THE PRINCESS AND MONSTER GAME ON THE CIRCLE AND THE INTERVAL

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1 Introduction

In his book *Differential Games* [6], Rufus Isaacs introduced a game called the 'princess and monster' game, which he described as follows:

> "The monster P searches for the princess E, the time required being the payoff. They are both in a totally dark room Q (of any shape) but they are each cognizant of its boundary (possibly through small light admitting perforations high in the walls). Capture means that the distance $PE \leq r$, a quantity small in comparison with the dimension of Q. The monster, supposed highly intelligent, moves at a known speed. We permit the princess full freedom of locomotion."

In this thesis, we will consider the cases that Q is a circle and Q is an interval. The princess and monster game on a circle is the only search game with a mobile princess that has been solved, and it was done by Alpern (1974) [1] and Zeliken (1972) [7]. In Section 3, we prove that, according to Zeliken, the value of this game exists, what this value is and indicate for which strategies that value is optimal. The princess and monster game on an interval was believed to be trivial. People thought that it was optimal for the monster to start at a random end and then move to the other end. For the search game with an immobile princess, this strategy is indeed optimal. In Section 4, we demonstrate that with a mobile princess, that strategy is not optimal and that the game $\Gamma(I)$ is not trivial. We achieve this by providing estimates on its value V based on the article of Alpern, Fokkink, Lindelauf and Olsder [4].

2 Search games

A search game is a two-player game with a searcher and a hider. This game takes place in a set that is called the search space Q. The searcher can choose any path inside Q, but his speed cannot exceed 1. The hider can, just like the searcher, choose any path inside Q, or he can remain immobile. It is assumed that the hider and the searcher have no knowledge about each other's movement. In general, the game is considered to be over when the distance between the searcher and the hider is less than or equal to ϵ. Usually we consider the case ϵ is zero, so the searcher and the hider must be in the same place for the game to end. The time that the searcher needs to catch the hider (the capture time) is the payoff of the game. Since the hider wants to maximize this capture time and the searcher wants to minimize it, a search game is a zero-sum game. Search games, as mathematical models, can be applied to hide-and-seek situations such as those played in our childhood or practiced in certain military situations.

2.1 The princess and monster game

In this thesis, we discuss a search game called the princess and monster game. In this game, the hider (princess) is mobile. We will denote a path of the searcher (monster) by M and a path of the princess by P. The payoff function (the capture time, i.e. the time at which the monster and princess are in the same place) is denoted by $T(M, P)$. The existence of a value $V = V(Q)$ for this game, an optimal searcher mixed strategy and an ϵ-optimal hider mixed strategy are proved in [3] and [5]. We define a strategy to be ϵ-optimal if the expected payoff is at least $V - \epsilon$ against any strategy of the opponent.

3 The game on a circle

For the princess and monster game on a circle, the space Q is the circumference of a circle. The only difference from the original princess and monster game is that we now assume that both M and P have a speed that does not exceed 1. The radius of the circle is r, and ϕ and ψ are the angles of the position vectors to M and P with some fixed line through the centre of the circle. We then have the system

$$\begin{aligned} \dot{\phi} &= v/r, \quad |v| \leq 1, \\ \dot{\psi} &= w/r, \quad |w| \leq 1. \end{aligned} \tag{1}$$

For the strategies of M and P, we take the paths of (1), i.e. the functions $\phi(t)$ and $\psi(t)$, respectively. Because the players have a maximal speed of 1, these functions satisfy the Lipschitz condition:

$$\begin{aligned} |\phi(t) - \phi(t')| &\leq |t - t'|/r, \\ |\psi(t) - \psi(t')| &\leq |t - t'|/r. \end{aligned} \tag{2}$$

Neither player has a dominant strategy, so they have to consider a mixed strategy, i.e. a probability distribution on the set of pure strategies. Each player chooses such a probability distribution and then we can compute the expected value of the capture time.

3.1 Optimal strategies and the payoff

In this section, we prove that the value of the game exists and equals $\frac{3}{2}\pi r$. The optimal strategies (which are the same for both players) belonging to this value are as follows: at $t = 0$, start randomly (so the initial distribution is uniform); with probability $\frac{1}{2}$ go to the antipodal point halfway around the circle clockwise (counterclockwise) with maximal speed until the time $t = \pi r$, which we call a period. Now, again with probability $\frac{1}{2}$, choose one of the two directions until the antipodal point is reached. In this way, one can introduce a measure on the space of functions (2) equivalent to such a movement.
Let us prove that these strategies are optimal. Let M follow the described strategy and let P choose an arbitrary acceptable strategy. In this case, we prove the payoff is constant, independent of the strategy of P, and that this payoff equals $\frac{3}{2}\pi r$.

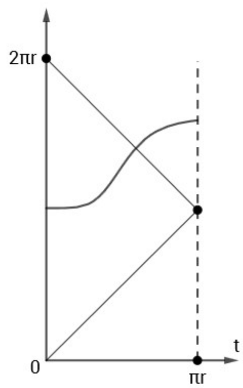

Figure 1:

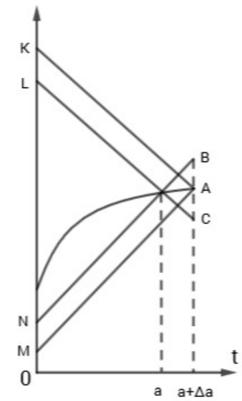

Figure 2:

Lemma 3.1. *The probability of capture in the first period is* $\frac{1}{2}$.

Proof. First, make a graph of the movement with the time t on the x-axis and the arc-length that P has passed through on the y-axis. At $t = 0$, we take the origin for the position of M. Now the two possible paths for M, beginning at the endpoints of the interval $[0, 2\pi r]$, are the lines with slopes 1 and -1 respectively. Figure 1 depicts these paths. Now, we consider some specific path of P on the interval $[0, \pi r]$. In Figure 1 that path is a curve that satisfies the Lipschitz condition with Lipschitz constant 1. Of course this curve starts at some point in the interval $[0, 2\pi r]$, as this interval represents the circumference of the circle. Because of the Lipschitz condition, the path of P intersects only one of the legs of the right triangle, not both (or for one point on the y-axis, the curve reaches the vertex of the triangle). So for all starting positions of P on the y-axis (except for the one where the curve reaches the vertex of the triangle), a choice of M in one direction leads to capture in the first period, while a choice in the other direction leads to avoidance of capture. This means that for a fixed path and a fixed starting position of P, the probability of capture is $\frac{1}{2}$. Hence, for a fixed path and a uniform distribution of the initial position of P, the probability of capture is $\frac{1}{2}$. We can take an arbitrary measure on the space of functions satisfying the Lipschitz condition (2). Now for any fixed function, the probability of capture in the first period is also $\frac{1}{2}$. ■

At this point, we calculate the expected value of the capture time. To do so, we first need to calculate the expected value of the capture time, subject to the condition that the princess is captured in the first period. The distance along the arc of the circle from M to P in the counterclockwise direction is denoted by x. Then, we call $\theta(x)$ the capture time, where P has a fixed path and begins at x, and M chooses the direction that ensures the capture of P in the first period.

