New D1/D5 Black Holes from Scherk-Schwarz Reductions of Type IIB String Theory

Author:

Koen Stemerdink BSc

Supervisor:

Prof. dr. Stefan Vandoren

Utrecht University

Institute for Theoretical Physics
Faculty of Science
Utrecht University

June, 2018

Abstract

We study D1/D5 black holes in the context of Scherk-Schwarz reductions of type IIB string theory. In particular, we are interested in reductions that break the supersymmetry partially. We derive constraints on the parameters of the Scherk-Schwarz twist that need to be satisfied in order to obtain valid 3-charge black hole solutions in the reduced theory. We find that these constraints can only be satisfied by reductions that break the supersymmetry to $\mathcal{N} \in \{0, 4, 8\}$, i.e. only these kinds of reductions lead to theories that support 3-charge D1/D5 black holes. The black holes that we construct in the $\mathcal{N} = 0$ and $\mathcal{N} = 4$ theories are unknown in previous classifications. We find valid D1/D5 solutions in $\mathcal{N} = 2$ and $\mathcal{N} = 6$ by taking the three charges to be equal. This results in the single charge Tangherlini black hole.

Contents

Chapter 1

Introduction

Black holes have, to some extent, always been surrounded by mystery. Among other things, they have given rise to the so-called black hole information paradox[1]. This is the paradox that emerges from the notion that black holes evaporate through Hawking radiation [2]. General relativity tells us that information that falls through a black hole event horizon can never escape the black hole. A logical consequence would be to assume that this information is lost, since (given enough time) the black hole will evaporate completely. However, the principles of quantum mechanics dictate that information must always be conserved. Hence, we have a paradox: is information lost in black hole evaporation, or does it survive somehow and come out with the Hawking radiation?

Another black hole related problem is the microscopic explanation of the Bekenstein-Hawking entropy. The formula for this (macroscopic) entropy was first conjectured by Bekenstein up to a constant of proportionality [3]. Two years later, Hawking derived the exact expression [2]. It reads

$$S_{\mathrm{BH}} = \frac{k_B\, c^3 A}{4\hbar\, G_N}\,, \tag{1.1}$$

where k_B is the Boltzmann constant, c is the speed of light, A is the area of the event horizon of the black hole, $\hbar$ is the Planck constant, and G_N is Newton's constant. The subscript BH conveniently stands both for 'black hole' and for 'Bekenstein-Hawking'.

As in any thermodynamic system, the entropy is a measure of the number of microstates that correspond to a certain macrostate. A proper theory of quantum gravity should be able to explain the macroscopic Bekenstein-Hawking entropy formula from a microscopic perspective. That is, by counting the microstates that correspond to black hole solutions. For a few specific

[1] For a non-academic review of the information paradox and of related topics in modern theoretical physics, I can highly recommend the book *The Black Hole War* by Leonard Susskind [1].

black hole solutions, this has been done successfully in the context of string theory (which makes it a promising candidate for a theory of quantum gravity). For these solutions, the microscopic entropy can be computed by counting D-brane configurations. This was done for the first time by Strominger and Vafa for extremal 5-dimensional black holes [4]. Later, similar calculations were completed for near-extremal and rotating 5-dimensional black hole solutions [5, 6, 7], as well as for 4-dimensional black hole solutions [8, 9, 10]. The computation of the microscopic entropy for more general black holes (for example, black holes that are far from extremal) remains an open problem. Nevertheless, the result that the Bekenstein-Hawking entropy can be verified microscopically for specific black hole solutions is often considered to be the biggest success of string theory (yet).

Superstring theory is only consistent in a 10-dimensional spacetime. Therefore, one or multiple spatial dimensions must be compactified, in order to find lower-dimensional black hole solutions in the context of string theory. There are many different ways, in which (string) theories can be compactified. In this thesis we will primarily focus on Scherk-Schwarz reduction. This type of compactification was originally proposed by Joël Scherk and John Henry Schwarz in the context of supergravity [11, 12]. Scherk-Schwarz reduction, also known as generalized reduction or twisted reduction, is characterized by the (partial) breaking of supersymmetry. Furthermore, it typically leads to various fields becoming massive. In untwisted Kaluza-Klein reduction, these fields would remain massless.

Both black holes and Scherk-Schwarz reductions have been studied extensively. However, they have rarely been studied together. The only published article in which black holes are constructed in Scherk-Schwarz reduced theories, is written in 2015, by Gaddam et al. [13]. In addition, there is a recent piece of unpublished work by Hull and Vandoren on this topic [14]. Substantial parts of this thesis are based on the latter.

As mentioned before, the only black holes that are understood both from a macroscopic and a microscopic perspective are either extremal or near-extremal. Black holes that are far from extremal, however, are not yet understood microscopically. Because such black holes preserve no supersymmetry, we are interested in the effects of supersymmetry breaking on both the macroscopic and the microscopic description of black holes. Scherk-Schwarz reduction provides us with a natural way of supersymmetry breaking. This is what motivates the combined study of black holes and Scherk-Schwarz reduction.

In this thesis, we study the 5-dimensional D1/D5 black hole in the context of Scherk-Schwarz reductions of type IIB string theory. We focus on Scherk-Schwarz reductions that partially break the supersymmetry. Our primary result is a set of constraints (on the parameters of the twist and on the black hole charges) that needs to be satisfied in order for the black hole to be a valid solution of the reduced theory.

1.1 Outline

In chapter 2, we discuss some basic aspects of string theory and supergravity. In particular, we will examine both p-branes and D-branes, as these extended objects play a central role in the construction of string theoretical black holes.

Subsequently, in chapter 3, we study compactification. We start with Kaluza-Klein reduction, the most basic type of toroidal compactification. Following that, we consider the various dualities that play a role in compactifications of string theories. Lastly, in section 3.3, we discuss Scherk-Schwarz reduction. This type of compactification is one of the main topics of interest in this thesis.

In chapter 4, we apply the formalism that we developed in chapter 3 by working out two reductions explicitly. We first reduce pure gravity on $T^2 \times S^1$, and after that we reduce type IIB string theory on $T^4 \times S^1$. In both cases, we include a Scherk-Schwarz twist in the final reduction on the circle.

Finally, in chapter 5, we arrive at the other main topic of this thesis: black holes. We construct and compactify the rotating D1/D5 black hole solution, and compute its macroscopic entropy with the Bekenstein-Hawking area law. In section 5.3, we analyze which kinds of Scherk-Schwarz twists leave this black hole solution intact.

1.2 Conventions

We use the 'mostly plus' convention for the signature of the metric tensor, i.e. the Minkowski metric is given by $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$.

We choose natural units such that $c = \hbar = k_B = 1$. However, we do not set Newton's constant equal to 1. The d-dimensional Newton constant is related to the 10-dimensional one by $G_N^{(d)} = G_N^{(10)}/V_{10-d}$, where V_{10-d} is the volume of the manifold that we compactify on. The Newton constant is related to the gravitational coupling by $8\pi G_N^{(10)} = \kappa_{10}^2$.

We use $G_{\mu\nu}$ for the string frame metric, and $g_{\mu\nu}$ for the Einstein frame metric. The dilaton can be decomposed as $\Phi = \tilde{\Phi} + \langle\Phi\rangle$, where $\tilde{\Phi}$ is its varying part and $\langle\Phi\rangle$ is its constant part.

On some d-dimensional manifold with metric $g_{\mu\nu}$, we denote the curved space Levi-Civita tensor by $\tilde{\varepsilon}_{\mu_1 \ldots \mu_d} = \sqrt{-g}\, \varepsilon_{\mu_1 \ldots \mu_d}$, where $\varepsilon_{\mu_1 \ldots \mu_d} = \pm 1$ depending on the order of the indices. The Hodge star operator (defined on this manifold) works on a p-form as

$$* F_p = \frac{1}{p!\,(d-p)!}\, \tilde{\varepsilon}^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_{d-p}}\, F_{\mu_1 \ldots \mu_p}\, \mathrm{d}x^{\nu_1} \wedge \ldots \wedge \mathrm{d}x^{\nu_{d-p}}. \tag{1.2}$$

For the kinetic terms of form-values fields, we use the following notation:

$$\left(|F_p|^2\right)^{ij} = \frac{1}{p!}\, F^i_{\mu_1\ldots\mu_p} \bar{F}^j_{\nu_1\ldots\nu_p}\, g^{\mu_1\nu_1} \ldots g^{\mu_p\nu_p}, \tag{1.3}$$

where $\bar{F}^i$ is the complex conjugate of F^i. The indices i and j are only needed when we consider a theory with multiple p-form field strengths. In other situations, we drop them. On a spherically symmetric manifold, we denote the 'ordinary' radial coordinate by ρ, and the isotropic radial coordinate by r. These two radii are related by a shift. In the context of black holes and black branes this shift is exactly the distance between the singularity and the event horizon.

Chapter 2

Aspects of String Theory

In this chapter, we cover some preliminary aspects of string theory. We do not present an in-depth study of most of these topics, as we assume that most readers will have encountered them before. Instead, we recall some results, that will be relevant for the main part of this thesis. For more comprehensive reviews of the topics covered in this chapter, we refer to [15, 16, 17, 18, 19, 20]. Specific information about supergravity can be found in [21].

In section 2.1, we start with some aspects of supergravity. In particular, we study the type II supergravity theories. We consider these theories both in string frame and in Einstein frame. Subsequently, in section 2.2, we study branes. These extended objects, are the building blocks of the black holes that we will construct in chapter 5.

2.1 Supergravity

There are two main approaches to supergravity as a research area. On the one hand, supergravity actions are the low-energy effective theories that follow from string theories. This is perhaps the context in which they are studied most frequently. On the other hand, supergravity has also been studied extensively as a subject on its own, without the notion of a more fundamental theory that it descends from. In this context, supergravities are theories that follow from combining the principles of supersymmetry and general relativity. In this thesis, we work from the former point of view. For many purposes, we work with supergravity actions (and compactifications thereof), but we always keep in mind that we do so in the context of string theory.

In section 2.1.1, we discuss the field contents of both of the type II supergravity theories. For type IIB, we write down the bosonic part of the action explicitly. Following that, we discuss the two frames in which supergravity actions are usually written down: the string frame and the Einstein frame. In section 2.1.2, we convert the type IIB action from string frame to Einstein frame. In this Einstein frame action we can recognize an $SL(2)$ symmetry (see section 2.1.3) called S-duality. This duality relates type IIB string theory at weak coupling and at strong coupling.

2.1.1 Type IIA and type IIB supergravity

Type II string theories are based on closed strings. Depending on the choice of the GSO projection, we either have type IIA (non-chiral) or type IIB (chiral) string theory. Both of these theories are maximally supersymmetric, i.e. they possess 32 supercharges.

The massless spectra that follow from the type II string theories are the field contents of the type IIA and type IIB supergravity actions. In the NS-NS sector, both theories contain a graviton $G_{\mu\nu}$, a 2-form field B_2 known as the Kalb-Ramond field, and a scalar field Φ called the dilaton. The R-R sectors of both theories are different. For type IIA, this sector contains forms with odd tensor rank, and for type IIB, it contains forms with even tensor rank. We denote these fields as

$$\begin{aligned} &\text{IIA:} \quad A_1,\, A_3 \\ &\text{IIB:} \quad C_0,\, C_2,\, C_4. \end{aligned} \tag{2.1}$$

Since we focus on type IIB string theory in this thesis, we write down the action of type IIB supergravity explicitly. It reads [20]

$$\begin{aligned} S_{\mathrm{IIB}} = &\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\,\sqrt{-G}\,e^{-2\Phi}\left(R + 4|\mathrm{d}\Phi|^2 - \frac{1}{2}|H_3|^2\right) \\ &- \frac{1}{4\kappa_{10}^2}\int \mathrm{d}^{10}x\,\sqrt{-G}\left(|F_1|^2 + |F_3|^2 + \frac{1}{2}|F_5|^2\right) \\ &- \frac{1}{4\kappa_{10}^2}\int C_4\wedge H_3\wedge F_3. \end{aligned} \tag{2.2}$$

Here, we use the following notation for the field strengths:

$$\begin{aligned} &H_3 = \mathrm{d}B_2, \qquad F_1 = \mathrm{d}C_0, \\ &F_3 = \mathrm{d}C_2 - C_0\,\mathrm{d}B_2, \\ &F_5 = \mathrm{d}C_4 - \frac{1}{2}C_2\wedge\mathrm{d}B_2 + \frac{1}{2}B_2\wedge\mathrm{d}C_2. \end{aligned} \tag{2.3}$$

For simplicity, we have only written down the bosonic part of the action and we have neglected the fermionic part (following common practice in the field of supergravity). The idea behind this is that the fermionic sector can be constructed from the bosonic sector without much difficulty by using supersymmetry transformations. Therefore, writing the fermionic action down explicitly is often considered to be unnecessary.

From the supersymmetry transformations one can derive a set of conditions that solutions of supergravity theories need to satisfy in order to preserve (a part of) the supersymmetry. These conditions are known as BPS conditions. In section 2.2, we will encounter objects that preserve exactly half of the available supersymmetry: branes.

The 5-form field strength is self-dual, i.e. $*F_5 = F_5$. As a consequence, the kinetic term of F_5 in the action (2.2) vanishes. It is customary, however, to keep this term and to implement the self-duality only on the level of the equations of motion.

2.1.2 String frame and Einstein frame

There are two common parameterizations, or 'frames', in which supergravity actions can be written down: the string frame and the Einstein frame. The latter is the parametrization that is usually worked with in the context of general relativity. Actions in this frame take the form $S = S_{\text{EH}} + S_{\text{M}}$, where S_{EH} is the Einstein-Hilbert action and S_{M} is the matter action. The string frame, however, is characterized by a factor $e^{-2\Phi_D}$ in front of the Einstein-Hilbert term in the action. Here Φ_D is the D-dimensional dilaton[1]. We denote the metric in string frame by $G_{\mu\nu}$ and the metric in Einstein frame by $g_{\mu\nu}$. The two are related by a Weyl rescaling:

$$G_{\mu\nu} = g_{\mu\nu}\, e^{\frac{4}{D-2}\Phi_D}. \tag{2.4}$$

The action of type IIB, as given in (2.2), is written in string frame. As an example, we now convert this action to Einstein frame. In $D = 10$, the string and Einstein frame are related by

$$G_{\mu\nu} = g_{\mu\nu}\, e^{\frac{1}{2}\Phi}. \tag{2.5}$$

It follows directly that $\sqrt{-G} = \sqrt{-g}\; e^{\frac{5}{2}\Phi}$. We should also compute the relation between the Ricci scalars in both frames. In general, under a Weyl transformation $g'_{\mu\nu} = g_{\mu\nu}\, e^{2\omega}$ of a D-dimensional metric, one has

$$R' = e^{-2\omega}\left(R - 2(D-1)\nabla^2\omega - (D-2)(D-1)\partial_\mu\omega\partial^\mu\omega\right), \tag{2.6}$$

where the contraction of indices on the right-hand side is done with the metric $g_{\mu\nu}$. Using this, we find that for the particular rescaling at hand

$$R(G) = e^{-\frac{1}{2}\Phi}\left(R(g) - \tfrac{9}{2}\,\nabla^2\Phi - \tfrac{9}{2}\,\partial_\mu\Phi\partial^\mu\Phi\right). \tag{2.7}$$

[1] In section 3.1.2, we will see how the dilatons in various dimensions is related to the 10-dimensional one.

In order to transform the entire action to Einstein frame, we should be aware that the metric is also present in the kinetic terms of the matter fields. We have

$$\left|F_p\right|^2 = \frac{1}{p!} F_{\mu_1\ldots\mu_p} \bar{F}_{\nu_1\ldots\nu_p} G^{\mu_1\nu_1}\ldots G^{\mu_p\nu_p}, \tag{2.8}$$

from which it follows that $\left|F_p\right|^2 = e^{-\frac{p}{2}\Phi} \left|F_p\right|^2_{\mathrm{E}}$. Here we use the subscript E to indicate that the indices of the field strength are contracted with the Einstein frame metric. By combining these results, we obtain the type IIB action in Einstein frame as

$$\begin{aligned} S_{\mathrm{IIB}} = &\frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-g} \left(R(g) - \frac{1}{2}|\mathrm{d}\Phi|^2 - \frac{1}{2} e^{-\Phi} |H_3|^2 \right) \\ &- \frac{1}{4\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-g} \left(e^{2\Phi} |F_1|^2 + e^{\Phi} |F_3|^2 + \frac{1}{2}|F_5|^2 \right) \\ &- \frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3. \end{aligned} \tag{2.9}$$

(All contractions of indices here are done with the Einstein frame metric. We have dropped the subscript E for notational convenience.)

2.1.3 S-duality

In order to uncover the $SL(2)$ symmetry of the type IIB action, we define

$$\tau = C_0 + ie^{-\Phi}, \qquad G_3 = F_3 - ie^{-\Phi} H_3. \tag{2.10}$$

In terms of these fields, we rewrite the action as

$$\begin{aligned} S_{\mathrm{IIB}} = &\frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-g} \left(R(g) - \frac{|\mathrm{d}\tau|^2}{2\,(\mathrm{Im}\,\tau)^2} - \frac{|G_3|^2}{2\,\mathrm{Im}\,\tau} - \frac{1}{4}|F_5|^2 \right) \\ &+ \frac{1}{8i\kappa_{10}^2} \int \frac{1}{\mathrm{Im}\,\tau} C_4 \wedge G_3 \wedge \bar{G}_3. \end{aligned} \tag{2.11}$$

This action has an $SL(2,\mathbb{R})$ symmetry. An element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2)$ acts on the type IIB fields as

$$\tau \;\rightarrow\; \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \;\rightarrow\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix}. \tag{2.12}$$

The graviton and the 4-form field transform as singlets under this group. It can be verified straightforwardly that these transformations leave the action (2.11) invariant. This symmetry of the type IIB action is known as S-duality. The various (equivalent) theories that are related by it are said to be 'S-dual' to one another.

On the level of the supergravity action, $SL(2,\mathbb{R})$ is a proper symmetry group. When we study type IIB supergravity from the context of string theory, however, this symmetry group is broken down to its discrete subgroup $SL(2,\mathbb{Z})$. By restricting ourselves to this discrete group, we make sure that the charges of D-branes remain quantized, as they should be in a quantum string theory.

Particularly interesting is the S-duality transformation for which $a = d = 0$ and $b = -c = 1$. Under this transformation, the string coupling constant transforms as $g_s \to 1/g_s$. In other words, S-duality relates type IIB string theory at weak coupling and at strong coupling.

2.2 Branes

Branes are extended objects that appear both in supergravity and in string theory. They generalize the notion of a point to higher dimensions. Branes play a central role in this thesis: we will use them to build black holes in chapter 5.

In supergravity, branes are usually called p-branes, while in string theory they are known as D-branes or Dp-branes. Since all known properties of p-branes are equivalent to those of D-branes, they are usually treated as the exact same object. In sections 2.2.1 and 2.2.2, we discuss the origin and some properties of p-branes and D-branes, respectively. Afterwards, in section 2.2.3, we consider arrays of branes.

2.2.1 In supergravity: p-branes

From the perspective of supergravity, we encounter branes as solutions of the field equations of the actions of type IIA and type IIB. We call these solutions p-branes, where p is the number of spatial directions in which the brane lies. The worldvolume that is swept out by a p-brane is a $(p+1)$-dimensional hypersurface that lies in p spatial directions and in the time direction. In the other directions, this worldvolume is pointlike.

The presence of a p-brane in a 10-dimensional Minkowski space $\mathbb{R}^{1,9}$ causes the Lorentz group to decompose as

$$SO(1,9) \;\to\; SO(1,p) \times SO(9-p). \tag{2.13}$$

We see that we have rotational symmetry in the $9-p$ directions perpendicular to the brane. In these directions, we will use isotropic spherical coordinates. We write our full coordinate system as $(t, x_1, \ldots, x_p, r, \varphi_1, \ldots, \varphi_{8-p})$, where the x_i denote the spatial coordinates parallel with the brane, and r and φ_i respectively denote the isotropic radius and the angles of the coordinates perpendicular to the brane.

