Utrecht University

A Mathematical Background to Cubic and Quartic Schilling Models

Author:
I.F.M.M.Nelen

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Supervisor: Prof. Dr. F. Beukers
Second Reader: Prof. Dr. C.F. Faber
Masters Programme: Mathematical Sciences
Department: Departement of Mathematics
University: Utrecht University

Abstract

At the end of the twentieth century plaster models of algebraic surface were constructed by the company of Schilling. Many universities have some series of these models but a rigorous mathematical background to the theory is most often not given. In this thesis a mathematical background is given for the cubic surfaces and quartic ruled surfaces on which two series of Schilling models are based, series VII and XIII.

The background consists of the classification of all complex cubic surface through the number and type of singularities lying on the surface. The real cubic surfaces are classified by which of the singularities are real and the number and configuration of the lines lying on the cubic surface. The ruled cubic and quartic surfaces all have a singular curve lying on them and they are classified by the degree of this curve.

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Chapter 1

Introduction

Many universities which offer a study in mathematics have some collection of mathematical models. These models are often on display at their department of mathematics or in the university museum. This is the case at the university of Utrecht as well. At Utrecht in the mathematical library there is a collection of mathematical models on display, ranging from curves and surfaces in plaster to thread models of ruled surfaces. At every model there is a little card with its description but there is not much of a mathematical background. This thesis will give some of that mathematical background for a group of models, particularly the cubic surfaces and the quartic ruled surfaces.
The first question which arises when looking at the models of the cubic surfaces is: why are these models chosen to illustrate cubic surfaces? Furthermore one can ask if there are more and how many. This is one of the main questions in this thesis and will be anwsered by classifying all cubic surfaces. The same will be done for every ruled surface of degree three or four.

Outline

First a history of mathematical models will be given. This will run from the start of projective and algebraic geometry until the modern collections. The theory of projective spaces, is heavily used in the creation of the models. The surface classification could not have been done without this. The line through history lies from perspective geometry to algebraic geometry and eventually to Klein, Brill and Schilling who made the collection of models.
Before a full classification of the cubic surfaces, and of the models as well, can be made, some theory is needed. This will be done in the chapter of preliminaries. In this chapter the mathematics needed for the classification is briefly summarized. From projective space to algebraic varieties and singularities, these are all briefly treated in that chapter.
The classification of the cubic surfaces is done in chapter 4. The smooth surfaces are first treated with the theorem of 27 lines on a smooth cubic surfaces as a start. The classical proof is a long calculation which underlines the techniques

used in classical algebraic geometry. A short proof with modern mathematics is given as well. To fully classify the different smooth cubic surfaces the 27 lines will be split between those which are defined over the reals and those which are not.
The classification of cubic surfaces with a singularity over an algebraically closed field is done by tranformations. When a singularity is isolated and occurs in the origin it has its normal form without linear terms. Every such singularity in the origin has its own normal form and determining which normal form corresponds to the euqation classifies the singularity. The result of this will help determine all possibilities of cubic surfaces as real mathematical models.
The difference between cubic surfaces over the reals, and thus the different models, is a refinement of the classification over the complex field. The normal forms of the isolated singularities will differ over the reals only in the units in frond of the terms. Limiting the options of cubic surfaces more untill at the end of chapter 4 there will be a complete list of all possible cubic surfaces and the models which they correspond too.
In chapter 5 the ruled surfaces of degree three and four are treated. Although there are not many different cubic ruled surfaces, the theory needed for the classification is non-trivial. The ruled surfaces of order four are more diverse and will be treated by a construction argument.
All results will be accompanied by pictures of the surfaces constructed as mathematical models by Schilling or surfaces created to illustrate the theory.

Chapter 2

History of Mathematical Models

In this chapter the history of mathematical models is treated. In the first half of this chapter the focus lies on the history of algebraic geometry, this eventually led to the need and creation of mathematical models. The second half of this chapter treats mathematical models in the Netherlands, specifically at Utrecht University. The subsection titled "Perspective Geometry" is based on Burtons "History of Mathematics" [2] the second part titled "Analytic Geometry and the birth of Algebraic Geometry" is based on parts of the "History of Mathematics" by Merzbach and Boyer [12].

2.1 History of Algebraic Geometry and Mathematical Models

The history of surface models starts at the birth of relatively modern branch of mathematics: algebraic geometry. The modern algebraic geometry first appeared in the 19th century with its basis lying in the projective geometry of the beginning of the 19th century. The basis of projective geometry comes from perspective geometry.

Perspective Geometry

Arround the year 1600 the french architect en engineer Girard Desargues (1593-1662) studied the subject of perspective. It was based on the Renaissance painters who tried to represent the three dimensional world, as accurately as possible, on a two-dimensional canvas. Desargues studied the geometric objects which remain unaltered under this projection, this treatise in *"Brouillon project d'une atteinte aux événemens des rencontres d'un cone avec un plan.* The difference between Desargues and the mathematicians before was the extension

of the Euclidean plane by infinitely distant points. In such an infinitely distant point two parallel straight lines would intersect, a direct development of the new technique of perspective used by painters. Because there are infinitely many parallel pairs of lines, in different directions, there will be infinitely many points at infinity. These infinitely many points will be considered to lie on a line, the line at infinity. The normal plane together with the line at infinity is called the projective plane.
Much of the special cases of Euclidean geometry, dealing with parallels, would be eliminated in projective geometry. For example with the notion of a line at infinity, any two lines will intersect in one and only one point.
Desargues was not a professional mathematician, therefore most of his mathematical papers were almost unreadable for mathematicians at the time. Through this unfortunate barrier most of his work ideas were not picked up on by mathematicians in the 17th century. Some of its works would eventually find its way into french universities but it would take arround 2 centuries before the theory of projective geometry would be picked up again.
This was done by another Frenchman namely Jean Victor Poncelet (1788-1867). He was captured by the russians after the retreat of Moscow in 1812. In prison he reconstructed the analytic geometry he learned at the École Polytechnique without books. He presented this, together with his own material, to his fellow prisoners.
Poncelet made, in his own material, extensive use of the then controversial use of geometric continuity. The principle of maintaining geomteric properties as one figure is transformed, by projection or distortion, into another. Although he got to some advanced results the principle was not well defined. As Cauchy indicated it was "capable of leading to manifest errors". Through his work Poncelet layed the groundwork for modern geometry. [2]

Analytic geometry and the birth of Algebraic Geometry

The first specialist of this modern geometry was Julius Plücker (1801-1868). Plücker believed the way to study geometry analytically was through use of algebraic methods. In contrast to the purely geometric way of Poncelet and Steiner. Plückers name survives in coordinate geometry by the Plücker coordinates, these coordinates give a one-to-one correspondance between the lines in $\mathbb{P}^3$ and the points on a quadric in $\mathbb{P}^5$. As well as the coordinate system which was named after him, Plücker rediscovered the homogeneous coordinate system, in Analytisch-geometrische Entwicklungen. This coordinate system, discovered by Möbius (1790-1860) and Bobillier (1798-1840), tied down the infinite elements of Desargues and Poncelet. This made is easier to work with the geometric objects in space, every object would now be given by a homogeneous polynomial. Plücker did not take advantage of developments in determinants and this could be a reason why he never systematically developed analytical geometry of more than three dimensions.
Meanwhile in 1843 Arthur Cayley (1821-1895) initiated the ordinary analytic geometry of n-dimensional space, using determinants. In his time Cayley was in

contact with the irish mathematician George Salmon (1819-1904). In a correspondence Cayley send Salmon a letter in which he detailed that smooth cubic surfaces have a finite number of line, on which Salmon replied that it should be exactly 27. The theory of the 27 lines on a smooth cubic surface is today known as the Cayley-Salmon theorem. [12]

Klein, Brill and Schilling

After the first results of Cayley and Salmon there was interest for the models of these results. Not only the cubic surfaces but for other degree surfaces as well. Two professors at the Polytechnic School in Münich, Felix Klein and Alexander Brill teamed up to constructed models to make the visualization of geometric objects easier. Klein believed strongly in the use of mathematical models to illustrate his lectures. The attention for these models was wider than mathematical. For example, Klein took an exhibition of "German mathematical models" to the world fair in Chicago in 1893.[20]
Klein was a student of Plücker from whom he got his enthousiasm for line geometry and geometric models. Plücker in turn got his from Faraday who used models to understand the mathematics he needed for his work. Klein saw his first model, the model of 27 lines on a smooth cubic surface, in 1868 in Bergstrasse. This incited Klein to pursue the modelling of surfaces himself and with the help of Brill he was proud to say at the end of his life that "no German university was without a proper collection of mathematical models".[21]
In the 1870s Ludwig Brill began to research, construct and sell some mathematical models. He founded a company in 1880 in Darmstadt. In 1899 it was taken over by Marten Schilling who renamed it and eventually moved the company to Leipzig. By 1904 Schilling had produced over 23 series of models, in that same year his company published a book containing some mathematical background of the models excisting at the time.
In 1911 the catalog of Schilling [17] contained 40 series of models. These 40 series containg about 400 different models and most mathematical models found in exhibits today are based on the models or are the models created by Schillings company. The catalog of Schilling does not only contain plaster models but thread models as well offering a wide range of mathematical models with a small bit of mathematical background.

