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Evolutionary Game Theory

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Abstract

This thesis provides an introduction into the basics of Evolutionary Game Theory. Evolutionary Game Theory is linked with the concept of natural selection, which deals with *selection* and *mutation*. After a small introduction in Game Theory, the definition of an Evolutionary Stable Strategy (ESS) is introduced. An ESS focusses on the concept of *mutation*, where the payoff is measured by the number of offspring. A point is an ESS if it is a better strategy than a mutant strategy, and it fares better against the mutation than the mutation does against itself. The main result is that an Evolutionary Stable Strategy is always a symmetric Nash equilibrium, but the converse is not true. Then, the concept of the Replicator Dynamic is introduced, which covers the concept of *selection*. The replicator dynamics is an ordinary differential equation, which is used to measure the change in the composition of the population over time. The main result is that a Lyapunov stable stationary state is a symmetric Nash equilibrium. Finally, an economic application of Evolutionary Game Theory is explained, where the replicator dynamic equation is calculated, equilibrium points are found and stability is measured.

Contents

Chapter 1

Introduction

Game theory is a growing subject with many sorts of applications. A broad, but accurate, definition of game theory is given by Hans Peters in his book *'Game Theory: A Multi-Levelled Approach'*[15], which will be the basis of this study. He states: 'Game theory is a formal, mathematical discipline which studies situations of competition and cooperation between several involved parties'.

Game theory was originally created when Von Neumann and Morgenstern published their book *'Theory of Games and Economic Behavior'* in 1944[24]. Nevertheless, the main revolution in game theory was accomplished by John Nash, who published his article *'Non-Cooperative Games'* in the early 1950s[12]. In the early 1970s this resulted in the realization of the powerful tool John Nash had provided in formulating the equilibrium concept. This caused a reawakening in game theory, since economists applied the idea to economic issues[25]. Nash's theories were mainly based on rationality of the players, which created ambiguousness about the definition of rationality. In 1973, Maynard Smith published his article *'The Logic of Animal Conflict'*, where he introduced evolutionary game theory[10][11][9]. Here is not assumed that the players are aware of the game and therefore act rationally.

The introduction of evolutionary game theory required a radical shift in perspective [18]. In evolutionary game theory, the players are interpreted as populations of individuals or animals [15]. Each individual chooses among alternative actions whose payoffs depend on the choices of other individuals. The distribution of observed behavior in a population and the possible actions evolve over time, as fitter and more efficient strategies become more prevalent. This prevalence of behaviors can make actions fitter or less fit. So, one can imagine that the dynamics can become quite complex. Evolutionary game theory focusses on the question which behaviors become extinct and which survive over time, and whether the system approaches a stable-steady state[2].

A fundamental point is that biologists often deal with the genetic mechanism of natural selection. This mechanism can be used in economics or other social sciences as well. In economics, the social mechanisms of learning and imitation are usually more important than the genetic mechanism. A wide variety of learning and imitation processes are conceivable and the appropriate dynamical representation is highly contextdependent. The results of evolutionary game theory could be interesting to economists in the sense that the areas of attractions are easily found[2]. Market competition pushes firms who are not maximizing their profits out of the market, and eventually reaches an equilibrium. Evolutionary game theory enables analysis in these interactive environments [25]. An example of such an interactive environment is illustrated in chapter 4.

Game theoretical concepts in evolutionary biology have been criticized by populations geneticists. The theory of long-term evolution neglects crucial aspects such as the mating system or the mode of inheritance. Genetic constrains, which may be dominant in a short-term perspective, will in the long run disappear due to the constant supply of mutations [26].

This thesis aims to provide an introduction into the basics of evolutionary game theory. This chapter will continue explaining the basics of standard game theory needed to understand evolutionary game theory. Chapter 2 will enlighten the concept of evolutionary stable strategies; Chapter 3 is about replicator dynamics and evolutionary stability. The last chapter will show an economical application of evolutionary game theory.

1.1 The basics

The models of game theory are highly abstract presentations of real life situations [14]. By providing different examples we will explain the basic definitions.

1.1.1 Time vs. Newsweek

The first example is a non-cooperative game involving two papers, Time Magazine and Newsweek. They need to decide which article to publish on the front-page. Both parties have two alternatives, 'Impeachment' or 'Financial Crisis'. Their payoffs are shown in the matrix below, which we call a *payoff matrix*:

		Newsweek	
		Impeachment	Financial Crisis
Time Magazine	Impeachment	(45,45)	(90,40)
	Financial Crisis	(40,90)	(20,20)

Time Magazine is referred to as player 1 and considers the rows of the matrix. In other words, they look at the 'first mentioned number' in each of the possibilities. Newsweek is referred to as player 2 and considers the columns of the matrix, or the 'second number mentioned'. It shows a 2×2-matrix. Such a game is called a *matrix game*, and in particular a *bimatrix game*, which are generally defined as follows:

Definition 1.1.1 (Matrix Game [15]). *A matrix game is a $m \times n$-matrix A of natural numbers where the number of rows m and the number of columns n are integers such that $m, n \in \mathbb{N}$ with $m, n \geq 1$. The matrix A looks like:*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Definition 1.1.2 (Bimatrix Game [15]). *A bimatrix game is a pair of $m \times n$-matrices (A,B), where $m, n \in \mathbb{N}$ with $m, n \geq 1$.*

The game in our example has a special format. It is called a *symmetric game*. We could define the payoff as $G = (A, A^T)$, where A is given by:

$$A = \begin{pmatrix} 45 & 90 \\ 40 & 20 \end{pmatrix}$$

Definition 1.1.3 (Symmetric Game [15]). *Let $G = (A, B)$ be an $m \times n$-bimatrix game. Then G is symmetric if $m = n$ and $B = A^T$, where A^T denotes the transpose of A. Hence, $b_{ij} = a_{ji}$ for all $i, j = 1, ..., m$.*

Non-cooperative game theory designed by John Nash includes some assumptions. We assume that both player have full knowledge about the payoffs of the game. They choose simultaneously, without the possibility of bargaining. Thereby, both player act rationally, which is often referred to as their 'best reply'. So, they should always maximize their expected payoff, given the knowledge of or conjecture about the strategies chosen by the other players. Both players take the other person's options into consideration.

In this example Time Magazine (player 1) has to choose between the two articles. If Newsweek (player 2) would choose to publish the impeachment article, Time Magazine has to choose between a payoff of 45 or 40. Since they act rationally they prefer to publish the impeachment article as well. If Newsweek would choose to publish the financial crisis article, Time Magazine has to choose between a payoff of 90 or 20. Again, they prefer to publish the article on impeachment. So, unregarded the choice of Newsweek, Time Magazine's best reply is always to publish the article on impeachment. This is called a *dominant strategy* or *pure strategy*. The same deduction can be performed for Newsweek, and again the result will show that their dominant strategy is to publish the article on impeachment. This game as an equilibrium in pure strategies, namely (impeachment, impeachment). Such a point is called a *saddle point* or *Nash equilibrium*. In such a saddlepoint no player has the incentive to deviate unilaterally.

Definition 1.1.4 (Saddlepoint [15]). *A position (i, j) in a $m \times n$-matrix game A is a saddlepoint if $a_{ij} \geq a_{kj}$ for all $k = 1, ..., m$ and $a_{ij} \leq a_{ik}$ for all $k = 1, ...n$. In other words, a_{ij} is a saddlepoint if it is maximal in its column j and minimal in its row i.*

The Nash equilibrium is the main tool in game theory. It is, for example, used in oligopolistic and political competition [14]. Before presenting the definition of a Nash equilibrium some insides on strategies are necessary.

In a bimatrix game (A, B) player 1 considers the rows of the $m \times n$-matrix A. A (mixed) strategy of player 1 is a probability distribution $\mathbf{p}$ over the m rows of A, so an element of the set

$$\Delta^m := \{\mathbf{p} = (p_1, ..., p_m) \in \mathbb{R}^m \mid \sum_{i=1}^{m} p_i = 1, p_i \geq 0 \text{ for all } i = 1, ..., m\}.$$

Player 2 considers the columns of the matrix B. Analogous, A (mixed) strategy of player 2 is a probability distribution $\mathbf{q}$ over the n columns of B, so an element of the set

$$\Delta^n := \{\mathbf{q} = (q_1, ..., q_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} q_j = 1, q_j \geq 0 \text{ for all } i = 1, ..., n\}.$$

Furthermore, a strategy $\mathbf{p}$ of player 1 is called *pure* if there is a row with $p_i = 1$. This is sometimes denote by $\mathbf{e}^i$ (analogous, a strategy $\mathbf{q}$ of player 2 is called pure if $q_j = 1$, denoted by $\mathbf{e}^j$). In our example, the pure strategy of Time Magazine is $\mathbf{p} = (1, 0)$, also denoted by $\mathbf{e}^1$. The same holds for Newsweek, their pure strategy is $\mathbf{q} = (1, 0)$, denoted by $\mathbf{e}^1$.The interpretation and computation of the payoff for player 1 is as follows:

$$\mathbf{p}A\mathbf{q} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j a_{ij} \tag{1.1}$$

Where A is an $m \times n$-matrix, $\mathbf{p} \in \Delta^m$ and $\mathbf{q} \in \Delta^n$. Analagous, for player 2 the payoff is calculated by

$$\mathbf{p}B\mathbf{q} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j b_{ij} \tag{1.2}$$

The definition of a Nash equilibrium was originally written by John Nash [12], but the next definition is cited from Peters' book [15]:

Definition 1.1.5 (Nash Equilibrium [15])**.** *A pair of strategies (**p**,**q**) in an $m \times n$-bimatrix game (A,B) is a Nash equilibrium if **p** is the best reply of player 1 to **q** and **q** is the best reply of player 2 to **p**. In other words, if both*

$$\mathbf{p}A\mathbf{q} \geq \mathbf{p}'A\mathbf{q} \quad \textit{for all } \mathbf{p}' \in \Delta^m$$

and

$$\mathbf{p}B\mathbf{q} \geq \mathbf{p}B\mathbf{q}' \quad \textit{for all } \mathbf{q}' \in \Delta^n$$

*hold. A Nash equilibrium (**p**,**q**) is called pure if both **p** and **q** are pure strategies. A Nash equilibrium $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ is symmetric if every player plays the same strategy, if $\boldsymbol{p}^* = \boldsymbol{q}^*$.*

We can conclude that our example has a symmetric Nash equilibrium in pure strategies, namely$((1, 0), (1, 0))$.