As we have already revealed, p-branes are solutions of the equations of motion of the type IIA and type IIB supergravity actions. The most general expression for a p-brane solution is quite complex (see for example [15]). However, the expression becomes much simpler under the assumption that the brane is extremal. As a bonus, extremal branes are BPS solutions, i.e. they preserve supersymmetry. The presence of extremal branes typically breaks half of the available supersymmetry. Hence, they are 1/2 BPS states. In this thesis, we only encounter extremal p-branes so we can suffice with stating only the (simpler) extremal solution. Written in string frame, it reads

$$\begin{cases} \mathrm{d}s^2 = H_p^{-\frac{1}{2}}\left[-\,\mathrm{d}t^2 + \mathrm{d}x_1^2 + \ldots + \mathrm{d}x_p^2\right] + H_p^{\frac{1}{2}}\left[\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{8-p}^2\right] \\ e^{\tilde{\Phi}} = H_p^{\frac{3-p}{4}} \\ C_{0\ldots p} = H_p^{-1} - 1. \end{cases} \tag{2.14}$$

Here H_p is an r-dependent harmonic function given by

$$H_p = 1 + \frac{Q_p}{r^{7-p}}, \tag{2.15}$$

where Q_p is the charge of the p-brane solution. In solutions where there can be multiple coinciding branes, this charge is often written as $Q_p = c_p N_p$, where c_p is the charge of a single brane and N_p is the number of branes. It can straightforwardly be checked that H_p is a proper harmonic function, which amount to checking that it is a solution of Laplace's equation (with the flat Laplacian defined on the space transverse on the brane).

It can readily be checked that the extremal p-brane solution, as defined above, has an event horizon. This event horizon is located at $r = 0$ (this is usually the case for solutions written in isotropic coordinates). Because of the presence of an event horizon, these branes are often called black p-branes. In particular, a black 1-brane is known as a black string.

The p-brane solution, (2.14), is written in terms of the metric, the dilaton and a $(p+1)$-form potential. This potential is one of the fields in the R-R sector of the type IIA or the type IIB spectrum (or the Hodge dual of one of these fields). The spectrum of type IIA only contains R-R potentials of odd degree, and so type IIA supergravity only allows p-brane solutions for even values of p. Equivalently, it can be proven that type IIB supergravity only permits p-brane solutions for odd values of p.

A p-brane is said to carry charge under the potential C_{p+1}. In the case that C_{p+1} is one of the R-R fields of the supergravity spectrum, this charge is called electric. When C_{p+1} is the Hodge dual of one of the R-R fields, however, the charge is called magnetic. In this way, each

p-brane is charged either electrically or magnetically by one of the R-R potentials[2].

In this thesis, it will not be checked explicitly that (2.14) is a valid solution of the field equations of the type IIA (for p even) and the type IIB (for p odd) supergravity actions. We refer the reader who prefers to see such calculations in more detail to [22]. In this article, the equations of motion of the most common supergravity actions are derived both in string frame and in Einstein frame. With the relevant equations of motion at hand, it is a matter of straightforward substitution to see that (2.14) indeed is a valid solution.

2.2.2 In string theory: D-branes

In string theory, one usually encounters branes in the context of open strings. The endpoints of open strings must satisfy either Neumann or Dirichlet boundary conditions. In a direction where an endpoint has Dirichlet boundary conditions, the constraint $\delta X^\mu = 0$ must be satisfied. This implies that in that direction the endpoint of the string is located at a fixed point in space.

If an endpoint of a string has Neumann boundary conditions in the time direction and in p spatial directions, and Dirichlet boundary conditions in the other $9-p$ spatial directions, the endpoint is confined on a $p+1$ dimensional hypersurface: a Dp-brane.

It was shown by Polchinski that D-branes are charged by the R-R fields of type II string theory [23]. This discovery is what eventually led to the identification of D-branes with the black p-brane solutions of type IIA and type IIB supergravity.

2.2.3 Arraying branes

In the next chapter, we study compactification. As we will see there, compactification on a circle in, for example, the z-direction requires the identification $z \sim z + 2\pi w R$ with $w \in \mathbb{Z}$. Suppose that we compactify in the presence of a brane, that is pointlike in the compact z-direction. In order to satisfy the periodicity, we need to replace the brane by an array of branes such that the distance between neighboring branes is $2\pi R$. This is known as the arraying, or smearing, of branes.

Similarly, if we compactify (on a torus) multiple directions $\vec{z}$ that are transverse on the brane, we obtain a lattice of branes in these directions. The harmonic function that corresponds to

[2]The Kalb-Ramond field B_2 that appears in both type IIA and type IIB, also charges certain extended objects. It couples electrically to the fundamental string (often denoted as the F1-string), and magnetically to the NS5-brane.

such a collection of branes is given by

$$H_p = 1 + \sum_i \frac{Q_p}{\left|\vec{r} - \vec{r}_i\right|^{7-p}} \,, \tag{2.16}$$

where the sum runs over all branes in the lattice and $\vec{r}_i$ denotes the location of the i-th brane. Typically, in compactification, the radii of the compact directions are taken to be so small that they effectively become invisible. The distances between the branes in the lattice are proportional to these radii, and so we can replace the sum over the branes by an integral:

$$H_p = 1 + \int \mathrm{d}^n\, \vec{z}\, \frac{Q_p}{r^{7-p}} \,. \tag{2.17}$$

Here n denotes the number of compact directions transverse on the brane. Each integral over one of the transverse coordinates effectively raises the power of r by one. We absorb the numerical prefactors that we obtain in the integration in the charge parameter. We thus find

$$H_p = 1 + \frac{Q_p}{r^{7-p-n}} \,, \tag{2.18}$$

where the total charge is now given by $Q_p = c_p^{(n)} N_p$. The constant $c_p^{(n)}$ denotes the charge of a single p-brane that is arrayed in n directions. For more details, see [24].

Chapter 3

Compactification

One of the central themes in this thesis is compactification. This is the mechanism that, in string theory, is used to reduce the number of spatial dimensions. One always compactifies on a certain compact manifold. The most canonical manifold for the reduction of n dimensions is an n-dimensional torus T^n. We will focus on such compactifications, often called toroidal compactifications, in this thesis.

Compactification on (untwisted) tori preserves all supersymmetry. There are other manifolds, however, that preserve supersymmetry only partially. Famous examples of such manifolds are Calabi-Yau manifolds. These are generalizations of K3-surfaces to an arbitrary (even) number of dimensions. For more information about compactification on such manifolds, see [18, 20].

In section 3.1, we study Kaluza-Klein reduction. This is the type of reduction that is most commonly used in toroidal compactification. We work out the reduction of the metric, the dilaton and form-valued potentials explicitly. After that, in section 3.2, we discuss some aspects of the dualities that appear in compactified theories: T-duality and U-duality. Finally, we turn to Scherk-Schwarz reduction in section 3.3. This is the type of reduction that we are most interested in in this thesis. Scherk-Schwarz reduction is sometimes called twisted reduction, because it adds a duality twist to an otherwise ordinary Kaluza-Klein reduction. This twist can affect the reduction of scalar fields and the reduction of form-values potentials. Therefore, we work out the Scherk-Schwarz reductions of these types of fields explicitly in sections 3.3.1 and 3.3.2.

3.1 Kaluza-Klein reduction

When we consider the compactification of a single dimension on a circle, we have a $(D+1)$-dimensional spacetime of topology $\mathbb{R}^{1,D-1} \times S^1$. We denote the coordinates on the D-

dimensional Minkowski space by x^μ ($\mu = 0, \ldots, D-1$) and the coordinate on the circle by z. Furthermore, we use $x^{\hat{\mu}} = (x^\mu, z)$ to denote the collection of all $D+1$ coordinates. The coordinate on the circle is periodic, so we can make the identification $z \sim z + 2\pi w R$, where R is the radius of the circle and $w \in \mathbb{Z}$.

In principle, all the fields in the $(D+1)$-dimensional theory depend on each of the coordinates $x^{\hat{\mu}}$. We consider, for simplicity, a massless scalar field $\hat{\phi}$ (obeying $\partial^{\hat{\mu}}\partial_{\hat{\mu}}\hat{\phi} = 0$), and expand its z-dependence as a Fourier series, so that

$$\hat{\phi}(x^\mu, z) = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu)\, e^{inz/R}. \tag{3.1}$$

We used that the momentum in the z-direction can be quantized as $p_z = n/R$, with $n \in \mathbb{Z}$, to ensure the wave function to be single-valued. It follows that

$$\partial^\mu \partial_\mu \phi_n - \frac{n^2}{R^2}\phi_n = 0, \tag{3.2}$$

which implies that we get a family of scalar fields ϕ_n with masses $m = |n|/R$. These scalar fields are often referred to as the tower of Kaluza-Klein modes.

The usual Kaluza-Klein philosophy demands us to take R to be extremely small (of the order of the Planck length), so that we can take our spacetime to be effectively D-dimensional on length scales much larger then R. In more colorful language, it is often stated that the extra dimension is so small that is becomes 'invisible'. We see that for small values of R, the masses of the scalar fields in (3.2) become large. In fact, when we take R to be of the order of the Planck length, the masses become so large (of the order of the Planck mass) that we can neglect all massive modes and truncate to the massless sector. Consequently, in D dimensions, we are left with a massless scalar field ϕ_0 that is independent of z. This argument holds for each field component, and so we see that in this dimensional reduction all fields effectively lose their dependence on the compactified coordinate z.

3.1.1 Reduction of the metric

We first consider the reduction of the metric. We can naturally decompose $\hat{G}_{\hat{\mu}\hat{\nu}}$ into $\hat{G}_{\mu\nu}$, $\hat{G}_{\mu z}$ and $\hat{G}_{zz}$. From a D-dimensional point of view, these appear to be a metric, a vector field and a scalar field, respectively. It would, however, be very inconvenient to identify these pieces of $\hat{G}_{\hat{\mu}\hat{\nu}}$ with the lower dimensional fields directly. In fact, a better parametrization is given by the Kaluza-Klein ansatz:

$$\hat{G}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e^{2\alpha\phi}G_{\mu\nu} + e^{2\beta\phi}\mathcal{A}_\mu\mathcal{A}_\nu & e^{2\beta\phi}\mathcal{A}_\mu \\ e^{2\beta\phi}\mathcal{A}_\nu & e^{2\beta\phi} \end{pmatrix}. \tag{3.3}$$

Here the proper D-dimensional fields can be recognized: $G_{\mu\nu}$ is the lower dimensional metric, $\mathcal{A}_\mu$ is a $U(1)$ gauge field called the graviphoton, and ϕ is a scalar field that carries many names (we will mostly refer to it as the Kaluza-Klein scalar in this thesis, but the names dilaton[1], radion and graviscalar are also used). We use $\mathcal{F}_{\mu\nu}$ to denote the field strength associated with the graviphoton.

In principle, the constants α and β in the Kaluza-Klein ansatz can be chosen freely, but two choices turn out to be particularly useful. First of all, if one starts with a theory in Einstein frame, the choice $\beta = -(D-2)\alpha$ makes sure that the compactified theory is also written in Einstein frame. This result is derived in detail in [25]. In this thesis, we will perform all compactifications entirely in string frame. For this, we need $\alpha = 0$ and $\beta = 1$, which simplifies the Kaluza-Klein ansatz to

$$\hat{G}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} G_{\mu\nu} + e^{2\phi}\mathcal{A}_\mu\mathcal{A}_\nu & e^{2\phi}\mathcal{A}_\mu \\ e^{2\phi}\mathcal{A}_\nu & e^{2\phi} \end{pmatrix}. \tag{3.4}$$

For future reference, we also state the inverse of this ansatz:

$$\hat{G}^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} G^{\mu\nu} & -\mathcal{A}^\mu \\ -\mathcal{A}^\nu & e^{-2\phi} + \mathcal{A}^2 \end{pmatrix}. \tag{3.5}$$

The metric naturally appears in supergravity actions in the volume element and in the Ricci scalar. From the parametrization given in (3.4), it can be seen that $\sqrt{-\hat{G}} = e^\phi\sqrt{-G}$. After a straightforward (though somewhat tiresome) calculation, the decomposition of the $(D+1)$-dimensional Ricci scalar can be found as [17]

$$\hat{R} = R - 2e^{-\phi}\nabla^2 e^\phi - \frac{1}{4}e^{2\phi}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}, \tag{3.6}$$

where the hat on R (or the absence thereof) indicates from which metric each Ricci scalar is constructed. The lower dimensional metric $G_{\mu\nu}$ is used for the contractions of indices on the right-hand side.

Suppose now that we compactify multiple dimensions on circles. Each of these steps yields one scalar that comes directly from the graviton. However, for multiple subsequent compactification steps, we also obtain scalars that come from the graviphotons. In section 5.3, it will turn out to be useful to distinguish between these two types of scalars. We refer to scalars

[1] The usage of the name dilaton for the scalar that comes from the graviton in dimensional reduction offers some clarification as to why the massless scalar in the NS-NS sector of string theory is called the dilaton. Indeed, this dilaton is the scalar that comes from the 11-dimensional graviton in the reduction from M-theory to 10-dimensions. In general, a dilaton parametrizes the volume of the compactified dimension.

that come directly form the graviton as dilatons, and to scalars that come from graviphotons as axions. When we use the term 'dilaton' it should always be clear from the context whether we mean the dilaton from the NS-NS sector of string theory or a generic dilatonic scalar that comes from the reduction of a metric.

3.1.2 Reduction of the dilaton

As we already mentioned in the previous section, we need the Kaluza-Klein ansatz given in (3.4) in order to perform a reduction in string frame. However, we see that for this ansatz we obtain an exponent of the Kaluza-Klein scalar in front of the Einstein-Hilbert term. This exponent comes from the reduction of the square root of the determinant of the metric: $\sqrt{-\hat{G}} = e^{\phi}\sqrt{-G}$. If we were to compactify more dimensions on circles with the ansatz (3.4), we would get such exponents for all of the Kaluza-Klein scalars.

In order to find the Einstein-Hilbert term exactly in the string frame form (with only an exponent $e^{-2\Phi_D}$ in front), we need to redefine the lower-dimensional dilaton as

$$e^{-2\Phi_D} = \sqrt{G_{DD}^{(10)} \ldots G_{99}^{(10)}}\, e^{-2\Phi_{10}}. \tag{3.7}$$

3.1.3 Reduction of form-valued potentials

We now consider the reduction of an $(n-1)$-form potential $\hat{A}_{n-1} = \frac{1}{(n-1)!}\hat{A}_{\hat{\mu}_1\ldots\hat{\mu}_{n-1}}\mathrm{d}x^{\hat{\mu}_1} \wedge \ldots \wedge \mathrm{d}x^{\hat{\mu}_{n-1}}$. Just by looking at the indices, we see that we can decompose this into two parts: either all indices lie in the x^μ-direction, or $(n-2)$ indices lie in this direction and one lies in the z-direction. From a D-dimensional perspective these are an $(n-1)$-form and an $(n-2)$-form. Explicitly, this decomposition is given by

$$\begin{aligned}\hat{A}_{n-1} &= A_{n-1} + A_{n-2} \wedge \mathrm{d}z,\\ \hat{F}_n &= \mathrm{d}A_{n-1} + \mathrm{d}A_{n-2} \wedge \mathrm{d}z,\end{aligned} \tag{3.8}$$

where A_{n-1} and A_{n-2} are the D-dimensional fields. In the expression for $\hat{F}_n$, we add and subtract a term $\mathrm{d}A_{n-2} \wedge \mathcal{A}_1$, and so we obtain

$$\hat{F}_n = F_n + F_{n-1} \wedge (\mathrm{d}z + \mathcal{A}_1), \tag{3.9}$$

where the D-dimensional field strengths are defined as

$$\begin{aligned}F_n &= \mathrm{d}A_{n-1} - F_{n-1} \wedge \mathcal{A}_1,\\ F_{n-1} &= \mathrm{d}A_{n-2}.\end{aligned} \tag{3.10}$$

At this point, the particular choice that we made for defining the lower dimensional field strengths appears unnecessarily complex. However, we will come to appreciate it, when we see how $|\hat{F}_n|^2$, the usual kinetic term of an $(n-1)$-form field in the $(D+1)$-dimensional theory, decomposes as we perform the compactification. We find

$$|\hat{F}_n|^2 = |F_n|^2 + e^{-2\phi}\,|F_{n-1}|^2. \tag{3.11}$$

This simple form is due to the fact that we defined the field strengths in D dimensions in such a way that $\hat{F}_n$ could be written as in (3.9).

3.2 Dualities

Compactifications of string theories typically result in theories that are subject to a number of dualities. These dualities relate equivalent theories that are, for example, obtained from compactification on circles with different radii. After the compactification of multiple dimensions, these dualities typically form a group structure. This group acts as a global symmetry group on the compactified theory.

First, in section 3.2.1, we study T-duality. This duality relates different types of superstring theories. In particular, T-duality gives an equivalence between compactifications of type IIA and type IIB string theory. In section 3.2.2, we discuss U-duality. Roughly speaking, this is the symmetry that comes from combining T-duality with S-duality.

We do not study these dualities in too much detail. Instead, we present some important results that we will use later on in this thesis. For more detailed accounts of T-duality and U-duality, we refer to [15] and [16], respectively.

3.2.1 T-duality

Consider a string theory, compactified on a circle (with radius R). As we already discussed in section 3.1, this implies that the momentum in this compact direction is quantized as $p_z = n/R$ with $n \in \mathbb{Z}$. Now recall that the compact coordinate is periodic, i.e. we have the identification $z \sim z + 2\pi R$. From this, it follows that a closed string is allowed to wind a certain number of times (say $w \in \mathbb{Z}$) around the compact direction. Such windings contribute to the mass of the string.

Taking into account both the momentum number n and the winding number w, the mass spectrum of a closed string can be found as [15]

$$M^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \text{oscillations}\,, \tag{3.12}$$

where the contributions due to oscillations on the string are the same as in the uncompactified case. The possible states are constraint by the condition $nw + N - \tilde{N} = 0$ (with N and $\tilde{N}$ the levels of the right and left-moving oscillations, respectively). We now note that this spectrum is invariant under

$$R \to \frac{\alpha'}{R}\,, \qquad\qquad n \leftrightarrow w. \tag{3.13}$$

This invariance of the spectrum suggests that the theories obtained from compactification on a circle with radius R and on a circle with radius α'/R (with momentum and winding numbers interchanged) are equivalent. This equivalence is known as T-duality, and indeed it turns out to be a true symmetry of the full interacting string theory[2]. It is a general feature of T-duality that it relates theories compactified on circles with small radii to theories compactified on circles with large radii.

When we apply T-duality to either of the type II string theories, we find that it switches the chirality of the right-moving worldsheet fermions. As a result, T-duality maps type IIA, compactified with radius R, to type IIB, compactified with radius α'/R, and vice versa. A more detailed version of this argument can be found in [18].

We now consider open strings in either of the type II string theories. We expect the T-duality transformations, that we found for closed strings, to apply to open strings as well. After all, the interior of an open string (sufficiently far away from the endpoints) is indistinguishable from that of a closed string. By applying T-duality to the endpoints of an open string, we find that it interchanges Neumann and Dirichlet boundary conditions in the direction in which we dualize. For a comprehensive derivation of this result, we again refer to [15]. Because of this switch of boundary conditions, the dimensionality of the branes on which the open string ends changes: if we T-dualize in a direction transverse on a p-brane, we obtain a $(p+1)$-brane, and if we dualize in a direction parallel to the brane, we obtain a $(p-1)$-brane[3].