2.2 Collections of Mathematical Models in the Netherlands

More than half of the universities in the Netherlands have a collection of mathematical models. The biggest of which is the collection of the University of Amsterdam.

Collection at the University of Amsterdam The university of Amsterdam has about 180 models arround the year 2006. Most of which are made by the company of Brill and Schilling. These models are restorded and are found in the university museum.[15]

Collection at the Rijksuniversiteit Groningen The rijksuniversiteit Groningen has about 130 different models most of Schilling and Brill. Catalogued on the internet by Drs. Irene Polo-Blanco and Lotte van der Zalm they can be found with a short mathematical background http://www.math.rug.nl/models/.

Collection at the University Leiden The university of Leiden has about 100 models which are mostely in good state. They can be found in the library and a room in the Mathematical Institute. The literature, catalogue or reference to the collection of models is lacking.[15]

Collection at Technical University of Delft The Technical University of Delft has around 70 mathematical models as of 2016 most of which are thread models. As of 2016 they are in the process of linking the models to the Schilling catalogue.

Collection at Utrecht University The university of Utrecht has about 90 different models, of Schilling, restored and catalogued. Part of the collection can be seen in the mathematical library in Utrecht. It consists of models from 22 different Schilling model series. Most are in good condition but there are some with missing parts.

One might wonder how the Dutch universities got these models, and why they bought them. There is no advertisment for the models, at the time, in dutch mathematical magazines. At the end of the 19th century, there where advertisements in the American Journal of Mathematics for the models of Klein and Schilling. The models where advertised for the purpose of higher education. Although it looks like most models where bought to support education on unviversities, there are not many sources which support this. [15]

2.2.1 Some Series of Models at Utrecht University

Utrecht university has done a renewed effort to classify and restore all its models. In 2015 most of the models at the Utrecht University lay in the vault in the mathematical building and where on display in the University museum. In 2015 the next step was matching the models to their respective Schilling catalogue number and restoring the models to their former form. The restauration has been done by Anite Koster and Jan Willem Pette.
In 2017 many mathematical models of Utrecht University are on display at the mathematical library. The collection in the library contains most of series VII of Schilling which is the collection of cubic surfaces in plaster and the main focus of this thesis.

Chapter 3

Preliminaries

3.1 Projective space

As been described in the historical notes perspective is a very important tool to describe which surfaces look different but are equivallent. It was told that this was done by adding points at infinity. This will be formalized here.
Start with the following definition of projective n-space.

Definition 3.1. *For $n \in \mathbb{Z}_{\geq 0}$ define the projective n-space, $\mathbb{P}^n$, as the quotient $k^{n+1} - \{0\}$ by the equivalence relation $\sim$. Here $a \sim b$ if $\exists \lambda \in k^{n+1},\ \lambda \neq 0$ such that $a = \lambda b$.*

Remark that the projective n-space can be seen as the set of different lines through the origin in affine $n+1$ space or as affine n-space with $n-1$ space at infinity. Every point in projective n-space can be written as $(x_1 : ... : x_{n+1})$ where colons are used to denote ratios. This gives an easy embedding of $\mathbb{A}^n$ into $\mathbb{P}^n$.

$$\phi : \mathbb{A}^n \to \mathbb{P}^n \tag{3.1}$$

$$\phi(x_1, ..., x_n) = (x_1 : ... : x_n : 1) \tag{3.2}$$

To find which surfaces are equivalent in projective space, the notion of projective equivalence is needed. Two objects in projective space are equivalent if one can be changed into the other by a projective transformation. Projective transformations will be the tool the calculate equivalent surfaces. For $n \in \mathbb{Z}_{\geq 0}$ the group of invertible $n+1 \times n+1$ matrices with coefficients in a field k is denoted by $GL_{n+1}(k)$. This is the automorphism group of k^{n+1}.

Definition 3.2. *A projective transformation in $\mathbb{P}^n(k)$ is an element of the quotient group $PGL_{n+1}(k) := GL_{n+1}(k)/k^{\times}$.*

Thus an invertible $(n+1) \times (n+1)$ matrix without scaling.

3.2 Algebraic Surfaces

3.2.1 Algebraic Varieties

In the theory of algebraic geometry objects defined by polynomial equations are studied. Throughout this section assume that k is an algebraically closed field, mostly $\mathbb{C}$, and $k[x_1, ..., x_n]$ is the polynomial ring in the variables $x_1, ..., x_n$. To understand the objects in this thesis the following general definition is needed.

Definition 3.3. *For a subset of polynomials $S \subset k[x_1, ..., x_n]$ the zero set or locus, $\mathcal{L}(S) \subset \mathbb{A}^n$, is defined as*

$$\mathcal{L}(S) = \{P = (a_1, ..., a_n) \in \mathbb{A}^n | f(P) = 0 \ \forall f \in S\} \tag{3.3}$$

Affine Varieties

An affine algebraic variety is a subset of $\mathbb{A}^n$ which can be given as the locus of a subset of polynomials. An affine algebraic curve or affine algebraic surface, are an algebraic variety of dimension 1 respectively 2. In $\mathbb{A}^3$ this means, because the dimension drops by 1 for every condition, that an algebraic surface is given by the zero set of a single polynomial and an algebraic curve is usually given by a set of 2 polynomials. In this thesis all algebaric curves can be given by a set of 2 polynomials. An example of an algebraic surface is the cone. One way a cone ican be defined is as the zero set of the following polynomial

$$f(x, y, z) = x^2 + y^2 - z^2 \tag{3.4}$$

Every irreducible surface is the zero set of an irreducible polynomial. If the polynomial would be reducible the corresponding algebraic surface will be reducible as well. For example the locus of the polynomial $xz - (x + z)y + y^2$ in $\mathbb{A}^3$ will give two intersecting planes. In $\mathbb{A}^3$ an irreducible algebraic surface can be given by a single irreducible polynomial, this polynomial will be called the defining polynomial of the surface.

Projective Varieties

In the same way as the affine varieties the projective varieties can be defined, only the polynomial set should be a set of homogeneous polynomials in $k[x_1, ..., x_n]$. In $\mathbb{P}^2$ lines are given by the zeroset of a homogeneous polynomial of degree 1. The general form of such a line in $\mathbb{P}^2$ is given by $\{aX + bY + cZ = 0 | (x : y : z) \in \mathbb{P}^2\}$.

Theorem 3.1 (Bezout)**.** *Let $f_1, f_2 \in k[x, y, z]$ be homogeneous polynomials of degree d_1 and d_2 with k a closed field. Assume $Z(f_1)$ and $Z(f_2)$ do not share irreducible components, then $\#\{Z(f_1) \cap Z(f_2)\} = d_1 d_2$ when counted with multiplicity.*

This theorem is proven in most textbooks for algebraic geometry for example in [7].

Generally a projective variety $V_1 \subset \mathbb{P}^n$ is equivalent to a projective variety $V_2 \in \mathbb{P}^n$, with respective defining polynomials f_1, f_2, if there exist a projective transformation $g \in PGL_n(k)$ such that $f_1 \circ g = f_2$.

3.2.2 Singularities

In the studies of surfaces, singularities take a prominent place. A singularity can be seen as the exception of smoothness at a point or curve on a surface. The example of the cone in subsection 3.2.1 has a singular point at $(0, 0, 0)$. This is the point where the tangent plane is not defined. To fully define a singularity first the definition of the tangent plane is needed. This is the plane which lies tangent to the surface at a point P.

