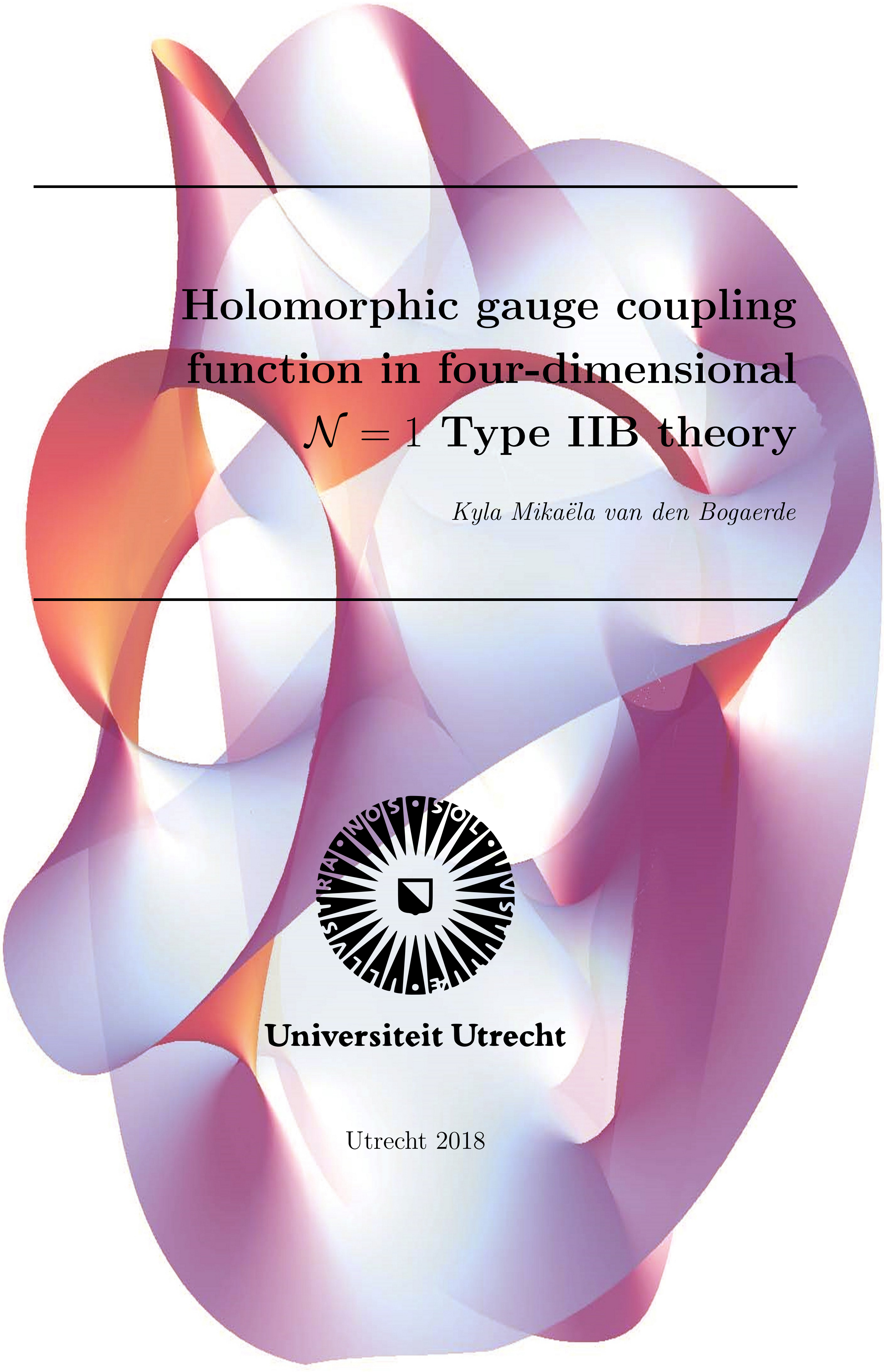

Holomorphic gauge coupling function in four-dimensional $\mathcal{N} = 1$ Type IIB theory

Kyla Mikaëla van den Bogaerde

Universiteit Utrecht

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Abstract

Using a Kaluza-Klein reduction, we determine the bosonic part of the $d = 4$, $\mathcal{N} = 1$ effective theory resulting from the ten-dimensional Type IIB supergravity action on a general compact Calabi-Yau orientifold, allowing for $O3/O7$-planes. Including a single spacetime filling D7-brane wrapped on a four-cycle, we consider the low-energy limit. We do not specify a specific orientifold or four-cycle on which the D7-brane is wrapped. However, using the general geometry of these objects, we give a detailed discussion regarding how to construct the $\mathcal{N} = 1$ gauge kinetic coupling function. In doing so, we extensively carry out the dualization procedure for the gauge fields, giving a clear outlook of how one should handle this technical aspect. By taking into consideration the Wilson line moduli arising from the higher-dimensional gauge vector on the brane, we obtain mixed kinetic interactions between the bulk and the brane gauge vectors. Furthermore, these Wilson lines give rise to an additional term in the D7-brane gauge interaction. We will emphasize on how the addition of Wilson lines alters the gauge kinetic coupling function and discuss how this effects its holomorphic property. As imposed by the $\mathcal{N} = 1$ supersymmetric representation, we show that indeed the bulk gauge coupling function is holomorphic in the chiral superfields. As a new result, we will show via M-theory that the mixed gauge kinetic coupling function is holomorphic in the complex structure moduli. As of yet, this result has not been presented anywhere in the academic literature. In addition, we argue the existence of two specific relations which together imply that the D7-brane gauge kinetic coupling function is holomorphic in the $\mathcal{N} = 1$ coordinates.

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Contents

Chapter 1

Introduction

The first section of this chapter is devoted to family and friends, to provide them with a brief historic introduction leading to what motivates this field of research. We continue with the main fundamentals underlying this thesis in string theory. After discussing the basic principles, we highlight how our work relates to other studies and comment on the setup we will analyze. Emphasizing on the aim of the research, we will briefly discuss our strategy. We conclude with the organization of this study.

1.1 A look back in history

For centuries, mankind has been intrigued by the world around us. Observing and learning from physical phenomena, theories have been developed. Starting in ancient Greece, where Plato and his student Aristotle laid out the basic principles of physics. It was Aristotle who first wrote down the thought that physical phenomena should lead to the laws of nature governing them. Hence, studying these observations could lead to discovering these laws. A field of study that he referred to in his work as "physics". The theory he developed contained the four elements; earth, air, water and fire, with which he attempted to explain phenomena like motion and gravity. Until the end of the Medieval Period Aristotle's work remained the mainstream theory in Europe. But approaching the era in which Galileo Galilei and thereafter Isaac Newton came into the picture things started to change.

Inspired by Nicolaus Copernicus' thoughts about the heliocentric model of the Solar System that the Earth revolves around the Sun, Galileo attempted to mathematically describe motion not only to support heliocentrism, but also other mechanical experiments. At the age of only 19, he discovered how the period of a pendulum is independent of the amplitude of the swing. Other experiments, enlightened how the path of a projectile is shaped in the form of a parabola or that the velocity with which bodies fall is not proportional to their weights. The experiments Galileo carried out and the results obtained were so contradictory to the theory Aristotle had developed, that a completely new theory of physics arose. This was the starting point of Newtonian mechanics, also known as classical mechanics, a theory which describes the macroscopic motion of objects, funded by Newton's laws of motion. On top of that Newton started an entirely new field within mathematics called calculus, which is still one of the most known and commonly used branches of mathematics today.

It is a long way from this changing point in physics that Newton and Galilei created to the modern physics studied today. A journey that is an extraordinary intellectual achievement to which several great researchers have contributed. To mention only a few of the most memorable findings; it was Michael Faraday who discovered the electromagnetic induction in 1831 or Wiliam Thomson, better known as Lord Kelvin, who formulated the first two laws of thermodynamics. Not to mention James Clerk Maxwell who was able to mathematically substantiate the experimental observations of Faraday and lay the foundation for electromagnetism. This theory, with in particular the establishment of the speed of light and the symmetry transformations of Maxwell's equations, led Albert Einstein to the development of special relativity, which he later extended to general relativity. Einstein is probably the most known scientist that has ever lived. In 1930 he began his work on a so-called unified theory. Just as Maxwell had provided a framework that unifies electric and magnetic phenomena, Einstein was convinced that gravity could be unified with the electromagnetic force, leading to a theory of all fundamental forces. He spent the rest of his live chasing this unified theory.

Nowadays, this hope of finding a "theory of everything" that constitutes a framework for all fundamental forces is still shared. However, instead of only considering gravity and the electromagnetic force, we have included the weak and strong nuclear interactions as fundamental forces. Hence, adding up to four fundamental forces, the three gauge interactions; the electromagnetic, weak and strong force are already brought

together in the same framework of quantum field theories. Even more, the Standard Model of particle physics unifies the electromagnetic and the weak force, which in itself is an extraordinary accomplishment. The experimental evidence underlying this theory has so extensively been tested at the LHC that the success is compelling.
The downside of all this is that the mathematical framework that has been proven so suitable at the subatomic scale to develop these quantum field theories is not applicable to the gravitational force. This is mainly due to the fact that the Einstein-Hilbert action is non-renormalizable. Even though the principles of quantum mechanics have not yet been reconciled with the gravitational force, Einstein's general relativity does provide a very well tested theory, on length scales of the Solar System, describing gravity. Though it cannot be completed to a theory of quantum gravity. In principle, any consistent quantum field theory that reduces to general relativity in the classical limit would qualify for a theory of quantum gravity.
One can imagine that combining two theories, of which on the one hand one describes phenomena on subatomic length scales ($\sim 10^{-19}m$) at the LHC, while on the other hand the second describes observables at the scale of the universe ($\sim 10^{27}m$), can be quite a tough task. However, despite all struggles yet to be overcome one of the leading candidates for setting a framework containing all four fundamental forces is string theory.

1.2 The basic concepts of string theory

The underlying fundamental thought behind string theory is quite elegant. Instead of considering interacting particles as points, we consider interacting strings propagating through space and time. The spin-2 metric field associated with the graviton is one of the infinitely many string excitations. This graviton is the elementary particle that mediates the gravitational force. All particles observed in nature should arise via the same mechanism in string theory. At low-energy scales string theory can be described as a field theory called supergravity. The two essential ingredients of supergravity are gravity and supersymmetry. In physics, two classes of particles are distinguished. An elementary particle can be either a fermion or a boson. Fermions are half-integer spin particles, which are described by spinors in relativistic field theories and cannot occupy the same quantum state. These particles constitute all matter we know of. Fields of integer spin describe bosons. These in contrast have the property that they can occupy the same state. Bosons are the mediators of forces. Supersymmetry relates these two classes in a one-to-one correspondence such that each fermion has its bosonic superpartner and vice versa. The superpartner of the graviton is the spin-$\frac{3}{2}$ particle named the gravitino. The number of copies of this gravitino in a supersymmetric theory is denoted with $\mathcal{N}$. This contributes to the number of supercharges, which is determined by multiplying the number of degrees of freedom of a d-dimensional spinor by $\mathcal{N}$. Therefore, every supergravity is characterized by two things; the number of dimensions of the theory d and the number of copies of the gravitini $\mathcal{N}$.

In this thesis we consider the low-energy description of string theories which gives rise to higher-dimensional supergravity actions. More specifically, we will discuss the ten-dimensional Type IIB superstring theory and the eleven-dimensional supergravity action following from M-theory in the low-energy limit at weak string coupling. We will restrict ourselves to the bosonic fields since supersymmetry relates this part of the theory in a one-to-one correspondence to the fermionic part of the action. Instead of only including the one-dimensional strings, one could also wonder about including higher-dimensional generalizations of the string. These dynamical objects are called Dp-branes, which are (p+1)-dimensional hypersurfaces embedded in the higher-dimensional theory. They couple to the Ramond-Ramond p-form potentials $C^{(p)}$. A p-form is an antisymmetric tensor forming a linear map from p vectors to the real numbers.
However, there is a mismatch between these higher-dimensional theories and the world we observe around us, since we only see three space dimensions and one time dimension. Nonetheless, that we do not observe any other dimensions does not necessarily mean they are not present. A thought of Kaluza [1] was to introduce a fifth dimension, which is so small it cannot be detected with the current accessible energies reached in modern day physics. This proposal seemed conceivable to Klein, who five years later in 1926 made an attempt to further elaborate on the idea [2]. The term Kaluza-Klein compactification is a widely used technique in string theory, which suites as a strategy to defend the unobservability of the extra dimensions and which we will apply extensively throughout this work. Therefore, we now elaborate on it.

Lets say for the moment we start from a D-dimensional supergravity theory, from which we seek to obtain the lower-dimensional effective field theory. The equations of motion, for a p-form potential with field strength $F^{(p+1)} = \mathrm{d}C^{(p)}$, following from the higher-dimensional supergravity action of massless fields result in

$$\Delta_D C^{(p)} = 0, \tag{1.2.1}$$

after gauge fixing. We denoted the Laplacian operator as Δ_D, acting on a p-form in D-dimensions, which we will discuss in section 2.1.1. We consider our observable four dimensions to be Minkowski spacetime. Therefore, splitting spacetime as $M_{(1,D-1)}(x,y) = \mathbb{R}^{1,3}(x) \times M_{D-4}(y)$, with $M_{D-4}(y)$ denoting the unobservable dimensions of a compact manifold expressed in local coordinates y, tells us that the Laplacian operator decomposes as

$$\Delta_D = \Delta_{1,3} + \Delta_{D-4}. \tag{1.2.2}$$

If we now also expand the p-form potential according to the decomposition of spacetime

$$C^{(p)} = \sum_{q+r=p<D-1} A^{(q)}(x) \wedge B^{(r)}(y) \qquad \text{with} \qquad \begin{array}{c} q = 0, 1, ..., 3, \\ r = p-3, ..., p \end{array} \tag{1.2.3}$$

in the forms $A^{(q)}$ on Minkowski spacetime and $B^{(r)}$ on the internal manifold, whose dimensions are hidden, we obtain information about the mass of the effective lower-dimensional fields

$$(\Delta_{1,3} A^{(q)}(x)) \wedge B^{(r)}(y) = -A^{(q)}(x) \wedge (\Delta_{D-4} B^{(r)}(y)). \tag{1.2.4}$$

Recall from field theory that the mass of a field $\phi(x)$ is given by $\Delta\phi(x) = \partial^\mu \partial_\mu \phi(x) = m^2 \phi(x)$, therefore the mass of the lower-dimensional q-form potential field $A^{(q)}$ is reflected by (1.2.4). However, since the mass of even the lightest massive four-dimensional fields is already of the order of the energy needed to measure the extra dimensions (which are unobservable), only massless modes are kept in the reduction. This means that the right hand side of (1.2.4) should vanish, which is exactly why

we expand the higher-dimensional fields in harmonics on the internal manifold. We will come back to this in section 2.1.1.
Shortly summarizing the above, when compactifying a theory from D to four dimensions, the D-dimensional fields have to be expanded into the zero mode eigenfunctions of the kinetic operator in the internal space, such that the effective lower-dimensional fields are massless. The interaction coefficients in the effective theory between these four-dimensional fields are determined by matrices resulting from the reduction of the internal space. Therefore, the choice of this manifold one compactifies on is highly non-trivial for the lower-dimensional effective theory obtained from the reduction.

What we did not take into account yet is how to cope with the (p+1)-dimensional hypersurfaces, stretching over a submanifold of the higher-dimensional spacetime. Since we will consider a spacetime filling Dp-brane, exactly (p-3)-dimensions of the brane extend in the internal directions. Therefore, when compactifying this part of the brane is wrapped on a cycle of the internal manifold. Thus, not only is the internal manifold crucial for the lower-dimensional interaction coefficients, but also for the allowed subspace on which the brane can be wrapped.

A commonly used choice, which we will also work with in this thesis, is to preform a Kaluza-Klein reduction on a Calabi-Yau manifold. In such a reduction, one expands all $\mathcal{N}$ higher-dimensional gravitini $\hat{\eta}$ schematically according to

$$\hat{\eta} = \sum_{i=1}^{M} \eta_i \, \xi_i, \tag{1.2.5}$$

where M denotes the number of Killing spinors ξ_i on the internal manifold and η_i are the lower-dimensional gravitini. Therefore, it follows that the total number of gravitini in the lower-dimensional theory is given by $M \cdot \mathcal{N}$.
Due to the property that a Calabi-Yau manifold allows for only one Killing spinor, the lower-dimensional theory results in the same number of gravitini $\mathcal{N}$ as the higher-dimensional theory started from. Thus, when reducing a $\mathcal{N} = 2$ supersymmetric ten-dimensional theory on a Calabi-Yau threefold, supersymmetry is broken from $\mathcal{N} = 8$ to $\mathcal{N} = 2$ in four dimensions.
Generally, including orientifold planes to a certain configuration of D-branes is required to cancel Ramond-Ramond tadpoles. The orientifold projection truncates the

four-dimensional $\mathcal{N} = 2$ sypersymmetric spectrum to $\mathcal{N} = 1$ by keeping precisely those states invariant under the projection.

1.3 The scope of the thesis

The amount of supersymmetry is an important phenomenological ingredient. With the current developments in research, the most promising choice of supersymmetry seems to be minimal $\mathcal{N} = 1$ supersymmetry in four dimensions. Even more, extended supersymmetry in four-dimensional theories does not allow for observed chiral fermions of the Standard Model [3, 4]. However, exact supersymmetry is not a correct theory of nature. If supersymmetry exists as an exact symmetry we should have already observed it in experimental data of particle physics. Though it may be possible that supersymmetry is non-linearly realized (i.e. spontaneously broken) in physics, meaning that it is spontaneously broken somewhat similar to the Higgs mechanism. Quite some studies have yet exploited this field of research [5–8]. Specifically within string theory, a promising set-up seems to be an orientifold bulk theory with supersymmetry $\mathcal{N} = 1$ and D-branes compatible to this supersymmetry. By including background fluxes to the bulk, supersymmetry is sponteneously broken [9–15].

It is necessary to derive the lower-dimensional $\mathcal{N} = 1$ theory in the low-energy limit in order to reliably deduce information about the supersymmetry breaking of such a theory. Therefore, in this work we confine our attention to the $d = 4$, $\mathcal{N} = 1$ low-energy supergravity action resulting from a Kaluza-Klein reduction on a general Calabi-Yau orientifold of the democratic version of the ten-dimensional $\mathcal{N} = 2$ supergravity action of Type IIB theory at tree level. We work in the weak string coupling limit, since we consider the tree level supergravity action.

Introducing D-branes to the theory can be studied from many different perspectives, for instance to improve the relation between string theory and cosmology in which branes have been incorporated to the theory to exploit cosmic inflation [16–18]. From the particle physics point of view, in general several spacetime filling D-branes are positioned in such a way that certain particles arise, for instance to give rise to models similar to the Standard Model [19–22]. If instead of a single spacetime filling

D-brane, which gives rise to a lower-dimensional $U(1)$ Abelian gauge theory, a stack of N spacetime filling D-branes is included to the theory, the $U(1)$ gauge group is enhanced to a non-Abelian $U(N)$ group. Thus, stacking spacetime filling D-branes gives rise to non-Abelian gauge theories with charged matter multiplets. Whenever the D-brane is wrapped on a cycle which allows for non-trivial one-cycles, Wilson line moduli are added to the theory.

In this thesis we will cope with the perspective which improves the relation between string theory and particle physics. For simplicity we consider a $U(1)$ Abelian gauge theory by including a single spacetime filling D7-brane that respects the supersymmetry of the bulk, wrapped on a (2,2)-cycle of the Calabi-Yau orientifold. In terms of the supergravity action, including a D7-brane implies including the Dirac-Born-Infeld and the Chern-Simons action to the supergravity bulk part of the action

$$S = S_{Bulk} + S_{DBI} + S_{CS_{D7}}. \tag{1.3.1}$$

As stated before one has to include orientifold planes, to a certain configuration of D-branes, to cancel Ramond-Ramond tadpoles [23–27]. Even though, in this work we do take into account an orientifold projection, in this specific model the configuration is not set in such a way that the orientifold and the D-brane cancel out each others tadpoles. Nevertheless, we are interested in the orientifold projection due to the fact that it reduces supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ in the compactification on the Calabi-Yau orientifold, which is preferred to obtain an effective theory closer to the phenomenology of particle physics. To give an example, apart from the chirality mentioned earlier, as the name already states the Minimal Supersymmetric Standard Model is an extension to the Standard Model that realizes $\mathcal{N} = 1$ supersymmetry. It constitutes a complete field of research [28–30].

In a $d = 4$, $\mathcal{N} = 1$ effective theory, the dynamics of both the gauge vectors arising from the brane and the Ramond-Ramond vector fields crucially depends on the gauge kinetic coupling function. Inherent to the $\mathcal{N} = 1$ supersymmetry representation, the gauge kinetic coupling function should be holomorphic in the chiral superfields treated as complex variables [31]. The purpose of this work is to show that the given set-up indeed yields a gauge kinetic function that is holomorphic in the complex scalars arising from the bosonic part of the chiral multiplets.

Without specifying the orientifold, apart from that it allows for $O3/O7$-planes, nor the (2,2)-cycle on which the brane is wrapped, except that we assume it to include non-trivial one-cycles, we will derive the gauge kinetic coupling function using the general geometry of the manifolds. Very similar analyses have been performed in [32, 33] or for D5-branes [34]. The work in [32] thoroughly carries out this derivation, neglecting the Wilson lines arising on the brane. However, since the Wilson lines are part of the $\mathcal{N}=1$ coordinates, the gauge coupling function should be holomorphic in these as well. For this reason we will be particularly interested in how the addition of these scalar fields influences the gauge kinetic coupling function. Therefore, we will derive the lower-dimensional theory resulting from our set-up in which we emphasize on the parts of the reduction contributing to the gauge coupling. We will stay close to the analysis performed in [32], but deviate from it towards the end of the reduction in which we define the $\mathcal{N}=1$ coordinates. The reason for this is that the result for the gauge coupling function including the Wilson lines is stated at the end of [32]. Though interestingly, they already mention the result does not seem to be holomorphic in the chiral coordinates. As pointed out in [33] there seems to be a mismatch regarding the Wilson line moduli, which is recovered at the open string one-loop level [35]. In addition to this, there seems to be a second problem regarding one of the derived $\mathcal{N}=1$ coordinates. In the presence of Wilson line moduli, one of these coordinates seems to be in conflict with the holomorphic property imposed by supersymmetry. This has recently been pointed out in [36], where a set of slightly different chiral coordinates is proposed. Therefore, in this thesis we will carefully try to combine the knowledge of these recent studies to work towards the point of showing that, in the presence of Wilson line moduli, the gauge kinetic coupling function is holomorphic in the $\mathcal{N}=1$ coordinates.

Extensive discussions of the holomorphic property of the kinetic coupling among the bulk gauge vector fields have yet been included in [32, 34]. Therefore, using an appropriate set of $\mathcal{N}=1$ coordinates, our main attempt is to show the mixed gauge kinetic coupling function and the gauge coupling amongst the D7-brane $U(1)$ vectors are holomorphic in the chiral superfields as well.

Our strategy will be to perform a Kaluza-Klein compactification on a Calabi-Yau orientifold including a single spacetime filling D7-brane such that we reduce the Type

IIB supergravity action, yielding a $d = 4$, $\mathcal{N} = 1$ supergravity. Just as in [32] we will take into account all terms up to second order in derivatives. By paying extra attention to those parts contributing to the gauge kinetic coupling function and extensively performing the dualization procedure inherent to starting from the democratic action, we carefully construct the gauge coupling function. We will discuss how the addition of Wilson lines has altered the gauge coupling. As a new way of approaching the objective, we will use an appropriate set of $\mathcal{N} = 1$ coordinates to express the D7-brane gauge coupling function in terms of the chiral coordinates in an attempt to manifestly show it is holomorphic. However, two mathematical identities should hold in order to obtain this manifestly holomorphic result. We present a possible approach to prove these equations, though we are unable to give a mathematically correct proof.
Additionally, we briefly review the holomorphic property of the gauge kinetic coupling function resulting from the bulk part of the action.

At last, to show that the mixed gauge kinetic coupling function is also holomorphic in the chiral superfields, we will shed light on this with an approach not yet presented elsewhere. We take an alternative route via M-theory in the weak coupling limit. Therefore, the reduction of M-theory on a general Calabi-Yau fourfold, leading to an $\mathcal{N} = 2$ supersymmetric three-dimensional theory, is explained to obtain the mixed gauge kinetic coupling function following from this theory. In order to compare the result with the mixed gauge kinetic coupling function obtained from the Type IIB compactification, we adjust the reduction performed for M-theory by considering an elliptically fibered Calabi-Yau fourfold and the weak string coupling limit when lifting to four dimensions, such that it suites our tree level Type IIB perspective. By showing the equivalence between the mixed gauge coupling obtained in M-theory and the one obtained in Type IIB, in combination with the fact that the mixed gauge kinetic coupling function is holomorphic in M-theory, we conclude it must be holomorphic in Type IIB theory as well.

1.4 The outline of the thesis

Specifically, the organization of this work is as follows. To become familiar with all mathematical definitions and conventions we start with the main basic properties of several manifolds in chapter 2. Starting briefly with real manifolds in section 2.1 including the differential geometry defined on them explained in sections 2.1.1-2.1.3. We will continue with complex manifolds in section 2.2 to build towards the Dolbeault cohomology discussed in 2.2.1, which will be of importance for the dependence of spacetime fields on the complex structure later on. After obtaining the basic notion of a real and complex manifold we can move toward the Kähler and Calabi-Yau manifold, respectively in sections 2.3 and 2.4. Both will have an important role in this work. We will compactify on a Calabi-Yau manifold in chapters 4 and 6. But before reaching the compactifications, first the moduli space of a Calabi-Yau threefold is discussed in chapter 3, which contains the geometry of the manifold and splits into two Kähler manifolds. We will elaborate on the Kähler class moduli space in section 3.1 and setup the framework for the complex structure moduli space in section 3.2.
In chapter 4 we define the orientifold projection and start the compactification of the ten-dimensional $\mathcal{N} = 2$ supergravity Type IIB action on a general Calabi-Yau orientifold. Thereby, analyzing the $\mathcal{N} = 1$ spectrum that results from the orientifold reduction. In chapter 5 we extend this by including the D7-brane action up to tree level and define the (2,2)-cycle on which the brane is wrapped. To reduce these terms of the action we include the spectrum of the $U(1)$ gauge vector on the brane and the normal coordinates, since the brane itself is a dynamical object, in section 5.1. First reducing the gauge vector part of the Chern-Simons action in detail in section 5.2, whereafter briefly reducing the rest of the action including the Dirac-Born-Infeld action in section 5.3. Since the considered supergravity action is formulated in the democratic version, the duality relations have to be imposed upon the reduced action. This dualization will be preformed in section 5.4. To eventually reach the conclusion that the gauge kinetic coupling function is holomorphic in the chiral fields, we define the $\mathcal{N} = 1$ chiral coordinates and vector multiplets in section 5.5 after which we give an argument showing the holomorphic property of the D7-brane gauge coupling function. To show that the mixed gauge kinetic coupling function is holomorphic, we first reduce M-theory on a general Calabi-Yau fourfold, in chapter 6. In order to compare the the mixed gauge kinetic coupling function obtained from M-theory

with the Type IIB result, we consider an elliptically fibered Calabi-Yau fourfold and the weak string coupling limit when lifting to four dimensions, in chapter 7. In this way we will show that the mixed gauge kinetic coupling between the bulk and the D7-brane gauge vectors must be holomorphic in the chiral superfields.

Chapter 2

Manifolds and differential geometry

Laying part of the mathematical foundation, we will start this thesis with a general introduction on certain types of manifolds and the differential geometry defined on these manifolds. We will only consider smooth Riemannian manifolds.

2.1 Real manifold

A real n-dimensional differential manifold is a topological space that locally looks like Euclidean space, meaning that coordinates and functions act in a similar way. This does not mean the metric is the same. It should be possible to construct a manifold by smoothly sewing together these locally flat regions, called patches. More precisely, the union of patches, U_i, is equal to the manifold, M. And every patch can be mapped one-to-one on Euclidean space, $\phi_i : U_i \to \mathbb{R}^n$ such that $\phi(U_i)$ is open in $\mathbb{R}^n$. The composite function, $\phi_i\phi_j^{-1}$, of any two overlapping patches, $U_i \cap U_j \neq \emptyset$, is smooth.
Before going into detail about other manifolds, we first want to use this definition of a manifold to introduce a special class of tensors defined on such a manifold known as differential forms, accompanied with some properties.

2.1.1 Differential forms

Differential p-forms are totally antisymmetric tensors of rank p, that form a linear map from p vectors to the real numbers, at a certain point on a manifold M. The definition of a p-form, $C^{(p)}$, in terms of local coordinates, $\{x^{\mu_i}\}$, on an n-dimensional manifold M is given by [37, 38]

$$C^{(p)} = \frac{1}{p!} C_{\mu_1,...,\mu_p} \mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_p}, \qquad \text{with} \qquad \mu_i = 1, ..., n \tag{2.1.1}$$

where $\wedge$ denotes the exterior product and we have used Einstein summation convention. Due to the antisymmetric property $A^{(p)} \wedge B^{(q)} = (-1)^{pq} B^{(q)} \wedge A^{(p)}$ holds, and the rank of two forms adds up.
Lastly, for the wedge product between two forms we use the convention

$$B^{(p)} \wedge C^{(q)} = \frac{1}{p!q!} B_{\mu_1,...,\mu_p} C_{\nu_{p+1},...,\nu_{p+q}} \mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_p} \wedge \mathrm{d}x^{\mu_{p+1}} \wedge ... \wedge \mathrm{d}x^{\mu_{p+q}}. \tag{2.1.2}$$

Exterior derivative

The *exterior derivative*, d, is an operator that maps a p-form to a (p+1)-form

$$\mathrm{d}C^{(p)} = \frac{1}{p!} \partial_{\mu_1} C_{\mu_2,...,\mu_{p+1}} \mathrm{d}x^{\mu_1} \wedge ... \wedge \mathrm{d}x^{\mu_{p+1}}. \tag{2.1.3}$$

It is a nilpotent operator meaning that $\mathrm{d}^2 C^{(p)} = 0$ for any p-form. Furthermore, the exterior derivative has the property

$$\mathrm{d}(A^{(p)} \wedge B^{(q)}) = \mathrm{d}A^{(p)} \wedge B^{(q)} + (-1)^p A^{(p)} \wedge \mathrm{d}B^{(q)}. \tag{2.1.4}$$

With a *closed* form we mean a form $C^{(p)}$ having the property $\mathrm{d}C^{(p)} = 0$ and a form that can be written as $C^{(p)} = \mathrm{d}D^{(p-1)}$ is called an *exact* form.

Interior product

The *interior product* or interior derivative is quite similar to the exterior derivative, except that it has the opposite property of lowering the degree of a form by one.