The field contents of type IIA and type IIB supergravity are different. However, toroidal compactification of both of these theories leads to identical lower-dimensional field contents. These spectra can be related to one another by T-duality. Explicit maps between the fields

[2]In order to demonstrate this full T-duality, we would need a more extensive set of transformations than the one given in (3.13). For more information on this, see for example [15].

[3]Let us be a bit more precise about the latter case. By T-dualizing in a direction parallel to a p-brane, we don't obtain just one $(p-1)$-brane, we obtain infinitely many: we find a $(p-1)$-brane that is arrayed in the dualized direction.

that come from the reduction of both of the type II theories, are derived in [26, 27].

3.2.2 U-duality

So far, we have seen two types of dualities: S-duality and T-duality. At first sight, these two dualities appear to be completely uncorrelated. Nevertheless, in 1995 it was conjectured by Hull and Townsend that both of these dualities can be unified in a larger symmetry group. They named this conjectured symmetry group U-duality, where the U stands for 'unified' [28].

This conjecture was motivated, amongst other things, by the global symmetry groups of maximal supergravity theories. These groups are exactly the maximally non-compact exceptional groups [29, 30]. That is, the symmetry group of maximal supergravity in $3 \leq d \leq 8$ dimensions is given by $E_{(11-d),(11-d)}$.

The S-duality and T-duality groups, in each of the dimensions $3 \leq d \leq 8$, are discrete subgroups of $E_{(11-d),(11-d)}$. The U-duality groups of Hull and Townsend are exactly the maximal discrete subgroups $E_{(11-d)}(\mathbb{Z})$. As a consequence, they have the groups corresponding to S-duality and T-duality as subgroups. In this sense, U-duality really unifies S and T-duality.

Exceptional groups are usually not very practical to work with. Luckily, for $d = 6, 7, 8$, we have so-called accidental isomorphisms between the exceptional groups $E_{(11-d),(11-d)}$ and other more convenient Lie groups (see e.g. [16]). For example, if we take $d = 6$, we have the isomorphism $E_{5,5} \cong SO(5,5)$. Thus, by compactifying type II string theory on a 4-torus, we obtain a 6-dimensional theory with the U-duality group $SO(5,5,\mathbb{Z})$. This particular isomorphism will be particularly useful for the purposes of this thesis.

3.3 Scherk-Schwarz reduction

So far, we have seen compactifications in which the D-dimensional fields were independent of the internal coordinate z. Clearly, this makes sure that we end up with a lower dimensional theory that is completely z-independent. This is something that we always demand from a proper dimensional reduction: the resulting theory should be independent of the compactified coordinate(s).

We will now consider Scherk-Schwarz reduction [11, 12], which is also known as generalized reduction or reduction with a duality twist. Suppose that our $(D+1)$-dimensional theory has a global symmetry group G and some field ψ that transforms as $\psi \to g\,\psi$ where $g \in G$. In contrast to what we did before, we no longer take our fields to be z-independent. Instead, we choose

$$\psi(x^\mu, z) = g(z)\,\psi(x^\mu), \tag{3.14}$$

where $g(z)$ is a z-dependent element of our symmetry group G. In general, we work with theories whose symmetry group is a matrix Lie group. We can then write $g(z)$ in terms of some Lie algebra element M as

$$g(z) = \exp\left(\frac{Mz}{2\pi R}\right). \tag{3.15}$$

The matrix M is called the mass matrix, as it introduces mass parameters into the theory. Later on, we will see in detail how this mass matrix can cause various fields in the theory to become massive. We see that, under a shift in the internal coordinate $z \to z + a$, our field changes as

$$\psi(x^\mu, z) \;\to\; g(a)\,\psi(x^\mu, z). \tag{3.16}$$

By noticing that $g(a)$ is simply an element of our global symmetry group G, we realize that this shift in z leaves the Lagrangian invariant. We see that, by choosing (3.14), we have given our field ψ dependence on the internal coordinate in such a way that the D-dimensional theory remains independent of z. This is a general feature of Scherk-Schwarz reductions. As a particular case of (3.16), we observe that our field is not periodic around the circle, but instead

$$\psi(x^\mu, z) \;\to\; e^M\,\psi(x^\mu, z), \tag{3.17}$$

as $z \to z + 2\pi R$. The factor that appears as we go around the circle, is called a monodromy, and it is denoted by $\mathcal{M} = e^M$.

3.3.1 Reduction of scalar fields

The scalar fields, also known as moduli fields, in a supergravity theory typically take values in a coset G/K, where K is the maximal compact subgroup of G. As usual in physics, we work with Lie groups. The coset G/K is called the moduli space, and on this space we can define a vielbein $\mathcal{V} \in G$ that transforms under global G transformations and under local K transformations. Explicitly, these transformations work as $\mathcal{V}(x) \to k(x)\,\mathcal{V}(x)\,g$, where $k(x) \in K$ and $g \in G$. From now on, we will assume that $\mathcal{V}$ is a real matrix in a real representation of G. This will prove to be a valid assumption in all cases that we encounter in this thesis.

The local (gauge) symmetry group K can be used to fix $\dim(K)$ degrees of freedom of $\mathcal{V} \in G$. Consequently, $\mathcal{V}$ has $\dim(G) - \dim(K)$ physical degrees of freedom, which matches with the number we would expect from a vielbein on the moduli space G/K. We can carry out this gauge fixing explicitly by constructing $\mathcal{H} = \mathcal{V}^T\eta\mathcal{V}$, where η is a K-invariant metric. This quantity is invariant under K and transforms as $\mathcal{H}(x) \to g^T\,\mathcal{H}(x)\,g$ under G. By looking at the way we have constructed $\mathcal{H}$ from vielbeins, we see that it is precisely the metric on the moduli space.

The Scherk-Schwarz reduction of scalar fields (taking values in a coset G/K) is most easily done in the notation that we have just introduced. Suppose that our $(D+1)$-dimensional theory has such a collection of scalars. Written in terms of $\mathcal{H}$, their kinetic term is then given by [31]

$$e^{-1}\mathcal{L}_{\rm s} = \frac{1}{2}\mathrm{Tr}\left[\partial_{\hat{\mu}}\mathcal{H}^{-1}\partial^{\hat{\mu}}\mathcal{H}\right]. \tag{3.18}$$

Note that this expression is indeed invariant under the transformation of $\mathcal{H}$ under the global symmetry group G. We give the scalar fields z-dependence through the ansatz

$$\mathcal{H}(x^\mu, z) = g^T(z)\,\mathcal{H}(x^\mu)\,g(z), \tag{3.19}$$

where $g(z) = e^{Mz}$ is a (local) element of G. By using (3.5), we can write out the scalar Lagrangian as

$$e^{-1}\mathcal{L}_{\rm s} = \frac{1}{2}\,\mathrm{Tr}\big[D_\mu\mathcal{H}^{-1}D^\mu\mathcal{H}\big] - e^{-2\phi}\,\mathrm{Tr}\big[M^2 + M^T\mathcal{H}M\mathcal{H}^{-1}\big]\,, \tag{3.20}$$

where we have defined $D_\mu\mathcal{H}$ as

$$D_\mu\mathcal{H} = \partial_\mu\mathcal{H} - \mathcal{A}_\mu\big(M^T\mathcal{H} + \mathcal{H}M\big)\,. \tag{3.21}$$

We can observe some major differences with the result we would have obtained from untwisted reduction (which can easily be retrieved from the expressions above by taking $M = 0$). First of all, in the twisted case we see that the partial derivative of $\mathcal{H}$ is replaced by a gauge covariant derivative. The gauge group is the 1-dimensional subgroup of G that is generated by the 'charge' M. The associated gauge field is the graviphoton $\mathcal{A}_\mu$, that we obtain as usual from the reduction of the metric. The emergence of gauged (supergravity) theories is a general feature of Scherk-Schwarz reduction.

Secondly, we observe that, in the Scherk-Schwarz twisted reduction, the scalar fields acquire a potential

$$V = e^{-2\phi}\,\mathrm{Tr}\big[M^2 + M^T\mathcal{H}M\mathcal{H}^{-1}\big]\,. \tag{3.22}$$

Here $\mathcal{H}$ contains the scalar fields that were present before compactification, and ϕ is the Kaluza-Klein scalar that comes from the reduction of the metric. Depending on the choice of M, this scalar potential may have stable minima where one or multiple moduli become massive.

In [31], it is derived that the potential (3.22) is positive definite. Therefore, we can find stable minima at points in the moduli space where the potential vanishes.

3.3.2 Reduction of form-valued potentials

Suppose that, in addition to the collection of scalar fields described in the previous section, the $(D+1)$-dimensional theory contains r $(n-1)$-form potentials $\hat{A}^i_{n-1}$ $(i = 1, \ldots, r)$ with r associated field strengths $\hat{F}^i_n$. We write the r-component vectors consisting of the potentials and the field strengths as $\hat{A}_{n-1} = (\hat{A}^1_{n-1}, \ldots, \hat{A}^r_{n-1})^T$ and $\hat{F}_n = (\hat{F}^1_n, \ldots, \hat{F}^r_n)^T$, respectively. Using this notation, the kinetic term of the form-valued potentials appears in the Lagrangian as

$$e^{-1}\mathcal{L}_{\mathrm{p}} = -\frac{1}{2n!}\,(\hat{F}_{\hat{\mu}_1\ldots\hat{\mu}_n})^t\,\mathcal{H}\,\hat{F}_{\hat{\nu}_1\ldots\hat{\nu}_n}\,G^{\hat{\mu}_1\hat{\nu}_1}\ldots G^{\hat{\mu}_n\hat{\nu}_n}. \tag{3.23}$$

Here $\mathcal{H}$ is the metric on the moduli space of the theory in a real r-dimensional representation of the global group G. This means that $\mathcal{H}$ is an $r \times r$ matrix, which makes the multiplication $\hat{F}^t_n\mathcal{H}\hat{F}_n$ well-defined. Under G, the potentials transform as $\hat{A}_{n-1} \to g^{-1}\hat{A}_{n-1}$ and $\mathcal{H}$ transforms as before. With these transformation rules, it is clear that (3.23) is indeed invariant under the global symmetry group G. In its current form, the Lagrangian of the potential fields is quite messy. We therefore write it as (recall the notation introduced in section 1.2)

$$e^{-1}\mathcal{L}_{\mathrm{p}} = -\frac{1}{2}\,\mathcal{H}_{ij}\left(|\hat{F}_n|^2\right)^{ij}. \tag{3.24}$$

Because of the transformation rule of the potentials $\hat{A}_{n-1}$ under the global symmetry group, we choose them to have z-dependence according to the ansatz

$$\hat{A}_{n-1}(x^\mu, z) = g^{-1}(z)\left[A_{n-1}(x^\mu) + A_{n-2}(x^\mu)\wedge \mathrm{d}z\right], \tag{3.25}$$

where $g(z) = e^{Mz}$ is a local element of G. In order to express the $(D+1)$-dimensional field strength in lower dimensional ones, we now follow the same approach as in section 3.1.3. From (3.25), we obtain the (vector of) field strengths by acting with an exterior derivative. We find

$$\hat{F}_n = e^{-Mz}\left[\mathrm{d}A_{n-1} + \mathrm{d}A_{n-2}\wedge\mathrm{d}z + (-1)^n M A_{n-1}\wedge\mathrm{d}z\right]. \tag{3.26}$$

We add and subtract $e^{-Mz}(\mathrm{d}A_{n-2} + (-1)^n M A_{n-1})\wedge\mathcal{A}_1$. This allows us to rewrite the $(D+1)$-dimensional field strength to

$$\hat{F}_n = e^{-Mz}\left[F_n + F_{n-1}\wedge(\mathrm{d}z + \mathcal{A}_1)\right], \tag{3.27}$$

where we have defined the lower dimensional field strengths as

$$\begin{aligned} F_n &= \mathrm{d}A_{n-1} - F_{n-1}\wedge\mathcal{A}_1, \\ F_{n-1} &= \mathrm{d}A_{n-2} + (-1)^n M A_{n-1}. \end{aligned} \tag{3.28}$$

Just like in section 3.1.3 it is the particular form of (3.27) that allows us to rewrite (3.24) to

$$e^{-1}\mathcal{L}_{\mathrm{p}} = -\frac{1}{2}\,\mathcal{H}_{ij}\left[\left(|F_n|^2\right)^{ij} + e^{-2\phi}\left(|F_{n-1}|^2\right)^{ij}\right]. \tag{3.29}$$

The D-dimensional field strengths, as given in (3.28), are invariant under the gauge transformations

$$\delta A_{n-1} = \mathrm{d}\Lambda, \qquad \delta A_{n-2} = -\,(-1)^n M\Lambda. \tag{3.30}$$

Here Λ is a local gauge parameter. These transformations are precisely the gauge transformations that we know from the (Abelian) Higgs mechanism. Motivated by this observation, we make the redefinition

$$A_{n-1} \;\rightarrow\; A'_{n-1} = A_{n-1} - (-1)^n M^{-1}\mathrm{d}A_{n-2}, \tag{3.31}$$

so that the D-dimensional field strengths reduce to

$$\begin{aligned} F_n &= \mathrm{D}A_{n-1}, \\ F_{n-1} &= (-1)^n M A_{n-1}. \end{aligned} \tag{3.32}$$

Here we have defined a covariant derivative

$$\mathrm{D}A_{n-1} = \mathrm{d}A_{n-1} + M\mathcal{A}_1 \wedge A_{n-1}. \tag{3.33}$$

With these simplified expressions for the field strengths, we can rewrite the Lagrangian of the potentials to

$$e^{-1}\mathcal{L}_{\mathrm{p}} = -\frac{1}{2}\left[\mathcal{H}_{ij}\left(|\mathrm{D}A_{n-1}|^2\right)^{ij} + e^{-2\phi}\left(M^T\mathcal{H}M\right)_{ij}\left(|A_{n-1}|^2\right)^{ij}\right]. \tag{3.34}$$

We see that the Lagrangian is now independent of A_{n-2}, so these fields have disappeared from the theory. Instead, we have a mass term for the fields A_{n-1}. In standard Higgs mechanism terminology, we say that the $(n-1)$-form fields have 'eaten' their $(n-2)$-form counterparts in order to become massive.

Furthermore, we observe the presence of the gauge covariant derivative in the Lagrangian. The associated gauge symmetry group is generated by M and the gauge field is the graviphoton $\mathcal{A}_1$. Note that these features are very similar to those of the gauged theory that we obtained from the twisted reduction of the scalar fields in the previous section. In particular, compare the two covariant derivatives, (3.21) and (3.33).

We should mention at this point that in some cases the procedure described here is not completely applicable. The reason for this is that, by using the field redefinition (3.31), we silently assumed the mass matrix M to be invertible. However, this is not generally true. For a reduction in which M is not invertible, a slightly modified analysis leads to the result that some (but not all) of the fields A^i_{n-1} absorb the corresponding A^i_{n-2} and acquire mass, while the rest remains massless. For more details, we refer to [32].

Chapter 4

Explicit Reductions

In this chapter we work out two examples of Scherk-Schwarz reductions in detail. First, in section 4.1, we consider pure gravity reduced on $T^2 \times S^1$. Subsequently, we examine the reduction of type IIB string theory on $T^4 \times S^1$ in section 4.2. In both examples, the reduction on the torus (T^2 and T^4, respectively) is performed first, after which we compactify further on a circle S^1 with a Scherk-Schwarz twist.

The first example, pure gravity on $T^2 \times S^1$, is included primarily to show how the formalism of section 3.3 works in practice. For this, we have intentionally chosen one of the easiest non-trivial examples of Scherk-Schwarz reduction. Most of the results that we work out in this section, have already been presented in [31].

The reduction of type IIB on $T^4 \times S^1$, that we study in section 4.2, is much more involved. Consequently, determining which fields remain massless and which fields acquire mass, becomes quite difficult to do by hand. Instead, we resort to group and representation theory in order to find these results. Some aspects of this reduction are new. Others have either been presented in [33], or are based on the unpublished work of Hull and Vandoren [14].

4.1 Pure gravity on $T^2 \times S^1$

The reduction of pure gravity in $D+3$ dimensions on an untwisted 2-torus is straightforward. We end up with a $(D+1)$-dimensional theory that contains gravity, two graviphotons and three scalar fields. We denote the graviphotons by $\mathcal{A}^1$ and $\mathcal{A}^2$, and the scalar fields by ψ, τ_1 and τ_2. All of these fields are massless because, as usual, we truncate the tower of massive Kaluza-Klein modes.

In general, the moduli space that results from reducing pure gravity on an n-dimensional torus is $GL(n,\mathbb{R})/SO(n,\mathbb{R})$. Accordingly, we find the moduli space after reduction on T^2 to be $GL(2,\mathbb{R})/SO(2,\mathbb{R})$. The area of the 2-torus is parametrized by e^{ψ} and its complex structure is parametrized by $\tau = \tau_1 + i\tau_2$.

We now reduce the theory further to D dimensions on a circle on which we include a Scherk-Schwarz twist. Our $(D+1)$-dimensional theory has a global $SL(2,\mathbb{R})$ symmetry, so we can choose a monodromy matrix in this group. Distinct reductions are characterized by the conjugacy classes of the symmetry group (see [31]). In this case, for $SL(2,\mathbb{R})$, there are three types of conjugacy classes, namely the hyperbolic, the elliptic and the parabolic conjugacy classes. They can be represented by the following monodromy matrices (see appendix A.1):

$$\mathcal{M}_h = \begin{pmatrix} e^m & 0 \\ 0 & e^{-m} \end{pmatrix}, \qquad \mathcal{M}_e = \begin{pmatrix} \cos m & -\sin m \\ \sin m & \cos m \end{pmatrix}, \qquad \mathcal{M}_p = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}. \tag{4.1}$$

These monodromies are generated by the mass matrices

$$M_h = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}, \qquad M_e = \begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix}, \qquad M_p = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}. \tag{4.2}$$

We have conveniently chosen to denote the constant in these matrices by m, as it will show up as a mass parameter in the reduced theory.

4.1.1 Reduction of the scalar fields

The scalar fields take values in the moduli space $GL(2,\mathbb{R})/SO(2,\mathbb{R})$, so we can write them in terms of the vielbein

$$\mathcal{V} = \frac{e^{\psi/2}}{\sqrt{\tau_2}} \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix}. \tag{4.3}$$

As before, we construct a metric on the moduli space from this as $\mathcal{H} = \mathcal{V}^T\mathcal{V}$. We find that $\mathcal{H} = e^{\psi} H(\tau)$, where

$$H(\tau) = \frac{1}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}. \tag{4.4}$$

It is now straightforward to find the scalar potential for each of the conjugacy classes of $SL(2,\mathbb{R})$. Using (3.22), we obtain

$$V_h = e^{-2\phi}\, m^2 \left(4 + \frac{4\tau_1^2}{\tau_2^2}\right), \tag{4.5}$$

$$V_e = e^{-2\phi}\, m^2 \left(-2 + \frac{1}{\tau_2^2}\left(1 + 2\tau_1^2 + |\tau|^4\right)\right), \tag{4.6}$$

$$V_p = e^{-2\phi}\, m^2 \,\frac{1}{\tau_2^2}. \tag{4.7}$$

Here ϕ is the Kaluza-Klein scalar from the reduction on the circle. We see that the potential is independent of ψ for each of the possible twists, so this field remains massless in the D-dimensional theory. In order to determine whether the other scalar fields acquire mass, we need to find critical points in the potentials. These are the points in the moduli space where the potential vanishes. Just by looking at the expressions for the scalar potentials, we see that (for finite values of the fields) the hyperbolic and the parabolic case have no such critical points. For the case with the elliptic twist, we find a minimum at $\tau = i$.