Definition 3.4. *Let V be a surface in $\mathbb{P}^3$ given by a polynomial in $F \in k[X : Y : Z : T]$. The tangent plane of V at the point P is called $T_P V$ and is given by*

$$\frac{\partial F}{\partial X} \cdot X + \frac{\partial F}{\partial Y} \cdot Y + \frac{\partial F}{\partial Z} \cdot Z + \frac{\partial F}{\partial T} \cdot T = 0 \tag{3.5}$$

When the tangent plane is defined the points is called smooth and when it is not defined then the point is singular. It is easy to see that the tangent plane will not be defined if all partial derivatives are zero. This gives the following definition for a singular point.

Definition 3.5. *Let $V \subset \mathbb{P}^3$ be a surface given by a polynomial $F(X : Y : Z : T) = 0$ then a point $P \in V$ is called singular iff $\frac{\partial F}{\partial X}(P) = \frac{\partial F}{\partial Y}(P) = \frac{\partial F}{\partial Z}(P) = \frac{\partial F}{\partial T}(P) = 0$.*

Not all singularities have the same properties. A classification of the properties has extensively been done by V.I. Arnold in the sixties and seventies [1]. The simpelest of the singularities are related to the Lie, Coxeter and Weyl groups A_k, D_k and E_k. The first step of Arnold was remarking that every function with a non-degenerate critical point has a neighbourhood in which the defining function can be represented in the Morse normal form:

$$f = \pm x_1^2 \pm ... \pm x_n^2 \tag{3.6}$$

This is called "the Morsification of the surface". Every degenerate normal form can, by a deformation, be made into a non-degenerate normal form. Two critical points are equivalent if, by a local diffeomorphism, it is possible to transform one critical point into the other. He found that every smooth function with a critical point can be observed locally. Because two critical points, of finite Milnor Number, are equivalent if their Taylor polynomials up to high enough order are equivalent. Thus the classification of singularities, with finite Milnor Number, is reduced to a sequence of algebraic problems. These problems deal with finite actions of Lie groups. The Milnor number can be seen as the number of non-degenerate points in which the singularity deforms by morsification and this will be treated in subsection 3.2.4. The first observed and classified of these singularities are called the simple or DuVal singularities and are shown in table 3.1.

3.2.3 Simple Singularities

The least intricate of Arnold's simple singularities is the ordinary double point. This double point occurs at the point $P = (0, 0, 1)$ on the quadratic cone X given by equation 3.4. The quadratic cone is the union of generating lines all meeting in the singular point P. When the "blow up" of X at P is taken the surface becomes a cylinder whose underlying set is the disjoint union of lines of X. The point P is thus blown up to a circle on the cylinder which is called the exceptional divisor of the blow up. This exceptional divisor is equivalent to $\mathbb{P}^1$. The surface X when blown up at P is non-singular, thus the singularity at P is resolved. Most isolated singularities will not be resolved after one blow up and will sometimes take more steps.

Blowing up singularities

The technique of blowing up singularities is done by specifying the ratio, it will add to every point on the surface it's ratio. The blow up of the A_1 and D_4 singularity are computed in a blog by Lewallen [11] and will be given here. Start with the blowing up of the singularity on the double cone. The blow up of the double cone X, given by $xy = z^2$, is the subset

$$\{((x_1, x_2, x_3), (a_1 : a_2 : a_3)) \in \mathbb{A}^3 \times \mathbb{P}^2 | (x, y, z) \in X, x_i a_j = x_j a_i\} \tag{3.7}$$

When for the points in the ratio the restriction $a_3 = 1$ is made, thus restricting the space $\mathbb{P}^2$ to an affine chart $(a_1 : a_2 : 1)$. Combining the restrictions given, and when $a_3 = 1$, gives the equation of the blow up on the affine chart $z^2(a_1 - a_2^2) = 0$. Thus it consists of two irreducible pieces: $z^2 = 0$ and $(a_1 - a_2^2) = 0$. Here z^2 is a copy of $\mathbb{P}^2$ lying above 0, and $(a_1 - a_2^2)$ is the exceptional divisor, which is smooth and is a copy of $\mathbb{P}^1$. Similarly on the charts $a_1 = 1$ and $a_2 = 1$ the exceptional divisor is smooth. The singularity is now resolved in one step with one exceptional divisor.

It is not always the case that a singularity is resolved after one blow up, sometimes the process needs to be repeated multiple times and the exceptional divisors will intersect. The way the exceptional divisors intersect is one way to classify them. This is seen in table 3.1, where the resolution graph shows the number of exceptional divisors as circles and lines between them if they intersect. This will be illustrated by the blow up of D_4.

The blow up of the normal form of a D_4 singularity X, given by $x^2+y^2z+z^3 = 0$, is the subset

$$\{((x_1, x_2, x_3), (a_1 : a_2 : a_3)) \in \mathbb{A}^3 \times \mathbb{P}^2 | (x, y, z) \in X, x_i a_j = x_j a_i\} \tag{3.8}$$

On the chart when $a_2 = 1$ and combining the restrictions gives the equation on the affine chart $y^2(a_1^2 + a_3 y + a_3^3 y^3) = 0$ the piece $y^2 = 0$ is the exceptional divisor. The other irreducible piece has exactly 3 singular points. Namely the point $(0, 0, 0), (0, 0, i)$ and $(0, 0, -i)$. Observe that these three singular points

Name:	Normal form:	Resolution Diagram
$A_k \quad k \geq 1$	$x_1^{k+1} + x_2^2 + x_3^2$	○−○····○
$D_k \quad k \geq 4$	$x_1^{k-1} + x_1 x_2^2 + x_3^2$	○−○−○····○ ○
E_6	$x_1^4 + x_2^3 + x_3^2$	○−○−○−○−○ ○
E_7	$x_1^3 x_2 + x_2^3 + x_3^2$	○−○−○−○−○−○ ○
E_8	$x_1^5 + x_2^3 + x_3^2$	○−○−○−○−○−○−○ ○

Table 3.1: Normal Form and Dynkin Diagram of DuVal Singularities

are A_1 singularities thus can be blown up by the principle above. This will give three extra exceptional divisors intersection the first.
On the chart where $a_1 = 1$ the affine equation is $x^2(1+a_2^2 a_3 x + a_3^3 x^3) = 0$ which has no singularities.
When $a_3 = 1$ the surface has equation $z^2(a_1^2 + a_2^2 z + z) = 0$ and 2 singularities namely $(0, i, 0)$ and $(0, -i, 0)$ but these correspond to the singularities $(0, 0, i)$ and $(0, 0, -i)$ on the affine chart where $a_2 = 1$. So after these four blow ups the singularity is completely resolved and the exceptional divisors will intersect as in the resolution diagram of 3.1.

3.2.4 Milnor Number

The Milnor number can be defined for all singularities in a n dimensional space and may be viewed as "the number of points infinitesimally glued to form the singularity". Which is the same as the number of S^2 curves of the deformation of the singularity. The following definition gives the easiest way to calculate the Milnor number for simple singularities. This will be done here for some simple singularities occuring in $\mathbb{P}^3$. For more information see Milnor [13].

Definition 3.6. *Let V be a surface in $\mathbb{C}^3$ and let V have an isolated singularity at $P = (0,0,0)$. If $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ the local normal form of the singularity at P then the Milnor number μ of the singularity is*

$$\mu(V, 0) = dim_{\mathbb{C}} \left(\mathbb{C}[[x, y, z]]/\nabla(f)\right) \tag{3.9}$$

Where $\nabla(f) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$, the ideal generated by the partial derivatives.

The ideal $\nabla(f)$ is called the Jacobi ideal. The Milnor number will now be calculated for the simple singularities A_3 and D_5.
Milnor number of A_3. The normal form of the A_3 type singularity is $f(x, y, z) = x^4+y^2+z^2$ in $\mathbb{C}^3$ then $\nabla(f) = (4x^3, 2y, 2z)$. The quotient field $\mathbb{C}[x, y, z]/(4x^3, 2y, 2z)$

is generated, as a vector space over $\mathbb{C}$, by $(1, x, x^2)$. The dimension of the quotient field is 3 and the Milnor number of A_3, $\mu(A_3) = 3$

Milnor number of D_5. The normal form of the D_5 types singularity is $f(x, y, z) = x^4 + x^2 y + z^2$ in $\mathbb{C}^3$ and in this case $\nabla(f) = (4x^3 + y^2, 2xy, 2z)$. The quotient field $\mathbb{C}/\nabla(f)$ is generated as a vector space by $(1, x, y, x^2, y^2)$ and the Milnor number of D_5 is 5.