Lemma 3.2. *The random variable $\theta(x)$ is uniformly distributed on $[0, \pi r]$ for a fixed path of P.*

Proof. Look at figure 2. Like in figure 1, we have trajectories for the monster, the lines with slopes 1 and -1 and a specific trajectory of the princess. Since the monster paths have slopes of 1 and -1, the line segment LN is equal to $2a$ and the segment KM is equal to $2a + 2\Delta a$. It is therefore apparent that $KL + MN = BC = 2\Delta a$, where $KM < 2\pi r$ and $a + \Delta a < \pi r$. Then $\mu \{x : a \leq \theta(x) \leq a + \Delta a, a \geq 0, a + \Delta a \leq \pi r, 0 \leq x \leq 2\pi r\} =$
$\mu \{[a, a + \Delta a] \cap [0, \pi r]\} \cup \mu[0, 2\pi r] = 2\pi r \cdot \mathbb{P}(\theta(x) \in [a, a + \Delta a])$. Since the initial position of P has a uniform distribution and P has a fixed path, $\theta(x)$ has a uniform distribution. ∎

Because $\theta(x)$ has a uniform distribution, the expected value of the capture time, subject to the condition that P is captured in the first period is $(0 + \pi r)/2 = \frac{1}{2}\pi r$. Note that $\theta(x)$ does not depend on the path of P, as it was fixed, so we can take any measure on the space of paths of P. Thus, the expected value of the capture is also equal to $\frac{1}{2}\pi r$.
We proved that the probability of capture in the first period is $\frac{1}{2}$, so the probability of capture in the second period is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ and in the third period is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. In this way, the expected value of the capture time, subject to the condition that P is captured in the second period is equal to $\pi r + \frac{1}{2}\pi r$. This is because if the capture does not occur in the first period, the first period passes by $(t = \pi r)$ and the expected value of the capture in the second period is $\frac{1}{2}\pi r$. Subject to the condition that P is captured in the third period, the expected value of the capture time is $2\pi r + \frac{1}{2}\pi r$ because capture does not take place in first two periods $(t = 2\pi r)$ and the expected value of the capture in the third period is $\frac{1}{2}\pi r$. Of course, we can also compute these values using the definition of the expected value. For example, the expected value of the capture time, subject to the condition that P is captured in the second period, is calculated as follows: $\mathbb{E}$(capture time | P captured in the second period) $= \int\limits_{\pi r}^{2\pi r} x \cdot \frac{1}{\pi r}\, dx = \left[\frac{1}{2\pi r} \cdot x^2\right]_{\pi r}^{2\pi r} = 2\pi r - \frac{1}{2}\pi r = \pi r + \frac{1}{2}\pi r.$

Now, we can use the law of total expectation, which provides

$$CT = \frac{1}{2}(\frac{1}{2}\pi r) + \frac{1}{4}(\pi r + \frac{1}{2}\pi r) + \frac{1}{8}(2\pi r + \frac{1}{2}\pi r) + \ldots$$

We can solve this using the series $\sum_{k=1}^{\infty} k \cdot x^k = \frac{x}{(x-1)^2}$, so

$$\begin{aligned} CT &= \frac{1}{2}(\frac{1}{2}\pi r) + \frac{1}{4}(\pi r + \frac{1}{2}\pi r) + \frac{1}{8}(2\pi r + \frac{1}{2}\pi r) + \ldots \\ &= \frac{2}{4}\pi r + \frac{3}{8}\pi r + \frac{4}{16}\pi r + \ldots \\ &= \pi r \cdot \sum_{k=2}^{\infty} \frac{k}{2^k} \\ &= \pi r \cdot \left(\sum_{k=1}^{\infty} \frac{k}{2^k} - \frac{1}{2} \right) \\ &= \pi r (2 - \frac{1}{2}) \\ &= \frac{3}{2}\pi r. \end{aligned} \tag{3}$$

Now, let P follow the strategy described above and let M choose any arbitrary acceptable strategy. Then by symmetry, we obtain the exact same payoff $\frac{3}{2}\pi r$. Thus, the monster has a strategy, the described optimal strategy, which gives him the value $\frac{3}{2}\pi r$ and the princess has a strategy, also the optimal strategy, which likewise gives her the value $\frac{3}{2}\pi r$. Therefore, that the value $\frac{3}{2}\pi r$ is the value of our game for the described optimal strategy.

4 The game on an interval

In this section, we discuss the princess and monster game on the interval $[-1,1]$. The monster will take a path $M = M(t)$ of speed 1, which will be his pure strategy. The princess' pure strategy will be any continuous path $P = P(t)$ since the princess has an arbitrary speed. As Section 2 has noted, the payoff for this game is $T = T(M,P) = \min\{t : M(t) = P(t)\}$. Now, take the topology of uniform convergence on compact subsets. Then the function T is upper semi-continuous and the mixed strategy space of the monster is compact Hausdorff. Now, the pure strategy space of P is $\mathcal{P} = \{P : [0,\infty) \to I = [-1,1], P \text{ continuous}\}$. The pure strategy space of M is

$$\mathcal{M} = \{M : [0,\infty) \to I = [-1,1], |M(t) - M(t')| \leq |t - t'| \forall t, t' \geq 0\},$$

or in other words, all the paths in $\mathcal{P}$ which satisfy the Lipschitz condition with Lipschitz constant 1.

4.1 Properties of the equilibrium solutions

First, the monster chooses a path that covers the entire interval $[-1,1]$; otherwise, the princess can hide at some endpoint and then the payoff is infinite. We demonstrate in this section that if the monster reaches an end, he should move directly to the other end and if the princess reaches an end, she should stay there. We then show that the princess should never exceed speed 1, despite being unrestricted in speed, and that both the monster and the princess optimally use mixed strategies such that they are invariant under the reflection $\phi(x) = -x$. Finally, we prove that the optimal response for the monster is to use a finite number of pure strategies whenever the princess uses a mixed strategy consisting of a finite number of pure strategies and that in an optimal mixed strategy of the princess, the pure strategies do not intersect. To prove this, we must first establish some definitions.

Definition 1. *A pure strategy $M \in \mathcal{M}$ is called* end-reflecting *if $M(t_0) = \pm 1 \Rightarrow |M(t) - M(t_0)| = t - t_0$ for $t_0 \leq t \leq t_0 + 2$; in other words, if the monster reaches an end, he should move straight to the other end.*

Definition 2. *A pure strategy $P \in \mathcal{P}$ is* end-absorbing *if the princess stays at an end after she reaches that end; thus, $P(t_0) = \pm 1 \Rightarrow P(t) = P(t_0)$ for $t \geq t_0$.*

Definition 3. *For either player, a mixed strategy is* symmetric *if the strategy is invariant under the reflection* $\phi(x) = -x$*. We denote a strategy that is symmetric to* $M \in \mathcal{M}$ *and to* $P \in \mathcal{P}$ *by* $-M$ *and* $-P$*, respectively.*

Definition 4. *The monster or the princess* runs *at time* t *if* $|M'(t)| = 1$ *or* $|P'(t)| = 1$*, i.e. if the player moves with speed 1 at time* t*.*

The following three lemmas concern pure strategies which are dominated. We note that a path is smooth if it is continuously differentiable. We can limit the pure hider strategies to any subset that is dense in $\mathcal{M}$. So we can consider smooth paths only without changing the value of the game.