In a stationary point, moduli fields can become massive. In general, we can determine the mass of a real scalar field σ from a potential V with the formula

$$m_\sigma^2 = \left.\frac{\partial^2 V}{\partial \sigma^2}\right|_{\text{stat}} \tag{4.8}$$

where the subscript 'stat' means that the derivative is evaluated at the stationary point. By applying this to the minimum of V_e, we find that τ_1 and τ_2 acquire the masses

$$m_{\tau_1}^2 = m_{\tau_2}^2 = 8m^2. \tag{4.9}$$

In addition, we find that ϕ remains massless. This is always true for Scherk-Schwarz reductions, as can be seen from the general form of the scalar potential (3.22).

4.1.2 Reduction of the 1-form fields

Our $(D+1)$-dimensional theory contains two 1-form fields (the graviphotons). For convenience, we introduce the notation $\vec{\mathcal{A}} = (\mathcal{A}^1, \mathcal{A}^2)^T$. Twisted reduction with an invertible mass matrix (this is the case for a hyperbolic or an elliptic monodromy), gives rise to a mass term for these fields like the one given in (3.34). Explicitly, we find

$$-\frac{1}{2}\, e^{-2\phi}\, \vec{\mathcal{A}}_\mu^t \left(M^T \mathcal{H} M\right) \vec{\mathcal{A}}_\nu \, G^{\mu\nu}. \tag{4.10}$$

In order to find the masses that the graviphotons acquire due to this term, we need to expand the scalar fields around their vacuum expectation values. Of this expansion, we only keep the zeroth order, because higher orders give interaction terms between the scalars and the 1-forms instead of mass terms.

We perform this calculation explicitly for the stationary point at $\tau = i$, that we found in the scalar potential for the elliptically twisted case. By substituting the values of the moduli at the minimum and the mass matrix M_e, we obtain the mass term

$$-\frac{1}{2}m^2\left(\mathcal{A}^1_\mu\right)^2-\frac{1}{2}m^2\left(\mathcal{A}^2_\mu\right)^2. \tag{4.11}$$

From this we see directly that both graviphotons become massive:

$$m^2_{\mathcal{A}^1}=m^2_{\mathcal{A}^2}=m^2. \tag{4.12}$$

By putting the results together, we find the spectrum of the D-dimensional theory resulting from reduction with an elliptic twist. The massless sector contains gravity, one 1-form (the graviphoton that comes from the reduction on S^1) and two scalars (ϕ and ψ), and the massive sector contains two 1-forms ($\mathcal{A}^1$ and $\mathcal{A}^2$) and two scalar fields (τ_1 and τ_2).

4.2 Type IIB on $T^4 \times S^1$

4.2.1 The $D = 6$ field content

We first compactify type IIB string theory on T^4. For most purposes, this comes down to the compactification the corresponding effective theory (type IIB supergravity). We do keep in mind, however, that our compactification originates from the full quantum string theory. Amongst other things, this will result in a discretized U-duality group, $SO(5,5,\mathbb{Z})$.

The compactification on the (untwisted) 4-torus preserves all supersymmetry of the original 10-dimensional theory. Therefore, we end up with maximal $D = 6$ supergravity [34], which has 32 supercharges. Confusingly, this theory is referred to by different authors as $\mathcal{N} = 8$, as $\mathcal{N} = (4,4)$, and as $\mathcal{N} = (2,2)$ supergravity.

The bosonic spectrum of type IIB supergravity consists of the graviton G, the Kalb-Ramond field B_2, the dilaton Φ, and the R-R fields C_0, C_2 and C_4. We can find the 6-dimensional spectrum by straightforward Kaluza-Klein reduction, following section 3.1. The R-R field C_4 is self-dual in $D = 10$, so we only keep half of the fields that descend from it after compactification. The (bosonic) $D = 6$ field content is summarized in the table below. This entire collection of fields makes up a single gravity multiplet [33].

Type IIB on T^4	G	B_2	Φ	C_0	C_2	C_4	Total
scalars	10	6	1	1	6	1	25
1-forms	4	4	-	-	4	4	16
2-forms	-	1	-	-	1	3	5
metric	1	-	-	-	-	-	1

The R-symmetry group of $\mathcal{N} = 8$, $D = 6$ supergravity is[1]

$$USp(4)_L \times USp(4)_R \cong SO(5)_L \times SO(5)_R, \tag{4.13}$$

where the subscripts L and R denote the chiralities of the gravitini that transform under each of the $USp(4)$'s.

As usual, the scalar fields parametrize a moduli space. For maximal $D = 6$ supergravity, this moduli space is given by [21]

$$\frac{SO(5,5)}{SO(5) \times SO(5)}\,. \tag{4.14}$$

The dimension of this group is 25, which is consistent with the number of scalars in the $D = 6$ spectrum. The 25 scalar fields sit in the $(\mathbf{5}, \mathbf{5})$ representation of the R-symmetry group. We denote them by $\phi^{a\dot{b}}$, with $a, \dot{b} = 1, \ldots, 5$. Indices without a dot transform under $USp(4)_L$, and indices with a dot transform under $USp(4)_R$.

The 16 one-form gauge fields form the **16** spinor representation of the U-duality group $SO(5,5)$. Under the decomposition to the R-symmetry group, $SO(5,5) \to SO(5)_L \times SO(5)_R$, this branches to the $(\mathbf{4}, \mathbf{4})$ representation. We denote the gauge fields by $\mathcal{A}_1^{A\dot{B}}$, with $A, \dot{B} = 1, \ldots, 4$.

The 6-dimensional spectrum contains 5 two-form potentials. We decompose these into self-dual and anti-self-dual parts, that we denote by B^a_{2+} and $B^{\dot{a}}_{2-}$ respectively. Together, these transform in the **10** vector representation of $SO(5,5)$, which branches to the $(\mathbf{5}, \mathbf{1}) + (\mathbf{1}, \mathbf{5})$ of the R-symmetry group. We see that the self-dual parts transform only under $USp(4)_L$, and that the anti-self-dual parts transform only under $USp(4)_R$.

The fermionic spectrum of $\mathcal{N} = 8$, $D = 6$ supergravity contains 8 chiral gravitini and 40 chiral dilatini. The left and right-handed gravitini sit in the $(\mathbf{4}, \mathbf{1}) + (\mathbf{1}, \mathbf{4})$ of $USp(4)_L \times USp(4)_R$, and we denote them by ψ_L^A and $\psi_R^{\dot{A}}$, respectively. The dilatini transform in the $(\mathbf{5}, \mathbf{4}) + (\mathbf{4}, \mathbf{5})$ representation. We denote the left-handed dilatini by $\chi_L^{a\dot{B}}$, and the right-handed ones by $\chi_R^{\dot{a}B}$.

4.2.2 Supersymmetry breaking

We reduce this 6-dimensional theory further to $D = 5$ on a circle with a Scherk-Schwarz twist. In order to break supersymmetry (partially), we choose a monodromy matrix in the R-symmetry group. In $D = 5$, only supergravity theories with an even number of supersymmetries are allowed [21], i.e. with $\mathcal{N} \in \{0, 2, 4, 6, 8\}$. Therefore, we decompose the R-symmetry

[1]Technically, the isomorphism in (4.13) is not correct. The correct version reads $SO(5) \cong USp(4)/\mathbb{Z}_2$. However, for many purposes the discrete group $\mathbb{Z}_2$ is irrelevant so in this thesis we often neglect it. For more information on this isomorphism, see appendix A.3.

group as

$$USp(4)_L \times USp(4)_R \;\rightarrow\; USp(2)_{L_1} \times USp(2)_{L_2} \times USp(2)_{R_1} \times USp(2)_{R_2} . \tag{4.15}$$

Each of these $USp(2)$'s embodies 2 of the 8 supersymmetries. Because of this, we can break supersymmetry in a very structured way: namely by twisting in a certain number of $USp(2)$'s.

In order to find out which fields of the $D = 6$ spectrum acquire mass due to such a Scherk-Schwarz twist, we need to know how the representations of these fields branch under the decomposition (4.15). Under $USp(4) \rightarrow USp(2) \times USp(2) \cong SU(2) \times SU(2)$, we have the branching rules [14]

$$\begin{aligned} \mathbf{4} &\rightarrow (\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2}) , \\ \mathbf{5} &\rightarrow (\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{1}) . \end{aligned} \tag{4.16}$$

With these simple rules, we can construct the branchings of the representations of the $D = 6$ fields under the decomposition of the R-symmetry group (4.15). We find

$$\begin{aligned} \text{scalars}: &\quad (\mathbf{5},\mathbf{5}) &&\rightarrow (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2}) + (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) , \\ \text{1-forms}: &\quad (\mathbf{4},\mathbf{4}) &&\rightarrow (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2}) , \\ \text{2-forms}: &\quad (\mathbf{5},\mathbf{1}) + (\mathbf{1},\mathbf{5}) &&\rightarrow (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) + 2\,(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) , \\ \text{gravitini}: &\quad (\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{4}) &&\rightarrow (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2}) , \\ \text{dilatini}: &\quad (\mathbf{5},\mathbf{4}) + (\mathbf{4},\mathbf{5}) &&\rightarrow (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2}) + (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{2}) \\ & && \quad + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2}) . \end{aligned} \tag{4.17}$$

The graviton is a singlet under the entire R-symmetry group, and therefore forms a $(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$ under the decomposition in $USp(2)$'s. These are the same branchings that were found in [14].

In general, twisting with respect to a certain $USp(2)$ will leave all the fields that are a singlet under that $USp(2)$ massless, and will make all the fields that are charged under it massive. It can now be seen explicitly that for each $USp(2)$ that we twist in, we break 2 (out of 8) supersymmetries. Indeed, we see from the decomposition of the $(\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{4})$ that two gravitini acquire mass for each $USp(2)$ in which we twist.

4.2.3 The $D = 5$ field contents for generic twists

By using the branchings from the previous section, we can derive the field contents of the $D = 5$ theories that follow from twisting with respect to different numbers of $USp(2)$'s. In what follows, we analyze the massless bosonic field contents that follow from these reductions, and we check that they fit in the relevant supermultiplets. The other fields either become massive, or are eaten by the Higgs mechanism.

For now, we assume that all fields that are charged under a certain $USp(2)$ will become massive if we twist in that $USp(2)$. In general, this is true, but as we will see in section 5.3, there are cases in which we can tune the mass parameters in such a way that some of these fields remain massless. For generic twists, however, we will end up with the spectra that we find here.

I) $\mathcal{N} = 8$

We start with the untwisted case. Here we twist in none of the $USp(2)$'s, and so all the fields remain massless. This results in maximal $\mathcal{N} = 8$ supergravity in $D = 5$, which is the same theory that we would have obtained by reducing type IIB on a 5-torus T^5 immediately. The $D = 5$ field content is summarized in the table below.

Type IIB on $T^4 \times S^1$	G	B_2	Φ	C_0	C_2	C_4	Total
scalars	15	10	1	1	10	5	42
1-forms	5	6	-	-	6	10	27
metric	1	-	-	-	-	-	1

In the construction of this spectrum, we used that 2-forms can be dualized to 1-forms in $D = 5$. This field content makes up (the bosonic sector of) a single gravity multiplet [35].

II) $\mathcal{N} = 6$

In order to end up with $\mathcal{N} = 6$, $D = 5$ supergravity, we twist in only one of the $USp(2)$'s. Without loss of generality, we choose $USp(2)_{R_2}$. The fields that remain massless, are those that are a singlet under $USp(2)_{R_2}$, i.e. fields that sit in representations of the form $(\cdot, \cdot, \cdot, \mathbf{1})$.

By looking at the representations of the $D = 6$ fields (4.17), we see that 5 scalars, 8 one-forms, and 6 (self-dual or anti-self-dual) two-forms are $USp(2)_{R_2}$ singlets. These yield 13 massless scalars and 14 massless one-forms in $D = 5$. Of course, the graviton also remains massless, so in total we end up with 14 scalars, 15 vectors and a graviton in the massless $D = 5$ spectrum. These fields form the bosonic sector of the gravity multiplet of $\mathcal{N} = 6$, $D = 5$ supergravity [36, 37].

IIIA) $\mathcal{N} = 4$: $(2, 0)$

There are two distinct ways to end up in $\mathcal{N} = 4$ supergravity in $D = 5$. We can either twist in two $USp(2)$'s that transform gravitini with the same chirality, or twist in two $USp(2)$'s that transform gravitini with different chiralities. The former choice results in $(2, 0)$, and the latter results in $(1, 1)$ supergravity. Because there is no notion of chirality in $D = 5$, these theories are equivalent. The different names are merely a relic of the 6-dimensional chiral theory that they descend from.

Here we twist with respect to $USp(2)_{R_1}$ and $USp(2)_{R_2}$, and so we arrive at $(2, 0)$ supergravity. From (4.17), we see that in $D = 6$, there are 5 scalars, and 6 two-form tensors, in addition to the graviton, that are not charged under $USp(2)_{R_1} \times USp(2)_{R_2}$. It follows that the massless $D = 5$ spectrum contains 6 scalars, 7 vectors and a graviton.

In $\mathcal{N} = 4$, $D = 5$ supergravity, the gravity multiplet contains a graviton, 6 vectors and 1 scalar field, and the vector multiplet contains 1 vector and 5 scalars [38, 39]. We see that our massless spectrum consists of a gravity multiplet coupled to one vector multiplet.

IIIb) $\mathcal{N} = 4$: $(1, 1)$

We now perform the other twisted reduction that will result in $\mathcal{N} = 4$ supergravity. We twist in $USp(2)_{L_2}$ and $USp(2)_{R_2}$ and therefore we end up with the $(1, 1)$ case. The singlets under such a twist are 1 scalar, 4 one-forms, 2 two-forms and the graviton. Therefore, the massless field content after reduction to $D = 5$ contains 6 scalar fields, 7 vector fields and gravity.

This is the same spectrum as the one that we found in case IIIa, and so it fits in the same $\mathcal{N} = 4$ multiplets: one gravity multiplet and one vector multiplet.

IV) $\mathcal{N} = 2$

We end up in $\mathcal{N} = 2$ supergravity, by twisting in three of the four $USp(2)$'s. Without loss of generality, we choose to leave only $USp(2)_{L_1}$ untwisted. This means that all $D = 6$ fields become massive, except the ones that transform in a representation of the form $(\cdot, \mathbf{1}, \mathbf{1}, \mathbf{1})$. We see from (4.17) that 1 scalar, 2 two-form tensors and the graviton remain massless in this twisted reduction. These yield 2 scalars, 3 vectors a graviton in $D = 5$.

The $\mathcal{N} = 2$, $D = 5$ gravity multiplet consists of a graviton and a vector, and the corresponding vector multiplet consists of a vector and a scalar [40]. Ergo, the massless field content that we find after twisting in three $USp(2)$'s forms a gravity multiplet coupled to two vector multiplets.

V) $\mathcal{N} = 0$

By twisting in all $USp(2)$'s, we make all the gravitini massive and therefore we break all supersymmetry ($\mathcal{N} = 0$). Just like in the $\mathcal{N} = 2$ case, the (bosonic) singlets under the twist are 1 scalar field, 2 two-form fields (one self-dual and one anti-self-dual) and the graviton. Therefore, the $D = 5$ massless field content consists of 2 scalars, 3 vectors and gravity. Because this theory has no supersymmetry, there is no need to fit this spectrum in supermultiplets.

4.2.4 The mass matrix

So far, we have deduced which fields become massive by arguments from group and representation theory. We now make the story more precise by choosing an explicit mass matrix and by deriving the masses that each of the fields acquire. We have two mass matrices, one in the Lie algebra corresponding to $USp(4)_L$ and another in the algebra corresponding to $USp(4)_R$. They read

$$M_A{}^B = \begin{pmatrix} 0 & 0 & -m_1 & 0 \\ 0 & 0 & 0 & -m_2 \\ m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \end{pmatrix}, \qquad \tilde{M}_{\dot{A}}{}^{\dot{B}} = \begin{pmatrix} 0 & 0 & -\tilde{m}_1 & 0 \\ 0 & 0 & 0 & -\tilde{m}_2 \\ \tilde{m}_1 & 0 & 0 & 0 \\ 0 & \tilde{m}_2 & 0 & 0 \end{pmatrix}, \tag{4.18}$$

where m_1, m_2, $\tilde{m}_1$ and $\tilde{m}_2$ are (real) mass parameters. Each of these parameters represents one of the $USp(2)$'s in the decomposition (4.15). By taking a mass parameter equal to zero, we leave the corresponding $USp(2)$ untwisted.

The monodromies, corresponding to the mass matrices (4.18), can be found by exponentiating:

$$\begin{aligned} \mathcal{M}_A{}^B = (e^M)_A{}^B &= \begin{pmatrix} \cos m_1 & 0 & -\sin m_1 & 0 \\ 0 & \cos m_2 & 0 & -\sin m_2 \\ \sin m_1 & 0 & \cos m_1 & 0 \\ 0 & \sin m_2 & 0 & \cos m_2 \end{pmatrix}, \\ \tilde{\mathcal{M}}_{\dot{A}}{}^{\dot{B}} = (e^{\tilde{M}})_{\dot{A}}{}^{\dot{B}} &= \begin{pmatrix} \cos \tilde{m}_1 & 0 & -\sin \tilde{m}_1 & 0 \\ 0 & \cos \tilde{m}_2 & 0 & -\sin \tilde{m}_2 \\ \sin \tilde{m}_1 & 0 & \cos \tilde{m}_1 & 0 \\ 0 & \sin \tilde{m}_2 & 0 & \cos \tilde{m}_2 \end{pmatrix}. \end{aligned} \tag{4.19}$$

It can easily be seen that both of these matrices consist of two $SO(2)$ rotation matrices. In fact, they are elements of the subgroup $SO(2) \times SO(2) \subset USp(2) \times USp(2) \subset USp(4)$. In appendix A.2, we discuss in detail how the subgroup $USp(2) \times USp(2)$ is embedded in $USp(4)$.

The monodromy matrices, as shown in (4.19), act with normal matrix multiplication on fields in the fundamental representation **4** of $USp(4)_L$ and $USp(4)_R$, respectively. Via the isomorphism $USp(4) \cong SO(5)$, we can map these matrices to their $SO(5)$ counterparts. These 5 by 5 matrices act (with matrix multiplication) on fields in the fundamental representation

5 of either $SO(5)_L$ or $SO(5)_R$. We can map the generators (4.18) to generators in $\mathfrak{so}(5)$ with the relations

$$M_{ab} = -\frac{1}{2}\,\mathrm{Tr}\big[M_A{}^B(\Gamma_{ab})_B{}^C\big]\,, \qquad \tilde{M}_{\dot{a}\dot{b}} = -\frac{1}{2}\,\mathrm{Tr}\Big[\tilde{M}_{\dot{A}}{}^{\dot{B}}(\Gamma_{ab})_{\dot{B}}{}^{\dot{C}}\Big]\,. \tag{4.20}$$

For the derivation of these expressions, and for an appropriate basis of Dirac matrices Γ_a, see appendix A.3. We find the mass matrices as

$$\begin{aligned} M_{ab} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(m_1+m_2) & 0 & 0 \\ 0 & m_1+m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(m_1-m_2) \\ 0 & 0 & 0 & m_1-m_2 & 0 \end{pmatrix}, \\ \tilde{M}_{\dot{a}\dot{b}} &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(\tilde{m}_1+\tilde{m}_2) & 0 & 0 \\ 0 & \tilde{m}_1+\tilde{m}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(\tilde{m}_1-\tilde{m}_2) \\ 0 & 0 & 0 & \tilde{m}_1-\tilde{m}_2 & 0 \end{pmatrix}. \end{aligned} \tag{4.21}$$

The corresponding monodromies (elements of $SO(5)_L$ and $SO(5)_R$, respectively) read

$$\begin{aligned} \mathcal{M}_{ab} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\big[m_1+m_2\big] & -\sin\big[m_1+m_2\big] & 0 & 0 \\ 0 & \sin\big[m_1+m_2\big] & \cos\big[m_1+m_2\big] & 0 & 0 \\ 0 & 0 & 0 & \cos\big[m_1-m_2\big] & -\sin\big[m_1-m_2\big] \\ 0 & 0 & 0 & \sin\big[m_1-m_2\big] & \cos\big[m_1-m_2\big] \end{pmatrix}, \\ \tilde{\mathcal{M}}_{\dot{a}\dot{b}} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\big[\tilde{m}_1+\tilde{m}_2\big] & -\sin\big[\tilde{m}_1+\tilde{m}_2\big] & 0 & 0 \\ 0 & \sin\big[\tilde{m}_1+\tilde{m}_2\big] & \cos\big[\tilde{m}_1+\tilde{m}_2\big] & 0 & 0 \\ 0 & 0 & 0 & \cos\big[\tilde{m}_1-\tilde{m}_2\big] & -\sin\big[\tilde{m}_1-\tilde{m}_2\big] \\ 0 & 0 & 0 & \sin\big[\tilde{m}_1-\tilde{m}_2\big] & \cos\big[\tilde{m}_1-\tilde{m}_2\big] \end{pmatrix}. \end{aligned} \tag{4.22}$$

The subgroup $SO(2)\times SO(2) \subset SO(5)$ can easily be recognized in each of these block diagonal matrices.

