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Master's Thesis

Strongly interacting anisotropic gauge theories and holography

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June 21, 2018

Universiteit Utrecht

Abstract

The AdS/CFT correspondence has become, since its development in the late 90's, one of the most usefull tools to study large N gauge theories. These theories share most of the key characteristics of quantum chromodynamics, and therefore holography can be used to explore strong interactions in the regimes where perturbative and lattice QCD fail. In recent years, a growing effort has been put into the study of anisotropic many body systems using these tools. In this work, we consider a generic class of such systems driven by a parametric family of dilaton and axion potentials in the gravity side, and look for the IR asymptotics of the bulk geometry. We also investigate the thermodynamics of the system and study the confinement-deconfinement phase transition.

Acknowledgements

Although this thesis has only my signature, there are many people who have helped me in the process of its making, and whom I need to sincerely thank.

First of all, I have to mention Matti Järvinen, my supervisor in the shadows, and Govert Nijs. I certainly wouldn't have been able to finish this thesis on time without their help.

Second, many thanks to my family, without whose effort I wouldn't be able to have done this Master in the first place.

Finally, I also want to mention my friends and colleagues from the master. Here is to our long and stressful (also fun) hours in the master room.

Contents

1 Introduction: Why anisotropic QCD?

Strongly interacting many body systems remain one of the most poorly understood parts of physics, despite the tremendous amount of work that has been put into them on recent decades. The reason is that the standard calculation techniques, based on perturbation theory and grounded on the weakly interacting regime, obviously don't apply. However, many phenomena in nature are modelled by strongly coupled systems, from fundamental particle physics (e.g. Quantum Chromodynamics at low energies) to condensed matter physics (e.g. Hubbard model, high temperature superconductors,...); and therefore it is required that we persevere in the exploration and resolution of such complicated problems.

Sometimes, it is possible to find exact solutions to particular cases, that do not make use of perturbation theory and thus are applicable also to the strong coupling regime. This solutions, however, are usually very dependant on the particular system under study, and very often rely heavily on it being highly symmetric.

Nevertheless, in many cases, one doesn't have the amount of symmetries to try and make use of these approaches. In particular, it is quite common that the 3d rotational symmetry is broken due to the presence of a special axis. This can be caused by an external electric or magnetic field, by intrinsic properties of the models (e.g. anisotropic spin systems) or by the geometry of one particular experimental setup.

The main motivation of our work is to understand the Quark-Gluon Plasma (QGP) in the presence of a non-specific anisotropy. The main constituents of the QGP are, as one might guess by its name, quarks and gluons. The fundamental theory that explains their interaction is QCD, which we know that is strongly coupled at low energies. Therefore, the QGP is one of the prime examples of strongly interacting many body systems that we would like to understand, among other things, because of its relevance in the evolution of the early universe. The anisotropy can have various origins. In heavy-ion collision experiments it can be due to the collisions being non-central (i.e. the ions don't collide exactly front-to-front but with a non-trivial impact parameter), which results in the symmetry around the beam axis to be broken. In neutron stars, whose core is also formed by the QGP, it can be a product e.g. of the rotation of the star.

There are not known exact results for this system –if they exist at all–. Thus, if progress is to be made in the theoretical understanding of the QGP, only one of two methods can be used: numerical simulations based on lattice QCD or holographic techniques.

The gauge-gravity duality, based on the AdS/CFT correspondence found by Maldacena [8] in the context of string theory, allows us to find analytic (although approximate) results that can be trusted precisely in the regime where standard QFT calculations cannot. The drawback is that this correspondence is supposed to work only in the large N limit (where the N indicates the gauge group $SU(N)$ of the Yang-Mills theory), which in principle is not applicable to QCD, where $N = 3$. However, it is known that all non-abelian gauge theories share their main characteristics, and therefore at the very least we will be able to extract some qualitative knowledge and gain insight into the Quark-Gluon Plasma.

The study of anisotropic strongly coupled plasmas by means of holographic techniques was initiated in recent years in [1, 2, 3]. In this work, we attempt to further their analysis by considering a very general theory on the gravity side that will be dual to an anisotropic, strongly interacting confining plasma on the field theory side. We achieve that by considering 5d gravity with two scalar fields: the dilaton (dual to the coupling strength) and axion (dual to the anisotropy), and a parametric family of potentials chosen so that the dual plasma will feature confinement in the infrared and an anisotropy with unspecified source.

This thesis is organized as follows:

- In chapter 2, we present the basic background of the framework we shall use: the original AdS/CFT correspondence. We start by reviewing the basics of the geometry of AdS spaces and of Conformal Field Theory, and then we expose the necessary elements of supergravity and string theory to understand the duality.

- In chapter 3, we explain how the duality of AdS/CFT can be adapted and extended to non-conformal field theories like QCD. In particular, we will justify why the choice of the potentials for the dilaton and axion is the most general one that can be dual to an anisotropic confining gauge theory.

- In chapter 4, the core of this work, we present the solution of the system under consideration in the gravity side, and analyse in detail the parameter space on the grounds of the phenomenological applicability of our model.

- In chapter 5, we make use of the results of chapter 4 and study the thermodynamics of the plasma, with focus on the study of the deconfinement phase transition.

2 The AdS/CFT correspondence

In this chapter, we attempt to describe and argue the AdS/CFT correspondence as was first proposed by Maldacena. In order to do so, we start by studying its two main ingredients: Anti de Sitter spacetime and Conformal Field Theory –in particular, maximally supersymmetric Yang-Mills theory–. After that, we will proceed to expose the necessary elements of supergravity and superstring theory and provide a sketch of the proof of the duality. We finish by commenting on Witten's prescription for the calculation of observables through the correspondence. Throughout the chapter, we will follow the exposition of [7, 9, 10].

2.1 Anti de Sitter space

In this section, we will present the main features of the AdS geometry. We will start by studying the more simple, euclidean version of Lobachevski space, and then proceed with the Lorentzian Anti de Sitter, with special attention to the different coordinate systems we will make use of throughout this work. After that, we will study the massive and massless geodesics in this manifold, with emphasis on the importance of its conformal boundary in order to be able to have well defined Cauchy problems.

2.1.1 Lobachevsky space

Lobachevsky space is the d-dimensional generalization of the 2-dimensional hyperboloid. This hypersurface is most easily introduced as the subset of $\mathbb{R}^{d+1}$ satisfying

$$-X_{d+1}^2 + \sum_{i=1}^{d} X_i^2 = -R^2, \tag{2.1}$$

with the metric

$$ds^2 = -dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2. \tag{2.2}$$

From (2.1) we can immediately extract the following two conclusions:

- The hyperbolic space has two connected components, one with $X_{d+1} > R$ and another with $X_{d+1} < -R$. Unless stated otherwise we will only consider the first branch.
- The action of the group $SO(d,1)$ in the obvious way,

$$X_i \mapsto A_i^{\ j} X_j, \quad A_i^{\ j} \in SO(d,1) \tag{2.3}$$

  leaves the manifold invariant. Moreover, from (2.2) we notice that these transformations are isometries.

There are several coordinate systems that allow us to better understand Lobachevsky without having to think of it as a subset of $\mathbb{R}^{d+1}$. Directly from the $SO(d,1)$ symmetry we can see that the following coordinate transformation is a sensible one:

$$\begin{aligned} X_{d+1} &= R\cosh\rho, \\ X_i &= R\sinh\rho\ \Omega_i, \quad \sum_{i=1}^{d}\Omega_i^2 = 1, \end{aligned} \tag{2.4}$$

so that the metric becomes

$$ds^2 = R^2\left(d\rho^2 + \sinh^2\rho\ d\Omega_p^2\right), \tag{2.5}$$

where $d\Omega_p^2$ is the standard metric on S^p. These are the so called *global coordinates*.

Also, we have the *upper half plane* coordinates, given by

$$u = X_d - X_{d+1}, \quad v = X_d + X_{d+1}, \tag{2.6}$$

with the metric

$$ds^2 = dudv + \sum_{i=1}^{d-1} X_i^2. \tag{2.7}$$

Lastly, one of the most useful parametrizations is in terms of the *Poincaré* coordinates,

$$X_i \mapsto x_i = R\frac{X_i}{u}, \qquad u \mapsto z = R^2/u, \tag{2.8}$$

such that the metric is

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + \sum_{i=1}^{d} dx_i^2\right). \tag{2.9}$$

2.1.2 Anti de Sitter space

Anti de Sitter spacetime is the natural generalization of the embedding (2.1) into a Minkowski spacetime, instead of an euclidean one. For reasons that will become apparent later on, we consider $(p+2)$-dimensional AdS inside $\mathbb{R}^{2,p+1}$:

$$-X_0^2 - X_{p+2}^2 + \sum_{i=1}^{p+1} X_i^2 = -R^2, \tag{2.10}$$

with the metric

$$ds^2 = -dX_0^2 - dX_{p+2}^2 + \sum_{i=1}^{p+1} dX_i^2. \tag{2.11}$$

Analogously to Lobachevsky space, the group of isometries is $SO(2,p+1)$, and we can parametrize

$$\begin{aligned} X_0 &= R\cosh\rho\ \cos\tau, \qquad X_{p+2} = R\cosh\rho\ \sin\tau, \\ X_i &= R\sinh\rho\ \Omega_i, \qquad \sum_{i=1}^{p+1}\Omega_i^2 = 1, \end{aligned} \tag{2.12}$$

so that (2.11) becomes

$$ds^2 = R^2 \left(-\cosh^2\rho \, d\tau^2 + d\rho^2 + \sinh^2\rho \, d\Omega_p^2\right). \tag{2.13}$$

Notice that from (2.12) we see that the coordinates ρ and τ take values in $[0,\infty)$ and $[0,2\pi)$ respectively. In particular, the periodicity of the timelike coordinate τ can give rise to the presence of closed timelike curves, and in turn cause problems with causality. In order to solve this, we need to *unwrap* the τ coordinate, making it run from 0 to ∞ without any identifications. Formally, what we do here is consider a covering space of the previously introduced AdS. From now on, unless stated otherwise, we will call Anti de Sitter this universal covering of our original manifold.

Another useful coordinate system are the so called *Poincaré* coordinates (z, t, x_i), given by

$$\begin{aligned}
X^0 &= \frac{1}{2z}\left(1 + z^2\left(R^2 + \sum_{i=1}^{p} x_i^2 - t^2\right)\right), \\
X_i &= Rzx_i, \qquad X_{p+2} = Rzt. \\
X_{p+1} &= \frac{1}{2z}\left(1 - z^2\left(R^2 - \sum_{i=1}^{p} x_i^2 + t^2\right)\right).
\end{aligned} \tag{2.14}$$

It's easy to see that in terms of these coordinates, the metric (2.11) becomes[1]

$$ds^2 = R^2\left(\frac{dz^2}{z^2} + z^2\left(-dt^2 + d\vec{x}^2\right)\right). \tag{2.15}$$

Poincaré coordinates cover only half of AdS space –called the *Poincaré patch*–. However, they are useful because with them one can easily see how the subgroup $SO(1,p)\times SO(1,1) \le SO(2,p+1)$ acts on our spacetime: $SO(1,p)$ are the Lorentz transformations in the coordinates $(t,\vec{x})$, while $SO(1,1)$ is the rescaling $(t,\vec{x},z) \mapsto (\lambda t, \lambda\vec{x}, \lambda^{-1}z)$ with $\lambda > 0$. This will be relevant later on, when we interpret the holographic coordinate z as an energy scale.

The last coordinate system which we will need is the so called *conformal coordiate system*, related to the Poincaré coordinates by $u \mapsto r = 1/u$, so that

$$ds^2 = \frac{R^2}{r^2}\left(dr^2 + d\vec{x}^2 - dt^2\right). \tag{2.16}$$

It should be noted that the metric (2.15) is a solution of Einstein equations with vanishing energy-momentum tensor,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \Lambda g_{\mu\nu}, \tag{2.17}$$

and with a negative cosmological constant

$$\Lambda = -\frac{(p+1)p}{R^2}. \tag{2.18}$$

[1]This is the reason why we started with (p+2)-dimensional AdS, so that now we can see that p is the number of spatial dimensions.

2.1.3 Causal structure and AdS boundary

It will be necessary for our later purposes to study the causal structure of Anti de Sitter spacetime, and in particular the relevance of its conformal boundary.

As is shown in detail in appendix A, the Penrose diagram of AdS is one half of Einstein's static universe $\mathbb{R} \times S^{p+1}$. Therefore, its conformal boundary is $\mathbb{R}$ times the equator of S^{p+1}, i.e. $\mathbb{R} \times S^p$. In general, we say that a spacetime is asymptotically AdS if its conformal boundary is $\mathbb{R} \times S^p$. The reason why the boundary of AdS is more relevant than the conformal boundary of other manifolds is that, as we set out to prove in what follows, massless geodesics can 'bounce back' from the boundary and back to the bulk in a finite amount of time.

Massive geodesics

We start by looking at the trajectory of massive particles in Anti de Sitter. To do so, we consider the action for a massive relativistic particle,

$$S = m \int ds = m \int \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}. \tag{2.19}$$

In our case, the metric is (2.13), and assuming trajectories with constant angular coordinates (2.19) becomes

$$S = mR \int d\tau \sqrt{\cosh^2 \rho - (\partial_\tau \rho)^2}. \tag{2.20}$$

It's easy to see that the solutions of this system oscillate around $\rho = 0$, never reaching the conformal boundary [13]. One can intuitively understand this by looking at the hyperbolic cosine in the action as a potential.

Massless geodesics

Massless geodesics can be studied directly from the metric (2.13) with the null condition $ds^2 = 0$. From this, and also considering the case of only radial trajectories, we find that the equations of motion are

$$d\tau = \pm \frac{d\rho}{\cosh \rho} = \pm d\theta, \tag{2.21}$$

where θ is the conformal coordinate (see appendix A), given by

$$\tan \theta = \tanh \rho, \qquad 0 \leq \theta < \frac{\pi}{2}. \tag{2.22}$$

Therefore, we see that the massless goedesics in conformal coordinates are straight lines

$$\theta = \pm \tau. \tag{2.23}$$

Since the conformal diagram of AdS is a "hyper-cylinder" with radius $\theta < \pi/2$, we check that light rays can arrive at the conformal boundary in a finite amount of proper time. Moreover, if we choose suitable boundary conditions, it is possible for those light rays to bounce back to the bulk (also in a finite amount of time). This poses the following problem: knowledge of all the degrees of freedom in a Cauchy hyper-surface only implies knowledge at all later times *for some given boundary conditions*. This is, as was advertised, the reason why the conformal boundary of AdS is more relevant than the conformal boundary of other spacetime manifolds.

2.2 Conformal Field Theory

In this section we attempt to introduce the second main ingredient of the holographic duality, Conformal Field Theory (CFT). Conformal symmetry is a generalization of Poincaré symmetry obtained by adding scale invariance[2]. We begin by studying the conformal group and algebra, and then proceed to present the main ingredients of conformal field theory: primary fields, Operator Product Expansions and the constraints in the form of the correlators. Finally, we discuss what happens when we add supersymmetry transformations to the conformal algebra, focusing on the example of $\mathcal{N} = 4$ Super Yang-Mills, the most relevant theory in what will follow.

2.2.1 Conformal algebra

The conformal group is the group of transformations of the spacetime coordinates that result in an overall scaling of the metric:

$$x^\mu \mapsto x'^\mu \quad \text{such that} \quad g_{\mu\nu}(x) \mapsto g_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x). \tag{2.24}$$

with $\mu, \nu = 0, 1, \dots, d-1$.

The elements of the conformal group are the Poincaré transformations, together with the scalings

$$x^\mu \mapsto \lambda x^\mu, \tag{2.25}$$

and the special conformal transformations

$$x^\mu \mapsto \frac{x^\mu + a^\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2}. \tag{2.26}$$

We denote the generators as follows:

- $M_{\mu\nu}$ generate Lorentz transformations (rotations and boosts).
- P_μ generates translations.
- D generates the scalings (2.25).
- K_μ generates the special conformal transformations (2.26).

They satisfy the commutation relations of the conformal algebra

$$\begin{aligned} [M_{\mu\nu}, P_\rho] &= -i\left(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu\right), & [M_{\mu\nu}, K_\rho] &= -i\left(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu\right), \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -i\eta_{\mu\rho} M_{\nu\sigma} \pm \text{permutations}, & [M_{\mu\nu}, D] &= 0, \\ [D, K_\mu] = iK_\mu, \qquad [D, P_\mu] &= -iP_\mu, & [P_\mu, K_\mu] &= 2iM_{\mu\nu} - 2i\eta_{\mu\nu} D, \end{aligned} \tag{2.27}$$

[2]While rigorously speaking separate Poincaré and scale invariance are just a subgroup of the full conformal group, it is conjectured that for unitary field theories the symmetry is automatically enlarged. This was proven in the case of 2 dimensions, and there are not known counterexamples [11].

and the rest of the commutators vanish.

This algebra is isomorphic to $SO(2,d)$. The isomorphism is given by

$$J_{\mu\nu} = M_{\mu\nu}, \quad J_{\mu d} = \frac{K_\mu - P_\mu}{2}, \quad J_{\mu(d+1)} = \frac{K_\mu + P_\mu}{2}, \quad J_{(d+1)d} = D. \tag{2.28}$$

Note that this group coincides with the isometry group of AdS_5. This fact will be relevant for the AdS/CFT correspondence.

2.2.2 Fields, correlators, OPEs and the state-operator map

We are interested in studying fields that live in representations of the conformal group. In particular, we look for fields that are eigenfunctions of the scaling operator D (with eigenvalue $-i\Delta$), since that implies that under a scaling transformation of the coordinates, they will transform as

$$\phi(x) \mapsto \phi'(x) = \lambda^\Delta \phi(\lambda x). \tag{2.29}$$

We know that the scaling dimension Δ of a field is always bounded from below by its classical mass dimension (e.g. for a scalar field, $\Delta \geq (d-2)/2$). We also know from the commutation relations (2.27) that the operators P_μ and K_μ raise and lower the dimension of the field. Therefore, each representation must have a field of the lowest possible scaling dimensions that is annihilated by K_μ (at $x = 0$). These are called *primary fields*. The action of the generators of the conformal group on a primary field $\Phi(x)$ is given by

$$\begin{aligned}
[P_\mu, \Phi(x)] &= i\partial_\mu \Phi(x), \\
[M_{\mu\nu}, \Phi(x)] &= [i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu}]\Phi(x), \\
[D, \Phi(x)] &= i(-\Delta + x^\mu \partial_\mu)\Phi(x), \\
[K_\mu, \Phi(x)] &= [i(x^2\partial_\mu - 2x_\mu x^\nu \partial_\nu + 2x_\mu \Delta) - 2x^\nu \Sigma_{\mu\nu}]\Phi(x),
\end{aligned} \tag{2.30}$$

where $\Sigma_{\mu\nu}$ are matrices of the corresponding representation of the Lorentz group. Thus, we see that starting from a primary field, we can build the whole representation of the conformal algebra, and that it is univocally determined by a representation of the Lorentz group and the scaling dimension Δ.

One important feature of conformal symmetry is that it greatly restricts the form of the correlation functions of primary fields. It can be shown that 2-point correlators of fields of different dimension have to vanish, while if they have the same dimension,

$$\langle \phi(0)\phi(x) \rangle \propto \frac{1}{x^{2\Delta}}. \tag{2.31}$$

Similarly, 3-point functions are also determined up to a constant,

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle \propto \frac{1}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1 - x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}}. \tag{2.32}$$

One of the most useful tools of CFT are the Operator Product Expansions (OPEs). They state that the correlation function between two operators that act on close-by spacetime points can be written as a sum of local operators acting at that point,

$$\mathcal{O}_1(x)\mathcal{O}_2(y) = \sum_n c_{12}^n(x-y)\mathcal{O}_n(y). \tag{2.33}$$

Usually the leading order terms in the OPE diverge when $x \to y$. As a matter of fact, those are the only relevant terms, and it can be seen that the coefficients contain a lot of information about the symmetries. For example, from the OPE of a primary operator with the energy-momentum tensor we can extract the scaling dimension,

$$T_{\mu\nu}(x)\mathcal{O}(y) = \Delta\mathcal{O}(y)\partial_\mu\partial_\nu\frac{1}{(x-y)^2} + \cdots . \tag{2.34}$$

Lastly, it should be mentioned that in CFTs there is a one-to-one map between local operators $\mathcal{O}$ and states $|\mathcal{O}\rangle$. To see this, one needs to use radial quantization, where the time coordinate corresponds to the radial direction in $\mathbb{R}^d$, with the origin corresponding to past infinity. Then, the state operator map is defined by

$$|\mathcal{O}\rangle = \lim_{x\to 0}\mathcal{O}(x)|0\rangle. \tag{2.35}$$

This map is invertible, because for a state $|\mathcal{O}\rangle$ defined at a given time (i.e. in a sphere of a given radius), we can use conformal invariance to shrink that ball to a point and obtain a local operator.

2.2.3 Superconformal field theories

Another relevant question is whether or not conformal symmetry can be extended to a larger group that also includes supersymmetric transformations (see Appendix B for a review of the basics of SUSY theories).

It turns out that this is not always possible, only for certain dimensions ($d \leq 6$) and some numbers of SUSY charges. In addition to the generators of the conformal group and the supercharges, one needs to include also some additional fermionic generators S (one for each supercharge) called conformal supersymmetries, and –if appropriate– $R-$symmetry generators.

The concrete commutation relations of the conformal algebra depend on the number of dimensions (since the spinor representations are sensitive to that) and on the $R-$symmetry group. However, they always look schematically as follows,

$$\begin{gathered}[D,Q] = -\frac{i}{2}Q, \qquad [D,S] = \frac{i}{2}S, \qquad [K,Q] \simeq S, \qquad [P,S] \simeq Q, \\ \{Q,Q\} \simeq P, \qquad \{S,S\} \simeq K, \qquad \{Q,S\} \simeq M + D + R. \end{gathered} \tag{2.36}$$

Here we can also see that Q and S also raise and lower the scaling dimension of the field. Therefore, we can build representations starting with fields that are annihilated by both K and S and proceeding as before.

However, we are mostly interested in one concrete example, $\mathcal{N} = 4$ Super Yang Mills theory in 4 dimensions, which is the one that appears explicitly in the original AdS/CFT correspondence.

2.2.4 $\mathcal{N} = 4$ Super Yang-Mills

The field content of $\mathcal{N} = 4$ SYM in 4d is completely determined by the SUSY algebra (see Appendix B). It consists of one gauge field A_μ, four Weyl fermions λ^a and six real scalars X^i. The lagrangian is

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta_I}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} - \sum_a i\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda_a - \sum_i D_\mu X^i D^\mu X^i \right. \\ \left. + \sum_{a,b,i} gC_i^{ab}\lambda_a[X^i,\lambda_b] + \sum_{a,b,i} g\bar{C}_{iab}\bar{\lambda}^a[X^i,\bar{\lambda}^b] + \frac{g^2}{2}\sum_{i,j}[X^i,X^j]^2\right], \tag{2.37}$$

where the constants C_i^{ab} are related to the Clifford algebra.

The supersymmetry transformations that leave this lagrangian invariant are

$$\begin{aligned} \delta X^i &= [Q^a_\alpha, X^i] = C^{iab}\lambda_{\alpha b}, \\ \delta\lambda_b &= \{Q^a_\alpha, \lambda_{\beta,b}\} = F_{\mu\nu}(\sigma^{\mu\nu})^\alpha{}_\beta + [X^i,X^j]\epsilon_{\alpha\beta}(C_{ij})^a{}_b, \\ \delta\bar{\lambda}^b_{\dot{b}} &= \{Q^a_\alpha, \bar{\lambda}^b_{\dot{b}}\} = C_i^{ab}\bar{\sigma}^\mu_{\alpha\dot{\beta}}D_\mu X^i, \\ \delta A_\mu &= [Q^a_\alpha, A_\mu] = (\sigma_\mu)_\alpha{}^{\dot{\beta}}\bar{\lambda}^a_{\dot{\beta}}, \end{aligned} \tag{2.38}$$

where the $(C_{ij})^a{}_b$ are related to bilinears of Clifford matrices.

This field theory is scale invariant at the classical level. This can be seen from the fact that performing standard dimensional analysis one finds

$$[A_\mu] = [X^i] = 1, \qquad [\lambda^a] = \frac{3}{2}, \qquad [g] = [\theta_I] = 0, \tag{2.39}$$

and therefore there aren't any energy scales introduced in the lagrangian.

Upon quantization, it can be seen that there are no UV divergences in the loop contributions. Also non-perturbative instanton contributions are finite, and it is believed that the theory is fully UV finite. Therefore, no new energy scale is introduced in the renormalization process, the $\beta-$functions vanish identically, and the scale invariance is preserved at the quantum level. Therefore, we conclude that $\mathcal{N} = 4$ SYM is indeed a superconformal field theory.

It should be noted, however, that this symmetry may be spontaneously broken, depending on the potential term of the lagrangian (2.37),

$$-\frac{g^2}{2}\sum_{i,j} \mathrm{Tr}\left[[X^i,X^j]^2\right]. \tag{2.40}$$

Each term of the sum is positive or zero (this is a very general statement, due to the fact that the trace of the commutator defines the Killing form of a Lie algebra, which is negative-definite if and only if the corresponding Lie group is compact), and therefore the minimum satisfies

$$[X^i,X^j] = 0 \qquad \forall\, i,j = 1,\ldots,6. \tag{2.41}$$

If all the vevs $\langle X^i\rangle = 0$, (2.41) is automatically satisfied, and we are in the *conformal phase*, where the symmetry is explicit. However, we can also have solutions such that some $\langle X^i\rangle \neq 0$. In that case, the vev introduces an energy scale, and conformal invariance is spontaneously broken. This is called the *Coulomb phase*.

2.3 Supergravity and superstring theory

Until now, we have studied AdS space and Conformal Field Theories, the main ingredients that appear at either side of the holographic duality. Now, we begin to work towards the proof of the correspondence. In this section, we will swiftly overview the aspects of supergravity and superstring theory necessary to discuss D-branes (and in particular D3-branes), which will take a paramount role in said proof.

2.3.1 $D = 10$ supergravity

Supergravity can be understood as the resulting theory of promoting a global supersymmetry to a local one. From the defining algebra of the SUSY transformations,

$$\{Q^a_\alpha, \bar{Q}_{\beta b}\} = 2\sigma^\mu_{\alpha\beta} P_\mu \delta^a_b, \tag{2.42}$$

we see that if the supercharges depend on the spacetime point, also P_μ will. But the momentum generates spacetime translations, and promoting them to local transformations is nothing else than considering general diffeomorphisms. Therefore, we easily see that for a consistent theory of local SUSY, one needs to include gravity: hence the name, *supergravity*.