1.1.2 The Battle of the Bismarck Sea

The next game is an example of a *zero-sum game*, because the sum of the payoffs equal zero in every case. The payoff matrix is given by [15]:

		Japan	
		North	South
America	North	(2,-2)	(2,-2)
	South	(1,-1)	(3,-3)

The story of this game is that America (player 1, rows) wants to bomb the transport of Japan to New Guinea (player 2, columns). America and Japan have two options: the shorter Northern route, which will take 2 days, or the longer Southern route, which will take 3 days. So, if they choose the same route America has either 2 or 3 days to bomb Japan's transport. If America chooses their route differently form Japan, they can call back the planes and send them to the other route. But, this will delay the attack a whole day. We assume that the number of bombing days is a positive payoff for America and a negative payoff for Japan, therefore the sum of the game is zero. We could summarize the payoffs into one matrix given by

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

Where the payoff is considered positive for America (player 1), and negative for Japan (player 2). From Japan's perspective we see that if America would choose the shorter Northern route, Japan needs to choose between a payoff of -2 or -2. Since these are the same, Japan is indifferent between the two routes. If America would choose the longer Southern route, Japan needs choose between a payoff of -1 or -3. In this case Japan's best reply is to choose the Northern route. Overall, Japan is slightly better of with the Northern route, regardless the choice of America. Japan's strategy is called *weakly dominant*. America has full knowledge about the payoff and therefore knows that Japan will probably choose the Northern route. America will choose the Northern route as well (a payoff of 2 versus 1). The combination (North, North) maximizes its minimal payoff in its row for America (because, $2 = \min\{2,2\} \geq 1 = \min\{1,3\}$) and minimizes the maximum payoff in its column for Japan (because, $2 = \max\{2,1\} \leq 3 = \max\{2,3\}$). This point is the saddlepoint or Nash equilibrium in this game.

1.1.3 Matching Pennies

Another often used example is the matching pennies game. This game is also an example of a zero-sum game. In this game, both players have a coin which they simultaneously throw. If the coins match player 1 wins both coins. If they do not match, both coins go to player 2. The payoff matrix is given by [15]:

		Player 2	
		Heads	Tails
Player 1	Heads	(1,-1)	(-1,1)
	Tails	(-1,1)	(1,-1)

If player 2 throws heads, player 1 wishes to throw heads as well, and if player 2 throws tails, player 1 wishes to throw tails as well. From player 2's perspective it is the other way around. If player 1 throws heads, player 2 wishes to throw tails, and if player 1 throws tails player 2 wishes to throw heads. This shows that neither player has a pure or dominant strategy in this game. The randomized choices of the players is called a *mixed strategy*. These choices are often interpreted as 'beliefs' of the players.

Theory about mixed strategy is, for example, used in describing the distribution of the tongue length of bees or the tube length in flowers [14]. It has been proven that every two-person matrix game has a *value* if mixed strategies are possible [23]. Suppose player 1 chooses heads or tails with probability $\frac{1}{2}$. Suppose player 2 chooses heads with probability q and tails with probability $1-q$, where $0 \leq q \leq 1$. The expected payoff for player 1 is equal to $\frac{1}{2}[q \cdot 1 + (1-q) \cdot -1] + \frac{1}{2}[q \cdot -1 + (1-q) \cdot 1]$ which is equal to 0 (so independent of q). We say that 0 is the *value* of the game.

1.1.4 Prisoners' Dilemma

Probably the most famous example is the prisoners' dilemma given by [15]:

		Prisoner 2	
		C	D
Prisoner 1	C	(-1,-1)	(-10,0)
	D	(0,-10)	(-9,-9)

This payoff matrix shows a symmetric 2×2-matrix, non-zero sum game. Here, two prisoners (players 1 and 2) have committed a crime together and are interrogated independently of each other. Both prisoners can choose to either 'cooperate' (C), so not betray his partner, or 'defect' (D), betray his partner. The punishment for their crime is 10 years in prison. Both betraying their partner reduces their punishment by 1 year. If they betray their partner and are not betrayed themselves they can go home, while the other still has to stay for 10 years in prison. If they both do not betray the other they are convicted to 1 year in prison for a minor offence. Their negative payoff (number of years in prison) is shown in the payoff matrix.

By the same sort of reasoning as in the previous examples we see that both players have a pure strategy in betraying their partner, regardless what the other prisoner does. So, the symmetric Nash equilibrium is (D, D) leading to a payoff of $(-9, -9)$. But, we see that both prisoners could be better of by both not betraying their partner and therefore choose strategy C. By playing (C,C) their payoff would become (-1,-1).

This shows that the result is not *Pareto Optimal*, since both player could obtain a better payoff[1]. This game is a great metaphor within economics and is used to analyse different situations [4]. Many scientists analysed the consequences of the possibility of cooperation in similar situations, for example [1].

[1] An equilibrium is *Pareto Optimal* if it is impossible to reallocate so as to make an individual better off without making at least one individual worse off.

Chapter 2

Evolutionary Stable Strategies

With the standard interpretation of non-cooperative games, as explained above, It is assumed that rational players are aware of the structure and constantly try to maximize their payoffs by predicting the moves of their opponents[25]. Evolutionarily game theory is motivated entirely different. Here, it is presumed that the players' strategies are biologically encoded and heritable. Individuals have no control over their strategy and need not be aware of the game. They reproduce and are subject to the forces of natural selection. These mutations create alternative options or strategies. Thereby, in the classical game theory a strategy is chosen for its payoff in that strategy. In evolutionary game theory the payoff is equivalent to the fitness function, where the payoff is measured as the number of offspring. Evolutionary game theory is a replication of a strategy so it can be carried out to the next generation [16].

A key concept in evolutionary game theory is the *Evolutionary Stable Strategies (ESS)*. An evolutionarily stable strategy is a strategy which, if adopted by a population in a given environment, cannot be invaded by any alternative or mutant strategy. It was originally developed by Maynard Smith and Price in 1973 [11]. The main conclusion is that such a strategy is robust to evolutionary mutation pressures, which means that a small mutation does not change the distribution of the population. The idea is as follows. Suppose you repeatedly draw individuals (player 1 and 2) from a large population to play a symmetric two person game. Further, assume that these individuals are 'programmed' or 'genetically encoded' to play a certain pure or mixed strategy. This can be seen as the genetic structure of an individual. The individuals are not in control of their strategy, since they inherited it from their parents. Evolutionary game theory is interested in the effects that occur when you inject a small population of individuals who are programmed to play a different pure or mixed strategy. They are interested if this mutation disrupts the stability of the current state of the distribution among the population. An evolutionary stable strategy occurs if the existing population has some barrier against the mutation, such that the payoff of the mutant strategy falls below the payoff of the current distribution [25].

We assume a large population (effectively infinite) so that we can reasonably model the proportions of the two strategies. If the population were sufficiently small, we would, for example, need to worry about the random extinction of strategies at low frequency. Weibull (1995) [25] argues in his book that for a positive invasion barrier to be effective, such a barrier should exceed $\frac{1}{n}$, where n is the size of the population. Thereby, this way the effect of current individuals' actions on others' future actions can be neglected. Neil (2004) [13] dedicated his research to explain why a large (infinite) population is necessary. He shows that a large population results in a better test of evolutionary stability.

Due to the Darwinian link it is an important subject of study within biological sciences. The ESS is used for predicting outcomes of long-term phenotypic evolution when fitness depends on the frequencies of the various phenotypes present in a population. The largest advantage of this strategy is that it can be resolved with only information about phenotypic aspects, so without the often unknown genetic details. Thereby, using ESS more complex ecological interactions and adaptations can be explored [3][19].

Despite the biological importance, economists use ESS as well. In such a social or economic environment, evolutionary stability requires that any small group of individuals who try a different strategy are less well off then those who stick to the status quo strategy. This way, the large populations have no incentive to change strategy. An ESS that occurs in a social or economical environment may be thought of as a *convention* [25]. An example of using evolutionary game theory in economics is presented in chapter 4.

In spite of the great use of ESS, it has a drawback. It is not certain that, during the course of evolution, the ESS will be reached. One might expect that ESS is similar to a *convergence stability*, which indicates that a process slowly approaches its point of stability over time. Nevertheless, evolutionary stable strategies and convergence stability are two independent stability concepts. Evolutionary stable strategies renders a population against invasion by any new mutant. Whereas, convergence stability is reached through small evolutionary steps. This has been acknowledged and researched by several scientists like Eshel, Taylor and Geritz et al. [3][21]. They concluded that a phenotype that is convergent stable can always be invaded by another phenotype. The significance of the ESS as long-term predictor depends on whether or not the phenotype is convergence stable[1]. A way to improve the model is to correct for preference in selection. This is further explained in chapter 3 about replicator dynamics.

This chapter will start by expanding the knowledge about symmetric two-player games. Then evolutionary stability is defined and some useful theories are provided. We end with a set of examples to get a better understanding of evolutionary stable strategies.

[1] A more detailed view on this matter can be found in Geritz et al. (1998) [3]

2.1 Evolutionary Stability

This section provides the definition of a evolutionary stable strategy and some useful theories. Let A be an $m \times n$-matrix, and let Δ^m denote the incumbent set of mixed strategies for player 1 or player 2. The following definition is from the book of Peters (2008), but is originally based on the definition given by Maynard Smith and Price (1973) [15][11].