The masses that the $D = 6$ fields obtain in the Scherk-Schwarz reduction are proportional to the absolute values of the eigenvalues of the mass matrices. A simple calculation yields the eigenvalues of the $\mathfrak{usp}(4)$ mass matrices as

$$\mu_A = \big(\pm\, i\, m_1,\, \pm\, i\, m_2\big), \qquad \tilde{\mu}_{\dot{A}} = \big(\pm\, i\, \tilde{m}_1,\, \pm\, i\, \tilde{m}_2\big), \tag{4.23}$$

and the eigenvalues of the $\mathfrak{so}(5)$ mass matrices as

$$\mu_a = \big(0,\, i\,(\pm\, m_1 \pm m_2)\big), \qquad \tilde{\mu}_{\dot{a}} = \big(0,\, i\,(\pm\, \tilde{m}_1 \pm \tilde{m}_2)\big). \tag{4.24}$$

At this point, we recall that the monodromy matrices are elements of the R-symmetry group, which is a subgroup of the U-duality group $SO(5,5,\mathbb{Z})$. In order to take the discrete nature of this group into account, our twist matrices must be discretized. The embedding of the R-symmetry group in the U-duality group goes via the $SO(5)$ matrices. In particular, we can combine our two $SO(5)$ monodromies (4.22) into a block diagonal $SO(5,5)$ matrix:

$$\begin{pmatrix} \mathcal{M}_{ab} & 0 \\ 0 & \tilde{\mathcal{M}}_{\dot{a}\dot{b}} \end{pmatrix} \in SO(5,5). \tag{4.25}$$

This matrix should be an element of the discrete U-duality group $SO(5,5,\mathbb{Z})$, which leads to the following quantization conditions for the mass parameters:

$$m_1 \pm m_2 \in \frac{\pi}{2}\,\mathbb{Z}, \qquad \tilde{m}_1 \pm \tilde{m}_2 \in \frac{\pi}{2}\,\mathbb{Z}. \tag{4.26}$$

For these quantized values of the mass parameters, each $SO(2)$ block in each of the $SO(5)$ matrices can take on one of four possible forms:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.27}$$

These matrices form the cyclic group $\mathbb{Z}_4$. Indeed, we have the isomorphism $SO(2,\mathbb{Z}) \cong \mathbb{Z}_4$.

4.2.5 The mass spectrum

As we already mentioned in the previous section, the masses that fields acquire due to the twisted reduction are proportional to the absolute values of the eigenvalues of the mass matrices. Before we apply this to the reduction at hand, we check that this claim is consistent with the mass spectrum that we found for the reduction of pure gravity on $T^2 \times S^1$ that we performed in section 4.1.

We calculated the mass spectrum of pure gravity on $T^2 \times S^1$ explicitly for case in which we twisted with the elliptic mass matrix M_e. This matrix has eigenvalues $\pm\, im$. Therefore, fields that transform in the fundamental representation **2** of the symmetry group $SL(2)$ should acquire masses proportional to $|\pm im| = |m|$. The $(D+1)$-dimensional spectrum contains two scalars (τ_1, τ_2) in the **2**, one scalar ψ in the **1**, and two graviphotons $(\mathcal{A}^1, \mathcal{A}^2)$ in the **2** of $SL(2)$. In sections 4.1.1 and 4.1.2, we computed explicitly that $m_\tau^2 = 8m^2$ and that $m_\mathcal{A}^2 = m^2$. We see that the masses of the fields that are charged under the twist are indeed proportional to $|m|$.

We now return to the reduction of type IIB on $T^4 \times S^1$. We can deduce which mass matrix eigenvalues correspond to which fields, by looking at the way they transform under the R-symmetry group. In this way, we construct the mass spectrum as given in the table below. These masses agree with the spectrum that was found in [33].

Fields	Representation	Masses (up to a prefactor)
$\phi^{a\dot{b}}$	$(\mathbf{5},\mathbf{5})$	$\lvert\mu_a + \tilde{\mu}_{\dot{b}}\rvert$
$\mathcal{A}_1^{A\dot{B}}$	$(\mathbf{4},\mathbf{4})$	$\lvert\mu_A + \tilde{\mu}_{\dot{B}}\rvert$
B_{2+}^{a}	$(\mathbf{5},\mathbf{1})$	$\lvert\mu_a\rvert$
$B_{2-}^{\dot{a}}$	$(\mathbf{1},\mathbf{5})$	$\lvert\tilde{\mu}_{\dot{a}}\rvert$
ψ_L^A	$(\mathbf{4},\mathbf{1})$	$\lvert\mu_A\rvert$
$\psi_R^{\dot{A}}$	$(\mathbf{1},\mathbf{4})$	$\lvert\tilde{\mu}_{\dot{A}}\rvert$
$\chi_L^{a\dot{B}}$	$(\mathbf{5},\mathbf{4})$	$\lvert\mu_a + \tilde{\mu}_{\dot{B}}\rvert$
$\chi_R^{\dot{a}B}$	$(\mathbf{4},\mathbf{5})$	$\lvert\tilde{\mu}_{\dot{a}} + \mu_B\rvert$

We now analyze how these masses (in particular the masses of the scalar fields and the 2-form fields) split up under the R-symmetry decomposition (4.15). We know that the **5** branches as

$$\mathbf{5} \;\rightarrow\; (\mathbf{2},\mathbf{2}) + (\mathbf{1},\mathbf{1})\,, \tag{4.28}$$

in the decomposition $USp(4) \rightarrow USp(2) \times USp(2)$. Logically, it follows that the five eigenvalues μ_a split up in such a way that the singlet $(\mathbf{1},\mathbf{1})$ corresponds to the eigenvalue 0 (and so it remains massless), and that the four fields in the $(\mathbf{2},\mathbf{2})$ acquire masses proportional to the four non-zero eigenvalues $|\pm m_1 \pm m_2|$. Of course, we have a similar decomposition of the five eigenvalues $\tilde{\mu}_{\dot{a}}$ for fields that transform in the **5** of $USp(4)_R$.

By using this way of distributing the eigenvalues, we can deduce the masses that the fields in each of the branches of the R-symmetry decomposition obtain. We only present the branched masses of the scalars and the 2-forms, because these will play a central role in the constraints on black hole existence that we derive in section 5.3. For these fields the masses are summarized in the table below.

Fields	Representation	Masses (up to a prefactor)
$\phi^{a\dot{b}}$	$(\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2})$	$\lvert \pm m_1 \pm m_2 \pm \tilde{m}_1 \pm \tilde{m}_2 \rvert$
	$(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$	$\lvert \pm m_1 \pm m_2 \rvert$
	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})$	$\lvert \pm \tilde{m}_1 \pm \tilde{m}_2 \rvert$
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0
B^a_{2+}	$(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$	$\lvert \pm m_1 \pm m_2 \rvert$
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0
$B^{\dot{a}}_{2-}$	$(\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2})$	$\lvert \pm \tilde{m}_1 \pm \tilde{m}_2 \rvert$
	$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$	0

In section 4.2.3, we constructed the massless spectra for generic twists by assuming that the fields that transform as a singlet under the twist remain massless. It can be checked that this assumption is consistent with the masses that we find here. Fields in the $(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$, for example, remained massless as long as we didn't twist in $USp(2)_{L_1} \times USp(2)_{L_2}$. In order to leave these two $USp(2)$'s untwisted, we have to take $m_1 = m_2 = 0$. We can see from the table above that, for these two mass parameters equal to zero, such fields indeed stay massless.

It is always true that a field that is a singlet under the twist, remains massless. The converse statement, however, is not always true. For generic twists, all fields that are charged under the twist matrix become massive. But in some cases, we can tune the mass parameters of the twist in such a way that some charged fields stay massless. For example, if we twist in $USp(2)_{L_1} \times USp(2)_{L_2}$ with mass parameters that satisfy $m_1 = m_2$, only two of the four fields in the $(\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1})$ acquire a non-zero mass. The other two remain massless. We will use this method of keeping certain fields massless in section 5.3.

Chapter 5

Black Holes

We start this chapter by refreshing some basic aspects of black hole physics in section 5.1. After that, we move on to string theoretical black holes. In section 5.2, we study the well-known D1/D5 black hole. At first, we construct this black hole by using untwisted Kaluza-Klein reduction. Finally, in section 5.3, we study the validity of this black hole solution in the twisted reductions that we carried out in section 4.2.

5.1 Preliminaries

In this section we recall some basic black hole solutions from Einstein-Maxwell theory: the Schwarzschild solution, the Reissner-Nordström solution, and the Tangherlini solution. After that, we discuss some aspects of black hole thermodynamics. Most importantly, we present the Bekenstein-Hawking area law for the entropy of a black hole.

5.1.1 Black holes in Einstein-Maxwell theory

The Schwarzschild solution

The most basic black hole solution in 4-dimensional general relativity is the Schwarzschild solution. This solution describes black holes that are characterized only by their mass M. They have no charge or angular momentum. The Schwarzschild solution is usually written as

$$\mathrm{d}s^2 = -\left(1 - \frac{2G_N M}{\rho}\right)\mathrm{d}t^2 + \left(1 - \frac{2G_N M}{\rho}\right)^{-1}\mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega_2^2. \tag{5.1}$$

A black hole solution typically has one or multiple event horizons. The locations of these horizons can be found by solving $g^{\rho\rho} = 0$. It can be seen directly from (5.1) that a Schwarzschild black hole has a single event horizon located at $\rho = 2G_N M$.

The Reissner-Nordström solution

We now consider 4-dimensional charged black holes. Such black holes can be described by the Reissner-Nordström solution. This solution is characterized by three parameters: the mass M, the electric charge Q and the magnetic charge P. Depending on the values of these parameters, a Reissner-Nordström black hole can have either zero, one or two event horizons. We find one event horizon if the condition $G_N M^2 = Q^2 + P^2$ is satisfied [41]. Black holes that satisfy this condition are called extremal, and their metric is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{G_N M}{\rho}\right)^2 \mathrm{d}t^2 + \left(1 - \frac{G_N M}{\rho}\right)^{-2} \mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega_2^2. \tag{5.2}$$

The event horizon of this extremal black hole solution is located at $\rho = G_N M$. We now rewrite this solution to isotropic coordinates. This amounts to shifting the radial coordinate to $r = \rho - G_N M$. In terms of this isotropic radius r, the Reissner-Nordström metric reads

$$\mathrm{d}s^2 = -\left(1 + \frac{G_N M}{r}\right)^{-2} \mathrm{d}t^2 + \left(1 + \frac{G_N M}{r}\right)^2 \left[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_2^2\right]. \tag{5.3}$$

In this isotropic form, the event horizon is located at $r = 0$.

The Tangherlini solution

The 5-dimensional analogue of the Reissner-Nordström black hole is called the Tangherlini black hole, or sometimes the Schwarzschild-Tangherlini black hole. Written in terms of isotropic coordinates, the metric of the extremal Tangherlini black hole reads

$$\mathrm{d}s^2 = -\left(1 + \frac{Q}{r^2}\right)^{-2} \mathrm{d}t^2 + \left(1 + \frac{Q}{r^2}\right) \left[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2\right]. \tag{5.4}$$

Here Q denotes the charge of the black hole. This solution has a single event horizon at $r = 0$, just like the extremal Reissner-Nordström black hole.

5.1.2 Black hole thermodynamics

Black hole thermodynamics is the field of study that describes the laws of thermodynamics and statistical physics on and near black hole event horizons. The most prominent result in this research area is the Bekenstein-Hawking area law for the entropy of a black hole.

According to this law, the entropy of a black hole is given by [2]

$$S_{\mathrm{BH}} = \frac{A}{4G_N}. \tag{5.5}$$

In section 5.2.4, we repeatedly use this formula for the computation of the entropy of the black holes that we find there.

In general, black holes have a non-zero temperature, called the Hawking temperature T_{H}. Consequently, they emit Hawking radiation. The black holes that we consider in this thesis, however, are all extremal which implies that their temperature is zero and that they don't emit radiation.

5.2 The D1/D5 black hole

In the context of string theory, black holes are made of D-branes. In order to construct a black hole, these branes need to be placed exclusively in spatial dimensions that we plan on compactifying. After compactification, such a configuration of D-branes becomes pointlike in all remaining spatial dimensions. This pointlike object is then recognized as the singularity of the black hole.

In this section, we will consider the so-called D1/D5 solution of type IIB string theory. This is one of the most basic configurations of branes that will lead to a black hole with non-zero entropy after compactification. The resulting black hole is 5-dimensional, and it emerges after compactification on $T^5 \cong T^4 \times S^1$.

We write our $D = 10$ coordinate system explicitly as $(t, r, \theta, \varphi_1, \varphi_2, x_5, x_6, x_7, x_8, x_9)$. Here $x_6, \ldots, x_9$ are the coordinates on the 4-torus, and x_5 is the coordinate on the circle. We can picture the compactification scheme schematically as

$$\begin{array}{l} D = 10 \\ \quad \Big\downarrow \; x_6, \ldots, x_9 \text{ on } T^4 \\ D = 6 \\ \quad \Big\downarrow \; x_5 \text{ on } S^1 \\ D = 5 \end{array} \tag{5.6}$$

The other spatial coordinates are non-compact. They are written in isotropic spherical coordinates (in particular, r is the isotropic radial coordinate).

5.2.1 The static solution

We first consider the static solution, i.e. the solution that results in a black hole without angular momentum. In order to construct this black hole in such a way that it has an event horizon with non-zero area, we combine D1-branes, D5-branes and a gravitational wave of momentum. If we would omit any of these components, the resulting black hole would have an event horizon with area $A = 0$. In other words, the singularity would be naked. Of course, this violates the cosmic censorship conjecture, and so we will be careful to take along all necessary components to make sure that $A \neq 0$.

We take the D5-branes to lie in all the compact spatial directions, i.e. in $x_5, \ldots, x_9$, and the D1-branes to lie in the x_5-direction. In order to make the compactification of $x_6, \ldots, x_9$ (the compact directions in which the D1-branes are pointlike) well-defined the D1-branes need to be arrayed in these directions. The harmonic functions corresponding to the D1-branes and the D5-branes, respectively, are therefore given by

$$H_1 = 1 + \frac{Q_1}{r^2}, \qquad H_5 = 1 + \frac{Q_5}{r^2}, \tag{5.7}$$

as can be seen from (2.15) and (2.18). We also add momentum in the x_5-direction via the (similar looking) harmonic function

$$H_K = 1 + K = 1 + \frac{Q_K}{r^2}. \tag{5.8}$$

We can now write down the solution that incorporates all the elements that are required for the construction of the black hole. Details on how multiple p-brane solutions can be combined to form a single solution, can be found in [24]. In string frame, the solution reads

$$\left\{ \begin{aligned} ds_{10}^2 &= H_1^{-\frac{1}{2}} H_5^{-\frac{1}{2}} \left[-\mathrm{d}t^2 + \mathrm{d}x_5^2 + K(\mathrm{d}t - \mathrm{d}x_5)^2 \right] \\ &\quad + H_1^{\frac{1}{2}} H_5^{-\frac{1}{2}} \left[\mathrm{d}x_6^2 + \mathrm{d}x_7^2 + \mathrm{d}x_8^2 + \mathrm{d}x_9^2 \right] \\ &\quad + H_1^{\frac{1}{2}} H_5^{\frac{1}{2}} \left[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2 \right] \\ e^{\tilde{\Phi}_{10}} &= H_1^{\frac{1}{2}} H_5^{-\frac{1}{2}} \\ C_{t5} &= H_1^{-1} - 1 \\ \bar{C}_{t56789} &= H_5^{-1} - 1, \end{aligned} \right. \tag{5.9}$$

where $\mathrm{d}\Omega_3^2 = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi_1^2 + \cos^2\theta\, \mathrm{d}\varphi_2^2$ is the metric on the 3-sphere of the non-compact spatial coordinates.

This solution imposes constraints on the spinors ϵ_L and ϵ_R that partially break the supersymmetry. The presence of D1 and D5-branes yields the constraints [42]

$$\Gamma^0\Gamma^5\epsilon_L = \epsilon_R, \qquad\qquad \Gamma^0\Gamma^5\Gamma^6\Gamma^7\Gamma^8\Gamma^9\epsilon_L = \epsilon_R. \tag{5.10}$$

Each of these constraints breaks half of the supersymmetry. Furthermore, the momentum in the x_5-direction imposes the additional constraints

$$\Gamma^0\Gamma^5\epsilon_L = \epsilon_L, \qquad\qquad \Gamma^0\Gamma^5\epsilon_R = \epsilon_R. \tag{5.11}$$

Together, the constraints preserve 4 out of the original 32 supercharges. Therefore, our configuration of D1-branes, D5-branes and momentum is 1/8 BPS.