Therefore, the interior product ι_X by a vector field X on the manifold M, is a map from the space of p-forms on the manifold, denoted by $A^p(M)$, to the set of (p-1)-forms on the manifold

$$\iota_X : A^p(M) \to A^{p-1}(M). \tag{2.1.5}$$

The interior product relates the Lie derivative to the exterior derivative, and has a similar property as the one given for the exterior derivative in (2.1.4)

$$\iota_X(C^{(p)} \wedge B^{(q)}) = (\iota_X C^{(p)}) \wedge B^{(q)} + (-1)^p C^{(p)} \wedge (\iota_X B^{(q)}). \tag{2.1.6}$$

Even more, by antisymmetry of forms we know that

$$\iota_X \iota_Y C^{(p)} = -\iota_Y \iota_X C^{(p)} \tag{2.1.7}$$

for a p-form $C^{(p)}$ and two vector fields X and Y on the manifold. Since the vector field X with respect to which we take the interior product ι_X will play an important role, we will emphasize it by denoting the interior product of a vector field X acting on a p-form $C^{(p)}$ as $X \lrcorner\, C^{(p)}$.

Hodge star operator

Whenever the manifold allows for a metric, one can define the *Hodge star operator*, $*$. This operator is not topological since it does not exist without a reference to a metric on the manifold. It maps a p-form to an (n-p)-form on an n-dimensional manifold with metric $g_{\mu\nu}$ as

$$* \, C^{(p)} = \frac{\sqrt{g}}{p!(n-p)!} C_{\mu_1,\ldots,\mu_p} \epsilon^{\mu_1,\ldots,\mu_p}{}_{\nu_{p+1},\ldots,\nu_n} \mathrm{d}x^{\nu_{p+1}} \wedge \ldots \wedge \mathrm{d}x^{\nu_n}, \tag{2.1.8}$$

where g is the determinant of the metric $g_{\mu\nu}$ and the Levi-Civita symbol is defined as

$$\epsilon_{\mu_1,\ldots,\mu_n} = \begin{cases} \pm 1 \text{ for } \mu_1, \ldots, \mu_n \text{ even/odd permutations of } 1, 2, \ldots, n \\ 0 \text{ else.} \end{cases} \tag{2.1.9}$$

Three important properties of the Hodge star operator defined on an n-dimensional manifold, M, are

$$* * C^{(p)} = (-1)^{p(n-p)+\delta} C^{(p)} \begin{cases} \delta = 1, & \text{for a Lorentzian signature metric} \\ \delta = 0, & \text{for a Euclidean signature metric} \end{cases} \tag{2.1.10}$$

$$\int_M C^{(p)} \wedge * D^{(p)} = \frac{1}{p!} \int_M C_{\mu_1,\ldots,\mu_p} D^{\mu_1,\ldots,\mu_p} \sqrt{g} \mathrm{d}^n x, \tag{2.1.11}$$

and on a Cartesian product of manifolds $M_D(x,y) = M_d(x) \times M_{D-d}(y)$, the Hodge star decomposition is given by[1]

$$\hat{*}_D (A_q \wedge B_p) = (-1)^{p(d-q)} (*_d A_q) \wedge (\star_{D-d} B_p), \tag{2.1.12}$$

where A_q and B_p are a q-form and a p-form on the manifolds $M_d(x)$ and $M_{D-d}(y)$, respectively.

With the Hodge star operator in n dimensions, one can define the Hodge duality

$$F^{(p+1)} = * \tilde{F}^{(n-p-1)} \tag{2.1.13}$$

since both field strengths, $F^{(p+1)} = \mathrm{d}C^{(p)}$ and $\tilde{F}^{(n-p-1)} = \mathrm{d}\tilde{C}^{(n-2-p)}$, describe the same number of degrees of freedom. Therefore, the Hodge dual relates a p-form potential $C^{(p)}$ to a dual (n-2-p)-form potential $\tilde{C}^{(n-2-p)}$ in n dimensions. This is the generalized version of the electro-magnetic duality which relates two vector potentials in four dimensions.

Harmonic forms

At last, we can define the adjoint of the exterior derivative acting on a p-form on an n-dimensional Riemannian manifold as $\mathrm{d}^\dagger = (-1)^{n(p-1)+1} * \mathrm{d} *$. Analogously to what we called a closed and exact form, we can define a co-closed and co-exact form as $\mathrm{d}^\dagger C^{(p)} = 0$ and $C^{(p)} = \mathrm{d}^\dagger D^{(p+1)}$ respectively. With this definition, we can state what

[1] To differentiate between the Hodge star on the internal and external manifold, we use $\star$ and $*$, respectively throughout the thesis.

it means for a form to be harmonic. A *harmonic form*, $C^{(p)}$, satisfies

$$\Delta C^{(p)} = 0 \tag{2.1.14}$$

with the *Laplacian* operator

$$\Delta = (\mathrm{d} + \mathrm{d}^\dagger)^2 = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}. \tag{2.1.15}$$

As a final remark, we want to state that a form is harmonic if and only if it is closed and co-closed.

2.1.2 Cycles and chains

Given an n-dimensional manifold M, we can define the set of p-dimensional submanifolds $\{\Gamma_p^i\}$. A *p-chain*, a_p, is defined as a linear combination of such a set of p-dimensional submanifolds

$$a_p = \sum_i c_i \Gamma_p^i \tag{2.1.16}$$

with coefficients c_i. A p-form can be integrated over such a p-chain, which is defined to be

$$\int_{a_p} C^{(p)} = \sum_i c_i \int_{\Gamma_p^i} C^{(p)} = \sum_i c_i \int_{\Gamma_p^i} \mathrm{d}^p x C_{1,\dots,p}. \tag{2.1.17}$$

Note that the form must be defined on the submanifold we integrate over, otherwise the integral vanishes.

The *boundary operator*, δ, for chains is the equivalent of the exterior derivative for forms. This operator maps a p-chain to a (p-1)-chain. Again we are dealing with a nilpotent operator. A *p-cycle* is a p-chain, h_p, without a boundary, hence $\delta h_p = 0$. What one would call a *trivial p-chain*, d_p, is a p-chain that is the boundary of (p+1)-chain, thus $d_p = \delta a_{p+1}$. Note that even though the exterior derivative raises the rank of the form, the boundary operator lowers the dimension of the manifold.

2.1.3 (Co)homology

We have now build up the basics to be able to discuss what the pth cohomology and homology groups of a manifold are. Starting with the p^{th} *de Rham cohomology* group, we consider the quotient group of the closed p-forms modulo all exact p-forms on a manifold M

$$H^p(M) = \frac{\{C^{(p)}|\, \mathrm{d}C^{(p)} = 0\}}{\{D^{(p)}|\, D^{(p)} = \mathrm{d}A^{(p-1)}\}}. \tag{2.1.18}$$

With this definition, an equivalence class of the pth cohomology exists of all closed p-forms that differ from each other only up to an exact form, hence $C^{(p)} \sim C^{(p)} - \mathrm{d}A^{(p-1)}$. Furthermore, the p^{th} *Betti number* is given by [39]

$$b^p = \mathrm{dim} H^p(M), \tag{2.1.19}$$

which is a topological invariant of the manifold M. For a compact manifold, the Betti number is finite. Finally, the Hodge decomposition theorem states that every p-form can uniquely be split into an exact part, a co-exact part and a harmonic part $\tilde{D}^{(p)}$

$$C^{(p)} = \mathrm{d}A^{(p-1)} + \mathrm{d}^\dagger B^{(p+1)} + \tilde{D}^{(p)}. \tag{2.1.20}$$

Given that $C^{(p)}$ is closed, we obtain $C^{(p)} = \mathrm{d}A^{(p-1)} + \tilde{D}^{(p)}$ and hence $C^{(p)} - \mathrm{d}A^{(p-1)} = \tilde{D}^{(p)}$, which we recognize as an element of the same equivalence class of the pth cohomology group that $C^{(p)}$ belongs to. Therefore, each equivalence class of the pth cohomology can be represented by a unique harmonic p-form. Thus, the pth Betti number corresponds to the number of distinct harmonic p-forms that exist on the manifold.

Now switching to what we call the p^{th} *homology* group, we consider the quotient group of p-cycles modulo all trivial p-chains. Therefore, in complete analogy to the pth cohomology, the pth homology of a manifold M is defined as

$$H_p(M) = \frac{\{h_p|\, \delta h_p = 0\}}{\{d_p|\, d_p = \delta a_{p+1}\}}, \tag{2.1.21}$$

with dimensions $b_p = \mathrm{dim} H_p(M)$. Therefore, two p-cycles are in the same equivalence class if they differ up to a boundary, meaning $h_p \sim h_p - \delta a_{p+1}$.

Poincaré duality

The fact that these two quotient groups are constructed with so much analogy, results in a duality between both spaces, the so-called *Poincaré duality.* The de Rham's theorem states that there exists a isomorphism between the cohomology and the homology group on a smooth manifold M. Due to this isomorphism, one can state that for any basis of closed p-forms $\{C^{(p),k}|k = 1, ..., \mathrm{dim}H^p(M)\}$ of $H^p(M)$ there exists a basis of p-cycles $\{h_p^k|k = 1, ..., \mathrm{dim}H_p(M)\}$ of $H_p(M)$ such that we can define the *period* of $C^{(p),k}$ along a dual basis of p-cycles h_p^l

$$\int_{h_p^l} C^{(p),k} = \delta_l^k, \tag{2.1.22}$$

which explicitly shows the one-to-one correspondence between both groups. Furthermore, the duality that maps $H^p(M)$ to $H^{n-p}(M)$, due to the Hodge star operator, ensures that for every basis of closed p-forms $\{C^{(p),k}|k = 1, ..., \mathrm{dim}H^p(M)\}$ of $H^p(M)$ there exists a dual basis of closed (n-p)-forms $\{A_l^{(n-p)}|l = 1, ..., \mathrm{dim}H^{n-p}(M)\}$ of $H^{n-p}(M)$ such that

$$\int_M C^{(p),k} \wedge A_l^{(n-p)} = \delta_l^k. \tag{2.1.23}$$

Therefore, on a manifold M we also have

$$\int_{h_p} C^{(p)} = \int_M C^{(p)} \wedge A^{(n-p)}, \tag{2.1.24}$$

for the p-cycle $h_p \subset M$ that is related to the closed (n-p)-form $A^{(n-p)}$, via the Poincaré duality.

At last, with these dualities we are able to define an intersection number between a p-cycle a_p and an (n-p)-cycle d_{n-p} on a manifold M as

$$a_p \cdot d_{n-p} = \int_M D^{(p)} \wedge A^{(n-p)}, \tag{2.1.25}$$

where $D^{(p)}$ and $A^{(n-p)}$ are the dual forms of the cycles d_{n-p} and a_p.

2.2 Complex manifold

In the previous section we gave a definition of a real manifold, on which we defined differential forms, cycles and (co)homology groups. We could have done this equally well for complex manifolds, on which we will focus in this section.

A complex manifold, M, is a manifold with 2n-dimensions which locally looks like $\mathbb{C}^n$. It allows an indexed collection of charts (U_i, f_i) with $U_i \subset M$ and $f_i : U_i \to \mathbb{C}^n$ a one-to-one map such that $f(U_i)$ is open in $\mathbb{C}^n$. For any two overlapping patches $U_i \cap U_j \neq \emptyset$, the composite function $f_i f_j^{-1}$ is holomorphic. This last demand is the crucial difference with a 2n-dimensional real manifold, where a smooth map between non-empty intersections of patches is sufficient.
Complex manifolds allow a complex structure on the manifold. A complex structure is an almost complex structure which is integrable over the manifold. Therefore, we discuss the latter. An almost complex structure, $\mathcal{J}$, is a (1,1)-tensor field (a multi-linear map from one vector and one 1-form to the real numbers) with the property

$$\mathcal{J}^2 = -\mathbb{1}. \tag{2.2.1}$$

Locally this can be expressed as

$$\mathcal{J}_\alpha{}^\beta \mathcal{J}_\beta{}^\gamma = -\delta_\alpha{}^\gamma \qquad \text{for } \alpha, \beta, \gamma = 1, ..., 2n, \tag{2.2.2}$$

which is used to construct the local complex coordinates on the manifold. Let x^α, y^α with $\alpha = 1, ..., n$ be a set of real coordinates on M, the locally defined complex coordinates can be written as

$$z^\alpha = x^\alpha + \mathcal{J}_\beta{}^\alpha y^\beta. \tag{2.2.3}$$

Now as mentioned above, this almost complex structure must be integrable in order to have globally defined complex coordinates on the manifold. To meet this integrability condition, the Nijenhuis tensor

$$N_{\alpha\beta}{}^\gamma = (\partial_\alpha \mathcal{J}_\beta{}^\epsilon)\mathcal{J}_\epsilon{}^\gamma - \mathcal{J}_\alpha{}^\epsilon(\partial_\epsilon \mathcal{J}_\beta{}^\gamma) - ((\partial_\beta \mathcal{J}_\alpha{}^\epsilon)\mathcal{J}_\epsilon{}^\gamma - \mathcal{J}_\beta{}^\epsilon(\partial_\epsilon \mathcal{J}_\alpha{}^\gamma)) \tag{2.2.4}$$

must vanish. Given this condition, the (1,1)-tensor field $\mathcal{J}$ represents the complex structure of the complex manifold.

2.2.1 Dolbeault cohomology

Similar to the cohomology group defined in section 2.1.3, one can define the so-called *Dolbeault cohomology* (or $\bar{\partial}$-cohomology) on a complex manifold using the complex structure. Let M be an n-dimensional complex manifold (hence 2n real dimensions), then one can always introduce a set of holomorphic/antiholomorphic complex coordinates $z^{\alpha}/\bar{z}^{\bar{\beta}}$ with $\alpha, \beta = 1, ..., n$. Given this set of coordinates, we can define a (p,q)-form on the manifold M as

$$\chi^{(p,q)} = \frac{1}{p!q!}\chi_{\alpha_1,...,\alpha_p,\bar{\beta}_1,...,\bar{\beta}_q}\mathrm{d}z^{\alpha_1} \wedge ... \wedge \mathrm{d}z^{\alpha_p} \wedge \mathrm{d}\bar{z}^{\bar{\beta}_1} \wedge ... \wedge \mathrm{d}\bar{z}^{\bar{\beta}_q}, \tag{2.2.5}$$

which is fully antisymmetric in both its p holomorphic indices and its q antiholomorphic indices. We now split the exterior derivative in two parts

$$\mathrm{d} = \partial + \bar{\partial} \qquad \text{with locally } \partial = dz^{\alpha}\partial_{\alpha} \text{ and } \bar{\partial} = \mathrm{d}\bar{z}^{\bar{\beta}}\partial_{\bar{\beta}}, \tag{2.2.6}$$

such that the operators act on the space of (p,q)-forms $A^{p,q}(M)$ on the manifold as

$$\partial : A^{p,q}(M) \rightarrow A^{p+1,q}(M) \qquad \text{and} \qquad \bar{\partial} : A^{p,q}(M) \rightarrow A^{p,q+1}(M). \tag{2.2.7}$$

The notion of closed and exact with respect to these two operators $\partial, \bar{\partial}$, called the Dolbeault operators, is in complete analogy to the exterior derivative. Both Dolbeault operators are nilpotent.

As the name indicates, the (p+q)th Dolbeault cohomology is defined with respect to the operator $\bar{\partial}$

$$H^{p,q}_{\bar{\partial}}(M) = \frac{\{C^{(p,q)} | \, \bar{\partial}C^{(p,q)} = 0\}}{\{D^{(p,q)} | \, D^{(p,q)} = \bar{\partial}A^{(p,q-1)}\}}. \tag{2.2.8}$$

Thus, we consider the quotient group of closed (p,q)-forms with respect to the $\bar{\partial}$ operator, modulo the exact forms. The complex dimensions of these groups are called *Hodge numbers*

$$h^{p,q} = \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}(M). \tag{2.2.9}$$

Any form $\chi^{(p,q)} \in B^{p,q}_{\bar{\partial}}(M)$, where $B^{p,q}_{\bar{\partial}}(M)$ denotes the space of $\bar{\partial}$-closed (p,q)-forms on the manifold M, can be written as $\chi^{(p,q)} = \bar{\partial}\alpha^{(p,q-1)} + \tilde{\chi}^{(p,q)}$, where $\tilde{\chi}^{(p,q)}$ is an harmonic form with respect to the operator

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^{\dagger} + \bar{\partial}^{\dagger}\bar{\partial} = \frac{1}{2}\Delta. \tag{2.2.10}$$

Therefore, any Dolbeault cohomology class can uniquely be represented by a harmonic with respect to the operator $\Delta_{\bar{\partial}}$. Note that the last equality of equation (2.2.10) only holds for compact Kähler manifolds, which will be discussed in section 2.3.

On a compact Kähler manifold, M, the kth cohomology and the (p+q)th Dolbeault cohomology can be related, using (2.2.10)

$$H^k(M) = \bigoplus_{k=p+q} H^{p,q}_{\bar{\partial}}(M). \tag{2.2.11}$$

Therefore, a similar expression holds for the relation between the kth Betti number and the Hodge numbers

$$b^k = \sum_{k=p+q} h^{p,q}. \tag{2.2.12}$$

These Hodge numbers are quite often arranged in a Hodge diamond which, using the duality relations of this chapter and complex conjugation which relates the left side to the right side, can be expressed as

$$\begin{array}{ccccccccc} & & & & h^{0,0} & & & & \\ & & & h^{1,0} & & h^{1,0} & & & \\ & & . & & . & & . & & \\ h^{n,0} & & & . & & . & & & h^{n,0} \\ & & . & & . & & . & & \\ & & & h^{1,0} & & h^{1,0} & & & \\ & & & & h^{0,0} & & & & \end{array} . \tag{2.2.13}$$

To conclude this section we want to end with how the interior product splits, similar to the exterior derivative (2.2.6), when acting on a Dolbeault cohomology class. Consider a vector field $\hat{X}$ on the manifold M which can be split into its holomorphic part and its anti-holomorphic part as

$$\hat{X} = X + \bar{X}. \tag{2.2.14}$$

Then the interior derivatives of these parts can act separately on the space of (p,q)-forms $A^{p,q}(M)$ on the manifold as [40]

$$\iota_X : A^{p,q}(M) \to A^{p-1,q}(M) \qquad \text{and} \qquad \iota_{\bar{X}} : A^{p,q}(M) \to A^{p,q-1}(M). \tag{2.2.15}$$

2.3 Kähler manifold

With the construction of section 2.2, we are now at a point to define a Kähler manifold. A Kähler manifold is a complex manifold that allows for a Hermitian metric. Whenever a complex manifold also allows for a Hermitian metric, one can always construct a symplectic (1,1)-form, using the complex structure. To show this, we start from the Hermiticity condition on the metric written in local coordinates

$$\begin{aligned} g_{\alpha\beta} &= g_{\bar{\alpha}\bar{\beta}} = 0, \\ g_{\alpha\bar{\beta}} &= \mathcal{J}_\alpha{}^\gamma \mathcal{J}_{\bar{\beta}}{}^{\bar{\epsilon}} g_{\gamma\bar{\epsilon}}. \end{aligned} \tag{2.3.1}$$

It therefore follows that one can construct the components of a (1,1)-form

$$J_{\alpha\bar{\beta}} = \mathcal{J}_\alpha{}^\gamma g_{\gamma\bar{\beta}}, \tag{2.3.2}$$

written in local coordinates

$$J = i g_{\alpha\bar{\beta}} \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^{\bar{\beta}}. \tag{2.3.3}$$

On top of this, the symplectic form should obey

$$\mathrm{d}J = 0, \tag{2.3.4}$$

in order for J to be the fundamental Kähler form. This is the final defining condition for a Kähler manifold. From this defining relation, a characteristic aspect of a Kähler manifold can be constructed. Writing (2.3.4) explicitly using the decomposition of the exterior derivative introduced in (2.2.6) one obtains

$$\mathrm{d}J = i\partial_{[\gamma}g_{\alpha]\bar{\beta}}\mathrm{d}z^{\gamma} \wedge \mathrm{d}z^{\alpha} \wedge \mathrm{d}\bar{z}^{\bar{\beta}} + i\partial_{[\bar{\gamma}}g_{\alpha\bar{\beta}]}\mathrm{d}z^{\alpha} \wedge \mathrm{d}\bar{z}^{\bar{\beta}} \wedge \mathrm{d}\bar{z}^{\bar{\gamma}} = 0, \tag{2.3.5}$$

which results in two constraints

$$\partial_{\gamma}g_{\alpha\bar{\beta}} = \partial_{\alpha}g_{\gamma\bar{\beta}} \qquad \text{and} \qquad \partial_{\bar{\gamma}}g_{\alpha\bar{\beta}} = \partial_{\bar{\beta}}g_{\alpha\bar{\gamma}}. \tag{2.3.6}$$

Therefore, the characteristic result of a Kähler manifold

$$g_{\alpha\bar{\beta}} = \partial_{\alpha}\partial_{\bar{\beta}}\mathcal{K} \qquad \Rightarrow \qquad J = i\partial\bar{\partial}\mathcal{K} \tag{2.3.7}$$

is obtained, where $\mathcal{K}$ is called the Kähler potential. Due to these relations the only non-vanishing Christoffel symbols turn out to be $\Gamma^{\gamma}_{\alpha\beta}$ and $\Gamma^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}}$. Therefore, the only non-trivial Riemann tensor is constraint to $R^{\rho}{}_{\alpha\beta\bar{\gamma}}$. When considering the Ricci curvature tensor, it follows that only $R_{\alpha\bar{\beta}}$ may have a non-vanishing value. With this tensor the Ricci (1,1)-form is constructed

$$R = iR_{\alpha\bar{\beta}}\mathrm{d}z^{\alpha} \wedge \mathrm{d}\bar{z}^{\bar{\beta}} = i\partial\bar{\partial}\log(\sqrt{g}\,), \tag{2.3.8}$$

which implies $\mathrm{d}R = 0$ and defines the first Chern class on the manifold

$$c_1(M) = \left[\frac{R}{2\pi}\right]. \tag{2.3.9}$$

2.4 Calabi-Yau manifold

The framework we have set up of defining complex and Kähler manifolds in the previous sections were needed to define a Calabi-Yau manifold. One can define a Calabi-Yau manifold in two ways. We will give both since they seem to emphasize on different properties of a Calabi-Yau manifold [41].

Definition 1: A 2n-dimensional *Calabi-Yau manifold*, M, is a compact Kähler manifold of n complex dimensions with a no where vanishing holomorphic n-form.

This holomorphic n-form can be represented in complex coordinates $z^{\alpha}, \bar{z}^{\bar{\alpha}}$ which constitute a basis on M

$$\Omega = \Omega_{\alpha_1,\dots,\alpha_n}(z)\,\mathrm{d}z^{\alpha_1} \wedge \dots \wedge \mathrm{d}z^{\alpha_n}. \tag{2.4.1}$$

That this form is holomorphic is reflected in the fact that the coefficient is a holomorphic function in the complex coordinates and that it is an (n,0)-form. Due to the holomorphic property, $\bar{\partial}\Omega = 0$ and since Ω is an (n,0)-form in n-dimensions $\partial\Omega = 0$. Thus, Ω is closed. One can also show that this holomorphic n-form is co-closed, therefore it is harmonic.
As a final property we state from this definition, Ω is unique up to constant rescalings.

Definiton 2: A 2n-dimensional Kähler manifold with vanishing first Chern class is called a Calabi-Yau manifold.

From the fact that the first Chern class vanishes, it is clear that the Ricci form must be trivial in cohomology. Therefore, the Ricci form is exact and globally defined on the manifold. This forms the link to the unique holomorphic n-form, connecting the two definitions. We will not prove the equivalence of both definitions here. We refer the reader to [41].

A consequence of this definition is that, given any Kähler metric g with a Kähler form J associated to it (2.3.7), there exists a unique Ricci-flat metric $\tilde{g}$ with associated Kähler form $\tilde{J}$ such that $\tilde{J}$ is in the same Kähler class as J, i.e. $[J] = [\tilde{J}]$. Thus, a Kähler manifold with vanishing first Chern class admits in every Kähler class a unique Ricci-flat metric.

Finally, for this work the restriction to three complex dimensions will be most relevant. An important property of a Calabi-Yau threefold is that its moduli space $\mathcal{M}$ splits into a direct product of two special Kähler manifolds, the Kähler class moduli space and the complex structure moduli space

$$\mathcal{M} = \mathcal{M}_{\text{Kähler class}} \times \mathcal{M}_{\text{complex structure}}. \tag{2.4.2}$$

Special Kähler manifolds are distinguished by the fact that the Kähler potential can be expressed in terms of a holomorphic prepotential [42]. We will elaborate on the moduli space of a Calabi-Yau threefold in later chapters.

Chapter 3

Moduli spaces

In Type IIB theory on a Calabi-Yau threefold Y_3, one starts from an $\mathcal{N} = 2$ ten-dimensional supergravity action and obtains after compactification an $\mathcal{N} = 2$ supersymmetric effective theory in four dimensions. In such a reduction there is a freedom of the metric on a Calabi-Yau threefold allowing for fluctuations of the metric, constrained to the defining properties of a Calabi-Yau manifold, discussed in section 2.4. Therefore, our interest goes to those infinitesimal fluctuations of the metric $g + \delta g$ allowing the manifold to remain Kähler and Ricci-flat. Expanding $R_{\mu\nu}(g + \delta g) = 0$ to first order in δg and recalling that $R_{\mu\nu}(g) = 0$, we arrive at [43]

$$\Delta_L \delta g_{\mu\nu} \equiv \nabla^2 \delta g_{\mu\nu} + 2R_{\mu\ \nu}^{\ \rho\ \sigma} \delta g_{\rho\sigma} = 0 \qquad (3.0.1)$$

after fixing the diffeomorphism invariance, since we are not interested in δg generated by a coordinate transformation. The operator Δ_L is known as the Lichnerowicz operator. Given the index structure of the metric, equation (3.0.1) decouples for fluctuations $\delta g_{\alpha\bar{\beta}}$ and $\delta g_{\alpha\beta}$. Hence, studying these deformations separately, we will start with fluctuations related to the Kähler form.

Metric variations $\delta g_{\alpha\bar{\beta}}$ lead to non-trivial cohomology changes of the corresponding Kähler form J, i.e. given J the Kähler form corresponding to $g_{\alpha\bar{\beta}}$ and J' corresponds to $g_{\alpha\bar{\beta}} + \delta g_{\alpha\bar{\beta}}$, then $[J] \neq [J']$. However, as stated in section 2.4 every Kähler class admits an unique Ricci-flat metric. Since we want to keep track of these Kähler class deformations in the lower-dimensional theory, we expand the Kähler form according to [44]

$$J_{\alpha\bar{\beta}} = v^i(\omega_i)_{\alpha\bar{\beta}}, \qquad i = 1, ..., h^{1,1}(Y_3), \tag{3.0.2}$$

where ω_i denote a basis of harmonic (1,1)-forms on the Calabi-Yau threefold. Combining this with the relation between the metric and the Kähler form (2.3.3) we can determine the massless modes arising from the Kähler class deformations

$$\delta g_{\alpha\bar{\beta}} = -iv^i(\omega_i)_{\alpha\bar{\beta}}, \tag{3.0.3}$$

where v^i are real scalar fields in the effective four-dimensional theory, called Kähler moduli.

In a similar fashion, one can also study the deformations of the metric disrupting the complex Hermitian metric structure (2.3.1). The massless modes arising from the complex structure deformations of Y_3 are encoded as [44]

$$\delta g_{\alpha\beta} = \frac{i}{||\Omega||^2}\bar{z}^{\mathfrak{s}}(\bar{\chi}_{\mathfrak{s}})_{\alpha\bar{\gamma}_1\bar{\gamma}_2}\Omega^{\bar{\gamma}_1\bar{\gamma}_2}{}_{\beta}, \qquad \mathfrak{s} = 1, ..., h^{1,2}(Y_3), \tag{3.0.4}$$

with the lower-dimensional scalar fields $z^{\mathfrak{s}}$ named complex structure moduli, Ω the unique three-form and $\chi_{\mathfrak{s}}$ constitute a basis of harmonic (2,1)-forms.

The lower-dimensional scalar fields resulting from these fluctuations are the moduli fields and the space they live on is a scalar manifold called the moduli space, which incorporates the geometry of the manifold. The moduli fields are viewed as coordinates on the scalar manifold. As stated before, the moduli space of a Calabi-Yau threefold, in $\mathcal{N} = 2$ Type IIB theory, splits into a direct product of the $(h^{1,1}(Y_3)+1)$-dimensional Kähler class, spanned by the scalars of the hypermultiplets and the $h^{2,1}(Y_3)$-dimensional complex structure moduli space describing the scalars of the vector multiplets [34, 45]

$$\mathcal{M} = \mathcal{M}_{\text{Kähler}}^{h^{1,1}(Y_3)+1} \times \mathcal{M}_{\text{complex structure}}^{h^{2,1}(Y_3)}. \tag{3.0.5}$$

Since the resulting lower-dimensional theory is a $d = 4$, $\mathcal{N} = 2$ supergravity, the Kähler class moduli space is a quaternionic Kähler manifold, while the complex structure moduli space is a (local) special Kähler manifold [46, 47]. A quaternionic Kähler manifold differs from a Kähler manifold by its holonomy group. Extremely simplified, the intuition of a holonomy group can be described with parallel transport. When parallel transporting a vector $\vec{v}$ along a closed curve on a manifold, one transforms the vector into $G\vec{v}$. The holonomy group is formed by collecting all elements G obtained in this way [43]. A Riemannian manifold of real dimension $4n$, with $n \geq 2$, whose holonomy group is a subgroup of $Sp(n)Sp(1)$[1] is called a quaternionic Kähler manifold. A Kähler manifold of complex dimension n has holonomy group $U(n)$. For further interest, we refer the reader to [48].

3.1 Kähler class moduli space

Starting with the Kähler class, we define the triple intersection number

$$\mathcal{K}_{ijk} = \int_{Y_3} \omega_i \wedge \omega_j \wedge \omega_k, \tag{3.1.1}$$

where ω_i are harmonic (1,1)-forms of $H^{1,1}(Y_3)$.

Since the Kähler form can be expanded as $J = v^i \omega_i$ we can write

$$\mathcal{K}_{ij} = \int_{Y_3} \omega_i \wedge \omega_j \wedge J = \mathcal{K}_{ijk} v^k \tag{3.1.2}$$

$$\mathcal{K}_{i} = \int_{Y_3} \omega_i \wedge J \wedge J = \mathcal{K}_{ijk} v^j v^k \tag{3.1.3}$$

$$6V_{Y_3} = \mathcal{K} = \int_{Y_3} J \wedge J \wedge J = \mathcal{K}_{ijk} v^i v^j v^k \tag{3.1.4}$$

in which V_{Y_3} denotes the volume of the Calabi-Yau threefold.