The presence of a spin 2 field together with supersymmetry automatically implies that in the theory a spin-3/2 field will appear. This field is called the *gravitino*. The classical equation of motion for a free gravitino es the Rarita-Schwinger equation,

$$\left(\epsilon^{\mu\kappa\rho\nu}\gamma_5\gamma_\kappa\partial_\rho - im\sigma^{\mu\nu}\right)\psi_\nu = 0. \tag{2.43}$$

Note that the field ψ_ν has both a vector index and a spinor index.

Similarly as what happens with global supersymmetry, imposing invariance under these transformations highly constraints the field content and form of the lagrangians. We will not work out the representations here (see [12] for a full treatment), but we will present the main results of $D = 10$ supergravity for future reference.[3]

The $\mathcal{N} = 2$, $D = 10$ type IIB SUGRA contains the following fields:

- $G_{\mu\nu}$, the metric/graviton, with 35 bosonic d.o.f.
- $C + i\Phi$, two scalar fields (axion and dilaton), with 2 bosonic d.o.f.
- $B_{\mu\nu} + iA_{2\mu\nu}$, two 2-forms with 56 bosonic d.o.f. ($B_{\mu\nu}$ is called the Kalb-Ramond field).
- $A_{4\mu\nu\rho\sigma}$, a 4-form with 35 bosonic d.o.f.
- $\psi^i_{\mu\alpha}$ with $i = 1, 2$, two Majorana-Weyl gravitinos with 112 fermionic d.o.f[4].
- λ^i_α with $i = 1, 2$, two Majorana-Weyl dilatinos with 16 fermionic d.o.f.

[3]Note that $D = 11$ is the maximal dimension for supergravity, since in other case dimensional reduction would lead to $\mathcal{N} > 8$ in four dimensions, which is not allowed by the Weinberg-Witten theorem. This also explains that in 10d we can have at most $\mathcal{N} = 2$.

[4]The Majorana and Weyl conditions for spinors are dependent on the number of the spacetime dimensions. The Weyl condition only makes sense for an even number of dimensions. The Majorana condition is only possible in $D = 0, 1, 2, 3, 4$ (mod 8). In dimensions $D = 0, 4$ (mod 8) Majorana and Weyl fermions are equivalent, while in $D = 2$ (mod 8) the two conditions can be imposed separately.

The action of type IIB supergravity is most easily written in the language of differential forms. One defines the following field strengths/curvatures,

$$F_1 = dC, \qquad H_3 = dB, \qquad F_3 = dA_2, \qquad F_5 = dA_4, \tag{2.44}$$

and from them, the combinations

$$\tilde{F}_3 = F_3 - CH_3, \qquad \tilde{F}_5 = F_5 - \frac{1}{2}A_2 \wedge H_3 + \frac{1}{2}B \wedge F_3. \tag{2.45}$$

Then, the action can be written as

$$\begin{aligned} S_{\text{IIB}} = &+ \frac{1}{4\kappa_B^2}\int \sqrt{G}e^{-2\Phi}\left(2R_G + 8\partial_\mu\Phi\partial^\mu\Phi - H_3 \wedge \star H_3\right) \\ &- \frac{1}{4\kappa_B^2}\int \left[\sqrt{G}\left(F_1 \wedge \star F_1 + \tilde{F}_3 \wedge \star \tilde{F}_3 + \frac{1}{2}\tilde{F}_5 \wedge \tilde{F}_5\right) + A_4 \wedge H_3 \wedge F_3\right] + \text{fermions}. \end{aligned} \tag{2.46}$$

An important symmetry of type IIB supergravity is the following. First one defines the complex fields

$$\tau := C + ie^{-\Phi}, \qquad G_3 := \frac{F_3 - \tau H_2}{\sqrt{\text{Im}\ \tau}}. \tag{2.47}$$

Then it can be checked that the action is left invariant by the $SL(2,\mathbb{R})$ transformations

$$\tau \mapsto \tau' = \frac{a\tau + b}{c\tau + d}, \qquad G_3 \mapsto G_3' = \frac{c\tau^* + d}{c\tau + d}G_3. \tag{2.48}$$

It should be mentioned that there is another consistent $\mathcal{N} = 2$, $D = 10$ supergavity theory, called type IIA. However, we will not need it here. The particular name is due to the fact that both theories can be understood as the low energy limit of the corresponding superstring theories, which we set to review in what follows.

2.3.2 Superstring theory

In bosonic string theory, one studies the vibration modes of strings propagating in Minkowski spacetime, which coincide with particle states from ordinary QFT. It can be undestood as a 2d scalar CFT, where the scalar fields are the embeddings of the worldsheet into spacetime. In superstring theory, one considers both worldsheet scalars $X^\mu(\sigma)$ and fermions $\psi^\mu_\pm(\sigma)$, and thus after quantization one has bosonic and fermionic creation and annihilation operators in order to build the Fock space. The boundary conditions on the worldsheet give rise to two different sectors of this space: the Neveu-Schwartz (NS) and Ramond (R) sectors. The ground state of the NS sector is bosonic (in spacetime), while the one of the R sector is fermionic. Supersymmetry requires imposing the GSO projection, which implies that we must have the same number of bosonic and fermionic d.o.f. at each mass level. Among other things, with this projection one gets rid of the tachyon from bosonic string theory. It turns out that there are several consistent truncations and selections of subsectors of the NS and R sectors, that lead to several different superstring theories –although they are related by dualities–. Here, we will be interested only in the type IIB one.

Interactions in string theory arise from the different embeddings one can have from the 2d worldsheet into 10d spacetime. In path integral language, one needs to integrate over the moduli spaces of all

Riemann surfaces (with fixed in and out states). This leads to the genus expansion, which in turn leads to understanding the interactions as the splitting and joining of string endpoints. In this way, one can assign to each 'vertex' a coupling that is dynamically determined by the theory,

$$g_s = e^{\langle\Phi\rangle}, \tag{2.49}$$

where Φ is the dilaton, a scalar field that appears in the massless level of the closed string.

One important feature of superstring theory is that its low energy limit is supergravity, as we shall now see by means of a simple example. At this limit, we can consistently consider the worldsheet fields X^μ and $\psi^\mu_\pm$ coupled to the lowest excitations of the string spectrum, instead of the full interacting theory. The resulting action is that of a non linear sigma model,

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma \sqrt{h}\left[\left(h^{mn}G_{\mu\nu}(X) + \epsilon^{mn}B_{\mu\nu}(X)\right)\partial_m X^\mu \partial_n X^\nu + \alpha' R_h^{(2)}\Phi(X)\right] + \text{ SUSY completion}, \tag{2.50}$$

where we only wrote the bosonic fields for simplicity. Σ is the 2d worldsheet, h_{mn} the metric on it, and $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ are the graviton, Kalb-Ramond and dilaton fields that appear as the lowest excitations of the NS-NS sector.

In the Fadeev-Popov quantization of string theory, conformal symmetry plays an important role in order to have the negative norm states decouple. This requires that, for consistency, no energy scales can be introduced upon renormalization. However, if one calculates the β-functions starting from the action (2.50), which can be done as a perturbative series in α', one finds

$$\begin{aligned}
\beta^G_{\mu\nu} &= \frac{1}{2}R_{\mu\nu} - \frac{1}{8}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + \partial_\mu\Phi\partial_\nu\Phi + \mathcal{O}(\alpha'),\\
\beta^B_{\mu\nu} &= -\frac{1}{2}D_\rho H^\rho{}_{\mu\nu} + \partial_\rho H^\rho{}_{\mu\nu} + \mathcal{O}(\alpha'),\\
\beta^\Phi &= \frac{1}{6}(D-10) + \alpha'\left(2\partial_\mu\Phi\partial^\mu\Phi - 2\nabla^\mu\partial_\mu\Phi + \frac{1}{2}R_G - \frac{1}{24}H_{\mu\nu\rho}H^{\mu\nu\rho}\right) + \mathcal{O}(\alpha'^2),
\end{aligned} \tag{2.51}$$

where $H_{\mu\nu\rho}$ is the field strength of the Kalb-Ramond field, ∇_μ is the covariant derivative including the Riemannian connection due to $G_{\mu\nu}$ and D_μ is the covariant derivative including the gauge connection due to $B_{\mu\nu}$.

Imposing that these β-functions vanish, we find, on the one hand, the critical dimension of superstring theory, D=10, and on the other hand the same equations of motion that are derived from the action of type IIB supergravity (2.46) in the case where all the A fields are zero (which corresponds to having ignored the fermionic and R-R sectors of the superstring spectrum), thus proving our claim that supergarvity is the low energy limit of string theory.

2.3.3 Branes in supergravity

Specially important for our purposes are the objects known as *branes*. In superstring theory, D-branes arise from the boundary conditions for the open strings, they are by definition the manifold where the endpoints can move. It turns out, however, that they are in fact dynamical objects that can interact with the different fields of the string spectrum. In supergravity one can also have objects known as p-branes (in principle different from D-branes), which we introduce now.

We have seen that the bosonic fields in supergravity can be generally written in terms of differential forms. A very natural thing to do when one has a $p+1$-form is to integrate it over a $p+1$-dimensional manifold, which defines a sensible action

$$S_{p+1} = T_{p+1} \int_{\Sigma_{p+1}} A_{p+1}. \tag{2.52}$$

This action is invariant under gauge transformations given by the shift $A_{p+1} \mapsto A_{p+1} + d\omega_p$ for any p-form ω_p. We also can define a gauge invariant quantity $F_{p+2} = dA_{p+1}$, whose flux is conserved due to Stoke's theorem,

$$\int_{S^{p+2}} F_{p+2} = \int_{\partial B^{p+3}} dA_{p+1} = \int_{B^{p+3}} d^2 A_{p+1} = 0. \tag{2.53}$$

In the case where $p = 0$ and A is the Maxwell field, this statement is equivalent to the inexistence of magnetic monopoles.

Solutions of supergravity that feature a manifold charged under A_{p+1} are referred to as p-branes, where p stands for the number of spacial dimensions. For every $p+1$-form one can define its magnetic dual, related by the Hodge duality of their respective field strengths, which is a $(D-3-p)$-form,

$$A_{p+1} \overset{d}{\to} F_{p+2} \overset{\star}{\to} \star F_{p+2} = F^{mag}_{D-p-2} \overset{d}{\leftarrow} A^{mag}_{D-3-p}, \tag{2.54}$$

and therefore one can also define the magnetic dual of a p-brane, which will be a $(D-4-p)$-brane.

For example, in $D = 11$ SUGRA it's possible to see that we only have a 3-form, and thus the only brane solution we can have is a 2-brane, denoted M2, and its magnetic dual M5. In $D = 10$ type IIB SUGRA, we have several forms: two 2-forms $B_{\mu\nu}$ (from the NS-NS sector in superstring theory) and $A_{2\mu\nu}$ (from the R-R sector), to which two different 1-branes will correspond, F1 (which happens to be the fundamental string) and D1 (which happens to be a D-brane); and a 4-form, and thus a 3-brane D3.

In the presence of a p-brane, Poincaré supersymmetry will in general break down. It can be seen that exactly half of the supersymmetries will be conserved. The Lorentz group will break into diagonal blocks corresponding to the directions parallel to the p-brane x^μ and directions perpendicular to the p-brane y^u. Translations in the perpendicular directions will always be broken, so the subgroup of the Poincaré group that will be preserved in total is $\mathbb{R}^{p+1} \times SO(1,p) \times SO(D-p-1)$.

This symmetry can be used to propose an ansantz for the metric in the supergavity equations of motion and easily solve them. We list the results for some of the examples above:

$$\begin{aligned}
&\text{Dp} \qquad ds^2 = H(\vec{y})^{-1/2} dx^\mu dx_\mu + H(\vec{y})^{1/2} d\vec{y}^2, \quad e^\Phi = H(\vec{y})^{(3-p)/4},\\
&\text{M2} \qquad ds^2 = H(\vec{y})^{-2/3} dx^\mu dx_\mu + H(\vec{y})^{1/3} d\vec{y}^2,\\
&\text{M5} \qquad ds^2 = H(\vec{y})^{-1/3} dx^\mu dx_\mu + H(\vec{y})^{2/3} d\vec{y}^2,
\end{aligned} \tag{2.55}$$

where the function H has the form

$$H(y) = 1 + \frac{L^{D-p-3}}{y^{D-p-3}}. \tag{2.56}$$

The parameter L has dimensions of length and therefore has to be proportional to $\sqrt{\alpha'}$ (the only dimensional parameter of string theory). It's also possible to see that it will be proportional to the string coupling g_s. Its concrete expression, however, depends on the concrete system under study.

One of the fundamental pillars of the proof of the AdS/CFT correspondence is the fact that p-branes (solutions of supergravity) and Dp-branes (manifolds where open strings can end) are two different descriptions of the same object. To see this, consider open strings propagating in the background of a p-brane. In the weak coupling limit $g_s \to 0$ (when the genus expansion makes sense and one is able to do calculations in string theory), the p-brane radius $L \to 0$, and the metric (2.56) becomes flat, except at $y = 0$, where there is a defect. Therefore, we have strings propagating in Minkowski space, and the interaction with the p-brane can be summarized by a suitable boundary condition, which turns out to be Dirichlet b.c. in the directions perpendicular to the p-brane and Neumann b.c. in the parallel ones.

2.3.4 The special case of D3-branes

The case of D3-branes is of particular interest for several reasons. First, the symmetry in the brane world-volume corresponds to the 4d Poincaré group. Second, the magnetic dual of a 3-brane is itself a 3-brane in 10d. Third, from (2.55) we see that the dilaton solution is a constant. The full D3-brane solution, including all the fields present in type IIB supergravity, is

$$\begin{aligned} \Phi = const_1, \qquad C = const_2, \qquad g_s = e^{\Phi}, \\ ds^2 = H(\vec{y})^{-1/2} dx^\mu dx_\mu + H(\vec{y})^{1/2} d\vec{y}^2, \\ B_{\mu\nu} = A_{2\mu\nu} = 0, \qquad F_{5\mu\nu\rho\sigma\tau} = \epsilon_{\mu\nu\rho\sigma\tau\alpha}\partial^\alpha H. \end{aligned} \tag{2.57}$$

We consider the particular case ofa stack of N D3-branes, localized at the coordinates $\vec{y}_I = 0$, $I = 4, \ldots, 10$. In this case, the function $H(y)$ is

$$H(y) = 1 + \frac{4\pi g_s N (\alpha')^2}{y^4}. \tag{2.58}$$

We can already extract information very relevant for the proof of the correspondence from (2.58), by comparing the characteristic length of the supergravity theory L and the length of the fundamental string $\ell_S = \sqrt{\alpha}$,

$$L^4 = 4\pi g_s N \ell_S^4. \tag{2.59}$$

If $g_s N \ll 1$, the supergravity length will be much smaller than the fundamental string length. We don't expect the supergravity approximation to work well in that case. However, in this case, since $N \in \mathbb{N}$, we have $g_s \ll 1$ and string perturbation theory is expected to be reliable. On the other hand, when $g_s N \gg 1$, $L \gg \ell_S$ and supergravity is a good approximation of the full string theory. Note that provided that N is large enough, we can still have in this case $g_s \ll 1$.

Far from the stack of D3-branes ($y \to \infty$), the metric (2.57) becomes flat. Close to the branes ($y \to 0$), however, it seems from (2.58) that there is a singularity. This is not the case, as can be seen by changing the coordinate $y \mapsto u = L^2/y$. In this frame, we can take the large u limit without any problem, and we obtain that

$$ds^2 = L^2 \left[\frac{du^2}{u^2} + \frac{1}{u^2}\eta_{ij} dx^i dx^j + d\Omega_5^2 \right]. \tag{2.60}$$

Comparing with (2.16) we see that the geometry corresponds to $AdS_5 \times S^5$, where the sphere and the Anti de Sitter spaces have the same radius L given by (2.59).

2.4 The AdS/CFT correspondence

We now have all the ingredients needed to sketch the proof of the original AdS/CFT correspondence as was proposed in the first place by Maldacena [8]. The strategy is to study the system of 3-branes from the perspective of string theory and supergravity respectively. At low energies, the first approach leads to a $\mathcal{N}=4$ SYM theory on the D-brane, while the second to supergravity on $AdS_5 \times S^5$.

We consider first string theory on a background given by the 3-brane SUGRA solution and take the limit $\alpha' \to 0$ keeping everything else constant. We will be interested in taking the low energy limit, and therfore we use the action of the non-linear sigma model for the string interactions. We start from (2.50) and (2.57), and do a rescaling of the metric $ds^2 = G_{\mu\nu}dx^\mu dx^\nu = L^2\tilde{G}_{\mu\nu}dx^\mu dx^\nu$, so that

$$S_G = \frac{L^2}{4\pi\alpha'}\int_\Sigma \sqrt{h}h^{mn}\tilde{G}_{\mu\nu}(x;L)\partial_m x^\mu \partial_n x^\nu, \tag{2.61}$$

$$ds^2 = \tilde{G}_{\mu\nu}(x;L)\ dx^\mu dx^\nu = \left(1+\frac{L^4}{u^4}\right)^{-1/2}\frac{1}{u^2}\eta_{ij}dx^i dx_j + \left(1+\frac{L^4}{u^4}\right)^{1/2}\left(\frac{du^2}{u^2}+d\Omega_5^2\right), \tag{2.62}$$

where we have ignored the part of the action corresponding to the F_5 field for simplicity.

From (2.58) we see that $L^2 \propto \alpha'$ and therefore the limit $\alpha' \to 0$ is well defined. If we denote

$$\lambda = g_s N, \tag{2.63}$$

we have

$$S_G = \sqrt{\frac{\lambda}{4\pi}}\int_\Sigma \sqrt{h}h^{mn}\tilde{G}_{\mu\nu}(x;0)\ \partial_m x^\mu \partial_n x^\nu, \tag{2.64}$$

$$ds^2 = \tilde{G}_{\mu\nu}(x;0)dx^\mu dx^\nu = \frac{1}{u^2}\eta_{ij}dx^i dx^j + \frac{du^2}{u^2} + d\Omega_5^2. \tag{2.65}$$

The geometry (2.65) is $AdS_5 \times S^5$ as in (2.60), but the length L has dropped out. This is the near horizon geometry of the 3-brane system, which allows us to reach the following conclusion: in the $\alpha' \to 0$ limit, the string modes in the bulk decouple, and the dynamics from (2.50) reduce to those of strings propagating arbitrarily close to the D-brane stack.

Now we need to take the low energy limit. From the conclusion above, we can ignore the fields due to closed strings away from the D-brane, they don't interact with our theory. We have only the fields due to open strings. In a situation where we have N separated D-branes, we will have in the lowest level of the spectrum $U(1)$ gauge fields labelled by Chan-Paton indices, and a $U(1)^N$ gauge symmetry in total. When the D-branes coincide, this symmetry is enhanced to $U(N)$. It is possible to see that the $U(1)$ subgroup corresponds to the overall position of the branes, and thus the gauge group in the world-volume is $SU(N)$. Also, we know that p-brane solutions always preserve part of the supersymmetry. At this point, we can make the educated guess that the theory must be $\mathcal{N}=4$ SYM, and indeed it can be checked that the massless fields of the open superstrings are the ones that are required. The physical parameters in the lagrangian (2.37) are also related to quantities in superstring theory: the coupling constant is the coupling constant of the open strings, and the instantonic angle in the Chern-Simmons term is given by the expectation value of the axion field (RR scalar),

$$g_{YM} = \sqrt{g_s} = e^{\langle\Phi\rangle/2}, \qquad \theta_I = \langle C\rangle. \tag{2.66}$$

Now we consider the same system from a different point of view, i.e. in the supergravity picture. We have already given the solution of the 3-brane system in this case (2.57), (2.58). For an observer far away from the branes, the energy measured will be related to the energy at finite y by the redshift factor

$$E \sim \sqrt{-G_{00}}E_y \sim H^{-1/4}E_y \sim yE_y. \tag{2.67}$$

Therefore, we see that we can have 2 kinds of low energy excitations: honest low energy modes in the bulk, and arbitrary energy modes close to the branes ($y \to 0$, where the metric becomes $AdS_5 \times S^5$). Then one needs to study the interactions between these two different kind of modes. This can be done using tools of QFT in curved spacetime, since the problem is somewhat similar to that of Hawking radiation (note from (2.56) that our metric is analogous to that of a black hole). In particular, one can compute the probability with which the bulk modes will be absorbed by the brane horizon. It turns out that this absorption cross section goes like $\sigma \sim \text{energy}^3$ [14, 15], so indeed at low energies they will decouple from the $AdS_5 \times S^5$ supergravity close to the branes.

Taking the low energy limit can be tricky in this situation, mainly due to the fact that, as we stated before, arbitrary energy modes will appear low energy to an observer at infinity. In order to keep these close-horizon-arbitrary-energy modes from disappearing at the same time as the ones in the bulk, the limit that we need to take is energy $\to 0$ and $y \to 0$ with y/α' constant, i.e. also $\alpha' \to 0$, like we did when doing the computation in the string theory side.

As a quick check, we can look at the global symmetries both sides of the correspondence. In the supergravity side we have $SO(2,4) \times SO(6)$ corresponding to the isometries of AdS_5 and S^5 respectively. On the field theory side, we have the conformal symmetry of Minkowski, $SO(2,4)$ times the R-symmetry (that rotates the different supercharges among themselves), which is $SU(4) \cong SO(6)$. Note that this requires that the conformal symmetry is explicit. The duality relates supergravity with $\mathcal{N} = 4$ SYM *in the conformal phase.*

Why do we say that the gauge theory lives in the boundary of AdS? This can be understood by means of a heuristic argument: notice that in the string theory calculation we didn't do any close-horizon approximation, (2.65) appeared purely because of the decoupling limit $\alpha' \to 0$, without

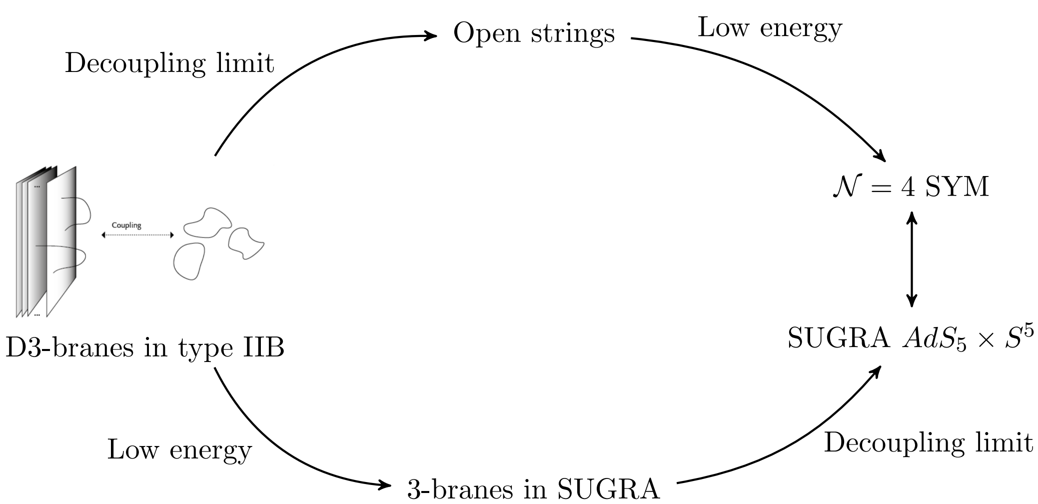


Figure 2.1: Basic idea of the AdS/CFT correspondence.

any involvement of y. Therefore, we can say that the YM theory lives in the full geometry (2.57), it's just that whatever takes place away from the branes doesn't affect it. On the other hand, in the supergravity calculation we did take the limit $y \to 0$. The full geometry, where the conformal theory lives, is found by "undoing" the limit, i.e. taking now $y \to \infty$ (or $u \to 0$ in the coordinates of (2.60)), which indeed corresponds to the conformal boundary of the AdS_5. This argument can be made fully precise, but for that one needs to study the system of a probe D3-brane in the background of the stack of branes. In that case, one can rigorously check that a string stretching from the probe brane to the stack corresponds in the supergravity picture to a string from the D3-brane to the AdS_5 boundary.

The remaining question is what is the regime of validity of the correspondence. We already argued in the previous section that if $g_s N \ll 1$, supergravity is not a good approximation. In this case, however, since $g_s = g_{YM}^2$ we can make use of perturbation theory in the field theory side. Conversely, if $g_s N \gg 1$ we won't be able to do calculations in the field theory, but supergravity will be reliable. Is enough for this to have g_s large and $N = 1, 2, 3, \dots$? The answer is no: making use of the $SL(2, \mathbb{R})$ transformations (2.48) one can effectively change $g_s \mapsto 1/g_s$ in the previous inequalities. To be able to do calculations in the gravity side of the correspondence, we need large N.

2.5 Holographic dictionary and Witten prescription

Now that we have a (at least partially) satisfactory proof of the AdS/CFT correspondence, we can begin to ask how can one make use of it to calculate different quantities and "translate" the results from one side of the duality to the other. In this section, we argue how this is done in the context of pure Conformal Field Theory and Anti de Sitter space. The next chapter will be devoted to extend these methods to confining QCD-like theories, which are obviously not conformal.

2.5.1 Field-operator correspondence and Witten prescription

The natural objects to relate at both sides of the AdS/CFT correspondence are fields and operators[5]. The key insight to relate them was given by Witten in [16] and Gubser, Klebanov and Polyakov in [17], and it is the fact that given a field configuration on the AdS_5 boundary, it can be uniquely extended to the full bulk geometry. For a massless scalar field, this can be easily proven:

Consider such a massless scalar, whose equation of motion is

$$D_i D^i \phi = 0 \tag{2.68}$$

in the bulk of the AdS space and with a given configuration in the boundary $\phi_0(\vec{x}) = \phi(\vec{x}, u = 0)$. If we add to ϕ any other solution to (2.68) that vanishes in the boundary, we would spoil the advertised uniqueness. Fortunately, one can prove that if a solution vanishes in the boundary it also vanishes everywhere else,

$$0 = \int_{AdS} d^5y \, \sqrt{g} \, \phi D_i D^i \phi = -\int_{AdS} d^5y \, \sqrt{g} \, |d\phi|^2, \tag{2.69}$$

and therfore $\phi = \text{constant} = 0$. Proving the same result for other fields (i.e. gauge fields, metrics, spinors) is significantly more complicated and we won't attempt to do it here.