From (5.9), we can see that the R-R field C_2 charges the D1-branes electrically, and that it charges the D5-branes magnetically (via its Hodge dual $\bar{C}_6$). It will turn out to be more convenient to rewrite $\bar{C}_{t56789}$ in terms of components of C_2. In order to do that, we first take the exterior derivative

$$(\mathrm{d}\bar{C})_{rt56789} = -H_5^{-2}\,\partial_r H_5. \tag{5.12}$$

On this expression, we act with the Hodge star operator. Since $\bar{C}_6$ is the Hodge dual of C_2, we have the relation[1] $*\,\mathrm{d}\bar{C}_6 = *\,*\,\mathrm{d}C_2 = \mathrm{d}C_2$. Using this, we can translate (5.12) to components of the R-R field C_2. We compute

$$\begin{aligned}
(\mathrm{d}C)_{\theta\varphi_1\varphi_2} &= (*\,\mathrm{d}\bar{C})_{\theta\varphi_1\varphi_2} \\
&= \tilde{\varepsilon}^{rt56789}{}_{\theta\varphi_1\varphi_2}(\mathrm{d}\bar{C})_{rt56789} \\
&= \tilde{\varepsilon}_{rt56789\theta\varphi_1\varphi_2}\,G^{rr}\big(G^{tt}G^{55} - G^{t5}G^{t5}\big)G^{66}G^{77}G^{88}G^{99}\,\big(-H_5^{-2}\,\partial_r H_5\big) \\
&= \varepsilon_{rt56789\theta\varphi_1\varphi_2}\sqrt{-G}\,\big(H_1^{-\frac{3}{2}}H_5^{\frac{1}{2}}\,\partial_r H_5\big) \\
&= \varepsilon_{tr\theta\varphi_1\varphi_2 56789}\;r^3\cos\theta\sin\theta\;(\partial_r H_5) \\
&= r^3\cos\theta\sin\theta\;(\partial_r H_5).
\end{aligned} \tag{5.13}$$

Here, we used that $\tilde{\varepsilon} = \varepsilon\sqrt{-G}$ is the curved space Levi-Civita tensor, and we chose the Levi-Civita symbol (the one without a tilde) such that $\varepsilon_{tr\theta\varphi_1\varphi_2 56789} = +1$. We now substitute our expression for H_5 and find

$$(\mathrm{d}C)_{\theta\varphi_1\varphi_2} = -2Q_5\cos\theta\sin\theta, \tag{5.14}$$

[1] In general, applying a Hodge star operator, defined on a d-dimensional manifold, twice on a p-form returns $*\,*\,A_p = (-1)^{p(d-p)+s}A_p$, where s is the signature of metric. Using this, we find that $*\,*\,\mathrm{d}C_2 = \mathrm{d}C_2$.

which allows us to rewrite the static D1/D5 solution (5.9) to

$$\begin{cases} \mathrm{d}s_{10}^2 = H_1^{-\frac{1}{2}} H_5^{-\frac{1}{2}} \left[-\,\mathrm{d}t^2 + \mathrm{d}x_5^2 + K(\mathrm{d}t - \mathrm{d}x_5)^2 \right] \\ \qquad\quad + H_1^{\frac{1}{2}} H_5^{-\frac{1}{2}} \left[\mathrm{d}x_6^2 + \mathrm{d}x_7^2 + \mathrm{d}x_8^2 + \mathrm{d}x_9^2 \right] \\ \qquad\quad + H_1^{\frac{1}{2}} H_5^{\frac{1}{2}} \left[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2 \right] \\ e^{\tilde{\Phi}_{10}} = H_1^{\frac{1}{2}} H_5^{-\frac{1}{2}} \\ C_2 = (H_1^{-1} - 1)\, \mathrm{d}t \wedge \mathrm{d}x_5 + Q_5 \cos^2\theta \; \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 . \end{cases} \tag{5.15}$$

5.2.2 The rotating solution

The solution from the previous section can be generalized by adding angular momentum. With this addition, we obtain a solution that is stationary, but not static. In order to obtain a rotating black hole in 5 dimensions, we should add the angular momentum in the directions transverse on the D-branes. This transverse space (consisting of the r, θ, φ_1 and φ_2-directions) has an $SO(4)$ rotational symmetry.

The rank of $SO(4)$ is 2, so in principle the angular momentum is characterized by two parameters, J_1 and J_2. It turns out, however, that in order to preserve supersymmetry, one linear combination of these parameters must vanish [43]. Because of this, supersymmetric black holes in 5 dimensions are only characterized by a single angular momentum parameter, that we simply denote by J.

The full rotating D1/D5 solution can be written as [44]

$$\begin{cases} \mathrm{d}s_{10}^2 = H_1^{-\frac{1}{2}} H_5^{-\frac{1}{2}} \left[-\,\mathrm{d}t^2 + \mathrm{d}x_5^2 + K(\mathrm{d}t - \mathrm{d}x_5)^2 - \dfrac{J}{r^2} (\mathrm{d}t - \mathrm{d}x_5) \left(\sin^2\theta \; \mathrm{d}\varphi_1 - \cos^2\theta \; \mathrm{d}\varphi_2 \right) \right] \\ \qquad\quad + H_1^{\frac{1}{2}} H_5^{-\frac{1}{2}} \left[\mathrm{d}x_6^2 + \mathrm{d}x_7^2 + \mathrm{d}x_8^2 + \mathrm{d}x_9^2 \right] + H_1^{\frac{1}{2}} H_5^{\frac{1}{2}} \left[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2 \right] \\ e^{\tilde{\Phi}_{10}} = H_1^{\frac{1}{2}} H_5^{-\frac{1}{2}} \\ C_2 = (H_1^{-1} - 1)\, \mathrm{d}t \wedge \mathrm{d}x_5 + Q_5 \cos^2\theta \; \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 \\ \qquad + \dfrac{J}{2r^2} H_1^{-1} (\mathrm{d}t - \mathrm{d}x_5) \wedge \left(\sin^2\theta \; \mathrm{d}\varphi_1 - \cos^2\theta \; \mathrm{d}\varphi_2 \right). \end{cases} \tag{5.16}$$

For $J = 0$, these expressions reduce to (5.15). Of course, this is in line with our expectations, because (5.15) is simply the non-rotating version of this more general solution.

5.2.3 Reduction of the rotating solution

In this section we reduce the rotating D1/D5 solution on $T^4 \times S^1$. We perform this reduction according to the scheme laid out in (5.6). For now, we assume that this reduction is untwisted. Consequently, all fields remain massless throughout this reduction.

First, we reduce the metric. We recall that, in order to remain in string frame throughout the compactification, we need the following ansatz:

$$\hat{G}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} G_{\mu\nu} + e^{2\phi}\mathcal{A}_\mu\mathcal{A}_\nu & e^{2\phi}\mathcal{A}_\mu \\ e^{2\phi}\mathcal{A}_\nu & e^{2\phi} \end{pmatrix}. \tag{5.17}$$

For notational clarity, we will from now on use $\hat{\hat{G}}$, $\hat{G}$ and G for the (string frame) metric in $D = 10$, $D = 6$ and $D = 5$, respectively. We see from (5.16) that the metric of the 10-dimensional solution does not contain cross terms between the coordinates on T^4 and those on $\mathbb{R}^{1,4} \times S^1$. Therefore the four graviphotons that we obtain in the compactification on T^4 are all equal to zero in this solution. The four Kaluza-Klein scalars, however, take non-zero values. We denote these scalar fields by ϕ_i, where the index $i = 6, \ldots, 9$ indicates that it is the Kaluza-Klein scalar corresponding to the reduction of the x_i-direction. We obtain

$$e^{2\phi_i} = \hat{\hat{G}}_{ii} = H_1^{\frac{1}{2}} H_5^{-\frac{1}{2}}, \qquad\qquad i = 6, \ldots, 9. \tag{5.18}$$

From (5.17), we see that because the graviphotons on T^4 are zero, we obtain the 6-dimensional metric simply as

$$\hat{G}_{\hat{\mu}\hat{\nu}} = \hat{\hat{G}}_{\hat{\mu}\hat{\nu}}. \tag{5.19}$$

We now perform the subsequent reduction on S^1. Just like before, we can read off the Kaluza-Klein scalar ϕ_5 directly from the metric. We find

$$e^{2\phi_5} = \hat{G}_{55} = H_1^{-\frac{1}{2}} H_5^{-\frac{1}{2}} (1 + K). \tag{5.20}$$

The 6-dimensional metric $\hat{G}_{\hat{\mu}\hat{\nu}}$ contains cross terms between x_5 and several of the non-compact coordinates. Because of this, some components of the graviphoton $\mathcal{A}_1^5$ (corresponding to the compactification of x_5) take on non-zero values. We compute these as follows:

$$\begin{aligned}
e^{2\phi_5}\mathcal{A}_t^5 &= \hat{G}_{5t} = H_1^{-\frac{1}{2}} H_5^{-\frac{1}{2}} (-K) && \Rightarrow \quad \mathcal{A}_t^5 = (1+K)^{-1} - 1, \\
e^{2\phi_5}\mathcal{A}_{\varphi_1}^5 &= \hat{G}_{5\varphi_1} = H_1^{-\frac{1}{2}} H_5^{-\frac{1}{2}} \left(\frac{J}{2r^2}\sin^2\theta\right) && \Rightarrow \quad \mathcal{A}_{\varphi_1}^5 = \frac{J}{2r^2}(1+K)^{-1}\sin^2\theta, \\
e^{2\phi_5}\mathcal{A}_{\varphi_2}^5 &= \hat{G}_{5\varphi_2} = H_1^{-\frac{1}{2}} H_5^{-\frac{1}{2}} \left(-\frac{J}{2r^2}\cos^2\theta\right) && \Rightarrow \quad \mathcal{A}_{\varphi_2}^5 = -\frac{J}{2r^2}(1+K)^{-1}\cos^2\theta.
\end{aligned} \tag{5.21}$$

The other components of $\mathcal{A}_1^5$ are zero. With these expressions, we can compute the non-zero components of the 5-dimensional metric as

$$\begin{aligned}
G_{tt} &= \hat{G}_{tt} - e^{2\phi_5}\mathcal{A}_t^5\mathcal{A}_t^5 &&= -H_1^{-\frac{1}{2}}H_5^{-\frac{1}{2}}(1+K)^{-1}, \\
G_{\varphi_1\varphi_1} &= \hat{G}_{\varphi_1\varphi_1} - e^{2\phi_5}\mathcal{A}_{\varphi_1}^5\mathcal{A}_{\varphi_1}^5 &&= H_1^{\frac{1}{2}}H_5^{\frac{1}{2}}\, r^2\sin^2\theta - H_1^{-\frac{1}{2}}H_5^{-\frac{1}{2}}(1+K)^{-1}\left(\frac{J^2}{4r^4}\sin^4\theta\right), \\
G_{\varphi_2\varphi_2} &= \hat{G}_{\varphi_2\varphi_2} - e^{2\phi_5}\mathcal{A}_{\varphi_2}^5\mathcal{A}_{\varphi_2}^5 &&= H_1^{\frac{1}{2}}H_5^{\frac{1}{2}}\, r^2\cos^2\theta - H_1^{-\frac{1}{2}}H_5^{-\frac{1}{2}}(1+K)^{-1}\left(\frac{J^2}{4r^4}\cos^4\theta\right), \\
G_{t\varphi_1} &= \hat{G}_{t\varphi_1} - e^{2\phi_5}\mathcal{A}_t^5\mathcal{A}_{\varphi_1}^5 &&= -H_1^{-\frac{1}{2}}H_5^{-\frac{1}{2}}(1+K)^{-1}\left(\frac{J}{2r^2}\sin^2\theta\right), \\
G_{t\varphi_2} &= \hat{G}_{t\varphi_2} - e^{2\phi_5}\mathcal{A}_t^5\mathcal{A}_{\varphi_2}^5 &&= -H_1^{-\frac{1}{2}}H_5^{-\frac{1}{2}}(1+K)^{-1}\left(-\frac{J}{2r^2}\cos^2\theta\right), \\
G_{\varphi_1\varphi_2} &= \hat{G}_{\varphi_1\varphi_2} - e^{2\phi_5}\mathcal{A}_{\varphi_1}^5\mathcal{A}_{\varphi_2}^5 &&= -H_1^{-\frac{1}{2}}H_5^{-\frac{1}{2}}(1+K)^{-1}\left(-\frac{J^2}{4r^4}\sin^2\theta\cos^2\theta\right),
\end{aligned} \tag{5.22}$$

and as

$$\begin{aligned}
G_{rr} &= \hat{G}_{rr} = H_1^{\frac{1}{2}}H_5^{\frac{1}{2}}, \\
G_{\theta\theta} &= \hat{G}_{\theta\theta} = H_1^{\frac{1}{2}}H_5^{\frac{1}{2}}\, r^2.
\end{aligned} \tag{5.23}$$

By combining all these components, we find the $D = 5$ metric as

$$\begin{aligned}
\mathrm{d}s_5^2 = &-H_1^{-\frac{1}{2}}H_5^{-\frac{1}{2}}(1+K)^{-1}\left[\mathrm{d}t + \frac{J}{2r^2}\left(\sin^2\theta\,\mathrm{d}\varphi_1 - \cos^2\theta\,\mathrm{d}\varphi_2\right)\right]^2 \\
&+ H_1^{\frac{1}{2}}H_5^{\frac{1}{2}}\left[\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2\right],
\end{aligned} \tag{5.24}$$

where, as before, the metric on the 3-sphere reads $\mathrm{d}\Omega_3^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi_1^2 + \cos^2\theta\,\mathrm{d}\varphi_2^2$.

In section 3.1.2, we discussed how the 10-dimensional dilaton is related to its lower-dimensional analogue. For this specific reduction, we find

$$\begin{aligned}
e^{\tilde{\Phi}_5} &= e^{-\frac{1}{2}\phi_5}\dots e^{-\frac{1}{2}\phi_9}\, e^{\tilde{\Phi}_{10}} \\
&= \left(H_1^{\frac{1}{8}}H_5^{\frac{1}{8}}(1+K)^{-\frac{1}{4}}\right)\left(H_1^{-\frac{1}{8}}H_5^{\frac{1}{8}}\right)^4\left(H_1^{\frac{1}{2}}H_5^{-\frac{1}{2}}\right) \\
&= H_1^{\frac{1}{8}}H_5^{\frac{1}{8}}(1+K)^{-\frac{1}{4}}.
\end{aligned} \tag{5.25}$$

Lastly, we consider the reduction of the 2-form field C_2. We recall from section 3.1.3 that the ansatz for the reduction of form-valued fields is

$$\hat{A}_{n-1} = A_{n-1} + A_{n-2}\wedge\mathrm{d}z, \tag{5.26}$$

with z the compact coordinate. In other words, we can simply take the components of C_2, whose indices lie in the compact directions, to be components of the forms with lower tensor rank that descend from C_2. From (5.16), we see that C_2 has no components on the 4-torus. There are non-zero components in the x_5-direction, however. Therefore, we find that the 1-form, that comes from C_2 in the reduction of x_5, acquires non-zero components. We denote this 1-form by C_1. We find the values of the $D = 5$ fields coming from the 2-form R-R potential as

$$\begin{aligned} C_2 &= Q_5 \cos^2\theta \, \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 + \frac{J}{2r^2} H_1^{-1} \, \mathrm{d}t \wedge (\sin^2\theta \, \mathrm{d}\varphi_1 - \cos^2\theta \, \mathrm{d}\varphi_2), \\ C_1 &= (H_1^{-1} - 1) \, \mathrm{d}t - \frac{J}{2r^2} H_1^{-1} (\sin^2\theta \, \mathrm{d}\varphi_1 - \cos^2\theta \, \mathrm{d}\varphi_2). \end{aligned} \tag{5.27}$$

We have now reduced the entire rotating D1/D5 solution, as given in (5.16). The 5-dimensional result reads

$$\begin{cases} \mathrm{d}s_5^2 = -H_1^{-\frac{1}{2}} H_5^{-\frac{1}{2}} H_K^{-1} \left[\mathrm{d}t + \frac{J}{2r^2} \sigma_1\right]^2 + H_1^{\frac{1}{2}} H_5^{\frac{1}{2}} \left[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2\right] \\ e^{\tilde{\Phi}_5} = H_1^{\frac{1}{8}} H_5^{\frac{1}{8}} H_K^{-\frac{1}{4}} \\ e^{\phi_i} = H_1^{\frac{1}{4}} H_5^{-\frac{1}{4}} \qquad (i = 6, \ldots, 9) \\ e^{\phi_5} = H_1^{-\frac{1}{4}} H_5^{-\frac{1}{4}} H_K^{\frac{1}{2}} \\ C_2 = Q_5 \cos^2\theta \, \mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 + H_1^{-1} \frac{J}{2r^2} \, \mathrm{d}t \wedge \sigma_1 \\ C_1 = (H_1^{-1} - 1) \, \mathrm{d}t - H_1^{-1} \frac{J}{2r^2} \sigma_1 \\ \mathcal{A}_1^5 = (H_K^{-1} - 1) \, \mathrm{d}t + H_K^{-1} \frac{J}{2r^2} \sigma_1, \end{cases} \tag{5.28}$$

where we have introduced the 1-form $\sigma_1 = \sin^2\theta \, \mathrm{d}\varphi_1 - \cos^2\theta \, \mathrm{d}\varphi_2$ for notational convenience.

5.2.4 Black hole solutions

The $D = 5$ solution that we found in the previous section, is a pointlike object with an event horizon, so indeed we have found a black hole. We can obtain some black hole solutions that are well-known in black hole related literature, by considering certain limits for the charge and angular momentum parameters.

The BMPV black hole

In it's most general form, the solution (as given in (5.28)) is known as the BMPV Black Hole, named after Breckenridge, Myers, Peet and Vafa, who were the first to study it microscopically [7]. It has three charges and one angular momentum parameter.

Usually, black holes are studied in Einstein frame. With the approach of section 2.1.2, we convert the black hole metric to this frame. We find

$$\mathrm{d}s_{\mathrm{E}}^2 = -\big(H_1 H_5 H_K\big)^{-\frac{2}{3}} \left[\mathrm{d}t + \frac{J}{2r^2}\,\sigma_1\right]^2 + \big(H_1 H_5 H_K\big)^{\frac{1}{3}} \left[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2\right]. \tag{5.29}$$

With this Einstein frame metric, we can compute the entropy of the black hole. For this, we need the area A of the event horizon. Remember that our metric is written in isotropic coordinates, so the event horizon is a 3-sphere located at $r = 0$. From (5.29), we deduce that the metric on a 3-sphere (at isotropic radius r) around the black hole is given by

$$\mathrm{d}s_{\text{3-sphere}}^2 = -\big(H_1 H_5 H_K\big)^{-\frac{2}{3}} \left[\frac{J}{2r^2}\,\sigma_1\right]^2 + \big(H_1 H_5 H_K\big)^{\frac{1}{3}} \left[r^2 \mathrm{d}\Omega_3^2\right]. \tag{5.30}$$

We now calculate the area of the event horizon as

$$\begin{aligned} A &= \oint_{S^3} \sqrt{g_{\text{3-sphere}}}\,\bigg|_{r=0} \\ &= 2\pi^2 \sqrt{Q_1 Q_5 Q_K - \frac{J^2}{4}}\,. \end{aligned} \tag{5.31}$$

As was already mentioned in section 2.2.1, the charges can be quantized as $Q_i = c_i N_i$, for $i = 1, 5, K$. For a derivation of the constants c_i for the three charges at hand, see for example [42]. The product of all three unit charges gives

$$c_1 c_5 c_K = \frac{16\,\big(G_N^{(5)}\big)^2}{\pi^2}. \tag{5.32}$$

Using this, we compute the Bekenstein-Hawking entropy of the black hole as

$$S_{\mathrm{BH}} = \frac{A}{4G_N^{(5)}} = 2\pi \sqrt{N_1 N_5 N_K - \tilde{J}^2}\,, \tag{5.33}$$

where we have defined $\tilde{J} \equiv J\pi \big/ \big(8G_N^{(5)}\big)$.

The static D1/D5 black hole

We now consider the static case, i.e. we demand $J = 0$. The Einstein frame metric for this non-rotating black hole reads

$$\mathrm{d}s_{\mathrm{E}}^2 = -\big(H_1 H_5 H_K\big)^{-\frac{2}{3}}\,\mathrm{d}t^2 + \big(H_1 H_5 H_K\big)^{\frac{1}{3}}\Big[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2\Big]. \tag{5.34}$$

This is the black hole that was originally studied by Strominger and Vafa, and shortly after that by Callan and Maldacena, in order to compute the microscopic entropy from degenerate D-brane configurations [4, 5]. For $J = 0$, the Bekenstein-Hawking entropy reduces to

$$S_{\mathrm{BH}} = 2\pi\,\sqrt{N_1 N_5 N_K}\,. \tag{5.35}$$

The Tangherlini black hole

We get the simplest possible 5-dimensional BPS black hole solution, by taking $J = 0$ and $Q_1 = Q_5 = Q_K \equiv Q$. This result in the following Einstein frame metric:

$$\mathrm{d}s_{\mathrm{E}}^2 = -H^{-2}\,\mathrm{d}t^2 + H\,\Big[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2\Big], \tag{5.36}$$

where $H = 1 + \frac{Q}{r^2}$ is the harmonic function corresponding to Q. This is the Tangherlini black hole, that we encountered in section 5.1.1. It has a single charge parameter, and no angular momentum. In terms of Q, the macroscopic entropy reads

$$S_{\mathrm{BH}} = \frac{\pi^2}{2G_N^{(5)}}\,Q^{\frac{3}{2}}. \tag{5.37}$$

Note that for equal charges, the scalars drop out of the black hole solution. That is, they are equal to zero (see (5.28)):

$$e^{\tilde{\Phi}_5} = e^{\phi_i} = e^{\phi_5} = 1 \qquad\qquad \Rightarrow \qquad\qquad \tilde{\Phi}_5 = \phi_i = \phi_5 = 0, \tag{5.38}$$

where, as usual, $i = 6, \ldots, 9$.

