{C.27}$$

Considering the expansion of Ω (C.19) we can then match coefficients to find,

$$-iz^B = z^A A_B{}^A - \mathcal{F}_A C^{AB} \tag{C.28}$$

$$i\mathcal{F}_B = z^A B_{AB} + \mathcal{F}_A A_B{}^A. \tag{C.29}$$

In the next step of our derivation we consider the Kodaira equation [23],

$$\frac{\partial}{\partial z^a}\Omega = k_a\Omega + i\eta_a. \tag{C.30}$$

Where η_a is a harmonic $(2,1)$ form. We can multiply both sides by $\overline{\Omega}$ and integrate over the Calabi-Yau manifold to find (where the term with the η_a vanishes),

$$\int_{CY_3} \frac{\partial}{\partial z^a}\Omega \wedge \overline{\Omega} = \int_{CY_3} k_a \Omega \wedge \overline{\Omega}. \tag{C.31}$$

Now we find from equation (C.20) that the RHS is equal $-ik_a e^{-K}$, with K the Kähler potential. This we can express according to equation (C.15) in the prepotential $\mathcal{F}$ to find,

$$= k_a \left(\overline{z}^A \mathcal{F}_A - z^A \overline{F}_A\right) = 2k_a \mathrm{Im}\left(\overline{z}^A \mathcal{F}_A\right) \tag{C.32}$$

We now quickly note the following fact about homogeneous functions[13]. Namely, let $f(x)$ be a homogeneous function, then

$$x^i \frac{\partial f}{\partial x^i} = 2f \tag{C.33}$$

$$x^i \frac{\partial f}{\partial x^i \partial x^j} = \frac{\partial f}{\partial x^j}, \tag{C.34}$$

which we can apply to $\overline{z}^A \mathcal{F}_A = \overline{z}^A z^B \mathcal{F}_{AB} =< z|\overline{z} >$ to rewrite the RHS to,

$$2k_a < z|\overline{z} > . \tag{C.35}$$

with

$$\mathcal{F}_{AB} = \frac{\partial \mathcal{F}_A}{\partial z^B} \tag{C.36}$$

$$< F|\overline{G} > = \mathrm{Im}\mathcal{F}_{AB} F^A \overline{G}^B. \tag{C.37}$$

This concludes our analysis of the RHS of equation (C.31). Rewriting the LHS as well we find an expression for k_a:

$$k_a = \frac{1}{< z|\overline{z}} \mathrm{Im}\mathcal{F}_{aB} \overline{z}^B = -\partial_a K. \tag{C.38}$$

We now note that we can use the Kodaira equation (C.30) and the expansion of our $(0,3)$-form (C.19) to find,

$$\alpha_A + \mathcal{F}_{AB} B^B = \partial_A \Omega = k_A \Omega + i\eta_A. \tag{C.39}$$

Taking the Hodge star on both sides, and using equation (C.26) we find,

$$-ik_A \Omega + i\eta = *(\partial_A \Omega) = -i\partial_A \Omega \tag{C.40}$$

$$= -2ik_A \Omega + i\partial_A \Omega \tag{C.41}$$

$$= i\left(\alpha_A - \mathcal{F}_{AB}\beta^B\right) - 2ik_A \left(Z^B \alpha_B - \mathcal{F}_B \beta^B\right) \tag{C.42}$$

$$= *\alpha_A - \mathcal{F}_{AB} * \beta^B \tag{C.43}$$

where in the third step we eliminated η_A by using the Kodaira equation (C.30) again. By integrating and using the normalization of the basis we can find these expressions for the matrices,

$$A_A^B = -\mathrm{Re}\mathcal{F}_{AC} \left(\mathrm{Im}\mathcal{F}^{-1}\right)^{CB} + \frac{\overline{z}^B \mathcal{F}_A + z^B \overline{\mathcal{F}}}{< z|\overline{z} >} \tag{C.44}$$

$$B_{AB} = \mathrm{Im}\mathcal{F}_{AB} + \mathrm{Re}\mathcal{F}_{AC} \left(\mathrm{Im}\mathcal{F}^{-1}\right) \mathrm{Re}\mathcal{F}_{DB} - \frac{\overline{\mathcal{F}_A}\mathcal{F}_B + \mathcal{F}_A \overline{\mathcal{F}_B}}{< z|\overline{z} >} \tag{C.45}$$

$$C^{AB} = -\left(\mathrm{Im}\mathcal{F}^{-1}\right)^{AB} + \frac{\overline{z}^A z^B + z^A \overline{z}^B}{< z|\overline{z} >}. \tag{C.46}$$

They can be simplified by defining the matrix $\mathcal{M}$,

$$\mathcal{M}_{AB} = \overline{F}_{AB} + \frac{2}{< z|\overline{z} >} \mathrm{Im}\mathcal{F}_{AC} z^C \mathrm{Im}\mathcal{F}_{BD} z^D \tag{C.47}$$

in terms of which we get

$$\begin{aligned} A &= (\mathrm{Re}\mathcal{M})\,(\mathrm{Im}\mathcal{M})^{-1} && \text{(C.48)} \\ B &= -\,(\mathrm{Im}\mathcal{M}) - (\mathrm{Re}\mathcal{M})\,(\mathrm{Im}\mathcal{M})^{-1}\,(\mathrm{Re}\mathcal{M})\,() && \text{(C.49)} \\ C &= \left((\mathrm{Im}\mathcal{M})^{-1}\right). && \text{(C.50)} \end{aligned}$$