[5]Note that in a CFT one does not have asymptotic states or an S-matrix, as opposed to a normal QFT. Thus, operators in different representations of the conformal algebra are the most natural object to consider.

Look, as an example, at the concrete case of a marginal operator $\mathcal{O}$ in the CFT that effectively changes the value of the coupling constant of the $\mathcal{N}=4$ SYM theory. By (2.66) this corresponds to changing the string theory coupling constant, that is, the classical configuration of the dilaton field $\phi(\vec{x},u)$. However, by the previous argument, this corresponds to changing the boundary value $\phi_0(\vec{x})$. Thus, it seems natural to add a term to the lagrangian

$$\int d^4x\ \phi_0(\vec{x})\mathcal{O}(\vec{x}) \tag{2.70}$$

In this case, by the correspondence of the couplings at both sides of the duality we immediately saw that a massless field corresponds to a operator with scaling dimension $\Delta = 4$ (marginal). However, we need to take into account that the space where the supergravity lives in is $AdS_5 \times S^5$. This means that even if in $D = 10$ all the SUGRA fields are massless, we will also have Kaluza-Klein massive modes in AdS_5 due to the compact manifold S^5 (see Appendix C for the details of the more simple S^1 case).

For the massive cases, the relation between the mass of the field in the gravity side and the scaling dimension of the operator it couples to in gravity can be obtained from the eigenvalues of the Laplace equation on the sphere, which will be related to the dependence on the radial coordinate in AdS. The relations that one finds are the following:

$$\begin{aligned} \text{scalar} \qquad & m^2 = \Delta\,(\Delta-4)\,, \\ \text{spin } 1/2, 3/2 \qquad & m^2 = (\Delta-2)^2\,, \\ p-\text{form} \qquad & m^2 = (\Delta-p)(\Delta+p-4), \\ \text{spin } 2 \qquad & m^2 = \Delta(\Delta-4). \end{aligned} \tag{2.71}$$

Once we have that, we can generalize (2.70) for each case, understanding that its meaning is that the fields in the AdS_5 space act as sources for the corresponding operators in the CFT. With this, one arrives at the so called *Witten prescription* that relates the partition functions at both sides of the correspondence,

$$\left\langle \exp\left\{\int d^4x\ \Phi_m(x)\mathcal{O}_\Delta(x)\right\}\right\rangle_{CFT} = \mathcal{Z}_{SUGRA}\left[\Phi_m(x,u)|_{u=0} = \Phi_{m,0}(x)\right], \tag{2.72}$$

from which we can calculate correlation functions in the CFT by solving classical equations of motion in the gavity side.

2.5.2 Holography

The AdS/CFT correspondence, in the terms we have just presented, provides a very powerful tool to do complicated calculations in Conformal Field Theory. The duality, however, is equally interesting in the other direction, since it provides a holographic description of gravity in Anti de Sitter space.

Before the advent of AdS/CFT, it was already believed in the community that holography had to play an important role in a theory of quantum gravity. This was in part due to the existence of several no-go theorems for theories of gravity coupled to matter, which could be circumvented if the gravity theory lived in a different numbers of spacial dimensions that the matter QFT. Moreover, standard techniques of QFT in curved spacetime gave rise to the Bekenstein-Hawking formula for a black hole entropy,

$$S_{BH} = \frac{A_h}{4G_N}, \tag{2.73}$$

where A_h is the area of the black hole horizon and G_N is Newton's constant.

This formula naturally leads to an upper bound for the entropy of any matter configuration. According to the second law of thermodynamics, the entropy always increases in any physical process. For any initial configuration, collapsing into a black hole is a possible physical process (note that here is where we use that the matter is coupled to gravity). Therefore, we always need to have

$$S_{matter} \leq S_{BH}, \tag{2.74}$$

i.e. all the degrees of freedom of the system should be able to be encoded on the surface.

The AdS/CFT correspondence was the first proposal of a theory which satisfied this bound in a natural way: all the degrees of freedom of the supergravity theory on AdS_{d+1} can be described by a field theory in d dimensions.

Then the question remains of what is the meaning of the radial coordinate in AdS. We already hinted towards it in our discussion of Anti de Sitter space, where we saw that in Poincaré coordintes, the metric looked like

$$ds^2 = R^2 \left(\frac{dz^2}{z^2} + z^2(-dt^2 + d\vec{x}^2) \right), \tag{2.75}$$

and the scaling transformation that leaves it invariant is

$$t \mapsto \lambda t, \qquad \vec{x} \mapsto \lambda \vec{x}, \qquad z \mapsto \frac{z}{\lambda}. \tag{2.76}$$

From this one can already see that z will be related to an energy scale, since it grows for smaller distances. Indeed, this can be rigorously seen from the matching of symmetries at both sides of the correspondence. In the CFT side, the scalings (generated by the operator D) clearly move us between different energy scales, by definition. The corresponding subgroup of $SO(2,4)$ gets mapped through the duality to the transformations (2.76).

Therefore, we can conclude that the AdS/CFT correspondence is more rich than just the field theory living in the boundary of AdS. The whole bulk of AdS can be foliated in terms of the radial coordinate, and in each one of the sheets (which have the same topology as the AdS boundary) lives the field theory at an energy scale given by z.

This statement is not very relevant for the case that we have so far considered, i.e. pure AdS and pure CFT (as in the original paper of Maldacena) because in any case the field theory is invariant under energy scalings. However, in the following chapter we will deal with how to break conformality and obtain in the QFT side a gauge theory that features a confinement-deconfinement phase transition. In that case, understanding the holographic coordinate as an energy scale will be of great importance, since it leads us to look at the gravity equations of motion with respect to said coordinate as Renormalization Group equations of the field theory, thus allowing us to infer physics in the IR from those in the UV.

3 Holographic theories for QCD

Now that we have presented the basic background regarding the AdS/CFT correspondence, we proceed to try to make use of it as a tool to study QCD, and in particular the Quark Gluon Plasma. In this chapter, we will show how to solve the main problems one has to face, mainly related to the fact that QCD is blatantly not conformal. First, we will comment on the two different strategies one can adopt to tackle this problem, called *top-down* and *bottom-up* approaches. Next, we will make use of the latter to build a phenomenological holographic theory for the strong interactions, and we will discuss how we can use it to calculate the thermodynamics of the QGP. Finally, we present the setup of the anisotropic QGP that we attempt to study in this thesis, and briefly review the work that has been done in that direction in recent years. Throughout this chapter, we will follow the exposition of [18, 19, 20, 21].

3.1 Introduction: *top-down* and *bottom-up* approaches

The AdS/CFT correspondence, as we have presented it in the previous chapter, cannot be directly applied to the study of strong interactions. This is due to the fact that there are some fundamental differences between $\mathcal{N} = 4$ SYM and QCD (see Appendix D for a brief review of QCD), among which we have:

- All of the fields in $\mathcal{N} = 4$ SYM are massless.
- In $\mathcal{N} = 4$ SYM the couplings don't run due to conformal invariance, i.e. there cannot be asymptotic freedom and IR confinement.
- The fermions in $\mathcal{N} = 4$ SYM transform in the adjoint of $SU(N)$ instead of the fundamental.
- In $\mathcal{N} = 4$ SYM there are 6 extra scalar fields, also in the adjoint representation.

There are two different ways to try to solve these issues. The first are the so called *top-down* approaches, where one considers a particular D-brane configuration in string theory and tries to derive another particular instance of the holographic correspondence that features a theory more similar to QCD in the field theory side. The second are the *bottom-up* approaches, where one applies the holographic dictionary and tries to build a theory on the gravity side such that the expected QFT dual shares as many characteristics with QCD as possible.

Top-down approaches have the obvious advantage that one can always track with precision the dictionary between quantities at both sides of the duality by using the complete string theory picture. They also have the drawback that finding a gauge theory that matches QCD with precision is extraordinarily difficult. Generally, it's only possible to find gauge theories that are qualitatively similar to QCD, but that never match it exactly (e.g. usually there is supersymmetry). As an example of this, in [22] they consider a stack of 5-branes and carry out the decoupling limit. On the gauge theory side, they find $\mathcal{N} = 1$ SYM, which is not conformal, features confinement in the IR and chiral symmetry breaking, but nevertheless has a different particle spectrum than QCD and a supersymmetry.

Very generally, one of the problems that arise in top-down approaches is the fact that starting with $D = 10$ superstring theory gives rise in the gravity side to a space that looks like $AdS \times K$ with K compact. This implies that we will have Kaluza-Klein modes that need to be mapped to some field through the correspondence. In the original case, these massive modes matched exactly the CFT operators with higher scaling dimension, but this is not the case with QCD.

An obvious way to try to circumvent the issue of the Kaluza-Klein modes is to start from string theory in a non-critical dimension, for example $D = 5$. However, this presents problems of its own, mostly regarding the decoupling limit. It can be seen that in these systems the curvature of the gravity solution is of the same order of magnitude as the string length, and thus said decoupling limit is incompatible with the low energy supergravity approximation.

On the other hand, bottom-up approaches are more direct and phenomenological ways to capture the IR dynamics of QCD with holographic tools. The main idea is to consider 5d gravity with a dilaton profile that can be used to model the behaviour of QCD. For example, the first models that used this kind of approach [23, 24], called *hard-wall* models, postulated a constant dilaton profile (like in the original type IIB SUGRA $\leftrightarrow \mathcal{N} = 4$ SYM correspondence), but with a IR cutoff in the deep interior of the AdS geometry dual to Λ_{QCD}. Imposing suitable boundary conditions at this IR cutoff then leads to the correct results for the masses of mesons and baryons, but also has problems, e.g. it can give non-sensical predictions for the behaviour of thermodynamic functions.

In order to attempt to overcome some of this problems, a slightly less crude model, called *soft-wall* model was introduced [25], where a dilaton profile was engineered and put by hand to give rise to realistic features.

The model that we will use in this work is the so-called *improved holographic QCD*, which can be understood as a variant of the soft-wall model but with a dynamical dilaton profile. This is motivated by the fact that in the non-critical 5d string theory picture, integrating out the 4-form that allows the existence of the D3-branes (and therefore of some $U(N)$ gauge theory in their world-volume) produces new terms for the dilaton field potential. In what follows, we will comment on how we can fix said dilaton potential with phenomenological arguments. This approach solves most of the problems that appeared in the hard or soft wall models, and gives quantitative predictions that seem to match very well with experimental results and lattice QCD calculations.

3.2 Construction of holographic QCD

To summarize, the ingredients that we have to build the holographic theory of QCD are the following:

- One extra dimension that we can understand as an RG scale. At least in the IR $g_s N \gg 1$ the 5d (non-critical) string theory should be approximately described by classical gravity.
- In the 5d gravitational setting we have the fields
  - Metric $g_{\mu\nu}$ that couples to the QFT energy momentum tensor $T_{\mu\nu}$. In the conformal boundary, this metric should asymptote to AdS_5.
  - Dilaton ϕ, whose asymptotic value is related to the coupling constant, and therefore couples to the QFT gauge kinetic term $\mathrm{Tr}[F \wedge \star F]$.
  - Axion χ, whose asymptotic value is related to the instantonic angle θ_I, and therefore couples to the QFT Chern-Simmons term $\mathrm{Tr}[F \wedge F]$. However, it can be seen that this term will be suppressed as $1/N$ with respect to the other two, so in the beginning we shall ignore this field.
- A dilaton potential $V(\phi)$ that gives rise to a dependence of ϕ in the holographic coordinate, so that we can break conformal symmetry and have a confining theory.

Taking this into account, our starting point is the action

$$S = M_p^3 N^2 \int d^5x \sqrt{-g}\left(R - \frac{4}{3}\partial_\mu\phi\partial^\mu\phi + V(\phi)\right) + 2M_p^3 \int_{\partial M} d^4x \sqrt{h}K + S_{ct}, \tag{3.1}$$

where the second term is the Gibbons-Hawking boundary term that one needs to add in the case of spacetime manifolds with boundary in order to give rise to the correct Einstein equations ($h_{\alpha\beta}$ is the induced metric in said boundary, and K its curvature); and S_{ct} are the counterterms necessary to have a finite on-shell action in the case of infinite volume spaces (which is necessary to be able to apply Witten's prescription).

For the metric, we will use the following ansatz in domain-wall coordinates,

$$ds^2 = du^2 + e^{2A(u)}\eta_{\mu\nu}dx^\mu dx^\nu, \tag{3.2}$$

where the boundary corresponds to $u \to -\infty$. Occasionally we will also use the conformal coordinate system, related to the domain-wall coordinates by

$$du = e^{A(r)}dr, \tag{3.3}$$

with which the boundary is now at $r = 0$, and the metric becomes

$$ds^2 = e^{2A(r)}\left(dr^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right), \tag{3.4}$$

The Einstein equations for this system are

$$A'' = -\frac{4}{9}\left(\phi'\right)^2, \tag{3.5}$$

$$3A'' + 12A'^2 = V(\phi), \tag{3.6}$$

where the prime denotes a derivative with respect to u.

3.2.1 Matching the UV: asymptotic freedom

Starting from this general setting we can now proceed to use arguments of a phenomenological nature to try to fix the potential $V(\phi)$. First we look at the UV asymptotics. We know that due to asymptotic freedom, at arbitrarily high energies QCD becomes conformal. Thus, we expect that in this limit, which corresponds to $u \to -\infty$, the space (3.2) becomes AdS_5. In other words, we require that

$$A \to -\frac{u}{\ell_{AdS}} \qquad \text{when } u \to -\infty, \tag{3.7}$$

where we rename the AdS radius as ℓ_{AdS} so that its not mistaken with the Ricci curvature R. Note that the resulting metric,

$$ds^2 = du^2 + e^{-2u/\ell_{AdS}}\eta_{\mu\nu}dx^\mu dx^\nu, \tag{3.8}$$

is related with the standard Poincaré coordiantes in AdS (2.15) by

$$e^{-u/\ell_{AdS}} = \ell_{AdS}z. \tag{3.9}$$

From these asymptotic behaviour we can already look at what the leading order behaviour of the potential in the UV will be. Plugging (3.7) into the equation of motion (3.6) we see that we must have

$$V(\phi) \to \frac{12}{\ell_{AdS}^2} \tag{3.10}$$

on the boundary, which matches the cosmological constant corresponding to AdS space.

In order to determine the subleading terms of the potential, we turn to the beta function of the $SU(N)$ Yang-Mills theory, which is well known. For small coupling g we know that it looks like

$$\beta(g) = \mu\frac{dg}{d\mu} = -b_0 g^2 - b_1 g^3 + \cdots . \tag{3.11}$$

Notice that in the UV, $1 \gg g = e^{\phi}$, so we must have $\phi(u) \to -\infty$ when $u \to -\infty$. Therefore, we propose an ansatz for the dilaton potential which is an expansion on e^{ϕ},

$$V(\phi) = \frac{12}{\ell_{AdS}}\left(1 + v_0 e^{\phi} + v_1 e^{2\phi} + \cdots\right). \tag{3.12}$$

The fact that we can identify $A(u)$ with the energy scale μ in the RG equation, which is motivated by its asymptotic behaviour, makes it easy to see that from (3.11) and the equations of motion (3.5), (3.6) we will be able to obtain v_0, v_1, etc. as functions of the known coefficients b_0, b_1,..., and thus determine the dilaton potential that will reproduce the correct behaviour for the dual QFT in the UV.

3.2.2 Matching the IR: linear confinement

Next, we proceed to perform a similar study of the IR asymptotics. In this case, we know that a QCD-like gauge theory should feature a linear confinement potential of the quarks when the distance between them is large enough,

$$V_{q\bar{q}}(L) = \sigma_0 L + \ldots, \tag{3.13}$$

where σ_0 is the tension of the string-like QCD flux tubes.

First, we can already extract some information from the fact that there needs to be confinement, linear or not. From the identification of the coupling constant with the exponential of the dilaton we see that we need to have that $\phi \to +\infty$ in the IR. Also, it's possible to see that we need to have $A \to -\infty$ in order to satisfy the Null Energy Condition, which in turn is related with the c-theorem in the field theory side [26].

The concrete behaviour of the confinement potential can be extracted from the Nambu-Goto action of the non-critical string theory. The endpoints of the string can be interpreted as the quarks, and we can look at the on-shell behaviour of the action as the separation between them grows $L \to \infty$. It can be seen that the effective potential will grow linearly with L as long as the function

$$A_s(u) = A(u) + \frac{2}{3}\phi(u) \tag{3.14}$$

has a minimum. In that case, it is possible to check that the coefficient σ_0 is (3.13) will be

$$\sigma_0 = \frac{e^{2A_s(u_{min})}}{2\pi\alpha'}. \tag{3.15}$$

It remains to see how the existence of the minimum of A_s translates into a requirement for the dilaton potential. We know that in the UV that combination diverges and starts decreasing towards the interior, so we know that if in the deep IR it also diverges towards $+\infty$, we must have a minimum for some value of u between both limits. Since $A(u)$ and $\phi(u)$ separately are monotonic, this means we have to require that

$$\frac{dA}{d\phi} > -\frac{2}{3} \qquad \text{in the IR.} \tag{3.16}$$

More precisely, we must have

$$\lim_{\phi\to\infty}\left(\frac{d\phi}{3dA}+\frac{1}{2}\right)\phi = const, \tag{3.17}$$

since in the IR we must require $\phi \to \infty$. Combining (3.17) with the Einstein equations (3.5), (3.6) it's possible to see that we must have

$$\frac{d\phi}{3dA} \to -\frac{3}{8}\frac{V'(\phi)}{V(\phi)} \qquad \phi \to \infty. \tag{3.18}$$

This fixes the IR behaviour of the potential to

$$V(\phi) \sim V_{IR} e^{\frac{4}{3}\phi}\phi^q, \qquad q > 0. \tag{3.19}$$

In this work, in an aim for generality, we will drop the requirement of linear confinement, and consider the most generic potential that features confinement in the IR,

$$V(\phi) = V_{IR} e^{2m\phi}\phi^n, \tag{3.20}$$

for two positive constants m and n.

3.3 Thermodynamics and the confinement-deconfinement phase transition

Having determined the IR and UV asymptotics of the dilaton potential that we will use in the following chapters of this thesis, we now present the techniques that allow the study the thermodynamics of the system and in particular of the confinement-deconfinement phase transition.

The most straight-forward way to do so would be to use Witten-prescription (2.72), from which we can obtain the partition function (and therefore the free energy) of the system by evaluating the action on the gravity side on the classical solution that we obtain by solving Einstein's equations. However, in order to do so, we would need to explicitly compute the Gibbons-Hawking boundary term as well as the counterterms in (3.1).

Less direct, although computationally easier, is to study these aspects directly from the perspective of black hole thermodynamics. Thus, in this section we will briefly review how we can obtain the full thermodynamics of the plasma solely from the (solution of the) metric (3.2).

3.3.1 Temperature

In order to study the thermodynamics, we need to modify our ansatz for the metric by adding a blackening factor $f(r)$ that can give rise to the existence of a black hole (or rather a black brane). In conformal coordinates,

$$ds^2 = e^{2A(r)}\left(-f(r)dt^2 + d\vec{x}^2 + \frac{dr^2}{f(r)}\right). \tag{3.21}$$

Assume that we have a black-brane solution, i.e. there is a point r_h in the holographic coordinate such that $f(r_h) = 0$. In this situation, the temperature of the black-brane can be calculated as the inverse of the period of euclidean time, necessary to avoid a conical singularity. The argument goes as follows: look only at the r and t part of the metric, with a Wick rotation to euclidean time,

$$ds^2 = f(r)d\tau^2 + \frac{dr^2}{f(r)}, \tag{3.22}$$

and do a Taylor expansion of $f(r)$ around r_h,

$$ds^2 = \xi f'(r_h)d\tau^2 + \frac{d\xi^2}{\xi f'(r_h)}, \tag{3.23}$$

where $\xi = r - r_h$. Then we can do the coordinate transformation

$$\frac{d\xi^2}{\xi f'(r_h)} = du^2 \qquad \Rightarrow \qquad \xi = \frac{f'(r_h)}{4}u^2, \tag{3.24}$$

with which the metric becomes

$$ds^2 = du^2 + u^2 d\left(\frac{f'(r_h)\tau}{2}\right)^2. \tag{3.25}$$

In order to avoid a conical singularity, we need that the timelike direction has a period of 2π,

$$\frac{f'(r_h)\tau}{2} = \frac{f'(r_h)\tau}{2} + 2\pi, \tag{3.26}$$

so our original euclidean time must behave like

$$\tau = \tau + \frac{4\pi}{|f'(r_h)|}. \tag{3.27}$$

Since we know that the period of euclidean time equals $\hbar\beta$, in natural units we get

$$T = \frac{|f'(r_h)|}{4\pi}. \tag{3.28}$$

3.3.2 Entropy and free energy

Another thermodynamic quantity that we can easily calculate if we know the metric is the entropy of the black brane solution. This is given by the Berkenstein-Hawking formula,

$$S_{BH} = \frac{A_h}{4G_N}, \tag{3.29}$$

where G_N is Newton's constant and A_h the area of the event horizon. This area can be calculated as the square root of the determinant of the metric on the surface determined by $r_h = 0$, i.e. starting from (3.21) with $dt = dr = 0$,

$$S = \frac{e^{3A(r_h)}}{4G_N}. \tag{3.30}$$

Once we know temperature and entropy we can calculate every other thermodynamic quantity by means of the standard techniques. In particular, we want to study the confinement deconfinement phase transition, so we will be interested on the free energy,

$$dF = -SdT \qquad \Rightarrow \qquad F = -\int SdT. \tag{3.31}$$

We have already seen how to obtain entropy and temperature as a function of the radius of the horizon r_h, so we will be able to calculate F as

$$F = -\int S(r_h)\frac{dT(r_h)}{dr_h}dr_h. \tag{3.32}$$

3.3.3 Phase transition

The existence of the phase transition in the holographic picture is understood by the fact that general systems of partial differential equations don't have unique solutions for a given boundary condition. In our case, this means that we can have solutions with $f(r) = 1$ or with $f(r) \neq 1$.

In the solution with $f(r) = 1$, there cannot be any conical singularity and therefore we can impose any compactification of the euclidean time. We call this phase the *thermal gas* phase, which exists from $T = 0$ (no compactification of τ at all) up to arbitrary T. This phase corresponds to the confined state of the large N plasma, which can be seen by studying the behaviour of the perturbations on top of the background metric and checking that they obey linear confinement.

In the other solution we have $f(r) \neq 1$. In this case the temperature will be fixed by the derivative of f at the black brane horizon. In particular, since f cannot be a constant, we have a minimum temperature T_{min} at which the black-brane solution starts to exist. This solution corresponds to the deconfined plasma.

We see that in general, for $T > T_{min}$, we will have more than one valid solution for the metric. In that case, which of the phases is preferred will be determined by the free energy. What will happen is that there is a critical temperature $T_C > T_{min}$ at which there is a Hawking-Page phase transition. In the gravity side, this phase transition occurs due to the instability of the AdS black-hole solution: for temperatures smaller that T_C the specific heat becomes negative. On the gauge theory side of the correspondence this corresponds to the confinement-deconfinement phase transition.

3.4 Anisotropic holographic QCD

As we have stated in the introduction to this thesis, the aim of this work is to investigate an anisotropic plasma, a direction of research that has seen increasing effort put into it in recent years mainly due to the broad range of possible applications, from heavy-ion collisions to condensed matter experiments. However, these systems have not yet been studied with any degree of generality. In particular, the two main strategies that have been used are:

- Starting from an holographic model coming from a top-down approach to AdS/QCD, introduce a mechanism that at the same time breaks conformality and isotropy. Examples of this are [2, 3].

- Consider a bottom-up approach for a charged plasma that couples to an external magnetic field that breaks the isotropy. Examples of this are [4, 5, 6].

It wasn't until very recently where an anisotropic system was studied without introducing any particular source for the breaking of the rotational symmetry. In [1], the authors introduced an axion field on the gravity side, dual to a generic marginal operator in the field theory side with a coupling constant dependant on one of the spatial coordinates. This field, which we denote χ, is introduced in the action coupled to the dilaton as follows:

$$S = M_p^3 N_c^2 \int d^5x \sqrt{-g}\,[R + \mathcal{L}_M] + S_{GH} + S_{ct}, \tag{3.33}$$

$$\mathcal{L}_M = -\frac{4}{3}(\partial\phi)^2 + V(\phi) - \frac{1}{2}Z(\phi)(\partial\chi)^2. \tag{3.34}$$

Due to the anisotropy, the ansatz for the metric has to change from (3.21) to

$$ds^2 = e^{2A(r)}\left(-f(r)dt^2 + d\vec{x}_\perp^2 + e^{2h(r)}dx_3^2 + \frac{dr^2}{f(r)}\right), \tag{3.35}$$

and a dilaton profile depending only in the holographic coordinate,

$$\phi = \phi(r). \tag{3.36}$$

It turns out that the equation of motion for the axion field is immediately satisfied by

$$\chi(x_3) = a x_3, \tag{3.37}$$

with a a constant.

In said paper, the authors considered dilaton and axion potentials $V(\phi)$ and $Z(\phi)$ given by exponentials of ϕ, and studied the thermodynamics of the QGP-like plasma. The main result of their analysis was that the temperature of the confinement-deconfinement phase transition depends on the "degree of anisotropy" a, and that in general it is always lower that the critical temperature in the isotropic case. The relevance of their calculation lies in that said difference was proven to be due to the anisotropy itself, and not to an intrinsically magnetic effect as was previously believed.

In this work, we attempt to generalise and extend the analysis in [1], by considering the most general axion and dilaton potentials that can give rise to (linear) confinement in the IR, given by (3.20), i.e. an exponential times a power law correction that, as argued in section 3.2, cannot be fixed by phenomenological arguments.

4 Infrared structure

We have presented, in the previous chapters, the framework of holographic QCD with which we attempt to study the IR dynamics of the general anisotropic QGP-like plasma. In this chapter, the core of this thesis, we show in detail the solution of this system in the gravity side. The outline of the strategy is to rewrite Einstein equations in a way that can be directly integrated. Due to the generality of the dilaton and axion potentials under consideration, an analytic expression for the functions of the metric can only be found as an asymptotic expansion in the IR limit, not for the whole range of the holographic coordinate. Fortunately, this is precisely the regime we are interested in, since it is where standard QCD calculations cannot be trusted.