The metric of the Kähler class moduli space turns out to be

$$g_{ij} = \frac{1}{4V_{Y_3}} \int_{Y_3} \omega_i \wedge \star\omega_j, \tag{3.1.5}$$

which does not yet show the characteristic form of equation (2.3.7) explicitly, but

[1] $Sp(n)$ refers to the symplectic group.

rewriting it with

$$\star\,\omega_j = -J\wedge\omega_i + \frac{\mathcal{K}_i}{4V_{Y_3}}J\wedge J \tag{3.1.6}$$

results in the expression [34]

$$g_{ij} = -\frac{1}{4}\Big(\frac{\mathcal{K}_{ij}}{V_{Y_3}} - \frac{\mathcal{K}_i\mathcal{K}_j}{4V_{Y_3}^2}\Big) = \partial_i\bar{\partial}_j(-\mathrm{ln}8V_{Y_3}). \tag{3.1.7}$$

The inverse metric can be expressed in terms of the dual (2,2)-forms $\tilde{\omega}^i$ [34]

$$g^{ij} = 4V_{Y_3}\int_{Y_3}\tilde{\omega}^i\wedge\star\tilde{\omega}^j = -4V_{Y_3}\Big(\mathcal{K}^{ij} - \frac{v^iv^j}{2V_{Y_3}}\Big), \tag{3.1.8}$$

in which we defined

$$\mathcal{K}^{ij}\mathcal{K}_{jk} = \delta^i_k. \tag{3.1.9}$$

From the metric in (3.1.7) we observe that the Kähler potential, $\mathcal{K}$, can be written

$$e^{-\mathcal{K}} = 8V_{Y_3}. \tag{3.1.10}$$

We have yet explained that to keep track of the Kähler class deformations in the lower-dimensional theory we expand these deformations according to (3.0.3). Likewise, a Type IIB theory comes with an anti-symmetric two tensor $\hat{B}$ which is expanded in a similar fashion as the metric $g_{\alpha\bar{\beta}}$,

$$\hat{B} = b^i\omega_i. \tag{3.1.11}$$

Combining the lower-dimensional fields resulting from the expansion of the Kähler class deformations and the anti-symmetric two-form, we can construct a complex coordinate $X^I = (1, t^i)$ in which t^i is a scalar in the hypermultiplets defined as

$$t^i = b^i + iv^i. \tag{3.1.12}$$

In terms of these coordinates X^I and a prepotential $\mathcal{F}$ we can write the Kähler potential as

$$e^{-\mathcal{K}} = i(\bar{X}^I\mathcal{F}_I - X^I\bar{\mathcal{F}}_I), \tag{3.1.13}$$

with prepotential [43]

$$\mathcal{F} = -\frac{1}{3!}\frac{\mathcal{K}_{ijk}X^iX^jX^k}{X^0} \qquad \text{and} \qquad \mathcal{F}_I \equiv \frac{\partial}{\partial X^I}\mathcal{F}, \tag{3.1.14}$$

which fully determines the Kähler potential, making it a special Kähler manifold [42]. This prepotential again appears in the coupling matrix N_{IJ} between the hypermultiplets in Type IIB as [44]

$$N_{IJ} = \bar{\mathcal{F}}_{IJ} + \frac{2i}{X^M \mathrm{Im}\mathcal{F}_{MN}X^N}\mathrm{Im}\mathcal{F}_{IK}X^K\mathrm{Im}\mathcal{F}_{JL}X^L, \tag{3.1.15}$$

with real and imaginary parts given by

$$\mathrm{Re}N_{00} = -\frac{1}{3}\mathcal{K}_{ijk}b^ib^jb^k, \qquad \mathrm{Im}N_{00} = -V_{Y_3} + \Big(K_{ij} - \frac{1}{4}\frac{\mathcal{K}_i\mathcal{K}_j}{V_{Y_3}}\Big)b^ib^j, \tag{3.1.16}$$

$$\mathrm{Re}N_{i0} = \frac{1}{2}\mathcal{K}_{ijk}b^jb^k, \qquad \mathrm{Im}N_{i0} = -\Big(\mathcal{K}_{ij} - \frac{1}{4}\frac{\mathcal{K}_i\mathcal{K}_j}{V_{Y_3}}\Big)b^j, \tag{3.1.17}$$

$$\mathrm{Re}N_{ij} = -\mathcal{K}_{ijk}b^k, \qquad \mathrm{Im}N_{ij} = \Big(\mathcal{K}_{ij} - \frac{1}{4}\frac{\mathcal{K}_i\mathcal{K}_j}{V_{Y_3}}\Big). \tag{3.1.18}$$

3.2 Complex structure moduli space

To construct the complex structure moduli space we need a basis of harmonic three-forms on the Calabi-Yau manifold. Hence, we start by introducing these. A Calabi-Yau threefold has one unique (3,0)-form Ω and $h^{2,1}(Y_3)$ different harmonic (2,1)-forms $\chi_{\mathfrak{s}}$ with $\mathfrak{s} = 1, ..., h^{2,1}(Y_3)$, which together with their complex conjugates span a space of all three-forms on Y_3, $H^3(Y_3) = H^{3,0} \bigoplus H^{2,1} \bigoplus H^{1,2} \bigoplus H^{0,3}$ [49]. Note that this basis was already used in (3.0.4) for the expansion of the pure metric deformations. Two important relations for these forms are [43, 45]

$$\star\,\Omega = -i\Omega \qquad \text{and} \qquad \star\,\chi_{\mathfrak{s}} = i\chi_{\mathfrak{s}}, \tag{3.2.1}$$

from which we see that $\chi_{\mathfrak{s}}$ is a primitive (2,1)-form, since in general one would expect [43]

$$\star\,\beta = i\beta - i\omega \wedge \star(\omega \wedge \beta) \tag{3.2.2}$$

for a (2,1)-form β. However, the contraction between $\chi_{\mathfrak{s}}$ and the Kähler form vanishes, making $\chi_{\mathfrak{s}}$ a so-called primitive form.
Furthermore, $\frac{\partial}{\partial z^{\mathfrak{s}}}\Omega$ is in $H^{3,0}\bigoplus H^{2,1}$ since the Kähler covariant derivative of Ω is given by [43]

$$D_{\mathfrak{s}}\Omega = \frac{\partial}{\partial z^{\mathfrak{s}}}\Omega - k_{\mathfrak{s}}\Omega = i\chi_{\mathfrak{s}}, \tag{3.2.3}$$

with $z^{\mathfrak{s}}$ the complex structure moduli fields (3.0.4) and $k_{\mathfrak{s}}$ given by

$$k_{\mathfrak{s}} = -\partial_{\mathfrak{s}}\mathcal{K}, \tag{3.2.4}$$

where $\mathcal{K}$ is the Kähler potential, we will come back to in a second.

Another basis that spans the space of $H^3(Y_3)$ is the symplectic basis of three-forms

$$\int_{Y_3} \alpha_I \wedge \beta^J = \delta_I^J, \qquad \int_{Y_3} \alpha_I \wedge \alpha_J = \int_{Y_3} \beta_I \wedge \beta_J = 0, \tag{3.2.5}$$

with $2(h^{2,1}+1)$ real three-forms α_I, β^J.

These two bases can be related through [43]

$$\Omega = Z^I\alpha_I - \mathcal{H}_I\beta^I, \tag{3.2.6}$$

where $\mathcal{H}_I$ and Z^I are periods of Ω defined by

$$\mathcal{H}_I = \int_{Y_3} \Omega \wedge \alpha_I, \qquad \text{and} \qquad Z^I = \int_{Y_3} \Omega \wedge \beta^I. \tag{3.2.7}$$

Analog to the Kähler class, the coordinates on the complex structure moduli space are $Z^I = (1, z^{\mathfrak{s}})$ and the first derivative of an $\mathcal{N} = 2$ sypersymmetric holomorphic prepotential $\mathcal{H}$ of degree 2 in the coordinates Z^I is represented by $\mathcal{H}_I = \partial_I\mathcal{H}$. From (3.2.6) we can deduce

$$\partial_I\Omega = \alpha_I - \mathcal{H}_{IJ}\beta^J, \tag{3.2.8}$$

defining $\mathcal{H}_{IJ} = \partial_I\partial_J\mathcal{H}$. Thus, combining (3.2.3) and (3.2.8) leads to[2]

$$k_I\Omega + i\chi_I = \alpha_I - \mathcal{H}_{IJ}\beta^J, \tag{3.2.9}$$

[2]Implicitly a coordinate redefinition has been made [43].

an important result that will be used later.

The metric on the complex structure moduli space that connects the scalars $z^{\mathfrak{s}}$ of the vector multiplets is given by [34]

$$\mathcal{G}_{\mathfrak{s}\bar{\mathfrak{r}}} = -\frac{i\int_{Y_3}\chi_{\mathfrak{s}}\wedge\bar{\chi}_{\bar{\mathfrak{r}}}}{\int_{Y_3}\Omega\wedge\bar{\Omega}} = -\frac{i}{V_{Y_3}||\Omega||^2}\int_{Y_3}\chi_{\mathfrak{s}}\wedge\bar{\chi}_{\bar{\mathfrak{r}}} = \partial_{\mathfrak{s}}\partial_{\bar{\mathfrak{r}}}(-\ln(i\int_{Y_3}\Omega\wedge\bar{\Omega})), \quad (3.2.10)$$

using (3.2.3) for the last equality. Hence, we obtain the Kähler potential

$$e^{-\mathcal{K}} = i\int_{Y_3}\Omega\wedge\bar{\Omega} = i(\bar{Z}^I\mathcal{H}_I - Z^I\bar{\mathcal{H}}_I), \quad (3.2.11)$$

where we have used equations (2.3.7), (3.2.5) and (3.2.6). Again, we observe that the Kähler potential is fully determined by the holomorphic prepotential. Therefore, the complex structure moduli space of a Calabi-Yau threefold is a special Kähler manifold [42].

In (3.1.15) the coupling matrix of the hypermultiplets was already stated, similarly we will now give the $\mathcal{N}=2$ gauge coupling matrix between the vector multiplets in four-dimensional Type IIB. In order to do so we have to determine the integrals in the reduction that contain Hodge duals of the symplectic basis of $H^3(Y_3)$. The latter are given by [49]

$$\begin{aligned} \star\alpha_I &= A_I^{\ J}\alpha_J + B_{IJ}\beta^J, \\ \star\beta^I &= C^{IJ}\alpha_J + D^I_{\ J}\beta^J, \end{aligned} \quad (3.2.12)$$

with matrices A, B, C and D

$$\begin{aligned} A_J^I &= -\int_{Y_3}\beta^I\wedge\star\alpha_J, & B_{IJ} &= \int_{Y_3}\alpha_I\wedge\star\alpha_J, \\ C^{IJ} &= -\int_{Y_3}\beta^I\wedge\star\beta^J, & D_{\ I}^J &= \int_{Y_3}\alpha_I\wedge\star\beta^J, \end{aligned} \quad (3.2.13)$$

which determine the $\mathcal{N}=2$ gauge kinetic coupling matrix of the vector multiplets

$$M_{IJ} = A_I^{\ K}C_{KJ} + iC_{IJ}. \quad (3.2.14)$$

We defined C_{IJ} to be the inverse of C^{IJ} and mention that these matrices fulfill the properties [50]

$$A^T = -D, \qquad B^T = B, \qquad C^T = C. \tag{3.2.15}$$

One can write the gauge coupling matrix M in terms of the prepotential, which results in [51]

$$M_{IJ} = \bar{\mathcal{H}}_{IJ} + \frac{2i}{Z^K \mathrm{Im}\mathcal{H}_{KL} Z^L} \mathrm{Im}\mathcal{H}_{IM} Z^M \mathrm{Im}\mathcal{H}_{JN} Z^N, \tag{3.2.16}$$

with the matrices A, B and C in terms of M [49]

$$\begin{aligned}
A &= (\mathrm{Re}M)(\mathrm{Im}M)^{-1}, \\
B &= -(\mathrm{Re}M)(\mathrm{Im}M)^{-1}(\mathrm{Re}M) - (\mathrm{Im}M), \\
C &= (\mathrm{Im}M)^{-1}.
\end{aligned} \tag{3.2.17}$$

Chapter 4

IIB reduction on Y_3 inlcuding O3/O7-planes

In this chapter we perform a dimensional reduction of the bulk action of the ten-dimensional $\mathcal{N} = 2$ Type IIB supergravity at tree level on a Calabi-Yau threefold including $O3/O7$ orientifold planes. We expand the ten-dimensional fields of the supergravity action according to their equations of motion in harmonics on the Calabi-Yau threefold, restricting the theory to the subspace invariant under the orientifold projection O, and integrate out the internal coordinate dependence. We choose the ten-dimensional spacetime background to be $M_{(1,9)} = \mathbb{R}^{1,3} \times Y_3/O$, with Y_3/O a compact Calabi-Yau orientifold.

4.1 The spectrum

Ignoring the orientifold projection for the moment, we write the metric according to the block diagonal spacetime[1]

$$\langle \mathrm{d}\hat{s}^2_{10} \rangle = \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + 2\breve{g}_{m\bar{n}} \mathrm{d}y^m \mathrm{d}\bar{y}^{\bar{n}}, \qquad (4.1.1)$$

where $\breve{g}_{m\bar{n}}$ is the background value of the metric on the Calabi-Yau threefold. Note

[1]Throughout the thesis we will denote the higher-dimensional objects with a hat.

that locally this means $\breve{g}_{mn} = \breve{g}_{\bar{m}\bar{n}} = 0$, in agreement to the Hermicity condition on the metric (2.3.1). Recall from section 1.2 that the effective four-dimensional theory we obtain after compactifying on the threefold Y_3 includes all massless fluctuations around the background. These fluctuations can be deduced from the Hodge diamond, stating the number of harmonic forms, of a Calabi-Yau threefold

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{4.1.2}$$

As discussed in chapter 3, the fluctuations around the background metric (4.1.1) can be divided into two types of deformations. The Kähler class deformations are the mixed forms $\delta g_{m\bar{n}}$ and the pure metric deformations δg_{mn} and $\delta g_{\bar{m}\bar{n}}$ break the complex Hermitian metric structure (2.3.1).

Recall from equations (3.0.3) and (3.0.4) that up to first order in the moduli fields, the metric on the threefold Y_3 reads

$$\begin{aligned} g_{m\bar{n}} &= \breve{g}_{m\bar{n}} - i v^i (\omega_i)_{m\bar{n}} \\ g_{mn} &= \bar{z}^{\mathfrak{s}} (\bar{b}_{\mathfrak{s}})_{mn}, \end{aligned} \tag{4.1.3}$$

with

$$(\bar{b}_{\mathfrak{s}})_{mn} = \frac{i}{||\Omega||^2} (\bar{\chi}_{\mathfrak{s}})_{m\bar{e_1}\bar{e_2}} \Omega^{\bar{e_1}\bar{e_2}}{}_n . \tag{4.1.4}$$

To include the orientifold projection, the states not invariant under the projection have to be modded out of the spectrum. The orientifold projection acting on the Type IIB states is given by

$$O = (-1)^{F_L} \Omega_p \sigma^* . \tag{4.1.5}$$

Here F_L is the spacetime left-moving fermion number, Ω_p the world-sheet parity operator that mods out the string orientation and σ^* is the pullback of an internal symmetry acting only on the Calabi-Yau threefold and leaving the four-dimensional

Minkowskian spacetime invariant. Due to the orientifold projection $\mathcal{N}=2$ supersymmerty is broken. However, maintaining $\mathcal{N}=1$ supersymmetry, σ is required to be an isometric and holomorphic involution of Y_3 [50, 52, 53].

The fixed point set (or each disconnected component) of this involution spans the O-planes. Note that since the involution does not act on the Minkowski space, all orientifold planes are spacetime filling. Since we will discuss D7-branes in this work, we want to include $O3/O7$-planes to the theory. Meaning, given the complex coordinates y^m with $m=1,2,3$ on the Calabi-Yau threefold, we choose the action of the involution to be[2] $\sigma y^m \sigma^{-1} = \pm y^m$ with either all three complex directions reflected creating $O3$-planes or just one complex direction reversed resulting in $O7$-planes [46]. This means the involution σ must be constrained to act on the unique (3,0)-form Ω as

$$\sigma^* \Omega = -\Omega. \tag{4.1.6}$$

We preform the reduction of the bulk $\mathcal{N}=2$ ten-dimensional democratic action of Type IIB supergravity given in the string frame by

$$S^{(10)}_{\text{Bulk IIB, SF}} = -\frac{1}{2\kappa_{10}^2}\int e^{-2\hat{\phi}}\hat{R}\hat{*}\mathbb{1} + \frac{1}{2}e^{-2\hat{\phi}}(8\mathrm{d}\hat{\phi}\wedge\hat{*}\mathrm{d}\hat{\phi} - \hat{H}\wedge\hat{*}\hat{H}) - \frac{1}{4}\sum_{\substack{p=1,3,\\5,7,9}}\hat{G}^{(p)}\wedge\hat{*}\hat{G}^{(p)}, \tag{4.1.7}$$

where κ_{10} is the ten-dimensional gravitational coupling constant, $\hat{H}$ is the field strength of the anti-symmetric two tensor $\hat{B}$, i.e. $\hat{H}=\mathrm{d}\hat{B}$, $\hat{\phi}$ the ten-dimensional dilaton and the field strengths $\hat{G}^{(p)}$ are defined to be

$$\hat{G}^{(p)} = \begin{cases} \mathrm{d}\hat{C}^{(0)} & p=1, \\ \mathrm{d}\hat{C}^{(p-1)} - \mathrm{d}\hat{B}\wedge\hat{C}^{(p-3)} & \text{else}, \end{cases} \tag{4.1.8}$$

with $\hat{C}^{(p-1)}$ for $p=1,3,5,7,9$ the anti-symmetric ten-dimensional potential fields from the open string Ramond-Ramond sector. We will expand according to the vanishing backgrounds $\langle\hat{C}^{(p)}\rangle = \langle\hat{B}\rangle = 0$. Since the democratic action contains all Ramond-Ramond forms of Type IIB supergravity, the equations of motion have to be supplemented by the duality constraints

[2] The minus sign denotes the reflection of orientation.

$$\hat{G}^{(1)} = \hat{*}\hat{G}^{(9)}, \qquad \hat{G}^{(3)} = (-1)\hat{*}\hat{G}^{(7)}, \qquad \hat{G}^{(5)} = \hat{*}\hat{G}^{(5)}. \tag{4.1.9}$$

Next, the Kaluza-Klein spectrum of this compactification is obtained by expanding all fields according to their equations of motion into harmonics on the Calabi-Yau threefold. The Hodge diamond of a Calabi-Yau threefold (4.1.2) contains the information necessary for this expansion. This results in the massless four-dimensional $\mathcal{N}=2$ Kaluza Klein spectrum. However, as mentioned before, truncating the spectrum further by keeping only the states invariant under the orientifold projection O, results in the four-dimensional massless $\mathcal{N}=1$ Kaluza Klein spectrum. We first note that the harmonic forms of $H^{p,q}_{\bar{\partial}}(Y_3)$ split into positive and negative eigenforms under σ^* [34, 52, 54]. Therefore, our chosen basis for all Dolbeault cohomology groups splits as $H^{p,q}_{\bar{\partial}}(Y_3) = H^{p,q}_{\bar{\partial},+}(Y_3) \times H^{p,q}_{\bar{\partial},-}(Y_3)$ and is shown in Table 4.1 [32].

space	basis	dimension	space	basis	dimension
$H^{1,1}_{\bar{\partial},+}(Y_3)$	ω_α	$\alpha = 1, ..., h^{1,1}_+$	$H^{1,1}_{\bar{\partial},-}(Y_3)$	ω_a	$a = 1, ..., h^{1,1}_-$
$H^{2,2}_{\bar{\partial},+}(Y_3)$	$\tilde{\omega}^\alpha$	$\alpha = 1, ..., h^{1,1}_+$	$H^{2,2}_{\bar{\partial},-}(Y_3)$	$\tilde{\omega}_a$	$a = 1, ..., h^{1,1}_-$
$H^3_+(Y_3)$	$\alpha_{\hat{\alpha}}, \beta^{\hat{\alpha}}$	$\hat{\alpha} = 1, ..., h^{2,1}_+$	$H^3_-(Y_3)$	$\alpha_{\hat{a}}, \beta^{\hat{a}}$	$\hat{a} = 0, ..., h^{2,1}_-$
$H^{2,1}_{\bar{\partial},+}(Y_3)$	$\chi_{\tilde{\alpha}}$	$\tilde{\alpha} = 1, ..., h^{2,1}_+$	$H^{2,1}_{\bar{\partial},-}(Y_3)$	$\chi_{\tilde{a}}$	$\tilde{a} = 1, ..., h^{2,1}_-$
$H^{1,2}_{\bar{\partial},+}(Y_3)$	$\bar{\chi}_{\tilde{\alpha}}$	$\tilde{\alpha} = 1, ..., h^{2,1}_+$	$H^{1,2}_{\bar{\partial},-}(Y_3)$	$\bar{\chi}_{\tilde{a}}$	$\tilde{a} = 1, ..., h^{2,1}_-$

Table 4.1 – Cohomology basis

In order to determine which states are projected out by the orientifold projection given in (4.1.5), we need to know how the world-sheet parity operator and the left-moving fermion number act on the states.
From the fact that the world-sheet parity operator acts on the world-sheet bosons, on a string of length l, as [46]

$$\Omega_p \hat{X}^\mu(\tau,\sigma)\Omega_p^{-1} = \hat{X}^\mu(\tau, l-\sigma) \tag{4.1.10}$$

and similarly for the world-sheet fermions. One can deduce the eigenvalues of the NS-NS fields and the Ramond-Ramond fields under this operator [55]

$$\begin{aligned} \Omega_p = +1: &\qquad \hat{\phi}, \hat{g}_{\mu\nu}, \hat{C}_2, \hat{C}_6 \\ \Omega_p = -1: &\qquad \hat{B}_{\mu\nu}, \hat{C}_0, \hat{C}_4, \hat{C}_8. \end{aligned} \tag{4.1.11}$$

Finally, since all NS-NS fields have eigenvalue +1 under the operator $(-1)^{F_L}$ and Ramond-Ramond fields eigenvalue -1, we conclude that in order for states to be invariant under the orientifold projection (4.1.5), and hence not to be projected out, they must obey to the eigenvalues

$$\begin{aligned} \sigma^* = +1 : &\qquad \hat{\phi}, \hat{g}_{\mu\nu}, \hat{C}_0, \hat{C}_4, \hat{C}_8 \\ \sigma^* = -1 : &\qquad \hat{B}_{\mu\nu}, \hat{C}_2, \hat{C}_6. \end{aligned} \tag{4.1.12}$$

under the involution operator. This brings us to the expansions [32]

$$\hat{J} = v^\alpha(x)\omega_\alpha, \qquad \hat{B} = b^a(x)\omega_a, \qquad \hat{\phi} = \phi(x)\mathbb{1}, \tag{4.1.13}$$

$$\begin{aligned} &\hat{C}^{(8)} = \tilde{l}^{(2)}(x) \wedge \frac{\Omega \wedge \bar{\Omega}}{\int_{Y_3} \Omega \wedge \bar{\Omega}}, \\ &\hat{C}^{(6)} = \tilde{c}_a^{(2)}(x) \wedge \tilde{\omega}^a, \\ &\hat{C}^{(4)} = D_{(2)}^\alpha(x) \wedge \omega_\alpha + V^{\hat{\alpha}}(x) \wedge \alpha_{\hat{\alpha}} + U_{\hat{\alpha}}(x) \wedge \beta^{\hat{\alpha}} + \rho_\alpha(x)\tilde{\omega}^\alpha, \\ &\hat{C}^{(2)} = c^a(x)\omega_a, \\ &\hat{C}^{(0)} = l(x)\mathbb{1}, \end{aligned} \tag{4.1.14}$$

with scalar fields $v^\alpha, b^a, \phi, \rho_\alpha, c^a$ and l, vector fields $V^{\hat{\alpha}}$ and $U_{\hat{\alpha}}$ and two-form tensor fields $\tilde{l}^{(2)}, \tilde{c}_a^{(2)}$ and $D_{(2)}^\alpha$.
Finally, since the metric has a positive eigenvalue under the operator $(-1)^{F_L}\Omega_p$ and the involution acts on the unique (3,0)-form as given in (4.1.6), the expansion of the complex structure deformations (4.1.3) becomes

$$g_{mn} = \bar{z}^{\tilde{a}}(\bar{b}_{\tilde{a}})_{mn}. \tag{4.1.15}$$

4.2 Bulk compactification

Before starting the dimensional reduction of the bulk ten-dimensional Type IIB action (4.1.7), we preform a Weyl rescaling $(\hat{g}_{MN})^{old} = e^{\phi/2}(\hat{g}_{MN})^{new}$ to switch from the string frame to the Einstein frame for convenience. Ignoring total derivatives the action in the Einstein frame becomes

$$S^{(10)}_{\text{Bulk IIB, EF}} = -\frac{1}{2\kappa^2_{10}} \int \hat{R}\hat{*}\mathbb{1} - \frac{1}{2}\mathrm{d}\hat{\phi} \wedge \hat{*}\mathrm{d}\hat{\phi} - \frac{1}{2}e^{-\hat{\phi}}\hat{H} \wedge \hat{*}\hat{H} - \frac{1}{4}\sum_{\substack{p=1,3,\\5,7,9}} e^{\frac{\hat{\phi}}{2}(5-p)}\hat{G}^{(p)} \wedge \hat{*}\hat{G}^{(p)}. \tag{4.2.1}$$

A more detailed discussion of how one should preform such a Weyl rescaling is included in appendix B.

We begin with the dimensional reduction of the Einstein-Hilbert term up to second order in the moduli fields

$$S^{(10)}_{\text{EH, EF}} = -\frac{1}{2\kappa^2_{10}} \int \hat{R}\hat{*}\mathbb{1}. \tag{4.2.2}$$

The ten-dimensional Ricci scalar is given by

$$\begin{aligned} \hat{R} = \hat{g}^{MN}\hat{R}^{P}{}_{MPN} = g^{\mu\nu}R^{\rho}{}_{\mu\rho\nu} + \Big[& g^{\mu\nu}R^{m}{}_{\mu m\nu} + g^{mn}(R^{\mu}{}_{m\mu n} + R^{p}{}_{mpn} + R^{\bar{p}}{}_{m\bar{p}n}) \\ & + g^{m\bar{n}}(R^{\mu}{}_{m\mu\bar{n}} + R^{p}{}_{mp\bar{n}} + R^{\bar{p}}{}_{m\bar{p}\bar{n}}) + c.c.\Big], \end{aligned} \tag{4.2.3}$$

with $\hat{R}^{R}{}_{MPN}$ the Riemann curvature tensor

$$\hat{R}^{R}{}_{MPN} = \partial_P\hat{\Gamma}^R_{NM} - \partial_N\hat{\Gamma}^R_{PM} + \hat{\Gamma}^R_{PL}\hat{\Gamma}^L_{NM} - \hat{\Gamma}^R_{NL}\hat{\Gamma}^L_{PM} \tag{4.2.4}$$

and Christoffel symbols

$$\hat{\Gamma}^R_{MN} = \frac{1}{2}\hat{g}^{RP}(\partial_M\hat{g}_{PN} + \partial_N\hat{g}_{PM} - \partial_P\hat{g}_{MN}). \tag{4.2.5}$$

Deriving all terms of the Ricci scalar (4.2.3) up to second order in moduli fields, using the background given in (4.1.1) expanded according to (4.1.13) and (4.1.15), results in the lower-dimensional Einstein-Hilbert action

$$\begin{aligned} S^{(4)}_{\text{EH, EF}} = -\frac{1}{2\kappa^2_{10}} \int d^{10}x\sqrt{-g_{10}}\,\Big[R + \Big(\frac{1}{2}(\omega_\alpha)^{\bar{n}m}(\omega_\beta)_{m\bar{n}} - (\omega_\alpha)_m{}^m(\omega_\beta)_n{}^n\Big)(\partial_\mu v^\alpha)\partial^\mu v^\beta \\ - \frac{1}{2}(b_{\tilde{a}})^{mn}(\bar{b}_{\tilde{b}})_{mn}(\partial_\mu z^{\tilde{a}})\partial^\mu \bar{z}^{\tilde{b}}\Big]. \end{aligned} \tag{4.2.6}$$

The details of this derivation are presented in appendix C.

When compactifying all other terms on the background (4.1.1), we use the Hodge star decomposition (2.1.12) combined with the expansions (4.1.13) and (4.1.14). The dimensional reduction of (4.2.1) results in

$$\int \hat{H} \wedge \hat{*}\hat{H} = \frac{2\mathcal{K}}{3} G_{ab} \int \mathrm{d}b^a \wedge *\mathrm{d}b^b, \tag{4.2.7}$$

$$\int \hat{G}^{(1)} \wedge \hat{*}\hat{G}^{(1)} = \frac{\mathcal{K}}{6} \int \mathrm{d}l \wedge *\mathrm{d}l, \tag{4.2.8}$$

$$\int \hat{G}^{(3)} \wedge \hat{*}\hat{G}^{(3)} = \frac{2\mathcal{K}}{3} G_{ab} \int (\mathrm{d}c^a - l\mathrm{d}b^a) \wedge *(\mathrm{d}c^b - l\mathrm{d}b^b), \tag{4.2.9}$$

$$\begin{aligned}\int \hat{G}^{(5)} \wedge \hat{*}\hat{G}^{(5)} = \int & \frac{2\mathcal{K}}{3} G_{\alpha\beta} \mathrm{d}D_2^\alpha \wedge *\mathrm{d}D_2^\beta + \frac{3}{2\mathcal{K}} G^{\alpha\beta} \mathrm{d}\rho_\alpha \wedge *\mathrm{d}\rho_\beta + B_{\hat{\alpha}\hat{\beta}} \mathrm{d}V^{\hat{\alpha}} \wedge *\mathrm{d}V^{\hat{\beta}} \\ & - C^{\hat{\alpha}\hat{\beta}} \mathrm{d}U_{\hat{\alpha}} \wedge *\mathrm{d}U_{\hat{\beta}} - 2A_{\hat{\beta}}^{\ \hat{\alpha}} \mathrm{d}U_{\hat{\alpha}} \wedge *\mathrm{d}V^{\hat{\beta}} - c^a \mathrm{d}b^b \wedge *\mathrm{d}\rho_\alpha \int_{Y_3} \omega_a \wedge \omega_b \wedge \star\tilde{\omega}^\alpha \\ & + (c^a \mathrm{d}b^b) \wedge *(c^c \mathrm{d}b^d) \int_{Y_3} \omega_a \wedge \omega_b \wedge \star(\omega_c \wedge \omega_d),\end{aligned} \tag{4.2.10}$$

$$\begin{aligned}\int \hat{G}^{(7)} \wedge \hat{*}\hat{G}^{(7)} = \int & \frac{3}{2\mathcal{K}} G^{ab} \mathrm{d}\tilde{c}_a^2 \wedge *\mathrm{d}\tilde{c}_b^2 - 2\mathrm{d}b^a \wedge D_2^\alpha \wedge *\mathrm{d}\tilde{c}_b^2 \int_{Y_3} \omega_a \wedge \omega_\alpha \wedge \star\tilde{\omega}^b \\ & + \mathrm{d}b^a \wedge D_2^\alpha \wedge *(\mathrm{d}b^b \wedge D_2^\beta) \int_{Y_3} \omega_a \wedge \omega_\alpha \wedge \star(\omega_b \wedge \omega_\beta),\end{aligned} \tag{4.2.11}$$

$$\int \hat{G}^{(9)} \wedge \hat{*}\hat{G}^{(9)} = \int \frac{6}{\mathcal{K}} \mathrm{d}\tilde{l}^2 \wedge *\mathrm{d}\tilde{l}^2 + \frac{6}{\mathcal{K}} \mathrm{d}b^a \wedge \tilde{c}_a^2 \wedge *(\mathrm{d}b^b \wedge \tilde{c}_b^2) - \frac{12}{\mathcal{K}} (\mathrm{d}b^a \wedge \tilde{c}_a^2) \wedge *\mathrm{d}\tilde{l}^2, \tag{4.2.12}$$

in which we used that there are no 1-forms on a Calabi-Yau threefold and that the unique 6-form is positive under the involution. Therefore, we are able to write

$$\omega_a \wedge \tilde{\omega}^b = \delta_a^b \frac{\Omega \wedge \bar{\Omega}}{\int_{Y_3} \Omega \wedge \bar{\Omega}} = \frac{1}{V_{Y_3}} \delta_a^b \Omega \wedge \bar{\Omega}, \tag{4.2.13}$$

where V_{Y_3} denotes the volume of the Calabi-Yau threefold and $6V_{Y_3} = \mathcal{K}$, with

$$\mathcal{K} = \int_{Y_3} J \wedge J \wedge J = \mathcal{K}_{\alpha\beta\gamma} v^\alpha v^\beta v^\gamma. \tag{4.2.14}$$

Note that the volume form $\mathrm{d}V_{Y_3}$ of the Calabi-Yau threefold is even under the involution, therefore $\mathcal{K}_{abc} = 0$ since $\omega_a \wedge \omega_b \wedge \omega_c$ is odd under the involution.