Lemma 4.1. *Every pure monster strategy* $M \in \mathcal{M}$ *is dominated by one that is end-reflecting.*

Proof. Assume that M is not end-reflecting and reaches, let's say, $+1$ at time t_0 (so $M(t_0) = +1$). Let M^* be end-reflecting and equal to M up to time t_0 and equal to $1 + t_0 - t$ for $t \geq t_0$. Since M^* is end-reflecting, $M^*(t_0) = +1 \Rightarrow |M^*(t) - M^*(t_0)| = t - t_0$ for $t_0 \leq t \leq t_0 + 2$. Let $P \in \mathcal{P}$ be arbitrary. If $T(M, P) \leq t_0$, then $T(M^*, P) = T(M, P)$. If $T(M, P) = t_1 > t_0$, we know that $P(t_1) - M(t_1) = 0$. Since $M(t) \geq M^*(t)$ for all $t \geq t_0$, this means that $P(t_1) = M(t_1) \geq M^*(t_1)$, so $P(t_1) - M^*(t_1) \geq 0$. Because $P(t) \leq 1$ for all $t \geq 0$, $P(t_0) - M^*(t_0) = P(t_0) - 1 \leq 0$. Since $P - M^*$ is a continuous function, the intermediate value theorem dictates that there exists a $t_2 \in [t_0, t_1]$ such that $P(t_2) - M^*(t_2) = 0$. Thus, $T(M^*, P) = t_2 \leq t_1 = T(M, P)$, so M^* dominates M. ■

Lemma 4.2. *Every pure princess strategy* $P \in \mathcal{P}$ *is dominated by one that is end-absorbing.*

Proof. Suppose P is not end-absorbing and reaches, let's say, $+1$ at time t_0 (so $P(t_0) = +1$). Let P^* be end-absorbing and equal to P for $t \leq t_0$ and then stays at $+1$. P^* is end-absorbing, so $P^*(t_0) = +1 \Rightarrow P^*(t) = P^*(t_0)$ for $t \geq t_0$. Now consider an arbitrary $M \in \mathcal{M}$. If $T(M, P^*) = t_1 > t_0$, then $M(t_1) = P^*(t_1) = +1$. Since, again, $P(t) \leq 1$ for all $t \geq 0$, $M(t_1) = 1 \geq P(t_1)$. Because $M(t_0) < 1 = P(t_0)$ and M is continuous, the intermediate value theorem states that there exists a $t_2 \in [t_0, t_1]$ such that $M(t_2) = P(t_2)$. We can conclude that $T(M, P) = t_2 \leq t_1 = T(M, P^*)$; thus, for any M, P is dominated by P^*. ■

Lemma 4.3. *Every smooth princess strategy $P \in \mathcal{P}$ is dominated by one which is in $\mathcal{M}$. Therefore, the princess should never exceed speed 1.*

Proof. Let P be a smooth princess, and let P^* be a princess that follows P but has speed bounded by 1. Let t_0 =inf$\{t \in [0, \infty) : |P'(t)| > 1\}$ and assume that t_0 is finite. So, t_0 is the first time at which P exceeds speed 1. For $t \leq t_0$, define $P^*(t) = P(t)$. For $t > t_0$, the princess P^* continues moving at speed 1 in the same direction as P. By the boundedness of the interval, P^* meets P again at time $\tau > t_0$. Now, suppose that M finds P at $T = T(M, P)$ when P and P^* are not in the same place and suppose that the capture takes place before time τ, or else let t_1 =inf$\{t \in [\tau, \infty) : |P'(t)| > 1\}$ and repeat the construction inductively. So, we now have $t_0 < T < \tau$. By symmetry, let P be the one that moves right at time t_0, so $P'(t_0) = +1$. Then P^* has speed 1 and $P^*(t) < P(t)$ $\forall t \in (t_0, t_1)$. $M(T) = P(T) > P^*(T)$, and the monster moves with speeds bounded by 1, so $M(t) > P^*(t)$ for all $t \in (t_0, T)$. We can conclude that $T(M, P^*) > T(M, P)$; that is, P^* cannot be caught from behind and she will not be caught by the monster before P, so the princess should never exceed speed 1. ■

As a result of this proposition, we only have to consider paths of the princess with speed ≤ 1. In the following propositions, we consider mixed strategies, i.e. probability measures on the Borel σ-algebra of $\mathcal{M}$.

Lemma 4.4. *There is an optimal monster mixed strategy, and for any ϵ, there is an ϵ-optimal princess mixed strategy. Both are invariant under $\phi(x) = -x$.*

Proof. This proof is a special case of theorem 3 of Alpern and Asic [2]. This theorem proves the existence of such strategies invariant under the isometry group of a network Q. If $Q = I$, that group consists of the identity and ϕ. ■

Lemma 4.5. *Suppose the princess uses a mixed strategy focused on a finite number of pure strategies $P_j \in \mathcal{M}$, j=1,...,J. Then the optimal response of the monster is also focused on a finite number of pure strategies. In particular, the monster chooses a random permutation σ of 1,...,J and after he has met $P_{\sigma(j)}$ he runs to $P_{\sigma(j+1)}$ until he has met all princesses with different pure strategies. So, for a pure strategy M of the monster and for pure princess strategies P_j, the capture times are $T(M, P_j) = t_j$, with $0 = t_1 \leq t_2 \leq ... \leq t_J$ and*

$$M'(t) = sign(P_{j+1}(t_j) - M(t_j)) \; \forall t \in (t_j, t_{j+1}) \tag{4}$$

Proof. Let M^* be any pure monster strategy which is an optimal response to the mixed princess strategy which fails (4) for some j. Choose P_j so that $T(M^*, P_j) = t_j^*$ is non-decreasing in j. Let k be the smallest j such that (4) fails. Without loss of generality, suppose that $P_{k+1}(t_k^*) > M^*(t_k^*)$. Now, define a new monster strategy $M(t)$ with $T(M, P_j) = t_j \forall j = 1, ..., J$. This strategy is given by:

$$M(t) = \begin{cases} M^*(t), & \text{if } t \leq t_k^* \\ M^*(t_k^*) + (t - t_k^*), & \text{if } t_k^* \leq t \leq t_{k+1} \\ P_{k+1}(t), & \text{if } t_{k+1} \leq t \leq t_{k+1}^* \\ M^*(t), & \text{if } t_{k+1}^* \leq t. \end{cases} \tag{5}$$

Now, $t_j = t_j^*$ except for $j = k+1$ and $t_{k+1}^* > t_{k+1}$. Therefore, M^* is not an optimal response. ■

This proposition indicates that if the monster uses an optimal strategy, we can focus on pure strategies in which the monster runs all the time. After Lemmas 4.4 and 4.5, we can conclude that it is sufficient to consider finite mixed strategies only.

Definition 5. *We call a pair of pure hider strategies P_1, P_2* non-crossing *if $P_1(t) \leq P_2(t)$ for all $t \geq 0$. P_1, P_2 are* non-intersecting *if the inequality holds strictly.*

For pure princess strategies P_1 and P_2, we can define pure strategies $P_1 \wedge P_2 = \min\{P_1, P_2\}$ and $P_1 \vee P_2 = \max\{P_1, P_2\}$. It is obvious that $P_1 \wedge P_2$ and $P_1 \vee P_2$ are non-crossing.