4.1 Holographic setup

To sum up the discussion of the previous sections, the system that we will study in this work is the one given by (3.33), (3.34),

$$S = M_p^3 N_c^2 \int d^5x \sqrt{-g}\,[R + \mathcal{L}_M]\,, \tag{4.1}$$

$$\mathcal{L}_M = -\frac{4}{3}(\partial\phi)^2 + V(\phi) - \frac{1}{2}Z(\phi)(\partial\chi)^2, \tag{4.2}$$

for a family of potentials with IR behaviour that looks like

$$V(\phi) = V_{IR}\; e^{2m\phi}\phi^n, \qquad Z(\phi) = Z_{IR}\; e^{2\beta\phi}\phi^\alpha. \tag{4.3}$$

As we saw in section 3.2, the constants m and n need to be positive. However, we don't have to make this requirement of β and α.

The ansatz for the metric is the same as in [1], that is,

$$ds^2 = e^{2A(r)}\left(-f(r)dt^2 + d\vec{x}_\perp^2 + e^{2h(r)}dx_3^2 + \frac{dr^2}{f(r)}\right), \tag{4.4}$$

$$\phi = \phi(r), \qquad \chi = a\; x_3. \tag{4.5}$$

Note that we are using conformal coordinates as in (3.4), so the UV and IR limits are

$$\text{UV} \qquad r \to 0, \tag{4.6}$$

$$\text{IR} \qquad r \to \infty. \tag{4.7}$$

4.1.1 Equations of motion

In order to find the functions $A(r)$, $h(r)$ and $\phi(r)$, we need to solve Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}, \qquad T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{4.8}$$

In our case, the energy momentum tensor that we obtain from the matter lagrangian (4.2) is

$$T_{\mu\nu} = \frac{8}{3}\partial_\mu\phi\partial_\nu\phi + Z(\phi)\partial_\mu\chi\partial_\nu\chi + g_{\mu\nu}\left(-\frac{4}{3}(\partial\phi)^2 + V(\phi) - \frac{1}{2}Z(\phi)(\partial\chi)^2\right). \tag{4.9}$$

On the other hand, the only non-vanishing components of the Einstein tensor (LHS of (4.8)) are

$$G_{tt} = -\frac{1}{2}f(r)\left[f'(r)\left(3A'(r)+h'(r)\right)+2f(r)\left(3A'(r)^2+3A'(r)h'(r)+h'(r)^2+3A''(r)+h''(r)\right)\right], \tag{4.10}$$

$$G_{11} = G_{22} = 3A'(r)f'(r)+f'(r)h'(r)+\frac{1}{2}f''(r)+f(r)\left(3A'(r)^2+3A'(r)h'(r)+h'(r)^2+3A''(r)+h''(r)\right), \tag{4.11}$$

$$G_{33} = \frac{1}{2}e^{h(r)}\left(6A'(r)f'(r)+6f(r)\left(A'(r)^2+A''(r)\right)+f''(r)\right), \tag{4.12}$$

$$G_{rr} = 3A'(r)\left(2A'(r)+h'(r)\right)+\frac{f'(r)}{2f(r)}\left(3A'(r)+h'(r)\right). \tag{4.13}$$

The corresponding components of (4.9) are

$$T_{tt} = -e^{2A(r)}f(r)V(\phi), \tag{4.14}$$

$$T_{11} = T_{22} = e^{2A(r)}V(\phi), \tag{4.15}$$

$$T_{33} = \frac{1}{2}Z(\phi)a^2+e^{2(A(r)+h(r))}V(\phi), \tag{4.16}$$

$$T_{rr} = \frac{1}{2}\phi'(r)^2+\frac{e^{2A(r)}}{f(r)}V(\phi). \tag{4.17}$$

From this, and after some simplifications, we obtain four independent equations of motion,

$$\frac{a^2}{2}Z(\phi)e^{-2h}+\left[f'h'+f\left(3A'h'+h'^2+h''\right)\right]=0, \tag{4.18}$$

$$-A'^2+\frac{h'^2}{3}+\frac{4\phi'^2}{9}+A''+\frac{h''}{3}=0, \tag{4.19}$$

$$f''+f'(3A'+h')=0, \tag{4.20}$$

$$e^{2A}V(\phi)-\frac{a^2}{2}Z(\phi)e^{-2h}-\left[f'(3A'+h')+f(12A'^2+6A'h'-\frac{4\phi'^2}{3})\right]=0. \tag{4.21}$$

Note that in principle, besides Einstein equations, that come from varying the action (4.1) with respect to $g^{\mu\nu}$, we should also consider the dilaton equation of motion. However, it turns out that this equation is not independent.

For now we are interested in finding the thermal gas solution, that is, the solution without a black hole. Therefore, we set $f(r)=1$, which automatically solves (4.20). The three differential equations that we will need to solve are therefore

$$\frac{a^2}{2}Z(\phi)e^{-2h}+3\dot{A}\dot{h}+\dot{h}^2+\ddot{h}=0, \tag{4.22}$$

$$-\dot{A}^2+\frac{\dot{h}^2}{3}+\frac{4\dot{\phi}^2}{9}+\ddot{A}+\frac{\ddot{h}}{3}=0, \tag{4.23}$$

$$e^{2A}V(\phi)-\frac{a^2}{2}Z(\phi)e^{-2h}-(12\dot{A}^2+6\dot{A}\dot{h}-\frac{4\dot{\phi}^2}{3})=0, \tag{4.24}$$

where we have changed the notation of the derivative with respect to r from a prime to a dot.

4.1.2 A useful change of coordinates

The equations (4.22)–(4.24), as they stand, are quite difficult to solve by brute force. The strategy that we will use instead is to manipulate them until we can reformulate the problem with a system of differential equations of the form $\vec{y}' = F(\vec{y})$. In order to do so, it is convenient to perform the following change of variables. We have seen in the previous chapter that the function $A(r)$ is decreasing and can be understood as the energy scale. Since it is a monotonic function in r, and therefore injective, it is a well defined coordinate transformation. In order to use A as a coordinate, however, we need to introduce new functions $q(A)$ and $p(A)$, defined by

$$-e^p = q = e^A \frac{dr}{dA}. \tag{4.25}$$

With this, the metric transforms to

$$ds^2 = -e^{2A} f(r(A))dt^2 + e^{2A} d\vec{x}_\perp^2 + e^{2(h(r(A))+A)} dx_3^2 + \frac{e^{2p(A)}}{f(r(A))} dA^2 \tag{4.26}$$

We can use (4.25) to properly substitute the derivatives of A with respect to r in the equations of motion,

$$\frac{d^2A}{dr^2} = \frac{d}{dA}\left(\frac{dA}{dr}\right)\frac{dA}{dr} = \frac{e^A}{q}\frac{d}{dA}\left(\frac{e^A}{q}\right) = e^{2A}\frac{q-q'}{q^3} = e^{2A}(1-p')e^{-2p}, \tag{4.27}$$

where now a prime denotes a derivative with respect to A. With this, the equations (4.22)–(4.24) transform to

$$\frac{a^2}{2} Z(\phi) e^{2p} e^{-2(h+A)} + 3h' + h'^2 + h'' + h'(1-p') = 0, \tag{4.28}$$

$$-1 + \frac{h'^2}{3} + \frac{4\phi'^2}{9} + (1-p') + \frac{h''}{3} + \frac{h'}{3}(1-p') = 0, \tag{4.29}$$

$$e^{2p} V(\phi) - \frac{a^2}{2} Z(\phi) e^{2p} e^{-2(h+A)} - (12 + 6h' + \frac{4\phi'^2}{3}) = 0. \tag{4.30}$$

Lastly, we also do a shift of the function h,

$$h \mapsto \tilde{h} = h + A. \tag{4.31}$$

With these changes, the metric (4.4) becomes

$$ds^2 = -e^{2A} f(A)dt^2 + e^{2A} d\vec{x}_\perp^2 + e^{2\tilde{h}(A)} dx_3^2 + \frac{e^{2p(A)}}{f(A)} dA^2, \tag{4.32}$$

and the equations,

$$\frac{a^2}{2} Z(\phi) e^{2p} e^{-2\tilde{h}} + (4-p')(\tilde{h}'-1) + (\tilde{h}'-1)^2 + \tilde{h}'' = 0, \tag{4.33}$$

$$(\tilde{h}'+2)(1-p') + \tilde{h}'' + (\tilde{h}'-1)^2 + \frac{4\phi'^2}{3} - 3 = 0, \tag{4.34}$$

$$e^{2p} V(\phi) - \frac{a^2}{2} Z(\phi) e^{2p} e^{-2\tilde{h}} - 12 - 6(\tilde{h}'-1) + \frac{4\phi'^2}{3} = 0. \tag{4.35}$$

These equations can be solved for $\tilde{h}''$, ϕ' and p', finding:

$$\tilde{h}'' = -\frac{1}{6}e^{2p}\left(3a^2e^{-2\tilde{h}}Z(\phi) + 2V(\phi)(\tilde{h}' - 1)\right), \tag{4.36}$$

$$8\phi'^2 = 36(\tilde{h}' + 1) - 6e^{2p}V(\phi) + 3a^2e^{-2\tilde{h}}e^{2p}Z(\phi), \tag{4.37}$$

$$p' = \frac{1}{3}\left(9 - e^{2p}V(\phi) + 3\tilde{h}'\right). \tag{4.38}$$

Being able to write Einstein equations in this way is the main reason why it's more convenient to use A instead of r as a coordinate.

4.2 IR fixed point and slow-roll

It is easy to see that the equations (4.36)–(4.38) have a fixed point, given by

$$\tilde{h}' = 0 \rightarrow \tilde{h}_* = 0, \tag{4.39}$$

$$3a^2Z(\phi_*) = 2V(\phi_*), \tag{4.40}$$

$$e^{2p_*}V(\phi_*) = 9, \tag{4.41}$$

$$a^2e^{2p_*}Z(\phi_*) = 18, \tag{4.42}$$

where we can set $\tilde{h}_* = 0$ by absorbing the exponential $e^{2\tilde{h}_*}$ in the constant a. Moreover, close to this fixed point we can see that there exists a slow-roll solution. By plugging in the ansatz

$$\phi(A) = \phi_* + c_\phi A, \tag{4.43}$$

$$\tilde{h}(A) = c_h A, \tag{4.44}$$

$$p(A) = p_* + c_p A, \tag{4.45}$$

into the equations (4.36)–(4.38), we obtain

$$\frac{3}{2}a^2e^{-2c_hA}Z(\phi(A)) - (1 - c_h)V(\phi(A)) = 0, \tag{4.46}$$

$$8c_\phi^2 + 3e^{2p(A)}V(\phi(A)) + \frac{54a^2e^{-2c_hA}Z(\phi(A))}{V(\phi(A))} - 3a^2e^{-2c_hA}e^{2p(A)}Z(\phi(A)) = 72, \tag{4.47}$$

$$c_p + \frac{1}{3}e^{2p(A)}V(\phi(A)) + \frac{3a^2e^{-2c_hA}Z(\phi(A))}{2V(\phi(A))} = 4. \tag{4.48}$$

Assuming the coefficients c_i are small and we can stay at linear order, these equations can be rewritten as

$$e^{2p(A)}V(\phi(A)) = 9\left(1 + \frac{c_h}{2}\right), \tag{4.49}$$

$$\frac{3}{2}a^2e^{-2c_hA}Z(\phi(A)) = (1 - c_h)V(\phi(A)), \tag{4.50}$$

$$\frac{1}{3}e^{2p(A)}V(\phi(A)) = 3 + c_h - c_p. \tag{4.51}$$

From this, we can conclude that such a slow-roll solution exists (we don't need the concrete expressions of c_h, c_p and c_ϕ for now). Motivated by this, and given the fact that we care only about the IR dynamics of the system, we proceed to look for more general solutions where the second derivative doesn't necessarily vanish, but is negligible in the deep IR (large r, large $-A$ limit).

4.3 Flow equations

As we have just stated, we now will look for solutions of the equations of motion that have $d^2/dA^2 \sim 0$ in the IR. This will allow us to rewrite the system of differential equations (4.36)–(4.38) into one of the form $\vec{y}' = F(\vec{y})$. These are what we call the *flow equations*, and as we will see, we will be able to integrate them easily to find a solution. Evidently, we will need to check that indeed the second derivative of said solution is small in the deep IR so that our approximation is consistent.

4.3.1 Derivation of the flow equations

Under the approximation of negligible second derivatives with respect to A, the resulting Einstein equations that we find are

$$a^2 e^{-2\tilde{h}+2p} Z(\phi) + 2(-1+\tilde{h}')(3+\tilde{h}'-p') = 0, \tag{4.52}$$

$$3\tilde{h}'^2 + 4\phi'^2 = 6p' + 3\tilde{h}'(1+p'), \tag{4.53}$$

$$6e^{2p}V(\phi) + 8\phi'^2 = 36 + 3a^2 e^{-2\tilde{h}+2p} Z(\phi) + 36\tilde{h}'. \tag{4.54}$$

Similarly to what we did before, now we can solve for $\tilde{h}'$, p' and ϕ', and we obtain

$$\tilde{h}' = 1 - \frac{3a^2 e^{-2\tilde{h}} Z(\phi)}{2V(\phi)}, \tag{4.55}$$

$$8(\phi')^2 = 72 - 6e^{2p}V(\phi) + 3a^2 e^{-2\tilde{h}} e^{2p} Z(\phi) - 54\frac{a^2 e^{-2\tilde{h}} Z(\phi)}{V(\phi)}, \tag{4.56}$$

$$p' = 4 - \frac{1}{3}e^{2p}V(\phi) - \frac{3a^2 e^{-2\tilde{h}} Z(\phi)}{2V\phi}. \tag{4.57}$$

These now have become first order differential equations, which means they can have solutions that only of class $\mathcal{C}^1$. In order to ensure that they are $\mathcal{C}^2$, we also impose that the derivatives of these equations are also satisfied. This results in two new equations (the third one is not independent),

$$-\frac{3a^2 e^{-2\tilde{h}}}{2V(\phi)^2}\left[Z(\phi)V'(\phi)\dot{\phi} + V(\phi)\left(2Z(\phi)\dot{\tilde{h}} - Z'(\phi)\dot{\phi}\right)\right] = 0, \tag{4.58}$$

$$\begin{aligned}\frac{3e^{-2\tilde{h}}}{(V(\phi))^2}\Big[&4e^{2\tilde{h}+2p}(V(\phi))^3\dot{p} - 18a^2 Z(\phi)V'(\phi)\dot{\phi} + 18a^2 V(\phi)\left(-2Z(\phi)\dot{\tilde{h}} + Z'(\phi)\dot{\phi}\right)\\ &+e^{2p}(V(\phi))^2\left(2a^2 Z(\phi)(\dot{\tilde{h}} - \dot{p}) + \dot{\phi}\left(2e^{2\tilde{h}}V'(\phi) - a^2 Z'(\phi)\right)\right)\Big] = 0,\end{aligned} \tag{4.59}$$

where we have again switched notation, so that a prime denotes a derivative with respect to ϕ and a dot with respect to A. Now we can solve for $\dot{\phi}(A)$, $\dot{\tilde{h}}(A)$, $\dot{p}(A)$, $e^{2p(A)}$ and $a^2 e^{-2\tilde{h}(A)}$ in equations (4.55)–(4.59). The result is

$$\dot{\phi} = \frac{6Z(\phi)\left(-3Z(\phi)V'(\phi) + V(\phi)Z'(\phi)\right)}{-3Z(\phi)V'(\phi)Z'(\phi) + V(\phi)\left(16(Z(\phi))^2 + 3(Z'(\phi))^2\right)}, \tag{4.60}$$

$$\dot{\tilde{h}} = \frac{3[3(Z(\phi))^2(V'(\phi))^2 - 4V(\phi)Z(\phi)V'(\phi)Z'(\phi) + (V(\phi))^2(Z'(\phi))^2]}{V(\phi)(-3Z(\phi)V'(\phi)Z'(\phi) + V(\phi)\left(16(Z(\phi))^2 + 3(Z'(\phi))^2\right)}, \tag{4.61}$$

$$\dot{p} = \frac{3Z(\phi)V'(\phi)\left(3Z(\phi)V'(\phi) - V(\phi)Z'(\phi)\right)}{V(\phi)\left(-3Z(\phi)V'(\phi)Z'(\phi) + V(\phi)\left(16(Z(\phi))^2 + 3(Z'(\phi))^2\right)\right)}, \tag{4.62}$$

$$e^{2p} = \frac{-54Z(\phi)V'(\phi)Z'(\phi) + 36V(\phi)(4(Z(\phi))^2 + (Z'(\phi))^2)}{V(\phi)(-3Z(\phi)V'(\phi)Z'(\phi) + V(\phi)\left(16(Z(\phi))^2 + 3(Z'(\phi))^2\right))}, \tag{4.63}$$

$$a^2 e^{-2\tilde{h}} = \frac{2[16(V(\phi))^2 Z(\phi) - 9Z(\phi)(V'(\phi))^2 + 9V(\phi)V'(\phi)Z'(\phi)]}{-9Z(\phi)V'(\phi)Z'(\phi) + V(\phi)\left(48(Z(\phi))^2 + 9(Z'(\phi))^2\right)}. \tag{4.64}$$

Finally, plugging the expressions of the IR behaviour of the potentials from (4.3),

$$\dot{\phi} = \frac{6\phi[3n - \alpha + (6m - 2\beta)\phi]}{3(n-\alpha)\alpha + 6[m\alpha + (n-2\alpha)\beta]\phi + 4(-4 + 3m\beta - 3\beta^2)\phi^2}, \tag{4.65}$$

$$\dot{\tilde{h}} = -\frac{3[3n^2 - 4n\alpha + \alpha^2 + 4(3mn - 2m\alpha - 2n\beta + \alpha\beta)\phi + 4(3m^2 - 4m\beta + \beta^2)\phi^2]}{3(n-\alpha)\alpha + 6[m\alpha + (n-2\alpha)\beta]\phi + 4(-4 + 3m\beta - 3\beta^2)\phi^2}, \tag{4.66}$$

$$\dot{p} = -\frac{3(n + 2m\phi)[3n - \alpha + (6m - 2\beta)\phi]}{3(n-\alpha)\alpha + 6[m\alpha + (n-2\alpha)\beta]\phi + 4(-4 + 3m\beta - 3\beta^2)\phi^2}, \tag{4.67}$$

$$e^{2p} = \frac{18}{V_{IR}} e^{-2m\phi}\phi^{-n} \frac{(3n - 2\alpha)\alpha + (6m\alpha + 6n\beta - 8\alpha\beta)\phi + 4(3m\beta - 2(1+\beta^2))\phi^2}{3(n-\alpha)\alpha + 6[m\alpha + (n-2\alpha)\beta]\phi + 4(-4 + 3m\beta - 3\beta^2)\phi^2}, \tag{4.68}$$

$$a^2 e^{-2\tilde{h}} = \frac{2V_{IR}}{3Z_{IR}} e^{2(m-\beta)\phi}\phi^{n-\alpha} \frac{9n(n-\alpha) + 18(2mn - m\alpha - n\beta)\phi + 4(-4 + 9m^2 - 9m\beta)\phi^2}{3(n-\alpha)\alpha + 6[m\alpha + (n-2\alpha)\beta]\phi + 4(-4 + 3m\beta - 3\beta^2)\phi^2}. \tag{4.69}$$

These are the advertised flow equations, in which we only have derivatives of the fields on the LHS of the system, and the RHS depends only on the dilaton. Equations (4.65)–(4.67) are dynamical, while (4.68), (4.69) are interpreted as constraints.

4.3.2 Large dilaton expansion

The flow equations (4.65)–(4.69), while easier than the original Einstein equations, are still non-linear and cannot be directly solved. However, we are interested in the IR dynamics of the system, i.e. the strongly coupled regime of the plasma. In this situation we will have that the dilaton is very large, so we can perform an expansion around $\phi \to \infty$. Up to corrections of order $\mathcal{O}(1/\phi^2)$ we have

$$\dot{\phi} = \frac{9m - 3\beta}{-4 + 3m\beta - 3\beta^2} - \frac{3[6n(2+\beta^2) + \alpha(-4 + 9m^2 - 18m\beta + 3\beta^2)]}{2(4 - 3m\beta + 3\beta^2)^2\phi}, \tag{4.70}$$

$$\begin{aligned}\dot{\tilde{h}} =& \frac{9m^2 - 12m\beta + 3\beta^2}{4 - 3m\beta + 3\beta^2} \\ &+ 3\frac{9m^3\alpha + 8\alpha\beta - 9m^2(n+2\alpha)\beta + m\alpha(-16 + 9\beta^2) - n\beta(16 + 9\beta^2) + 6mn(4 + 3\beta^2)}{2(4 - 3m\beta + 3\beta^2)^2\phi},\end{aligned} \tag{4.71}$$

$$\dot{p} = \frac{3m(-3m + \beta)}{-4 + 3m\beta - 3\beta^2} + 3\frac{9m^3\alpha - 9m^2(n+2\alpha)\beta - n\beta(4 + 3\beta^2) + m\alpha(-4 + 3\beta^2) + 6mn(4 + 3\beta^2)}{2(4 - 3m\beta + 3\beta^2)^2\phi}, \tag{4.72}$$

$$e^{2p} = e^{-2m\phi}\frac{\phi^{-n}}{V_{IR}}\left[\frac{36 - 54m\beta + 36\beta^2}{4 - 3m\beta + 3\beta^2} + 9\frac{4\alpha\beta + 3m\alpha(-2 + \beta^2) - 3n\beta(2 + \beta^2)}{(4 - 3m\beta + 3\beta^2)^2\phi}\right], \tag{4.73}$$

$$\begin{aligned}a^2 e^{-2\tilde{h}} =& e^{2(m-\beta)\phi}\frac{V_{IR}\phi^{n-\alpha}}{Z_{IR}}\left[\frac{2(-4 + 9m^2 - 9m\beta)}{3(-4 + 3m\beta - 3\beta^2)}\right. \\ &\left. - \frac{9m^3\alpha + 8\alpha\beta - 9m^2(n+2\alpha)\beta + m\alpha(-16 + 9\beta^2) - n\beta(16 + 9\beta^2) + 6mn(4 + 3\beta^2)}{(4 - 3m\beta + 3\beta^2)^2\phi}\right].\end{aligned} \tag{4.74}$$

Note that from the 0th order term in the first three equations we can read directly the coefficients of the slow-roll solution of the previous section,

$$c_\phi = \frac{9m - 3\beta}{-4 + 3m\beta - 3\beta^2}, \tag{4.75}$$

$$c_h = \frac{9m^2 - 12m\beta + 3\beta^2}{4 - 3m\beta + 3\beta^2}, \tag{4.76}$$

$$c_p = \frac{3m(-3m + \beta)}{-4 + 3m\beta - 3\beta^2}. \tag{4.77}$$

See also that in the case $\alpha = n = 0$, that is, where the dilaton and axion potentials consist of just an exponential without a power law correction, the terms of order $\mathcal{O}(1/\phi)$ in (4.70)–(4.72) vanish, and the solutions of the flow equations are linear in A. For general α and n, however, there will be a subleading correction, which we set to calculate in what follows.

4.4 Generic solution

Now that we have recast the equations of motion in a way that can easily be solved analytically, we proceed to do so and find the IR expansion of the metric functions and the dilaton. First we solve the system of equations in terms of the coordinate A, and then we undo the variable change to obtain the solution in terms of the conformal coordinate r.

4.4.1 Integration of the flow equations

The RHS of the flow equations (4.70)–(4.72) depends only on the dilaton ϕ. Therefore, we have to solve first (4.70) and then plug the result back into the equations for $\tilde{h}$ and p. After the IR expansion, the equation that we need to solve looks like

$$\frac{d\phi}{dA} = -q_1 + \frac{q_2}{\phi}, \tag{4.78}$$

where we abbreviate

$$q_1 = \frac{9m - 3\beta}{4 - 3m\beta + 3\beta^2}, \tag{4.79}$$

$$q_2 = -\frac{3[6n(2 + \beta^2) + \alpha(-4 + 9m^2 - 18m\beta + 3\beta^2)]}{2(4 - 3m\beta + 3\beta^2)^2}. \tag{4.80}$$

Equation (4.78) can be easily integrated to give

$$A = -\frac{\phi}{q_1} - \frac{q_2 \log(-q_2 + q_1\phi)}{q_1^2}. \tag{4.81}$$

This equation cannot be solved for ϕ in general. However, it is possible to obtain its asymptotic behaviour in the IR. When $\phi \gg 1$, we have

$$q_1^2 A \simeq \underbrace{-q_1\phi}_{leading} - \underbrace{q_2 \log(q_1\phi)}_{small}. \tag{4.82}$$

Thus, we can look for a solution of the form $\phi(A) = -q_1 A + \lambda$ with λ small. This leads to

$$\begin{aligned}\phi(A) &= -q_1 A - \frac{q_2}{q_1}\log(-q_1^2 A) - \frac{q_2}{q_1}\log\left(1 + \frac{q_2}{q_1^2 A}\log(-q_1^2 A)\right) + \cdots \\ &= -q_1 A - \frac{q_2}{q_1}\log\left(-q_1^2 A - q_2\log(-q_1^2 A + \cdots)\right). \end{aligned} \tag{4.83}$$

We will keep only the first subleading term, and also ignore the overall constant that will be negligible in the large A limit,

$$\phi(A) = -q_1 A - \frac{q_2}{q_1}\log(-A). \tag{4.84}$$

Now we can plug this result into the equations for $\tilde{h}$ and p. In an aim to be able to obtain analytical expressions, we will only keep the linear term for this purpose. The result that we obtain is

$$\frac{d\tilde{h}}{dA} = c_1 + \frac{c_2}{\phi} = c_1 - \frac{c_2}{q_1 A}, \tag{4.85}$$

$$\frac{dp}{dA} = k_1 + \frac{k_2}{\phi} = k_1 - \frac{k_2}{q_1 A}, \tag{4.86}$$

where we have denoted

$$c_1 = \frac{9m^2 - 12m\beta + 3\beta^2}{4 - 3m\beta + 3\beta^2}, \tag{4.87}$$

$$c_2 = 3\frac{9m^3\alpha + 8\alpha\beta - 9m^2(n+2\alpha)\beta + m\alpha(-16+9\beta^2) - n\beta(16+9\beta^2) + 6mn(4+3\beta^2)}{2(4-3m\beta+3\beta^2)^2}, \tag{4.88}$$

$$k_1 = \frac{3m(-3m+\beta)}{-4+3m\beta-3\beta^2}, \tag{4.89}$$

$$k_2 = 3\frac{9m^3\alpha - 9m^2(n+2\alpha)\beta - n\beta(4+3\beta^2) + m\alpha(-4+3\beta^2) + 6mn(4+3\beta^2)}{2(4-3m\beta+3\beta^2)^2}. \tag{4.90}$$

Equations (4.85) and (4.86) can be easily integrated, and we find

$$\tilde{h}(A) = c_1 A - \frac{c_2}{q_1}\log(-A), \tag{4.91}$$

$$p(A) = k_1 A - \frac{k_2}{q_1}\log(-A). \tag{4.92}$$

Note that for the three functions $\phi(A)$, $\tilde{h}(A)$ and $p(A)$, the second derivative is proportional to $\propto A^{-2}$. Thus, we check that the approximation that we made of negligible second derivatives is indeed valid in the IR.