Hence, all non vanishing intersection numbers are

$$\begin{aligned} \mathcal{K}_{\alpha\beta\gamma} &= \int_{Y_3} \omega_\alpha \wedge \omega_\beta \wedge \omega_\gamma, \\ \mathcal{K}_{\alpha\beta} &= \mathcal{K}_{\alpha\beta\gamma} v^\gamma, \\ \mathcal{K}_{\alpha} &= \mathcal{K}_{\alpha\beta\gamma} v^\beta v^\gamma, \\ \mathcal{K}_{ab} &= \int_{Y_3} \omega_a \wedge \omega_b \wedge J = \mathcal{K}_{ab\gamma} v^\gamma. \end{aligned} \tag{4.2.15}$$

Furthermore, we defined the metrics on the space of harmonic two-forms

$$G_{ab} = \frac{1}{4V_{Y_3}} \int_{Y_3} \omega_a \wedge \star\omega_b = -\frac{3}{2}\frac{\mathcal{K}_{ab}}{\mathcal{K}}, \tag{4.2.16}$$

$$G_{\alpha\beta} = \frac{1}{4V_{Y_3}} \int_{Y_3} \omega_\alpha \wedge \star\omega_\beta = -\frac{3}{2}\Big(\frac{\mathcal{K}_{\alpha\beta}}{\mathcal{K}} - \frac{3}{2}\frac{\mathcal{K}_\alpha \mathcal{K}_\beta}{\mathcal{K}^2}\Big) \tag{4.2.17}$$

and denoted their inverse metrics by G^{ab} and $G^{\alpha\beta}$ respectively.
The metric on the complex structure deformations $z^{\tilde{a}}$ is defined to be

$$\mathcal{G}_{\tilde{a}\tilde{b}} = \frac{1}{4V_{Y_3}} \int_{Y_3} b_{\tilde{a}} \wedge \star\bar{b}_{\tilde{b}} = \frac{\partial^2}{\partial z^{\tilde{a}} \partial \bar{z}^{\tilde{b}}} \mathcal{K}_{CS}(z,\bar{z}), \tag{4.2.18}$$

with Kähler potential

$$\mathcal{K}_{CS}(z,\bar{z}) = -\ln(i \int_{Y_3} \Omega \wedge \bar{\Omega}). \tag{4.2.19}$$

Finally, the introduced coefficients of the vector fields $U_{\hat{\alpha}}$ and $V^{\hat{\alpha}}$ are given by [49]

$$A_{\hat{\beta}}^{\ \hat{\alpha}} = -\int_{Y_3} \beta^{\hat{\alpha}} \wedge \star\alpha_{\hat{\beta}}, \qquad B_{\hat{\alpha}\hat{\beta}} = \int_{Y_3} \alpha_{\hat{\alpha}} \wedge \star\alpha_{\hat{\beta}}, \qquad C^{\hat{\alpha}\hat{\beta}} = -\int_{Y_3} \beta^{\hat{\alpha}} \wedge \star\beta^{\hat{\beta}}, \tag{4.2.20}$$

which is consistent with

$$\star\,\alpha_{\hat{\alpha}} = A_{\hat{\alpha}}{}^{\hat{\beta}}\alpha_{\hat{\beta}} + B_{\hat{\alpha}\hat{\beta}}\beta^{\hat{\beta}}, \qquad\qquad \star\beta^{\hat{\alpha}} = C^{\hat{\alpha}\hat{\beta}}\alpha_{\hat{\beta}} + D^{\hat{\alpha}}{}_{\hat{\beta}}\beta^{\hat{\beta}}. \tag{4.2.21}$$

Furthermore, defining the matrix

$$M_{\hat{\alpha}\hat{\beta}} = A_{\hat{\alpha}}^{\hat{\gamma}}C_{\hat{\gamma}\hat{\beta}} + iC_{\hat{\alpha}\hat{\beta}} \tag{4.2.22}$$

implies the identities

$$A = \mathrm{Re}M(\mathrm{Im}M)^{-1}, \qquad B = -\mathrm{Im}M - \mathrm{Re}M(\mathrm{Im}M)^{-1}\mathrm{Re}M, \qquad C = (\mathrm{Im}M)^{-1}, \tag{4.2.23}$$

where we have written $(C^{\hat{\alpha}\hat{\beta}})^{-1} = C_{\hat{\alpha}\hat{\beta}} = (\mathrm{Im}M)_{\hat{\alpha}\hat{\beta}}$.

Note that all the above definitions (4.2.14) - (4.2.17) correspond exactly to what is defined for the Kähler class moduli space in section 3.1, if one takes into account that the orientifold projection (4.1.5) mods out the states not invariant under the projection.
The same holds for the comparison between the complex structure moduli space defined in section 3.2 and the definitions (4.2.18) - (4.2.23).
Therefore, judging from the way the full Kähler potential, which follows from the reduction of $\mathcal{N}=2$ Type IIB supergravity on a Calabi-Yau threefold including $O3/O7$-planes, decomposes into two parts [34], the moduli space becomes block diagonal and reads

$$\mathcal{M} = \mathcal{M}^{h^{1,1}(Y_3)+1}_{\text{Kähler class}} \times \mathcal{M}^{h^{2,1}_{-}(Y_3)}_{\text{complex structure}}. \tag{4.2.24}$$

This scalar manifold is restricted to be Kähler since the $\mathcal{N}=1$ supersymmetry imposes this constraint in four dimensions. Even more, the moduli space is required to be a Hodge Kähler manifold, due to the fact that the chiral supermultiplets are a representation of $d=4$, $\mathcal{N}=1$ supergravity [47]. A Hodge Kähler manifold is a Kähler manifold with a Hodge metric, meaning that the fundamental Kähler form defines a cohomology class of integral forms. The latter are forms defined on the manifold whose integral over any cycle of this manifold is integer [56]. The Kähler class moduli space is a Kähler manifold and the complex structure moduli space is even a special Kähler manifold [34].

Furthermore, for orientifold compactifications with $O3/O7$-planes, relation (3.2.16) alters due to the fact that the fields $z^{\tilde{\alpha}}, \bar{z}^{\tilde{\alpha}}$ are projected out by the orientifold, resulting in [34]

$$\bar{M}_{\hat{\alpha}\hat{\beta}} \equiv \bar{M}_{IJ}\Big|_{z^{\tilde{\alpha}}=\bar{z}^{\tilde{\alpha}}=0} = \mathcal{H}_{IJ}\Big|_{z^{\tilde{\alpha}}=\bar{z}^{\tilde{\alpha}}=0}. \tag{4.2.25}$$

From here on we will denote $\mathcal{H}_{\hat{\alpha}\hat{\beta}} = \mathcal{H}_{IJ}\Big|_{z^{\tilde{\alpha}}=\bar{z}^{\tilde{\alpha}}=0}$.

From (4.2.6) it is clear that we should preform a second Weyl-rescaling in order to obtain the standard four-dimensional Einstein-Hilbert term. Thus, we rescale the action with $(g_{\mu\nu})^{old} = V_{Y_3}^{-1}(g_{\mu\nu})^{new}$ and rewrite the coefficient matrices using

$$\mathrm{dln}(V_{Y_3}) = \frac{1}{6V_{Y_3}}\mathrm{d}(\mathcal{K}_{\alpha\beta\gamma}v^\alpha v^\beta v^\gamma) = \frac{3}{6V_{Y_3}}\mathcal{K}_{\alpha\beta\gamma}v^\alpha v^\beta \mathrm{d}v^\gamma = \frac{1}{2V_{Y_3}}\mathcal{K}_\alpha \mathrm{d}v^\alpha \tag{4.2.26}$$

to obtain the low-energy four-dimensional effective $\mathcal{N}=1$ action [32]

$$\begin{aligned} S^{(4)}_{Bulk,\,\mathrm{EF}} = \frac{1}{2\kappa_4^2}\int \Big[& -R*\mathbb{1} + 2\mathcal{G}_{\tilde{a}\tilde{b}}\mathrm{d}z^{\tilde{a}}\wedge *\mathrm{d}\bar{z}^{\tilde{b}} + 2G_{\alpha\beta}\mathrm{d}v^\alpha\wedge *\mathrm{d}v^\beta \\ & + \frac{1}{2}\mathrm{d}(\ln V_{Y_3})\wedge *\mathrm{d}(\ln V_{Y_3}) + \frac{1}{2}\mathrm{d}\phi\wedge *\mathrm{d}\phi + \frac{1}{4}e^{2\phi}\mathrm{d}l\wedge *\mathrm{d}l + 2e^{-\phi}G_{ab}\mathrm{d}b^a\wedge *\mathrm{d}b^b \\ & + \frac{\mathcal{K}^2}{36}G_{\alpha\beta}\mathrm{d}D^\alpha_{(2)}\wedge *\mathrm{d}D^\beta_{(2)} + \frac{9}{4\mathcal{K}^2}G^{\alpha\beta}\mathrm{d}\rho_\alpha\wedge *\mathrm{d}\rho_\beta - \frac{9}{2\mathcal{K}^2}\mathcal{K}_{ab\alpha}G^{\alpha\beta}(c^a\mathrm{d}b^b)\wedge *\mathrm{d}\rho_\beta \\ & + \frac{9}{4\mathcal{K}^2}(\mathcal{K}_{ab\alpha}\mathcal{K}_{cd\beta}G^{\alpha\beta} + \mathcal{K}_{abe}\mathcal{K}_{cdf}G^{ef})(c^a\mathrm{d}b^b)\wedge *(c^c\mathrm{d}b^d) + \frac{1}{16}e^{-\phi}G^{ab}\mathrm{d}\tilde{c}^{(2)}_a\wedge *\mathrm{d}\tilde{c}^{(2)}_b \\ & + \frac{1}{16}e^{-\phi}\mathcal{K}_{ab\alpha}\mathcal{K}_{cd\beta}G^{bd}\big(\mathrm{d}b^a\wedge D^\alpha_{(2)}\big)\wedge *\big(\mathrm{d}b^c\wedge D^\beta_{(2)}\big) \\ & - \frac{1}{8}e^{-\phi}\mathcal{K}_{ab\alpha}G^{bc}\big(\mathrm{d}b^a\wedge D^\alpha_{(2)}\big)\wedge *\mathrm{d}\tilde{c}^{(2)}_c + e^\phi G_{ab}(\mathrm{d}c^a - l\mathrm{d}b^a)\wedge *(\mathrm{d}c^b - l\mathrm{d}b^b) \\ & + \frac{1}{4}e^{-2\phi}\mathrm{d}\tilde{l}^{(2)}\wedge *\mathrm{d}\tilde{l}^{(2)} + \frac{1}{4}e^{-2\phi}\mathrm{d}b^a\wedge\tilde{c}^{(2)}_a\wedge *(\mathrm{d}b^b\wedge\tilde{c}^{(2)}_b) - \frac{1}{2}e^{-2\phi}\big(\mathrm{d}b^a\wedge\tilde{c}^{(2)}_a\big)\wedge *\mathrm{d}\tilde{l}^{(2)} \\ & + \frac{1}{4}B_{\hat{\alpha}\hat{\beta}}\mathrm{d}V^{\hat{\alpha}}\wedge *\mathrm{d}V^{\hat{\beta}} - \frac{1}{4}C^{\hat{\alpha}\hat{\beta}}\mathrm{d}U_{\hat{\alpha}}\wedge *\mathrm{d}U_{\hat{\beta}} - \frac{1}{2}A_{\hat{\beta}}{}^{\hat{\alpha}}\mathrm{d}U_{\hat{\alpha}}\wedge *\mathrm{d}V^{\hat{\beta}}\Big]. \end{aligned} \tag{4.2.27}$$

An important remark here is that (4.2.27) has double the amount of degrees of freedom that we would expect. This is due to the fact that we have compactified the democratic version of the Type IIB action and have yet to impose the duality relations (4.1.9).

Chapter 5

IIB reduction on $\mathbf{Y_3}$ including a D7-brane

We will include a single spacetime filling D7-brane in this chapter. Meaning, we will be discussing a $U(1)$ Abelian gauge theory. After generally defining the (2,2)-cycle on which the brane is wrapped, we consider the spectrum of the brane consisting of the $U(1)$ gauge field on the brane and the fluctuations in the directions normal to the brane. Hereafter, compactifying and collecting all lower-dimensional terms obtained so far, we impose the self-duality relations upon the action. Finally, we write the obtained four-dimensional theory in the $\mathcal{N} = 1$ supergravity representation and read of the gauge kinetic coupling function of which we discuss its holomorphic property.

5.1 D7-brane

The internal part of the D7-brane worldvolume, $\mathcal{W}$, has four legs on Y_3. Therefore, in reducing the action, it is wrapped on a (2,2)-cycle of the Calabi-Yau threefold, due to restrictions on Y_3, which are captured in the Hodge diamond (4.1.2).
Consider a four-cycle $S_1 \subset Y_3$ on which the D7-brane is wrapped. Since we discuss an orientifold theory $M_{(1,9)} = \mathbb{R}^{1,3} \times Y_3/O$, we must include its image D7-brane wrapped on the four-cycle $\sigma(S_1)$. We will assume that the D7-brane and its image are disjoint, meaning $S_1 \cap \sigma(S_1) = \emptyset$. However, we only wanted to include a single spacetime

filling D7-brane. Therefore, we construct the internal four-cycle

$$S_+ = S_1 \cup \sigma(S_1) \tag{5.1.1}$$

on which the brane is wrapped, such that $\mathcal{W} = \mathbb{R}^{1,3} \times S_+$, the worldvolume of the D7-brane, is invariant under the involution, making it an O-plane. In this way, we prevent the problem of having to include the image brane by construction. By requiring the empty intersection $S_1 \cap \sigma(S_1) = \emptyset$, we implicitly required that the involution does not have any fixed points in the four-cycle S_1. If this would have been the case, we would expect extra massless states in the twisted open string sector [57]. For later convenience we define

$$S_- = S_1 \cup -\sigma(S_1), \tag{5.1.2}$$

where the minus sign denotes the flipping of orientation of the cycle. Thus, the two cycles we have defined obey $\sigma(S_\pm) = \pm S_\pm$.

The spectrum of the D7-brane consists of two parts, namely the degrees of freedom due to the $U(1)$ gauge field on the brane and secondly the fluctuations of the embedding of the four-cycle into the two directions of the Calabi-Yau threefold, normal to the internal cycle. Both type of degrees of freedom belong to the bosonic part of the action. We will start by discussing the former.

5.1.1 $U(1)$ gauge field on the brane

Since the $U(1)$ gauge boson $\hat{A}$ on the worldvolume of the brane is negative under the world-sheet parity operator Ω_p, we expand it in harmonics of the four-cycle S_+

$$\hat{A} = A(x)P_-(y) + a_I(x)A^I(y) + \bar{a}_{\bar{J}}(x)\bar{A}^{\bar{J}}(y), \tag{5.1.3}$$

where $\{A^I\}$ and $\{\bar{A}^{\bar{J}}\}$ form a basis of $H^{0,1}_{\bar{\partial},-}(S_+)$ and $H^{1,0}_{\bar{\partial},-}(S_+)$ respectively.Furthermore, $P_-(y)$ is an element of $H^0_-(S_+)$ obeying the relation

$$P_-(y) = \begin{cases} +1 & y \in S_1, \\ -1 & y \in \sigma(S_1). \end{cases} \tag{5.1.4}$$

The lower-dimensional Minkowski fields resulting from this expansion are a four-dimensional $U(1)$ gauge vector $A(x)$ and the Minkowski scalars a_I and $\bar{a}_{\bar{J}}$, called Wilson lines. The four-dimensional gauge vectors A^I and $\bar{A}^{\bar{J}}$ are on the cycle S_+. The lower-dimensional vector boson $A(x)$ contains less degrees of freedom than the higher-dimensional gauge vector $\hat{A}$. However, all gauge degrees of freedom of the original eight-dimensional vector boson $\hat{A}$ should be captured in the lower-dimensional theory. Those gauge degrees of freedom captured by the Wilson lines give rise to a shift symmetry of both $a_I(x)$ and $\bar{a}_{\bar{J}}(x)$. This is demonstrated with a simple example of a circle reduction in appendix D.
Due to the expansion of the ten-dimensional gauge vector $\hat{A}$ in the forms A^I which constitute a basis of $H^{0,1}_{\bar{\partial},-}(S_+)$, their coefficients a_I are holomorphic in the complex structure moduli fields. Similarly, the fields $\bar{a}_{\bar{J}}$ are anti-holomorphic in the complex structure.
Additionally D-branes may carry lower-dimensional Ramond-Ramond charges, distributed over the brane, which appear as background fluxes f within the field strength. Therefore, the field strength $\hat{F}$ on the brane in the presence of background fluxes is defined as

$$\hat{F} = f + d\hat{A}, \tag{5.1.5}$$

such that $\langle\hat{F}\rangle = f$ an harmonic two-form on the worldvolume of the D7-brane which reads

$$f = f^a \iota^* \omega_a. \tag{5.1.6}$$

Note that $\iota^*\omega_a$ is an element of $H^2_-(S_+)$ which is in accord with the fact that the field strength $\hat{F}$ is negative under the involution, which was already displayed in the expansion in (5.1.3). We defined ι to be the map $\iota : S_+ \hookrightarrow Y_3$ which embeds the cycle S_+ into the Calabi-Yau threefold, therefore ι^* is the pullback of this embedding. An important remark here is that we have implicitly assumed the pullback of the harmonics in the cohomology class $H^{1,1}_{\bar{\partial},-}(Y_3)$ to be the only two-forms on the cycle, negative under the involution.

5.1.2 Normal coordinates

Apart from that the brane is accompanied with $U(1)$ gauge fields that live on it, as described in the previous section 5.1.1, the brane itself is a dynamical object, which

can vibrate in the directions normal to the worldvolume of the brane, in the Calabi-Yau threefold. These fluctuations are encoded in the map $\varphi : \mathcal{W} \hookrightarrow M_{(1,9)}$ embedding the worldvolume of the brane into the ten-dimensional spacetime. Since we are describing the fluctuations of the brane in the directions normal to the worldvolume, these fluctuations are sections of the real normal bundle of the cycle in the Calabi-Yau threefold. We denote the fluctuations with $\hat{\zeta} \in H^0_+(S_+, N^{\mathbb{R}}_{Y_3}S_+)$, where $N^{\mathbb{R}}_{Y_3}S_+$ is notation for the real normal bundle of the cycle S_+ in the Calabi-Yau threefold Y_3. Note that because the world-sheet parity operator Ω_p acts with a plus on states normal to the brane, we consider the positive eigenspace of $H^0(S_+, N^{\mathbb{R}}_{Y_3}S_+)$ which does not get projected out by the orientifold (4.1.5).
Now decomposing these real sections $\hat{\zeta}$ into a holomophic part ζ, coming from the holomorphic normal bundle $N_{Y_3}S_+$, and an anti-holomorphic part $\bar{\zeta}$, from the anti-holomorphic normal bundle $\overline{N_{Y_3}S_+}$, we can expand [54]

$$\hat{\zeta} = \hat{\zeta}^{\mathcal{A}}(x)\hat{s}_{\mathcal{A}}(y) = \zeta + \bar{\zeta} = \zeta^A(x)s_A(y) + \bar{\zeta}^{\bar{A}}(x)\bar{s}_{\bar{A}}(y). \tag{5.1.7}$$

In this notation $\{s_A\}$ constitutes a basis of $H^0_+(S_+, N_{Y_3}S_+)$ and their complex conjugates $\{\bar{s}_{\bar{A}}\}$ for $H^0_+(S_+, \overline{N_{Y_3}S_+})$. Furthermore, the coefficients $\zeta^A, \bar{\zeta}^{\bar{A}}$ turn out to be scalar fields in the effective lower-dimensional theory, after compactification.

The decomposition of the real normal bundle into an holomorphic and an anti-holomorphic part, depends on the background complex structure of the Calabi-Yau threefold. Therefore, when considering the vector field s_A it is natural to explore the interior product with the complex structure forms $\Omega, \chi_{\tilde{a}}$ and their complex conjugates. Consider the background complex structure of the Calabi-Yau threefold to be z_0, then we obtain

$$s_A \lrcorner \Omega(z_0) = 0, \qquad s_A \lrcorner \bar{\Omega}(z_0) = 0, \qquad s_A \lrcorner \bar{\chi}_{\tilde{a}}(z_0) = 0, \tag{5.1.8}$$

in the cohomology of Y_3. This is a logical result since $h^{2,0} = 0$ for a Calabi-Yau threefold. However, on the (2,2)-cycle S_+ we do not have this information. If we would have considered for instance a D5-brane wrapped around a (1,1)-cycle on the Calabi-Yau threefold, we would have been able to obtain more specified information. For the sake of getting a notion on what is relevant to keep in mind, we will explore this situation briefly.

Consider an identical situation but with a spacetime filling D5-brane wrapped around the (1,1)-cycle Σ, instead of a D7-brane. For a supersymetrically embedded D5-brane, any two-form that is pulled back to the cycle Σ has to be proportional to the (1,1)-Kähler form. Therefore, in the background z_0 the only non-trivial two-form, resulting from the interior product of the vector field s_A with a holomorphic form $\Omega, \chi_{\tilde{a}}$ or their complex conjugates, which can be pulled back to Σ is $s_A \lrcorner \chi_{\tilde{a}}(z_0)$. In other words, equation (5.1.8) also holds for the cohomology of the cycle Σ.
When in addition taking into account that the complex structure may vary around its supersymmetric background value z_0, we can expand the complex structure as $z = z_0 + \delta z$. To understand how this differs, we consider $\iota^*(s_A \lrcorner \Omega(z))$ and expand to linear order in δz [54]

$$\iota^*(s_A \lrcorner \Omega(z)) = \iota^*(s_A \lrcorner \Omega(z_0)) + \delta z^{\tilde{a}} \iota^*(k_{\tilde{a}} s_A \lrcorner \Omega(z_0) + i s_A \lrcorner \chi_{\tilde{a}}(z_0)) = i \delta z^{\tilde{a}} \iota^*(s_A \lrcorner \chi_{\tilde{a}}(z_0)), \tag{5.1.9}$$

where we have made use of (3.2.3) and that (5.1.8) holds for the cohomology of the (1,1)-cycle. Equation (5.1.9) explicitly shows that even though $s_A \lrcorner \Omega$ is a (2,0)-form on the (1,1)-cycle Σ in the complex structure z, it is a (1,1)-form on Σ in the background complex structure z_0, up to first order fluctuations.

Back to the D7-brane, unfortunately we cannot derive such equalities that easily due to the fact that we do not know a similar relation to (5.1.8) for the complex structure background on the (2,2)-cycle S_+. Therefore, the best we can do for now is to phrase that, in the complex structure background z_0, $s_A \lrcorner \Omega(z_0)$ is an element of $H^{2,0}_{\bar{\partial},-}(S_+)$, $s_A \lrcorner \bar{\Omega}(z_0) = 0$, $s_A \lrcorner \chi_{\tilde{a}}(z_0) \in H^{1,1}_{\bar{\partial},-}(S_+)$, $s_A \lrcorner \bar{\chi}_{\tilde{a}}(z_0) \in H^{0,2}_{\bar{\partial},-}(S_+)$ in the cohomology of the (2,2)-cycle and similar for $\bar{s}_{\bar{A}}$.

5.2 Compactification Chern-Simons action

In the Abelian case, the general form of the Chern-Simons action of a Dp-brane is given by [50]

$$S^{(10)}_{CS} = \mu_p \int_{\mathcal{W}} \sum_q \phi^*_\zeta(\hat{C}^{(q)}) e^{\ell \hat{F} - \varphi^*_\zeta \hat{B}}. \tag{5.2.1}$$

This part of the action captures the interactions between the gauge fields on the brane coupled to the Ramond-Ramond fields on the bulk. The integral is over the (p+1)-dimensional worldvolume $\mathcal{W} = \mathbb{R}^{1,3} \times S_+$, which is allowed to fluctuate since it represents a dynamical object. Furthermore, the exponential is meant to be a wedged power series of the form $\ell\hat{F} - \varphi_\zeta^*\hat{B}$, wedged with the pullback of $\hat{C}^{(q)}$ to the brane such that the total integrand adds up to an 8-form that is compatible with the worldvolume. With the latter we imply that the integrand should be an 8-form split into a 4-form defined on Minkowski spacetime and another 4-form on the (2,2)-cycle S_+.

Therefore, one needs to know how to cope with the pullback to the worldvolume of the brane, given by the inverse map of $\varphi : \mathcal{W} \hookrightarrow M_{(1,9)}$. This is done by expanding the ten-dimensional forms according to a Kaluza-Klein expansion on the manifold $M_{(1,9)} = \mathbb{R}^{1,3} \times Y_3/O$ and pull back the forms on the Calabi-Yau orientifold to the cycle with the inverse map of $\iota : S_+ \hookrightarrow Y_3$.

The normal coordinate expansion of the anti-symmetric two-form $\hat{B}$, that captures the fluctuations on the D7-brane in the two directions normal to the brane, is given by [32, 45]

$$\varphi_\zeta^*\hat{B} = b^a\iota^*\omega_a + b_{m\bar{n}}\partial_\mu\zeta^m\partial_\nu\bar{\zeta}^{\bar{n}}\mathrm{d}x^\mu\mathrm{d}x^\nu, \tag{5.2.2}$$

up to second order in derivatives.

Since we are mainly interested in the gauge coupling between the gauge vectors, both arising from the bulk and the brane, we will explicitly compactify only the specific part of the Chern-Simons action involving the gauge vector fields

$$\frac{\mu_7}{2}\int_{\mathcal{W}} \varphi_\zeta^*(V^{\hat{\alpha}}(x) \wedge \alpha_{\hat{\alpha}} + U_{\hat{\alpha}}(x) \wedge \beta^{\hat{\alpha}}) \wedge (\ell\hat{F} - \varphi_\zeta^*\hat{B}) \wedge (\ell\hat{F} - \varphi_\zeta^*\hat{B}), \tag{5.2.3}$$

coming from the ten-dimensional four-form potential $\hat{C}^{(4)}$. Using (5.2.2) and the expansion of the gauge field strength $\hat{F}$, while redefining

$$\mathcal{B} = \mathcal{B}^a\iota^*\omega_a = (b^a - \ell f^a)\iota^*\omega_a, \tag{5.2.4}$$

we obtain up to second order in derivatives

$$\ell\hat{F} - \varphi_\zeta^*\hat{B} = -\mathcal{B}^a\iota^*\omega_a + \ell(\mathrm{d}A \wedge P_- + \mathrm{d}a_I \wedge A^I + \mathrm{d}\bar{a}_{\bar{J}} \wedge \bar{A}^{\bar{J}}) - b_{m\bar{n}}\partial_\mu\zeta^m\partial_\nu\bar{\zeta}^{\bar{n}}\mathrm{d}x^\mu\mathrm{d}x^\nu. \tag{5.2.5}$$

Furthermore, to compactify the Chern-Simons action (5.2.1), one needs to know how to pullback a q-form from the Calabi-Yau threefold to the worldvolume of the brane. Incorporating fluctuations in the directions normal to the brane, one can write up to second order in derivatives

$$\begin{aligned}\varphi_\zeta^*(\hat{C}^{(q)}) =& \Big(\frac{1}{q!}C^{(q)}_{\mu_1\dots\mu_q} + \frac{1}{q!}\zeta^n\partial_n(C^{(q)}_{\mu_1\dots\mu_q}) - \frac{1}{(q-1)!}\nabla_{\mu_1}\zeta^n C^{(q)}_{n\mu_2\dots\mu_q} \\ &+ \frac{1}{2q!}\zeta^n\partial_n(\zeta^m\partial_m(C^{(q)}_{\mu_1\dots\mu_q})) - \frac{1}{(q-1)!}\nabla_{\mu_1}\zeta^n\zeta^m\partial_m C^{(q)}_{n\mu_2\dots\mu_q} \\ &+ \frac{1}{2(q-2)!}\nabla_{\mu_1}\zeta^n\nabla_{\mu_2}\zeta^m C^{(q)}_{nm\mu_3\dots\mu_q} + \frac{q-2}{2q!}R_n{}^k{}_{\mu_1 m}\zeta^n\zeta^m C^{(q)}_{k\mu_2\dots\mu_q}\Big) \\ &\mathrm{d}x^{\mu_1}\wedge\dots\wedge\mathrm{d}x^{\mu_q}.\end{aligned} \tag{5.2.6}$$

Note that in this expression the directions normal to the worldvolume of the brane are denoted with Roman indices and the directions tangent to the brane are denoted with Greek indices. Furthermore, ∇ denotes a covariant derivative of the normal bundle.