Definition 2.1.1 (Evolutionary Stable Strategy[15]). *A strategy $\boldsymbol{x} \in \Delta^m$ is an evolutionary stable strategy (ESS) in A if for every strategy $\boldsymbol{y} \in \Delta^m$, $\boldsymbol{y} \neq \boldsymbol{x}$, there exists some $\boldsymbol{\epsilon_y} \in (0,1)$ such that for all $\boldsymbol{\epsilon} \in (0, \boldsymbol{\epsilon_y})$ we have*

$$\mathbf{x}A(\boldsymbol{\epsilon}\mathbf{y} + (1-\boldsymbol{\epsilon})\mathbf{x}) > \mathbf{y}A(\boldsymbol{\epsilon}\mathbf{y} + (1-\boldsymbol{\epsilon})\mathbf{x}) \tag{2.1}$$

This indicates that when considering a small *mutation* $\boldsymbol{\epsilon}\mathbf{y} + (1-\boldsymbol{\epsilon})\mathbf{x}$ of $\mathbf{x}$, the original strategy $\mathbf{x}$ is better than the mutant strategy $\mathbf{y}$ if 2.1 holds. Thereby, if the population $\mathbf{x}$ is invaded by a small part of the mutant population $\mathbf{y}$, it will survive since it fares better against the mutation than $\mathbf{y}$ itself does. The following proposition shows that an ESS results in a symmetric Nash equilibrium.

Theorem 2.1.1 ([15]). *Let A be an $m \times m$-matrix and let $\mathbf{x} \in \Delta^m$ be an ESS in A (so equation 2.1 holds). Then $(\mathbf{x}, \mathbf{x})$ is a Nash equilibrium in $G = (A, A^T)$.*

Proof. Let $\mathbf{y} \in \Delta^m$, then we need to show that $\mathbf{x}A\mathbf{x} \geq \mathbf{y}A\mathbf{x}$. Take $\boldsymbol{\epsilon_y} \in (0,1)$ such that for all $\boldsymbol{\epsilon} \in (0, \boldsymbol{\epsilon_y})$ we have

$$\mathbf{x}A(\boldsymbol{\epsilon}\mathbf{y} + (1-\boldsymbol{\epsilon})\mathbf{x}) > \mathbf{y}A(\boldsymbol{\epsilon}\mathbf{y} + (1-\boldsymbol{\epsilon})\mathbf{x})$$

for all $0 < \boldsymbol{\epsilon} < \boldsymbol{\epsilon_y}$ as in theorem 2.1.1. By taking $\lim_{\epsilon \to 0}$, we see that $\mathbf{x}A\mathbf{x} \geq \mathbf{y}A\mathbf{x}$. $\square$

This theorem shows that evolutionary stable strategies result in symmetric Nash equilibria. It is then sufficient to restrict our attention to symmetric Nash equilibria. It is easily verified that every ESS is optimal against itself. If a strategy $\mathbf{x}$ is not optimal against itself, then there exists some other strategy $\mathbf{y}$ that obtains a higher payoff against $\mathbf{x}$ than $\mathbf{x}$ does. Hence, if the population share of such a mutant strategy $\mathbf{y}$ is small enough, then, by continuity of 2.1 it will earn more against the population mixture than the incumbent strategy $\mathbf{x}$ will, and thus $\mathbf{x}$ is not evolutionarily stable [25].

This is not the only condition which should hold for $\mathbf{x}$ to be an ESS. The strategy $\mathbf{x}$ should be a better reply to the mutant strategy $\mathbf{y}$ then $\mathbf{y}$ is to itself. To see this, suppose, on the contrary, that an alternative best reply $\mathbf{y}$ to $\mathbf{x}$ earns at least as much against itself as $\mathbf{x}$ does. Then $\mathbf{y}$ has a payoff at least as much as $\mathbf{x}$, also against the mixture $\boldsymbol{\epsilon}\mathbf{y} + (1-\boldsymbol{\epsilon})\mathbf{x}$ (irrespective of $\boldsymbol{\epsilon}$), so $\mathbf{x}$ is not evolutionarily stable. The converse of this also holds: if $(\mathbf{x}, \mathbf{x})$ is a symmetric Nash equilibrium and every alternative best reply $\mathbf{y}$ earns less against itself than $\mathbf{x}$ earns against it, then such mutants do worse than $\mathbf{x}$ in the post-entry population [25]. These insights are stated in the next theorem.

Theorem 2.1.2. *Let A be an $m \times m$-matrix. If $\mathbf{x} \in \Delta^m$ is an ESS in A, then for all $\mathbf{y} \in \Delta^m$ with $\mathbf{y} \neq \mathbf{x}$ we have:*

$$\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x} \longrightarrow \mathbf{x}A\mathbf{y} > \mathbf{y}A\mathbf{y} \tag{2.2}$$

Conversely, if $(\mathbf{x}, \mathbf{x}) \in \Delta^m \times \Delta^m$ is a Nash equilibrium in $G = (A, A^T)$ and 2.2 holds, then $\mathbf{x}$ is an ESS.

Proof. The first part can be proved using contradiction. Let $\mathbf{x} \in \Delta^m$ be an ESS in A. Let $\mathbf{y} \in \Delta^m$ with $\mathbf{y} \neq \mathbf{x}$ and $\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x}$. Suppose that $\mathbf{x}A\mathbf{y} \leq \mathbf{y}A\mathbf{y}$. Then, for any $\epsilon \in [0, 1]$, $\mathbf{x}A(\epsilon\mathbf{y} + (1-\epsilon)\mathbf{x}) \leq \mathbf{y}A(\epsilon\mathbf{y} + (1-\epsilon)\mathbf{x})$, which contradicts 2.1, since it cannot be an ESS.

Conversely, let $(\mathbf{x}, \mathbf{x}) \in \Delta^m \times \Delta^m$ be a Nash equilibrium in $G = (A, A^T)$ and let 2.2 hold for $\mathbf{x}$. If $\mathbf{x}A\mathbf{x} > \mathbf{y}A\mathbf{x}$, then also $\mathbf{x}A(\epsilon\mathbf{y} + (1-\epsilon)\mathbf{x}) > \mathbf{y}A(\epsilon\mathbf{y} + (1-\epsilon)\mathbf{x})$ for small enough $\epsilon \in (0, 1]$. If $\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x}$, then $\mathbf{x}A\mathbf{y} > \mathbf{y}A\mathbf{y}$, hence 2.1 holds for any $\epsilon \in (0, 1]$. □

With these two theorems we conclude that evolutionary stable strategies $\mathbf{x}$ occur in symmetric Nash equilibria, and perform strictly better against any alternative best reply $\mathbf{y}$ than that alternative best reply performs against itself. Combining the two theorems above we can denote the set fo all ESS by $ESS(A)$, which is characterized as follows:

Definition 2.1.2 ([15]). *Let A be a symmetric $m \times m$ game. Then*

$$ESS(A) = \{\mathbf{x} \in NE(A) | \forall \mathbf{y} \in \Delta^m, \mathbf{y} \neq \mathbf{x}\ [\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x} \longrightarrow \mathbf{x}A\mathbf{y} > \mathbf{y}A\mathbf{y}]\}$$

2.2 Symmetric Two-Player Games

Evolutionary game theory is often concerned with symmetric two-player games. We are especially interested in symmetric Nash equilibria. We denote the set of Nash equilibria of (A, A^T) by $NE(A, A^T)$ and the set of all symmetric Nash equilibrium that occur in (A, A^T) by

$$NE(A) = \{\mathbf{x} \in \Delta^m | (\mathbf{x}, \mathbf{x}) \in NE(A, A^T)\}$$

We can prove that this set is non-empty.

Lemma 2.2.1 ([15]). *For any $m \times m$-matrix A, $NE(A) \neq \emptyset$. In other words, every symmetric $m \times m$-matrix game has a symmetric Nash equilibrium.*

Proof. Some definitions in this proof have not been addressed earlier in the thesis. They are mentioned to complete the proof. For more information see Peters' book [15]
Let $\mathbf{x} \in \Delta^m$, viewed as a strategy of player 2 in (A, A^T), also referred to as $(q, 1-q)$ for $0 \leq q \leq 1$. Let $\beta_1(\mathbf{x})$ be the set of best replies of player 1 in (A, A^T), called $(p, 1-p)$. For a set of best replies holds that for some $0 \leq a \leq 1$ the set looks like:

$$\beta_1(\mathbf{x}) = \begin{cases} \{(1,0)\}(\text{ or } \{(0,1)\}) & \text{if } 0 \leq q < a \\ \{(p, 1-p) | 0 \leq p \leq 1\} & \text{if } q = a \\ \{(0,1)\}(\text{ or } \{(1,0)\}) & \text{if } a < q \leq 1 \end{cases}$$

(For methods to derive such a function see the 'Hawk-Dove' example in section 2.3). This set is an example of a *upper semi-continuous set*. Thereby, the function can take on every value for any $0 \leq p, q \leq 1$ within the set, which indicates that the set is *convex*[2]. So, the correspondence $\mathbf{x} \mapsto \beta_1(\mathbf{x})$ is upper semi-continuous and convex valued, then by the Kakutani Fixed Point Theorem[3] we know that $\beta_1(\mathbf{x})$ has a fixed point we call $\mathbf{x}^*$. So, $\mathbf{x}^* \in \Delta^m$ with $\mathbf{x}^* \in \beta_1(\mathbf{x}^*)$. Since player 2's payoff matrix is the transpose of A, it follows that also $\mathbf{x}^* \in \beta_2(\mathbf{x}^*)$. Thus, $(\mathbf{x}^*, \mathbf{x}^*) \in NE(A, A^T)$, and therefore $\mathbf{x}^* \in NE(A)$. □

Weibull (1995) [25] proves in his book (section 1.3.3) that payoff differences between any two strategies for a player, given other players' strategies, are invariant under local shifts of payoff functions. Since Nash equilibria are defined in terms of such payoff differences, the set $NE(A)$ is invariant under transformations which are shown below. Evolutionary stability is defined in terms of such individual payoff differences, and therefore is the set $NE(A)$ invariant under local payoff shifts. Thus, for the following payoff matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{2.3}$$

we may consider (without loss of generality) the following:

$$A' = \begin{pmatrix} a_{11} - a_{21} & a_{12} - a_{12} \\ a_{21} - a_{21} & a_{22} - a_{12} \end{pmatrix} = \begin{pmatrix} a_{11} - a_{21} & 0 \\ 0 & a_{22} - a_{12} \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

where $a_1 := a_{11} - a_{21}$ and $a_2 := a_{22} - a_{12}$ [15]. For a generic matrix A[4], with $a_1, a_2 \neq 0$, there are essentially three different cases.