All other simple singularities can be done in the same way and the first thing which stands out is that the milnor number μ is the same as the subscript of the singularity.

Chapter 4

Cubic Surfaces

In this chapter the cubic surfaces in $\mathbb{P}^3(\mathbb{C})$ and $\mathbb{P}^3(\mathbb{R})$ are classified. A short introduction of cubic surfaces is given, after which the classification in $\mathbb{P}^3(\mathbb{C})$ will occur. The classification starts with the smooth cubic surfaces and continues with the cubic surfaces which have only isolated singularities. The cubic surfaces with non isolated singularities are treated in chapter 5. The last part of this chapter will be dedicated to cubic surfaces in $\mathbb{R}^3$.

4.1 Smooth Cubic Surface

A cubic surface in $\mathbb{P}^3$ is defined as the zeroset V of a homogeneous polynomial F of degree three in $\mathbb{P}^3$.

$$V = \{(X : Y : Z : T) \in \mathbb{P}^3 | F(X : Y : Z : T) = 0\} \tag{4.1}$$

When a cubic surface has no singularities we call it smooth. This means there is no point $P = (X : Y : Z : T)$ for which all first order partial derivatives are zero at P. The following theorem was first proved by a collaboration of Cayley and Salmon in 1849 [3], which determined that there are finitely many lines on a smooth cubic surface.

Theorem 4.1. *Let k be an algebraically closed field and V a smooth cubic surface in $\mathbb{P}^3(k)$, then V contains precisely 27 lines.*

There is no modern algebraic geometry needed to prove theorem 4.1. A sketch of the proof will be given below, for the full classical proof see Reid [16]. The next part will outline the proof given by Reid with the full calculations omitted.

1. *At most three lines of V go through a point P, and every intersection between a plane and V will not give a multiple line.* If $l \subset V$ a line on V through P then the tangent plane at P, called $T_P V$, will contain l. Because V is a cubic $T_P V \cap V$ will consist of at most 3 lines. Giving a

maximum of three lines through a point P. If the intersection of a plane Π and V would give a double line the surface would be singular.

2. *There exists at least one line l on V.* Look at an arbitary point P on V and take the intersection of V with the tangent plane at P named $T_P V$. This gives a curve $C = T_P V \cap V$ with a nodal or cuspidal singularity at P when C is irreducible, or C is reducible and contains a line lying on V. There exist a linear change of coordinates such that $P = (0 : 0 : 1 : 0)$ and $T_P V = (T = 0)$. Now $C := XYZ = X^3 + Y^3$ if P is nodal or $C := X^2 Z = Y^3$ if P is cuspidal. Both cases are similar, but assume we work with the cuspidal case. Now $F = X^2 Z - Y^3 + g_2(X, Y, Z, T)T$ where g_2 is homogeneous of degree two in the coordinates X, Y, Z, T. By nonsingularity at P it follows that $g_2(0 : 0 : 1 : 0) \neq 0$ thus assume $g_2(0 : 0 : 1 : 0) = 1$.
   Every line through $P_\alpha = (1 : \alpha : \alpha^3 : 0)$ on C goes through a point $Q = (0, y, z, t)$ on the plane $X = 0$. The line through P_α and Q, $P_\alpha Q$, can be parametrized by writing out $F(\lambda P + \mu Q) = A(y, z, t) + B(y, z, t) + C(y, z, t)$, where $g_i \in k(\alpha)$ is homogeneous of degree i in the variables (y, z, t). Then:
   $$P_\alpha Q \subset V \iff A(y, z, t) = B(y, z, t) = C(y, z, t) = 0 \tag{4.2}$$
   Claim: a resultant polynomial $R_{27}(\alpha)$ exists, such that
   $$R_{27}(\alpha) = 0 \iff A, B, C \text{ have a common zero } (\eta, \zeta, \tau) \text{ in } \mathbb{P}^2 \tag{4.3}$$
   Define the polar of f as a form in two points $(X, Y, Z, T), (X', Y', Z', T')$ as:
   $$f_1(X, Y, Z, T; X', Y', Z', T') = \frac{\partial f}{\partial X} X' + \frac{\partial f}{\partial Y} Y' + \frac{\partial f}{\partial Z} Z' + \frac{\partial f}{\partial T} T' \tag{4.4}$$
   Then
   $$F(\lambda P + \mu Q) = \lambda^3 f(P) + \lambda^2 \mu f_1(P, Q) + \lambda \mu^2 f_1(Q, P) + \mu^3 F(Q) \tag{4.5}$$
   Then $A = f_1(P; Q)$, $B = f_1(Q; P)$ and $C = F(Q)$. Calculating the points for which $f(P) = f_1(P; Q) = f_1(Q; P) = F(Q)$ will give a relation given by a resultant polynomial of degree 27 in α.
   Every root of this polynomial will give a line on the surface V. There are at most 27 roots if all roots are unique. This gives a maximum of 27 lines on a smooth cubic surface.

3. *Given a line l on V, then there exist exactly five pairs of lines (l_i, l_i') such that every pair of lines (l_i, l_i') is coplaner with l and $(l_i \cup l_i') \cap (l_j \cup l_j') = \emptyset$ for all $i \neq j$.* Given a line l on V and a plane Π such that $l \subset \Pi$ then the intersection $\Pi \cap V$ is a line and a conic. When this intersection is singular

$\Pi \cap V$ consists of three lines. Suppose $l = (Z = T = 0)$, thus every term of the polynomial is divisible by Z or T, then F may be written out as:

$$F(X,Y,Z,T) = AX^2 + BXY + CY^2 + DX + EY + F \tag{4.6}$$

Where A, B, C are linear forms in Z, T, D, E are quadric forms in Z, T and F is a cubic form in Z, T. Considering this equation as a conic in X and Y, this is singular when the discriminant is zero. This discriminant is a polynomial g of degree five in (Z, T):

$$\Delta(Z,T) = 4ACF + BDE - AE^2 - B^2F - CD^2 = 0 \tag{4.7}$$

Proving that it has only simple roots will prove the claim. Every such root α will give a plane Π_α through l such that $\Pi_\alpha \cap V$ consists of three lines. Five such planes exist by the simpleness of the roots of g. Thus every line l on V intersects with exactly 10 lines.

4. *There are at least five disjoint pairs of lines (l_i, l_i') which intersect with l, and any other line $n \subset V$ will meet exactly one of l_i and l_i' for $i = 1, ..., 5$.* In $\mathbb{P}^3$ the line n will intersect the plane Π_i, where $\Pi_i \cap V = l \cup l_i \cup l_i'$, thus n will intersect one of the lines. It cannot intersect l, because all lines which intersect l are found,hence n must intersect l_i or l_i'. It cannot intersect both because then it will lie on the plane Π_i and the intesection of Π_i and V will give four lines.

5. *If there are four disjoint lines $l_1, ..., l_4$ in $\mathbb{P}^3$ then they lie on a quadric and have an infinite number of lines intersecting all lines, or they do not lie on a quadric and have one or two.* Through three disjoint lines $l_1, ..., l_3$ there always passes a smooth quadric Q. This quadric has 2 sets of lines S_1, S_2. Because $l_1, ..., l_3$ are disjoint they belong to one set of lines, say S_1, and every lines which intersects al three lies on Q and belongs to S_2. Thus if l_4 is disjoint and lies on Q then it belongs to S_1 and the infinite family of lines S_2 will all intersect the four lines. If l_4 does not lie on Q then it intersects Q in one or two points and the lines from S_2 passing through these points intersect all four lines.
   To end the proof, take two disjoint lines l, m on V. Then for every pair (l_i, l_i') for $i = 1, ..., 5$ which intersect l, one of them intersects m as well. Assume the l_i intersect both l and m. Then m intersects with the pairs (l_i, l_i'') for $i = 1, .., 5$. This gives 17 lines on V. Namely l, m the five lines intersecting both, five which intersect only l and five which intersect only m of the pair.

6. *Any line $n \subset V$ which is not one of the* 17 *above will intersect* 3 *of the lines $l_1, ..., l_5$.* No four of the lines will lie on a quadric because then V would be reducible. The line n cannot meet more than three of the l_i because then it would be l or m by point 5. If it would intersect with less than three of the l_i it will intersect with three or more of the l_i'. So it meets lets say either l_2', l_3', l_4', l_5' or l_1, l_3', l_4', l_5'. But than l and l_1'' intersect

these four, and by the same argument as in point 5, n cannot intersect all four as well. Thus n intersects three of the l_i.