Lemma 4.6. *Let P be the princess strategy that consists of the pure strategies P_1, P_2 with equal probability. Then P is dominated by the non-crossing princess strategy that consists of the strategies $P_1 \wedge P_2$, $P_1 \vee P_2$ with equal probability. As a consequence, any finite mixed princess strategy can be assumed to consist of two non-crossing pure strategies.*

Proof. From the definition of $P_1 \wedge P_2$ and $P_1 \vee P_2$, it is clear that $\{P_1 \wedge P_2, P_1 \vee P_2\} = \{P_1, P_2\}$ for all t. If M catches the first of the two princesses P_1, P_2, let us say P_1, at time t_1, then M simultaneously catches the first of the non-crossing princesses $P_1 \wedge P_2$, $P_1 \vee P_2$. Reflecting the interval, let M approach from the left side. He thus catches $P_1 \wedge P_2$ at time t_1, so $M(t_1) = P_1 \wedge P_2(t_1) =$

$P_1(t_1) \leq P_2(t_1) = P_1 \vee P_2(t_1)$. So at time t_1, P_2 and $P_1 \vee P_2$ are at the same place and M is to their left. Also, $P_1 \vee P_2(t) \geq P_2(t)$ for all t, so $T(M, P_2) \leq T(M, P_1 \vee P_2)$ so M cannot catch $P_1 \vee P_2$ before he catches P_2. We can conclude that $P_1 \vee P_2$ dominates P_2. Since $P_1 \wedge P_2$ and $P_1 \vee P_2$ are non-crossing, it is possible to construct a finite mixed princess strategy that consist of non-crossing pure strategies. ■

A collection of non-intersecting paths is used to approximate any finite collection of non-crossing paths arbitrarily close. This means that there exist ϵ-optimal princess strategies that are finite, symmetric and non-intersecting.

Lemma 4.7. *Any pure princess strategy P in a non-intersecting symmetric princess strategy is contained in half of the interval $[-1, 1]$, so $P(t) \in [-1, 0]$ or $P(t) \in [0, 1]$ for all t.*

Proof. If the princess uses the pure strategy P, then she also uses $-P$. If $P(t) = 0$, then P and $-P$ would intersect, but this cannot occur since the mixed strategy is non-intersecting. Therefore, either $P(t) \neq 0\ \forall t$ or P is immobile and remains in 0 for the rest of the game. When $P(t) \neq 0\ \forall t$, neither P nor $-P$ can cross the middle of the interval. So, in both cases, P is contained in half of the interval $[-1, 1]$. ■

4.2 Estimates on the value of the game

The game appears to be trivial, since there are apparently some optimal strategies for the interval that are dominant. In this section, we show that these strategies are not dominant and that the game is therefore not trivial. For the monster, an obvious approach is to start at a random end and then run to the other end. Let us call this strategy A. In this strategy the monster starts in 1 and runs to -1 such that $A(t) = 1 - t$, or symmetrically $-A(t) = -1 + t$ for all $t \in [0, 2]$. These strategies A and $-A$ are called 'the *sweepers*'. The optimal response for the princess would be to wait at the middle until time $1 - \epsilon$ and then run to 1 for A or -1 for $-A$. We denote these strategies with F and $-F$, so $F(t) = \begin{cases} \max\limits_{t \geq 0} \{0, \epsilon - 1 + t\} & \text{for } \ 0 \leq t \leq 2 - \epsilon \\ 1 & \text{for } \ t > 2 - \epsilon \end{cases}$, and respectively $-F(t) = \begin{cases} \min\limits_{t \geq 0} \{0, -\epsilon + 1 - t\} & \text{for } \ 0 \leq t \leq 2 - \epsilon \\ -1 & \text{for } \ t > 2 - \epsilon \end{cases}$. This gives an

estimate $V \leq \frac{3}{2}$ because if the monster equiprobably uses strategies A and $-A$, he catches the princess such that, at worst, the capture times are 1 and 2.
For the princess, it would be obvious to start at a random end and then stay there the whole game. This gives strategies E and $-E$, satisfying $E(t) = 1$ and $-E(t) = -1$, respectively. The optimal response to the strategy that equiprobably uses E and $-E$ is for the monster to use A and $-A$ with equal probability. The expected capture time for this optimal response is 1 because $T = \frac{1}{2}T(A, E) + \frac{1}{2}T(-A, E) = \frac{1}{2}(0 + 2) = 1$. This gives an estimate $V \geq 1$. If $V = 1$ or $V = \frac{3}{2}$, the game would be trivial, so by presenting simple strategies, we demonstrate that $1 < \frac{97}{75} < V < \frac{47}{32} < \frac{3}{2}$. It is optimal against the strategies A and $-A$ to use strategies F and $-F$, so to wait around 0 until time $1 - \epsilon$ and then run to one of the endpoints. The monster can now ambush the princess by adding a search path that keeps watch over the centre. Let B be the strategy that starts at 0, runs to the left, joins sweeper $-A$ at time $\frac{1}{2}$ when he meets him until he reaches 1, then goes to -1 by Lemma 4.1. The symmetric counterpart of B is $-B$. Now, the mixed strategy of the monster consists of strategies $\pm A$ each with probability $\frac{7}{16}$ and $\pm B$ each used with probability $\frac{1}{16}$. A space-time diagram $[-1, 1] \times [0, \infty)$ in Figure 3 depicts the strategies $\pm A$ and $\pm B$ of the monster.

Theorem 4.8. *$V \leq \frac{47}{32} = 1.4688$.*

Proof. Let P be any princess strategy and let the monster adopt the mixed strategy in which he uses $\pm A$ each with probability $\frac{7}{16}$ and $\pm B$ each with probability $\frac{1}{16}$. The probability that $T \leq t$ is denoted by $\mathbb{P}(t)$. We consider two cases: (i) $|P(\frac{1}{2})| \leq \frac{1}{2}$ and (ii) $|P(\frac{1}{2})| > \frac{1}{2}$.
(i) $|P(\frac{1}{2})| \leq \frac{1}{2}$, so $-\frac{1}{2} \leq P(\frac{1}{2}) \leq \frac{1}{2}$. This means that at time $t = \frac{1}{2}$, B of $-B$ has been met, so $\mathbb{P}(\frac{1}{2}) = \frac{1}{16}$. Then, the princess has met A of $-A$ at time $t = 1$, so $\mathbb{P}(1) = \frac{8}{16}$. Finally, at time $t = 2$, all have been met, which means that $\mathbb{P}(2) = 1$. Thus,

$$T \leq \frac{1}{16} \cdot \frac{1}{2} + \left(\frac{8}{16} - \frac{1}{16}\right) \cdot 1 + \left(1 - \frac{8}{16}\right) \cdot 2 = \frac{1}{16} \cdot \frac{1}{2} + \frac{7}{16} \cdot 1 + \frac{1}{2} \cdot 2 = \frac{47}{32}.$$

(ii) Because of symmetry, we can assume that $P(\frac{1}{2}) > \frac{1}{2}$. The princess has then met A at time $t = \frac{1}{2}$, so $\mathbb{P}(\frac{1}{2}) = \frac{7}{16}$. Subsequently, she can maximally avoid $-A$ and B until time $t = 2$. This yields $\mathbb{P}(2) = \frac{15}{16}$. At time $t = 4$, all