5.3 The D1/D5 black hole from Scherk-Schwarz reductions

In this section, we bring the two main topics of this thesis, black holes and Scherk-Schwarz reduction, together. Our goal is to find out whether the D1/D5 black hole, as described in the previous section, remains a valid solution if we add a Scherk-Schwarz twist in the final reduction on S^1. We already studied such Scherk-Schwarz twists in some detail in section 4.2, in particular for twists in the Cartan subalgebra of the R-symmetry group.

The compactified D1/D5 black hole solution (5.28) will remain a valid solution of the compactified theory as long as non of the fields that source the black hole become massive. Because the $D = 6$ graviton is a singlet under the entire (U-duality) symmetry group, the $D = 5$ fields $G_{\mu\nu}$, $\mathcal{A}_1^5$ and ϕ_5 stay massless no matter how we choose our twist. We need to be more careful, however, with the other fields that charge the black hole. We need to choose our reduction in such a way that the $D = 5$ fields C_2, C_1, $\tilde{\Phi}_5$ and ϕ_i remain massless.

In section 4.2, we deduced the masses that the scalars and 2-forms acquire due to the Scherk-Schwarz twist. We did this by looking at their representations under the decomposition of the R-symmetry group (4.15). Therefore, we now need to identify the fields (that charge the black hole) in the R-symmetry representations, in order to find out under what conditions they remain massless.

5.3.1 Identifying fields in representations

The identification of the scalar fields in the R-symmetry representation, is done most easily from the perspective of the M-theory and the type IIA supergravity spectra. By compactifying M-theory on T^5, or equivalently by compactifying type IIA on T^4, we obtain maximal supergravity in $D = 6$. This theory can be related to type IIB on T^4 by T-duality. The field contents of each of these $D = 6$ theories is the same, but the higher-dimensional fields that they descend from are different. In the tables below, we show the origin of the scalar fields from the perspective of type IIA and M-theory. The equivalent table for type IIB is given on page 28.

IIA on T^4	G	B_2	Φ	A_1	A_3	Total
scalars	10	6	1	4	4	25

M-theory on T^5	G^M	A_3^M	Total
scalars	15	10	25

The collection of scalar fields $\phi^{a\dot{b}}$ $(a, \dot{b} = 1, \dots, 5)$ can be seen as a 5 by 5 matrix, of which the columns transform as vectors under $SO(5)_L$ and the rows transform as vectors under $SO(5)_R$. From the perspective of M-theory, 15 of the scalar fields originate from the 11-dimensional metric. As we already discussed in section 3.1.1, we obtain two types of scalar fields in the reduction of a metric: dilatons and axions. Altogether, we find 5 dilatons, 10 axions, and 10 scalars coming from the M-theory 3-form A_3^M.

In what follows, we attempt to fit these 5+10+10 scalars in the matrix $\phi^{a\dot{b}}$. Subsequently, we will deduce which fields acquire which masses from the way they are organized in this 5 by 5 structure. We would like to point out that the grouping of the scalars that we find here is substantiated mostly by it being the most 'logical' way of ordering the scalars. We assign each scalar one-to-one to a position in the matrix, but this is probably an oversimplification: the matrix elements $\phi^{a\dot{b}}$ can just as well be given by (functions of) multiple scalar fields. For the means of this thesis, we conjecture that the structure that we present in this section is the correct way of ordering at least the degrees of freedom of the scalar fields[2].

The scalar fields (from the perspective of M-theory) are embedded in the matrix $\phi^{a\dot{b}}$ in such a way that the 5 dilatons are located on the diagonal, i.e. they are the components ϕ^{11}, ϕ^{22}, ϕ^{33}, ϕ^{44}, ϕ^{55}. We choose our embedding such that the 10 axions fill up the strictly upper triangular matrix, and the 10 scalars from A_3 fill up the strictly lower triangular matrix. Schematically, the matrix $\phi^{a\dot{b}}$ then has the following structure:

$$\phi^{a\dot{b}} = \begin{pmatrix} G^M_{\text{dil}} & & & G^M_{\text{ax}} & \\ & G^M_{\text{dil}} & & & \\ & & G^M_{\text{dil}} & & \\ & A^M_3 & & G^M_{\text{dil}} & \\ & & & & G^M_{\text{dil}} \end{pmatrix} . \tag{5.39}$$

Compactifying M-theory on a circle returns type IIA supergravity. In this compactification step, the 5 dilatons (coming from G^M) decompose to the type IIA dilaton Φ and to the 4 dilatonic scalars coming from the type IIA metric G. Without loss of generality, we choose $\Phi = \phi^{11}$. The 10 axions from G^M decompose to the 6 axions coming from G and the 4 scalars coming from A_1. Similarly, the 10 scalars from A^M_3 also decompose as 4+6. From the type IIA perspective these come from A_3 and B_2 respectively. These decompositions lie in the matrix $\phi^{a\dot{b}}$ as

$$\phi^{a\dot{b}} = \left(\begin{array}{c|cccc} \Phi & & A_1 & & \\ \hline & G_{\text{dil}} & & G_{\text{ax}} & \\ A_3 & & G_{\text{dil}} & & \\ & & & G_{\text{dil}} & \\ & & B_2 & & G_{\text{dil}} \end{array}\right) . \tag{5.40}$$

Note that by making the choice $\Phi = \phi^{11}$, we have selected one row ($a = 1$) and one column ($\dot{b} = 1$) that split off from the rest of the matrix in the reduction from M-theory to type IIA. By 'splitting off', we mean that the scalar fields in that row and column descend from other type IIA fields than the ones in the rest of the matrix.

[2] In order to prove (or disprove) this conjecture, one would have to to derive a vielbein $\mathcal{V} \in SO(5,5)$ on the moduli space given in (4.14). With this vielbein at hand, the formalism of section 3.3.1 can be applied in order to find the exact mass that each of the scalar fields acquires. This (more rigorous) approach is outside the scope of this thesis.

We now map these scalars through T-duality to the ones we would have obtained from the reduction of type IIB. In terms of the type IIB fields, we find the structure of $\phi^{a\dot b}$ to be

$$\phi^{a\dot b} = \left(\begin{array}{c|cccc} \Phi & \multicolumn{4}{c}{C_0+\frac{1}{2}C_2} \\ \hline & G_{\rm dil} & & G_{\rm ax} & \\ C_4 & & G_{\rm dil} & & \\ + & & & G_{\rm dil} & \\ \frac{1}{2}C_2 & & B_2 & & \\ & & & & G_{\rm dil} \end{array}\right). \tag{5.41}$$

5.3.2 Conditions for black hole existence

In sections 4.2.4, we found that the $\mathfrak{so}(5)$ mass matrices are block diagonal matrices, consisting of one 0 and two $\mathfrak{so}(2)$ blocks. Accordingly, we found that fields transforming in the $\mathbf{5}$ of $SO(5)_L$ obtain masses proportional to 0 and $|\pm m_1 \pm m_2|$, and fields in the $\mathbf{5}$ of $SO(5)_R$ obtain masses proportional to 0 and $|\pm \tilde m_1 \pm \tilde m_2|$.

By altering our twist matrices, we can choose an embedding of the $\mathfrak{so}(2)$ blocks in the $\mathfrak{so}(5)$ matrices. Such modifications of the twist matrices are related to the specific choice that was presented in section 4.2.4 by a change of basis. The eigenvalues of the mass matrices are not affected by this choice.

By choosing an embedding of the $\mathfrak{so}(2)$'s in $\mathfrak{so}(5)_L$, we effectively designate each of the 5 masses, 0 and $|\pm m_1 \pm m_2|$, to each of the 5 values that the index $a = 1, \ldots, 5$ can take. In this way we can choose, for example, which of the 5 self-dual tensors B^a_{2+} remains massless, and which four acquire the masses $|\pm m_1 \pm m_2|$. Similarly, by choosing the embedding of the $\mathfrak{so}(2)$ blocks in $\mathfrak{so}(5)_R$, we determine which of the masses 0 and $|\pm \tilde m_1 \pm \tilde m_2|$ correspond to each of the values that the index $\dot a = 1, \ldots, 5$ can take.

In what follows, we choose the embeddings of the $\mathfrak{so}(2)$'s in $\mathfrak{so}(5)_L$ and in $\mathfrak{so}(5)_R$ in the same way. That is, we choose the embeddings such that if $a = \dot a$, then fields with an index a obtain the same mass contribution as fields with an index $\dot a$ up to the exchange of $m_1 \leftrightarrow \tilde m_1$ and $m_2 \leftrightarrow \tilde m_2$. In particular, this means that the fields $\tilde\Phi_5$ and ϕ_i (the diagonal elements of $\phi^{a\dot b}$) obtain masses 0 and $|\pm(m_1 + \tilde m_1) \pm (m_2 + \tilde m_2)|$, and that the self-dual and anti-self-dual part of C_2 obtain the same mass (up to the tildes on the mass parameters).

For these kinds of embeddings, we still have the freedom to choose which of the 5 values of the indices $a = \dot a$ is coupled to the mass equal to 0. The fields that correspond to those values of the indices are the singlets $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$, that we found in the branchings of the representations (4.17). In other words, these fields are not charged under the Scherk-Schwarz twist.

We choose our embedding in such a way that we leave the $D = 6$ field C_2 (both its self-dual part and its anti-self-dual part) uncharged under the twist. For this choice, the $D = 5$ form-valued fields C_2 and C_1 remain massless. In addition, one of the five scalars $\{\tilde{\Phi}_5, \phi_i\}$ remains massless. The other four acquire masses that are proportional to $|\pm(m_1 + \tilde{m}_1) \pm (m_2 + \tilde{m}_2)|$. We now present two different ways in which we can make sure that our black hole solution remains valid after this Scherk-Schwarz reduction:

- We tune the mass parameters such that $|\pm(m_1 + \tilde{m}_1) \pm (m_2 + \tilde{m}_2)| = 0$. This leads to the constraints $m_1 = -\tilde{m}_1$ and $m_2 = -\tilde{m}_2$. These constraints can only be satisfied by reductions that break the supersymmetry to $\mathcal{N} \in \{0, 4, 8\}$.
- We choose the black hole charges such that the massive fields, four out of the five $\{\tilde{\Phi}_5, \phi_i\}$, drop out of the solution (5.28). This can be done by choosing $Q_1 = Q_5 = Q_K$, i.e. we end up with the single charge Tangherlini black hole (or a rotating version thereof). Here we don't need any constraints on the mass parameters, so this choice is consistent with reductions that break any amount of supersymmetry: $\mathcal{N} \in \{0, 2, 4, 6, 8\}$.

5.3.3 The 3-charge black hole in $\mathcal{N} = 4$

We now pay some extra attention to the 3-charge black hole that we found in the previous section at the first dot. In particular, we study this black hole in the reduction that breaks supersymmetry to $\mathcal{N} = 4$. This black hole is unknown in previous classifications of $\mathcal{N} = 4$, $D = 5$ black holes, i.e. it is a new black hole solution.

In order to break supersymmetry to $\mathcal{N} = 4$, we need to set two of the mass parameters equal to zero, and the other two unequal to zero. Without loss of generality, we choose $m_1 = -\tilde{m}_1 = 0$ and $m_2 = -\tilde{m}_2 \neq 0$, i.e. we twist in $USp(2)_{L_2}$ and in $USp(2)_{R_2}$. This is precisely the reduction that we analyzed in section 4.2.3 under IIIB, except that we now have the additional constraint $m_2 = -\tilde{m}_2$ on the mass parameters.

The massless spectrum that we found for this reduction in section 4.2.3 consisted of a gravity multiplet coupled to a vector multiplet. We now find a larger massless field content due to the constraint $m_2 = -\tilde{m}_2$. In addition to the original multiplets, we find that 8 of the 16 scalar fields in the $(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2})$ and 2 of the 4 vector fields in the $(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})$ remain massless. These fields are charged under the Scherk-Schwarz twist, but the mass that they acquire is equal to zero. This results in 10 additional scalars and 2 additional vectors in the massless $D = 5$ field content. These fields make up exactly two vector multiplets. Therefore, the total massless field content that follows from this reduction consists of one gravity multiplet and three vector multiplets.

Chapter 6

Conclusion

In this thesis, we studied Scherk-Schwarz reductions and the string theoretical D1/D5 black hole. As we already mentioned in the introduction, a great deal of research has been done on both of these topics. Nevertheless, they have rarely been studied together. In particular, the question whether the D1/D5 black hole can be obtained from Scherk-Schwarz reductions, is yet to be answered in the academic literature. This is the question that we addressed in this thesis.

We studied the Scherk-Schwarz reduction of type IIB string theory on $T^4 \times S^1$ in section 4.2. Here the reduction on T^4 was untwisted, and we added a Scherk-Schwarz twist in the reduction on S^1. In particular, we were interested in reductions that broke the supersymmetry partially. Therefore, we chose a twist matrix in the CSA of the R-symmetry group. From the perspective of supergravity, these kinds of reductions have previously been examined in [33].

In section 5.2, we reviewed the rotating D1/D5 black hole solution. This solution is known as the BMPV black hole. It has been studied microscopically in [7] and macroscopically in e.g. [44]. We saw that, for zero angular momentum and equal charges, this solution reduces to the extremal Tangherlini black hole. Both for the general BMPV solution, and for the Tangherlini solution, we computed the macroscopic Bekenstein-Hawking entropy.

The new research from this thesis is presented in section 5.3. There, we analyzed the validity of the (rotating) D1/D5 solution after the Scherk-Schwarz reductions described in section 4.2. We found a set of constraints on the mass parameters of the twist that needs to be satisfied in order to obtain the full rotating D1/D5 black hole as a valid solution of the reduced theory. Finally, we observed that these constraints can only be satisfied by Scherk-Schwarz twists that break the supersymmetry to $\mathcal{N} \in \{0, 4, 8\}$.

The 3-charge D1/D5 black hole cannot be found as a valid solution of theories that are Scherk-Schwarz reduced to $\mathcal{N} = 2$ or $\mathcal{N} = 6$. In order to find a valid D1/D5 black hole from such reductions, the three charges must be chosen to be equal. Therefore, we obtain the single charge Tangherlini black hole (either rotating or static) in $\mathcal{N} = 2$ or $\mathcal{N} = 6$.

6.1 Outlook

A suitable topic for follow-up research is applying the more rigorous vielbein approach to the computation of the scalar masses. The method presented in this thesis is, up to some level, conjectured. In a more meticulous analysis, this conjecture should be checked.

In this thesis, we considered the macroscopic description of the D1/D5 black hole. A logical next step would be to study the effects of the Scherk-Schwarz twist on the microscopic description. An interesting question would be whether the microscopic entropy is affected by the Scherk-Schwarz twist. We already found that the macroscopic entropy is identical in both the twisted and the untwisted reductions, so we expect that the same holds for the microscopic entropy. However, for a complete description of the D1/D5 black hole from Scherk-Schwarz reduction, this needs to be checked.

In further research, the methodology from this thesis can be applied to the near-extremal version of the D1/D5 black hole. This black hole solution has been studied microscopically in [5]. The same Scherk-Schwarz twist that we used in this thesis, could also be applied to this near-extremal black hole. Does this twist preserve the validity of the near-extremal solution, as it did for the extremal one?

Alternatively, one could choose to focus on entirely different black hole solutions and check their validity in generalized reductions. Examples for such choices are the type IIA black hole constructed from D2, D6 and NS5-branes, and the M-theory black hole constructed from M2 and M5-branes. Again one could ask the question under what constraints on the Scherk-Schwarz twist these black hole solutions remain valid.

Appendix A

Group Theory

In this appendix, we discuss some properties of certain groups that we need in this thesis. First, in section A.1, we classify the conjugacy classes of $SL(2, \mathbb{R})$. Then, in section A.2, we study the way in which the subgroup $USp(2) \times USp(2)$ is embedded in $USp(4)$. Lastly, we analyze the isomorphism $USp(4) \cong Spin(5)$ in section A.3. We are particularly interested in how the generators of $USp(4)$ map to the generators of $SO(5) \cong Spin(5)/\mathbb{Z}_2$. In section 4.2.4, we choose such a generator as the mass matrix of our Scherk-Schwarz twist.

A.1 The conjugacy classes of $SL(2, \mathbb{R})$

Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an arbitrary $SL(2, \mathbb{R})$ matrix. The eigenvalues of this matrix can be found by solving $\det(A - x\mathbb{1}) = 0$, i.e. they are the roots of the characteristic polynomial

$$\begin{aligned} p_A(x) &= \det(A - x\mathbb{1}) \\ &= x^2 - (a+d)\,x + 1 \\ &= x^2 - t\,x + 1. \end{aligned} \tag{A.1}$$

Here $t = a + d$ is the trace of A. The roots of the characteristic polynomial p_A can straightforwardly be found as

$$x = \tfrac{1}{2}t \pm \tfrac{1}{2}\sqrt{t^2 - 4}. \tag{A.2}$$

We see that for $t^2 > 4$, $t^2 = 4$ and $t^2 < 4$ the matrix A respectively has 2 real, 1 real and 2 complex eigenvalues. Two $SL(2, \mathbb{R})$ matrices A and B are conjugate, if there is a matrix $C \in SL(2, \mathbb{R})$ such that $A = CBC^{-1}$. By taking the trace of this equality, we find

$$\mathrm{Tr}[A] = \mathrm{Tr}[CBC^{-1}] = \mathrm{Tr}[B]. \tag{A.3}$$

In other words, all matrices that are members of the same conjugacy class have the same trace. From (A.2), we see that the eigenvalues of an $SL(2,\mathbb{R})$ matrix only depend on its trace. Therefore, all matrices in a conjugacy class also have the same eigenvalues.

In order to classify the conjugacy classes of $SL(2,\mathbb{R})$, we distinguish between the cases $t^2 > 4$, $t^2 = 4$ and $t^2 < 4$.

I) $t^2 > 4$

For $t^2 > 4$, the matrix A has two distinct real eigenvalues λ_1 and λ_2. We denote the eigenvectors that correspond to these eigenvalues by v_1 and v_2 respectively. Now consider the invertible matrix $V = (v_1 \;\, v_2)$. With an appropriate non-zero scaling of the eigenvectors, we can make sure that this matrix has determinant 1. Then $V \in SL(2,\mathbb{R})$. We can now deduce that

$$V^{-1}A\,V \;=\; V^{-1}\left(\lambda_1 v_1 \;\; \lambda_2 v_2\right) \;=\; \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{A.4}$$

Hence, A is conjugate to diag(λ_1, λ_2). Of course, this matrix should have determinant 1, so we have $\lambda_2 = 1/\lambda_1$. Therefore, for each trace t (with $t^2 > 4$), we find a conjugacy class that can be represented by the matrix diag$(\lambda,\, 1/\lambda)$, where λ is such that $\lambda + 1/\lambda = t$. These conjugacy classes are known as the hyperbolic conjugacy classes.