7. *All combinations of three of the l_i will give a line which is not contained in the* 17 *found lines.* By part 5 there are 10 lines intersecting l_1. Only four, l, l_1', l_1'', m, are found, so there must be 6 more. Every one of them by point 6 will be intersect 2 of l_2, l_3, l_4, l_5. There are only 6 possibilities hence they all occur.

8. Counting all the found lines gives the 2 lines l, m the 15 lines l_i, l_i', l_i'' for $i = 1, ..., 5$ and the $\begin{pmatrix} 5 \\ 3 \end{pmatrix} = 10$ of the intersecting l_i, l_j, l_k lines with $i \neq j \neq k$ and $i, j, k = 1, ..., 5$. This will give 27 lines of a maximum of 27 thus all are found.

This is a very classical proof which gives the configuration as well. It uses a lot of linear algebra instead of the modern algebraic geometry. There is a modern proof as well. Therefore we need the following:

Theorem 4.2. *Every smooth cubic surface $V \subset \mathbb{P}^3$ is the blow-up of the projective plane at six points $P_1, P_2, P_3, P_4, P_5, P_6$ in general position, thus no 3 on a line or 6 on a conic.*

Proof. Looking at the proof above, 2 skew lines exist on the cubic surface n, m. Trough a point x on V there is a unique line l which will intersect both n and m in unique points $(x_1 : x_2)$ on n and $(y_1 : y_2)$ on m. This leads to a birational map ϕ from $V \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

$$x \dashrightarrow ((x_1 : x_2), (y_1 : y_2)) \tag{4.8}$$

The 5 lines which intersect both n and m will be blown down to 5 points $z_1, ..., z_5$. Looking at the space generated by $x_1y_1, x_1y_2, x_2y_1, x_2y_2$ which has dimension 4. The subspace of all forms which vanish at z_1 is of dimension 3. Thus there is a base consisting of f_1, f_2, f_3 and there is a map

$$\Phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2 \tag{4.9}$$

$$p \to (f_1(p) : f_2(p) : f_3(p)) \tag{4.10}$$

Then $\Phi \circ \phi$ should be a map which blows down 6 lines and is an isomorphism at every other point. The five lines which intersect both n and m will be blown down. But if we look at the morphism from $\mathbb{P}^2$ tot $\mathbb{P}^1 \times \mathbb{P}^1$ then $\mathbb{P}^1 \times \mathbb{P}^1 = (\mathbb{A}^1 \times \infty) \times (\mathbb{A}^1 \times \infty)$ and $\mathbb{P}^2 = \mathbb{A}^2 \times \mathbb{P}^1$. So an axis of $\mathbb{P}^1 \times \mathbb{P}^1$ needs to be blown up from $\mathbb{P}^2$ which is the sixth point.
The 27 lines now rise easily and are found as:The six exeptional divisors, the six conics trough 5 of the six points and the forms of the 15 lines trough each pair of the points.
In figure 4.1 the model of a smooth cubic surface is shown. This surface has exactly 27 real lines which are scratched into the surface.

Figure 4.1: Clebsch Smooth Cubic Surface, Schilling VII-01

4.2 Cubic Surface with an Isolated Singularity

This section will give the classification the singularities lying on cubic surfaces over algebraically closed fields. The cubic surfaces will be distinguished by the types of singularity occuring on the cubic surface. From now on assume that the cubic surface V is given by $F(X : Y : Z : T) = 0$ has a singularity. By a linear change of coordinates this singularity may be assumed to be at $P = (0 : 0 : 0 : 1)$. Which means $\frac{\delta F}{\partial X}(P) = \frac{\partial F}{\partial Y}(P) = \frac{\partial F}{\partial Z}(P) = \frac{\partial F}{\partial T}(P) = 0$. Thus no terms will be divisible by T^2.
To define these singularities by type, it is only needed to look locally around the singularity. Therefore we may assume that we work on the affine piece U_T of $\mathbb{P}^3$, which is given by $T = 1$. Call the singular point on the affine part $P_T = (0, 0, 0)$ hence the singularity lies at the origin on this affine space. On the affine piece U_T, the equation defining the cubic surface may be written as:

$$F(X : Y : Z : 1) = f_2(x, y, z) + f_3(x, y, z) \tag{4.11}$$

Here f_i is a homogeneous polynomial of degree i in (x, y, z). Not all possible singularities will occur on the cubic surface V and not all configurations are possible. The following results limit the configuration of singularities on cubic surfaces, as there will be no more than four singularities on V and no three singularities will lie on a line.

Theorem 4.3. *Let V be a cubic surface defined by a homogeneous polynomial $F(X : Y : Z : T)$, with three singular points P_1, P_2, P_3 lying on a line $l \subset V$. Then l is a singular line, meaning that every point on l is a singular point.*

Proof: Assume the line $l = (Z = 0, T = 0)$ and the singular points are as follows $P_1 = (1 : 0 : 0 : 0), P_2 = (0 : 1 : 0 : 0), P_3 = (1 : 1 : 0 : 0)$. For $F(X : Y : Z : T)$ to be singular at P_i we have $F_X(P_i) = 0, F_Y(P_i) =$

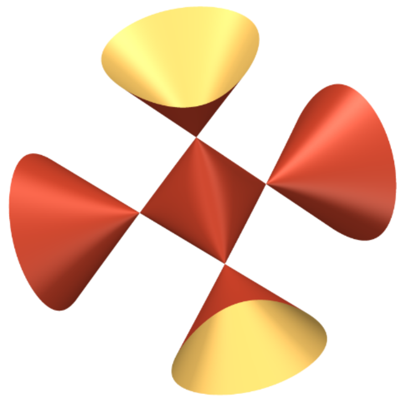

Figure 4.2: Cubic Surface with four isolated singularities

$0, F_Z(P_i) = 0, F_T(P_i) = 0$. These give linear constraints on the coefficients of F and we get:

$$F = Z^2 \cdot f_1(X : Y : Z : T) + ZT \cdot f_2(X : Y : Z : T) + T^2 \cdot f_3(X : Y : Z : T) \quad (4.12)$$

With f_i homogeneous of degree 1. For every point Q on the line $l = (Z = T = 0)$, Q is a singular point and the line l is a singular line. □

Theorem 4.4. *A cubic surface with only isolated singularities has at most* 4 *isolated singularities.*

Proof By a change of coordinates the singular points are $(0 : 0 : 0 : 1), (0 : 0 : 1 : 0), (0 : 1 : 0 : 0)$ and $(1 : 0 : 0 : 0)$. Then by simple calculation the function defining a cubic surface is of the form:

$$F(X : Y : Z : T) = a_1 XYZ + a_2 XYT + a_3 XZT + a_4 YZT \quad (4.13)$$

By a scale change of the coordinates assume $a_1 = ... = a_4 = 1$. Then the only singular points are the four above. This may be done for every such cubic surface with four isolated singularities, thus every such surface is projectively equivalent to the zero set of equation 4.13. Every such surface has, just like the surface given by the equation in 4.13, 4 singular points. □

Such a cubic surface with 4 singular points is given in figure 4.2. The surface given by equation 4.13 has not all its singularities on the affine piece $(X : Y : Z : 1)$ but by a translation $T = X + Y + Z + T$ this gives a surface with all isolated singularities easily seen.

To classify the cubic surfaces by their isolated singularities on them the local normal form of the simple singularities is needed. Recall that the simple singularities have the following local normal forms arround $(0, 0, 0)$ when working in a 3 dimensional affine space.

$$\begin{aligned}
&A_n && x^{n+1}+y^2+z^2, && (n \geq 1) && (4.14)\\
&D_n && x^{n-1}+xy^2+z^2, && (n \geq 4) && (4.15)\\
&E_6 && x^2+y^4+z^3 && && (4.16)\\
&E_7 && x^2+y^3+yz^3 && && (4.17)\\
&E_8 && x^2+y^3+z^5 && && (4.18)
\end{aligned}$$

To classify the cubic surfaces the first step is to look at the rank of $f_2(X:Y:Z)$, the homogeneous part of degree 2 in equation 4.11. The rank of $f_2(X:Y:Z)$ can only be $3, 2, 1, 0$. The second step is defining of which type the singularity at $P = (0:0:0:1)$ will be. This classification, of cubic surfaces with isolated singularities, is based on the work of Bruce and Wall [9]. Because the singularities are defined by their normal forms locally, analytical transformation may be used without changing the type of singularity.