Combining all the above and noting that the cycle S_+ on which the D7-brane is wrapped is a (2,2)-cycle, we obtain from (5.2.3) up to second order in derivatives the non-vanishing terms

$$\begin{aligned}&\mu_7\int_{\mathcal{W}}\ell^2(V^{\hat{\alpha}}\wedge\iota^*\alpha_{\hat{\alpha}} + U_{\hat{\alpha}}\wedge\iota^*\beta^{\hat{\alpha}})\wedge\mathrm{d}A\wedge P_-\wedge(\mathrm{d}a_I\wedge A^I + \mathrm{d}\bar{a}_J\wedge\bar{A}^J) \\ &= -\mu_7\int\ell^2(a_I\mathcal{A}^I_{\hat{\alpha}} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\bar{J}}_{\hat{\alpha}})\mathrm{d}V^{\hat{\alpha}}\wedge\mathrm{d}A + (a_I\mathcal{A}^{\hat{\alpha}I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\alpha}\bar{J}})\mathrm{d}U_{\hat{\alpha}}\wedge\mathrm{d}A.\end{aligned} \tag{5.2.7}$$

Here we have repeatedly used some of the conventions of section 2.1.1 and defined the matrices

$$\begin{aligned}\mathcal{A}^{\hat{\alpha}I} &= \int_{S_-}\iota^*\beta^{\hat{\alpha}}\wedge A^I, &\qquad \mathcal{A}^I_{\hat{\alpha}} &= \int_{S_-}\iota^*\alpha_{\hat{\alpha}}\wedge A^I, \\ \bar{\mathcal{A}}^{\hat{\alpha}\bar{J}} &= \int_{S_-}\iota^*\beta^{\hat{\alpha}}\wedge\bar{A}^{\bar{J}}, &\qquad \bar{\mathcal{A}}^{\bar{J}}_{\hat{\alpha}} &= \int_{S_-}\iota^*\alpha_{\hat{\alpha}}\wedge\bar{A}^{\bar{J}}.\end{aligned} \tag{5.2.8}$$

The other terms in the action (5.2.1) are compactified in a similar fashion with expansions in massless Kaluza-Klein modes given in (4.1.13), (4.1.14) and (5.1.3), which results in the lower-dimensional effective action [32]

$$
\begin{aligned}
S_{CS}^{(4)} = \mu_7 \int \Big(&\frac{1}{4}\mathrm{d}\tilde{l}^{(2)} - \mathrm{d}(\tilde{c}_a^{(2)}\mathcal{B}^a) + \frac{1}{2}\mathcal{K}_{\alpha bc}\mathrm{d}(D_{(2)}^{\alpha}\mathcal{B}^b\mathcal{B}^c)\Big) \wedge \mathcal{L}_{A\bar{B}}\Big(\mathrm{d}\zeta^A\bar{\zeta}^{\bar{B}} - \mathrm{d}\bar{\zeta}^{\bar{B}}\zeta^A\Big) \\
&- \ell\mathrm{d}\Big(\tilde{c}_P^{(2)} - \mathcal{K}_{\alpha bP}D_{(2)}^{\alpha}\mathcal{B}^b\Big) \wedge A + \ell^2\Big[-\frac{1}{2}C_{\alpha}^{I\bar{J}}\mathrm{d}D_{(2)}^{\alpha} \wedge \Big(\mathrm{d}a_I\bar{a}_{\bar{J}} - \mathrm{d}\bar{a}_{\bar{J}}a_I\Big) \\
&+ \frac{1}{2}\Big(\rho_\Lambda - \mathcal{K}_{\Lambda ab}c^a\mathcal{B}^b + \frac{1}{2}\mathcal{K}_{\Lambda ab}\mathcal{B}^a\mathcal{B}^b l\Big)F \wedge F \\
&- \Big((a_I\mathcal{A}_{\hat{\alpha}}^I + \bar{a}_{\bar{J}}\bar{\mathcal{A}}_{\hat{\alpha}}^{\bar{J}})\mathrm{d}V^{\hat{\alpha}} \wedge F + (a_I\mathcal{A}^{\hat{\alpha}I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\alpha}\bar{J}})\mathrm{d}U_{\hat{\alpha}} \wedge F\Big)\Big]
\end{aligned}
\tag{5.2.9}
$$

Important to note here is that integration over the function P_- switches the integration domain from S_+ to S_- and visa versa. This follows directly from the definition of these cycles (5.1.1) and (5.1.2) and the definition of the function (5.1.4).

Note that in this derivation we have repeatedly used (2.1.22) and (2.1.24) to obtain

$$\tilde{c}_P^{(2)} \equiv \tilde{c}_a^{(2)} \wedge \int_{S_-} \iota^*\tilde{\omega}^a = \tilde{c}_a^{(2)} \wedge \int_{S_+} \iota^*\tilde{\omega}^a P_-, \tag{5.2.10}$$

$$\rho_\Lambda \equiv \rho_\alpha \wedge \int_{S_+} \iota^*\tilde{\omega}^\alpha. \tag{5.2.11}$$

This notation will be used throughout the thesis, denoting the subscripts Λ and P for a contraction with an integral over the cycles S_+ and S_- respectively of their corresponding pulled back (2,2)-forms $\tilde{\omega}^a P_-$ / $\tilde{\omega}^\alpha$ and $\tilde{\omega}^a$ / $\tilde{\omega}^\alpha P_-$.

Lastly, we have defined the matrices $\mathcal{L}_{A\bar{B}}$ and $C_\alpha^{I\bar{J}}$ to be

$$\mathcal{L}_{A\bar{B}} = \frac{\int_{S_+} s_A \lrcorner \Omega \wedge \bar{s}_{\bar{B}} \lrcorner \bar{\Omega}}{\int_{Y_3} \Omega \wedge \bar{\Omega}}, \tag{5.2.12}$$

and

$$C_\alpha^{I\bar{J}} = \int_{S_+} \iota^*\omega_\alpha \wedge A^I \wedge \bar{A}^{\bar{J}}. \tag{5.2.13}$$

5.3 Compactification Dirac-Born-Infeld action

When including a single Dp-brane to the Type IIB orientifolded theory, one should add the Abelian Dirac-Born-Infeld action which captures the degrees of freedom of the kintetic terms of the D7-brane. The Dirac Born Infeld action can in general be displayed in the string frame as

$$S^{(10)}_{DBI,\,\mathrm{SF}} = -\mu_7 \int_{\mathcal{W}} \mathrm{d}^8 x e^{-\hat{\phi}} \sqrt{-\det(\varphi^*_\zeta(\hat{g} + \hat{B}_2) - \ell \hat{F})}. \tag{5.3.1}$$

In this section we will not go through the compactification in detail, but only give the main techniques and state the result. For a more detailed description of the reduction we refer the reader to [32].

The action (5.3.1) shows that we need to know how to pull back forms from the Calabi-Yau threefold to the worldvolume of the brane. Equation (5.2.5) already views how to cope with $\varphi^*_\zeta(\hat{B}_2) - l\hat{F}$ in the action. Hence, at this point we are interested in the normal coordinate expansion of the metric (4.1.1), which is given by

$$\varphi^*_\zeta \hat{g} = \frac{6}{\mathcal{K}} e^{\phi/2} \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + 2e^{\phi/2} g_{m\bar{n}} \mathrm{d}y^m \mathrm{d}\bar{y}^{\bar{n}} + 2e^{\phi/2} g_{m\bar{n}} \mathrm{d}\zeta^m \mathrm{d}\bar{\zeta}^{\bar{n}}, \tag{5.3.2}$$

after a Weyl rescaling. To be able to drop the square root in the Dirac-Born-Infeld action, the action is expanded according to the Taylor series

$$\sqrt{\det(\mathfrak{U} + \mathfrak{B})} = \sqrt{\det \mathfrak{U}} \Big[1 + \frac{1}{2} \mathrm{Tr} \mathfrak{U}^{-1} \mathfrak{B} + \frac{1}{8} \Big((\mathrm{Tr} \mathfrak{U}^{-1} \mathfrak{B})^2 - 2\mathrm{Tr}(\mathfrak{U}^{-1} \mathfrak{B})^2 \Big) + \mathcal{O}(\mathfrak{B}^3) \Big] \tag{5.3.3}$$

up to second order. Applying this Taylor series to the action (5.3.1), we split the determinant into two parts. Namely, with $\mathfrak{U}$ we represent the background configuration, both of Minkowski spacetime and the Calabi-Yau threefold including the spacetime filling D7-brane. Furthermore, $\mathfrak{B}$ encodes the fluctuations around this background. These fluctuations are coming from three different causes. First, of course we have expanded the NS-NS fields $\hat{g}$ and $\hat{B}$ in perturbations around their background. Second, we have included the fluctuations of the embedding ι of the cycle into the Calabi-Yau threefold, which are parametrized by the normal coordinates and result in the four-dimensional fields $\zeta^{\mathcal{A}}$ and $\bar{\zeta}^{\bar{\mathcal{A}}}$. And third, we consider the fluctuations around the background of the $U(1)$ vector boson resulting in the inclusion of the Wilson lines $a^I, \bar{a}^{\bar{J}}$. Using (5.3.3) and inserting the calibration condition [32]

$$\mathrm{d}^4x\sqrt{-\det(e^{\hat{\phi}/2}\varphi^*\hat{g}+\mathcal{B}^a\iota^*\omega_a)} = \frac{1}{2}e^{\hat{\phi}}\varphi^*J\wedge\varphi^*J - \frac{1}{2}\mathcal{B}^a\iota^*\omega_a\wedge\mathcal{B}^b\iota^*\omega_b, \tag{5.3.4}$$

which ensures supersymmetry and the fact that the spacetime filling D7-brane is a BPS state, one obtains the effective four-dimensional Dirac-Born-Infeld action [32]

$$\begin{aligned} S^{(4)}_{DBI,\,\mathrm{EF}} &= \mu_7\ell^2\int\Big[\frac{1}{4}(\mathcal{K}_\Lambda - e^{-\phi}\mathcal{K}_{\Lambda ab}\mathcal{B}^a\mathcal{B}^b)F\wedge *F + \frac{12}{\mathcal{K}}iC^{I\bar{J}}_\alpha v^\alpha \mathrm{d}a_I\wedge *\mathrm{d}\bar{a}_{\bar{J}}\Big] \\ &\quad + \mu_7\int\Big[i\mathcal{L}_{A\bar{B}}(e^\phi - G_{ab}\mathcal{B}^a\mathcal{B}^b)\mathrm{d}\zeta^A\wedge *\mathrm{d}\bar{\zeta}^{\bar{B}} + \frac{18}{\mathcal{K}^2}(e^\phi\mathcal{K}_\Lambda - \mathcal{K}_{\Lambda ab}\mathcal{B}^a\mathcal{B}^b) * \mathbb{1}\Big]. \end{aligned} \tag{5.3.5}$$

We will not elaborate on the calibration condition of a Dp-brane. However, a more detailed discussion is given in [32].

5.4 Imposing self-dual relations on gauge vectors

The not yet dualized action obtained so far still has to many degrees of freedom. Recall from chapter 4 that the vector fields $U_{\hat{\alpha}}$ and $V^{\hat{\alpha}}$ are related to each other via the self-duality relation given in (4.1.9), which has to be imposed upon the action. Since we are mainly interested in the lower-dimensional gauge vector fields arising from the D7-brane and the four-form potential $\hat{C}^{(4)}$ of the bulk, we will explicitly carry out the dualization for this part of the action. We first collect only the terms relevant for the gauge kinetic coupling functions [32]

$$\begin{aligned} S^{(4)}_{gauge} = \frac{1}{2\kappa_4^2}\int & \frac{1}{4}B_{\hat{\alpha}\hat{\beta}}\mathrm{d}V^{\hat{\alpha}}\wedge *\mathrm{d}V^{\hat{\beta}} - \frac{1}{4}C^{\hat{\alpha}\hat{\beta}}\mathrm{d}U_{\hat{\alpha}}\wedge *\mathrm{d}U_{\hat{\beta}} - \frac{1}{2}A_{\hat{\beta}}{}^{\hat{\alpha}}\mathrm{d}U_{\hat{\alpha}}\wedge *\mathrm{d}V^{\hat{\beta}} \\ & + \kappa_4^2\mu_7 l^2(\frac{1}{2}\mathcal{K}_\Lambda - \frac{1}{2}e^{-\phi}\mathcal{K}_{\Lambda ab}\mathcal{B}^a\mathcal{B}^b)F\wedge *F \\ & + \kappa_4^2\mu_7 l^2(\rho_\Lambda - \mathcal{K}_{\Lambda ab}c^a\mathcal{B}^b + \frac{1}{2}l\mathcal{K}_{\Lambda ab}\mathcal{B}^a\mathcal{B}^b)F\wedge F \\ & - 2\kappa_4^2\mu_7 l^2((a_I\mathcal{A}^I_{\hat{\alpha}} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\bar{J}}_{\hat{\alpha}})\mathrm{d}V^{\hat{\alpha}}\wedge F + (a_I\mathcal{A}^{\hat{\alpha}I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\alpha}\bar{J}})\mathrm{d}U_{\hat{\alpha}}\wedge F). \end{aligned} \tag{5.4.1}$$

We repeatedly use relations (4.2.20) - (4.2.22) and (2.1.12) and the fact that

$$*_d *_d F^{(p)} = (-1)^{p(d-p)+\delta} \quad \text{with} \begin{cases} \delta = 1 \text{ for a Lorentzian metric,} \\ \delta = 0 \text{ for an Euclidean metric,} \end{cases} \tag{5.4.2}$$

for $F^{(p)}$ a p-form in d dimensions, to rewrite the self-duality relation obtaining new constraints. Collecting all terms that correspond to the same basis form $\alpha_{\hat{\alpha}}$ of the self-duality relation $\hat{G}^{(5)} = \hat{*}\hat{G}^{(5)}$ results in the constraint

$$-\,\mathrm{d}U_{\hat{\alpha}} = C_{\hat{\alpha}\hat{\beta}} * \mathrm{d}V^{\hat{\beta}} + \mathrm{d}V^{\hat{\gamma}} A_{\hat{\gamma}}^{\ \hat{\beta}} C_{\hat{\beta}\hat{\alpha}}, \tag{5.4.3}$$

which needs to be imposed upon dualization. Similarly collecting all terms corresponding to $\beta^{\hat{\alpha}}$ results in another self-duality constraint

$$\mathrm{d}V^{\hat{\alpha}} = -B^{\hat{\alpha}\hat{\beta}} * \mathrm{d}U_{\hat{\beta}} + B^{\hat{\alpha}\hat{\gamma}} A_{\hat{\gamma}}^{\ \hat{\beta}} \mathrm{d}U_{\hat{\beta}}, \tag{5.4.4}$$

defining $B^{\hat{\gamma}\hat{\beta}}$ to be the inverse matrix of B, i.e. $B_{\hat{\alpha}\hat{\gamma}} B^{\hat{\gamma}\hat{\beta}} = \delta_{\hat{\alpha}}^{\hat{\beta}}$.

For simplicity, we will first dualize the action omitting the D7-brane gauge vector field. To do so we introduce a vector field $\tilde{V}^{\hat{\alpha}}$ functioning as a Lagrange multiplier and include the term $\frac{1}{2}\mathrm{d}\tilde{V}^{\hat{\alpha}} \wedge H_{\hat{\alpha}}$ to the action

$$\begin{aligned} S^{(4)}_{gauge} = \frac{1}{2\kappa_4^2} \int & \frac{1}{4} B_{\hat{\alpha}\hat{\beta}} \mathrm{d}V^{\hat{\alpha}} \wedge *\mathrm{d}V^{\hat{\beta}} - \frac{1}{4} C^{\hat{\alpha}\hat{\beta}} H_{\hat{\alpha}} \wedge *H_{\hat{\beta}} - \frac{1}{2} A_{\hat{\beta}}^{\ \hat{\alpha}} H_{\hat{\alpha}} \wedge *\mathrm{d}V^{\hat{\beta}} \\ & + \frac{1}{2} \mathrm{d}\tilde{V}^{\hat{\alpha}} \wedge H_{\hat{\alpha}}. \end{aligned} \tag{5.4.5}$$

In this way the equation of motion for $\tilde{V}^{\hat{\alpha}}$ incorporates the trivial Bianchi identity $\mathrm{d}H_{\hat{\alpha}} = 0$ of the field, to maintain the information that locally $H_{\hat{\alpha}} = \mathrm{d}U_{\hat{\alpha}}$. Thus, we constructed an action that is a functional of $V^{\hat{\alpha}}$ and $H_{\hat{\alpha}}$ instead of $\mathrm{d}U_{\hat{\alpha}}$.

Within this limit of neglecting the brane vector fields,

$$H_{\hat{\alpha}} = -(\mathrm{Re}M)_{\hat{\alpha}\hat{\beta}} \mathrm{d}V^{\hat{\beta}} - (\mathrm{Im}M)_{\hat{\alpha}\hat{\beta}} * \mathrm{d}\tilde{V}^{\hat{\beta}} \tag{5.4.6}$$

are the equations of motion for the field $H_{\hat{\alpha}}$. Note the striking similarity with the self-duality constraint given in (5.4.3). Of course the equations of motion of the dualized field have to be in accord with the self-duality constraints, since these constraints need to be imposed upon the action. This is done by substituting the equations of motion into the action. To obtain equations of motion that match the self-duality constraint, we conclude that the Lagrange multiplier vector field $\tilde{V}^{\hat{\alpha}}$, must correspond to the bulk gauge vector field $V^{\hat{\alpha}}$. Substituting these equations of motion into the action, which corresponds to imposing the duality constraints upon the action, results in

$$\begin{aligned} S^{(4)}_{gauge} &= \frac{1}{2\kappa_4^2}\int -\frac{1}{4}(I + RI^{-1}R)\mathrm{d}\tilde{V}\wedge *\mathrm{d}\tilde{V} \\ &\qquad -\frac{1}{4}I(-\mathrm{d}\tilde{V} + RI^{-1} * \mathrm{d}\tilde{V})\wedge(- * \mathrm{d}\tilde{V} - RI^{-1}\mathrm{d}\tilde{V}) \\ &= \frac{1}{2\kappa_4^2}\int -\frac{1}{2}I\mathrm{d}\tilde{V}\wedge *\mathrm{d}\tilde{V} - \frac{1}{2}R\mathrm{d}\tilde{V}\wedge\mathrm{d}\tilde{V}, \end{aligned} \tag{5.4.7}$$

where we have used matrices (4.2.23), switched from Einstein summation convention to regular matrix multiplication and for convenience denoted ReM and ImM by R and I, respectively.

Thus, we have obtained the dualized action when neglecting the D7-brane gauge vector fields. For these fields to be included into the dualization, one still has to impose the self-duality constraint (5.4.3). However, due to the D7-brane vectors, the action now takes the form (5.4.1), from which we observe that couplings between the gauge vectors from the bulk and brane are added. We refer to these terms as source terms. For the sake of keeping a clean overview we denote these sources with $J^{\hat{\alpha}} = 4\kappa_4^2\mu_7 l^2(a_I\mathcal{A}^{\hat{\alpha}I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\alpha}\bar{J}})F$ and $J_{\hat{\alpha}} = 4\kappa_4^2\mu_7 l^2(a_I\mathcal{A}_{\hat{\alpha}}^{\ I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}_{\hat{\alpha}}^{\ \bar{J}})F$. Due to these source terms, the Bianchi identities for both fields $U_{\hat{\alpha}}$ and $V^{\hat{\alpha}}$ have been altered and are not trivial anymore. Therefore, the Lagrange multiplier that previously included the trivial Bianchi identities for the fields can no longer be blindly added. Though in dualizing the action (5.4.1), which is a functional of $U_{\hat{\alpha}}$ and $V^{\hat{\alpha}}$, the main techniques are not altered thus we still need to adjust the action to being a functional of the two-form $H_{\hat{\alpha}}$ and the vector field $V^{\hat{\alpha}}$. In doing so we include a Lagrange multiplier term that does incorporates the correct Bianchi identities of the fields in the action. We derive these Bianchi identities by substituting the duality constraints into the equations of motion of the fields. The equations of motion of $U_{\hat{\alpha}}$ and $V^{\hat{\alpha}}$ following from (5.4.1) are respectively

$$-\frac{1}{2}C^{\hat{\alpha}\hat{\beta}}\mathrm{d}*\mathrm{d}U_{\hat{\beta}} - \frac{1}{2}A_{\hat{\beta}}^{\hat{\alpha}}\mathrm{d}*\mathrm{d}V^{\hat{\beta}} - \frac{1}{2}\mathrm{d}J^{\hat{\alpha}} = 0, \tag{5.4.8}$$

$$\frac{1}{2}B_{\hat{\alpha}\hat{\beta}}\mathrm{d}*\mathrm{d}V^{\hat{\beta}} - \frac{1}{2}A_{\hat{\alpha}}^{\hat{\beta}}\mathrm{d}*\mathrm{d}U_{\hat{\beta}} - \frac{1}{2}\mathrm{d}J_{\hat{\alpha}} = 0. \tag{5.4.9}$$

Substituting the self-duality constraints (5.4.3) and (5.4.4) into the equations of motion results in the Bianchi identities

$$\frac{1}{2}C^{\hat{\alpha}\hat{\beta}}\mathrm{d}*(C_{\hat{\beta}\hat{\gamma}}*\mathrm{d}V^{\hat{\gamma}} + \mathrm{d}V^{\hat{\gamma}}A_{\hat{\gamma}}^{\hat{\kappa}}C_{\hat{\kappa}\hat{\beta}}) - \frac{1}{2}A_{\hat{\beta}}^{\hat{\alpha}}\mathrm{d}*\mathrm{d}V^{\hat{\beta}} - \frac{1}{2}\mathrm{d}J^{\hat{\alpha}} \tag{5.4.10}$$

$$= -\frac{1}{2}\mathrm{d}(\mathrm{d}V^{\hat{\alpha}}) - \frac{1}{2}\mathrm{d}J^{\hat{\alpha}} = 0, \tag{5.4.11}$$

$$\frac{1}{2}B_{\hat{\alpha}\hat{\beta}}\mathrm{d}*(-B^{\hat{\beta}\hat{\kappa}}*\mathrm{d}U_{\hat{\kappa}} + B^{\hat{\beta}\hat{\gamma}}A_{\hat{\gamma}}^{\hat{\kappa}}\mathrm{d}U_{\hat{\kappa}}) - \frac{1}{2}A_{\hat{\alpha}}^{\hat{\beta}}\mathrm{d}*\mathrm{d}U_{\hat{\beta}} - \frac{1}{2}\mathrm{d}J_{\hat{\alpha}} \tag{5.4.12}$$

$$= \frac{1}{2}\mathrm{d}(\mathrm{d}U_{\hat{\alpha}}) - \frac{1}{2}\mathrm{d}J_{\hat{\alpha}} = 0. \tag{5.4.13}$$

Since we eventually will express the action in fields having a trivial Bianchi identity, we define the new fields

$$\mathrm{d}\tilde{U}_{\hat{\alpha}} = \mathrm{d}U_{\hat{\alpha}} - J_{\hat{\alpha}} \qquad \Longrightarrow \qquad \mathrm{d}(\mathrm{d}\tilde{U}_{\hat{\alpha}}) = 0, \tag{5.4.14}$$

$$\mathrm{d}\tilde{V}^{\hat{\alpha}} = \mathrm{d}V^{\hat{\alpha}} + J^{\hat{\alpha}} \qquad \Longrightarrow \qquad \mathrm{d}(\mathrm{d}\tilde{V}^{\hat{\alpha}}) = 0. \tag{5.4.15}$$

Now incorporating these Bianchi identities as a Lagrange multiplier in the action, we include the term $+\frac{1}{4\kappa_4^2}(H_{\hat{\alpha}} - J_{\hat{\alpha}}) \wedge (\mathrm{d}V^{\hat{\alpha}} + J^{\hat{\alpha}})$. Note that because of this the information $\mathrm{d}H_{\hat{\alpha}} = \mathrm{d}J_{\hat{\alpha}}$ is captured in the action. Due to this alteration of the action, the equations of motion slightly differ and are given by

$$-\frac{1}{2}C^{\hat{\alpha}\hat{\beta}}*H_{\hat{\beta}} - \frac{1}{2}A_{\hat{\beta}}^{\hat{\alpha}}*\mathrm{d}V^{\hat{\beta}} + \frac{1}{2}\mathrm{d}V^{\hat{\alpha}} = 0, \tag{5.4.16}$$

$$\frac{1}{2}B_{\hat{\alpha}\hat{\beta}}\mathrm{d}*\mathrm{d}V^{\hat{\beta}} - \frac{1}{2}A_{\hat{\alpha}}^{\hat{\beta}}\mathrm{d}*H_{\hat{\beta}} - \frac{1}{2}\mathrm{d}J_{\hat{\alpha}} + \frac{1}{2}\mathrm{d}(H_{\hat{\alpha}} - J_{\hat{\alpha}}) = 0. \tag{5.4.17}$$

An important remark here is that these new equations of motion of $H_{\hat{\alpha}}$ and $V^{\hat{\alpha}}$ are in accord with the equation of motion of $V^{\hat{\alpha}}$ (5.4.9), upon imposing the Bianchi identity of $H_{\hat{\alpha}}$, and with the self-duality constraint of $U_{\hat{\alpha}}$ (5.4.3) as they should for consistency.

At this point the obtained action is a functional of $H_{\hat{\alpha}}$ and $V^{\hat{\alpha}}$

$$\begin{aligned}S^{(4)}_{gauge} = \frac{1}{2\kappa_4^2}\int & \frac{1}{4}B_{\hat{\alpha}\hat{\beta}}\mathrm{d}V^{\hat{\alpha}}\wedge *\mathrm{d}V^{\hat{\beta}} - \frac{1}{4}C^{\hat{\alpha}\hat{\beta}}H_{\hat{\alpha}}\wedge *H_{\hat{\beta}} - \frac{1}{2}A_{\hat{\beta}}{}^{\hat{\alpha}}H_{\hat{\alpha}}\wedge *\mathrm{d}V^{\hat{\beta}} \\ & + \kappa_4^2\mu_7 l^2(\frac{1}{2}\mathcal{K}_\Lambda - \frac{1}{2}e^{-\phi}\mathcal{K}_{\Lambda ab}\mathcal{B}^a\mathcal{B}^b)F\wedge *F \\ & + \kappa_4^2\mu_7 l^2(\rho_\Lambda - \mathcal{K}_{\Lambda ab}c^a\mathcal{B}^b + \frac{1}{2}l\mathcal{K}_{\Lambda ab}\mathcal{B}^a\mathcal{B}^b)F\wedge F \\ & - \mathrm{d}V^{\hat{\alpha}}\wedge J_{\hat{\alpha}} + \frac{1}{2}H_{\hat{\alpha}}\wedge \mathrm{d}V^{\hat{\alpha}} - \frac{1}{2}J_{\hat{\alpha}}\wedge J^{\hat{\alpha}},\end{aligned} \tag{5.4.18}$$

which we can now dualize by substituting the equation of motion of $H_{\hat{\alpha}}$ (5.4.16) into the action, i.e. imposing the duality constraint and eliminating this field from the action, together with half of the degrees of freedom. Note that the part of the action relevant for the dualization is identical to (5.4.5). Even more, the equations of motion for $H_{\hat{\alpha}}$ are in accord to those obtained neglecting the D7-brane gauge vector fields. Using the result of the dualization previously carried out (5.4.7), we obtain the dual action

$$\begin{aligned}S^{(4)}_{gauge} = \frac{1}{2\kappa_4^2}\int & -\frac{1}{2}(\mathrm{Im}M)_{\hat{\alpha}\hat{\beta}}\mathrm{d}V^{\hat{\alpha}}\wedge *\mathrm{d}V^{\hat{\beta}} - \frac{1}{2}(\mathrm{Re}M)_{\hat{\alpha}\hat{\beta}}\mathrm{d}V^{\hat{\alpha}}\wedge \mathrm{d}V^{\hat{\beta}} \\ & + \kappa_4^2\mu_7 l^2(\frac{1}{2}\mathcal{K}_\Lambda - \frac{1}{2}e^{-\phi}\mathcal{K}_{\Lambda ab}\mathcal{B}^a\mathcal{B}^b)F\wedge *F \\ & + \kappa_4^2\mu_7 l^2(\rho_\Lambda - \mathcal{K}_{\Lambda ab}c^a\mathcal{B}^b + \frac{1}{2}l\mathcal{K}_{\Lambda ab}\mathcal{B}^a\mathcal{B}^b)F\wedge F \\ & - \mathrm{d}V^{\hat{\alpha}}\wedge J_{\hat{\alpha}} - \frac{1}{2}J_{\hat{\alpha}}\wedge J^{\hat{\alpha}}.\end{aligned} \tag{5.4.19}$$

Since we aim for the action in terms of the new defined field $\tilde{V}^{\hat{\alpha}}$ (5.4.15) with trivial Bianchi identity, we substitute $\mathrm{d}V^{\hat{\alpha}} = \mathrm{d}\tilde{V}^{\hat{\alpha}} - J^{\hat{\alpha}}$ obtaining

$$\begin{aligned}S^{(4)}_{gauge} = \frac{1}{2\kappa_4^2}\int & -\frac{1}{2}(\mathrm{Im}M)_{\hat{\alpha}\hat{\beta}}\mathrm{d}\tilde{V}^{\hat{\alpha}}\wedge *\mathrm{d}\tilde{V}^{\hat{\beta}} - \frac{1}{2}(\mathrm{Re}M)_{\hat{\alpha}\hat{\beta}}\mathrm{d}\tilde{V}^{\hat{\alpha}}\wedge \mathrm{d}\tilde{V}^{\hat{\beta}} \\ & + \kappa_4^2\mu_7 l^2(\frac{1}{2}\mathcal{K}_\Lambda - \frac{1}{2}e^{-\phi}\mathcal{K}_{\Lambda ab}\mathcal{B}^a\mathcal{B}^b)F\wedge *F \\ & + \kappa_4^2\mu_7 l^2(\rho_\Lambda - \mathcal{K}_{\Lambda ab}c^a\mathcal{B}^b + \frac{1}{2}l\mathcal{K}_{\Lambda ab}\mathcal{B}^a\mathcal{B}^b)F\wedge F \\ & + (\mathrm{Im}M)_{\hat{\alpha}\hat{\beta}}\mathrm{d}\tilde{V}^{\hat{\alpha}}\wedge *J^{\hat{\beta}} + (\mathrm{Re}M)_{\hat{\alpha}\hat{\beta}}\mathrm{d}\tilde{V}^{\hat{\alpha}}\wedge J^{\hat{\beta}} - \mathrm{d}\tilde{V}^{\hat{\alpha}}\wedge J_{\hat{\alpha}} \\ & - \frac{1}{2}(\mathrm{Im}M)_{\hat{\alpha}\hat{\beta}}J^{\hat{\alpha}}\wedge *J^{\hat{\beta}} - \frac{1}{2}(\mathrm{Re}M)_{\hat{\alpha}\hat{\beta}}J^{\hat{\alpha}}\wedge J^{\hat{\beta}} + \frac{1}{2}J_{\hat{\alpha}}\wedge J^{\hat{\alpha}}.\end{aligned} \tag{5.4.20}$$