In the first case, a_1 and a_2 are of opposite sign. This is a variety of the Prisoners' Dilemma. If, for example, $a_1 = 1$ and $a_2 = -1$ then A' becomes;

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Since we are discussing symmetric games the payoff matrix can be viewed as (A, A^T):

		Player 2	
		Option 1	Option 2
Player 1	Option 1	(1,1)	(0,0)
	Option 2	(0,0)	(-1,-1)

[2] A set S is called *convex* if whenever $a, b \in S$ the segment $[a, b]$ is also contained in S (for example, discs and rectangles are convex) [7].

[3] *Kakutani Fixed Point Theorem:* Let $Z \subseteq \mathbb{R}^n$ be a non-empty compact and convex set and let $F : Z \mapsto Z$ be an upper semi-continuous and convex valued correspondence. Then F has a fixed point[15] *The proof of this theorem goes beyond the complexity of this thesis, a proof of this theorem can be found in [5]*

[4] The main characteristic of a generic matrix A is that it has non-zero and distinct eigenvalues.

By the same reasoning as explained in chapter 1, we see that both players have a dominant strategy for the option 1. The symmetric Nash equilibrium is for $\mathbf{x} = ((1,0),(1,0))$ or $(\mathbf{e}^1, \mathbf{e}^1)$. Such games posses exactly one evolutionary stable strategy; $NE(A) = \{\mathbf{e}^1\}$ if $a_1 > 0, a_2 < 0$ and $NE(A) = \{\mathbf{e}^2\}$ if $a_1 < 0, a_2 > 0$ [25] These equilibria are evolutionary stable since they are pure.

In the second case, a_1 and a_2 are both positive. In this case, we have three symmetric Nash equilibria. If for example $a_1 = a_2 = 1$ then

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The game has two symmetric equilibria in pure strategies, namely $((1,0),(1,0))$ and $((0,1),(0,1))$. Thereby, an equilibrium occurs in mixed strategies. Since both parties are indifferent between the two options an extra symmetric Nash equilibrium is $((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$. In general, $NE(A) = \{\mathbf{e}^1, \mathbf{e}^2, \hat{\mathbf{x}}\}$, where $\hat{\mathbf{x}} = (\frac{a_2}{(a_1+a_2)}, \frac{a_1}{(a_1+a_2)})$. The two equilibria $\mathbf{e}^1$ and $\mathbf{e}^2$ are pure and thus evolutionary stable. However, $\hat{\mathbf{x}}$ is not evolutionary stable since all $\mathbf{y} \in \Delta^2$ are best replies to $\hat{\mathbf{x}}$ [25].

The third and final case is when a_1 and a_2 are both negative. We have a variety of the Hawk-Dove problem, explained later this chapter. Such a game has two pure asymmetric equilibria and one symmetric equilibrium. If, for example, $a_1 = a_2 = -1$ then A' becomes;

$$A' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

The two asymmetric equilibria are $((1,0),(0,1))$ and $((0,1),(1,0))$. Again, the players are indifferent between the two strategies, therefore the only symmetric equilibria is $((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$. So, $NE(A) = \{\hat{\mathbf{x}}\}$ with $\hat{\mathbf{x}} = (\frac{a_2}{(a_1+a_2)}, \frac{a_1}{(a_1+a_2)})$. Only this time, $\hat{\mathbf{x}}$ is evolutionary stable, because there are no pure strategies which can intrude the stability. [25].

2.3 Examples

2.3.1 The Hawk-Dove Game

One of the most well known examples in evolutionary game theory is the *Hawk-Dove game* [14][15][17][19]. The game is about individuals of the same large populations who meet at random, in pairs, and behave either aggressively (Hawk) or passively (Dove). Their behavior is genetically determined, so they are not able to choose between the two modes of behavior. The payoffs shown below reflect the Darwinian fitness, in other words, the number of offspring.
It is immediately seen that it is a symmetric matrix game $G = (A, A^T)$ with:

$$A = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$$

		Player 2	
		Hawk	Dove
Player 1	Hawk	(0,0)	(3,1)
	Dove	(1,3)	(2,2)

A mixed strategy $\mathbf{p} = (p_1, p_2)$ is interpreted as expressing the population shares of individuals characterized by the same type of behavior. So, $p_1 \times 100\%$ of the populaion are Hawks and $p_2 \times 100\%$ op the population are Doves. We would like to find a symmetric Nash equilibrium, since at this point the players have the same strategy. We observe that this game has two Nash equilibria in pure strategies, namely (Hawk, Dove) and (Dove, Hawk). To find all Nash equilibria we determine the best replies of both players. We first consider the strategy $(q, 1-q)$ of player 2. If the expected payoff from playing Hawk is higher than the expected payoff from playing Dove, the best reply for player 1 is playing Hawk so $(q, 1-q) = (1, 0)$. The best reply for player 1 is Hawk if:

$$0q + 3(1-q) > 1q + 2(1-q)$$

which holds for $q < \frac{1}{2}$. Therefore, Dove is the best reply if $q > \frac{1}{2}$, and player 1 is indifferent between Hawk and Dove for $q = \frac{1}{2}$. Summarizing, if we call the set of best replies for player 1 $\beta_1(q, 1-q)$ we have:

$$\beta_1(q, 1-q) = \begin{cases} \{(1,0)\} & \text{if } 0 \leq q < \frac{1}{2} \\ \{(p, 1-p) | 0 \leq p \leq 1\} & \text{if } q = \frac{1}{2} \\ \{(0,1)\} & \text{if } \frac{1}{2} < q \leq 1 \end{cases}$$

Since the game is symmetric we find, by analogous reasoning, that the playing Hawk is best strategy for player 2 if $p < \frac{1}{2}$. Therefore, if we call the set of best replies for player 2 $\beta_2(p, 1-p)$ we also have:

$$\beta_2(p, 1-p) = \begin{cases} \{(1,0)\} & \text{if } 0 \leq p < \frac{1}{2} \\ \{(q, 1-q) | 0 \leq q \leq 1\} & \text{if } p = \frac{1}{2} \\ \{(0,1)\} & \text{if } \frac{1}{2} < p \leq 1 \end{cases}$$

We can conclude that there are three Nash equilibria: $((1,0),(0,1))$, $((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$, and $((0,1),(1,0))$. Obviously, only $((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$ is a symmetric Nash equilibrium. To make sure it is an ESS, equation 2.2 must hold. So consider again $\mathbf{x} = ((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2}))$. Let $\mathbf{y} = (y, 1-y)$ be an arbitrary strategy, then $\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x}$ is always satisfied since $(\mathbf{x}, \mathbf{x})$ is a Nash equilibrium[5]. We check whether $\mathbf{x}A\mathbf{y} > \mathbf{y}A\mathbf{y}$ holds for all $\mathbf{y} = (y, 1-y) \neq \mathbf{x}$.

$$\begin{gathered}
\mathbf{x}A\mathbf{y} > \mathbf{y}A\mathbf{y} \\
\sum_{i=1}^{2}\sum_{j=1}^{2} x_i y_j a_{ij} > \sum_{i=1}^{2}\sum_{j=1}^{2} y_i y_j a_{ij}{}^{6} \\
x_1y_1a_{11} + x_1y_2a_{12} + x_2y_1a_{21} + x_2y_2a_{22} > y_1y_1a_{11} + y_1y_2a_{12} + y_2y_1a_{21} + y_2y_2a_{22} \\
\frac{1}{2}(1-y)\cdot 3 + \frac{1}{2}y\cdot 1 + \frac{1}{2}(1-y)\cdot 2 > y(1-y)\cdot 3 + (1-y)y\cdot 1 + (1-y)^2\cdot 2 \\
4y^2 - 4y + 1 > 0
\end{gathered}$$

Which holds for all $y \neq \frac{1}{2}$. We can now conclude that $\mathbf{x} = (\frac{1}{2}, \frac{1}{2})$ the unique evolutionary stable strategy is in A.

2.3.2 Rock-Paper-Scissors

We can extend the theory of evolutionary stable strategies to a symmetric 3×3-matrix game. A well-known childrens game is *rock-paper-scissors*, shown in the following symmetric matrixgame:

		Player 2		
		Rock	Paper	Scissors
	Rock	(1,1)	(2,0)	(0,2)
Player 1	Paper	(0,2)	(1,1)	(2,0)
	Scissors	(2,0)	(0,2)	(1,1)

The matrix B obtained from the payoff matrix is

$$B = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

The rules are simple. Rock (strategy 1) beats scissors, scissors (strategy 2) beats paper, and paper (strategy 3) beats rock. Intuitively we see that this game has no Nash equilibrium in pure strategies. Using the methods from the previous example it is easily shown that there is exactly one Nash equilibrium in mixed strategies. Namely when both players randomize their choice, so for $(\mathbf{x}, \mathbf{x})$ with $\mathbf{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Obviously this is a symmetric Nash equilibrium, but it can be shown that it is not an ESS. The strategy $(\mathbf{x},\mathbf{x})$ is be an ESS if 2.2 holds. If we use the mutant strategy $\mathbf{y} = \mathbf{e}^1$ we see that

[5] for $\mathbf{y} = (y, 1-y)$ we see $\mathbf{x}A\mathbf{x} = \frac{3}{2}$, but also $\mathbf{y}A\mathbf{x} = \frac{3}{2}$. So $\mathbf{y}A\mathbf{x}$ is independent of $\mathbf{y}$ for $(\mathbf{x}, \mathbf{x})$ a Nash equilibrium.

[6] See equation 1.1

$$\mathbf{x}A\mathbf{x} = \mathbf{y}A\mathbf{x}$$
$$\sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j a_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} y_i x_j a_{ij}$$
$$1 = 1$$

which is in line with the assumption of theorem 2.1.2, but then $\mathbf{x}A\mathbf{y} \not> \mathbf{y}A\mathbf{y}$ can never hold, since both sides are equal to 1. From theorem 2.1.1 we know that every ESS is a symmetric Nash equilibrium. This example shows that the converse is not true. A symmetric Nash equilibrium is not necessarily an ESS.