4.2.1 A Conic Node

When the homogeneous part $f_2(x, y, z)$ is of rank 3, then the singular points P is called an conic node and $f_2(x, y, z)$ may be written as:

$$f_2(x, y, z) = x^2 + y^2 + z^2 \tag{4.19}$$

All terms of f_3 are of degree 3 so locally at P we have $f = f_2(x, y, z) = x^2 + y^2 + z^2$ and the point P is an $\mathbf{A}_1$ type singularity.

4.2.2 A Binode

When $f_2(x, y, z)$ has rank 2 then P is called a binode. The part f_2 is reducible into two linear factors. By a change of coordinates the homogeneous part can be written as $f_2 = x^2 - y^2 = (x-y)(x+y)$. The singularity at $P = (0:0:0:1)$ is determined by the intersection of $f_3(x:y:z) = 0$ with $f_2(x:y:z) = 0$ in the point $(0:0:1) \in \mathbb{P}^2$. For every situation the 6 points, counted with multiplicity, of intersection $f_2 = f_3 = 0$ may be changed by a analytic coordinate change to put f_3 in a normal form in $\mathbb{P}^2$.

Type $\mathbf{A}_2$ When $f_3(0:0:1) \neq 0$, thus f_3 does not intersect f_2 at $(0:0:1)$, then the coefficient of z^3 in f_3 is nonzero. By an analytic change of coordinates $f_3 = z^3$. Then $f(x, y, z)$, the affine representation of F around P, is of the form:

$$f(x, y, z) = x^2 - y^2 + z^3 \tag{4.20}$$

This gives, P is of the form $\mathbf{A}_2$.

Figure 4.3: Cubic Surface with an A_3 singularity, Schilling VII-13

Type $\mathbf{A}_3$ When $f_3(0:0:1)=0$ then the point $(0:0:1)$ lies on both $f_2=0$ and $f_3=0$. Look at the multiplicity of the intersection of $f_3(x:y:z)$ with the lines $(x-y)$ and $(x+y)$ at $(0:0:1)$. When the multiplicity of $f_3=0$, with both $(x-y)=0$ and $(x+y)=0$, is 1 at $(0:0:1)$, then by an analytic change of coordinates $f_3=2z^2x$. The normal form of f_3 is

$$f(x,y,z)=x^2-y^2+2z^2x \tag{4.21}$$

By a translation $\tilde{x}=(x+z^2)$, thus completing the squares, this is equivalent to

$$f(\tilde{x},y,z)=\tilde{x}^2-y^2-z^4 \tag{4.22}$$

Then P is of the form $\mathbf{A}_3$. A cubic surface with an A_3 singularity is seen in figure 4.3.

Type $\mathbf{A}_4$ Suppose the multiplicity of $f_3(x:y:z)=0$ at the point $(0:0:1)$ with one of the lines is 2, assume the multiplicity with $(x+y)=0$ is two, and the multiplicity with the line $(x-y)=0$ is one. By an analytic change of coordinates the function f_3 is of the form $f_3(x:y:z)=z^2(x+y)+zy^2$. Then:

$$f(x,y,z)=x^2-y^2+z^2(x+y)+zy^2 \tag{4.23}$$

By the first translation $\tilde{x}=(x+z^2)$ we get

$$f(\tilde{x},y,z)=\tilde{x}^2-y^2+z^2y+zy^2-\frac{1}{4}z^4 \tag{4.24}$$

To find the translation to get rid of the linear part of y by a translation A the following equation needs to be 0:

$$2(z-1)A+z^2=0 \tag{4.25}$$

Thus by the following translation $\tilde{y} = y + \frac{-z^2}{2(z-1)}$ the linear part of y will not occur.

$$f(\tilde{x}, \tilde{y}, z) = \tilde{x}^2 - \tilde{y^2} + z\tilde{y^2} - \frac{1}{4}z^4 - \frac{z^4}{2(z-1)} \tag{4.26}$$

Calculating the power series of $\frac{z^4}{2(z-1)}$ we get:

$$\frac{z^4}{2(z-1)} = -\frac{z^4}{4} - \frac{z^5}{4} + O(z^6) \tag{4.27}$$

Thus:

$$f(\tilde{x}, \tilde{y}, z) = \tilde{x}^2 - \tilde{y}^2 + z\tilde{y^2} - \frac{z^5}{4} + O(z^6) \tag{4.28}$$

Locally around $P_T = (0,0,0)$ this is equivalent to $x^2 - y^2 - \frac{1}{5}z^5$ thus f has an isolated singularity of type $\mathbf{A}_4$.

Type $\mathbf{A}_5$ The last situation with an isolated singularity is when $f_3 = 0$ intersects one line, say $(x - y) = 0$, with multiplicity 3 and $(x + y) = 0$ with multiplicity 1. By an analytic change of coordinates $f_3 = z^2(x + y) + x^3$. The transformation $\tilde{y} = y + \frac{z^2}{2}$ gives $f(x, \tilde{y}, z) = x^2 - \tilde{y}^2 + z^2x + \frac{z^4}{2} + x^3$. To find the transformation A such that f has no linear term in x, the following equation has to be solved:

$$2A + z^2 + 3A^2 = 0 \quad \Longrightarrow \quad A = \frac{\sqrt{1 - 3z^2} - 1}{3} \tag{4.29}$$

The Maclauren series of the square root is:

$$\sqrt{1 - 3z^2} = 1 - \frac{3z^2}{2} - \frac{9z^4}{8} - \frac{27z^6}{16} + O(z^7) \tag{4.30}$$

Then $\tilde{x} = x + A$ gives f is equivalent to:

$$f = \tilde{x}^3 + \tilde{x}^2\sqrt{1 - 3z^2} - \frac{z^2}{3} + \frac{2}{9}\sqrt{1 - 3z^2} - \frac{2}{27}\sqrt{1 - 3z^2} + \frac{2}{27} - \tilde{y}^2 - \frac{z^2}{4} \tag{4.31}$$

Using the Maclauren series for $\sqrt{1 - 3z^2}$ and calculating per power in z we get

$$f(\tilde{x}, \tilde{y}, z) = \tilde{x}^2 + \tilde{x}^3 + \tilde{y}^2 + \frac{z^6}{8} + \tilde{x}^2 \cdot O(z^2) + O(z^7) \tag{4.32}$$

Which locally arround $P_T = (0,0,0)$ is equivallent to $x^2 - y^2 + \frac{1}{8}z^6$, thus f has an $\mathbf{A}_5$ isolated singularity at P.

These are the possible ways for $f_3 = 0$ to intersect $f_2 = 0$ in $(0,0,1)$ with P still be an isolated singularity. If f_3 would intersect $x - y$ and $x + y$ both with multiplicity greater or equal than 2 then f_3 has a singular point at $(0 : 0 : 1)$ this means has the function f has the following form:

$$f(x, y, z) = x^2 \cdot a(y, z) + y^2 \cdot b(x, z) \tag{4.33}$$

where a and b are linear forms. This means that cubic surface defined by the function f has a singular line on $x = y = 0$. Thus these are the only configurations for which P is isolated and f_2 is of rank 2.

4.2.3 A Unode

When $f_2(x:y:z)$ is of rank 1 then we call P a unode. Which corresponds with a double linear factor in f_2. By a linear change of coordinates we may write the linear factor as x, then $f_2(x:y:z) = x^2$. When looked at the multiplicity of the three intersections of $f_3(x:y:z) = 0$ with the line $x = 0$ in $\mathbb{P}^2$, this is split up into three different cases:

- Three intersections of $f_3 = 0$ with $x = 0$ with multiplicity 1.
- Two intersections, one with multiplicity 1 and one with multiplicity 2, of $f_3 = 0$ with $x = 0$.
- One intersection with multiplicity 3 of $f_3 = 0$ with $x = 0$.

Three intersections When $x = 0$ and $f_3(x:y:z) = 0$ have three distinct intersections, then $f(x:y:z) = x^2 + zy^2 + yz^2$ for some analytic change of coordinates. The transformation $\tilde{z} = z - \frac{1}{2}y$ gives:

$$f(x, y, \tilde{z}) = x^2 + yz^2 - \frac{1}{4}y^3 \tag{4.34}$$

This gives a singularity of type D_4 at P.

Two intersections Here $f(x:y:z) = x^2 + xz^2 + zy^2$ the term xz^2 is needed or else there would be a line of singularities $x = y = 0$. Then by the transformation $\tilde{x} = x - \frac{1}{2}z^2$ this becomes:

$$f(\tilde{x}, y, z) = x^2 - \frac{1}{4}z^4 + zy^2 \tag{4.35}$$

This gives a singularity of type D_5 at P.