4.4.2 Solution in conformal coordinates

Having found the solution for the metric functions in terms of A, we now want to undo the coordinate change and write them in terms of r. We can do it using the definition of the function p,

$$e^{p(A)} = -e^A \frac{dr}{dA}. \tag{4.93}$$

Substituting (4.92) here, we find the following differential equation,

$$e^{(k_1-1)A-\frac{k_2}{q_1}\log(-A)} = -dr, \tag{4.94}$$

which we can integrate by making use once again of the large $|A|$ (IR) limit,

$$\begin{aligned} r &= \int_{\infty}^{A(r)} dA e^{(k_1-1)A} A^{-k_2/q_1} \\ &= \frac{1}{k_1-1}\int_{\infty}^{A(r)} dA \frac{d}{dA}\left(e^{(k_1-1)A}A^{-k_2/q_1}\right)\left[1+\mathcal{O}\left(\frac{1}{A}\right)\right] \\ &\simeq \frac{1}{k_1-1}\left(e^{(k_1-1)A(r)}A(r)^{-k_2/q_1}\right). \end{aligned} \tag{4.95}$$

Therefore, we obtain

$$\log r = (k_1-1)A - \frac{k_2}{q_1}\log A. \tag{4.96}$$

We are in a situation similar to that of (4.82), where the $\log A$ term is subleading with respect to the linear one. Thus, we propose an ansatz

$$A(r) = \frac{1}{k_1-1}\log r + \lambda, \tag{4.97}$$

and solve approximately for small λ. As we did before, we keep only the first subleading term, which in this case amounts to

$$\boxed{A(r) = \frac{1}{k_1-1}\log r + \frac{k_2}{q_1(k_1-1)}\log\log r.} \tag{4.98}$$

Once we have $A(r)$, we can directly substitute into (4.84) and (4.91),

$$\boxed{\phi(r) = -\frac{q_1}{k_1-1}\log r - \left(\frac{q_2}{q_1}+\frac{k_2}{k_1-1}\right)\log\log r,} \tag{4.99}$$

$$\boxed{h(r) = \frac{c_1-1}{k_1-1}\log r + \left(\frac{k_2(c_1-1)}{q_1(k_1-1)} - \frac{c_2}{q_1}\right)\log\log r,} \tag{4.100}$$

where we have already taken into account the shift $\tilde{h}(r) \mapsto h(r) = \tilde{h}(r) - A(r)$.

It will be useful for later purposes to write the coefficients directly in terms of the original parameters in the potentials. Doing so, we find

$$-\frac{q_1}{k_1-1} = \frac{9m-3\beta}{4-9m^2+3\beta^2}, \tag{4.101}$$

$$\frac{1}{k_1-1} = \frac{4-3m\beta+3\beta^2}{-4+9m^2-3\beta^2}, \tag{4.102}$$

$$\frac{c_1-1}{k_1-1} = \frac{4-9m^2+9m\beta}{4-9m^2+3\beta^2}, \tag{4.103}$$

$$-\left(\frac{q_2}{q_1}+\frac{k_2}{k_1-1}\right) = \frac{\alpha(4-9m^2+18m\beta-3\beta^2)-3n(4+9m^2-6m\beta+3\beta^2)}{2(3m-\beta)(-4+9m^2-3\beta^2)}, \tag{4.104}$$

$$\frac{k_2}{q_1(k_1-1)} = \frac{9m^3\alpha - 9m^2(n+2\alpha)\beta + m\alpha(-4+3\beta^2) + 6mn(4+3\beta^2) - n\beta(4+3\beta^2)}{2(3m-\beta)(-4+9m^2-3\beta^2)}, \quad (4.105)$$

$$\frac{k_2(c_1-1)}{q_1(k_1-1)} - \frac{c_2}{q_1} = \frac{27m^3\alpha + 8\alpha\beta - 9m^2(3n+2\alpha) - 3n\beta(4+3\beta^2) + 3m(6n\beta^2 + \alpha(-4+3\beta^2))}{2(3m-\beta)(-4+9m^2-3\beta^2)}. \quad (4.106)$$

4.5 Checks of the generic solution

Now that we have found the solution for the background metric, given by the functions (4.98), (4.99) and (4.100), we can proceed to calculate the blackening factor and the thermodynamics of the system. Before that, however, there are some checks that need to be performed to ensure that our solution is indeed correct. The tests that we will perform are two: first, with the fourth and fifth flow equations (4.73) and (4.74); and second, with the original Einstein equations (4.22)–(4.24).

4.5.1 Consistency check with the flow equations

During the derivation of the flow equations, we had to impose that the derivatives of our original equations were satisfied in order to make sure that our solutions were regular enough. This gave rise to two new equations, that among other things were fundamental to be able to algebraically solve the system of equations and obtain the flow equations with only the dilaton on the RHS (4.60)–(4.64). However, then we only needed the first three equations to find the solutions. Therefore, we will proceed to check that indeed (4.98), (4.99) and (4.100) also solve (4.73) and (4.74).

The first of these two equations, at zero-th order in ϕ, reads

$$e^{2p} = e^{-2m\phi}\phi^{-n}K. \quad (4.107)$$

Ignoring the overall constant (which is negligible in the large r limit), we get

$$p = -m\phi - \frac{n}{2}\log\phi, \quad (4.108)$$

and substituting the solution for ϕ (4.99) up to $\log\log$ order,

$$\begin{aligned} p(r) =& -m\left[-\frac{q_1}{k_1-1}\log r - \left(\frac{q_2}{q_1} + \frac{k_2}{k_1-1}\right)\log\log r\right] - \frac{n}{2}\log\left(-\frac{q_1}{k_1-1}\right)\log r \\ =& \frac{mq_1}{k_1-1}\log r + \left[m\left(\frac{q_2}{q_1} + \frac{k_2}{k_1-1}\right) - \frac{n}{2}\right]\log\log r. \end{aligned} \quad (4.109)$$

On the other hand, from (4.92) and (4.98), we get

$$\begin{aligned} p(r) =& k_1\left[\frac{1}{k_1-1}\log r + \frac{k_2}{q_1(k_1-1)}\log\log r\right] - \frac{k_2}{q_1}\log\left(-\frac{1}{k_1-1}\log r\right) \\ =& \frac{k_1}{k_1-1}\log r + \left[\frac{k_1k_2}{q_1(k_1-1)} - \frac{k_2}{q_1}\right]\log\log r. \end{aligned} \quad (4.110)$$

Therefore, we see that in order for our solution to be consistent with the constraint equations of the system, (4.73), these identities should be satisfied,

$$\frac{k_1}{k_1-1} = \frac{mq_1}{k_1-1} \quad \to \quad k_1 = mq_1, \tag{4.111}$$

$$\frac{k_2}{q_1(k_1-1)} = m\left(\frac{q_2}{q_1} + \frac{k_2}{k_1-1}\right) - \frac{n}{2}. \tag{4.112}$$

The first of these equations is obviously satisfied, as can be seen from the definitions (4.79) and (4.89). The second one is trickier, but one can simplify

$$m\left(\frac{q_2}{q_1} + \frac{k_2}{k_1-1}\right) - \frac{n}{2} = \frac{9m^3\alpha - 9m^2(n+2\alpha)\beta + m\alpha(-4+3\beta^2) + 6mn(4+3\beta^2) - n\beta(4+3\beta^2)}{2(3m-\beta)(-4+9m^2-3\beta^2)}, \tag{4.113}$$

and comparing with (4.105) check that indeed it is also satisfied.

We also need to do the corresponding check starting from (4.74),

$$a^2 e^{-2\tilde{h}} = e^{2(m-\beta)\phi}\phi^{n-\alpha}C, \tag{4.114}$$

and from this,

$$\tilde{h} = (\beta - m)\phi - \frac{n-\alpha}{2}\log\phi. \tag{4.115}$$

Substituting $\phi(r)$ (4.99),

$$\tilde{h}(r) = \frac{q_1(m-\beta)}{k_1-1}\log r + \left[(m-\beta)\left(\frac{q_2}{q_1} + \frac{k_2}{k_1-1}\right) - \frac{n-\alpha}{2}\right]\log\log r. \tag{4.116}$$

Then, taking into account that $h(r) = \tilde{h}(r) - A(r)$, and using (4.98),

$$h(r) = \frac{q_1(m-\beta)-1}{k_1-1}\log r + \left[(m-\beta)\left(\frac{q_2}{q_1} + \frac{k_2}{k_1-1}\right) - \frac{n-\alpha}{2} - \frac{k_2}{q_1(k_1-1)}\right]\log\log r. \tag{4.117}$$

Comparing with the result (4.100), we see that also the following identities need to be satisfied,

$$\frac{c_1-1}{k_1-1} = \frac{q_1(m-\beta)-1}{k_1-1} \quad \to \quad c_1 = q_1(m-\beta), \tag{4.118}$$

$$(m-\beta)\left(\frac{q_2}{q_1} + \frac{k_2}{k_1-1}\right) - \frac{n-\alpha}{2} - \frac{k_2}{q_1(k_1-1)} = \frac{k_2(c_1-1)}{q_1(k_1-1)} - \frac{c_2}{q_1}. \tag{4.119}$$

As happened before, the first identity is easy to check just with the definitions (4.87) and (4.79). For the second, one needs to compute

$$\begin{aligned}(m-\beta)\left(\frac{q_2}{q_1} + \frac{k_2}{k_1-1}\right) - \frac{n-\alpha}{2} - \frac{k_2}{q_1(k_1-1)} &= \\ = \frac{27m^3\alpha + 8\alpha\beta - 9m^2(3n+2\alpha) - 3n\beta(4+3\beta^2) + 3m(6n\beta^2 + \alpha(-4+3\beta^2))}{2(3m-\beta)(-4+9m^2-3\beta^2)}&,\end{aligned} \tag{4.120}$$

and comparing with (4.106) see that it is indeed satisfy. Thus, we conclude that our solution (4.98)–(4.100) does satisfy the constraint equations and is consistent with our derivation of the flow equations.

4.5.2 Check with the full equations of motion

The second check that we want to make is with our original Einstein equations (4.22)–(4.24), i.e. we want to make sure that (4.98)–(4.100) are indeed solutions of the original equations of motion, at least in the large r limit that corresponds to the approximations that we have made in the derivation of the flow equations and their resolution.

What we will do is use the functional form of the solutions of the flow equations as an ansatz for the full equations of motion, that is,

$$\phi(r) = \phi_0 + \phi_1 \log r + \phi_2 \log\log r, \tag{4.121}$$

$$A(r) = A_0 + A_1 \log r + A_2 \log\log r, \tag{4.122}$$

$$h(r) = h_0 + h_1 \log r + h_2 \log\log r. \tag{4.123}$$

Plugging this and the form of the potentials (4.3) into (4.22)–(4.24), we find

$$\begin{aligned} 0 =& \frac{\log r(h_2(3A_1+2h_1-1)+h_1(3A_1+h_1-1)\log r+3A_2h_1)+h_2(3A_2+h_2-1)}{r^2\log^2 r} \\ &+\frac{9Z_{IR}r^{-2h_1}(\beta-3m)^2\log^{-2h_2} r(\phi_1\log r+\phi_2\log\log r+\phi_0)^{\alpha}e^{2\beta(\phi_1\log r+\phi_2\log\log r+\phi_0)-2h_0}}{2(3\beta(\beta-m)+4)^2}, \end{aligned} \tag{4.124}$$

$$\begin{aligned} 0 =& \frac{1}{9r^2\log^2 r}\left[\log^2 r\left(-9A_1^2-9A_1+3h_1^2-3h_1+4\phi_1^2\right)\right. \\ &\left.+\log r(-9(2A_1+1)A_2+6h_1h_2-3h_2+8\phi_1\phi_2)-9A_2^2-9A_2+3h_2^2-3h_2+4\phi_2^2\right], \end{aligned} \tag{4.125}$$

$$\begin{aligned} 0 =& V_{IR}(\phi_1\log r+\phi_2\log\log r+\phi_0)^n e^{2(A_0+\log r(A_1+m\phi_1)+\log\log r(A_2+m\phi_2)+m\phi_0)} \\ &-\frac{6(A_1\log r+A_2)(h_1\log r+h_2)}{r^2\log^2 r}-\frac{12(A_1\log r+A_2)^2}{r^2\log^2 r}+\frac{4(\phi_1\log r+\phi_2)^2}{3r^2\log^2 r} \\ &-\frac{Z_{IR}(9m-3\beta)^2(\phi_1\log r+\phi_2\log\log r+\phi_0)^{\alpha}e^{2\beta(\phi_1\log r+\phi_2\log\log r+\phi_0)-2(h_0+h_1\log r+h_2\log\log r)}}{2(3\beta(\beta-m)+4)^2}. \end{aligned} \tag{4.126}$$

We are interested in the deep IR dynamics, so we can make a large r expansion, which leads to

$$\begin{aligned} 0 =& \frac{1}{2}a^2e^{-2h_0+2\beta\phi_0}(\log r)^{-2h_2+2\beta\phi_2}r^{-2h_1+2\beta\phi_1}Z_{IR}(\phi_0+\phi_1\log r+\phi_2\log(\log r))^{\alpha} \\ &+\frac{h_1(-1+3A_1+h_1)}{r^2}+\frac{3A_2h_1+(-1+3A_1+2h_1)h_2}{r^2\log r}+\mathcal{O}\left(\frac{1}{r^3}\right)+\mathcal{O}\left(\frac{1}{r^2(\log r)^2}\right), \end{aligned} \tag{4.127}$$

$$0 = -\frac{9A_1+9A_1^2+3h_1-3h_1^2-4\phi_1^2}{6r^2}+\frac{-9(1+2A_1)A_2+(-3+6h_1)h_2+8\phi_1\phi_2}{9r^2\log r}+\mathcal{O}\left(\frac{1}{r^3}\right)+\mathcal{O}\left(\frac{1}{r^2(\log r)^2}\right), \tag{4.128}$$

$$\begin{aligned} 0 =& e^{2A_0+2m\phi_0}(\log r)^{2A_2+2m\phi_2}r^{2A_1+2m\phi_1}V_{IR}(\phi_0+\phi_1\log r+\phi_2\log(\log r))^n \\ &-\frac{1}{2}a^2e^{-2h_0+2\beta\phi_0}(\log r)^{-2h_2+2\beta\phi_2}r^{-2h_1+2\beta\phi_1}Z_{IR}(\phi_0+\phi_1\log r+\phi_2\log(\log r))^{\alpha} \\ &-\frac{2(18A_1^2+9A_1h_1-2\phi_1^2)}{3r^2}-\frac{2(9A_2h_1+9A_1(4A_2+h_2)-4\phi_1\phi_2}{3r^2\log r}+\mathcal{O}\left(\frac{1}{r^3}\right)+\mathcal{O}\left(\frac{1}{r^2(\log r)^2}\right). \end{aligned} \tag{4.129}$$

Furthermore, we do the approximation

$$(\phi_0 + \phi_1 \log r + \phi_2 \log\log r)^\alpha = (\phi_1 \log r)^\alpha \left(1 + \frac{\phi_0}{\phi_1 \log r} + \frac{\phi_2 \log\log r}{\phi_1 \log r}\right)^\alpha \simeq (\phi_1 \log r)^\alpha, \tag{4.130}$$

in the terms coming from the power law of the potentials. With that,

$$\begin{aligned} 0 \simeq& \frac{1}{2} a^2 e^{-2h_0+2\beta\phi_0} \phi_1^\alpha \, (\log r)^{\alpha-2h_2+2\beta\phi_2} \, r^{-2h_1+2\beta\phi_1} Z_{IR} \\ &+ \frac{h_1(-1+3A_1+h_1)}{r^2} + \frac{3A_2h_1 + (-1+3A_1+2h_1)h_2}{r^2 \log r}, \end{aligned} \tag{4.131}$$

$$0 \simeq -\frac{9A_1 + 9A_1^2 + 3h_1 - 3h_1^2 - 4\phi_1^2}{6r^2} + \frac{-9(1+2A_1)A_2 + (-3+6h_1)h_2 + 8\phi_1\phi_2}{9r^2 \log r}, \tag{4.132}$$

$$\begin{aligned} 0 \simeq& e^{2A_0+2m\phi_0} \phi_1^n \, (\log r)^{n+2A_2+2m\phi_2} \, r^{2A_1+2m\phi_1} V_{IR} \\ &- \frac{1}{2} a^2 e^{-2h_0+2\beta\phi_0} \phi_1^\alpha \, (\log r)^{\alpha-2h_2+2\beta\phi_2} \, r^{-2h_1+2\beta\phi_1} Z_{IR} \\ &- \frac{2(18A_1^2 + 9A_1h_1 - 2\phi_1^2)}{3r^2} - \frac{2(9A_2h_1 + 9A_1(4A_2+h_2) - 4\phi_1\phi_2}{3r^2 \log r}. \end{aligned} \tag{4.133}$$

From this, we can try to find the coefficients that solve these equations. First, we look at the r dependences. Imposing that in the first terms of (4.131) and (4.133) there are no logarithms gives the two conditions

$$-2h_2 + 2\beta\phi_2 + \alpha = 0, \tag{4.134}$$

$$2A_2 + 2m\phi_2 + n = 0. \tag{4.135}$$

Imposing that in those same terms, the exponent of r is -2 gives

$$-2h_1 + 2\beta\phi_1 + 2 = 0, \tag{4.136}$$

$$2A_1 + 2m\phi_1 + 2 = 0. \tag{4.137}$$

Then, the $1/r^2$ terms cancel if we have

$$\frac{1}{2} a^2 e^{-2h_0+2\beta\phi_0} Z_{IR} \phi_1^\alpha + h_1(-1+3A_1+h_1) = 0, \tag{4.138}$$

$$e^{2A_0+2m\phi_0} V_{IR} \phi_1^n + h_1(-1+3A_1+h_1) - \frac{2}{3}(18A_1^2 + 9A_1h_1 - 2\phi_1^2) = 0. \tag{4.139}$$

Lastly, both terms in (4.132) have to cancel separately, so we find

$$9A_1 + 9A_1^2 + 3h_1 - 3h_1^2 - 4\phi_1^2 = 0, \tag{4.140}$$

$$-9(1+2A_1)A_2 + (-3+6h_1)h_2 + 8\phi_1\phi_2 = 0. \tag{4.141}$$

In conclusion, we need to solve (4.134)–(4.141). Note that we have eight equations and nine variables, so one of them will remain unfixed. It's easy to see that it must be one of the $\#_0$ coefficients, which we choose to be ϕ_0 so that we can set it equal to 0.

From equations (4.136), (4.137) and (4.140) we find

$$\phi_1 = \frac{9m-3\beta}{4-9m^2+3\beta^2}, \tag{4.142}$$
$$A_1 = \frac{4-3m\beta+3\beta^2}{-4+9m^2-3\beta^2}, \tag{4.143}$$
$$h_1 = \frac{4-9m^2+9m\beta}{4-9m^2+3\beta^2}. \tag{4.144}$$

Comparing this with (4.101)–(4.103) we see that indeed they are the same coefficients that we obtained from the flow equations. Next, we use (4.134), (4.135) and (4.141), together with the results (4.142)–(4.144) to find the $\#_2$ coefficients. The result is

$$\phi_2 = \frac{\alpha(4-9m^2+18m\beta-3\beta^2)-3n(4+9m^2-6m\beta+3\beta^2)}{2(3m-\beta)(-4+9m^2-3\beta^2)}, \tag{4.145}$$
$$A_2 = \frac{9m^3\alpha-9m^2(n+2\alpha)\beta+m\alpha(-4+3\beta^2)+6mn(4+3\beta^2)-n\beta(4+3\beta^2)}{2(3m-\beta)(-4+9m^2-3\beta^2)}, \tag{4.146}$$
$$h_2 = \frac{27m^3\alpha+8\alpha\beta-9m^2(3n+2\alpha)-3n\beta(4+3\beta^2)+3m(6n\beta^2+\alpha(-4+3\beta^2))}{2(3m-\beta)(-4+9m^2-3\beta^2)}. \tag{4.147}$$

These results also coincide with the ones from the flow equations (4.104)–(4.106). Lastly, from (4.138) and (4.139) we can fix A_0 and h_0,

$$\phi_0 = 0, \tag{4.148}$$
$$A_0 = \log\left[\frac{3\sqrt{2}\left(\frac{9m-3\beta}{4-9m^2+3\beta^2}\right)^{-\frac{n}{2}}\sqrt{\frac{8+14\beta^2+9m^2\beta^2+6\beta^2-3m\beta(6+5\beta^2)}{4-9m^2+3\beta^2}}}{\sqrt{V_{IR}}\sqrt{4-9m^2+3\beta^2}}\right], \tag{4.149}$$
$$h_0 = -\frac{1}{2}\log\left[\frac{12\left(\frac{9m-3\beta}{4-9m^2+3\beta^2}\right)^{-\alpha}\left(27m^3\beta+8(1+\beta^2)-9m^2(2+5\beta^2)+6m(\beta+3\beta^3)\right)}{aZ_{IR}(4-9m^2+3\beta^2)^2}\right]. \tag{4.150}$$

In conclusion, we see that the solutions of the flow equations indeed satisfy both of the consistency checks, and therefore we can be certain that they are indeed the correct IR expansion of the metric for the family of potentials (4.3).

4.6 Analysis of the parameter space

Despite the fact that the functions (4.98)–(4.100) have been shown to be a consistent solution of both the flow and Einstein equations, we still have to investigate whether or not they are valid in the whole parameter space of (β, m, α, n). In particular, we will look at the following possible issues:

- In Chapter 3, we saw that in order for the holographic theory to correctly reproduce a QCD-like large N gauge theory, we needed the metric functions to satisfy certain properties. For example, the dilaton $\phi \to \infty$ in the IR. Also, A represents the energy scale, so we must have $A \to -\infty$ in the same limit. Such conditions will imply that not all possible parameters are allowed, and indeed the possible values of β and m are quite restricted.

- During our resolution of the problem, we have continuously been working with quotients of polynomials in the variables (β, m, α, n). This awakes the obvious concern that for some values of the parameters we might have divided by zero or taken other similarly unwise steps. Indeed, we will find that for some singular subspaces of the parameter space, the generic solution (4.98)–(4.100) is not valid, and we will need to solve again the flow equations for each particular case.

4.6.1 Allowed parameter space

We begin by studying which regions of the parameter space are allowed by sensible physical arguments if the generic solution is to be trusted. The considerations that we make are:

- In order for the theory in the gravity side to be strongly coupled in the IR, we need that the dilaton $\phi \to \infty$ when $r \to \infty$. This translates into

$$\phi_1 > 0, \tag{4.151}$$

  where we are using the functional forms (4.121)–(4.123).

- In order for A to represent a suitable energy scale, we need that $A \to -\infty$ when $r \to \infty$, which amounts to

$$A_1 < 0. \tag{4.152}$$

  Note that in the gravity picture this corresponds to the spatial dimensions decreasing in size as r grows, which is consistent with the coarse-graining picture of the RG flows.

- The are two sensible conditions that we can impose regarding the anisotropy $h(r)$. We can either say that the whole spatial hyper-surface has to decrease in the deep IR[6], which amounts to

$$3A_1 + h_1 < 0, \tag{4.153}$$

  or we can look at the more restrictive condition that the x_3 direction doesn't blow up in size of its own,

$$A_1 + h_1 < 0. \tag{4.154}$$

  For now, we will consider the second, more restrictive condition.

- Lastly, we must require that all the coefficients are real numbers, since we don't want complex solutions. In the log and log log terms the coefficients are just quotients of polynomials and therefore are always real, so we only get constrains from A_0 and h_0.

In conclusion, we see that our parameters have to satisfy the following inequalities,

$$\phi_1 = \frac{9m - 3\beta}{4 - 9m^2 + 3\beta^2} > 0, \tag{4.155}$$

$$A_1 = \frac{4 - 3m\beta + 3\beta^2}{-4 + 9m^2 - 3\beta^2} < 0, \tag{4.156}$$

$$A_1 + h_1 = \frac{-3(3m^2 - 4m\beta + \beta^2)}{4 - 9m^2 + 3\beta^2} < 0. \tag{4.157}$$

[6]Note that this will be the area of the black brane solution after we include the blackening factor $f(r)$. Therefore, this condition is equivalent to the requirement of finite entropy in the deep IR.

Looking at (4.155), we find that we must have

$$3m-\beta>0 \quad \text{and} \quad 4-9m^2+3\beta^2>0 \tag{4.158}$$

or

$$3m-\beta<0 \quad \text{and} \quad 4-9m^2+3\beta^2<0. \tag{4.159}$$

However, the conditions in (4.159) are contradictory with one another, so we are left with (4.158). Since the denominators are equal, (4.156) reduces to

$$4-3m\beta+3\beta^2>0, \tag{4.160}$$

which is automatically satisfied in the case $3m>\beta$.