Chapter 3

Replicator Dynamics and Evolutionary Stability

Central in evolutionary theories are the concepts of *mutation* and *selection*. We have discussed the concept of mutation in the chapter about evolutionary stable strategies. As mentioned before, this theory is not complete. To capture the concept of selection and thereby search for convergence stability, we introduce replicator dynamics [15]. Basically, the replicator dynamic is a dynamic which describes how the frequencies of strategies within a population change in time, according to the strategies' success [6]. The replicator dynamics is a system of ordinary differential equations, which do not cover the mutation mechanism at all. Robustness against mutations is indirectly taken care of by the criteria of dynamic stability[25]. Dynamic stability is achieved when small shifts in the shares of the population always move back towards the equilibrium, this indicates that mutations are taken care of by natural selection.

We assumed with evolutionary stability that individuals were 'programmed' for pure and mixed strategy. The basic replicator dynamics assumes that each individuals can only be programmed to pure strategies. So, we interpreted a mixed strategy $\mathbf{x}$ as a population state, where each component $\mathbf{x}_i$ represents the population's share of individuals who are programmed to the corresponding pure strategy i. The interpretation of the payoff remains the same, random pairwise matchings in a large population creates the level of fitness measured by the number of offspring. Each offspring inherits a parent's strategy[25].

In 1978 Taylor and Jonker [21] published their article *'Evolutionary Stable Strategies and Game Dynamics'*, where they introduced the replicator dynamics. They assumed that the population develops according to an ordinary differential equation which will be presented in this chapter. The replicators are the pure strategies, which can be copied without error from parent to child, with the individuals in the population being their hosts. When the population state changes, the payoff to the pure strategies also changes from fitness[25].

In the analysis of the next chapter we give few extra definitions. Let $I = \{1, 2, ..., n\}$ be a set consisting of all players, with $n \in \mathbb{N}$. Let S_i be the set of pure strategies for player i. The set of pure strategies of player i is denoted by $S_i = \{1, 2, ..., m_i\}$, with $m_i \geq 2$. The set of pure strategies that is assigned positive probabilities by some mixed strategy $\mathbf{x}_i$ is called the *support* of $\mathbf{x}_i$, and is denoted by $C(\mathbf{x}) = \{h \in S_i | x_{ih} > 0\}$ The set $C(\mathbf{x})$ denotes the set of all interior points. The mixed strategies in this set are called *interior* or *completely mixed*. The strategies have positive probability for all the pure strategies and therefore have full support. $C(\mathbf{x}) = S$ for all $i \in I$ [25].

3.1 Replicator Dynamics

Consider again a large but finite population of individuals, and consider a symmetric $m \times m$-matrix game A. We can interpreted a mixed strategy $\mathbf{x} \in \Delta^m$ as a vector of population shares over pure strategies, evolving over time. For example in the Hawk-Dove game, $\mathbf{x} = (\frac{1}{2}, \frac{1}{2})$ indicates that half of the population has the pure strategy 'Hawk', and half of the population has the pure strategy 'Dove'. These population shares change over time. So, $\mathbf{x}$ is dependent on time, we write $\mathbf{x} = \mathbf{x}(t)$ [15]. The expected payoff for playing the pure strategy i equals $\mathbf{e}^i A\mathbf{x}$, therefore the average payoff equals

$$\sum_{i=1}^{m} x_i \mathbf{e}^i A\mathbf{x} = \mathbf{x}A\mathbf{x}. \tag{3.1}$$

As mentioned before, Taylor and Jonker [21] assumed in their article that population shares develop according to a differential equation. This equation is defined as:

$$\dot{x}_i = \frac{dx_i(t)}{dt} = [\mathbf{e}^i A\mathbf{x} - \mathbf{x}A\mathbf{x}]x_i \tag{3.2}$$

for $i = 1, 2, ..., m$ pure strategies. This equation is called the *replicator dynamics*. The share of the population playing the pure strategy i changes with a rate proportional to the difference between the expected payoff of i and the average population payoff[15].

To analyse replicator dynamics and therefore equation 3.2 we use theory off differential equations and dynamical systems. We will restrict our attention to a few basic concepts.

For each *initial state* $\mathbf{x}(0) = \mathbf{x}^0 \in \Delta^m$, equation 3.2 induces a solution $\boldsymbol{\xi}(t, \mathbf{x}^0) \in \Delta^m$. We call $\mathbf{x}$ a *stationary point* of the dynamics in 3.2 if $\dot{\mathbf{x}} = (\dot{x}_1, ..., \dot{x}_m) = (0, ..., 0)$, in other words $\sum_{i=1}^{m} \dot{x}_i = 0$. For example, $\mathbf{e}^i$ is a stationary point for all i [15].

Definition 3.1.1 ([15]). *A state $\mathbf{x}$ is Lyapunov stable if every open neighborhood B of $\mathbf{x}$ contains an open neighborhood B^0 of $\mathbf{x}$ such that $\boldsymbol{\xi}(t, \mathbf{x}^0) \in B$ for all $\mathbf{x}^0 \in B^0$ and $t \geq 0$.*

Definition 3.1.2 ([15]). *A state $\mathbf{x}$ is asymptotically stable if it is Lyapunov stable and it has an open neighborhood B^* such that $\lim_{t\to\infty} \boldsymbol{\xi}(t, \mathbf{x}^0) = \mathbf{x}$ for all $\mathbf{x}^0 \in B^*$.*

This implies that if a state $\mathbf{x}$ is Lyapunov stable, then it is a stationary point.

3.2 Symmetric Two-Player Games

We again consider two-player games, but this time to analyse the replicator dynamics for symmetric 2×2 games corresponding to A. Without loss of generality we can restrict our attention to the normalized game

$$A' = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \tag{3.3}$$

Recall that $a_1 = a_{11} - a_{21}$ and $a_2 = a_{22} - a_{12}$. Using $\mathbf{x} = (x_1, x_2)$ with $0 \leq x_1, x_2 \leq 1$ and $x_1 + x_2 = 1$, the replicator dynamics of equation 3.2 is can be reduced to

$$\begin{aligned}
\dot{x}_1 &= [\mathbf{e}^1 A\mathbf{x} - \mathbf{x}A\mathbf{x}]x_1 \\
&= [x_1 a_1 - \sum_{i=1}^{2}\sum_{j=1}^{2} x_i x_j a_{ij}]x_1 \\
&= [x_1 a_1 - (x_1^2 a_1 + x_2^2 a_2)]x_1 \\
&= x_1^2 a_1 - x_1^3 a_1 - x_1 x_2^2 a_2 \\
&= x_1^2 a_1 (1 - x_1) - x_1 x_2^2 a_2 \\
&= x_1^2 x_2 a_1 - x_1 x_2^2 a_2 \\
&= [a_1 x_1 - a_2 x_2]x_1 x_2
\end{aligned}$$

Since $x_1 + x_2 = 1$, it follows that $\dot{x}_1 + \dot{x}_2 = 0 \Longrightarrow \dot{x}_2 = -\dot{x}_1$. We consider the same three cases as before:

In the first case, a_1 and a_2 are of opposite sign. If, for example $a_1 = -1$ and $a_2 = 1$, then wee see that;

$$\begin{aligned}
\dot{x}_1 &= [x_1 a_1 - x_2 a_2]x_1 x_2 \\
&= (-x_1 - x_2)x_1 x_2 \\
&= -x_1 x_2
\end{aligned}$$

This shows that the population always declines. We conclude that the population either always decreases (for $a_1 < 0$ and $a_2 > 0$) or always increases (for $a_1 > 0$ and $a_2 < 0$). Therefore, the stationary points of the dynamics are $\mathbf{x} = \mathbf{e}^1$ and $\mathbf{x} = \mathbf{e}^2$. Starting from any interior initial position, the population share converges to the unique ESS [15][25].

In the second case both payoffs a_1 and a_2 are positive. The sign of the growth rate x_1 changes when $a_1 x_1 = a_2 x_2$, so when $x_1 = \frac{a_2}{(a_1+a_2)}$. If both payoffs are positive, x_1 tend towards 0 from any point below the 'switch point', and x_1 increases to 1 for any point above this point. These results show that from any initial interior position, the population share converges to one of the two ESS's [25], and that $x_1 = \frac{a_2}{(a_1+a_2)}$ is a saddle-point.

For the final case, both payoffs a_1 and a_2 are negative. Since we know that $\hat{\mathbf{x}} = (\frac{a_2}{(a_1+a_2)}, \frac{a_1}{(a_1+a_2)})$ is an ESS, we see that x_1 increases towards $\frac{a_2}{(a_1+a_2)}$ from any lower initial value and decreases towards this fraction from any higher initial value. So, the game has one unique ESS and the game always converges to this point, independent of the interior initial position [25].

This information leads to the following theorem.

Theorem 3.2.1. *Let A be a generic 2×2 matrix and let $\mathbf{x} \in \Delta^2$. Then $\mathbf{x} \in ESS(A)$ if and only if $\mathbf{x}$ is an asymptotically stable state of the replicator dynamics.*

3.3 Nash Equilibrium Strategies

This information brings up questions about the linkage between replicator dynamics and Nash equilibirum strategies. The next theorems answers these questions. We consider the 2×2 payoff matrix A mentioned in the previous section. Thereby, the set denoted by Δ_0^m is the interior of the set Δ^m. In other words, $\Delta_0^m = \{\mathbf{x} \in \Delta^m | \mathbf{x} > 0\}$ is the set of completely mixed strategies [15]. In the next theorem the set *ST(A)* consists of all the stationary states, so when $\dot{x}_i = 0$:

$$ST(A) = \{\mathbf{x} \in \Delta^m | \forall i \in C(\mathbf{x}) \, [\mathbf{e}^i A\mathbf{x} = \mathbf{x}A\mathbf{x}]\} \tag{3.4}$$

Theorem 3.3.1 **([15]).** *For any finite symmetric two-player game with payoff matrix A we have:*

1. $\{\mathbf{e}^1, ..., \mathbf{e}^m\} \cup NE(A) \subseteq ST(A)$
2. $ST(A) \cap \Delta_0^m = NE(A) \cap \Delta_0^m$
3. $ST(A) \cap \Delta_0^m$ *is a convex set and if* $\mathbf{z} \in \Delta^m$ *is a linear combination of states in this set, then* $\mathbf{z} \in NE(A)$

Proof. (1) From theorem 2.1.1 we see for every pure strategy i that $\mathbf{e}^i A\mathbf{x} = \mathbf{x}A\mathbf{x}$ and thus $\mathbf{e}^i \in ST(A)$. If we take $\mathbf{x} \in NE(A)$, then every $i \in C(\mathbf{x})$ is a pure best reply, which indicates that $\mathbf{e}^i A\mathbf{x} = \mathbf{x}A\mathbf{x}$. For $i \notin C(\mathbf{x})$ we see $x_i = 0$. So we conclude $\mathbf{x} \in ST(A)$. This proves the first statement

(2) We know for $\mathbf{x} \in ST(A)$, and thus for $\mathbf{x} \in ST(A) \cap \Delta_0^m$, that $\mathbf{e}^i A\mathbf{x} = \mathbf{x}A\mathbf{x}$. Since the same holds for $\mathbf{x} \in NE(A) \cap \Delta_0^m$ we can conclude the second statement.