Recalling the definitions of the sources

$$J^{\hat{\alpha}} = 4\kappa_4^2\mu_7 l^2(a_I\mathcal{A}^{\hat{\alpha}I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\alpha}\bar{J}})F \quad \text{and} \quad J_{\hat{\alpha}} = 4\kappa_4^2\mu_7 l^2(a_I\mathcal{A}_{\hat{\alpha}}^{\ I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}_{\hat{\alpha}}^{\ \bar{J}})F,$$

the final result for the dualized action is given by

$$\begin{aligned} S_{gauge}^{(4)} = \frac{1}{2\kappa_4^2}\int &-\frac{1}{2}(\mathrm{Im}M)_{\hat{\alpha}\hat{\beta}}\mathrm{d}V^{\hat{\alpha}}\wedge *\mathrm{d}V^{\hat{\beta}} - \frac{1}{2}(\mathrm{Re}M)_{\hat{\alpha}\hat{\beta}}\mathrm{d}V^{\hat{\alpha}}\wedge \mathrm{d}V^{\hat{\beta}} \\ &+ \kappa_4^2\mu_7 l^2\Big(\frac{1}{2}\mathcal{K}_\Lambda - \frac{1}{2}e^{-\phi}\mathcal{K}_{\Lambda ab}\mathcal{B}^a\mathcal{B}^b \\ &\quad - 8\kappa_4^2\mu_7 l^2 C_{\hat{\alpha}\hat{\beta}}(a_I\mathcal{A}^{\hat{\alpha}I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\alpha}\bar{J}})(a_I\mathcal{A}^{\hat{\beta}I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\beta}\bar{J}})\Big)F\wedge *F \\ &+ \kappa_4^2\mu_7 l^2\Big(\rho_\Lambda - \mathcal{K}_{\Lambda ab}c^a\mathcal{B}^b + \frac{1}{2}l\mathcal{K}_{\Lambda ab}\mathcal{B}^a\mathcal{B}^b \\ &\quad + i8\kappa_4^2\mu_7 l^2 C_{\hat{\alpha}\hat{\beta}}(a_I\mathcal{A}^{\hat{\alpha}I} - \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\alpha}\bar{J}})(a_I\mathcal{A}^{\hat{\beta}I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\beta}\bar{J}})\Big)F\wedge F, \\ &+ 4\kappa_4^2\mu_7 l^2 C_{\hat{\alpha}\hat{\beta}}\Big((a_I\mathcal{A}^{\hat{\alpha}I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\alpha}\bar{J}})\mathrm{d}V^{\hat{\beta}}\wedge *F \\ &\quad - i(a_I\mathcal{A}^{\hat{\alpha}I} - \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\alpha}\bar{J}})\mathrm{d}V^{\hat{\beta}}\wedge F\Big), \end{aligned} \tag{5.4.21}$$

where we have used (3.2.9) to construct the equalities

$$\begin{aligned} \int_{S_-}\iota^*\alpha_{\hat{\alpha}}\wedge\bar{A}^{\bar{J}} = \mathcal{H}_{\hat{\alpha}\hat{\beta}}\int_{S_-}\iota^*\beta^{\hat{\beta}}\wedge\bar{A}^{\bar{J}} \quad &\Longrightarrow \quad \bar{a}_{\bar{J}}\bar{\mathcal{A}}_{\hat{\alpha}}^{\ \bar{J}} = \mathcal{H}_{\hat{\alpha}\hat{\beta}}\bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\beta}\bar{J}}, \\ \int_{S_-}\iota^*\alpha_{\hat{\alpha}}\wedge A^I = \bar{\mathcal{H}}_{\hat{\alpha}\hat{\beta}}\int_{S_-}\iota^*\beta^{\hat{\beta}}\wedge A^I \quad &\Longrightarrow \quad a_I\mathcal{A}_{\hat{\alpha}}^{\ I} = \bar{\mathcal{H}}_{\hat{\alpha}\hat{\beta}}a_I\mathcal{A}^{\hat{\beta}I}. \end{aligned} \tag{5.4.22}$$

Important to mention here is that it is crucial that $A^I \in H^{0,1}_{\bar{\partial},-}$ and $\chi_{\tilde{a}} \in H^{2,1}_{\bar{\partial},-}$ in order for the integrals over the (2,2)-cycle to vanish. In combination with (4.2.22) and (4.2.25), this is used to obtain

$$a_I\mathcal{A}_{\hat{\alpha}}^{\ I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}_{\hat{\alpha}}^{\ \bar{J}} = (\mathrm{Re}M)_{\hat{\alpha}\hat{\beta}}(a_I\mathcal{A}^{\hat{\beta}I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\beta}\bar{J}}) + iC_{\hat{\alpha}\hat{\beta}}(a_I\mathcal{A}^{\hat{\beta}I} - \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\beta}\bar{J}}). \tag{5.4.23}$$

For the sake of completeness we state the full action after imposing the self-duality relations, inherent to starting from a democratic action. The dualization of the total action is analog to the dualization for the gauge vectors, explicitly carried out in the previous paragraph. Hence, imposing the constraints (4.1.9) to the action, we reduce the degrees of freedom by half and obtain the four-dimensional action in the Einstein frame [32]

$$
\begin{aligned}
S^{(4)}_{\mathrm{EF}} &= S^{(4)}_{Bulk} + S^{(4)}_{DBI} + S^{(4)}_{CS} \\
&= \frac{1}{2\kappa_4^2}\int \Big[- R * \mathbb{1} + 2\mathcal{G}_{\tilde{a}\tilde{b}} \mathrm{d}z^{\tilde{a}} \wedge *\mathrm{d}z^{\tilde{b}} + 2G_{\alpha\beta}\mathrm{d}v^{\alpha} \wedge *\mathrm{d}v^{\beta} \\
&\quad + \frac{1}{2}\mathrm{d}(\ln V_{Y_3}) \wedge *\mathrm{d}(\ln V_{Y_3}) + \frac{1}{2}\mathrm{d}\phi \wedge *\mathrm{d}\phi + 2e^{\phi}G_{ab}\mathrm{d}b^a \wedge *\mathrm{d}b^b \\
&\quad + 2i\kappa_4^2\mu_7\mathcal{L}_{A\bar{B}}(e^{\phi} + G_{ab}\mathcal{B}^a\mathcal{B}^b)\mathrm{d}\zeta^A \wedge *\mathrm{d}\bar{\zeta}^{\bar{B}} + \frac{24}{V_{Y_3}}\kappa_4^2\mu_7\ell^2 i C^{IJ}_{\alpha} v^{\alpha}\mathrm{d}a_I \wedge *\mathrm{d}\bar{a}_{\bar{J}} \\
&\quad + \frac{e^{2\phi}}{2}\Big(\mathrm{d}l + \kappa_4^2\mu_7\mathcal{L}_{A\bar{B}}\Big(\mathrm{d}\zeta^A\bar{\zeta}^{\bar{B}} - \mathrm{d}\bar{\zeta}^{\bar{B}}\zeta^A\Big)\Big)\wedge \\
&\qquad\qquad * \Big(\mathrm{d}l + \kappa_4^2\mu_7\mathcal{L}_{A\bar{B}}\Big(\mathrm{d}\zeta^A\bar{\zeta}^{\bar{B}} - \mathrm{d}\bar{\zeta}^{\bar{B}}\zeta^A\Big)\Big) \\
&\quad + 2e^{\phi}G_{ab}\Big(\nabla c^a - l\mathrm{d}b^a + \kappa_4^2\mu_7\mathcal{B}^a\mathcal{L}_{A\bar{B}}\Big(\mathrm{d}\zeta^A\bar{\zeta}^{\bar{B}} - \mathrm{d}\bar{\zeta}^{\bar{B}}\zeta^A\Big)\Big)\wedge \\
&\qquad\qquad * \Big(\nabla c^b - l\mathrm{d}b^b + \kappa_4^2\mu_7\mathcal{B}^b\mathcal{L}_{A\bar{B}}\Big(\mathrm{d}\zeta^A\bar{\zeta}^{\bar{B}} - \mathrm{d}\bar{\zeta}^{\bar{B}}\zeta^A\Big)\Big) \\
&\quad + \frac{9}{2V^2_{Y_3}}G^{\alpha\beta}\Big[\nabla\rho_{\alpha} - \frac{1}{2}\kappa_4^2\mu_7\mathcal{K}_{\alpha bc}\mathcal{B}^b\mathcal{B}^c\mathcal{L}_{A\bar{B}}\Big(\mathrm{d}\zeta^A\bar{\zeta}^{\bar{B}} - \mathrm{d}\bar{\zeta}^{\bar{B}}\zeta^A\Big) \\
&\qquad\qquad - \mathcal{K}_{\alpha bc}c^b\mathrm{d}b^c + 2\kappa_4^2\mu_7\ell^2 C^{I\bar{J}}_{\alpha}(a_I\mathrm{d}\bar{a}_{\bar{J}} - \bar{a}_{\bar{J}}\mathrm{d}a_I)\Big]\wedge \\
&\qquad * \Big[\nabla\rho_{\beta} - \frac{1}{2}\kappa_4^2\mu_7\mathcal{K}_{\beta bc}\mathcal{B}^b\mathcal{B}^c\mathcal{L}_{A\bar{B}}\Big(\mathrm{d}\zeta^A\bar{\zeta}^{\bar{B}} - \mathrm{d}\bar{\zeta}^{\bar{B}}\zeta^A\Big) \\
&\qquad\qquad - \mathcal{K}_{\beta bc}c^b\mathrm{d}b^c + 2\kappa_4^2\mu_7\ell^2 C^{I\bar{J}}_{\beta}(a_I\mathrm{d}\bar{a}_{\bar{J}} - \bar{a}_{\bar{J}}\mathrm{d}a_I)\Big] \\
&\quad - \frac{1}{2}(\mathrm{Im}M)_{\hat{\alpha}\hat{\beta}}\mathrm{d}V^{\hat{\alpha}} \wedge *\mathrm{d}V^{\hat{\beta}} - \frac{1}{2}(\mathrm{Re}M)_{\hat{\alpha}\hat{\beta}}\mathrm{d}V^{\hat{\alpha}} \wedge \mathrm{d}V^{\hat{\beta}} \\
&\quad + 4\kappa_4^2\mu_7 l^2 C_{\hat{\alpha}\hat{\beta}}\Big((a_I\mathcal{A}^{\hat{\alpha}I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\alpha}\bar{J}})\mathrm{d}V^{\hat{\beta}} \wedge *F - i(a_I\mathcal{A}^{\hat{\alpha}I} - \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\alpha}\bar{J}})\mathrm{d}V^{\hat{\beta}} \wedge F\Big) \\
&\quad + \kappa_4^2\mu_7 l^2\Big(\frac{1}{2}\mathcal{K}_{\Lambda} - \frac{1}{2}e^{-\phi}\mathcal{K}_{\Lambda ab}\mathcal{B}^a\mathcal{B}^b \\
&\qquad - 8\kappa_4^2\mu_7 l^2 C_{\hat{\alpha}\hat{\beta}}(a_I\mathcal{A}^{\hat{\alpha}I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\alpha}\bar{J}})(a_I\mathcal{A}^{\hat{\beta}I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\beta}\bar{J}})\Big)F \wedge *F \\
&\quad + \kappa_4^2\mu_7 l^2\Big(\rho_{\Lambda} - \mathcal{K}_{\Lambda ab}c^a\mathcal{B}^b + \frac{1}{2}l\mathcal{K}_{\Lambda ab}\mathcal{B}^a\mathcal{B}^b \\
&\qquad + i8\kappa_4^2\mu_7 l^2 C_{\hat{\alpha}\hat{\beta}}(a_I\mathcal{A}^{\hat{\alpha}I} - \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\alpha}\bar{J}})(a_I\mathcal{A}^{\hat{\beta}I} + \bar{a}_{\bar{J}}\bar{\mathcal{A}}^{\hat{\beta}\bar{J}})\Big)F \wedge F + \frac{1}{2}V_D * \mathbb{1}\Big],
\end{aligned}
\tag{5.4.24}
$$

where a scalar potential V_D, related to spontaneous supersymmetry breaking, is included. For a detailed background of the origin of this scalar potential, we refer the reader to [58].

5.5 $\mathcal{N} = 1$ supersymmetric representation

Any $\mathcal{N} = 1$ supergravity theory can be written in a standard form in which all fields are grouped into the chiral multiplets C^N and vector multiplets V^Φ. In the dualization we have seen that in a four-dimensional theory all fields can be written as either a scalar or a vector. Since there is only one supercharge in the effective theory, bosonic scalars and vectors cannot be grouped into the same supermultiplet. Therefore, all scalars fit into the chiral multiplets and all vectors in the theory are grouped as vector multiplets. The interactions in this $\mathcal{N} = 1$ supermultiplet representation are completely determined by the Kähler potential K, the superpotential W and the gauge kinetic coupling functions $f_{\Phi\Sigma}$. Generally speaking a four-dimensional $\mathcal{N} = 1$ supergravity action is of the form [32]

$$\begin{aligned} S^{(4)}_{\mathcal{N}=1} = \frac{1}{2\kappa_4^2} \int & - R * \mathbb{1} + 2K_{M\bar{N}} \mathrm{d}C^M \wedge * \mathrm{d}\bar{C}^{\bar{N}} + (V_F + V_D) * \mathbb{1} \\ & + (\mathrm{Re} f)_{\Phi\Sigma} F^\Phi \wedge * F^\Sigma + (\mathrm{Im} f)_{\Phi\Sigma} F^\Phi \wedge F^\Sigma, \end{aligned} \tag{5.5.1}$$

with the field strength of the vector multiplets $F^\Phi = (\mathrm{d}V^{\hat{\alpha}}, F)$ and the Kähler metric $K_{M\bar{N}} = \partial_M \bar{\partial}_{\bar{N}} K$.
Guided by [11, 36, 45, 59], the correct Kähler variables which are part of the $\mathcal{N} = 1$ chiral superfields are

$$\begin{aligned} & z^{\tilde{a}}, \qquad \zeta^A, \qquad a_I, \\ & S = \tau + \kappa_4^2 \mu_7 \mathcal{L}_{A\bar{B}} \zeta^A \bar{\zeta}^{\bar{B}}, \\ & G^a = c^a - \tau \mathcal{B}^a, \\ & T_\alpha = \frac{1}{2}\mathcal{K}_\alpha + \frac{i}{2(\tau - \bar{\tau})} \mathcal{K}_{\alpha ab} G^a (G^b - \bar{G}^b) \\ & \qquad + i\Big(\rho_\alpha - \frac{1}{2}\mathcal{K}_{\alpha ab} c^a \mathcal{B}^b\Big) + 2i\kappa_4^2 \mu_7 l^2 \mathcal{C}_\alpha^{I\bar{J}} a_I (a_J + \bar{a}_{\bar{J}}), \end{aligned} \tag{5.5.2}$$

in which we defined the original complex Type IIB axion-dilaton field $\tau = l + ie^{-\phi}$. However, the coupling to the open string sector leads to a shift in this field. Therefore, the shifted new axion-dilaton field is represented by S.

Finally, the potentials V_F and V_D are also fully determined by the gauge coupling functions, the superpotential and the Kähler potential. The scalar potential V_F is expressed as [50]

$$V_F = e^K (K^{M\bar{N}} D_M W D_{\bar{N}} \bar{W} - 3|W|^2), \tag{5.5.3}$$

in terms of the inverse Kähler metric and Kähler covariant derivatives of the superpotential, $D_M W = \partial_M W + (\partial_M K) W$.

The gauge kinetic coupling function is a complex holomorphic function of the chiral superfields. These chiral superfields are treated as complex variables. Under the interchange of its two indices, represented in the adjoint representation of the gauge group, the coupling function is symmetric. Finally, the mass dimension is zero and the function encodes the couplings of the chiral supermultiplets to the gauge supermultiplets [3].
Similar to the gauge kinetic coupling function, the superpotential is also a complex function holomorphic in the chiral superfields. The function has mass dimension three and must be invariant under the gauge symmetries of the theory.
The Kähler potential is a function of both the chiral superfields and their anti-chiral partners. It is a real function with mass dimension two. At tree level, the Kähler potential is always linear in the term $C^M \bar{C}^{\bar{N}}$.

Since the part of interest for this work is the gauge kinetic coupling function we will not go in to this in more depth and refer the reader to [3, 32, 60] where a thorough explanation is given.

Combining equations (5.4.21) and (5.5.1) and the knowledge that all gauge vectors fit into the vector multiplets of the $\mathcal{N} = 1$ supersymmetry representation, we read of the gauge kinetic coupling function

$$f_{\Phi\Sigma} = \begin{pmatrix} -\frac{i}{2}\bar{M}_{\hat{\alpha}\hat{\beta}} & 4\kappa_4^2\mu_7 l^2 C_{\hat{\alpha}\hat{\beta}} a_I \mathcal{A}^{\hat{\alpha}I} \\ 4\kappa_4^2\mu_7 l^2 C_{\hat{\alpha}\hat{\beta}} a_I \mathcal{A}^{\hat{\alpha}I} & \kappa_4^2\mu_7 l^2 \Big(L_\Lambda - 16\kappa_4^2\mu_7 l^2 C_{\hat{\alpha}\hat{\beta}} a_I \mathcal{A}^{\hat{\alpha}I} (a_I \mathcal{A}^{\hat{\beta}I} + \bar{a}_{\bar{J}} \bar{\mathcal{A}}^{\hat{\beta}\bar{J}}) \Big) \end{pmatrix}, \tag{5.5.4}$$

in which, for a more convenient notation, we defined

$$L_\Lambda = \frac{1}{2}\mathcal{K}_\Lambda + \frac{i}{2(\tau - \bar{\tau})}\mathcal{K}_{\Lambda ab} G^a (G^b - \bar{G}^b) + i\Big(\rho_\Lambda - \frac{1}{2}\mathcal{K}_{\Lambda ab} c^a \mathcal{B}^b\Big). \tag{5.5.5}$$

It follows from (4.2.25) that we can write the bulk gauge kinetic coupling function as

$$f_{\hat{\alpha}\hat{\beta}} = -\frac{i}{2}\mathcal{H}_{\hat{\alpha}\hat{\beta}}, \tag{5.5.6}$$

which explicitly shows the holomorphic nature of the gauge coupling function $f_{\hat{\alpha}\hat{\beta}}$, since the prepotential is a holomorphic function in the complex structure moduli.

The gauge kinetic coupling matrix (5.5.4) displays that whenever the Wilson line moduli are switched off, there is no mixed interaction between the bulk and the D7-brane gauge vector fields. Furthermore, neglecting the Wilson line moduli, the D7-brane gauge coupling function reduces to $f_{D7} = \kappa_4^2\mu_7 l^2 T_\Lambda$, which is manifestly holomorphic in the chiral superfields.

However, if the relations

$$-8C_{\hat{\alpha}\hat{\beta}}\mathcal{A}^{\hat{\alpha}I}\bar{\mathcal{A}}^{\hat{\beta}\bar{J}} = iC_\Lambda^{I\bar{J}} \tag{5.5.7}$$

and

$$-8C_{\hat{\alpha}\hat{\beta}}\mathcal{A}^{\hat{\alpha}I}\mathcal{A}^{\hat{\beta}J} = iC_\Lambda^{I\bar{J}} \tag{5.5.8}$$

hold, we can rewrite the last term of the D7-brane gauge coupling (5.5.4) involving the Wilson lines. This results in writing the D7-brane gauge coupling function as

$$f_{D7} = \kappa_4^2\mu_7 l^2 T_\Lambda, \tag{5.5.9}$$

which implies that regardless of the Wilson line moduli fields, the D7-brane gauge coupling function is holomorphic in the $\mathcal{N} = 1$ coordinates. Even though we have not proven (5.5.7) and (5.5.8), based on [32] we have reason to believe there must be such relations. This is due to the fact that [32] uses relation (5.5.7) in one of the last steps of their calculation to obtain the gauge kinetic coupling function. However, they do not mention using it, which leaves us a bit uncertain. Therefore, we have included suggestions for a closer look into the validity of these equations in appendix A.

Finally, the mixed gauge kinetic coupling function $f_{\hat{\alpha}D7}$ between the bulk and the brane vector bosons should be holomorphic in the chiral superfields in order to complement the $\mathcal{N} = 1$ supersymmetric representation. At a first glance, judging by its form $f_{\hat{\alpha}D7} = 4\kappa_4^2\mu_7 l^2 C_{\hat{\alpha}\hat{\beta}} a_I \mathcal{A}^{\hat{\alpha}I}$, it looks holomorphic in the $\mathcal{N} = 1$ coordinates since the only explicit dependence on the chiral fields seems to be a_I. However, this expression is misleading due to the fact that the matrices $C_{\hat{\alpha}\hat{\beta}}$ and $\mathcal{A}^{\hat{\alpha}I}$ depend on yet another one of the superfields; the complex structure, though this dependence is not manifest. It is hidden in the basis forms A^I of the Dolbeault cohomology $H^{0,1}_{\bar{\partial},-}(S_+)$ and in the Hodge star operator due to the Hermitian metric. Therefore, the holomorphic nature of the mixed gauge coupling function does not explicitly show. We will elaborate on this in the following chapters.

Chapter 6

M-theory reduction on a general Y_4

The final aim is to show that the gauge coupling function (5.5.4) of Type IIB theory is holomorphic in the chiral superfields. For the mixed gauge kinetic coupling function between the bulk and the brane gauge vectors, we will take an alternative route. Since there exists an indirect relation between Type IIB theory and M-theory, further elaborated on in the next chapter, we will reduce the eleven-dimensional supergravity action resulting from M-theory on a general Calabi-Yau fourfold in this chapter. Therefore, later on we can compare the results of the mixed gauge coupling function found in M-theory with the one resulting from Type IIB theory, to clarify that it should indeed be holomorphic in the complex structure moduli fields.

6.1 The spectrum

We start from the bosonic part of the eleven-dimensional supergravity action

$$S_{\mathrm{M}}^{(11)} = \frac{1}{2}\int \hat{R}\hat{*}\mathbb{1} - \frac{1}{2}\hat{G}\wedge\hat{*}\hat{G} - \frac{1}{6}\hat{C}\wedge\hat{G}\wedge\hat{G}, \tag{6.1.1}$$

with $\hat{R}$ the eleven-dimensional Ricci scalar and $\hat{G} = \mathrm{d}\hat{C}$ the field strength of the three-form potential $\hat{C}$. We perform a reduction on a Calabi-Yau fourfold Y_4. Analog to the reduction performed in [36], we will decompose according to the backgrounds

$$\begin{aligned} \langle \mathrm{d}\hat{s}_{11}^2 \rangle &= \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + 2\breve{g}_{m\bar{n}} \mathrm{d}y^m \mathrm{d}\bar{y}^{\bar{n}}, \\ \langle \mathrm{d}\hat{C} \rangle &= 0, \end{aligned} \tag{6.1.2}$$

where $\breve{g}_{m\bar{n}}$ is the background value of the Calabi-Yau metric on Y_4, thus locally $\breve{g}_{mn} = \breve{g}_{\bar{m}\bar{n}} = 0$. The background in equation (6.1.2) implies that the eleven-dimensional spacetime can be written in the irreducible form $M_{(1,10)} = M_{(1,2)} \times Y_4$. So far, this setup is in complete analogy to the derivation of the spectrum in section 4.1, when ignoring the orientifold projection and realizing that all forms are defined on Y_4 in this compactification instead of Y_3. Therefore, the relevant Hodge diamond reads

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & h^{1,1} & & 0 & & \\ & 0 & & h^{2,1} & & h^{1,2} & & 0 & \\ 1 & & h^{3,1} & & h^{2,2} & & h^{1,3} & & 1 \\ & 0 & & h^{2,1} & & h^{1,2} & & 0 & \\ & & 0 & & h^{1,1} & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array} . \tag{6.1.3}$$

We briefly state the massless spectrum arising from the fluctuations of the metric on a Calabi-Yau fourfold. Using the information encoded in the Hodge diamond (6.1.3), the metric on this manifold reads

$$\begin{aligned} g_{m\bar{n}} &= \breve{g}_{m\bar{n}} - i v^\Sigma (\omega_\Sigma)_{m\bar{n}}, \qquad \Sigma = 1, ..., h^{1,1}(Y_4), \\ g_{mn} &= \bar{z}^{\mathcal{K}} (\bar{b}_{\mathcal{K}})_{mn}, \end{aligned} \tag{6.1.4}$$

up to first order in the moduli fields, where we defined

$$(\bar{b}_{\mathcal{K}})_{mn} = \frac{i}{||\Omega||^2} (\chi_{\mathcal{K}})_{m\bar{e}_1\bar{e}_2\bar{e}_3} \Omega^{\bar{e}_1\bar{e}_2\bar{e}_3}{}_m, \qquad \mathcal{K} = 1, ..., h^{1,3}(Y_4). \tag{6.1.5}$$

Here ω_Σ denote a basis of harmonic two-forms on Y_4 and v^Σ are real scalar fields in the effective three-dimensional theory. Furthermore, $\chi_{\mathcal{K}}$ constitute a basis of harmonic (1,3)-forms on the Calabi-Yau fourfold and Ω is the unique holomorphic 4-form.

We use the Hodge diamond of Y_4 (6.1.3) to construct the massless modes arising from fluctuations of the three-form $\hat{C}$

$$\hat{C} = A^{\Sigma} \wedge \omega_{\Sigma} + N_{\mathcal{A}} \Psi^{\mathcal{A}} + \bar{N}_{\mathcal{A}} \bar{\Psi}^{\mathcal{A}}, \qquad \mathcal{A} = 1, ..., h^{1,2}(Y_4), \tag{6.1.6}$$

where A^{Σ} and $N_{\mathcal{A}}$ are vector fields and complex scalars respectively in the three-dimensional effective theory. Furthermore, the basis of harmonic (1,2)-forms is chosen to be

$$\Psi^{\mathcal{A}} = \frac{1}{2} \mathrm{Re} f^{\mathcal{A}\mathcal{B}} (\alpha_{\mathcal{B}} - i \bar{f}_{\mathcal{B}\mathcal{C}} \beta^{\mathcal{C}}), \tag{6.1.7}$$

with $(\alpha_{\mathcal{A}}, \beta^{\mathcal{B}})$ the real basis of harmonic three-forms on a Y_4 and $f_{\mathcal{A}\mathcal{B}}$ a function holomorpic in the complex structure, i.e. $f_{\mathcal{A}\mathcal{B}}(z^{\mathcal{K}}(x))$. We denote $\mathrm{Re} f^{\mathcal{A}\mathcal{B}}$ to be the inverse of $\mathrm{Re} f_{\mathcal{A}\mathcal{B}}$. Note that due to this choise of basis $\Psi^{\mathcal{A}}$, the scalar fields $N_{\mathcal{A}}$ are holomorphic in the complex structure moduli.