Next, looking at (4.157), we find

$$3m^2-4m\beta+\beta^2=(3m-\beta)(m-\beta)>0, \tag{4.161}$$

and therefore we conclude that from (4.155)–(4.157) the allowed parameter space is

$$m>\beta, \qquad 4-9m^2+3\beta^2>0. \tag{4.162}$$

Lastly, we have to look at the reality conditions for A_0 and h_0. Since they are given in terms of logarithms (4.149), (4.150), we have to require that the argument of said logarithm is real and positive. Taking into account the restrictions that we already found, we have two more inequalities that need to be satisfied,

$$27m^3\beta+8(1+\beta^2)-9m^2(2+5\beta^2)+6m(\beta+3\beta^3)>0, \tag{4.163}$$
$$8+14\beta^2+9m^2\beta^2+6\beta^2-3m\beta(6+5\beta^2)>0. \tag{4.164}$$

First we look at (4.163) and find the roots of the polynomial,

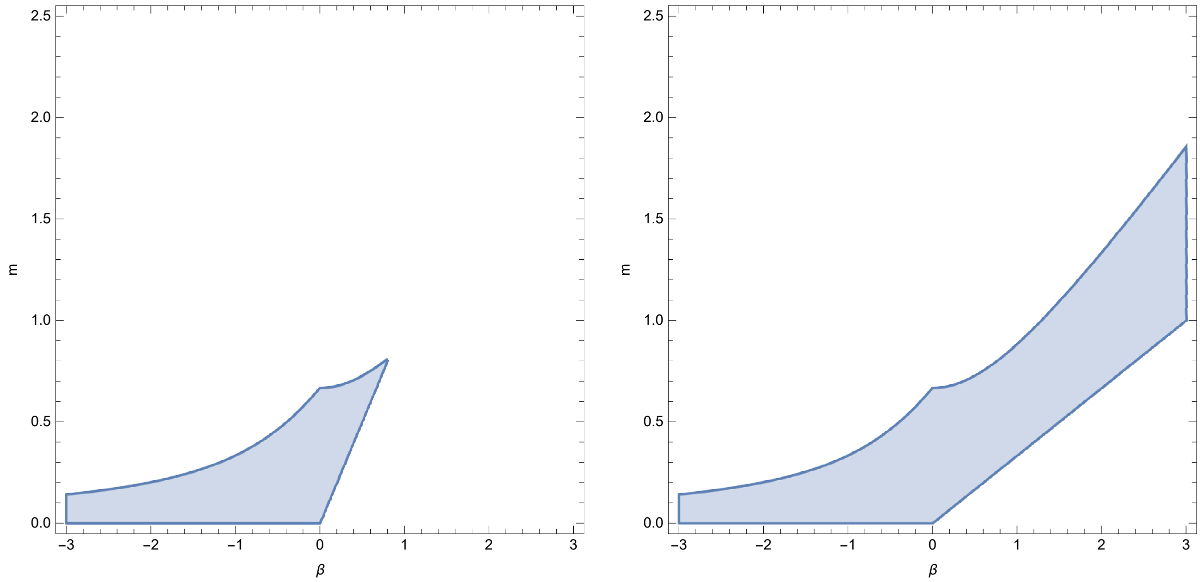
$$27m^3\beta+8(1+\beta^2)-9m^2(2+5\beta^2)+6m(\beta+3\beta^3)=-3\beta\left(m-\frac{2(1+\beta^2)}{3\beta}\right)\left(4-9m^2+9m\beta\right). \tag{4.165}$$

The first factor is always greater than zero in the region delimited by (4.162). Therefore, the new condition that we find is

$$4-9m^2+9m\beta>0. \tag{4.166}$$

It is easy to check that (4.164) is always satisfied in the already delimited region. Therefore, we conclude that the allowed parameter space is (see Figure 4.1a):

$$m-\beta>0, \tag{4.167}$$
$$4-9m^2+3\beta^2>0, \tag{4.168}$$
$$4-9m^2+9m\beta>0. \tag{4.169}$$



(a) (b)

Figure 4.1: According to the conditions (4.151)–(4.154), only a certain subregion of the (β, m) plane is allowed. In subfigure (a), we imposed that the anisotropic direction x_3 has a finite warp factor in the IR (4.154). In subfigure (b), we only consider the restriction of non-divergent entropy as $r \to \infty$ (4.153).

The analysis that we have performed in completely analogous in the case that we consider the less restrictive condition (4.153) instead of (4.154). The only change in the final result is that in (4.167) we need to swap $m \to 3m$. Thus, the allowed region in this case is (see Figure 4.1b):

$$3m - \beta > 0, \tag{4.170}$$

$$4 - 9m^2 + 3\beta^2 > 0, \tag{4.171}$$

$$4 - 9m^2 + 9m\beta > 0. \tag{4.172}$$

It is interesting to see that even though we started with completely general potentials, these simple arguments allow us to conclude that a sizeable region of the parameter space leads to unphysical results. Note also that we only find restriction for the parameters in the exponentials, the power laws of the potentials (4.3) remain completely unfixed.

4.6.2 Singular points

Now, we proceed to study in which regions of the parameter space our solution in terms of a $\log + \log\log$ expansion in the deep IR might not be valid. These might be because of various reasons, such as a division by zero at some point, a vanishing coefficient in the flow equations that leads to a completely different integral, etc. In order to find these singular points, we look for:

- Points where the coefficients of the generic solution (4.142)–(4.150) might diverge.
- Points where the coefficients in the large dilaton expansion of the flow equations (4.70)–(4.74) vanish (which would lead to a different functional form after integration).
- Points where the coefficients in the large dilaton expansion of the flow equations (4.70)–(4.74)

diverge.

It is easy to see that the singular points that we will need to look at are

- $m = \frac{1}{3}\sqrt{4+3\beta^2}$, where the coefficients A_1, h_1, ϕ_1, A_2, h_2 and ϕ_2 diverge.
- $m = \frac{1}{3}\beta$, where the zero-th order coefficients of the flow equations vanish, and also the coefficients A_2, h_2 and ϕ_2 diverge.
- $m = \frac{4+3\beta^2}{3\beta}$, where various coefficients of the flow equations diverge.
- Lastly, we will also look at the point $\beta = 3m$ and $\alpha = 3n$, which stands to attention due to the fact that some of the divergences that we found in the case $\beta = 3m$ cancel and give a finite result for A_2, h_2 and ϕ_2. From (4.145)–(4.147),

$$A_2(\beta = 3m, \alpha = 3n) = \frac{\left(9m^2-2\right)n}{18m^2+4}, \tag{4.173}$$

$$h_2(\beta = 3m, \alpha = 3n) = -\frac{3\left(9m^2-2\right)n}{18m^2+4}, \tag{4.174}$$

$$\phi_2(\beta = 3m, \alpha = 3n) = -\frac{9mn}{9m^2+2}. \tag{4.175}$$

In each of these points, the solution that we found in sections 4.4 and 4.5 is not valid. Therefore, now we proceed to solve the problem from the beginning in each of the cases separately.

4.7 Solution at singular points

In section 4.5 we have thoroughly checked that the solution that we obtained from the flow equations in the generic case was indeed the correct one, since it also approximately solved the original Einstein equations. Therefore, for this section, we assume that also the solution from the flow equations in the special cases can be trusted, and we will not redo all the checks once again.

The strategy is to look separately to each of the singular points that we found in subsection 4.6.2, find where the generic calculation goes wrong in each case, and find the corresponding solution.

Case I: $m = \beta/3$

In this case, some of the terms in the large dilaton expansion of the flow equations vanish, which means that the result of their integration will be, in principle, different. Setting $m = \beta/3$ in equations (4.70)–(4.72) results in the new flow equations,

$$\dot{\phi}(A) = \frac{3\alpha - 9n}{4\left(\beta^2+2\right)\phi}, \tag{4.176}$$

$$\dot{\tilde{h}}(A) = \frac{\beta(\alpha - 3n)}{2\left(\beta^2+2\right)\phi}, \tag{4.177}$$

$$\dot{p}(A) = \frac{3\beta n - \alpha\beta}{\left(4\beta^2+8\right)\phi}. \tag{4.178}$$

The equation for $\phi(A)$ can be easily solved,

$$\phi(A) = \sqrt{\frac{3A(\alpha - 3n)}{2(\beta^2+2)}} \tag{4.179}$$

Then we can use this result to solve the equations for h and p. Substituting into (4.177) and (4.178),

$$\dot{\tilde{h}}(A) = \sqrt{\frac{(\alpha - 3n)\beta^2}{6(2+\beta^2)A}}, \tag{4.180}$$

$$\dot{p}(A) = -\sqrt{\frac{(\alpha - 3n)\beta^2}{24(2+\beta^2)A}}, \tag{4.181}$$

and integrating,

$$\tilde{h}(A) = \sqrt{\frac{2(\alpha - 3n)\beta^2}{3(2+\beta^2)}A}, \tag{4.182}$$

$$p(A) = -\sqrt{\frac{(\alpha - 3n)\beta^2}{6(2+\beta^2)}A}. \tag{4.183}$$

Again we are neglecting integration constants that will be negligible in the deep IR limit.

As we did in the case of the generic solution, we are interested in finding the metric functions in conformal coordinates. From the definition of $p(A)$ (4.25),

$$dr = -e^{-A-\sqrt{\frac{(\alpha-3n)\beta^2}{6(2+\beta^2)}A}}dA. \tag{4.184}$$

This integral can be solved, and the result is

$$r = e^{-A-\sqrt{\frac{(\alpha-3n)\beta^2}{6(2+\beta^2)}A}}\left[1 + \mathcal{O}\left(\frac{1}{\sqrt{A}}\right)\right], \tag{4.185}$$

so in the large A limit we have

$$\log r = -A - \sqrt{\frac{(\alpha - 3n)\beta^2}{6(2+\beta^2)}A}. \tag{4.186}$$

This is the same type of equation as (4.82) and (4.96), where the linear term in A is leading with respect to the square root. Again we can solve it approximately by proposing an ansatz of the form $A = -\log r + \lambda$ with small λ. This leads to

$$A(r) = \log\frac{1}{r} - \sqrt{\frac{(3n - \alpha)\beta^2}{6(2+\beta^2)}\log r}. \tag{4.187}$$

Then we just need to put this result into (4.182) and (4.179),

$$\phi(r) = \frac{1}{2}\sqrt{\frac{(3n-\alpha)}{\beta^2+2}\left(6\log r + \sqrt{\frac{6\beta^2(3n-\alpha)}{\beta^2+2}\log r}\right)}, \tag{4.188}$$

$$h(r) = \log r + \sqrt{\frac{(3n-\alpha)\beta^2}{6(2+\beta^2)}\log r}\left[1 + 2\sqrt{1 + \frac{1}{\log r}\sqrt{\frac{(3n-\alpha)\beta^2}{6(2+\beta^2)}\log r}}\right], \tag{4.189}$$

where we have already taken into account the redefinition of h from $\tilde{h}$.

However, in this result, coming from directly substituting $A(r)$ into the metric functions, we notice that we have more subleading terms (of order $(\log r)^{1/4}$) than in the expression that we found for $A(r)$. This is an inconsistency, so we need to throw these further corrections. Finally, the result that we find is

$$\phi(r) = \frac{1}{2}\sqrt{\frac{6(3n-\alpha)}{\beta^2+2}\log r}, \tag{4.190}$$

$$h(r) = \log r + \sqrt{\frac{3(3n-\alpha)\beta^2}{2(2+\beta^2)}\log r}. \tag{4.191}$$

Case II: $m = \sqrt{4+3\beta^2}/3$

In this case, the problem with the generic solution was that all the coefficients A_1, h_1, ϕ_1, A_2, h_2, ϕ_2 diverge. However, none of the terms in the flow equations cancel or diverge, and therefore we should expect that our solution has the same functional form. The problem is that the variable k_1 (4.89) becomes 1, and therefore the change of variables back to conformal coordinates (4.95) is not valid.

Plugging $m = \sqrt{4+3\beta^2}/3$ into the flow equations (4.70)–(4.72) results in

$$\dot{\phi}(A) = -\frac{3}{\sqrt{3\beta^2+4}} - \frac{9\left(\alpha\beta\left(\beta-\sqrt{3\beta^2+4}\right)+\left(\beta^2+2\right)n\right)}{\left(\beta\left(\sqrt{3\beta^2+4}-3\beta\right)-4\right)^2\phi}, \tag{4.192}$$

$$\begin{aligned}\dot{\tilde{h}}(A) &= 1 - \frac{3\beta}{\sqrt{3\beta^2+4}} \\ &\quad - \frac{3\left(\alpha\left(-3\beta^4+2\beta^2+4\sqrt{3\beta^2+4}\beta+8\right)+\left(\beta\left(3\beta\left(\beta\sqrt{3\beta^2+4}-4\right)+2\sqrt{3\beta^2+4}\right)-16\right)n\right)}{2\left(\beta^2+2\right)\left(3\beta^2+4\right)^{3/2}\phi},\end{aligned} \tag{4.193}$$

$$\dot{p}(A) = 1 + \frac{3\left(\beta\left(\sqrt{3\beta^2+4}+3\beta\right)+4\right)n - 3\alpha\beta\left(\sqrt{3\beta^2+4}+\beta\right)}{2\left(\beta^2+2\right)\sqrt{3\beta^2+4}\phi}. \tag{4.194}$$

Solving these equations in terms of the coordinate A follows the same steps that we have done before: first solve the differential equation for $\phi(A)$, then plug the result in the other to equations, and find $\tilde{h}(A)$ and $p(A)$ by ordinary integration. The results are

$$\phi(A) = -\frac{3A}{\sqrt{3\beta^2+4}} - \frac{3\sqrt{3\beta^2+4}\left(\alpha\beta\left(\beta-\sqrt{3\beta^2+4}\right)+\left(\beta^2+2\right)n\right)}{\left(\beta\left(\sqrt{3\beta^2+4}-3\beta\right)-4\right)^2}\log(-A), \tag{4.195}$$

$$\begin{aligned}\tilde{h}(A) &= \frac{A\left(\sqrt{3\beta^2+4}-3\beta\right)}{\sqrt{3\beta^2+4}} \\ &\quad + \frac{\left(-\alpha\sqrt{3\beta^2+4}\beta^2+2\alpha\sqrt{3\beta^2+4}+4\alpha\beta+3\beta^3 n-4\sqrt{3\beta^2+4}n+2\beta n\right)}{2\left(\beta^2+2\right)\sqrt{3\beta^2+4}}\log(-A),\end{aligned} \tag{4.196}$$

$$p(A) = A + \frac{\left(\alpha\beta\left(\sqrt{3\beta^2+4}+\beta\right)-\left(3\beta^2+\sqrt{3\beta^2+4}\beta+4\right)n\right)}{2\left(\beta^2+2\right)}\log(-A). \tag{4.197}$$

In order to find the solution in conformal coordinates again we use the definition of $p(A)$ (4.25). In this case, however, the exponential part of the integral disappears, and actually the equation can be integrated exactly, without further deep IR approximations. The result is

$$r=-\frac{2A\left(\beta^2+2\right)(-A)^{\frac{\alpha\beta\left(\sqrt{3\beta^2+4}+\beta\right)-\left(3\beta^2+\sqrt{3\beta^2+4}\beta+4\right)n}{2(\beta^2+2)}}}{(\alpha+2)\beta^2+\alpha\sqrt{3\beta^2+4}\beta-\left(3\beta^2+\sqrt{3\beta^2+4}\beta+4\right)n+4}. \tag{4.198}$$

From (4.198), analogously to what we did before, we can find $\phi(r)$, $A(r)$, $h(r)$. We arrive at

$$A(r)=-\left(\frac{r\left((\alpha+2)\beta^2+\alpha\sqrt{3\beta^2+4}\beta-\left(3\beta^2+\sqrt{3\beta^2+4}\beta+4\right)n+4\right)}{2(\beta^2+2)}\right)^{\frac{2\left(\beta^2+2\right)}{(\alpha+2)\beta^2+\alpha\sqrt{3\beta^2+4}\beta-\left(3\beta^2+\sqrt{3\beta^2+4}\beta+4\right)n+4}} \tag{4.199}$$

$$\begin{aligned}\phi(r)=&\frac{3}{\sqrt{3\beta^2+4}}\left(-\frac{\left(\alpha\beta\left(\beta-\sqrt{3\beta^2+4}\right)+\left(\beta^2+2\right)n\right)}{\beta\left((\alpha-2)\sqrt{3\beta^2+4}+(4-\alpha)\beta\right)+\left(\beta\left(\sqrt{3\beta^2+4}-3\beta\right)-4\right)n+4}\log r\right.\\&\left.+\left(\frac{r\left((\alpha+2)\beta^2+\alpha\sqrt{3\beta^2+4}\beta-\left(\beta\left(\sqrt{3\beta^2+4}+3\beta\right)+4\right)n+4\right)}{2(\beta^2+2)}\right)^{\frac{2\left(\beta^2+2\right)}{(\alpha+2)\beta^2+\alpha\sqrt{3\beta^2+4}\beta-\left(\beta\left(\sqrt{3\beta^2+4}+3\beta\right)+4\right)n+4}}\right)\end{aligned} \tag{4.200}$$

$$\begin{aligned}h(r)=&\frac{3\beta\left(\frac{r\left((\alpha+2)\beta^2+\alpha\sqrt{3\beta^2+4}\beta-\left(3\beta^2+\sqrt{3\beta^2+4}\beta+4\right)n+4\right)}{4(\beta^2+2)}\right)^{\frac{2\left(\beta^2+2\right)}{(\alpha+2)\beta^2+\alpha\sqrt{3\beta^2+4}\beta-\left(3\beta^2+\sqrt{3\beta^2+4}\beta+4\right)n+4}}}{\sqrt{3\beta^2+4}}\\&-\frac{\left(\alpha\left(\sqrt{3\beta^2+4}\beta^2-2\sqrt{3\beta^2+4}-4\beta\right)+\left(-3\beta^3+4\sqrt{3\beta^2+4}-2\beta\right)n\right)}{\sqrt{3\beta^2+4}\left((\alpha+2)\beta^2+\alpha\sqrt{3\beta^2+4}\beta-\left(3\beta^2+\sqrt{3\beta^2+4}\beta+4\right)n+4\right)}\log r.\end{aligned} \tag{4.201}$$

Case III: $m=\frac{4+3\beta^2}{3\beta}$

In this case, the problem that arises is that various of the coefficients in the large dilaton expansion of the flow equations (4.70)–(4.72) diverge. Therefore, we cannot use these equations to solve the problem. Going one step back to the full flow equations (4.65)–(4.67), we rapidly notice that the problem is that the quadratic term in the denominator of each of the equations vanishes, and we are left with

$$\dot{\phi}(A)=\frac{6\phi(A)\left(4\left(\beta^2+2\right)\phi(A)+\beta(3n-\alpha)\right)}{\phi(A)\left(\alpha\left(8-6\beta^2\right)+6\beta^2n\right)+3\alpha\beta(n-\alpha)}, \tag{4.202}$$

$$\dot{h}(A)=-\frac{32\left(\beta^2+2\right)\phi(A)^2+4\beta\phi(A)\left(3\left(\beta^2+4\right)n-\alpha\left(3\beta^2+8\right)\right)+3\beta^2\left(\alpha^2+3n^2-4\alpha n\right)}{\beta\left(\phi(A)\left(\alpha\left(8-6\beta^2\right)+6\beta^2n\right)+3\alpha\beta(n-\alpha)\right)}, \tag{4.203}$$

$$\dot{p}(A)=-\frac{\left(\left(6\beta^2+8\right)\phi(A)+3\beta n\right)\left(4\left(\beta^2+2\right)\phi(A)+\beta(3n-\alpha)\right)}{\beta\left(\phi(A)\left(\alpha\left(8-6\beta^2\right)+6\beta^2n\right)+3\alpha\beta(n-\alpha)\right)}. \tag{4.204}$$

Performing the large dilaton expansion in this case gives (up to $1/\phi$ terms),

$$\dot{\phi}(A) = -\frac{12\left(\beta^2+2\right)\phi}{3\alpha\beta^2-4\alpha-3\beta^2 n} + \frac{3\left(9\alpha^2\beta^3+8\alpha^2\beta+9\beta^3 n^2-18\alpha\beta^3 n\right)}{\left(-3\alpha\beta^2+4\alpha+3\beta^2 n\right)^2}, \tag{4.205}$$

$$\dot{\tilde{h}}(A) = \frac{16\left(\beta^2+2\right)\phi}{\beta\left(3\alpha\beta^2-4\alpha-3\beta^2 n\right)} - \frac{2\left(9\alpha^2\beta^4+24\alpha^2\beta^2-8\alpha^2+9\beta^4 n^2+36\beta^2 n^2-18\alpha\beta^4 n-60\alpha\beta^2 n+24\alpha n\right)}{\left(-3\alpha\beta^2+4\alpha+3\beta^2 n\right)^2}, \tag{4.206}$$

$$\dot{p}(A) = \frac{4\left(\beta^2+2\right)\left(3\beta^2+4\right)\phi}{\beta\left(3\alpha\beta^2-4\alpha-3\beta^2 n\right)} + \frac{-27\alpha^2\beta^4-60\alpha^2\beta^2-32\alpha^2-45\beta^4 n^2-72\beta^2 n^2+72\alpha\beta^4 n+84\alpha\beta^2 n-48\alpha n}{\left(-3\alpha\beta^2+4\alpha+3\beta^2 n\right)^2}. \tag{4.207}$$

These equations can now be solved by the same process as in the previous cases. We arrive at

$$\phi(A) = \frac{e^{\frac{12A\left(\beta^2+2\right)}{\alpha\left(4-3\beta^2\right)+3\beta^2 n}}}{12\left(\beta^2+2\right)\left(\alpha\left(4-3\beta^2\right)+3\beta^2 n\right)}, \tag{4.208}$$

$$\tilde{h}(A) = -\frac{54A\beta\left(\beta^2+2\right)^2(n-\alpha)+e^{\frac{12A\left(\beta^2+2\right)}{\alpha\left(4-3\beta^2\right)+3\beta^2 n}}}{9\beta\left(\beta^2+2\right)\left(\alpha\left(4-3\beta^2\right)+3\beta^2 n\right)}, \tag{4.209}$$

$$p(A) = -\frac{\left(3\beta^2+4\right)e^{\frac{12A\left(\beta^2+2\right)}{\alpha\left(4-3\beta^2\right)+3\beta^2 n}}+216A\beta\left(\beta^2+2\right)^2 n}{36\beta\left(\beta^2+2\right)\left(\alpha\left(4-3\beta^2\right)+3\beta^2 n\right)}. \tag{4.210}$$

As usual, from $p(A)$ we can find the relation between A and the conformal coordinate r,

$$r = \frac{\left(\alpha\left(4-3\beta^2\right)+3\beta^2 n\right)e^{\frac{A\left(\alpha\left(3\beta^2-4\right)-3\left(3\beta^2+4\right)n\right)}{\alpha\left(4-3\beta^2\right)+3\beta^2 n}}}{\alpha\left(4-3\beta^2\right)+3\left(3\beta^2+4\right)n}, \tag{4.211}$$

and solving for A and plugging the result back into (4.208), (4.209),

$$A(r) = \frac{\left(3\alpha\beta^2-4\alpha-3\beta^2 n\right)}{-3\alpha\beta^2+4\alpha+9\beta^2 n+12n}\log r, \tag{4.212}$$

$$\phi(r) = -\frac{\left(\frac{3\alpha\beta^2-4\alpha-3\beta^2 n}{r\left(3\alpha\beta^2-4\alpha-9\beta^2 n-12n\right)}\right)^{-\frac{12\left(\beta^2+2\right)\left(3\alpha\beta^2-4\alpha-3\beta^2 n\right)}{\left(-3\alpha\beta^2+4\alpha+3\beta^2 n\right)\left(-3\alpha\beta^2+4\alpha+9\beta^2 n+12n\right)}}}{12\left(\beta^2+2\right)\left(3\alpha\beta^2-4\alpha-3\beta^2 n\right)}, \tag{4.213}$$

$$h(r) = \frac{\left(3\left(3\beta^2+4\right)n-\alpha\left(9\beta^2+8\right)\right)}{\alpha\left(4-3\beta^2\right)+3\left(3\beta^2+4\right)n}\log r - \frac{\left(\frac{-3\alpha\beta^2+4\alpha+3\beta^2 n}{r\left(9\beta^2 n+12n-3\alpha\beta^2+4\alpha\right)}\right)^{\frac{12\left(\beta^2+2\right)}{\alpha\left(4-3\beta^2\right)+3\left(3\beta^2+4\right)n}}}{9\beta\left(\beta^2+2\right)\left(\alpha\left(4-3\beta^2\right)+3\beta^2 n\right)}. \tag{4.214}$$

Case IV: $\beta = 3m$ **and** $\alpha = 3n$

The last point of parameter space that we look at is a sub-case of the already considered $\beta = 3m$. As we can see from (4.176)–(4.178), if $\alpha = 3n$ all the coefficients vanish. Moreover, we can see that the full equations of motion (4.65)–(4.67) vanish for this choice of parameters.

It turns out that this is a very general statement. Even before substituting the IR behaviour of the potentials $Z(\phi)$ and $V(\phi)$ (4.3), the flow equations (4.60)–(4.64) can be rewritten as

$$\dot{\phi}(A) = \frac{6\left(\frac{d}{d\phi}\log Z(\phi) - 3\frac{d}{d\phi}\log V(\phi)\right)}{16 - 3\frac{d}{d\phi}\log V(\phi)\frac{d}{d\phi}\log Z(\phi) + 3\left(\frac{d}{d\phi}\log Z(\phi)\right)^2}, \tag{4.215}$$

$$\dot{\tilde{h}}(A) = \frac{3\left(\frac{d}{d\phi}\log V(\phi) - \frac{d}{d\phi}\log Z(\phi)\right)\left(3\frac{d}{d\phi}\log V(\phi) - \frac{d}{d\phi}\log Z(\phi)\right)}{16 - 3\frac{d}{d\phi}\log V(\phi)\frac{d}{d\phi}\log Z(\phi) + 3\left(\frac{d}{d\phi}\log Z(\phi)\right)^2}, \tag{4.216}$$

$$\dot{p}(A) = \frac{3\frac{d}{d\phi}\log V(\phi)\left(3\frac{d}{d\phi}\log V(\phi) - \frac{d}{d\phi}\log Z(\phi)\right)}{16 - 3\frac{d}{d\phi}\log V(\phi)\frac{d}{d\phi}\log Z(\phi) + 3\left(\frac{d}{d\phi}\log Z(\phi)\right)^2}. \tag{4.217}$$

The point in the parameter space under consideration, $\beta = 3m$ and $\alpha = 3n$ corresponds precisely to

$$Z(\phi) = (V(\phi))^3, \tag{4.218}$$

and thus

$$\dot{\phi}(A) = 0, \qquad \dot{\tilde{h}}(A) = 0, \qquad \dot{p}(A) = 0. \tag{4.219}$$

The only solution in this case is given by constant functions

$$\phi(r) = \phi_0, \qquad A(r) = A_0, \qquad h(r) = h_0. \tag{4.220}$$

Note that from (4.216) we can see that the case $\alpha = n$ and $\beta = m$ will also have one of the flow equations cancel. However, in this case the dilaton $\phi(r)$ and scale function $A(r)$ will not be constant, only $h(r)$. Since the motivation for our study are anisotropic systems, this case is not interesting to us.

This concludes our study of the thermal gas solution for our general choice of dilaton and axion potentials. In the following chapter, we will proceed to study the black-hole solutions and the thermodynamics of the corresponding plasma.

5 Thermodynamics

In Chapter 4, we have performed the most cumbersome part of the calculation that our study involves, consisting on finding the metric functions $A(r)$ and $h(r)$ and dilaton $\phi(r)$ for the case of the thermal gas solution. As we saw in Chapter 3, this solution describes the confined phase of the anisotropic QGP-like plasma in the field theory side of the holographic correspondence.