(3) Let $\mathbf{x}$ and $\mathbf{y}$ be two completely mixed and stationary points. In other words, $\mathbf{x}, \mathbf{y} \in ST(A) \cap \Delta_0^m$. Let $\alpha, \beta \in \mathbb{R}$ and let $\mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y} \in \Delta^m$. Then for any pure strategy i we have

$$\mathbf{e}^i A\mathbf{z} = \alpha\mathbf{e}^i A\mathbf{x} + \beta\mathbf{e}^i A\mathbf{y} = \alpha\mathbf{x}A\mathbf{x} + \beta\mathbf{y}A\mathbf{y} = \mathbf{z}A\mathbf{z}$$

This implies that $\mathbf{z}$ is stationary. If $\mathbf{z}$ is completely mixed, statement (2) shows that if $\mathbf{z} \in ST(A) \cap \Delta_0^m$ then also $\mathbf{z} \in NE(A) \cap \Delta_0^m$. If $\mathbf{z}$ is a boundary point of $ST(A) \cap \Delta_0^m$, then it is also a boundary point of $NE(A) \cap \Delta_0^m$. In this case, $\mathbf{z} \in NE(A)$ since $NE(A)$ is a closed set. Since Δ^m is convex and $\mathbf{z} \in ST(A) \cap \Delta_0^m$ for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, we can also conclude that $ST(A) \cap \Delta_0^m$ is a convex set[15]. □

The main result of the theorem above is that every symmetric Nash equilibrium is stationary. Using Lyapunov stability we can refine the theorem about Nash equilibria.

Theorem 3.3.2. *Let* $\mathbf{x} \in \Delta^m$ *be a Lyapunov stable stationary state. Then* $\mathbf{x} \in NE(A)$

Proof. We prove this by contradiction. Suppose $\mathbf{x} \notin NE(A)$. Then the first statement of theorem 3.3.1 shows that $\mathbf{x}$ is not stationary, so $\mathbf{e}^i A\mathbf{x} - \mathbf{x}A\mathbf{x} > 0$ for some $i \notin C(\mathbf{x})$. We may assume continuity, so there is a $\delta > 0$ and an open neighborhood U of $\mathbf{x}$ such that $\mathbf{e}^i A\mathbf{y} - \mathbf{y}A\mathbf{y} \geq \delta$ for all $\mathbf{y} \in U \cap \Delta^m$. But then $\xi_i(t, \mathbf{x}^0) \geq x_i^0 exp(\delta t)$ for all $\mathbf{x}^0 \in U \cap \Delta^m$ and $t \geq 0$ such that $\xi(t, \mathbf{x}^0) \in U \cap \Delta^m$. This is the result of the fact that the solution of the differential equation $\dot{y} = \delta y$ with initial condition $(y0) = y^0$, is $y(t) = y^0 exp(\delta t)$. Now we see that $\xi(t, \mathbf{x}^0)$ increases exponentially from any $\mathbf{x}^0 \in U \cap \Delta_0^m$ with $x_i^0 > 0$ whereas $x_i = 0$. This result contradicts Lyupanov stationary stability.[15]. □

The final subject to consider is asymptotic stability. The theory implies that asymptotic stability implies Lyapunov stability and therefore by theorem 3.3.2 also Nash equilibrium. Then again, a Nash equilibrium implies stationarity by theorem 3.3.1. Therefore, an asymptotic Nash equilibrium is *isolated*, which means that in a small neighborhood around the Nash equilibrium no other equilibrium can be found. The next section considered examples to apply the previous theories [15].

3.4 Examples

3.4.1 The Hawk-Dove Game

The easiest way to illustrate the replicator dynamic is by continuing with the Hawk-Dove example [15]. Recall that matrix A is defined as:

$$A = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$$

Consider a vector of population shares $\mathbf{x} = (x, 1-x)$, and an arbitrary individual within the population. We know from 3.1 that the average fitness equals

$$\begin{aligned}
\sum_{i=1}^{2} x_i \mathbf{e}^i A\mathbf{x} &= \mathbf{x}A\mathbf{x} \\
&= \sum_{i=1}^{2}\sum_{j=1}^{2} x_i x_j a_{ij} \\
&= x_1 x_1 a_{11} + x_1 x_2 a_{12} + x_2 x_1 a_{21} + x_2 x_2 a_{22} \\
&= 2 - 2x^2
\end{aligned}$$

We assume that the share of the population develops over time. So, x is a function of time t. The change in x is described by the time derivative $\dot{x} = \dot{x}(t) = \frac{dx(t)}{dt}$, which is proportional to the difference with the average fitness. So, using equation 3.2 for the pure strategy 'Hawk' ($\mathbf{e}^1$) this equals,

$$\begin{aligned}\dot{x}(t) &= x(t)[\mathbf{e}^1 A\mathbf{x} - \mathbf{x}A\mathbf{x}] \\ &= x(t)[3(1-x(t)) - (2-2x(t)^2)] \\ &= x(t)(x(t)-1)(2x(t)-1)\end{aligned}$$

This equation is the replicator dynamics for the Hawk-Dove game. It indicates that the population of Hawks continuously changes over time and that this change is proportional to the difference of the fitness at time t and the average fitness of the population. Simplifying the equation (and writing x instead of $x(t)$) makes is possible to create a diagram of $\frac{dx}{dt}$ as a function of x, called a phase diagram, shown in figure 3.1[1] [15].

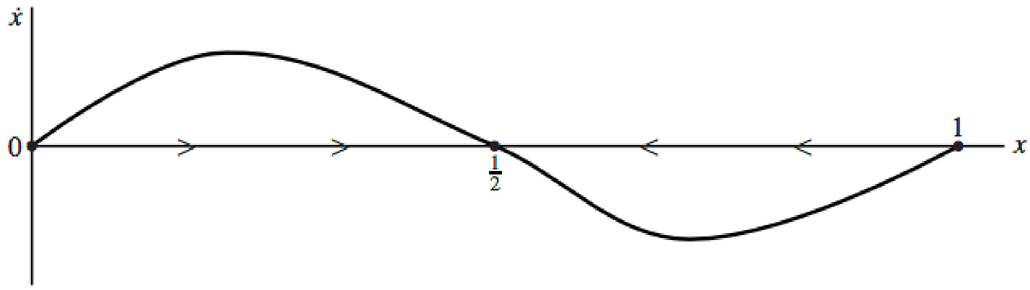

Figure 3.1: Replicator dynamics of the Hawk-Dove game. Source:[15]

In the figure we see that the replicator dynamic has three different roots, also called *rest point*. When $x = 0$, $x = \frac{1}{2}$ or $x = 1$ the derivative $\frac{dx}{dt}$ equals zero, so the population shares do not change at these moments. In the case that $x = 0$ all individuals are Doves. The fitness or payoff equals the average fitness and therefore nothing changes. However, this rest point is not stable. A slight disturbance, like a small mutation, increases the amount of Hawks, seen by the positivity of $\dot{x} = \frac{dx}{dt}$. This increase will continue until the second rest point is reached in $x = \frac{1}{2}$. From the same reasoning follows that $x = 1$, when the population only consists of Hawks, is not a stable rest point. So we consider the rest point $x = \frac{1}{2}$. As shown in the figure, a small change in more Dove of Hawks in the population, results in a movement back to the rest point where half the population is a Dove and half is a Hawk. This shows that this rest point is stable [15].

It is no coincidence that $x = \frac{1}{2}$ is the stable rest point. Recall that $\mathbf{x} = (\frac{1}{2}, \frac{1}{2})$ is the unique evolutionary stable strategy of this game. This result is stated in the next theorem:

Theorem 3.4.1 ([15]). *Let A be a 2×2-matrix. Then:*

1. *A has at least one evolutionary stable strategy*
2. *$\mathbf{x} = (x, 1-x)$ is an evolutionary stable strategy of A, if and only if $\mathbf{x}$ is a stable rest point of the replicator dynamics.*

[1]The figure is copied out of Peters' book *Game Theory*, see his book for more details.