One intersection When there is one intersection the function $f(x, y, z)$ can be written as $f(x, y, z) = x^2 + xz^2 + y^3$ which by the transformation $\tilde{x} = x + \frac{1}{2}z$ is

$$f(\tilde{x}, y, z) = x^2 - \frac{1}{4}z^4 + y^3 \tag{4.36}$$

This givs a singularity of type E_6 at P.
Every cubic surface with an isolated singularity at $P = (0:0:0:1)$ and which rank of $f_2(x:y:z)$ is one is equivalent to one of the surfaces given above. All of these surfaces given above only have one isolated singularity. Thus they cannot have multiple singularities when the singularity at P is of the form D_4, D_5 or E_6. A algebaric model of the $D5$ model can be seen in figure 4.4.

Figure 4.4: Cubic Surface with D_5 singularity, Schilling VII-18

4.2.4 P a triple point

When the rank of $f_2(x:y:z)$ is zero then the defining function is $F(X:Y:Z:T) = f_3(x:y:z)$. When the surface V defined by F has an isolated singularity at $P = (0:0:0:1)$ then $f_3(x:y:z) = 0$ defines a non singular curve. The surface is a cone over a non singular cubic curve and the singularity at P is called an $\hat{E}_6$ singularity. This is a ruled surface and will be treated in chapter 5.

4.2.5 Ruled Cubic Surfaces

When the cubic surface V is irreducible and has non isolated singularities. A generic plane section of such a surface is irreducible, thus only has a singular point. This means that the singular set of V is a line. When V has a singular line it is a ruled surface. This means every point X on V lies on a line $l \subset V$. The proof of this and other characteristics of cubic ruled surfaces will be treated in chapter 5.

4.2.6 Reducible cubic surfaces

When we have a reducible cubic surface we have a non-degenerate quadric and a plane, or three planes. When we have three planes this is certainly a ruled surface. If we look at the non-degenerate quadric over a algebraically closed field we have a ruled surface as well. When we look at the real representation this may not be the case. The classification of quadrics in $\mathbb{P}^3(\mathbb{C})$ falls not under the scope of this thesis.

4.3 Classification of Multiple Isolated Singularities

In this section the cubic surface V has a singularity at $P = (0:0:0:1)$ thus de defining polynomial of V is written as $F(x:y:z:1) = f_2(x:y:z) + f_3(x:y:z)$. The type of this singularity can be determined by section 4.2. It is possible for V to have isolated singularities away from P. In this section the different combinations will be treated. This will only be done for cubic surfaces with only isolated singularities.

4.3.1 Possible other singularities when f_2 is of rank 3.

The singular point is now found to be an A_1 singularity. To do this it is easier to write $f_2(x, y, z)$ in a different but equivalent way. Through a linear change of coordinates the homogeneous part of degree 2 can be written as $f_2(x:y:z) = y^2 - xz$. This is of rank 3 as well, so P is still an A_1 singularity. Common roots of $f_2 = f_3 = 0$ correspond to a line trough P on V, and conversely. When $f_2(x, y, z) = y^2 - xz$ every root of f_2 is of the form $x = \theta^2, y = \phi\theta, z = \phi^2$. Thus a line trough P on V corresponds to a root (ϕ, θ) of $f_3(\theta^2, \phi\theta, \phi^2)$. Here $f_3(\theta^2, \phi\theta, \phi^2)$ is homogeneous polynomial of degree 6 in the variables (θ, ϕ).
The defining polynomial of V in $\mathbb{P}^3$ can be changed to a polynomial $F(x:y:z:t) = t \cdot f_2(x, y, z) + f_3(x, y, z)$ where f_3 has no terms divisible by x^2z, xz^2 or xyz. This can be done by linear coordinate changes, assume $f_3(x, y, z)$ has a term c_1xz^2 with c_1 a constant. Rewrite $c_1xz^2 = c_1z(xz - y^2) + c_1zy^2$, and use the linear coordinate change on $t' = t - c_1z$ then the term c_1xz^2 will not occur in the function. This can be done for x^2z and xyz as well. These coordinate changes will put f_3 in the following normal form.

$$f_3(x, y, z) = a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3 + a_4y^2z + a_5yz^2 + a_6z^3 \qquad (4.37)$$

The following lemma will explain when two cubic surfaces with an A_1 singularity at P are equivalent and find which other isolated singularities may occur on a cubic surface with an A_1 singularity at P.

Lemma 4.5. *Let:*

$$F = T(Y^2 - XZ) + f_3(X, Y, Z),\ G = T(Y^2 - XZ) + g_3(X, Y, Z) \qquad (4.38)$$

a) $F = 0$ and $G = 0$ give projectively equivalent cubic surfaces, the equivalence fixing P, iff $f_3(\theta^2, \theta\phi, \phi^2)$ and $g_3(\theta^2, \theta\phi, \phi^2)$ are equivalent binary sextics.
b) For each singularity $Q \neq P$ the line QP lies on V with defining polynomial $F = 0$. The line will be given by a common root of $f_2 = f_3 = 0$, which is equivalent to a root of $f_3(\theta^2, \theta\phi, \phi^2) = 0$. This root of $f_3(\theta^2, \theta\phi, \phi^2)$ with multiplicity greater than 1.
c) Each root of $f_3(\theta^2, \theta\phi, \phi^2) = 0$ determines a line l trough P on V. If the root

Figure 4.5: Cubic Surface with an A_1 and A_5 singularity, Schilling VII-15

has multiplicity greater than 1 *there is precisely one other singular point on* l.
d) A k-tuple root of $f_3(\theta^2, \theta\phi, \phi^2) = 0$ *corresponds to an* A_{k-1} *singularity.*

Proof. ***a)*** If $F = 0$ and $G = 0$ are projectively equivalent then the change of coordinates $(x, y, z) \to (x', y', z')$ will take the lines on F through P to those on G trough P preserving multiplicities, which means that the change of coordinates will map the roots of f_3 into the roots of g_3. This linear change of coordinates x, y, z will induce a change of the coordinates $(\theta, \phi) \to (\theta', \phi')$ for which $g_3(\theta'^2, \theta'\phi', \phi'^2) = f_3(\theta^2, \theta\phi, \phi^2)$ because both sextics have the same six roots.
When the binary sextics, $f_3(\theta^2, \theta\phi, \phi^2)$, $g_3(\theta^2, \theta\phi, \phi^2)$ are equivalent, then the change of coordinates $(\theta, \phi) \to (\theta', \phi')$ induces a coordinate change of $(x, y, z) \to (x', y', z')$ which will be an automorphism of $y^2 - xz$, by definition. The resulting cubic g_3' has the same intersections with this conic as f_3. Thus f_3 and g_3 are equivalent modulo this conic. By Hilberts theorem this means that $f_3 - \lambda g_3' \in \sqrt{y^2 - xz}$ where $\sqrt{y^2 - xz}$ is the radical of the ideal generated by $(y^2 - xz)$. Thus

$$f_3 = \lambda g_3' + (y^2 - xz)l \tag{4.39}$$

Where λ is a non zero scalar and l is a linear form. The linear coordinate change $(X, Y, Z, T) = (X, Y, Z, T - l)$ and a scale change will prove that f_3 and g_3 are projectively equivalent.
b) By an analytic change of coordinates the singularity lies at $Q = (0 : 0 : 1 : 0)$. Because Q is singular this means that the terms xz^2, yz^2, z^3 are zero in f_3. Then θ^2 divides the binary sextic $f_3(\theta^2, \theta\phi, \phi^2)$, thus the multiplicity of the root is greater or equal than 2. This root of $f_3(\theta^2, \theta\phi, \phi^2)$ will correspond to a line on V.