$(1,1)$ forms on a Calabi-Yau

Besides the spaces of 3-forms The only other relevant Dolbeault cohomology is $H^{1,1}(CY_3)$. We will use a basis of harmonic $(1,1)$-forms given by $(\omega_i)_{\alpha\overline{\beta}}$ with $i \in \{1, ..., h^{1,1}\}$. In the process of compactifying we will run into several integrals, the notation for which we will outline here,

$$\begin{aligned} \mathcal{K} &= \frac{1}{6}\int_{CY_3} J \wedge J \wedge J, \quad \mathcal{K}_i = \int_{CY_3} \omega_i \wedge J \wedge J && \text{(C.51)} \\ \mathcal{K}_{ij} &= \int_{CY_3} \omega_i \wedge \omega_j \wedge J, \quad \mathcal{K}_{ijk} = \int_{CY_3} \omega_i \wedge \omega_j \wedge \omega_k. && \text{(C.52)} \end{aligned}$$

In the equation above J denotes the Kähler form, which is a $(1,1)$-form and can hence be expanded in terms of ω to give,

$$J = v^i \omega_i. \tag{C.53}$$

The quantity $\mathcal{K}$ gives the volume of the Calabi-Yau manifold. The expansion of J implies the following identities,

$$\mathcal{K}_{ijk} v^k = \mathcal{K}_{ij}, \quad \mathcal{K}_{ij} v^j = \mathcal{K}_i, \quad \mathcal{K}_i v^i = \mathcal{K}. \tag{C.54}$$

Using the combined Kähler and supersymmetric geometries we can define the metric on the *moduli space* as[26]

$$g_{ij} = \frac{1}{4\mathcal{K}} \int_{CY_3} \omega_i \wedge *\omega_j. \tag{C.55}$$

Using [33]

$$*\omega_i = -J \wedge + \frac{\mathcal{K}_i}{4\mathcal{K}} J \wedge J, \tag{C.56}$$

the moduli space metric can be rewritten to find,

$$g_{ij} = -\frac{1}{4\mathcal{K}} \left(\mathcal{K}_{ij} - \frac{1}{4\mathcal{K}} \mathcal{K}_i \mathcal{K}_j \right). \tag{C.57}$$

It further follows that

$$K_{ij} = \int_{CY_3} \left(-(\omega_i g)(\omega_j g) + (\omega_i \omega_j)\right), \tag{C.58}$$

$$g_{ij} = -\frac{1}{V_{CY_3}} \int_{CY_3} (\omega_i \omega_j). \tag{C.59}$$

We now define $\overline{\omega}^i$ to be the basis of the space $H^{2,2}(CY_3)$ which is dual to $H^{1,1}(CY_3)$, with a normalization,

$$\int_{CY_3} \omega_i \wedge \overline{\omega}^j = \delta_i^j. \tag{C.60}$$

From this normalization we derive the relations,

$$g^{ij} = 4\mathcal{K} \int_{CY_3} \overline{\omega}^i \wedge *\overline{\omega}^j, \quad *\omega_i = 4\mathcal{K} g_{ij} \overline{\omega}^i, \quad *\overline{\omega}^i = \frac{1}{4\mathcal{K}} g^{ij} \omega_j, \quad \omega_i \wedge \omega_j \sim \mathcal{K}_{ijk} \tilde{\omega}^k. \tag{C.61}$$

With the $\sim$ we mean that the two expressions are equivalent up to an exact form, i.e. they lie in the same cohomology class.

Appendix D

Calculating the perturbation matrix E

We work out the derivatives in $g = -\partial_i \partial_{\bar{j}} \log\left(\tilde{Q}\right) = -\partial_i \partial_{\bar{j}} \left(\log(p+f)\right)$,

$$\begin{aligned}
-\partial_i \partial_{\bar{j}} \left(\log(p+f)\right) &= -\partial_i \frac{\partial_{\bar{j}} p + \partial_{\bar{j}} f}{p+f} \\
&= \frac{-\partial_i \partial_{\bar{j}} p - \partial_i \partial_{\bar{j}} f}{p+f} + \frac{\partial_i (p+f) \partial_{\bar{j}} (p+f)}{(p+f)^2} \\
&= \frac{-(p+f)\partial_i \partial_{\bar{j}} p + \partial_i p \partial_{\bar{j}} p}{(p+f)^2} + \underbrace{\frac{-(p+f)\partial_i \partial_{\bar{j}} f + \partial_i p \partial_{\bar{j}} f + \partial_i f \partial_{\bar{j}} p + \partial_i f \partial_{\bar{j}} f}{(p+f)^2}}_{E_{i\bar{j}}} .
\end{aligned} \tag{D.1}$$

We defined the second term on the last line to be $E_{i\bar{j}}$, as the first term is equal to $-\partial_i \partial_{\bar{j}} \log(p)$ in the limit of large y_1, y_2; where $p + f \sim p$. This results in the following expression for $\tilde{Q}$,

$$\tilde{Q} = -\partial\overline{\partial} \log(p) + E. \tag{D.2}$$

Note that $E \sim f$, up to some derivatives.

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