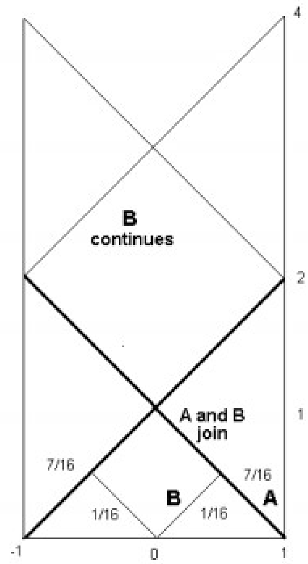

Figure 3: The monster strategy for $V < 47/32$ in a space-time diagram

have been met by the princess, so $\mathbb{P}(4) = 1$. This gives the value

$$T \leq \frac{7}{16} \cdot \frac{1}{2} + \left(\frac{15}{16} - \frac{7}{16}\right) \cdot 2 + \left(1 - \frac{15}{16}\right) \cdot 4 = \frac{7}{16} \cdot \frac{1}{2} + \frac{8}{16} \cdot 2 + \frac{1}{16} \cdot 4 = \frac{47}{32}.$$

■

At this point, we consider a lower bound on V. We consider the monster strategies A and B. Against A and B, the optimal princess strategy is to start at $\pm\frac{1}{2}$ and stay there until time $\frac{1}{2} - \epsilon$. Then, she should either run to the middle and back to the end or run straight to the closest end. We respectively denote these strategies by $\pm G$ and $\pm H$. These strategies, together with $\pm E$ and $\pm F$, are drawn in figure 4. Notice that these paths do not cross the centre because of Lemma 4.7. Combining these strategies, the princess obtains a mixed strategy in which she uses E, F, G, H and their counterparts. Then, according to Lemma 4.5, the monster starts in 0, $\pm\frac{1}{2}$ or ± 1 and runs between all possible princess paths. When the monster starts at an end, he must use strategy $\pm A$. If he starts in 0, the monster runs to $\pm\frac{1}{2}$, at which point he turns (B) or continues to an end and then runs back, which we call strategy M. Finally, if the monster starts in $\pm\frac{1}{2}$, he either runs to the closest end and back (strategy $\pm C$) or to the farthest end and back (strategy $\pm D$). Figure 4 also presents strategies C, D and M.

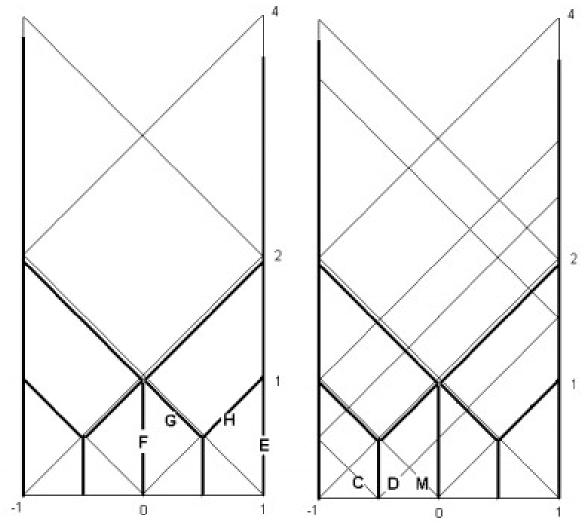

Figure 4: Left: E, F, G and H are the pure paths in an optimal mixed princess strategy against A, B. Right: C, D and M are the monster's response to $\{E, F, G, H\}$. C, D and M are, apart from A, B and up to symmetry, the only relevant paths.

Lemma 4.9. *For the above mixed strategies for the monster and the princess, the value of the corresponding matrix game gives the lower bound* $V \geq \frac{97}{75} =$ *1.293.*

Proof. The zero-sum game matrix, whereby each pure searcher strategy is a row and each pure princess strategy is a column, is as follows(ignoring ϵ):

$$
\begin{array}{c}
 \\ \pm A \\ \pm B \\ \pm C \\ \pm D \\ \pm M
\end{array}
\begin{array}{c}
\begin{array}{cccc} \pm E & \pm F & \pm G & \pm H \end{array} \\
\begin{pmatrix}
\frac{1}{2}(0+2) & \frac{1}{2}(1+2) & \frac{1}{2}(1+2) & \frac{1}{2}(\frac{1}{2}+2) \\
\frac{1}{2}(2+4) & \frac{1}{2}(0+0) & \frac{1}{2}(\frac{1}{2}+2) & \frac{1}{2}(2+4) \\
\frac{1}{2}(\frac{1}{2}+\frac{5}{2}) & \frac{1}{2}(\frac{5}{4}+\frac{5}{2}) & \frac{1}{2}(0+\frac{5}{2}) & \frac{1}{2}(0+\frac{5}{2}) \\
\frac{1}{2}(\frac{3}{2}+\frac{7}{2}) & \frac{1}{2}(\frac{1}{2}+\frac{1}{2}) & \frac{1}{2}(0+\frac{3}{4}) & \frac{1}{2}(0+\frac{3}{2}) \\
\frac{1}{2}(1+3) & \frac{1}{2}(0+0) & \frac{1}{2}(\frac{1}{2}+3) & \frac{1}{2}(1+3)
\end{pmatrix}
\end{array}
=
\begin{array}{c}
 \\ \pm A \\ \pm B \\ \pm C \\ \pm D \\ \pm M
\end{array}
\begin{array}{c}
\begin{array}{cccc} \pm E & \pm F & \pm G & \pm H \end{array} \\
\begin{pmatrix}
1 & \frac{3}{2} & \frac{3}{2} & \frac{5}{4} \\
3 & 0 & \frac{5}{4} & 3 \\
\frac{3}{2} & \frac{15}{8} & \frac{5}{4} & \frac{5}{4} \\
\frac{5}{2} & \frac{1}{2} & \frac{3}{8} & \frac{3}{4} \\
2 & 0 & \frac{7}{4} & 2
\end{pmatrix}
\end{array}
$$

Solving this game matrix gives a value of $\frac{97}{75}$, and this is a lower bound on V. ∎

In theory, it is possible to compute the value of the game. This can be done by adding finitely many pure strategies to the princess and then add finitely many pure strategies to the monster to optimize his response. However, the princess can then optimize her response by a finite number of pure strategies, etc. This increase in the number of pure strategies is exponential and unfortunately, the convergence to the value of the game appears to be slow and not effectively computable. In the following two sections, we improve the upper and lower bounds by assigning the princess and the monster pure strategies with continuous initial distributions.

4.3 A monster strategy with continuous initial distribution

In this section, we improve the upper bound $V < \frac{47}{32}$. To do this, we extend the mixed monster strategy which uses A, B. Replace B with a continuous mixed strategy m_Φ. In m_Φ the monster picks a point x on the interval $\Phi(x)$ according to a continuous distribution, and he runs to the right until he meets sweeper A, joins sweeper A until -1 and then he runs back to 1. $-m_\Phi$ is the symmetric counterpart of m_Φ where the monster starts according to $\Phi(-x)$.