Our goal is to be able to describe the full thermodynamics of the system and in particular the deconfinement phase transition. Therefore, we also need to solve Einstein's equations in the case where the blackening factor is $f(r) \neq 1$. Once we find that solution, corresponding to the deconfined phase, we will be able to calculate the temperature, entropy and free energy as we indicated in section 3.3.

It turns out that the integration constants that remained unfixed in the calculation of the metric functions via the flow equations, despite giving a negligible contribution in the large r limit, are quite important for the thermodynamics. In particular, the degree of anisotropy a that comes from the axion field solution (4.5) only enters the thermodynamic quantities through the constant term h_0 of $h(r)$, and one of the main goals that we have is to study the dependence in this parameter.

Therefore, we will use the results that we obtained in section 4.5 coming directly from Einstein equations, instead of the solutions that we found from the flow equations, in which the integration constant remained unfixed. This means that will only able to calculate the thermodynamics for the generic solution, and for the $Z \propto V^3$ case, of which we have the complete solution from Einstein equations, but not for the rest of the special cases that we considered in section 4.7.

5.1 Blackening factor

We need to find the solutions to Einstein equations (4.18)–(4.21) for $f(r) \neq 1$. The working assumption is that, since in the UV we always must have $f(r) \to 1$ as $r \to 0$ (so that the metric asymptotes to AdS), the deviation from $f(r) = 1$ is small enough so that we can neglect the backreaction of the blackening factor onto the rest of the metric functions. Thus, we only need to solve the equation

$$f'' + f'(3A' + h') = 0, \tag{5.1}$$

for the solutions of $A(r)$ and $h(r)$ found in the previous chapter.

5.1.1 $f(r)$ for the generic solution

The generic solution, valid in the region of parameter space studied in secion 4.6.1 (see Figure 4.1) is given by

$$\begin{aligned} A(r) = \log\left(\frac{3\sqrt{2}\sqrt{\frac{6\beta^4+14\beta^2+9\beta^2m^2-3(5\beta^2+6)\beta m+8}{3\beta^2-9m^2+4}}\left(\frac{9m-3\beta}{3\beta^2-9m^2+4}\right)^{-n/2}}{\sqrt{V_{IR}}\sqrt{3\beta^2-9m^2+4}}\right) + \frac{\left(3\beta^2-3\beta m+4\right)\log(r)}{-3\beta^2+9m^2-4} \\ + \frac{\log(\log(r))\left(9\alpha m^3-9\beta m^2(2\alpha+n)+\alpha\left(3\beta^2-4\right)m+6\left(3\beta^2+4\right)mn-\beta\left(3\beta^2+4\right)n\right)}{2(3m-\beta)\left(-3\beta^2+9m^2-4\right)}, \end{aligned} \tag{5.2}$$

$$h(r) = -\frac{1}{2}\log\left(\frac{12\left(8\left(\beta^2+1\right)+27\beta m^3-9\left(5\beta^2+2\right)m^2+6\left(3\beta^3+\beta\right)m\right)\left(\frac{9m-3\beta}{3\beta^2-9m^2+4}\right)^{-\alpha}}{aZ_{IR}\left(3\beta^2-9m^2+4\right)^2}\right)$$
$$+\frac{\left(-9m^2+9\beta m+4\right)\log(r)}{3\beta^2-9m^2+4}$$
$$+\frac{\log(\log(r))\left(8\alpha\beta+27\alpha m^3-9\beta m^2(2\alpha+3n)+3m\left(\alpha\left(3\beta^2-4\right)+6\beta^2 n\right)-3\beta\left(3\beta^2+4\right)n\right)}{2(3m-\beta)\left(-3\beta^2+9m^2-4\right)}. \tag{5.3}$$

Plugging these into (5.1), the differential equation for $f(r)$ that we will need to solve is

$$f''(r) + \frac{f'(r)}{r}\left(\lambda_1 - \frac{\lambda_2}{\log r}\right) = 0, \tag{5.4}$$

where

$$\lambda_1 = \frac{8+9m^2-18m\beta+9\beta^2}{-4+9m^2-3\beta^2}, \tag{5.5}$$

$$\lambda_2 = \frac{\alpha(-4+9m^2-9m\beta)+n(12-9m\beta+9\beta^2)}{-4+9m^2-3\beta^2}. \tag{5.6}$$

The solution of this equation is

$$f(r) = C_2 - C_1(\lambda_1-1)^{\lambda_2-1}\int_{(\lambda_1-1)\log r}^{\infty} dt\, e^{-t}\, t^{-\lambda_2}, \tag{5.7}$$

where C_1 and C_2 are integration constants. This can be checked using the fundamental theorem of calculus. If we call $F(t)$ a primitive of the integrand, then

$$f'(r) = -C_1(\lambda_1-1)^{\lambda_2-1}\frac{d}{dr}\left(F(\infty)-F((\lambda_1-1)\log r)\right) \tag{5.8}$$

$$= \frac{C_1}{r^{\lambda_1}}(\log r)^{-\lambda_2}. \tag{5.9}$$

Then, taking one more derivative is trivial, and given that

$$3A'(r) + h'(r) = \frac{1}{r}\left(\lambda_1 + \frac{\lambda_2}{\log r}\right). \tag{5.10}$$

one immediately checks that the equation (5.1) is indeed satisfied by (5.7).

Next, we need to fix the integration constants. They can be fixed by requiring that the blackening factor tends to 1 at the UV and that it vanishes at the horizon radius r_h,

$$f(r_h) = 0, \tag{5.11}$$

$$f(r) \xrightarrow[r\to 0]{} 1, \tag{5.12}$$

A key point is that for the region of validity of our generic solution, we always have $\lambda_1 < 1$. Therefore, as $r \to 0$, the lower integration limit in (5.7) goes to $+\infty$. Therefore, we can fix one of the integration constants,

$$C_2 = 1. \tag{5.13}$$

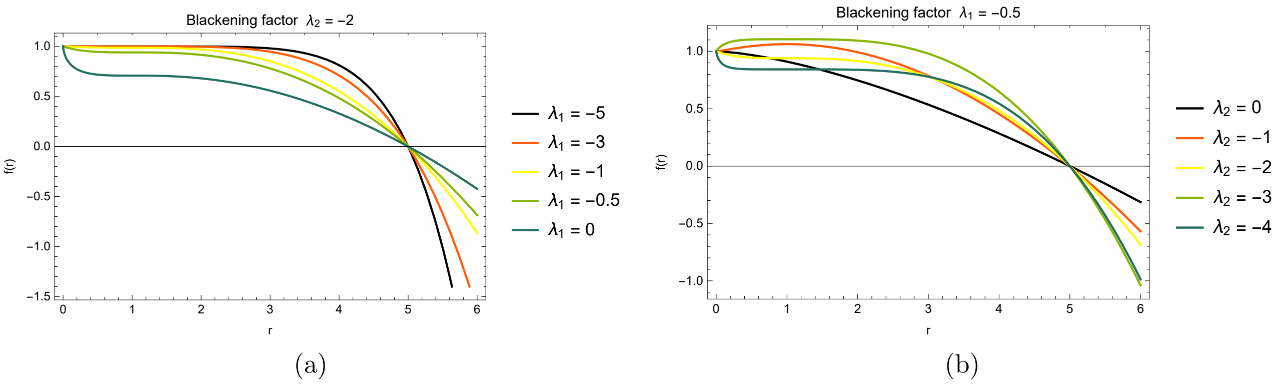


Figure 5.1: Plot of the blackening factor corresponding to the generic solution for different values of the parameters λ_1 and λ_2, with fixed horizon radius $r_h = 5$. In subfigure (a) we fixed $\lambda_2 = -2$. In subfigure (b) we fixed $\lambda_1 = -0.5$.

The second constant is trivial to put in terms of the horizon radius r_h. The result is that the blackening factor for the generic solution is

$$\boxed{f(r) = 1 - \frac{\int_{(\lambda_1-1)\log r}^{\infty} dt\ e^{-t}\ t^{-\lambda_2}}{\int_{(\lambda_1-1)\log r_h}^{\infty} dt\ e^{-t}\ t^{-\lambda_2}}.} \tag{5.14}$$

For ease of visualization, in Figure 5.1 we plot the function (5.14) for different values of the two parameters λ_1 and λ_2.

5.1.2 $f(r)$ for the special solution

As we have indicated previously, the only one of the special cases that we can obtain trustable results for the thermodynamics (while keeping track of the effect of the anisotropy) is the simplest one, $\beta = 3m$ and $\alpha = 3n$. In this case, we saw that the metric functions had to be constants,

$$A(r) = A_0, \qquad h(r) = h_0. \tag{5.15}$$

In this case, the equation for the blackening factor is simply $f'' = 0$, and it can be solved immediately,

$$f(r) = 1 - \frac{r}{r_h}. \tag{5.16}$$

5.2 Free energy and phase transition

Now that we also have the solution of Einstein equations in the case of $f(r) \neq 1$, that is, in the deconfined phase; we can proceed to investigate the confinement-deconfinement phase transition.

As we indicated in Chapter 3, there are two possible ways to do the calculation. One is to use Witten's prescription (2.72) and find the partition function by evaluating the gravity action (4.1) on the classical solution found in Chapter 4. This, however, requires both the Gibbons-Hawking term of the action, as well as the counterterms. Instead, we will compute the temperature and entropy and from them find the free energy using standard thermodynamic relations.

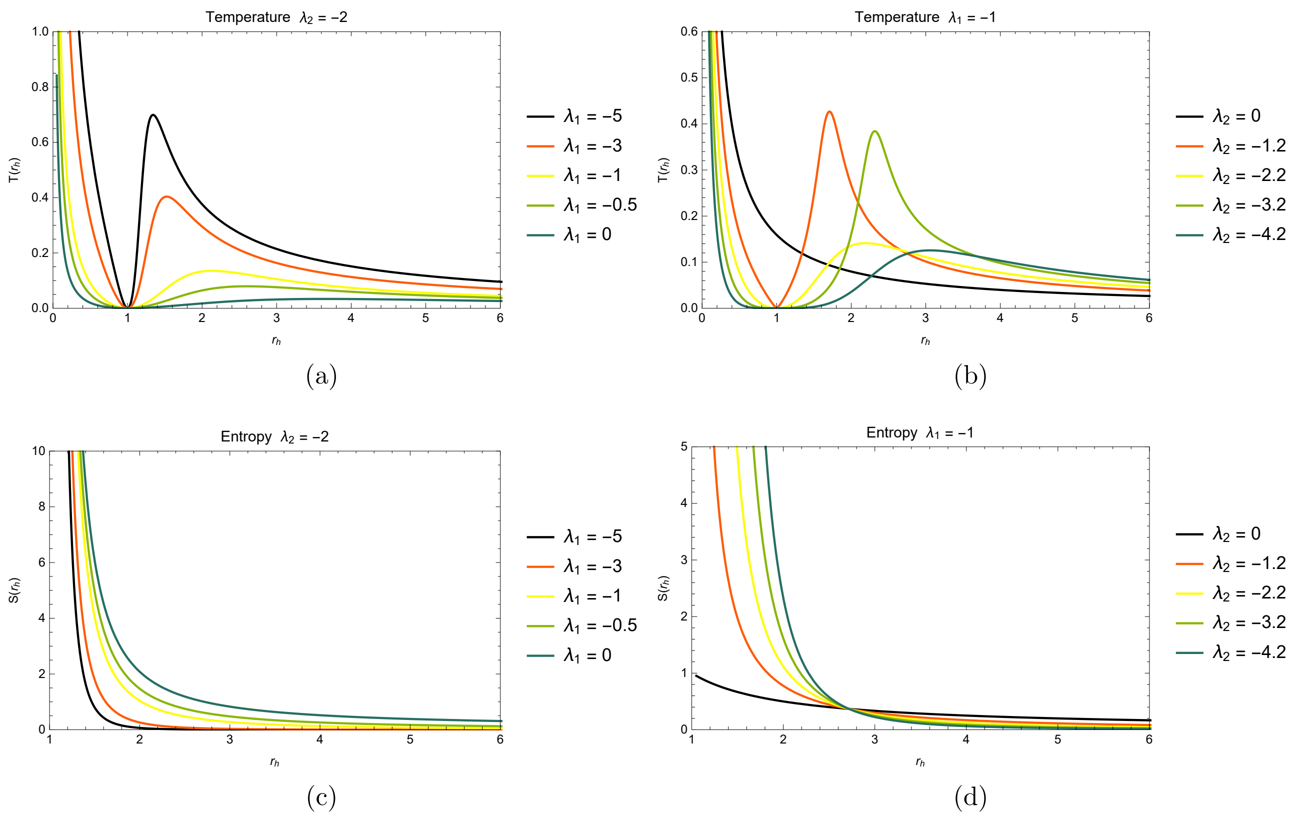


Figure 5.2: Plot of the temperature and the entropy as a functions of the radius horizon r_h, corresponding to the generic solution for different values of the parameters λ_1 and λ_2. In subfigure (a) we plot the temperature for fixed $\lambda_2 = -2$. In subfigure (b) we plot the temperature for fixed $\lambda_1 = -1$. In subfigure (c) we plot the entropy for fixed $\lambda_2 = -2$ and $a = 1$. In subfigure (d) we plot the entropy for fixed $\lambda_1 = -1$ and $a = 1$. Note that the only case where $S(r_h)$ doesn't diverge as $r_h \to 1$ is for $\lambda_2 = 0$. This case is also the only one in which $T(r_h)$ doesn't have a minimum.

5.2.1 Thermodynamics of the generic solution

First we investigate the thermodynamics for the generic solution. The starting point of the calculation is the blackening factor (5.14). As we argued in section 3.3, the temperature is the inverse of the period of euclidean time necessary to avoid a conical singularity near the horizon (3.28),

$$T(r_h) = \frac{|f'(r_h)|}{4\pi}. \tag{5.17}$$

We have already computed the derivative of the blackening factor in (5.9), so we find the temperature as a function of r_h without any additional effort,

$$\boxed{T = -\frac{(\lambda_1 - 1)^{1-\lambda_2}}{4\pi} \frac{(\log r_h)^{-\lambda_2}}{r_h^{\lambda_1}} \left(\int_{(\lambda_1 - 1)\log r_h}^{\infty} dt \; e^{-t} \; t^{-\lambda_2} \right)^{-1}}. \tag{5.18}$$

In Figures 5.2a and 5.2b we represent $T(r_h)$ for different values of the parameters λ_1 and λ_2. Notice that in all the cases, except when $\lambda_2 = 0$, the temperature is not monotonic: it reaches a local maximum at some finite value of r_h before decreasing and vanishing as $r_h \to \infty$. As we will soon see, this fact is deeply related to the existence of a phase transition.

Next, we can find the entropy as the area of the brane horizon (3.29). This area can be calculated as the square root of the determinant of the metric evaluated at the horizon. In our case, as opposed to the discussion in section 3.3, there is a contribution from $h(r)$, due to the anisotropy,

$$S = \frac{e^{3A(r_h)+h(r_h)}}{4G_N}. \tag{5.19}$$

Using our results (5.2) and (5.3), we find

$$\boxed{S(r_h) = K_{IR}\frac{r_h^{\lambda_1}}{(\log r_h)^{-\lambda_2}},} \tag{5.20}$$

where K_{IR} is a constant depending on the anisotrpy parameter a, the plank mass M_p, the number of colours of the gauge group N and the parameters Z_{IR} and V_{IR} of the potentials (4.3). In Figures 5.2c and 5.2d we represent this function for different values of λ_1 and λ_2.

Now that we have the temperature and the entropy, we can calculate the free energy by means of the standard thermodynamic relation,

$$dF = -SdT. \tag{5.21}$$

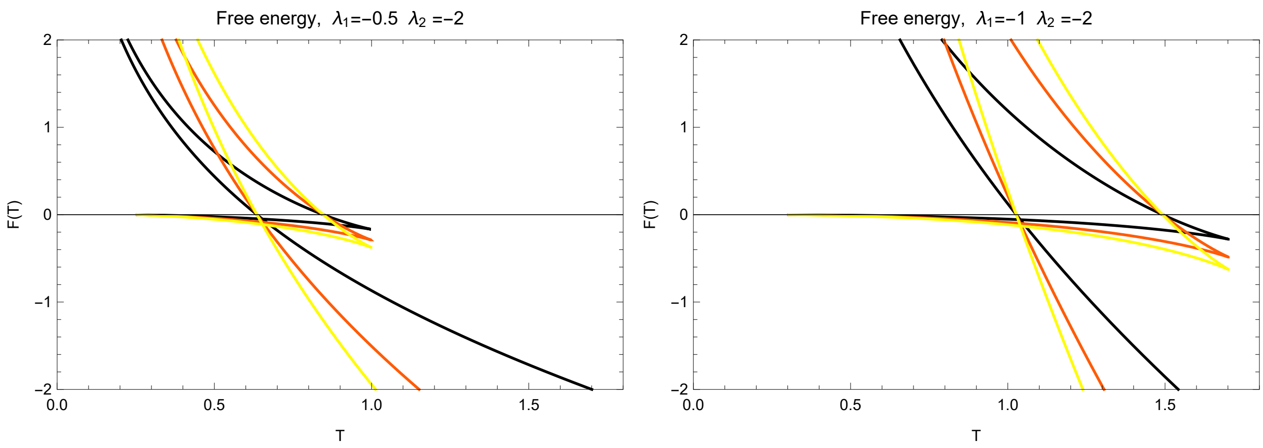
Substituting (3.28) and (5.20), we arrive to

$$dF = K_{IR} r_h^{\lambda_1 - 1} (\log r_h)^{\lambda_2 - 2} \frac{-r_h + r_h^{\lambda_1}\left(\int_{(\lambda_1-1)\log r_h}^{\infty} dt\; e^{-t}\; t^{-\lambda_2}\right)(\lambda_2 + \lambda_1 \log r_h)}{\left(\int_{(\lambda_1-1)\log r_h}^{\infty} dt\; e^{-t}\; t^{-\lambda_2}\right)^2} dr_h, \tag{5.22}$$

which cannot be integrated analytically.

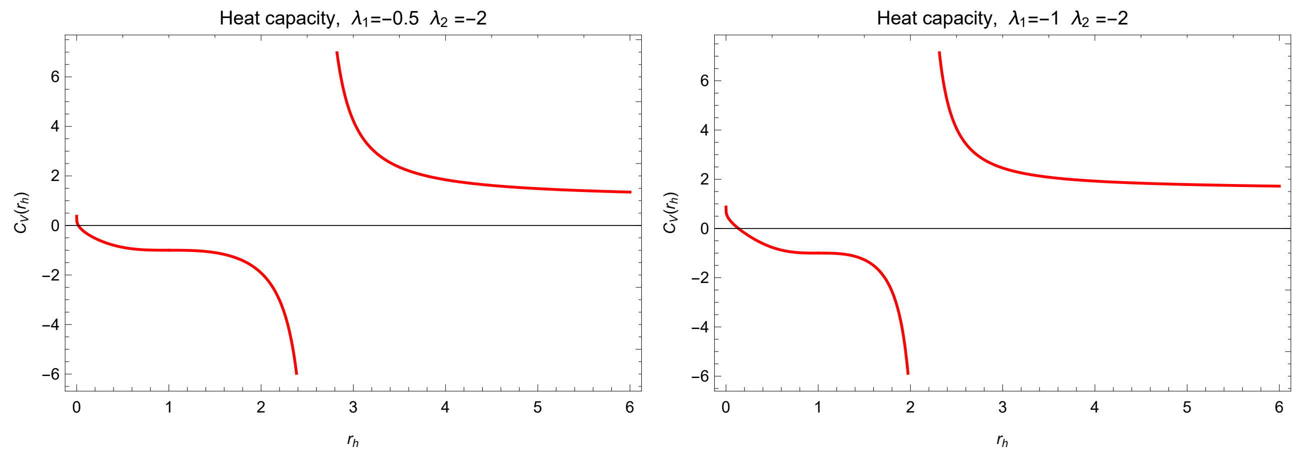
In this situation, we have two possible ways to proceed, either integrate (5.22) numerically or to do yet another deep IR approximation.

The numerical integration leads to the result shown in Figure 5.3, where we have normalized the free energy so that the axis $F = 0$ corresponds to the thermal gas solution (confined phase). Indeed we see that there is a critical temperature up from which the black-hole solution minimizes F, which corresponds to the deconfinement phase transition. Note that it will be a first order phase transition. We also see that there is a second branch (another black hole solution) that is never the one with lowest free energy, as was indeed expected [19, 20, 21].



(a) (b)

Figure 5.3: Free energy as a function of the temperature for different values of λ_1, λ_2 and a.



(a) (b)

Figure 5.4: Heat capacity as a function of r_h for different values of the parameters λ_1 and λ_2.

However, we also see that there appears to be a problem with the branches corresponding to the black hole solutions: they seem to have the wrong sign for the second derivative, which means that the specific heat will be negative and that the black-hole solution (corresponding to the deconfined phase) is always unstable. This can be checked from (3.28) and (5.20) directly,

$$C_V(r_h) = \frac{d\log S}{d\log T} = \frac{T(r_h)}{S(r_h)}\frac{S'(r_h)}{T'(r_h)} \tag{5.23}$$

$$= \frac{r_h}{r_h - r_h^{\lambda_1}\left(\int_{(\lambda_1-1)\log r_h}^{\infty} dt\ e^{-t}\ t^{-\lambda_2}\right)(\lambda_2 + \lambda_1 \log r_h)} - 1, \tag{5.24}$$

which indeed is smaller than 0 for the values of r_h corresponding to the black-hole branches (see Figure 5.4). This result implies that the deconfined phase of the plasma is always unstable. However, this is likely an artifice of our approximations and not a trustable physical conclusion.

The solution of the background metric that we found in chapter 4 is valid only on the IR, both because of the choice of the potentials and the large r approximations that we performed. Also treating the blackening factor $f(r)$ as a perturbation on top of the background metric is grounded on its being $f(r) \sim 1$ for small values of r. Therefore, we can conclude that our results for the thermodynamic functions $T(r_h)$ and $S(r_h)$ are only trustable in the large r_h limit.

To understand this asymptotic behaviour, we look at the integral in the temperature (5.18) and perform an approximation analogous to the one in (4.95),

$$\frac{d}{dt}\left(e^{-t}\ t^{-\lambda_2}\right) = \underbrace{-e^{-t}t^{-\lambda_2}}_{leading} - \underbrace{\lambda_2 e^{-t}t^{-\lambda_2-1}}_{negligible}, \tag{5.25}$$

since $t > \log r_h \gg 1$. Then we have

$$\left(\int_{(\lambda_1-1)\log r_h}^{\infty} dt\ e^{-t}\ t^{-\lambda_2}\right)^{-1} \simeq -\left(e^{-t}t^{-\lambda_2}\Big|_{(\lambda_1-1)\log r_h}^{\infty}\right)^{-1} = (\lambda_1-1)^{\lambda_2}(\log r_h)^{\lambda_2}\, r_h^{\lambda_1-1}, \tag{5.26}$$

and the temperature (5.18) becomes

$$\boxed{T(r_h) = -\frac{\lambda_1 - 1}{4\pi}\frac{1}{r_h}.} \tag{5.27}$$

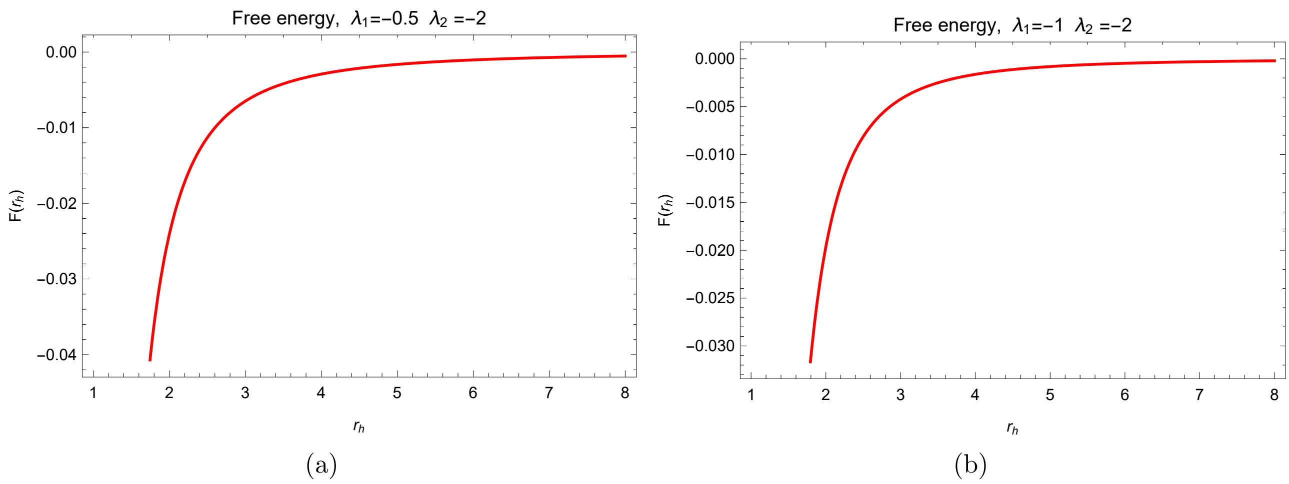


Figure 5.5: Free energy as a function of r_h for different values of the parameters λ_1 and λ_2.

In this limit, the free energy can be integrated analytically, and the result is

$$\boxed{F(r_h) = K_{IR}\frac{(1-\lambda_1)^{-\lambda_2}}{4\pi}\int_{(-\lambda_1+1)\log r_h}^{\infty} dt\; e^{-t}\; t^{\lambda_2}.} \tag{5.28}$$

However, with this approximation both the temperature (5.27) and the free energy (5.28) are monotonic as functions of r_h (see Figure 5.5), which means that there appears to be no phase transition. The conclusion is that the approximations that we have used throughout chapter 4 –that permitted us carry on with the analytic calculations– are not valid in the energy regime in which the deconfinement phase transition takes place.

5.2.2 Thermodynamics of the special solution

The thermodynamics of the special point in parameter space given by $\alpha = 3n$ and $\beta = 3m$ are very simple. Since the metric functions $A(r)$ and $h(r)$ are constants, the entropy is also a constant,

$$S(r_h) = S_0, \tag{5.29}$$

while the temperature will be

$$T(r_h) = \frac{1}{4\pi r_h}. \tag{5.30}$$

The free energy can be integrated directly, and we find,

$$F = -\frac{S_0}{4\pi r_h} = -S_0 T. \tag{5.31}$$

In this case, we also see no phase transition. This is not surprising, since from (4.215)–(4.217) we saw that this particular choice of the parameters was always a fixed point of the RG equations, and therefore conformal symmetry is recovered. The conclusion in this case is that this particular choice of parameters is of no use for the phenomenological description of strongly coupled plasmas.