Proof. Theorem 3.2.1 implies the second statement. A full proof can be found in the book of Weibull [25] □

3.4.2 Rock-Paper-Scissors

We consider the Rock-Paper-Scissors example again [25]. This time we analyse the generalized version, which is defined as follows:

		Player 2		
		Rock	Paper	Scissors
	Rock	(1,1)	(2+a,0)	(0,2+a)
Player 1	Paper	(0,2+a)	(1,1)	(2+a,0)
	Scissors	(2+a,0)	(0,2+a)	(1,1)

with $a \in \mathbb{R}$. In the example of chapter 2 we used $a = 0$ (the interior Nash equilibrium $\mathbf{x}^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is independent of a). We define the matrix B as:

$$B = \begin{pmatrix} 1 & 2+a & 0 \\ 0 & 1 & 2+a \\ 2+a & 0 & 1 \end{pmatrix}$$

Using equation 3.2 we see that the replicator dynamics becomes:

$$\begin{aligned} \dot{x}_1 &= [(1 \cdot x_1 + (2+a) \cdot x_2 + 0 \cdot x_3) - \mathbf{x}B\mathbf{x}]x_1 \\ &= [x_1 + (2+a)x_2 - \mathbf{x}B\mathbf{x}]x_1 \end{aligned}$$

$$\begin{aligned} \dot{x}_2 &= [(0 \cdot x_1 + 1 \cdot x_2 + (2+a) \cdot x_3) - \mathbf{x}B\mathbf{x}]x_2 \\ &= [x_2 + (2+a)x_3 - \mathbf{x}B\mathbf{x}]x_2 \end{aligned}$$

$$\begin{aligned} \dot{x}_3 &= [((2+a)\dot{x}_1 + 0 \cdot x_2 + 1 \cdot x_3) - \mathbf{x}B\mathbf{x}]x_3 \\ &= [(2+a)x_1 + x_3 - \mathbf{x}B\mathbf{x}]x_3 \end{aligned}$$

Where chapter 2 illustrates the basic principal, it can be shown that the logarithm of $x_1 \cdot x_2 \cdot x_3$ can increase, decrease or remain constant over time. We define $h(x) = \log(x_1 x_2 x_3) = \log(x_1) + \log(x_2) + \log(x_3)$, where $x_1 + x_2 + x_3 = 1$. If we differentiate $h(x)$ over time we get:

$$\begin{aligned} \dot{h}(x) &= \frac{\dot{x}_1}{x_1} + \frac{\dot{x}_2}{x_2} + \frac{\dot{x}_3}{x_3} \\ &= (x_1 + x_2 + x_3) + (2+a)(x_1 + x_2 + x_3) - 3\mathbf{x}B\mathbf{x} \\ &= 3 + a - 3\mathbf{x}B\mathbf{x} \end{aligned}$$

To calculate $\mathbf{x}B\mathbf{x}$ we first note that:

$$\begin{aligned} 1 &= x_1 + x_2 + x_3 \\ &= (x_1 + x_2 + x_3)^2 \\ &= (x_1^2 + x_2^2 + x_3^2) + 2(x_1x_2 + x_1x_3 + x_2x_3) \\ &= \|x\|^2 + 2(x_1x_2 + x_1x_3 + x_2x_3) \end{aligned}$$

where $\|x\|^2 = (x_1^2 + x_2^2 + x_3^2)$. This also indicates that $\frac{a}{2}(1 - \|x\|^2) = a(x_1x_2 + x_1x_3 + x_2x_3)$ Then follows:

$$\begin{aligned} \mathbf{x}B\mathbf{x} &= \sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j b_{ij} \\ &= x_1^2 b_{11} + x_1x_2b_{12} + x_1x_3b_{13} + x_2x_1b_{21} + x_2^2b_{22} + x_2x_3b_{23} + x_3x_1b_{31} + x_3x_2b_{32} + x_3^2b_{33} \\ &= x_1^2 + x_2^2 + x_3^2 + (2 + a)(x_1x_2 + x_2x_3 + x_1x_3) \\ &= \|x\|^2 + 2(x_1x_2 + x_1x_3 + x_2x_3) + a(x_1x_2 + x_1x_3 + x_2x_3) \\ &= 1 + a(x_1x_2 + x_1x_3 + x_2x_3) \\ &= 1 + \frac{a}{2}(1 - \|x\|^2) \end{aligned}$$

So, we conclude

$$\dot{h}(x) = \frac{a}{2}(3\|x\|^2 - 1) \tag{3.5}$$

Consider a convex unit triangle D. Then $\|x\|^2$ is maximal at each of the three vertices with value 1, and it is minimal at $\mathbf{x}^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. At its minimum x_i has the value $\frac{1}{3}$ for $i = 1, 2, 3$, which gives $\dot{h}(x) = 0$. For all $\mathbf{x} \neq \mathbf{x}^*$ we have $\dot{h}(x) > 0$ on D.

Weibull (2004) [25] shows in his book that if $a = 0$ all interior solutions to the replicator dynamics in D are periodic. If $a < 0$ the paths inside D move outwards, towards hyperbolas for which the value $x_1x_2x_3$ is low. For $a > 0$ the paths inside D move inwards, towards hyperbolas where the value $x_1x_2x_3$ is high. Weibull's main conclusion about the generalized game is that for $a > 0$ the unique Nash equilibrium strategy $\mathbf{x}^*$ is asymptotically stable and is Lyapunov stable. Therefore, in this case $\mathbf{x}^*$ meets the criteria for evolutionary stability by theorem 3.3.2. For $a = 0$ he shows that $\mathbf{x}^*$ is Lyapunov stable but not asymptotically stable. So, it is neutrally stable but not evolutionary stable. Finally, for $a < 0$, $\mathbf{x}^*$ is unstable. For more details and extensive reasoning, see Weibull's book.

Chapter 4

Pricing Electric Vehicles

This final chapter shows an example on how evolutionary game theory is used for economic purposes. It is based on the article *'A Discrete Two-Level Model for Charge Pricing of Electric Vehicles Based on Evolutionary Game Theory'*, written by D. Lian et al. in 2016 [8]. They mention the growing importance of vehicle-to-grid (V2G) technology in the future Smart Grid. The marketization of the V2G system is important, and therefore also the pricing strategies of the demand response management on electric vehicles. Using evolutionary game theory, they optimize the pricing strategy of Power Grid Corp (PGC) for electric vehicles. The next sections show the main mathematical developments of their research and state their main conclusions.

4.1 The Simple Evolution Game, Electric Vehicles Users vs Power Grid Corp

One can imagine that a two level game is not a perfect representation of reality. Individuals do not always act purely rational in situations. Due to this complexity of the actual economic environment, the model is expected to have incomplete information and limited rationality of the participants [8]. Therefore, we start off with a simplified model of the situation. The next section deals with an expanded version.

We introduce the model by providing background information. Firstly, in the future open market environment, Power Grid Corp can use different subsidy strategies to affect the level of participation of the electric vehicle users (EVU).

Secondly, both Power Grid Corp and the customers have two types of strategies. Power Grid Corp can choose to either actively adopt high subsidies to support their electric vehicles and encourage participation in load-shifting, or do not take high subsidy policy to support the electric vehicle market. The customers can choose for high participation and actively participate in load-shifting, or low participation and thus passive participation in load-shifting.

Third, we say that the probability of the strategy 'high subsidy' for the Power Grid Corp is $\alpha = \alpha(t)$. Then, the probability of the strategy 'no high subsidy' is $1 - \alpha =$

$1 - \alpha(t)$. We say that the probability of 'active participation' for consumers of electric vehicles is $\beta = \beta(t)$. Then, the probability of 'passive participation' is $1 - \beta = 1 - \beta(t)$.

Fourth, the payoff matrix in simple electric vehicle evolution game, looks as follows:

		EVU	
		Active participation	Passive Participation
PGC	High Subsidy	$(R_1 + \delta + Y_R - Z_1, R_2 + \delta + B_T - S_{un})$	$(R_1 - Z_1, R_2 + B_T)$
	No High Subsidy	$(R_1 + Y_R, R_2 - S_{un})$	(R_1, R_2)

Here, R_1 and R_2 are the two levels of game revenues when Power Grid Corp does not take the high subsidy and the customers choose passive participation. In this case, Power Grid Corp has less money to spend on improving their electric vehicles. Although both sides of the game could suffer from social and economic loss, there is still the original mechanism of intrinsic benefit. Y_R stands for the capacity gain and extra benefit for Power Grid in case of active participation of the consumers. S_{un} is the beneficial loss, like battery loss, caused by high participation of electric vehicle users. Z_1 is the quantitative benefit loss due to the return of investment. B_T is the additional subsidy income for electric vehicle consumers if the Power Grid Crop chooses the high subsidy strategy. δ is the possible extra income due to positive interaction benefits under a win-win situation.

To find the replicator dynamics, the revenue formulas need to be expressed and the average revenues for both parties must be measured. In this part we stick to the notation of Liang et al. [8]. U_1 is the average revenue for Power Grid corporation and U_2 is the average quantitative revenue for electric vehicle customers.

For the Power Grid Crop we see that

$$U_1 = \alpha U_{11} + (1 - \alpha)U_{12}$$

where U_{11} is the average revenue for Power Grid corporation if they choose to take the high subsidy and U_{12} is the average revenue if they do not take the high subsidy. In other words,

$$\begin{aligned} U_{11} &= \beta(R_1 + \delta + Y_R - Z_1) + (1 - \beta)(R_1 - Z_1) \\ U_{12} &= \beta(R_1 + Y_R) + (1 - \beta)R_1 \end{aligned}$$

The replicator dynamics changes the variation of the probability of the analysis according to the evolutionary direction [8]. Using equation 3.2 the replicator dynamic equation (called $R_f(\alpha)$ for Power Grid corporation) is obtained as follows

$$\begin{aligned} R_f(\alpha) = \frac{d\alpha(t)}{dt} &= (U_{11} - U_1)\alpha \\ &= (U_{11} - \alpha U_{11} - (1 - \alpha)U_{12})\alpha \\ &= \alpha(1 - \alpha)(U_{11} - U_{12}) \\ &= \alpha(1 - \alpha)(\beta(R_1 + \delta + Y_R - Z_1) + (1 - \beta)(R_1 - Z_1) - (\beta(R_1 + Y_R) + (1 - \beta)R_1) \\ &= \alpha(1 - \alpha)(\beta\delta - Z_1) \end{aligned}$$

We proceed by doing the same calculations for electric vehicle consumers, with average quantitative revenue U_2 given by

$$U_2 = \beta U_{21} + (1-\beta)U_{22}$$

where U_{21} is the average quantitative revenue for electric vehicle consumers if they choose to actively participate in consuming, and U_{22} is the average quantitative revenue if they passively participate. The equations are as follows

$$\begin{aligned} U_{21} &= \alpha(R_2 + \delta + B_T - S_{un}) + (1-\alpha)(R_2 - S_{un}) \\ U_{22} &= \alpha(R_2 + B_T) + (1-\alpha)R_2 \end{aligned}$$

Again, using equation 3.2, the replicator dynamic equation (called $R_f(\beta)$ for electric vehicle users) is obtained as follows

$$\begin{aligned} R_f(\beta) = \frac{d\beta(t)}{dt} &= (U_{21} - U_2)\beta \\ &= (U_{21} - \beta U_{21} - (1-\beta)U_{22})\beta \\ &= \beta(1-\beta)(U_{21} - U_{22}) \\ &= \beta(1-\beta)(\alpha(R_2 + \delta + B_T - S_{un}) + (1-\alpha)(R_2 - S_{un}) - (\alpha(R_2 + B_T) + (1-\alpha)R_2)) \\ &= \beta(1-\beta)(\alpha\delta - S_{un}) \end{aligned}$$