Lemma 4.10. *Suppose that the monster uses m_Φ. Let P be a pure princess strategy and denote the first time that the princess meets a sweeper by $y = y(P)$. Then the monster finds the princess before time $y \Leftrightarrow$ he starts in $(P(y) - y, P(0)]$ and runs to the right, or if he starts in $[P(0), P(y) + y)$ and runs to the left.*

Proof. According to Lemma 4.7, we can assume that $P(t) \geq 0 \; \forall t$, so $P \geq 0$. Then P meets sweeper A first, and $P(y) = 1 - y$. Because of the Lipschitz condition, $|P(y) - P(0)| \leq y$, so $1 - 2y \leq P(0)$. Let M be the pure strategy in m_Φ, so he initially runs to the right until he meets A. The argument for M running to the left is the same.
" $\Rightarrow$ " Assume that M does not start in $(P(y) - y, P(0)] = (1 - 2y, P(0)]$. Then either $M(0) > H(0)$ or $M(0) < 1 - 2y$. If $M(0) > P(0)$, M runs to the right until he meets sweeper A and finds P at time y and not before. In the second case, M cannot meet A before P does, so the paths of M and P do not cross before time y.
" $\Leftarrow$ " Suppose that M starts in $(P(y) - y, P(0)] = (1 - 2y, P(0)]$. Accordingly, M meets A at time $t = \frac{1-M(0)}{2} < y$. Since for $t < y$ holds that $M(t) = A(t) > P(t)$, it yields $M(t) > P(t)$ and since $M(0) < P(0)$, M finds P between 0 and t. ■

Lemma 4.11. *Let $f = \Phi'$ be the probability density of Φ. If the monster starts in $(P(y) - y, P(0)]$ and runs to the right, he catches the princess with expected capture time*

$$\int_0^y t f(P(t) - t)(1 - P'(t)) \, dt. \tag{6}$$

If the monster starts in $[P(0), P(y) + y)$ and runs to the left, he catches the princess with expected capture time

$$\int_0^y t f(-P(t) - t)(1 + P'(t)) \, dt. \tag{7}$$

Proof. We consider a small time interval $[t, t + \Delta t]$ in which the princess moves from $P(t)$ to $P(t) + \Delta P$. Since we assume that the princess does not move faster than 1, we can use that $|\Delta P| \leq \Delta t$. If the monster meets the princess in the interval $[t, t + \Delta t]$ and has started out from the right, then he started in the interval $[P(t) - t + \Delta P - \Delta t, P(t) - t]$. The probability that the monster starts in that interval is equivalent to the probability that the hider is caught in the interval $[t, t + \Delta t]$ and is approximated by the length of $[P(t) - t + \Delta P - \Delta t, P(t) - t]$ multiplied by the density f in $P(t) - t$. Therefore, this approximation is $f(P(t) - t)(\Delta t - \Delta P)$. If $\varphi(t)$ is the probability density describing the likelihood that the monster catches the princess, and the time of capture is t up to the first order, then the expected capture time is $\int_0^y t \cdot \varphi(t)\, dt$. Now $f(P(t) - t)(\Delta t - \Delta P)$ is equal to the approximation on $\varphi(t)$ over $[t, t + \Delta t]$, i.e. $\varphi(t) \cdot \Delta t$. By taking the limit of $\Delta t \to 0$, we get $\varphi(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \cdot f(P(t) - t)(\Delta t - \Delta P) = \lim_{\Delta t \to 0} f(P(t) - t) \cdot (1 - \frac{P(t+\Delta t) - P(t)}{\Delta t}) = f(P(t) - t)(1 - P'(t))$. So the expected capture time is $\int_0^y t \cdot \varphi(t)\, dt = \int_0^y t \cdot f(P(t) - t)(1 - P'(t))\, dt$. By symmetry we can obtain (7) in the same way. ■

The princess has chosen $x = P(0)$ and y and the monster uses m_Φ and $-m_\Phi$ with the same probability $\frac{1}{2}$, so the expected capture time is then represented by

$$\frac{1}{2}\int_0^y t \cdot [f(P(t) - t)(1 - P'(t)) + f(-P(t) - t)(1 + P'(t))]\, dt. \tag{8}$$

This is equivalent with maximizing the integral without the constant $\frac{1}{2}$ and can be simplified by partial integration:

$$\int_0^y t\cdot[f(P(t)-t)(1-P'(t))+f(-P(t)-t)(1+P'(t))]\,dt$$

$$= \left[t\int_0^y [f(P(t)-t)(1-P'(t))+f(-P(t)-t)(1+P'(t))]\right]_0^y$$

$$-\int_0^y\left(\int_0^y [f(P(t)-t)(1-P'(t))+f(-P(t)-t)(1+P'(t))]\,dt\right)$$

$$= [-t\,(\Phi(P(t)-t)+\Phi(-P(t)-t))]_0^y+\int_0^y [\Phi(P(t)-t)+\Phi(-P(t)-t)]\,dt$$

$$= -y\,(\Phi(P(y)-y)+\Phi(-P(y)-y))+\int_0^y [\Phi(P(t)-t)+\Phi(-P(t)-t)]\,dt$$

The first term results in the constant $-y\Phi(1-2y)$ by using $|P(y)|=1-y$. The second term is the integral in the following lemma.

Lemma 4.12. *Let y be the first time that the princess meets a sweeper. Then the optimal princess path from $P(0)$ to $P(y)$ maximizes*

$$\int_0^y [\Phi(P(t)-t)+\Phi(-P(t)-t)]\,dt. \tag{9}$$

The integral (9) is a variational problem. Let $\int_0^y [\Phi(P(t)-t)+\Phi(-P(t)-t)]\,dt = \int_0^y L(t,P(t),P'(t))\,dt$. Then its Euler-Lagrange equation is $\frac{\partial L}{\partial P(t)}-\frac{d}{dt}\frac{\partial L}{\partial P'(t)} = \frac{\Phi'(P(t)-t)}{P(t)}+\frac{\Phi'(-P(t)-t)}{-P(t)}-\frac{d}{dt}(0)=0$. This gives the equation $f(P(t)-t)=f(-P(t)-t)$, where $P(t)=0$ is a stationary value. If Φ is the uniform distribution, the equation is satisfied for any path of the princess, so the integral (9) is independent of P. It follows that $\Phi(x)=\frac{x+1}{2}$, so the integral becomes $\int_0^y \frac{P(t)-t+1}{2}+\frac{-P(t)-t+1}{2}\,dt = \int_0^y -t+1\,dt = \left[-\frac{1}{2}t^2+t\right]_0^y = -\frac{1}{2}y^2+y$.

The integral in (9) represents the payoff against the monster that starts in the interval $(P(0)-y, P(0)+y)$ and runs to $P(0)$. So, if the princess has met a sweeper, she should run to the end, because when the monster using $\pm m_\Phi$ meets a sweeper, he joins him and runs away from the end where the princess hides.