6 Summary and outlook

In this work we have studied in detail the infrared behaviour of an anisotropic, strongly interacting plasma characterized in the gravity side by a dilaton and axion potentials with an exponential times a power law term. The fact that these are the most general potentials that give rise to confinement at low energies imply that our results can be applied to the phenomenological study of many strongly coupled plasmas, and not just the Quark-Gluon Plasma that motivated our investigation.

Our investigation builds upon and extends the analysis of [1]. In their work, they considered potentials that behaved like an exponential of the dilaton field in the IR, and found that the metric functions at low energies depended on the holographic coordinate via a logarithm. The presence of a power law in said potentials adds a log(log) correction to their result, in the more generic case of the choice of parameters. However, in order to find closed analytic expressions for our solution, we had to make several approximations that render our results useful only in the deep IR limit:

- From the starting point, the form of the potentials under consideration is only valid in the infrared. As we have indicated in section 3.2, at high energies we would have a polynomial of the dilaton whose coefficients can be fixed by the β-function of the gauge theory.
- In the derivation of the flow equations and their resolution, we have used in several occasions limits justified only in the deep IR (negligible second derivatives, large ϕ and $|A|$, etc.)
- Also when finding the solution directly from Einstein equations, we saw that our expressions were only approximate solutions, valid only in the regime of the holographic coordinate corresponding to the deep IR, $r \to \infty$. It's easy to see from section 4.5 that the terms we have neglected are of order $\mathcal{O}\left(\frac{\log(\log r)}{r^2 \log r}\right)$.
- Finally, we treated the blackening factor $f(r)$ as a perturbation upon the background metric of the thermal gas phase. This is only consistent if $f(r) \sim 1$ for a wide range of the holographic coordinate, which in turn requires that the horizon radius r_h has to be large.

We stress the performed approximations because they imply that our results should not be trusted away from the deep IR. In particular, we have seen that the energy scale at which the deconfinement phase transition takes place is already too high for our results to be trusted, since they would predict a negative heat capacity for the black hole solutions and hence the instability of the deconfined phase of the plasma.

We cannot help but conclude that in order to obtain quantitative results regarding the phase transition one should abandon the hope of analytical results and just solve numerically the flow equations, or directly the full Einstein's equations.

Apart from the generic solution, we have also computed the IR behaviour of the background metric functions in all the particular points of parameter space where the generic solution was not applicable. In general, the problems that emerged were due to the fact that all the coefficients we were working with were more or less complicated rational functions of the parameters, and thus have poles that we should avoid.

Having extended our analysis to each of these particular choices of parameters greatly extends the phenomenological applicability of this calculation. For example, there are reasons to believe that the particular case of the Quark-Gluon Plasma (where the confinement is linear) is better modelled with the $\beta = 3m$ case.

Therefore, one of the most obvious directions in which this work can be extended is by looking in depth into these particular solutions. For each of them, it should be checked that they indeed solve the full Einstein equations (at least in the IR limit), which is the parameter (sub)space where they are valid, and of course, how the thermodynamics look for each of the cases.[7]

Nevertheless, it is also true that we should not expect the special solutions to work better when away from the IR than the generic solution, since they are subject to the same approximations. Arguably, it would be preferable to directly do a numerical calculation in these cases too, at least insofar as the goal is the understanding of the deconfinement phase transition.

[7]Note that due to the nature of this project, a master's thesis, time constraints play an important role and in particular explain why the mentioned analysis isn't part of this work.

Appendices

A Conformal diagrams in general relativity

General spacetime metrics can be in principle quite complicated, making an intuitive understanding of is global properties nearly impossible. Conformal diagrams, or Carter-Penrose diagrams, are a tool that attempt to circumvent this, allowing direct insight into the causal structure of a given –sufficiently symmetric– manifold.

The idea is to make a conformal transformation of the coordinates that renders the space finite while keeping the light-cones invariant. This can be done thanks to conformal transformations being local rescalings that preserve angles. However, in general, finding the desired coordinate change can be a difficult task. In what follows we will construct the conformal diagrams of Minkowski space and Anti de Sitter space, the solutions of general relativity we are most interested on.

A.1 Minkowski spacetime

- $\mathbb{R}^{1,1}$

We begin with two-dimensional Minkowski space,

$$ds^2 = -dt^2 + dx^2, \tag{A.1}$$

where both coordinates range from $-\infty$ to ∞.

First, we do the coordinate transformation

$$(t, x) \mapsto u_\pm = t \pm x, \quad ds^2 = -du_+ du_- \tag{A.2}$$

so that the coordinate axis are the light cones.

Secondly, we do the scaling transformation that brings the coordinates to a finite range,

$$u_\pm \mapsto \tilde{u}_\pm = \tan^{-1} u_\pm, \quad |\tilde{u}_\pm| < \frac{\pi}{2}, \tag{A.3}$$

$$ds^2 = \frac{1}{4\cos^2 \tilde{u}_+ \cos^2 \tilde{u}_-} \left(-d\tilde{u}_+ d\tilde{u}_-\right). \tag{A.4}$$

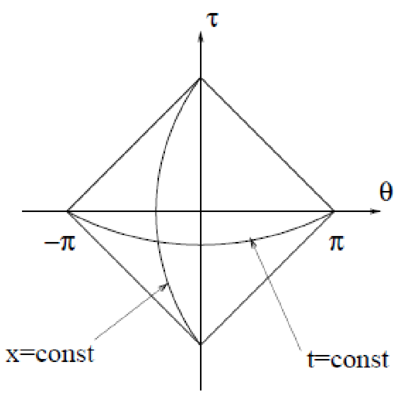


Figure A.1: Two-dimensional Minkowski space is mapped into the interior of the rectangle, preserving the light-cones at 45 degrees.

Lastly, we recover the original timelike and spacelike directions with

$$\tilde{u}_\pm \mapsto (\tau, \theta) = (\tilde{u}_+ + \tilde{u}_-, \tilde{u}_+ - \tilde{u}_-), \tag{A.5}$$

$$ds^2 = \frac{1}{(\cos\tau + \cos\theta)^2}\left(-d\tau^2 + d\theta^2\right). \tag{A.6}$$

We know that the overall conformal factor doesn't affect the causal structure of the spacetime, so we immediately see that the light-cones will look exactly like the ones from our original Minkowski space, but with coordinates that only take values on a finite interval (Figure A.1).

•$\mathbb{R}^{1,d}$

Now we consider Minkowski space with d spatial dimensions,

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-1}^2 \tag{A.7}$$

where $d\Omega_{d-1}^2$ is the metric on the sphere S^{d-1}.

To find the conformal diagram, we perform the same coordinate transformations as in the $d = 1$ case, so that (A.7) becomes

$$ds^2 = \frac{1}{(\cos\tau + \cos\theta)^2}\left(-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}^2\right) \tag{A.8}$$

The only difference is that, since $r > 0$, also $0 < \theta < \pi$, and Minkowski space is mapped –ignoring the angular coordinates– into a tringle (Figure A.2). Note also that this conformal diagram is topologically equivalent to $\mathbb{R} \times S^p$, which can be seen taking the maximal extension of the coordinate domain, $0 < \tau < \infty$ –the points $\theta = 0, \pi$ correspond to the poles of S^p–.

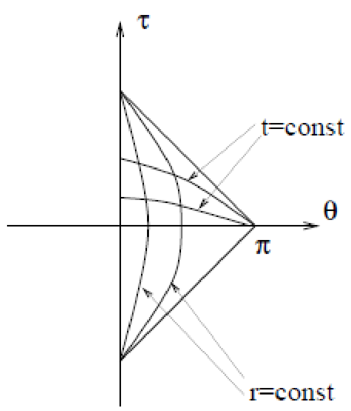


Figure A.2: (d+1)-dimensional Minkowski space is mapped into the interior of the triangle, preserving the light-cones at 45 degrees.

A.2 AdS spacetime

Now we attempt to find the conformal diagram of Anti de Sitter spacetime. In order to do this, we start from the metric in global coordinates,

$$ds^2 = R^2 \left(-\cosh^2\rho \; d\tau^2 + d\rho^2 + \sinh^2\rho \; d\Omega_p^2\right), \tag{A.9}$$

and do the change of coordinates

$$\tan\theta = \tanh\rho, \qquad 0 \le \theta < \frac{\pi}{2}. \tag{A.10}$$

With this, (A.9) becomes

$$ds^2 = \frac{R^2}{\cos^2\theta}\left(-d\tau^2 + d\theta^2 + \sin^2\theta \; d\Omega_p^2\right). \tag{A.11}$$

This metric coincides with (A.8) up to a conformal factor that doesn't affect the causal structure of our spacetime. However, in this case, due to the range of the coordinate θ, a slice of constant τ is mapped to *one hemisphere* of S^{p+1}. Therefore, we can conclude that the conformal diagram of AdS can be visualized as a cylinder (see Figure A.3).

Notice that the conformal boundary of AdS, $\mathbb{R} \times S^p$, corresponds to the conformal diagram of Minkowski space.

B Basics of supersymmetry

Usually in high energy physics, one studies systems with certain symmetries, mainly under the Poincaré group and under some other internal (compact, simply connected) Lie group. Supersymmetry enlarges these groups, adding new fermionic generators to the previously existing symmetries. Obviously, this new transformations will send a bosonic field to a fermionic one –and viceversa– and therefore a necessary condition for supersymmetry is that we have the same number of bosonic and fermionic degrees of freedom at each mass level.

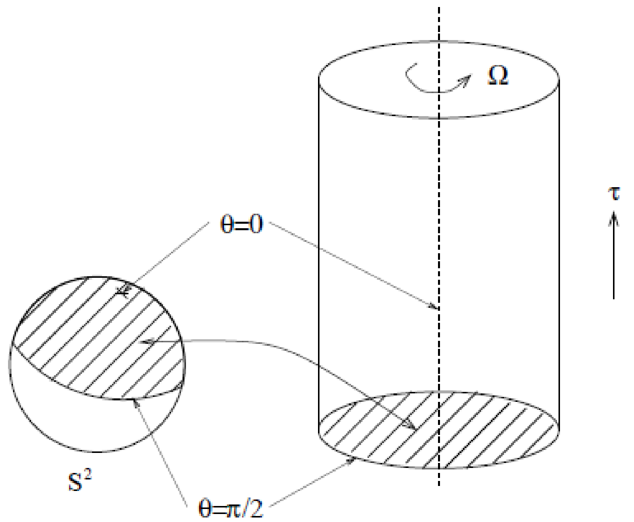


Figure A.3: Conformal diagram on Anti de Sitter spacetime in the case $p = 1$.

In this appendix, we will present the most basic aspects of supersymmetry that are necessary to discuss the super Yang-Mills theory relevant to the AdS/CFT correspondence.

B.1 SUSY algebra

The Poincaré group is generated by the momentum P_μ corresponding to the spacetime translations –and to the subgroup $\mathbb{R}^{1,3}$– and the Lorentz generators $M_{\mu\nu}$ corresponding to rotations and boosts –group $SO(1,3)$–. In SUSY, we add to these the new generators Q^a_α and $\bar{Q}_{\dot{\alpha}a} = (Q^a_\alpha)^\dagger\}$ with $\alpha = 1, 2$ and $a = 1, \ldots, \mathcal{N}$; which are called *supercharges*. The supercharges are Weyl spinors: they transform under the (1/2,0) or (0,1/2) representation of the Lorentz group, respectively[8],

$$[M_{\mu\nu}, Q^a_\alpha] = -\frac{i}{2}\,(\sigma_{\mu\nu})^\beta_\alpha\, Q^a_\beta. \tag{B.1}$$

To define the supersymmetry algebra, we set that they commute with the translations and that among themselves they satisfy

$$\{Q^a_\alpha, \bar{Q}_{\beta b}\} = 2\sigma^\mu_{\alpha\beta} P_\mu \delta^a_b, \qquad \{Q^a_\alpha, Q^b_\beta\} = 2\epsilon_{\alpha\beta} Z^{ab}, \tag{B.2}$$

where σ^μ are the Pauli matrices, $\epsilon_{\alpha\beta}$ is the antisymmetric tensor and, by construction, the so called *central charges* Z^{ab} are also antisymmetric and commute with everything (note that this implies that for minimal supersymmetry $\mathcal{N} = 1$ we must have $Z = 0$).

One can easily see that the transformations

$$Q^a \mapsto U^a{}_b Q^b, , \tag{B.3}$$

with $U^a{}_b \in SU(\mathcal{N})_R$ always leave the SUSY algebra invariant. This symmetry is called R-symmetry, and part of it –or all of it– can be anomalous, depending on the QFT under consideration.

B.2 Representations of the SUSY algebra

In order to study the possible supersymmetric theories that we can have for different values of $\mathcal{N}$, we need to look at the possible representations of the SUSY algebra.

Massless representations

We know that, for any massless particle, we can choose a Lorentz observer such that the momentum becomes $P_\mu = (E, 0, 0, E)$. With this, (B.2) becomes

$$\left\{Q^a_\alpha, \bar{Q}_{\beta b}\right\} = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\beta} \delta^a_b. \tag{B.4}$$

From the $\alpha = \beta = 2$ component we find

$$\left\{Q^a_2, (Q^b_2)^\dagger\right\} = 0, \tag{B.5}$$

[8] It should be noted that this is a matter of convention, it is also possible to build the SUSY algebra starting with Majorana spinors. See [12] as an example.

and hence for any element of the vector space of the representation, $|\psi\rangle$,

$$||Q_2^a|\psi\rangle||^2 = 0 \quad \Rightarrow \quad Q_2^a = 0, \tag{B.6}$$

where we have used that the representations that we are looking for are unitary.

As a consequence, we must have that $Z^{ab} = 0$, and the only nontrivial relation that we have can be written as

$$\left\{ \frac{Q^a}{2\sqrt{E}}, \frac{Q_b^\dagger}{2\sqrt{E}} \right\} = \delta_b^a, \tag{B.7}$$

where we have suppressed the subindex '1'. From (B.1) we can see that acting with Q^a lowers the helicity by 1/2 (and $(Q^a)^\dagger$ raises it by 1/2). From (B.7) we see that our system behaves like an ordinary fermionic oscillator, and therefore we can build the Hilbert space by acting with Q^a, $a = 1, \ldots, \mathcal{N}$ on a highest weight state $|h\rangle$. However, since this 'weight' in our case is the helicity, we have a constraint imposed by the Wienberg-Witten theorem –that states that a theory without gravity can only contain states with helicity $|h| < 1$, or $|h| < 2$ if there is gravity–, namely that we can have at most $\mathcal{N} = 4$ (or $\mathcal{N} = 8$ with gravity), or else the state $Q^1Q^2Q^3Q^4Q^5|h\rangle$ would violate said theorem. The dimension of the representation will be $2^{\mathcal{N}}$.

Massive representations

In the massive case we proceed similarly to what we did before: we choose a Lorentz frame such that $P_\mu = (M, 0, 0, 0)$ and (B.2) becomes

$$\{Q_\alpha^a, (Q_\beta^b)^\dagger\} = 2M\delta^{ab}\delta_{\alpha\beta}. \tag{B.8}$$

In the other anticommutator, we can use the R-symmetry to block diagonalize the central charges Z^{ab} so that

$$Z^{ab} = \begin{pmatrix} 0 & Z_1 & & & \\ -Z_1 & 0 & & & \\ & & 0 & Z_2 & \\ & & -Z_2 & 0 & \\ & & & & \ddots \end{pmatrix}. \tag{B.9}$$

Now we rename our indices $a \to (\hat{a}, \bar{a})$ so that $\hat{a} = 1, 2$ denotes the row inside each block and $\bar{a} = 1, \ldots, [\mathcal{N}/2]$ denotes the block. Then, we can define

$$\mathcal{Q}_{\alpha\pm}^{\bar{a}} = \frac{1}{2}\left(Q_\alpha^{(1,\bar{a})} \pm \sigma_{\alpha\dot{\beta}}^0 (Q_\beta^{(2,\bar{a})})^\dagger\right). \tag{B.10}$$

With this, the supersymmetry algebra (B.2) reduces to

$$\left\{\mathcal{Q}_{\alpha\pm}^{\bar{a}}, (\mathcal{Q}_{\beta\pm}^{\bar{b}})^\dagger\right\} = \delta^{\bar{a}\bar{b}}\delta_{\alpha\beta}\left(M \pm Z_{\bar{a}}\right), \tag{B.11}$$

and all the rest of the anticommutators vanish. Note that if the RHS of (B.11) is negative, we can obtain negative norm states. Since we are looking for unitary representations, we must require that

$$M \geq |Z_{\bar{a}}| \quad \forall\, \bar{a} = 1, \ldots, [\mathcal{N}/2] \tag{B.12}$$

which is known as the BPS bound. Whenever the equality is satisfied for a certain $\bar{a}$, the corresponding anticommutator vanishes and one of the supercharges $\mathcal{Q}^{\bar{a}}_{\alpha+}$ or $\mathcal{Q}^{\bar{a}}_{\alpha-}$ is represented by the zero operator. In general, if n_0 of the BPS bounds are saturated, the representation will have dimension $2^{2\mathcal{N}-2n_0}$.

B.3 Field content and lagrangians

Now we are in a situation to be able to study the spectrum of the different supersymmetric theories for any value of $\mathcal{N}$.

We begin by looking at the possible theories of massless particles with $\mathcal{N} = 1$ SUSY. In this case, as we saw in the previous section, we have only one operator that lowers the helicity by 1/2 and that squares to zero. Therefore, there are only two possible theories: one with a gauge field (of helicity 1) and a Majorana fermion –called the gaugino, its supersymmetric partner– and one with one scalar and a Weyl fermion.

Let's look, as an example, to the first theory, called the *gauge multiplet*. The field content is a gauge field A_μ and a spinor λ, as we just stated. The most simple lagrangian that we can write is

$$\mathcal{L} = -\frac{1}{g^2} Tr\left[F \wedge \star F\right] + \frac{\theta_I}{8\pi^2} Tr\left[F \wedge F\right] - \frac{i}{2} Tr\left[\bar{\lambda}\bar{\sigma}^\mu D_\mu \lambda\right], \tag{B.13}$$

where the star indicates the Hodge dual and D_μ is the usual covariant derivative. This lagrangian, made from just the standard kinetic term, the standard Chern-Simmons term, and the minimal coupling in terms of the covariant derivative, indeed exhibits a $\mathcal{N} = 1$ supersymmetry, given by the explicit transformations

$$\begin{aligned} \delta_\epsilon A_\mu &= i\bar{\epsilon}\bar{\sigma}_\mu \lambda - i\bar{\lambda}\bar{\sigma}_\mu \epsilon \\ \delta_\epsilon \lambda &= \sigma^{\mu\nu} F_{\mu\nu} \epsilon \end{aligned} \tag{B.14}$$

where the parameter of the transformation ϵ is a Weyl spinor. Note that indeed it is required that the fermion is massless, the massive gauge multiplet is more complicated.

We won't write here the lagrangians of all the supersymmetric theories. However, we will list their field content for future reference:

- $\mathcal{N} = 1$ gauge multiplet, formed by one gauge field A_μ and one Majorana fermion λ.
- $\mathcal{N} = 1$ chiral multiplet, formed by one Weyl fermion ψ and one complex scalar ϕ.
- $\mathcal{N} = 2$ gauge multiplet, formed by one gauge field A_μ, two Weyl fermions $\psi_\pm$ and one complex scalar ϕ.
- $\mathcal{N} = 2$ hypermultiplet, formed by two Weyl fermions $\psi_\pm$ and two complex scalars $H_\pm$.
- $\mathcal{N} = 4$ gauge multiplet, formed by one gauge field A_μ, four Weyl fermions λ^a and six real scalars X^i.

C Kaluza-Klein compactification on a circle

For the purposes of the AdS/CFT correspondence, as well as many other applications of String Theory, one needs to compactify a higher dimensional field theory into a 4 dimensional one. The way to do so is called the Kaluza-Klein mechanism, and here we explain it in the simplest case: the compactification of one dimension in a circle of radius R.

Scalar field

We begin by considering a free scalar field with equation of motion

$$\left(-\Box_d + m^2\right)\phi = \left[-\Box_{d-1} - \frac{\partial^2}{\partial y^2} + m^2\right]\phi = 0. \tag{C.1}$$

We consider that the scalar field obeys periodic boundary conditions on the S^1 corresponding to the direction y, so that we can Fourier expand,

$$\phi(x^{\bar{\mu}}, y) = \sum_{n\in\mathbb{Z}} \phi_n(x^{\bar{\mu}}) e^{2\pi i \frac{ny}{R}}, \tag{C.2}$$

where the index $\bar{\mu} = 0, 1, \ldots, d-2$.

With this decomposition, (C.1) becomes

$$\left(-\Box_{d-1} + m^2 + \frac{4\pi^2 n^2}{R^2}\right)\phi_n = 0 \qquad \forall\, n \in \mathbb{Z}. \tag{C.3}$$

When the compact dimension is small, $R \to 0$, and all the modes except $n = 0$ acquire an infinitely heavy mass and decouple. Thus, we conclude that the dimensional reduction of a scalar field is another scalar field with the same mass.

Tensor and spinor fields

The case of fields living in non-trivial representations of the Lorentz group is slightly more complicated that for the scalar field. The core of the previous calculation still is valid: higher modes in the Fourier expansion will decouple. However, now we will have more than one field. In the case of a vector field A_μ (i.e. a field belonging to the fundamental representation of $SO(1, d-1)$) we will find two fields, a vector $A_{\bar{\mu}}$ belonging to the fundamental of $SO(1, d-2)$ and a scalar A_y. In general, for a boson field living in a given representation R of $SO(1, d-1)$, one needs to study the restriction of R to $SO(1, d-2)$. In general, one studies what is the multiplicity with which each representation of the subgroup occurs in R. This restrictions receive the name of *branching rules* of $SO(1, d-1)$ to $SO(1, d-2)$.

Spinor fields are no different in this regard: the representation of $SO(1, d-1)$ will be written as some direct sum of representations of $SO(1, d-2)$. It can be shown that in the presence of supersymmetry, one can organize the boson and vector fields into SUSY multiplets, and that dimensional reduction will preserve all the supersymmetries, although one has to take into account that the supercharges will also behave as the spinor fields described above.

An important example of a tensor field is the metric/graviton $G_{\mu\nu}$. After dimensional reduction, we will find a scalar, the dilaton G_{yy}; a vector field $G_{\bar{\mu}y}$ and a $(d-1)$-dimensional metric $G_{\bar{\mu}\bar{\nu}}$.

D Brief review of QCD

Quantum Chromodynamics (QCD) was developed in 1973 as a way of putting together non-abelian gauge theories (which had been very successful in the study of electroweak interactions) and parton models (which also had been very successful as phenomenological models in particle physics). Here we review its main features, namely UV asymptotic freedom and IR confinement, that we will need to take into account when building a holographic model for the strong interactions.

D.1 Perturbative QCD

The lagrangian of QCD is that of a $SU(3)$ Yang-Mills theory,

$$\mathcal{L}_{QCD} = -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a + i\bar{q}_i\gamma^\mu D_\mu q_i - m\bar{q}_i q_i, \tag{D.1}$$

where the q_i are the spinor fields that describe the quarks, each of them a three component vector in the fundamental of $SU(3)$, and $i = u, d, c, s, t, b$ a flavour index. The covariant derivative and field strength tensor are,

$$D_\mu = \partial_\mu - igA^a_\mu\lambda_a, \tag{D.2}$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^a_{bc}A^b_\mu A^c_\mu. \tag{D.3}$$

Here, A_μ is the gluon field, g the coupling constant for the strong interactions, λ_a the Gell-Mann matrices and $[\lambda_a, \lambda_b] = f^c_{ab}\lambda_c$.

If one proceeds with perturbative quantization starting from (D.1), and extracts the Feynman rules, one finds that there are three different vertices: a 3-point gluon-quark interaction due to the covariant derivative term, that is proportional to g; a 3-point gluon self-interaction due to the gauge kinetic term, proportional to $p_\mu g$; and a 4-point gluon self-interaction proportional to g^2.

Loop contributions in QCD present both UV and IR divergences. The IR ones are solved by taking into account all the diagrams that lead to experimentally indistinguishable products. Thus, for example, a diagram with a quark external leg together with a diagram with a quark and a very soft gluon will partially cancel and give a finite contribution. This is related to the fact that in experiments of particle physics, what one measures are jets and not individual particles. On the other hand, the UV divergences require regularization and renormalization. After this procedure, one arrives at the following $\beta-$function for the coupling g,

$$\frac{\beta(g)}{g} = -\frac{33-2N_f}{3}\frac{g^2}{16\pi^2} - \frac{306-38N_f}{3}\left(\frac{g^2}{16\pi^2}\right)^2 + \mathcal{O}\left(g^6\right), \tag{D.4}$$

where $N_f = 6$ is the number of flavours.

D.2 Confinement and asymptotic freedom

The $\beta-$function (D.4) contains a lot of interesting information. Usually, one writes it in terms of the strong structure constant,

$$\alpha_s = \frac{g^2}{4\pi}. \tag{D.5}$$

In this terms, the resulting RG equation is

$$\frac{d\alpha_s(\mu)}{d\mu} = -\frac{33 - 2N_f}{6\pi}\alpha_s^2(\mu) - \frac{153 - 19N_f}{12\pi^2}\alpha_s^3(\mu) + \mathcal{O}\left(\alpha_s^4(\mu)\right), \tag{D.6}$$

This equation can be integrated between μ and another energy scale Q for which we know $\alpha_s(Q^2)$. The result is that

$$\alpha_s(\mu^2) = \frac{\alpha_s(Q^2)}{1 + \frac{33-2N_f}{6\pi}\alpha_s(Q^2)\log\frac{\mu^2}{Q^2}}\left[1 + \mathcal{O}\left(\frac{\log\log\mu^2}{\log\mu^2}\right)\right]. \tag{D.7}$$

From this expression of the running coupling constant we see two of the most important features of QCD: asymptotic freedom in the UV and confinement in the IR. At very high energies, the logarithm in the denominator grows and α_s tends to zero. In this situation the interaction between the quarks and gluons is very small and we may see them as free particles. However, at low energies, we can have that the logarithm becomes negative and the coupling constant diverges. The interaction between the elementary particles becomes strong, and we cannot observe them freely at low energies, only as composite particles that form colour singlets. This process of confinement cannot be studied directly from the QCD lagrangian, though, because for $g \gtrsim 1$ perturbation theory in term of Feynman diagrams is no longer valid.

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