The equilibrium of the replicator dynamic equations is established when $R_f(\alpha) = 0$ and $R_f(\beta) = 0$. Immediately is seen that this holds for $\alpha = 0, \alpha = 1, \beta = 0$ and $\beta = 1$. thereby, both equations are zero when $(\beta\delta - Z_1) = 0$ and $(\alpha\delta - S_{un}) = 0$. This gives us five points of equilibrium for (α, β), namely $(0,0), (0,1), (1,0), (1,1)$ and $(\frac{S_{un}}{\delta}, \frac{Z_1}{\delta})$. To analyse the local stability of these equilibrium points, we calculate the eigenvalues of the Jacobi matrix for each point of equilibrium. The Jacobi matrix is given by

$$\begin{aligned} J &= \begin{pmatrix} \frac{\partial R_f(\alpha)}{\partial \alpha} & \frac{\partial R_f(\alpha)}{\partial \beta} \\ \frac{\partial R_f(\beta)}{\partial \alpha} & \frac{\partial R_f(\beta)}{\partial \beta} \end{pmatrix} \\ &= \begin{pmatrix} (1-2\alpha)(\beta\delta - Z_1) & \alpha(1-\alpha)\delta \\ \beta(1-\beta)\delta & (1-2\beta)(\alpha\delta - S_{un}) \end{pmatrix} \end{aligned}$$

We see for $(\alpha, \beta) = (0,0)$ that:

$$J = \begin{pmatrix} -Z_1 & 0 \\ 0 & -S_{un} \end{pmatrix}$$

Where the eigenvalues are found as follows

$$\begin{aligned} det(J - \lambda I) &= \begin{vmatrix} -Z_1 - \lambda & 0 \\ 0 & -S_{un} - \lambda \end{vmatrix} = 0 \\ &= (-Z_1 - \lambda)(-S_{un} - \lambda) = 0. \end{aligned}$$

Which shows $\lambda_1 = -Z_1$ and $\lambda_2 = -S_{un}$. Since $\lambda_1, \lambda_2 < 0$, we conclude that $(0,0)$ is locally stable. By the same reasoning for $(0,1), (1,0)$ and $(1,1)$, we conclude that only $(0,0)$ and $(1,1)$ are locally stable. We only need the check for $(\alpha, \beta) = (\frac{S_{un}}{\delta}, \frac{Z_1}{\delta})$. These eigenvalues are given by

$$det(J - \lambda I) = \begin{vmatrix} 0 & S_{un} - \frac{1}{\delta}S_{un}^2 \\ Z_1 - \frac{1}{\delta}Z_1^2 & 0 \end{vmatrix} = 0$$
$$= \lambda^2 - (Z_1 - \frac{1}{\delta}Z_1^2)(S_{un} - \frac{1}{\delta}S_{un}^2) = 0.$$

Therefore, $\lambda_1, \lambda_2 = \pm\sqrt{(Z_1 - \frac{1}{\delta}Z_1^2)(S_{un} - \frac{1}{\delta}S_{un}^2)}$. We see that one lambda is positive and the other is negative, either in the real or the complex dimension. This shows that the equilibrium is a saddle point. The results on stability indicate that by repeating the game numerous times, the position of the game will converge either to 0 or to 1, depending on the initial position of α and β [8][1].

4.2 The Expanded Evolutionary Game Model, the Up-Level Model

The model discussed in the previous section is not enough to fully describe the relationship between the Power Grid corporation and the electric vehicle users. Not only Power Grid corporation should be taken into account, also some private companies involved in the sale of electricity should be considered. This section draws the up-level model of the evolution game. The problems of incomplete information and limited rationality of participants are still accurate. Player 1 represents a small corporation, which can also be considered the local government. Player 2 represents a big corporation, which can also be considered the central government. The payoff matrix of the up-level evolution game is as follows:

		Big Corp	
		Base Price	High Price
Small Corp	Base Price	(u_1, u_2)	$(u_1 + a_1v_1, (\frac{u_2}{v_1} - a_1)v_2)$
	High Price	$((\frac{u_1}{v_1} - a_2)v_2, u_2 + a_2v_1)$	$(u_1 + d, u_2 + d)$

where v_1 stands for the base price and v_2 for the high price in the electric vehicle market. If the small corporation and the big corporation both choose the base price, the revenue remains status quo and equals u_1 and u_2 respectively. Again, both types of corporations lose some social and economic effects, but there is still the original mechanism of intrinsic benefit. Furthermore, a_1 and a_2 represents the number of extra customers if the small corporation chooses the base price and the big corporation chooses the high price, and the other way around. Finally, d reflects the increase in revenues if both players choose the high price.

[1]Liang et al. (2016) performed some numerical simulations where they used example payoffs. The movement to either 0 or 1 is nicely illustrated. Check their article for images and more information.

Using the same reasoning as in the previous section we say that the probability for the strategy of a base price for the small corporation is $p = p(t)$, and the strategy for a high price is $1 - p = 1 - p(t)$. The probability for the strategy of a base price for the big corporation is $q = q(t)$, and thus the strategy for a high price is $1 - q = 1 - q(t)$.

From the payoff matrix we see that the average revenue for the small corporation equals

$$U_1 = pU_{11} + (1 - p)U_{12}.$$

Here, U_{11} is the average revenue if the small corporation chooses the base price and U_{12} is the average revenue if the small corporation chooses the high price. U_{11} and U_{12} are given by

$$\begin{aligned} U_{11} &= qu_1 + (1-q)(u_1 + a_1 v_1), \\ U_{12} &= q(\frac{u_1}{v_1} - a_2)v_2 + (1-q)(u_1 + d). \end{aligned}$$

Then, the replicator dynamics equation, called $R_f(p)$ becomes:

$$\begin{aligned} R_f(p) = \frac{dp(t)}{dt} &= (U_{11} - U_1)p \\ &= p(1-p)(U_{11} - U_{12}) \\ &= p(1-p)[(qu_1 + (1-q)(u_1 + a_1 v_1)) - (q(\frac{u_1}{v_1} - a_1)v_2 + (1-q)(u_1 + d))] \\ &= p(1-p)[a_1 v_1 - d - q(a_1 v_1 + (\frac{u_1}{v_1} - a_2)v_2 - (u_2 + d))] \\ &= p(1-p)(\gamma_1 - q\eta_1) \end{aligned}$$

where $\gamma_1 = a_1 v_1 - d$ and $\eta_1 = a_1 v_1 + (\frac{u_1}{v_1} - a_2)v_2 - (u_2 + d)$. Performing the same calculations for the big corporation gives an average revenue of

$$U_2 = qU_{21} + (1 - q)U_{22}$$

where U_{21} is the average revenue if the small corporation chooses the base price, and U_{22} is the average revenue if the small corporation chooses the high price. They are given by

$$\begin{aligned} U_{21} &= pu_2 + (1-p)(u_2 + a_2 v_1) \\ U_{22} &= p(\frac{u_2}{v_1} - a_1)v_2 + (1-p)(u_2 + d) \end{aligned}$$

The replicator dynamics equation $R_f(q)$ for the small corporation becomes:

$$\begin{aligned}
R_f(q) = \frac{dq(t)}{dt} &= (U_{21} - U_2)q \\
&= q(1-q)(U_{21} - U_{22}) \\
&= q(1-q)(pu_2 + (1-p)(u_2 + a_2 v_1) - (p(\frac{u_2}{v_1} - a_1)v_2 + (1-p)(u_2 + d))) \\
&= q(1-q)(a_2 v_1 - d - p[a_2 v_1 + (\frac{u_2}{v_1} - a_1)v_2 - (u_2 + d)]) \\
&= q(1-q)(\gamma_2 - p\eta_2)
\end{aligned}$$

where $\gamma_2 = a_2 v_1 - d$ and $\eta_2 = a_2 v_1 + (\frac{u_2}{v_1} - a_1)v_2 - (u_2 + d)$. The points of equilibrium of the replicator dynamics are established when $R_f(p) = 0$ and $R_f(q) = 0$. By same reasoning as in the previous section we get five equilibria for(p, q), namely $(0,0), (0,1), (1,0), (1,1)$ and $(\frac{\gamma_2}{\eta_2}, \frac{\gamma_1}{\eta_1})$. The stability of the point of equilibrium is checked using the Jacobi matrix given by

$$\begin{aligned}
J &= \begin{pmatrix} \frac{\partial R_f(p)}{\partial p} & \frac{\partial R_f(p)}{\partial q} \\ \frac{\partial R_f(q)}{\partial p} & \frac{\partial R_f(q)}{\partial q} \end{pmatrix} \\
&= \begin{pmatrix} (1-2p)(\gamma_1 - q\eta_1) & -p(1-p)\eta_1 \\ -q(1-q)\eta_2 & (1-2q)(\gamma_2 - p\eta_1) \end{pmatrix}
\end{aligned}$$

For further analysis of the stability of the points of equilibrium, Liang et al. (2016) performed simulations. The graphics of these simulations can be found in their article [8]. They concluded that with the increase of time t, the game will, after continuously repeating, converge separately to either 0 or 1.

These results have a lot in common with the simplified version of the evolution game. The up-level game and the basic evolutionary game are connected in an interactive game. The economic premise of the double layer evolutionary game theory is based on basic economic theories like the cost-benefit theory, the utility theory and risk aversion of subjects. Therefore, the following conditions have to be met for the double layer the game.

First of all, every subject has limited rationality and follows the principle of risk minimization and efficiency maximization. In other words, every player will choose the strategy which maximizes their utility function. Secondly, the strategy for Power Grid Corp becomes the bridge between the basic model and the up-level game. Their choice for taking subsidy or not affects the number of electric vehicle users, and therefore their pricing strategy. We assume that $\alpha \propto (1-p)(1-q)$, so we can say $\alpha = k_1(1-p)(1-q)$ where k_1 represents a weighted coefficient. Finally, we assume that every player has both continuity and timeliness in the process of evolution and strategy selection. A full explanation on how to derive the replicator dynamic equations for the double layer model and locate the stable points of equilibria can be found in the article of Liang et al. (2017) [8]. They conclude that strategy orientation is more beneficial when applied on the double layer model.

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