Thus, the four-form field strength $\hat{G}$ is obtained by

$$\hat{G} = \mathrm{d}\hat{C} = \mathrm{d}A^{\Sigma} \wedge \omega_{\Sigma} + \mathrm{d}N_{\mathcal{A}} \wedge \Psi^{\mathcal{A}} + N_{\mathcal{A}} \wedge \mathrm{d}\Psi^{\mathcal{A}} + \mathrm{d}\bar{N}_{\mathcal{A}} \wedge \bar{\Psi}^{\mathcal{A}} + \bar{N}_{\mathcal{A}} \wedge \mathrm{d}\bar{\Psi}^{\mathcal{A}}. \tag{6.1.8}$$

When writing $\mathrm{d}\Psi^{\mathcal{A}}$ explicitly we obtain

$$\begin{aligned} \mathrm{d}\Psi^{\mathcal{A}}(z^{\mathcal{L}}(x), \bar{z}^{\mathcal{M}}(x), y) = \frac{1}{2} \big[& (\partial_{z^{\mathcal{K}}} \mathrm{Re} f^{\mathcal{A}\mathcal{B}} \mathrm{d}z^{\mathcal{K}} + \bar{\partial}_{\bar{z}^{\mathcal{K}}} \mathrm{Re} f^{\mathcal{A}\mathcal{B}} \mathrm{d}\bar{z}^{\mathcal{K}}) \\ & \wedge (\alpha_{\mathcal{B}} - i \bar{f}_{\mathcal{B}\mathcal{C}} \beta^{\mathcal{C}}) + \mathrm{Re} f^{\mathcal{A}\mathcal{B}} (\mathrm{d}\alpha_{\mathcal{B}} - i \bar{f}_{\mathcal{B}\mathcal{C}} \mathrm{d}\beta^{\mathcal{C}}) \\ & - i \mathrm{Re} f^{\mathcal{A}\mathcal{B}} \bar{\partial}_{\bar{z}^{\mathcal{K}}} \bar{f}_{\mathcal{B}\mathcal{C}} \mathrm{d}\bar{z}^{\mathcal{K}} \wedge \beta^{\mathcal{C}} \big]. \end{aligned} \tag{6.1.9}$$

Note that implicitly $\mathrm{d}\bar{f}_{\mathcal{B}\mathcal{C}}(\bar{z}^{\mathcal{L}}(x)) = \bar{\partial}_{\bar{z}^{\mathcal{K}}} \bar{f}_{\mathcal{B}\mathcal{C}}(\bar{z}^{\mathcal{L}}(x)) \mathrm{d}\bar{z}^{\mathcal{K}}$ and recall that $\alpha_{\mathcal{A}}, \beta^{\mathcal{B}}$ are harmonics on Y_4. For convenience the dependence of the fields on the spacial coordinates is dropped. In addition, we use

$$\mathrm{d}\mathrm{Re} f^{\mathcal{A}\mathcal{B}} = -\mathrm{Re} f^{\mathcal{A}\mathcal{C}} \mathrm{d}\mathrm{Re} f_{\mathcal{C}\mathcal{D}} \mathrm{Re} f^{\mathcal{D}\mathcal{B}}, \tag{6.1.10}$$

in combination with the holomorphic property of $f_{\mathcal{A}\mathcal{B}}$ leading to

$$\partial_{z^{\mathcal{K}}} \mathrm{Re} f_{\mathcal{A}\mathcal{B}} = \frac{1}{2} \partial_{z^{\mathcal{K}}} (f_{\mathcal{A}\mathcal{B}} + \bar{f}_{\mathcal{A}\mathcal{B}}) = \frac{1}{2} \partial_{z^{\mathcal{K}}} f_{\mathcal{A}\mathcal{B}}, \tag{6.1.11}$$

to derive

$$\mathrm{d}\Psi^{\mathcal{A}} = -\frac{1}{2}\mathrm{Re}f^{\mathcal{AC}}\mathrm{d}f_{\mathcal{CD}} \wedge \Psi^{\mathcal{D}} - \frac{1}{2}\mathrm{Re}f^{\mathcal{AC}}\mathrm{d}\bar{f}_{\mathcal{CD}} \wedge \bar{\Psi}^{\mathcal{D}} = \mathrm{d}\bar{\Psi}^{\mathcal{A}}. \tag{6.1.12}$$

Therefore, we can now obtain the field strength by substituting this back into equation (6.1.8) and defining $DN_{\mathcal{A}}$ to be

$$DN_{\mathcal{A}} = \mathrm{d}N_{\mathcal{A}} - \mathrm{Re}N_{\mathcal{B}}\mathrm{Re}f^{\mathcal{BC}}\mathrm{d}f_{\mathcal{CA}}, \qquad \text{and} \qquad D\bar{N}_{\mathcal{A}} = \overline{DN_{\mathcal{A}}}. \tag{6.1.13}$$

Given this definition the eleven-dimensional four-form field strength reads

$$\hat{G} = \mathrm{d}A^{\Sigma} \wedge \omega_{\Sigma} + DN_{\mathcal{A}} \wedge \Psi^{\mathcal{A}} + D\bar{N}_{\mathcal{A}} \wedge \bar{\Psi}^{\mathcal{A}}. \tag{6.1.14}$$

6.2 Compactification

Having set up the spectrum of the lower-dimensional effective theory in the previous section, we will reduce the eleven-dimensional supergravity action (6.1.1) starting with the kinetic and the Chern-Simons term.
Substituting (6.1.14) into the second term of the eleven-dimensional action (6.1.1) and using the Hodge star decomposition (2.1.12), yields

$$\begin{aligned} -\frac{1}{4}\int \hat{G} \wedge \hat{*}\hat{G} = & -\frac{1}{4}\int \mathrm{d}A^{\Sigma} \wedge *\mathrm{d}A^{\Omega} \wedge (\omega_{\Sigma} \wedge \star\omega_{\Omega}) + iDN_{\mathcal{A}} \wedge *D\bar{N}_{\mathcal{B}} \wedge (\Psi^{\mathcal{A}} \wedge \bar{\Psi}^{\mathcal{B}} \wedge J) \\ & - iD\bar{N}_{\mathcal{A}} \wedge *DN_{\mathcal{B}} \wedge (\bar{\Psi}^{\mathcal{A}} \wedge \Psi^{\mathcal{B}} \wedge J) \\ = & -\int \frac{G_{\Sigma\Omega}}{\hat{V}}\mathrm{d}A^{\Sigma} \wedge *\mathrm{d}A^{\Omega} + \frac{1}{2}\hat{V}L^{\Sigma}\mathrm{d}_{\Sigma}{}^{\mathcal{AB}}DN_{\mathcal{A}} \wedge *D\bar{N}_{\mathcal{B}}, \end{aligned} \tag{6.2.1}$$

in which we have made use of

$$\star\, \Psi^{\mathcal{B}} = -i\Psi^{\mathcal{B}} \wedge J. \tag{6.2.2}$$

The coefficients in the effective three-dimensional action are defined to be

$$G_{\Sigma\Omega} = \frac{\hat{V}}{4}\int_{Y_4} \omega_{\Sigma} \wedge \star\omega_{\Omega} = -\frac{1}{8V}(\mathcal{K}_{\Sigma\Omega} - \frac{1}{18V}\mathcal{K}_{\Sigma}\mathcal{K}_{\Omega}) = -\frac{1}{4}\partial_{L^{\Sigma}}\partial_{L^{\Omega}}\mathrm{log}V, \tag{6.2.3}$$

$$d_\Sigma{}^{\mathcal{AB}} = i \int_{Y_4} \omega_\Sigma \wedge \Psi^{\mathcal{A}} \wedge \bar{\Psi}^{\mathcal{B}}, \tag{6.2.4}$$

where we denoted

$$\hat{V} = \frac{1}{4!}\int_{Y_4} J^4, \qquad L^\Sigma = \frac{v^\Sigma}{\hat{V}}, \qquad V = \frac{1}{4!}\mathcal{K}_{\Sigma\Omega\Gamma\Lambda}L^\Sigma L^\Omega L^\Gamma L^\Lambda \tag{6.2.5}$$

and the intersection numbers

$$\mathcal{K}_{\Sigma\Omega\Gamma\Lambda} = \int_{Y_4} \omega_\Sigma \wedge \omega_\Omega \wedge \omega_\Gamma \wedge \omega_\Lambda, \qquad \mathcal{K}_\Sigma = \mathcal{K}_{\Sigma\Omega\Gamma\Lambda}L^\Omega L^\Gamma L^\Lambda \qquad \mathcal{K}_{\Sigma\Omega} = \mathcal{K}_{\Sigma\Omega\Gamma\Lambda}L^\Gamma L^\Lambda. \tag{6.2.6}$$

Finally, to obtain equation (6.2.3) we additionally used

$$\star\, \omega_\Sigma = -\frac{1}{2}J \wedge J \wedge \omega_\Sigma + \frac{\hat{V}^2}{36}\mathcal{K}_\Sigma J \wedge J \wedge J. \tag{6.2.7}$$

Performing a dimensional reduction on the third term in the eleven-dimensional supergravity action (6.1.1), we substitute expansion (6.1.6) and define the lower-dimensional field strength $F^\Sigma = \mathrm{d}A^\Sigma$. As a result, the Chern-Simons term reads

$$\begin{aligned}
-\frac{1}{12}\int \hat{C} \wedge \hat{G} \wedge \hat{G} = &-\frac{1}{4}\int \mathrm{d}A^\Sigma \wedge \omega_\Sigma \wedge \big[N_{\mathcal{A}}\Psi^{\mathcal{A}} \wedge (\mathrm{d}(N_{\mathcal{B}}\Psi^{\mathcal{B}}) + \mathrm{d}(\bar{N}_{\mathcal{B}}\bar{\Psi}^{\mathcal{B}}))\\
&+ \bar{N}_{\mathcal{A}}\bar{\Psi}^{\mathcal{A}} \wedge (\mathrm{d}(N_{\mathcal{B}}\Psi^{\mathcal{B}}) + \mathrm{d}(\bar{N}_{\mathcal{B}}\bar{\Psi}^{\mathcal{B}}))\big]\\
&= -\frac{1}{4}\int F^\Sigma \wedge \omega_\Sigma \wedge \big[N_{\mathcal{A}}\Psi^{\mathcal{A}} \wedge (N_{\mathcal{B}}\mathrm{d}\Psi^{\mathcal{B}} + \mathrm{d}\bar{N}_{\mathcal{B}}\bar{\Psi}^{\mathcal{B}} + \bar{N}_{\mathcal{B}}\mathrm{d}\bar{\Psi}^{\mathcal{B}})\\
&+ \bar{N}_{\mathcal{A}}\bar{\Psi}^{\mathcal{A}} \wedge (\mathrm{d}N_{\mathcal{B}}\Psi^{\mathcal{B}} + N_{\mathcal{B}}\mathrm{d}\Psi^{\mathcal{B}} + \bar{N}_{\mathcal{B}}\mathrm{d}\bar{\Psi}^{\mathcal{B}})\big].
\end{aligned} \tag{6.2.8}$$

By using $\mathrm{d}\Psi^{\mathcal{A}} = \mathrm{d}\bar{\Psi}^{\mathcal{A}}$, shown in equation (6.1.12), we conclude that

$$\begin{aligned}
-\frac{1}{12}\int \hat{C}\wedge\hat{G} \wedge \hat{G} = &-\frac{1}{4}\int F^\Sigma \wedge \omega_\Sigma \wedge \big[(N_{\mathcal{A}}\mathrm{d}\bar{N}_{\mathcal{B}} - \bar{N}_{\mathcal{B}}\mathrm{d}N_{\mathcal{A}}) \wedge \Psi^{\mathcal{A}} \wedge \bar{\Psi}^{\mathcal{B}}\\
&+ (N_{\mathcal{A}}\Psi^{\mathcal{A}} + \bar{N}_{\mathcal{A}}\bar{\Psi}^{\mathcal{A}}) \wedge 2\mathrm{Re}N_{\mathcal{B}}\mathrm{d}\Psi^{\mathcal{B}}\big]\\
&= -\frac{1}{4}\int F^\Sigma \wedge \omega_\Sigma \wedge \big[(N_{\mathcal{A}}\mathrm{d}\bar{N}_{\mathcal{B}} - \bar{N}_{\mathcal{B}}\mathrm{d}N_{\mathcal{A}}) \wedge \Psi^{\mathcal{A}} \wedge \bar{\Psi}^{\mathcal{B}}
\end{aligned} \tag{6.2.9}$$

$$+ \bar{N}_{\mathcal{A}}\mathrm{Re}N_{\mathcal{B}}\mathrm{Re}f^{\mathcal{BC}}\mathrm{d}f_{\mathcal{CD}} \wedge \Psi^{\mathcal{D}} \wedge \bar{\Psi}^{\mathcal{A}} - N_{\mathcal{A}}\mathrm{Re}N_{\mathcal{B}}\mathrm{Re}f^{\mathcal{BC}}\mathrm{d}\bar{f}_{\mathcal{CD}} \wedge \Psi^{\mathcal{A}} \wedge \bar{\Psi}^{\mathcal{D}}]$$
$$= -\frac{1}{4i}\int d_{\Sigma}{}^{\mathcal{AB}}F^{\Sigma} \wedge (N_{\mathcal{A}}D\bar{N}_{\mathcal{B}} - \bar{N}_{\mathcal{B}}DN_{\mathcal{A}}).$$

The reduction of the Einstein-Hilbert term is in complete analogy with the reduction in appendix C except that now the metric deformations are expanded according to (6.1.4), which results in

$$\frac{1}{2}\int \hat{R}\hat{*}\mathbb{1} = \int \frac{1}{2}\hat{V}R * \mathbb{1} - \hat{V}\mathcal{G}_{\mathcal{K}\bar{\mathcal{L}}}\mathrm{d}z^{\mathcal{K}} \wedge *\mathrm{d}\bar{z}^{\bar{\mathcal{L}}} - \hat{V}G_{\Sigma\Omega}\mathrm{d}L^{\Sigma} \wedge *\mathrm{d}L^{\Omega} \qquad (6.2.10)$$

in the three-dimensional effective theory, with $\mathcal{G}_{\mathcal{K}\bar{\mathcal{L}}}$ a Kähler metric defined by

$$\mathcal{G}_{\mathcal{K}\bar{\mathcal{L}}} = -\frac{\int_{Y_4}\chi_{\mathcal{K}} \wedge \chi_{\bar{\mathcal{L}}}}{\int_{Y_4}\Omega \wedge \bar{\Omega}} = -\partial_{z^{\mathcal{K}}}\partial_{\bar{z}^{\bar{\mathcal{L}}}}\log\left(\int_{Y_4}\Omega \wedge \bar{\Omega}\right). \qquad (6.2.11)$$

Collecting all terms after the dimensional reduction, the three-dimensional effective action has become

$$\begin{aligned} S_M^{(3)} = \int \frac{1}{2}\hat{V}R * \mathbb{1} &- \hat{V}\mathcal{G}_{\mathcal{K}\bar{\mathcal{L}}}\mathrm{d}z^{\mathcal{K}} \wedge *\mathrm{d}\bar{z}^{\bar{\mathcal{L}}} - \hat{V}G_{\Sigma\Omega}\mathrm{d}L^{\Sigma} \wedge *\mathrm{d}L^{\Omega} - \frac{G_{\Sigma\Omega}}{\hat{V}}\mathrm{d}A^{\Sigma} \wedge *\mathrm{d}A^{\Omega} \\ &- \frac{1}{2}\hat{V}L^{\Sigma}d_{\Sigma}{}^{\mathcal{AB}}DN_{\mathcal{A}} \wedge *D\bar{N}_{\mathcal{B}} - \frac{1}{4i}d_{\Sigma}{}^{\mathcal{AB}}F^{\Sigma} \wedge (N_{\mathcal{A}}D\bar{N}_{\mathcal{B}} - \bar{N}_{\mathcal{B}}DN_{\mathcal{A}}). \end{aligned} \qquad (6.2.12)$$

Performing a Weyl rescaling of the form $(\eta_{\mu\nu})^{old} = \hat{V}^{-2}(\eta_{\mu\nu})^{new}$, explained in appendix B, on the three-dimensional external spacetime metric, results in the low-energy effective theory

$$\begin{aligned} S_M^{(3)} = \int \frac{1}{2}R * \mathbb{1} &- G_{\mathcal{K}\bar{\mathcal{L}}}\mathrm{d}z^{\mathcal{K}} \wedge *\mathrm{d}\bar{z}^{\bar{\mathcal{L}}} - G_{\Sigma\Omega}\mathrm{d}L^{\Sigma} \wedge *\mathrm{d}L^{\Omega} - G_{\Sigma\Omega}\mathrm{d}A^{\Sigma} \wedge *\mathrm{d}A^{\Omega} \\ &- \frac{1}{2}L^{\Sigma}d_{\Sigma}{}^{\mathcal{AB}}DN_{\mathcal{A}} \wedge *D\bar{N}_{\mathcal{B}} - \frac{1}{4i}d_{\Sigma}{}^{\mathcal{AB}}F^{\Sigma} \wedge (N_{\mathcal{A}}D\bar{N}_{\mathcal{B}} - \bar{N}_{\mathcal{B}}DN_{\mathcal{A}}). \end{aligned} \qquad (6.2.13)$$

Recall that the whole point of reducing the eleven-dimensional supergravity action, following from M-theory, was to relate the mixed gauge kinetic coupling function obtained from the M-theory reduction to the one resulting from the Type IIB compactification. Therefore, our main interest is this gauge kinetic coupling term.

Although it is at this point a priori not obvious which terms in the three-dimensional action resulting from M-theory will be related to the bulk and brane gauge vector fields of Type IIB theory, we rewrite the object $d_\Sigma{}^{\mathcal{AB}}$ introduced in (6.2.4), for later convenience of dualizing the mixed kinetic coupling. Thereby, using the intersection numbers

$$M_\Sigma^{\mathcal{AB}} = \int_{Y_4} \omega_\Sigma \wedge \beta^{\mathcal{A}} \wedge \beta^{\mathcal{B}}, \quad M_{\Sigma\mathcal{AB}} = \int_{Y_4} \omega_\Sigma \wedge \alpha_{\mathcal{A}} \wedge \alpha_{\mathcal{B}}, \quad M_{\Sigma\mathcal{A}}{}^{\mathcal{B}} = \int_{Y_4} \omega_\Sigma \wedge \alpha_{\mathcal{A}} \wedge \beta^{\mathcal{B}}, \tag{6.2.14}$$

which are independent of the complex structure moduli, we are able to state

$$\begin{aligned}
d_\Sigma{}^{\mathcal{AB}} &= i \int_{Y_4} \omega_\Sigma \wedge \Psi^{\mathcal{A}} \wedge \bar{\Psi}^{\mathcal{B}} \\
&= \frac{i}{4} \mathrm{Re} f^{\mathcal{AC}} \mathrm{Re} f^{\mathcal{BE}} \big[M_{\Sigma\mathcal{CE}} + (i\mathrm{Re} f + \mathrm{Im} f)_{\mathcal{CD}} M_{\Sigma\mathcal{E}}{}^{\mathcal{D}} + (i\mathrm{Re} f - \mathrm{Im} f)_{\mathcal{EG}} M_{\Sigma\mathcal{C}}{}^{\mathcal{G}} \\
&+ (\mathrm{Re} f - i\mathrm{Im} f)_{\mathcal{CD}} (\mathrm{Re} f + i\mathrm{Im} f)_{\mathcal{EG}} M_\Sigma^{\mathcal{DG}} \big] \\
&= \frac{i}{4} \mathrm{Re} f^{\mathcal{AC}} \mathrm{Re} f^{\mathcal{BE}} \big[M_{\Sigma\mathcal{CE}} + 2i\mathrm{Re} f_{(\mathcal{CD}} M_{\Sigma\mathcal{E})}{}^{\mathcal{D}} + 2\mathrm{Im} f_{[\mathcal{CD}} M_{\Sigma\mathcal{E}]}{}^{\mathcal{D}} + \mathrm{Re} f_{\mathcal{CD}} \mathrm{Re} f_{\mathcal{EG}} M_\Sigma^{\mathcal{DG}} \\
&+ \mathrm{Im} f_{\mathcal{CD}} \mathrm{Im} f_{\mathcal{EG}} M_\Sigma^{\mathcal{DG}} + 2i\mathrm{Re} f_{(\mathcal{CD}} \mathrm{Im} f_{\mathcal{E})\mathcal{G}} M_\Sigma^{\mathcal{DG}} \big] \\
&= -\frac{1}{2} \mathrm{Re} f^{\mathcal{BC}} M_{\Sigma\mathcal{C}}{}^{\mathcal{A}} - \frac{1}{2} \mathrm{Re} f^{\mathcal{BC}} \mathrm{Im} f_{\mathcal{CD}} M_\Sigma^{\mathcal{AD}} + \frac{i}{2} M_\Sigma^{\mathcal{AB}} \\
&= -\frac{1}{2} \mathrm{Re} f^{\mathcal{BC}} Q_{\Sigma\mathcal{C}}{}^{\mathcal{A}},
\end{aligned} \tag{6.2.15}$$

defining new matrices

$$Q_{\Sigma\mathcal{C}}{}^{\mathcal{A}} = M_{\Sigma\mathcal{C}}{}^{\mathcal{A}} + i f_{\mathcal{CD}} M_\Sigma^{\mathcal{DA}}, \tag{6.2.16}$$

which will prove to be of great importance in chapter 7 when dualizing to the Type IIB perspective. Note that these matrices (6.2.16) only depend on the complex structure through the holomorphic function $f_{\mathcal{CD}}$. Furthermore, in the third step we used

$$0 = \int_{Y_4} \omega_\Sigma \wedge \beta^{\mathcal{A}} \wedge \beta^{\mathcal{B}}, \tag{6.2.17}$$

which we have split into its real part

$$\begin{aligned}
0 = \mathrm{Re}\Big[\int_{Y_4} \omega_\Sigma \wedge \beta^{\mathcal{A}} \wedge \beta^{\mathcal{B}} \Big] = \frac{1}{4} \mathrm{Re} f^{\mathcal{AC}} \mathrm{Re} f^{\mathcal{BE}} \big[& M_{\Sigma\mathcal{CE}} - \mathrm{Re} f_{\mathcal{CD}} \mathrm{Re} f_{\mathcal{EG}} M_\Sigma^{\mathcal{DG}} \\
& + \mathrm{Im} f_{\mathcal{CD}} \mathrm{Im} f_{\mathcal{EG}} M_\Sigma^{\mathcal{DG}} + 2\mathrm{Im} f_{[\mathcal{CD}} M_{\Sigma\mathcal{E}]}{}^{\mathcal{D}} \big]
\end{aligned} \tag{6.2.18}$$

and the imaginary part

$$0 = \mathrm{Im}\Big[\int_{Y_4} \omega_\Sigma \wedge \beta^{\mathcal{A}} \wedge \beta^{\mathcal{B}}\Big] = -\frac{1}{2}\mathrm{Re} f^{[\mathcal{A}\mathcal{C}} M_{\Sigma\mathcal{C}}{}^{\mathcal{B}]} + \frac{1}{2}\mathrm{Re} f^{[\mathcal{A}\mathcal{C}}\mathrm{Im} f_{\mathcal{C}\mathcal{D}} M_\Sigma{}^{\mathcal{D}\mathcal{B}]}. \qquad (6.2.19)$$

Chapter 7

Holomorphic mixed gauge kinetic function

In the previous chapters we have shown the full reduction of the eleven-dimensional supergravity action on Y_4 following from M-theory and Type IIB theory on a Calabi-Yau orientifold including D7-branes. The reduction preformed for M-theory is valid for any smooth Calabi-Yau fourfold. M-theory is the geometrical origin of Type IIA theory in the strong coupling limit. Additionally, Type IIA can be dualized to Type IIB theory via a T-duality, which implies an implicit relation between our Type IIB reduced action and the effective lower-dimensional action resulting from M-theory. Generally speaking Type IIB theory can be related to M-theory on an elliptically fibered Calabi-Yau fourfold, when lifted on a circle to four dimensions. An important remark here is that we discuss the weak string coupling limit in M-theory to match the theory with the tree level supergravity Type IIB action. We will explore this relation in a bit more detail within this chapter, to reach the conclusion that M-theory shows the holomorphic property of the gauge kinetic coupling function in the chiral superfields.

7.1 Compatible Type IIB theory

Before beginning with the lift from three to four dimensions, we first turn to the Type IIB theory again and define an explicit basis of $H^{0,1}_{\bar{\partial},-}(S_+)$ and $H^{1,0}_{\bar{\partial},-}(S_+)$ denoted with $\{\gamma^p\}$ and $\{\bar{\gamma}^p\}$ respectively, such that the higher-dimensional gauge vector on the D7-brane decomposed in (5.1.3) can be expanded as

$$\hat{A} = A(x)P_-(y) + a_p(x)\gamma^p(y) + \bar{a}_p(x)\bar{\gamma}^p(y), \tag{7.1.1}$$

in the explicit basis [36]

$$\gamma^p = \frac{1}{2}\mathrm{Re}f^{pq}(\hat{\alpha}_q - i\bar{f}_{qr}\hat{\beta}^r), \tag{7.1.2}$$

where $(\hat{\alpha}_p, \hat{\beta}^p)$ is a symplectic basis of $H^1(S_+)$. All complex structure dependence is captured in the function f_{pq}, which is holomorphic in the complex structure moduli $z^{\tilde{a}}$. While not a priori obvious why one would choose such a basis, this will turn out to be convenient in translating from M-theory to Type IIB theory, due to the remarkable similarity with the choice of basis of the three-forms (6.1.7) in the M-theory reduction. Note that $a_I A^I = a_p\gamma^p$ since both span the full space $H^{0,1}_{\bar{\partial},-}(S_+)$ and γ^p is just a specific choice for A^I. Furthermore, note that we could have equally well chosen the basis $(\hat{\alpha}_p, \hat{\beta}^p)$ itself to expand the gauge vector boson. An expansion in these symplectic three-forms would have been [36]

$$\hat{A} = A(x)P_-(y) + \tilde{c}^p\hat{\alpha}_p + c_p\hat{\beta}^p, \tag{7.1.3}$$

which implies the form $a_p = ic_p + f_{pq}\tilde{c}^q$, explicitly showing that the complex structure dependence of the Wilson lines is captured by the function f_{pq}, since the coefficients $\tilde{c}^p$ and c_p are real. This manifestly shows the holomorphic property of a_p in the complex structure moduli fields.

7.2 M-theory perspective

Now turning to the M-theory side. To bring structure to the set of lower-dimensional scalars and vectors obtained in the M-theory reduction, we state how to group them into the three-dimensional $\mathcal{N} = 2$ supersymmetry multiplet representation. Therefore, we denote the scalar and vector multiplets of the representation respectively with

$$(z^{\mathcal{K}}, N_{\mathcal{A}}), \qquad (L^{\Sigma}, A^{\Sigma}). \tag{7.2.1}$$

To obtain the three-dimensional theory after reduction that is dual to the lower-dimensional Type IIB theory, we compactify M-theory on an elliptically fibered Calabi-Yau fourfold. Splitting the three-forms and two-forms according to the number of legs they have on the fiber we write

$$\Psi^{\mathcal{A}} = (\Psi^{A}, \Psi^{\kappa}), \qquad \omega_{\Sigma} = (\omega_0, \omega_i, \omega_\alpha), \tag{7.2.2}$$

where Ψ^{κ} corresponds to three-forms that live on the base of the fibration, while Ψ^{A} are three-forms that may have one leg on the fiber. Likewise, ω_α are (1,1)-forms on the base, ω_i may have one or no legs on the fiber and ω_0 are forms that have either two, one or zero legs on the fiber. Note that since we are now compactifying on a elliptically fibered Calabi-Yau fourfold, each non-vanishing integral over the internal space must have an integrand which is an eight-form with exactly two legs on the fiber and the other six on the base.
Due to the decomposition of the forms on the Calabi-Yau manifold, we include the corresponding decomposition of the scalars and vectors related to these forms

$$N_{\mathcal{A}} = (N_A, N_\kappa), \qquad L^{\Sigma} = (L^0, L^i, L^\alpha), \qquad A^{\Sigma} = (A^0, A^i, A^\alpha). \tag{7.2.3}$$

Since the higher-dimensional supergravity action started from in the Type IIB reduction was at tree level, we must consider the weak coupling limit in dualizing the low-energy effective result from M-theory to the fields of Type IIB theory. The N_A scalars lift to both the G^a moduli and the Wilson lines a_p on the D7-brane, while the complex scalars N_κ lift to the Ramond-Ramond vector fields $V^{\hat{\alpha}}$. From the three-dimensional $\mathcal{N} = 2$ vector multiplets, (L^0, A^0) lifts to the four-dimensional Kaluza-Klein vector that follows from the reduction of the metric, whereas (L^i, A^i) includes the D7-brane gauge vectors A. Finally, the vector multiplets (L^α, A^α) translate to the complex scalars T_α in four dimensions, therefore the vectors A^α have to be dualized into scalars. Considering the complex structure moduli fields $z^{\mathcal{K}}$ arising from the eleven-dimensional supergravity, they lift to the complex structure moduli in Type IIB $z^{\tilde{a}}$ as well as the axion-dilaton field and the normal coordinates $\zeta^{\mathcal{A}}$.

Since we are mainly interested in the mixed gauge kinetic coupling function of the bulk and brane gauge fields, the relevant four-dimensional Type IIB theory terms are of the form $\mathrm{d}V^{\hat{\alpha}} \wedge *F$ and $\mathrm{d}V^{\hat{\alpha}} \wedge F$. The coupling functions of these two terms in the action represent the real and imaginary part of the gauge kinetic coupling function respectively, as shown in equation (5.5.1). In the reduced three-dimensional M-theory, the parts of the action corresponding to this include the fields N_κ and (L^i, A^i), since these are dualized into $V^{\hat{\alpha}}$ and A. In [36] an explicit explanation of this duality map is reviewed and they furthermore state that within M-theory the gauge coupling function between N_κ and (L^i, A^i) is given by

$$f_{i\kappa} = Q_{i\kappa}{}^{\mathcal{A}} N_{\mathcal{A}} \tag{7.2.4}$$

up to a prefactor, where the matrix $Q_{\Sigma\mathcal{C}}{}^{\mathcal{A}}$ was defined in (6.2.16) and shown to be holomorphic. Thus, when writing out the gauge coupling function we obtain

$$f_{i\kappa} = (M_{i\kappa}{}^{\mathcal{A}} + if_{\kappa\mathcal{D}} M_i{}^{\mathcal{D}\mathcal{A}})N_{\mathcal{A}} = (M_{i\kappa}{}^{A} + if_{\kappa\lambda} M_i{}^{\lambda A})N_A, \tag{7.2.5}$$

from which it follows that the subscript $\mathcal{D}$ should turn into a subscript corresponding to the field N_λ, under the assumption that a gauge coupling of the form $f_{\kappa A}$ between N_A and N_κ does not exist. Even more, the matrices $M_{i\kappa}{}^{\mathcal{A}}$ and $M_i{}^{\lambda\mathcal{A}}$ are only non-trivial whenever $N_{\mathcal{A}}$ corresponds to N_A such that the integrals in (6.2.14) have precisely two legs on the fiber and thus are non-vanishing.

To obtain an expression closer to the form of the mixed gauge kinetic coupling function derived in the Type IIB reduction (5.5.4), one can use the particular property that

$$d_\Sigma{}^{\mathcal{A}\mathcal{B}} = \overline{d_\Sigma{}^{\mathcal{B}\mathcal{A}}}, \tag{7.2.6}$$

with $d_\Sigma{}^{\mathcal{A}\mathcal{B}}$ defined in (6.2.15) which implies the equality $\mathrm{Re} f^{\mathcal{A}\mathcal{B}} \mathrm{Re} f_{\mathcal{C}\mathcal{D}} \overline{Q_{\Sigma\mathcal{B}}{}^{\mathcal{D}}} = Q_{\Sigma\mathcal{C}}{}^{\mathcal{A}}$ [36]. Using this result and similar arguments to the ones stated above, one can write the gauge coupling between N_κ and (L^i, A^i)

$$\begin{aligned} f_{i\kappa} &= (\mathrm{Re} f^{A\mathcal{B}} \mathrm{Re} f_{\kappa\mathcal{D}} \overline{Q_{i\mathcal{B}}{}^{\mathcal{D}}}) N_A = \mathrm{Re} f^{AB} \mathrm{Re} f_{\kappa\lambda} (M_{iB}{}^{\lambda} - i\bar{f}_{BC} M_i{}^{C\lambda}) N_A \\ &= \mathrm{Re} f^{AB} \mathrm{Re} f_{\kappa\lambda} \Big(\int_{Y_4} \omega_i \wedge \alpha_B \wedge \beta^\lambda - i\bar{f}_{BC} \int_{Y_4} \omega_i \wedge \beta^C \wedge \beta^\lambda \Big) N_A, \end{aligned} \tag{7.2.7}$$

where we have split the symplectic basis of three-forms on the Calabi-Yau fourfold $(\alpha_{\mathcal{A}}, \beta^{\mathcal{B}})$ in $(\alpha_\kappa, \beta^\lambda)$ and (α_A, β^B) analog to the devision of the basis $\Psi^{\mathcal{A}}$ in (7.2.2). Therefore, a Greek index denotes a form without any legs on the fiber and a form with either one or no legs on the fiber is represented with a Latin subscript.
When dualizing from M-theory to Type IIB theory, the internal forms on the manifolds also have a correspondence to each other according to their expansion coefficients in Minkowski spacetime. Therefore, the correspondences

$$\begin{array}{lcl} (A^i, L^i) \longrightarrow A & \Longrightarrow & \omega_i \longrightarrow P_-, \\ N_\kappa \longrightarrow V^{\hat{\alpha}} & \Longrightarrow & (\alpha_\kappa, \beta^\lambda) \longrightarrow (\alpha_{\hat{\alpha}}, \beta^{\hat{\beta}}), \\ N_A \longrightarrow (G^a, a_p) & \Longrightarrow & (\alpha_A, \beta^B) \longrightarrow \begin{cases} (\hat{\alpha}_p, \hat{\beta}^p), \\ \omega_a, \end{cases} \end{array} \tag{7.2.8}$$

relate the mixed gauge kinetic coupling function in M-theory (7.2.7) to

$$f_{i\kappa} \xrightarrow{\text{weak coupling}} f_{\hat{\alpha}D7} = \mathrm{Re} f^{pq} \mathrm{Re} f_{\hat{\alpha}\hat{\beta}} \Big(\int_{S_+} P_- \wedge \hat{\alpha}_q \wedge \iota^* \beta^{\hat{\beta}} - i \bar{f}_{qr} \int_{S_+} P_- \wedge \hat{\beta}^r \wedge \iota^* \beta^{\hat{\beta}} \Big) a_p, \tag{7.2.9}$$

which states the corresponding expression in the Type IIB theory. Hence, this should be equivalent to the expression of the mixed gauge kinetic coupling function (5.5.4), obtained in the reduction of Type IIB theory. To explicitly show this equivalence, we recognize the basis form γ^p in expression (7.2.9), such that combined with $a_I A^I = a_p \gamma^p$ we can write

$$f_{\hat{\alpha}D7} = 2\mathrm{Re} f_{\hat{\alpha}\hat{\beta}} \int_{S_-} A^I \wedge \iota^* \beta^{\hat{\beta}} a_I = -2\mathrm{Re} f_{\hat{\alpha}\hat{\beta}}\, a_I \mathcal{A}^{\hat{\beta} I}, \tag{7.2.10}$$

using the matrices defined in (5.2.8).