Lemma 4.13. *Denote $x = P(0)$. The payoff of the monster using $\pm m_\Phi$ against any princess strategy is as follows:*

$$V(\pm m_\Phi, P) = 1 - \Phi(-P(0)) + 2\Phi(1-2y) + \frac{y}{2}(1-\Phi(P(0))) - \frac{y}{2}\Phi(1-2y) + \frac{1}{2}\int_0^y [\Phi(P(t)-t) + \Phi(-P(t)-t)]\, dt \tag{10}$$

Proof. We consider all possible positions where the monster can start as well as the direction in which he moves. So, we can assume that $P \geq 0$. The first option for the monster is to start left from $x = P(0)$ and run to the left. He does this with probability $\frac{1}{2}(1-\Phi(-P(0)))$ and he catches the princess at time 2. This gives the first term $\frac{1}{2} \cdot (1-\Phi(-P(0))) \cdot 2 = (1-\Phi(-P(0)))$. If the monster that starts out left from $1-2y$ and runs to the right with probability $\frac{1}{2} \cdot \Phi(1-2y)$, he ultimately catches the princess at time 4, which gives the second term. If the monster starts right from $x = P(0)$ and then runs to the right with probability $\frac{1}{2} \cdot (1-\Phi(P(0)))$, he joins the sweeper A and catches the princess at time y, which gives the third term. If the monster behaves as he does in 4.11, we obtain the two final terms. ■

If $\Phi(x) = \frac{x+1}{2}$, then (10) is equal to $1 - \frac{-P(0)+1}{2} + 2 \cdot \frac{2-2y}{2} + \frac{y}{2} \cdot (1 - \frac{P(0)+1}{2}) - \frac{y}{2} \cdot \frac{2-2y}{2} + \frac{1}{2} \cdot (-\frac{1}{2}y^2 + y) = \frac{10+2P(0)-y(7-P(0))+y^2}{4}$. This payoff is maximal at $P(0) = 1$ for any $0 \leq y \leq 1$ and we get $V(\pm m_\Phi, P) = \frac{12-8y+y^2}{4}$. We now consider a mixed monster strategy σ in which the monster uses the sweeper strategy $\pm A$ with probability p and uses $\pm m_\Phi$ with probability $1-p$. Sweeper $\pm A$ meets the princess at time y or at time 2, so $V(\pm A, P) = \frac{1}{2} \cdot (2+y)$. Now, $V(\sigma, P) = p \cdot V(\pm A, P) + (1-p) \cdot V(\pm m_\Phi, P) = p \cdot \frac{1}{2} \cdot (2+y) + (1-p) \cdot \frac{12-8y+y^2}{4}$. If the monster takes $p = \frac{7}{9}$, then $V(\sigma, P) = \frac{13}{9} - \frac{1}{18}y + \frac{1}{18}y^2$. This is maximal at $y = 0$ or $y = 1$. For the princess, it is therefore optimal to start at an end and stay there, or to start at an end, run to the middle and then run back to the end. For $y = 0$ or $y = 1$, the payoff is $\frac{13}{9}$. This is an upper bound for the value of the game and results in the following lemma.

Lemma 4.14. *If the monster uses the mixed strategy σ, then for any princess strategy we obtain $V \leq \frac{13}{9}$.*

4.4 A princess strategy with continuous initial distribution

In this section, we improve the lower bound $V > \frac{97}{75}$. To do so, we extend the mixed princess strategy which uses E, F, G. Replace F and G with a continuous mixed strategy p_Ψ. In p_Ψ, the princess picks a point $x \in [\epsilon, 1-\epsilon]$ according to a continuous distribution $\Psi(x)$ and waits in x until sweeper A is ϵ close. She then runs to ϵ, where she turns back to run back to 1. $-p_\Psi$ is the symmetric counterpart of p_Ψ where the princess starts in $[-1+\epsilon, -\epsilon]$.

Lemma 4.5 suggests that the monster M either starts at an end (i.e. M is a sweeper), or starts in $[-1+\epsilon, -\epsilon] \cup [\epsilon, 1-\epsilon]$. The first time that M gets ϵ close to a sweeper is denoted by y. By symmetry, we can assume that this sweeper is A. Sweeper A runs all the time, so for $t < y$, we know that $M(t) < A(t)$. Then M approaches A from the left, so until time y, he catches immobile princesses that start in $x = P(0) > M(0)$ and that wait for A or that are running ϵ in front of A. The monster wants to catch as many immobile princesses as possible and he therefore seeks to maximize the interval $[M(0), M(y)]$. So, M starts in $1-2y-\epsilon$ and runs to $1-y-\epsilon$. At time y, the monster now has two possibilities: he can continue and run to 1 to catch the princess E and then run back to -1 to catch all other princesses (strategy M_1), or he can turn and run to -1 to catch the princess with $x = P(0) < M(0)$ and turn back in -1 to return to 1 (strategy M_2). Against the end point strategies, M_1 catches princess E at time $2y$ and princess $-E$ at time $2y+2$, which gives the expected payoff $V(\pm M_1, \pm E) = 1 + 2y$. $V(\pm M_2, \pm E) = 3$ since M_2 catches $-E$ at time 2 and E at time 4. Now, we have the following lemma, where $\psi = \Psi'$ is the probability density of Ψ.

Lemma 4.15. *If we ignore ϵ, then the expected payoff for $\pm M_1$ against $\pm p_\Psi$*

is:

$$V(\pm M_1, \pm p_\Psi) = (1+y)(1-\Psi(2y-1)) + \frac{1}{2}\int\limits_0^{2y-1}(-t-1+2y)\cdot\psi(t)\,dt$$
$$+\frac{1}{2}(1+y)\Psi(1-2y) + \frac{1}{2}\int\limits_{1-2y}^{1-y}(t-1+2y)\cdot\psi(t)\,dt + \frac{1}{2}y(1-\Psi(1-y)) \tag{11}$$

If we again ignore ϵ, then the expected payoff for $\pm M_2$ against $\pm p_\Psi$ is:

$$V(\pm M_2, \pm p_\Psi) = 1-\Psi(2y-1) + \frac{1}{2}\int\limits_0^{2y-1}(-t-1+2y)\cdot\psi(t)\,dt$$
$$+\frac{1}{2}\Psi(1-2y) + \frac{1}{2}\int\limits_{1-2y}^{1-y}(t-1+2y)\cdot\psi(t)\,dt + \frac{1}{2}y(1-\Psi(1-y)) \tag{12}$$

Proof. Because of symmetry, we only have to consider M_1 and M_2 against $\pm p_\Psi$. First, we look at the payoff in (11). The first two terms concern princesses who start out from $x < 0$. In the first case, the princess starts to the left of M_1, so she starts in the interval $[-1, 1-2y)$. The princess picks her strategy with probability $\frac{1}{2}$ and her initial position x with probability $\Psi(-x)$, so the total probability is $\frac{1}{2}(1-\Psi(2y-1))$. M_1 finds this princess at time $2+2y$, so this gives the first term $(2+2y)\cdot\frac{1}{2}(1-\Psi(2y-1)) = (1+y)(1-\Psi(2y-1))$. The probability of a princess who starts in $x \in [1-2y, 0)$ is $\frac{1}{2}\int\limits_{1-2y}^{0}\psi(-t)\,dt = \frac{1}{2}\int\limits_0^{2y-1}\psi(t)\,dt$. The princess waits for $-A$, so she gets caught before she meets $-A$ by M_1 at time $t(x) = -x-(1-2y)$ for all $y \in [\frac{1}{2}, 1]$. This gives the second term $\frac{1}{2}\int\limits_0^{2y-1}(-t-1+2y)\cdot\psi(t)\,dt$. The other three terms concern princesses who start out from $x > 0$. In the third case, the princess begins in $x \in [0, 1-2y)$ with probability $\frac{1}{2}\Psi(1-2y)$. The monster starts in $1-2y$, runs to 1 and needs time $2y$ to do this while the princess stays immobile in x. When the princess almost meets A, she runs to the middle and the monster is distance $1-2y$ behind her. Thus, when she reaches the middle, time $t = 1$ has passed by, and the monster is $2y$ away from her. Now, the princess and the monster meet halfway, which is the half of $2y$. So the capture time is $1+y$. This gives the third term $\frac{1}{2}(1+y)\Psi(1-2y)$. The probability that the princess starts in $x \in [1-2y, 1-y)$ is $\frac{1}{2}\int\limits_{1-2y}^{1-y} \psi(t)\, dt$. The princess waits for A, which is similar to the second case, so the monster catches her at time $t(x) = x-(1-2y)$ for all $y \in [0, \frac{1}{2}]$. This gives the fourth term $\frac{1}{2}\int\limits_{1-2y}^{1-y} (t-1+2y)\cdot\psi(t)\, dt$. In the last case, the princess starts right from $1-y$ with probability $\frac{1}{2}(1-\Psi(1-y))$. The monster catches this princess at time y which gives the final term $\frac{1}{2}y(1-\Psi(1-y))$. For the payoff in (12), there are have two differences. In the first case, M_2 catches the princess at time 2 instead of time $2+2y$ because he joins A and therefore, in the third case, he catches the princess at time 1 instead of time $1+y$. ■