II) $t^2 = 4$

In this case, the matrix A has one real eigenvalue. From (A.2), we see that this eigenvalue is either $\lambda = 1$ or $\lambda = -1$. We denote the single eigenvector that corresponds to this eigenvalue by v. Furthermore, we choose an arbitrary real vector w such that the matrix $V = (v \;\, w)$ has determinant 1. By conjugating A with this matrix V, we find

$$V^{-1}A\,V \;=\; V^{-1}\left(\lambda v \;\; w'\right) \;=\; \begin{pmatrix} \lambda & x \\ 0 & y \end{pmatrix}, \tag{A.5}$$

where $x, y \in \mathbb{R}$. Again, this matrix has determinant 1, so we know that $y = 1/\lambda$. Since we have $\lambda = \pm 1$, it follows that $y = \lambda$. For $x = 0$, these matrices are $\pm\mathbb{1}$, which are conjugacy classes on their own. For $x \neq 0$, we can 'scale' x to ± 1 with the conjugation

$$\begin{pmatrix} \sqrt{|x|} & 0 \\ 0 & 1/\sqrt{|x|} \end{pmatrix}^{-1} \begin{pmatrix} \lambda & x \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \sqrt{|x|} & 0 \\ 0 & 1/\sqrt{|x|} \end{pmatrix} \;=\; \begin{pmatrix} \lambda & x/|x| \\ 0 & \lambda \end{pmatrix}. \tag{A.6}$$

We find that there are three conjugacy classes with $t = 2$. These can be represented by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \tag{A.7}$$

There are also three distinct conjugacy classes with $t = -2$. They can be represented by the three matrices in (A.7) with a minus sign in front of them. The six conjugacy classes with $t^2 = 4$ are known as the parabolic conjugacy classes.

III) $t^2 < 4$

In the last case, for $t^2 < 4$, the matrix A has two complex eigenvalues: $\lambda_\pm = \frac{1}{2}t \pm \frac{i}{2}\sqrt{4 - t^2}$. It can easily be checked that $\left|\lambda_\pm\right| = 1$. From this, it follows that we can write $\lambda_\pm = e^{\pm i\theta}$, for some value of $\theta \in (0, \pi)$. Now let v and $\bar{v}$ be the eigenvectors that correspond to the eigenvalues λ_+ and λ_- respectively. Then

$$\begin{aligned} A\,(v + \bar{v}) &= e^{i\theta}\, v + e^{-i\theta}\, \bar{v} = \cos\theta\,(v + \bar{v}) + i\sin\theta\,(v - \bar{v}), \\ A\,i(v - \bar{v}) &= ie^{i\theta}\, v - ie^{-i\theta}\, \bar{v} = -\sin\theta\,(v + \bar{v}) + i\cos\theta\,(v - \bar{v}). \end{aligned} \tag{A.8}$$

We now conjugate with the matrix $V = \big((v + \bar{v}) \quad i(v - \bar{v})\big)$, where we again scale the eigenvectors in order to make sure that $\det(V) = 1$. We find

$$V^{-1} A\, V = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{A.9}$$

so A is conjugate to a rotation matrix. For each t (with $t^2 < 4$), we find a conjugacy class that can be represented by a rotation matrix with the angle $\theta \in (0, \pi)$ chosen such that $2\cos\theta = t$. These conjugacy classes are known as the elliptic conjugacy classes.

We have now classified all the conjugacy classes of $SL(2, \mathbb{R})$. For the derivation of the conjugacy classes of the discrete group $SL(2, \mathbb{Z})$, we refer to [45].

A.2 The embedding of $USp(2) \times USp(2) \subset USp(4)$

We choose our conventions such that elements g (in index notation: ${g_A}^B$) of $USp(4)$ can be represented by 4×4 matrices satisfying

$$g^\dagger = g^{-1}, \qquad \Omega\, g\, \Omega^{-1} = (g^{-1})^T, \tag{A.10}$$

where Ω is the symplectic metric, given by the block matrix

$$\Omega^{AB} = \begin{pmatrix} 0_{2\times 2} & \mathbb{1}_{2\times 2} \\ -\mathbb{1}_{2\times 2} & 0_{2\times 2} \end{pmatrix}. \tag{A.11}$$

As usual, we can write g in terms of a $\mathfrak{usp}(4)$ generator via the exponential map. We denote this generator by ${M_A}^B$, so that we have $g = e^M$. Using this, we can translate (A.10) to the conditions on the generators of $USp(4)$. We find

$$M^\dagger = -M, \qquad \Omega\, M\, \Omega^{-1} = -M^T. \tag{A.12}$$

In order to satisfy these conditions, we deduce that a general $\mathfrak{usp}(4)$ generator M can be written in terms of the 2×2 block matrices A and B as

$$M = \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix}, \qquad \text{where } A^\dagger = -A, \text{ and } B^T = B. \tag{A.13}$$

In a similar way, we can derive that generators of $USp(2)$ also take the form (A.13), where we take A and B to be complex numbers instead of matrices.

We can embed elements of the subalgebra $\mathfrak{usp}(2) \times \mathfrak{usp}(2)$ in elements of $\mathfrak{usp}(4)$ by choosing

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \qquad \text{where } a_1^* = -a_1, \text{ and } a_2^* = -a_2. \tag{A.14}$$

Here the complex numbers a_1 and b_1 parametrize one of the $\mathfrak{usp}(2)$'s, and the numbers a_2 and b_2 parametrize the other. It can easily be checked that for this embedding both of the $\mathfrak{usp}(2)$ subalgebras are closed under the Lie bracket (the matrix commutator).

A.3 The isomorphism $USp(4) \cong Spin(5)$

The isomorphism $USp(4) \cong Spin(5)$ can be made explicit by introducing five 4×4 Dirac matrices, that satisfy the Euclidean Clifford algebra

$$\{\Gamma_a, \Gamma_b\}_A{}^B = 2\, \delta_{ab}\, \delta_A^B. \tag{A.15}$$

Here $a, b = 1, \ldots, 5$ are the indices corresponding to $Spin(5)$, and $A, B = 1, \ldots, 4$ are the indices corresponding to $USp(4)$. An explicit basis of (Hermitian and traceless) gamma matrices, that satisfies (A.15), is given by

$$\Gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \Gamma_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{A.16}$$

It can easily be checked that the gamma matrices with upper indices, defined as $(\tilde{\Gamma}_a)^{AB} = \Omega^{AC}(\Gamma_a)_C{}^B$, are antisymmetric[1], i.e. $(\tilde{\Gamma}_a)^T = -\tilde{\Gamma}_a$. Using this, we deduce that

$$(\Gamma_a)^T = (\Omega^{-1}\tilde{\Gamma}_a)^T = -\tilde{\Gamma}_a\,(\Omega^{-1})^T = \Omega\,\Gamma_a\,\Omega^{-1}. \tag{A.17}$$

Hence, the symplectic metric Ω acts on the gamma matrices as a charge conjugation matrix. We now define $\Gamma_{ab} = \frac{1}{2}[\Gamma_a, \Gamma_b]$. From (A.17) and the Hermitian property of the Dirac matrices, it follows directly that Γ_{ab} satisfies the conditions (A.12). Furthermore, with exhaustive use of the Clifford algebra, we can show that the commutator of Γ_{ab} reads

$$\big[\Gamma_{ab}, \Gamma_{cd}\big] = -2\,\delta_{ac}\Gamma_{bd} + 2\,\delta_{ad}\Gamma_{bc} + 2\,\delta_{bc}\Gamma_{ad} - 2\,\delta_{bd}\Gamma_{ac}. \tag{A.18}$$

This is exactly the commutator of the generators of the $\mathfrak{spin}(5)$ algebra [46]. We conclude that the 10 matrices Γ_{ab} form a legitimate basis of generators of $USp(4) \cong Spin(5)$.

From the matrices defined above, we can build a basis in which we can express a generic element of the $\mathfrak{usp}(4)$ algebra. Written in terms of this basis, a matrix $P_A{}^B \in \mathfrak{usp}(4)$ is given by [39]

$$P_A{}^B = Q\,\delta_A^B + Q^a\,(\Gamma_a)_A{}^B + Q^{ab}\,(\Gamma_{ab})_A{}^B\,, \tag{A.19}$$

for a particular set of values of the coefficients Q, Q_a and Q_{ab} (indices can raised and lowered freely with the flat metric δ_{ab}). In order to find a map from the generators of $USp(4)$ to the generators of $SO(5)$, we need to invert (A.19). That is, we need to find expressions for Q, Q^a and Q^{ab} in terms of $P_A{}^B$. In order to do this, we will need the following trace identities for

[1] This property will be essential in what follows, but it is not generally true for any choice of Ω and Γ_a. The reader should be careful in choosing an appropriate basis, when using the results of this appendix.

the gamma matrices:

$$\begin{aligned} \mathrm{Tr}\,[\Gamma_a] &= 0, \\ \mathrm{Tr}\,[\Gamma_a\Gamma_b] &= 4\,\delta_{ab}, \\ \mathrm{Tr}\,[\Gamma_a\Gamma_b\Gamma_c] &= 0, \\ \mathrm{Tr}\,[\Gamma_a\Gamma_b\Gamma_c\Gamma_d] &= 4\,\delta_{ab}\,\delta_{cd} - 4\,\delta_{ac}\,\delta_{bd} + 4\,\delta_{ad}\,\delta_{bc}. \end{aligned} \tag{A.20}$$

First, we take the trace of (A.19). This results in

$$\mathrm{Tr}\,[P] = 4\,Q + Q^a\,\mathrm{Tr}\,[\Gamma_a] + Q^{ab}\,\mathrm{Tr}\,[\Gamma_{ab}] \quad \Rightarrow \quad Q = \frac{1}{4}\,\mathrm{Tr}\,[P]\,. \tag{A.21}$$

Secondly, we multiply (A.19) by Γ_c from the right, and then take the trace. We find

$$\mathrm{Tr}\,[P\,\Gamma_c] = Q\,\mathrm{Tr}\,[\Gamma_c] + Q^a\,\mathrm{Tr}\,[\Gamma_a\Gamma_c] + Q^{ab}\,\mathrm{Tr}\,[\Gamma_{ab}\Gamma_c] \quad \Rightarrow \quad Q_c = \frac{1}{4}\,\mathrm{Tr}\,[P\,\Gamma_c]\,. \tag{A.22}$$

The remaining coefficients, Q_{ab}, can be found by taking the trace after multiplying (again from the right) by Γ_{cd}. A straightforward calculation yields

$$\mathrm{Tr}\,[P\,\Gamma_{cd}] = Q\,\mathrm{Tr}\,[\Gamma_{cd}] + Q^a\,\mathrm{Tr}\,[\Gamma_a\Gamma_{cd}] + Q^{ab}\,\mathrm{Tr}\,[\Gamma_{ab}\Gamma_{cd}] \quad \Rightarrow \quad Q_{cd} = -\frac{1}{8}\,\mathrm{Tr}\,[P\,\Gamma_{cd}]\,. \tag{A.23}$$

These expressions for Q, Q_a and Q_{ab} map the **1**, the **5** and the **10** of $USp(4)$ to those of $SO(5)$ [39]. The generators ${M_A}^B$ sit in the **10** of $USp(4)$, so we can use the map (A.23) to find the corresponding generators of $SO(5)$. We denote these generators by M_{ab}. The isomorphism between the $USp(4)$ and the $SO(5)$ generators is given by[2]

$$M_{ab} = -\frac{1}{2}\,\mathrm{Tr}\big[{M_A}^B(\Gamma_{ab})_B{}^C\,\big]\,. \tag{A.24}$$

The special orthogonal Lie algebra $\mathfrak{so}(5)$ consists of real antisymmetric matrices. In order to verify that (A.24) indeed gives a proper $SO(5)$ generator, we will check these two properties. The antisymmetry $M_{ab} = -M_{ba}$ follows directly from (A.24), because by definition $\Gamma_{ab} = -\Gamma_{ba}$. To prove the reality of M_{ab}, we recall that both ${M_A}^B$ and $(\Gamma_{ab})_A{}^B$ satisfy the conditions (A.12). Therefore, we have

$$\begin{aligned} M^* &= \Omega\,M\,\Omega^{-1}, \\ (\Gamma_{ab})^* &= \Omega\,\Gamma_{ab}\,\Omega^{-1}, \end{aligned} \tag{A.25}$$

[2] The prefactor $\frac{1}{2}$ instead of $\frac{1}{8}$ is necessary to ensure that the isomorphism $SO(5) \cong USp(4)/\mathbb{Z}_2$ is bijective.

which allows us to compute

$$\begin{aligned}(M_{ab})^* &= -\frac{1}{8}\,\mathrm{Tr}\big[M^*(\Gamma_{ab})^*\big]\\ &= -\frac{1}{8}\,\mathrm{Tr}\big[\Omega\, M\, \Omega^{-1}\,\Omega\,\Gamma_{ab}\,\Omega^{-1}\big]\\ &= -\frac{1}{8}\,\mathrm{Tr}\big[M\,\Gamma_{ab}\big] \;=\; M_{ab}.\end{aligned} \tag{A.26}$$

Thus we find that M_{ab}, as given in (A.24), is a real antisymmetric matrix, and therefore a suitable generator of $SO(5)$. Note that this argument builds on the antisymmetric property of $\tilde{\Gamma}_a = \Omega\,\Gamma_a$. Consequently, it would not hold for a choice of Ω and Γ_a for which $\tilde{\Gamma}_a$ is not antisymmetric.

References

[1] L. Susskind. *The Black Hole War*. Little, Brown & Company, 2009.

[2] S.W. Hawking. Particle Creation by Black Holes. *Communications in Mathematical Physics*, 43(3):199–220, 1975.

[3] J.D. Bekenstein. Black Holes and Entropy. *Physical Review D*, 7(8):2333, 1973.

[4] A. Strominger and C. Vafa. Microscopic Origin of the Bekenstein-Hawking Entropy. *Physics Letters B*, 379(1-4):99–104, 1996.

[5] C.G. Callan Jr and J.M. Maldacena. D-brane Approach to Black Hole Quantum Mechanics. *Nuclear Physics B*, 472(3):591–608, 1996.

[6] G.T. Horowitz and A. Strominger. Counting States of Near-Extremal Black Holes. *Physical Review Letters*, 77(12):2368, 1996.

[7] J.C. Breckenridge, R.C. Myers, A.W. Peet, and C. Vafa. D-branes and spinning black holes. *Physics Letters B*, 391(1):93–98, 1997.

[8] C.V. Johnson, R.R. Khuri, and R.C. Myers. Entropy of 4D Extremal Black Holes. *Physics Letters B*, 378(1-4):78–86, 1996.

[9] J.M. Maldacena and A. Strominger. Statistical Entropy of Four-Dimensional Extremal Black Holes. *Physical Review Letters*, 77(3):428, 1996.

[10] J.M. Maldacena, A. Strominger, and E. Witten. Black Hole Entropy in M-Theory. *Journal of High Energy Physics*, 1997(12):002, 1998.

[11] J. Scherk and J.H. Schwarz. Spontaneous Breaking of Supersymmetry Through Dimensional Reduction. *Physics Letters B*, 82(1):60–64, 1979.

[12] J. Scherk and J.H. Schwarz. How to Get Masses from Extra Dimensions. *Nuclear Physics B*, 153:61–88, 1979.

[13] N. Gaddam, A. Gnecchi, S.J.G. Vandoren, and O. Varela. Rholography, Black Holes and Scherk-Schwarz. *Journal of High Energy Physics*, 2015(6):58, 2015.

[14] C.M. Hull and S.J.G. Vandoren. Notes on Black Holes ans Scherk-Schwarz. manuscript, 2018.

[15] C.V. Johnson. *D-Branes*. Cambridge university press, 2002.

[16] K. Becker, M. Becker, and J.H. Schwarz. *String Theory and M-Theory: A Modern Introduction*. Cambridge University Press, 2006.

[17] J. Polchinski. *String Theory: Volume 1, An Introduction to the Bosonic String*. Cambridge university press, 1998.

[18] J. Polchinski. *String Theory: Volume 2, Superstring Theory and Beyond*. Cambridge university press, 1998.

[19] T. Ortín. *Gravity and Strings*. Cambridge University Press, 2004.

[20] R. Blumenhagen, D. Lüst, and S. Theisen. *Basic Concepts of String Theory*. Springer Science & Business Media, 2012.

[21] D.Z. Freedman and A. Van Proeyen. *Supergravity*. Cambridge University Press, 2012.

[22] M.J.D. Hamilton. The field and Killing spinor equations of M-theory and type IIA/IIB supergravity in coordinate-free notation. *arXiv preprint arXiv:1607.00327*, 2016.

[23] J. Polchinski. Dirichlet Branes and Ramond-Ramond Charges. *Physical Review Letters*, 75(26):4724, 1995.

[24] A.W. Peet. TASI Lectures on Black Holes in String Theory. In *Strings, branes and gravity*, pages 353–433. World Scientific, 2001.

[25] C.N. Pope. *Kaluza-Klein Theory*. Centre Emile Borel, Institut Henri Poincaré, 2000.

[26] T.H. Buscher. A symmetry of the string background field equations. *Physics Letters B*, 194(1):59–62, 1987.

[27] E. Bergshoeff, C. Hull, and T. Ortín. Duality in the Type-II Superstring Effective Action. *Nuclear Physics B*, 451(3):547–575, 1995.

[28] C.M. Hull and P.K. Townsend. Unity of Superstring Dualities. *Nuclear Physics B*, 438(1-2):109–137, 1995.

[29] E. Cremmer, B. Julia, H. Lü, and C.N. Pope. Dualisation of Dualities I. *Nuclear Physics B*, 523(1-2):73–144, 1998.

[30] E. Cremmer, B. Julia, H. Lü, and C.N. Pope. Dualisation of Dualities II: Twisted self-duality of doubled fields and superdualities. *Nuclear Physics B*, 535(1-2):242–292, 1998.

[31] A. Dabholkar and C. Hull. Duality Twists, Orbifolds, and Fluxes. *Journal of High Energy Physics*, 2003(09):054, 2003.

[32] A. Çatal-Özer. *Scherk-Schwarz Reductions of Effective String Theories in Even Dimensions*. unpublished doctoral thesis, 2003.

[33] L. Andrianopoli, S. Ferrara, and M.A. Lledó. No-scale $D = 5$ supergravity from Scherk-Schwarz reduction of $D = 6$ theories. *Journal of High Energy Physics*, 2004(06):018, 2004.

[34] Y. Tanii. $N = 8$ Supergravity in Six dimensions. In *Supergravities in Diverse Dimensions: Commentary and Reprints (In 2 Volumes)*, pages 368–371. World Scientific, 1989.

[35] B. de Wit, H. Samtleben, and M. Trigiante. The maximal D=5 supergravities. *Nuclear physics B*, 716(1-2):215–247, 2005.

[36] E. Cremmer. Supergravities in 5 Dimensions. In *Supergravities in Diverse Dimensions: Commentary and Reprints (In 2 Volumes)*, pages 422–437. World Scientific, 1989.

[37] G. Villadoro. $\mathcal{N} = 6$ gauged supergravities from generalized dimensional reduction. *Physics Letters B*, 602(1-2):123–129, 2004.

[38] M. Awada and P.K. Townsend. $N = 4$ Maxwell-Einstein supergravity in five dimensions and its SU(2) gauging. *Nuclear Physics B*, 255:617–632, 1985.

[39] G. Villadoro and F. Zwirner. The minimal $\mathcal{N} = 4$ no-scale model from generalized dimensional reduction. *Journal of High Energy Physics*, 2004(07):055, 2004.

[40] A. Ceresole and G. Dall'Agata. General matter coupled $\mathcal{N} = 2$, $D = 5$ gauged supergravity. *Nuclear Physics B*, 585(1-2):143–170, 2000.

[41] S.M. Carroll. *Spacetime and Geometry. An Introduction to General Relativity*. 2004.

[42] J.M. Maldacena. Black Holes in String Theory. *arXiv preprint hep-th/9607235*, 1996.

[43] J.P. Gauntlett, R.C. Myers, and P.K. Townsend. Black Holes of D=5 Supergravity. *Classical and Quantum Gravity*, 16(1):1, 1999.

[44] C.A.R. Herdeiro. Special properties of five-dimensional BPS rotating black holes. *Nuclear Physics B*, 582(1-3):363–392, 2000.

[45] O. DeWolfe, T. Hauer, A. Iqbal, and B. Zwiebach. Uncovering the Symmetries on [p,q] 7-branes: Beyond the Kodaira Classification. *Adv. Theor. Math. Phys.*, 3:1785–1833, 1999.

[46] C. Lüdeling. Group Theory for Physicists. *Lecture Notes, Bonn*, 2010.