Recall from equations (4.2.22) and (5.5.4) that

$$f_{\hat{\alpha}\hat{\beta}} = -\frac{i}{2} \overline{M}_{\hat{\alpha}\hat{\beta}} = -\frac{1}{2} \Big(C^{-1} + i A C^{-1} \Big)_{\hat{\alpha}\hat{\beta}} \tag{7.2.11}$$

in Type IIB theory. Thus, indeed the mixed gauge kinetic coupling function between the bulk and the brane bosonic gauge vectors turns out to be

$$f_{\hat{\alpha}D7} = C_{\hat{\alpha}\hat{\beta}} a_I \mathcal{A}^{\hat{\beta}I}, \tag{7.2.12}$$

up to a prefactor, as was already shown (5.5.4) from the reduction of the Type IIB theory on a Calabi-Yau threefold including D7-branes. Hence, as expected in the weak string coupling limit dualizing M-theory compactified on an elliptically fibered Calabi-Yau fourfold results in the same mixed gauge kinetic coupling function as obtained from the Type IIB reduction. However, an important result is deduced from this M-theory path. This path has shown that the mixed gauge coupling function, obtained from the Type IIB reduction, is holomorphic in the complex structure moduli fields, since the mixed gauge kinetic coupling function resulting directly from the reduction of the eleven-dimensional supergravity action on a Calabi-Yau fourfold (7.2.5) turned out to be holomorphic, due to the holomorphic property of the function $f_{\mathcal{AB}}$.

Chapter 8

Conclusion

In this thesis, we have investigated in a mathematical proof that the $\mathcal{N} = 1$ gauge kinetic coupling function resulting from our setup is holomorphic in the chiral coordinates. To reach this point, we have first compactified the democratic version of the ten-dimensional Type IIB supergravity in the weak string coupling limit on a Calabi-Yau threefold. Focussing on orientifold reductions admitting $O3/O7$-planes such that part of the spectrum is projected out, we obtained the lower-dimensional effective theory. Next, we have included a single spacetime filling D7-brane wrapped on a (2,2)-cycle of the Calabi-Yau orientifold. We have realized this by adding the Dirac-Born-Infeld and the Chern-Simons action for a D7-brane. Compactifying both has led to a $U(1)$ Abelian gauge theory. In this reduction we have taken into account the fluctuations of the embedding of the four-cycle into the two directions of the Calabi-Yau orientifold normal to the cycle and assumed the (2,2)-cycle which wrappes the brane allows for non-trivial one-cycles leading to Wilson lines. The final ingredient of this reduction was the inclusion of a background flux on the D7-brane, which gives rise to lower-dimensional Ramond-Ramond charges distributed over the brane. For this flux we have assumed the pullback of the harmonics in the cohomology $H^{(1,1)}_{\bar{\partial},-}(Y_3)$ to be the only non trivial two-forms on the four-cycle.

After reducing all parts of the democratic action, we worked out the technical aspect of imposing the self-dual constraints on the gauge vectors. Defining the correct $\mathcal{N} = 1$ chiral and vector multiplets, it can be shown that the four-dimensional theory obtained indeed suites the $\mathcal{N} = 1$ supergravity action, from which we can read of the gauge kinetic coupling function. This function is of our main interest. We briefly review that the coupling function among the bulk gauge vectors is holomorphic in the $\mathcal{N} = 1$ coordinates, as yet shown in [32, 34].

Apart from the $U(1)$ bulk gauge vectors we obtain another sector of $U(1)$ gauge vectors resulting from the brane. These D7-brane gauge vectors interact amongst themselves but also couple to the bulk vectors in the presence of Wilson line moduli. As a new result, we discuss the indications we have found that point towards the fact that the D7-brane gauge kinetic coupling function is indeed holomorphic in the chiral coordinates, taking into account the existence of Wilson lines. Even though we do not provide a mathematical proof, we do include recommendations for the discovery of such a proof in appendix A which could be supporting for further studies. Lastly, we note that the Wilson lines give rise to mixed kinetic coupling terms between the bulk and the brane gauge vectors. To show that this mixed kinetic coupling function is holomorphic, we first compactify the eleven-dimensional supergravity resulting from M-theory on a elliptically fibered Calabi-Yau fourfold. Hereafter, we explain which fields in M-theory correspond to the Type IIB fields in the weak coupling limit, to show that the mixed gauge kinetic coupling in M-theory corresponds to the function obtained in our Type IIB reduction upon matching the correct harmonics. Therefore, from the knowledge that the mixed gauge coupling function is holomorphic in M-theory, we conclude it must also be holomorphic from the Type IIB perspective.

As yet stated, both sectors of vectors are the gauge bosons of two different $U(1)$ groups. One possibility for extending the Standard Model with a new $U(1)$ Abelian gauge group is to interpret it as the very weak coupling of the dark photon with the electrically charged particles of the Standard Model through kinetic mixing with the regular photon arising from electromagnetism [61, 62]. The dark photon is the force carrier of the hidden/dark sector [63], similar to the photon. The coupling of the photon to the dark photon might provide the only non-gravitational window to the existence of the hidden sector. Within string theory it has been studied that placing hidden branes which are geometrically separated from the visible branes can give rise to these additional hidden $U(1)$ groups [64] with massive fields arising on strings stretching between the branes, with masses of the order ℓ^{-2}. The interactions appearing here are α'[1] suppressed. In our reduction, we have not taken into account higher orders of α' which can be interesting for phenomenological applications [11, 65]. Finally, we observe something quite similar to these force carriers of the hidden sector. Due to the mixed kinetic coupling we obtain between the two $U(1)$ gauge groups, we could possibly view the two gauge vectors as the photon and the dark photon, which could be an interesting point of view for further research.

[1] $2\pi\alpha' = \ell$

Appendix A

Recommendations for D7-brane coupling

In this chapter we focus more extensively on equations (5.5.7) and (5.5.8). We suggest some point of views that, if explored, could lead to a mathematical proof of these relations, which we will leave for future research.

One possible approach would be to search for how these terms appear in M-theory. We have only looked into this briefly, though we will now explain our findings to construct a starting point for further investigations.

First, we want to remind the reader where the Type IIB harmonics on the Calabi-Yau orientifold originate from in M-theory. Recall that

$$\begin{array}{lcl} \text{M-theory} & & \text{Type IIB theory} \\ \Psi^{\kappa} & & \alpha_{\hat{\alpha}}, \beta^{\hat{\alpha}}, \\ \Psi^{A} & \Longrightarrow & A^{I}, \bar{A}^{\bar{J}}, \\ \omega_{\alpha} & & \omega_{\alpha}, \\ \omega_{i} & & P_{-}. \end{array} \tag{A.0.1}$$

We will assume that the (1,2)-forms Ψ^{κ} on the Calabi-Yau fourfold take a similar form when dualized to the Type IIB perspective as $\psi^{\hat{\alpha}} = \frac{1}{2}\mathrm{Re} f^{\hat{\alpha}\hat{\epsilon}}(\alpha_{\hat{\epsilon}} - i\bar{f}_{\hat{\epsilon}\hat{\beta}}\beta^{\hat{\beta}})$.

Next we want to emphasize that ${d_\Sigma}^{\mathcal{AB}}$ was given in (6.2.4) as

$$ {d_\Sigma}^{\mathcal{AB}} = i \int_{Y_4} \omega_\Sigma \wedge \Psi^{\mathcal{A}} \wedge \bar{\Psi}^{\mathcal{B}}, \tag{A.0.2} $$

though, using (6.2.2), it can also be written as

$$ L^\Sigma {d_\Sigma}^{\mathcal{AB}} = \int_{Y_4} \Psi^{\mathcal{A}} \wedge \star \bar{\Psi}^{\mathcal{B}}. \tag{A.0.3} $$

With the above information, it becomes clear which matrices in M-theory correspond to the required Type IIB matrices

$$ \begin{aligned} C_\Lambda^{I\bar{J}} &\Longleftrightarrow L^\Sigma {d_\Sigma}^{AB}, && \Sigma \text{ must be } \alpha, \\ C^{\hat{\alpha}\hat{\beta}} &\Longleftrightarrow L^\Sigma {d_\Sigma}^{\kappa\lambda}, && \Sigma \text{ must be } 0, \\ \mathcal{A}^{\hat{\alpha}I} &\Longleftrightarrow L^\Sigma {d_\Sigma}^{\kappa\mathcal{A}}, && \Sigma \text{ must be } i. \end{aligned} $$

Note that the matrices in both theories are not equal, we only want to stress that there are certain relations between these expressions.

When writing both equation (5.5.7) and (5.5.8) in a slightly different form

$$ -8\mathcal{A}^{\hat{\alpha}I} \bar{\mathcal{A}}^{\hat{\beta}\bar{J}} = i C_\Lambda^{I\bar{J}} C^{\hat{\alpha}\hat{\beta}}, \tag{A.0.4} $$

$$ -8\mathcal{A}^{\hat{\alpha}I} \mathcal{A}^{\hat{\beta}J} = i C_\Lambda^{I\bar{J}} C^{\hat{\alpha}\hat{\beta}}, \tag{A.0.5} $$

it becomes more explicit that both result from similar parts in M-theory, namely

$$ L^\Sigma {d_\Sigma}^{\mathcal{AB}} L^\Omega {d_\Omega}^{\mathcal{CD}}. \tag{A.0.6} $$

Though, taking into consideration that every non-vanishing integral over the Calabi-Yau fourfold must have two legs on the elliptical fiber, we obtain that the left hand side of (A.0.4) and (A.0.5) originates from

$$ L^i {d_i}^{\kappa A} L^j {d_j}^{\lambda B} \tag{A.0.7} $$

in M-theory, while the right hand side from

$$L^0 {d_0}^{\kappa\lambda} L^\alpha {d_\alpha}^{AB}. \tag{A.0.8}$$

Hence, to show explicitly which parts of $L^\Sigma {d_\Sigma}^{\mathcal{AB}} L^\Omega {d_\Omega}^{\mathcal{CD}}$ result in the required Type IIB matrices, we write out both parts

$$\begin{aligned}
& L^0 {d_0}^{\kappa\lambda} {d_\alpha}^{AB} = {d_\alpha}^{AB} \int_{Y_4} \Psi^\kappa \wedge \star \bar{\Psi}^\lambda \Longrightarrow \\
& i \int_{S_+} \iota^* \omega_\alpha \wedge A^I \wedge \bar{A}^{\bar{J}} \int_{Y_3} \frac{1}{4} \mathrm{Re} f^{\hat{\alpha}\hat{\gamma}} (\alpha_{\hat{\gamma}} - i \bar{f}_{\hat{\gamma}\hat{\epsilon}} \beta^{\hat{\epsilon}}) \wedge \star (\alpha_{\hat{\delta}} + i \beta^{\hat{\kappa}} f_{\hat{\kappa}\hat{\delta}}) \mathrm{Re} f^{\hat{\delta}\hat{\beta}} \\
& = \frac{i}{4} {C_\Lambda}^{I\bar{J}} \mathrm{Re} f^{\hat{\alpha}\hat{\gamma}} \mathrm{Re} f^{\hat{\delta}\hat{\beta}} \Big[B_{\hat{\gamma}\hat{\delta}} - \bar{f}_{\hat{\gamma}\hat{\epsilon}} C^{\hat{\epsilon}\hat{\kappa}} f_{\hat{\kappa}\hat{\delta}} + i \bar{f}_{\hat{\gamma}\hat{\epsilon}} A^{\hat{\epsilon}}_{\hat{\delta}} + i D_{\hat{\gamma}}^{\hat{\kappa}} f_{\hat{\kappa}\hat{\delta}} \Big] \\
& = i {C_\Lambda}^{I\bar{J}} C^{\hat{\alpha}\hat{\gamma}} C^{\hat{\beta}\hat{\delta}} \Big[- C_{\hat{\gamma}\hat{\kappa}} A^{\hat{\kappa}}_{\hat{\epsilon}} A^{\hat{\epsilon}}_{\hat{\delta}} - C_{\hat{\gamma}\hat{\delta}} + \frac{1}{4} (i C_{\hat{\gamma}\hat{\zeta}} A^{\hat{\zeta}}_{\hat{\epsilon}} - C_{\hat{\gamma}\hat{\epsilon}}) C^{\hat{\epsilon}\hat{\kappa}} (i C_{\hat{\kappa}\hat{\xi}} A^{\hat{\xi}}_{\hat{\delta}} + C_{\hat{\kappa}\hat{\delta}}) \\
& + \frac{i}{2} (i C_{\hat{\gamma}\hat{\zeta}} A^{\hat{\zeta}}_{\hat{\epsilon}} - C_{\hat{\gamma}\hat{\epsilon}}) A^{\hat{\epsilon}}_{\hat{\delta}} + \frac{i}{2} (i C_{\hat{\delta}\hat{\zeta}} A^{\hat{\zeta}}_{\hat{\kappa}} + C_{\hat{\delta}\hat{\kappa}}) A^{\hat{\kappa}}_{\hat{\gamma}} \Big] \\
& = i {C_\Lambda}^{I\bar{J}} C^{\hat{\beta}\hat{\delta}} \Big[- \frac{9}{4} A^{\hat{\alpha}}_{\hat{\epsilon}} A^{\hat{\epsilon}}_{\hat{\delta}} - \frac{5}{4} \delta^{\hat{\alpha}}_{\hat{\delta}} \Big].
\end{aligned} \tag{A.0.9}$$

In this derivation, we repeatedly used equations (3.2.13), (3.2.15) and (3.2.17) and derived $\mathrm{Re} f^{\hat{\alpha}\hat{\beta}} = -2C^{\hat{\alpha}\hat{\beta}}$ from the result obtained in (7.2.11). Note that the M-theory equivalent of the four-dimensional Kaluza-Klein vector is involved in this derivation.

When aiming to obtain the left hand side of relation (A.0.4), we deduce

$$\begin{aligned}
& {d_i}^{A\kappa} {d_j}^{\lambda B} \Longrightarrow \\
& \frac{1}{4} \int_{S_+} P_- \wedge \iota^* (\alpha_{\hat{\delta}} + i \beta^{\hat{\kappa}} f_{\hat{\kappa}\hat{\delta}}) \mathrm{Re} f^{\hat{\delta}\hat{\alpha}} \wedge A^I \int_{S_+} P_- \wedge \mathrm{Re} f^{\hat{\beta}\hat{\gamma}} \iota^* (\alpha_{\hat{\gamma}} - i \bar{f}_{\hat{\gamma}\hat{\epsilon}} \beta^{\hat{\epsilon}}) \wedge \bar{A}^{\bar{J}} \\
& = \int_{S_-} i \iota^* \beta^{\hat{\kappa}} (f_{\hat{\kappa}\hat{\delta}} - 2 \bar{f}_{\hat{\kappa}\hat{\delta}}) C^{\hat{\delta}\hat{\alpha}} \wedge A^I \int_{S_-} C^{\hat{\beta}\hat{\gamma}} i (2 f_{\hat{\gamma}\hat{\epsilon}} - \bar{f}_{\hat{\gamma}\hat{\epsilon}}) \iota^* \beta^{\hat{\epsilon}} \wedge \bar{A}^{\bar{J}} \\
& = \int_{S_-} \iota^* \beta^{\hat{\kappa}} (- \frac{3i}{2} A^{\hat{\alpha}}_{\hat{\kappa}} + \frac{1}{2} \delta^{\hat{\alpha}}_{\hat{\kappa}}) \wedge A^I \int_{S_-} (\frac{3i}{2} A^{\hat{\beta}}_{\hat{\epsilon}} + \frac{1}{2} \delta^{\hat{\beta}}_{\hat{\epsilon}}) \iota^* \beta^{\hat{\epsilon}} \wedge \bar{A}^{\bar{J}} \\
& = \mathcal{A}^{\hat{\kappa} I} \bar{\mathcal{A}}^{\hat{\epsilon}\bar{J}} \Big[\frac{9}{4} A^{\hat{\alpha}}_{\hat{\kappa}} A^{\hat{\beta}}_{\hat{\epsilon}} + \frac{1}{4} \delta^{\hat{\alpha}}_{\hat{\kappa}} \delta^{\hat{\beta}}_{\hat{\epsilon}} \Big],
\end{aligned} \tag{A.0.10}$$

frequently using (5.4.22) and (5.5.6). The left hand side of equation (A.0.5) can be derived in a similar fashion. Judging from the similarity between both results (A.0.9) and (A.0.10), this seems to head in a promising direction, however to precisely deduce relations (A.0.4) and (A.0.5) this should be further explored, but we will leave this to the interested reader.

Another suggestion that could quite possibly lead to the mathematical discovery of equations (5.5.7) and (5.5.8) is to take a closer look into the derivation of the $\mathcal{N} = 1$ chiral coordinates. This is likely to reveal similar relations to the ones we are aiming for. A careful construction of these coordinates has been performed in [36]. Therefore, we will not go into further detail about this.

Appendix B

Weyl rescaling

Within string theory Weyl transformations are a commonly applied technique since they form a symmetry of the Polyakov worldsheet action. Such a transformation is a local rescaling of the metric

$$g_{\mu\nu}^{\text{old}} = e^{2\omega} g_{\mu\nu}^{\text{new}}. \tag{B.0.1}$$

These rescalings transform the original metric into another metric of the same conformal class. We will explicitly show how terms of the form

$$F^{(p)} \wedge *F^{(p)} \tag{B.0.2}$$

frequently appearing in supergravity actions transform under a Weyl rescaling. We assume a d-dimensional theory with a rescaling (B.0.1) in all d dimensions. The crucial assumption, is that we assume $F^{(p)}$ to be a p-form that itself is invariant under the Weyl rescaling. Writing (B.0.2) in Einstein summation convention shows the explicit dependence on the metric

$$F^{(p)} \wedge *_{\text{old}} F^{(p)} = \frac{1}{p!} F_{\mu_1,\ldots,\mu_p} F_{\nu_1,\ldots,\nu_p} g_{\text{old}}^{\mu_1\nu_1} \ldots\, g_{\text{old}}^{\mu_p\nu_p} \sqrt{g_{\text{old}}}\, \mathrm{d}^n x. \tag{B.0.3}$$

From (B.0.1) it follows that

$$g_{\text{old}}^{\mu\nu} = e^{-2\omega} g_{\text{new}}^{\mu\nu} \qquad \text{and} \qquad \sqrt{g_{\text{old}}} = \sqrt{(e^{2\omega})^d g_{\text{new}}} = (e^{\omega})^d \sqrt{g_{\text{new}}}, \tag{B.0.4}$$

which results in

$$\frac{1}{p!}F_{\mu_1,\ldots,\mu_p}F_{\nu_1,\ldots,\nu_p}(e^{-2\omega})^p g_{\mathrm{new}}^{\mu_1\nu_1}\ldots g_{\mathrm{new}}^{\mu_p\nu_p}(e^{\omega})^d\sqrt{g_{\mathrm{new}}}\,\mathrm{d}^n x = e^{\omega(d-2p)}F^{(p)}\wedge *_{\mathrm{new}}F^{(p)}. \quad \text{(B.0.5)}$$

However, if $F^{(p)}$ does transform under the Weyl rescaling, this has to be incorporated in addition to the result derived above. The most known example of such a situation is the Ricci scalar. The general transformation rule of the Ricci scalar under a Weyl rescaling (B.0.1) is explicitly derived in [55] to be

$$R^{\mathrm{old}} = e^{-2\omega}\Big(R-(d-1)(d-2)\nabla^\alpha\omega\nabla_\alpha\omega - 2(d-1)\nabla^\alpha\nabla_\alpha\omega\Big)^{\mathrm{new}}. \quad \text{(B.0.6)}$$

Appendix C

Einstein-Hilbert term reduction

In section 4.2 we reduced the ten-dimensional Einstein-Hilbert term (4.2.2) on the background (4.1.1) up to second order in moduli fields. However, before arriving at the reduced form given in equation (4.2.6), we will emphasize to some extent on the intermediate steps of this derivation.

Recalling that the ten-dimensional Ricci scalar is given by

$$\begin{aligned}\hat{R} = \hat{g}^{MN}\hat{R}^{P}{}_{MPN} = g^{\mu\nu}R^{\rho}{}_{\mu\rho\nu} + \Big[& g^{\mu\nu}R^{m}{}_{\mu m\nu} + g^{mn}(R^{\mu}{}_{m\mu n} + R^{p}{}_{mpn} + R^{\bar{p}}{}_{m\bar{p}n}) \\ & + g^{m\bar{n}}(R^{\mu}{}_{m\mu\bar{n}} + R^{p}{}_{mp\bar{n}} + R^{\bar{p}}{}_{m\bar{p}\bar{n}}) + c.c.\Big],\end{aligned} \tag{C.0.1}$$

with $\hat{R}^{R}{}_{MPN}$ the Riemann curvature tensor

$$\hat{R}^{R}{}_{MPN} = \partial_P \hat{\Gamma}^{R}_{NM} - \partial_N \hat{\Gamma}^{R}_{PM} + \hat{\Gamma}^{R}_{PL}\hat{\Gamma}^{L}_{NM} - \hat{\Gamma}^{R}_{NL}\hat{\Gamma}^{L}_{PM} \tag{C.0.2}$$

and Christoffel symbols

$$\hat{\Gamma}^{R}_{MN} = \frac{1}{2}\hat{g}^{RP}(\partial_M \hat{g}_{PN} + \partial_N \hat{g}_{PM} - \partial_P \hat{g}_{MN}). \tag{C.0.3}$$

Given the background (4.1.1) expanded in (4.1.13) and (4.1.15), the only non-vanishing ten-dimensional Christoffel symbols are

$$\begin{aligned}
\Gamma^m_{\mu n} &= \frac{1}{2}(\omega_\beta)_{n\bar{p}}(\omega_\alpha)^{\bar{p}m} v^\alpha \partial_\mu v^\beta - \frac{1}{2}(b_{\tilde{a}})^{mp}(\bar{b}_{\tilde{b}})_{pn} z^{\tilde{a}} \partial_\mu \bar{z}^{\tilde{b}} - \frac{i}{2}(\omega_\alpha)_n{}^m \partial_\mu v^\alpha, \\
\Gamma^{\bar{m}}_{\mu n} &= \frac{1}{2}(\bar{b}_{\tilde{a}})^{\bar{m}}{}_n \partial_\mu \bar{z}^{\tilde{a}} + \frac{i}{2}(\omega_\alpha)^{\bar{m}p}(\bar{b}_{\tilde{a}})_{pn} v^\alpha \partial_\mu \bar{z}^{\tilde{a}} + \frac{i}{2}(\omega_\alpha)_{n\bar{p}}(\bar{b}_{\tilde{a}})^{\bar{p}\bar{m}} \bar{z}^{\tilde{a}} \partial_\mu v^\alpha, \\
\Gamma^\mu_{mn} &= -\frac{1}{2}(\bar{b}_{\tilde{a}})_{mn} \partial^\mu \bar{z}^{\tilde{a}}, \\
\Gamma^\mu_{m\bar{n}} &= \frac{i}{2}(\omega_\alpha)_{m\bar{n}} \partial^\mu v^\alpha
\end{aligned} \tag{C.0.4}$$

and their complex conjugates.

As an example we explicitly calculate the third term of the Ricci scalar (C.0.1) up to second order in moduli fields

$$\begin{aligned}
\int d^{10}x\sqrt{-g_{10}}\, g^{mn} R^\mu{}_{m\mu n} &= -\int d^{10}x\sqrt{-g_{10}}\, z^{\tilde{a}}(b_{\tilde{a}})^{mn} \partial_\mu \Gamma^\mu_{mn} \\
&= \int d^{10}x\sqrt{-g_{10}}\, \frac{1}{2}(b_{\tilde{a}})^{mn}(\bar{b}_{\tilde{b}})_{mn} z^{\tilde{a}} \partial_\mu \partial^\mu \bar{z}^{\tilde{b}} \\
&= -\int d^{10}x\sqrt{-g_{10}}\, \frac{1}{2}(b_{\tilde{a}})^{mn}(\bar{b}_{\tilde{b}})_{mn}(\partial_\mu z^{\tilde{a}}) \partial^\mu \bar{z}^{\tilde{b}}.
\end{aligned} \tag{C.0.5}$$

Note that the derivatives do not act on the harmonics since these are closed and furthermore we performed a partial integration in which we drop the total derivatives and used metric compatibility and the mathematical trick $\det(A) = e^{\mathrm{Tr}[\ln(A)]}$ for a general matrix A to obtain

$$\partial_\mu\sqrt{-g_{10}} = \sqrt{-g_4}\partial_\mu\sqrt{g_6} = \frac{1}{2}\sqrt{-g_4}\sqrt{g_6}\,\mathrm{Tr}[g^{mn}\partial_\mu\, g_{np}] = -i\sqrt{-g_{10}}(\omega_\alpha)_m{}^m \partial_\mu v^\alpha \tag{C.0.6}$$

up to first order in moduli fields. Calculating all other terms of the Ricci scalar in a similar fashion results in

$$\begin{aligned}
\int d^{10}x\sqrt{-g_{10}}\, g^{\mu\nu} R^m{}_{\mu m\nu} = -\int d^{10}x\sqrt{-g_{10}}\, \frac{1}{2}\big[(\omega_\alpha)_m{}^m(\omega_\beta)_n{}^n \\
- \frac{1}{2}(\omega_\alpha)^{\bar{n}m}(\omega_\beta)_{m\bar{n}}\big](\partial_\mu v^\alpha)\partial^\mu v^\beta + \frac{1}{4}(b_{\tilde{a}})^{mn}(\bar{b}_{\tilde{b}})_{mn}(\partial_\mu z^{\tilde{a}})\partial^\mu \bar{z}^{\tilde{b}},
\end{aligned} \tag{C.0.7}$$

$$\int d^{10}x\sqrt{-g_{10}}\, g^{m\bar{n}} R^{\mu}{}_{m\mu\bar{n}} = -\int d^{10}x\sqrt{-g_{10}}\, \frac{1}{2}\big[(\omega_\alpha)_m^{\ m}(\omega_\beta)_n^{\ n} - \frac{1}{2}(\omega_\alpha)^{\bar{n}m}(\omega_\beta)_{m\bar{n}}\big](\partial_\mu v^\alpha)\partial^\mu v^\beta - \frac{1}{4}(b_{\tilde{a}})^{mn}(\bar{b}_{\tilde{b}})_{mn}(\partial_\mu z^{\tilde{a}})\partial^\mu \bar{z}^{\tilde{b}}, \tag{C.0.8}$$

$$\int d^{10}x\sqrt{-g_{10}}\, g^{m\bar{n}} R^{p}{}_{mp\bar{n}} = \frac{1}{4}\int d^{10}x\sqrt{-g_{10}}\big[(\omega_\alpha)_m^{\ m}(\omega_\beta)_n^{\ n} - (\omega_\alpha)^{\bar{n}m}(\omega_\beta)_{m\bar{n}}\big] (\partial_\mu v^\alpha)\partial^\mu v^\beta, \tag{C.0.9}$$

$$\int d^{10}x\sqrt{-g_{10}}\, g^{m\bar{n}} R^{\bar{p}}{}_{m\bar{p}\bar{n}} = \frac{1}{4}\int d^{10}x\sqrt{-g_{10}}\big[(\omega_\alpha)_m^{\ m}(\omega_\beta)_n^{\ n}(\partial_\mu v^\alpha)\partial^\mu v^\beta + (b_{\tilde{a}})^{mn}(\bar{b}_{\tilde{b}})_{mn}(\partial_\mu z^{\tilde{a}})\partial^\mu \bar{z}^{\tilde{b}}\big]. \tag{C.0.10}$$

Finally, combining the terms $g^{mn}(R^{p}{}_{mpn} + R^{\bar{p}}{}_{m\bar{p}n}) = g^{mn}R_{mn}$ yields the Ricci tensor on the internal space. Up to second order in moduli fields this term vanishes.

Collecting all terms, including the complex conjugates, we obtain the reduced Einstein-Hilbert action

$$S^{(4)}_{\text{EH, EF}} = -\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g_{10}}\,\Big[R + \Big(\frac{1}{2}(\omega_\alpha)^{\bar{n}m}(\omega_\beta)_{m\bar{n}} - (\omega_\alpha)_m^{\ m}(\omega_\beta)_n^{\ n}\Big)(\partial_\mu v^\alpha)\partial^\mu v^\beta - \frac{1}{2}(b_{\tilde{a}})^{mn}(\bar{b}_{\tilde{b}})_{mn}(\partial_\mu z^{\tilde{a}})\partial^\mu \bar{z}^{\tilde{b}}\Big]. \tag{C.0.11}$$

Appendix D

Shift symmetries from circle reduction

In this chapter we will discuss how shift symmetries arise when compactifying an Abelian gauge theory. Therefore, we discuss the most basic example of a D-dimensional theory reduced on a circle, considering the manifold $M_D = M_d \times S_1$. As a starting point, we will take the higher-dimensional action

$$S = \int \hat{F} \wedge \hat{*}\hat{F}, \tag{D.0.1}$$

with the field strength $\hat{F} = \mathrm{d}\hat{A}$ of the D-dimensional gauge vector $\hat{A}$. By construction the field strength is invariant under a gauge transformation

$$\hat{A} \to \hat{A} + \mathrm{d}\hat{\Lambda}. \tag{D.0.2}$$

Expanding the higher-dimensional gauge vector $\hat{A} = A \wedge \mathbb{1} + a \wedge \mathrm{d}y$ where y is the coordinate on the circle, $y \sim y + 2\pi$, results in a lower-dimensional gauge vector A and a d-dimensional scalar a. When also expanding the D-dimensional gauge transformation as $\mathrm{d}\hat{\Lambda} = \mathrm{d}\Lambda \wedge \mathbb{1} + p \wedge \mathrm{d}y$, we conclude that indeed the transformation (D.0.2) leads to a d-dimensional gauge vector transforming as $A \to A + \mathrm{d}\Lambda$, with a field strength $F = \mathrm{d}A$ invariant under the transformation. In addition there is a d-dimensional scalar a that enjoys a shift symmetry $a \to a + p$, obtained from the fact that the higher-dimensional gauge vector $\hat{A}$ includes more gauge degrees of freedom than the lower-dimensional gauge vector A. Therefore, the gauge degrees of freedom are spread over the d-dimensional scalar and vector and the vector on the internal space. Note that $\mathrm{d}\hat{\Lambda} = 0$ implies $\mathrm{d}p = 0$ which makes $\mathrm{d}a$ invariant under the shift symmetry.

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