Now let Ψ again be the uniform distribution. We then consider the mixed strategy γ in which the princess uses $\pm E, \pm p_\Psi$ where Ψ is the uniform distribution. We know that $V(\pm M_2, \pm E)$ does not depend on y, so it is optimal for the monster to pick a y such that the payoff $V(\pm M_2, \pm p_\Psi)$ is minimal.

Lemma 4.16. *Let the princess use the mixed strategy γ. Then, the best response of the monster gives a matrix game with value $\frac{15}{11}$. We conclude that $V \geq \frac{15}{11}$.*

Proof. Since Ψ is the uniform distribution, we have $\Psi(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \leq x \leq 1 \\ 1 & \text{for } x > 1 \end{cases}$ and $\psi(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$. If we put this into equations (11) and (12) we obtain expected payoffs $T_1(y) = (1+y)+0+\frac{1}{2}(1+y)(1-2y)+\frac{1}{2}\int\limits_{1-2y}^{1-y}(t-1+2y)\, dt+\frac{1}{2}y^2$ and $T_2(y) = 1+0+\frac{1}{2}(1-2y)+\frac{1}{2}\int\limits_{1-2y}^{1-y}(t-1+2y)\, dt+\frac{1}{2}y^2$ for $0 \leq y < \frac{1}{2}$ and $T_1(y) = (1+y)(2-2y)+\frac{1}{2}\int\limits_{0}^{2y-1}(-t-1+2y)\, dt+0+\frac{1}{2}\int\limits_{0}^{1-y}(t-1+2y)\, dt+\frac{1}{2}y^2$ and $T_2(y) = (2-2y)+\frac{1}{2}\int\limits_{0}^{2y-1}(-t-1+2y)\, dt+0+\frac{1}{2}\int\limits_{0}^{1-y}(t-1+2y)\, dt+\frac{1}{2}y^2$ for $\frac{1}{2} \leq y \leq 1$. This results in the following:

$$T_1(y) = \begin{cases} -\frac{1}{4}y^2 + \frac{1}{2}y + \frac{3}{2} & \text{for } 0 \le y < \frac{1}{2} \\ -\frac{5}{4}y^2 + 2 & \text{for } \frac{1}{2} \le y \le 1 \end{cases} \text{ and } T_2(y) = \begin{cases} \frac{3}{4}y^2 - y + \frac{3}{2} & \text{for } 0 \le y < \frac{1}{2} \\ \frac{3}{4}y^2 - 2y + 2 & \text{for } \frac{1}{2} \le y \le 1 \end{cases}.$$

Since $V(\pm M_2, \pm E)$ does not depend on y, it is optimal for the monster to pick a y with $0 \le u \le 1$ such that $T_2(y)$ is minimal. The minimum is at $y = 1$, so $T_2(1) = \frac{3}{4}$. If the monster only uses the mixed strategy $\pm A, \pm M_2$,

we get the game matrix
$$\begin{array}{c} \\ \pm A \\ \pm M_2 \end{array} \begin{array}{c} \pm E \quad \pm p_\Psi \\ \begin{pmatrix} 1 & \frac{3}{2} \\ 2 & \frac{3}{4} \end{pmatrix} \end{array}.$$
This matrix has value $\frac{15}{11}$, where the princess uses $\pm E$ with probability $\frac{3}{11}$ and $\pm p_\Psi$ with probability $\frac{8}{11}$. The monster cannot improve by including the strategy $\pm M_1$ because $\frac{3}{11}V(\pm M_1, \pm E) + \frac{8}{11}V(\pm M_1, \pm p_\Psi) = \frac{3}{11} \cdot (1 + 2y) + \frac{8}{11}T_1(y)$ is minimal for $y = 0$ and $y = 1$. Therefore, $\frac{3}{11} \cdot (1+2y) + \frac{8}{11}T_1(y) \ge \frac{15}{11}$. So, if the princess uses the mixed strategy γ, the monster cannot do better than expected capture time $\frac{15}{11}$, which makes $\frac{15}{11}$ a lower bound on V. ■

5 Conclusion

In this thesis we handled the princess and monster game in which we present the best way for a monster to search for a mobile princess who is restricted to the circumference of a circle and to the interval $[-1, 1]$. The value of this game exists, which follows from a minimax theorem by Alpern and Gal. For the circumference of a circle, the value of the game is $\frac{3}{2}\pi r$. The optimal strategies belonging to this value are as follows: start randomly anywhere on the circle; move with probability $\frac{1}{2}$ to the antipodal point halfway around the circle clockwise or counterclockwise with maximal speed 1.
For the interval $[-1, 1]$, it seems rather difficult to determine the value of the game. We have established many properties of optimal princess and monster paths; for example, the princess should never go faster than speed 1 despite being unrestricted in speed. By presenting some simple strategies, we have obtained the bounds $1 < \frac{97}{75} < V < \frac{47}{32} < \frac{3}{2}$ on the value of the game. Finally, we have improved these bounds by using mixed strategies which start according to a continuous distribution, and we subsequently obtained $\frac{15}{11} \leq V \leq \frac{13}{9}$.

References

[1] S. Alpern (1974), *The search game with mobile hider on the circle*, In Differential Games and Control Theory (E. O Roxin, P. T. Liu, and R. L. Sternberg, eds, Marcel Dekker, New York, pp 181-200.

[2] S. Alpern and M. Asic(1985), *The search value of a network*, Networks, 15, pp 229-238.

[3] S. Alpern, S. Gal (1998), *A mixed strategy minimax theorem without compactness*, SIAM J. Control Optim. 26, pp1357–1361

[4] S. Alpern, R. Fokkink, R. Lindelauf, G.J. Olsder (2008), *The Princess and Monster Game on an Interval*, S.I.A.M. J. Control Optim. 47, No 3, pp 1178-1190.

[5] S. Gal (1980), *Search Games*. Academic Press, New York

[6] R. Isaacs (1965), *Differential Games*, John Wiley, New York.

[7] M. I. Zeliken (1972), *On a differential game with incomplete information*, Soviet Math. Doklady 13, pp 228-231.