c) Suppose that $(\theta : \phi) = (0 : 1)$ is a multiple root of $f_3(\theta^2, \theta\phi, \phi^2) = 0$. Then $f_3(\theta^2, \theta\phi, \phi^2) = 0$ thus this determines a line l trough $P = (0 : 0 : 0 : 1)$ on V, namely the line $l := (z = t = 0)$. If f_3 has the form as in equation 4.37, the coefficients of the terms yz^2, z^3 are zero. A short calculation shows that $(0 : 0 : 1 : 0)$ is a singular point. The partial derivative to x is

$$\frac{\partial F}{\partial x} = -zt + \frac{\partial f_3}{\partial x} \tag{4.40}$$

which only vanishes at $P = (0 : 0 : 0 : 1)$ and $Q = (0 : 0 : 1 : 0)$ on l thus Q and P are the only singular points on the line l.
d) Again suppose the root of $f_3(\theta^2, \theta\phi, \phi^2) = \sum_0^6 a_i \theta^{6-i}\phi^i = 0$ is at $(\theta, \phi) = (0 : 1)$ and has multiplicity k. Then there is a k-tuple root if and only if $a_6 = ... = a_{7-k} = 0$ and $a_{6-k} \neq 0$, and the singularity is at $(0 : 0 : 1 : 0)$. The normal form of the function f_3 is given in equationEQNF3. This is used to define the function locally around the point $Q = (0 : 0 : 1 : 0)$. Then

$$F(x : y : 1 : t) = ty^2 - tx + a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3 + a_4y^2 \tag{4.41}$$

By a coordinate change:

$$t = t' + a_0(x^2 + xy^2 + y^4) + a_1(xy + y^3) + a_2y^2, \quad x = x' + y^2 \tag{4.42}$$

this gives

$$-x't' + a_0y^6 + a_1y^5 + a_2y^4 + a_3y^3 + a_4y^2 \tag{4.43}$$

The first term $-x't'$ is locally projectively equivalent to $x^2 + y^2$. Then for $a_6 = ... = a_{7-k} = 0$ and $a_{6-k} \neq 0$ the point Q is an A_{k-1} singularity. □

This gives an easy way to describe the singularities occurring when $P = (0 : 0 : 0 : 1)$ is an $\mathbf{A}_1$ singularity. There are at most 6 points where f_2 and f_3 intersect, counted with multiplicity. These six points can be partitioned by multiplicity in 11 different ways. Table 4.1 gives the different types of isolated singularities on V when P is a singularity of type A_1. In table 4.1 the number stands for the multiplicity of the intersetion, and the subscript the number of points with this multiplicity. Thus the notation $1^2 2^2$ stands for a partition of 2 intersections with multiplicity 1 and 2 intersections with multiplicity 2 giving the total of 6 intersections counted with multiplicities.
An example of a surface model with an A_1 and A_5 singularity is given in 4.5.

4.3.2 Possible other singularities when f_2 is of rank 2.

The different types of isolated singularities at other points then P, when $f_2(x, y, z)$ is of rank 2, are found. This means that the singularity is of type A_k for $k \geq 2$ as described in subsection 4.2.2. The different types of other singularities will be treated in this section. For this $f_2 = xy$ instead of $f_2 = x^2 - y^2$ to simplify the calculations. This is a linear change of coordinates in $\mathbb{P}^3$ and the rank of f_2 still is 2. The lines in $\mathbb{P}^2$ of which $f_2 = 0$ consists are $x = 0$ and $y = 0$. The

Partition intersections f_2, f_3:	1^6	$1^4 2$	$1^3 3$	$1^2 4$	$1^2 2^2$	15
Singularities occuring on V:	A_1	$2A_1$	$A_1 A_2$	$A_1 A_3$	$3A_1$	$A_1 A_4$
Partition intersections f_2, f_3:	123	2^3	42	3^2	6	
Singularities occuring on V:	$2A_1 A_2$	$4A_1$	$2A_1 A_3$	$A_1 2A_2$	$A_1 A_5$	

Table 4.1: Partition of the six intersections of f_2 and f_3 on a cubic surface with an A_1 singularity.

type of singularity at P can now be found by the classification above. But there may still lie other singularities on V. Which types of singularities this may be, is treated next.
The defining function of the surface V is of the form $F(X : Y : Z : T) = TXY + f_3(X, Y, Z)$. By a linear change of coordinates on T we can put F in a normal form where no term in $f_3(X, Y, Z)$ is divisible by XY. Take a term divisible by XY for example $cXYZ$ where c a non zero constant. The linear change of coordinats $(X : Y : Z : T) \to (X : Y : Z : T - cZ)$ will give an equivalent surface without the term $cXYZ$. Thus every such surface can be changed to a surface defined by normal form given in 4.44.

$$F = TXY + X(a_0X^2 + a_1XZ + a_2Z^2) + Y(a_3Y^2 + a_4YZ + a_5Z^2) + a_6Z^3 \quad (4.44)$$

This normal form is used in the following lemma which will explain which other types of singularities may occur on a cubic surface with a singularity at $P = (0 : 0 : 0 : 1)$ and for which the defining polynomial has a homogeneous quadratic part of degree 2 in X, Y, Z.

Lemma 4.6. *For a cubic surface V have defining polynomial:*

$$F(X : Y : Z : T) = TXY + f_3(X, Y, Z) \quad (4.45)$$

the following holds: a) Isolated singularities of V away from $P = (0 : 0 : 0 : 1)$ correspond to multiple intersections of $xy = 0$ with $f_3(X, Y, Z) = 0$ away from $(0 : 0 : 1)$ in $\mathbb{P}^2$.
b) A k-tuple intersection away from $(0 : 0 : 1)$ corresponds to an A_{k-1} singularity.

Proof: ***a)*** Assume $y = 0$ has a multiple intersection with $f_3 = 0$. By a change of coordinates assume the intersection occurs at $Q = (1 : 0 : 0)$. When F is put in the normal form described in equation 4.44:

$$F = TXY + X(a_0X^2 + a_1XZ + a_2Z^2) + Y(a_3Y^2 + a_4YZ + a_5Z^2) + a_6Z^3 \quad (4.46)$$

Then the multiple intersection corresponds to $a_0 = a_1 = 0$. Thus F has no terms divisible by X^2 which means it has a singularity at $Q' = (1 : 0 : 0 : 0)$.
For the reverse if there is another singularity then at $P = (0 : 0 : 0 : 1)$ say at

Partition:	$1^3.1^3$	$1^3.21$	$1^3.3$	21.21	21.3	3.3	$1^2.1^2$
Singularities:	A_2	A_2, A_1	$2A_2$	$2A_1, A_2$	$A_1, 2A_2$	$3A_2$	A_3
Partition:	$1^2.2$	2.2	$1^2.1$	$2, 1$	$1^2.0$	2.0	
Singularities:	A_1, A_3	$2A_1, A_3$	A_4	A_1, A_4	A_5	A_1, A_5	

Table 4.2: Partition of the six intersections of a cubic with A_k singularity.

$Q' = (1 : 0 : 0 : 0)$ then F has no terms divisible by X^2 and $a_0 = a_1 = 0$. Thus $Y = 0$ has a multiple intersection with $f_3 = 0$ at $Q = (1 : 0 : 0)$.
b) Assume the multiple singularity occurs at $Q = (1 : 0 : 0 : 0)$. To check which type of isolated singularity Q is, switch the role of X and T thus working on the affine part U_X. The surface on this affine part is given by $F(1 : Y : Z : T) = 0$ the defining function writin in the normal form as in equation (4.44):

$$F(1, y, z, t) = ty + a_2 z^2 + a_3 y^3 + a_4 y^2 z + a_5 y z^2 + a_6 z^3 \tag{4.47}$$

The quadratic part of F is $t \cdot y + a_2 z^2$. If a_2 is non zero the quadratic part is of rank 3 and by subsection 4.2.1 the singular point Q is of type A_1.
When $a_2 = 0$ then a_6 is certainly not zero otherwise the equation is reducible and the corresponding cubic surface is reducible too. In this case $f_2(t, y, z) = ty$ and $f_3(t, y, z) = a_6 z^3 + O(y)$. To find which singularity this is we have to look trough the criteria in the subsection 4.2.2. The critical point for determining the type of singularity at Q is the intersection of $f_3(t, y, z)$ at $(0 : 0 : 1)$. Because a_6 is non zero, no such intersection occurs. Thus looking at the classification in subsection 4.2.2 the point Q is a singularity of type A_2 and there are no more possibilities for the singularities to occur. □

To summarize: when $f_2(x : y : z)$ consists of two different linear forms $l_1(x : y : z), l_2(x : y : z)$ the singularity at $P = (0 : 0 : 0 : 1)$ is of type A_k for $k \geq 2$. Here k depends on the intersection of $f_3(x : y : z) = 0$ with $f_2 = 0$. Let Q be the point $l_1 = l_2 = 0$ in $\mathbb{P}^2$. If $f_3 = 0$ does not contain Q then $k = 2$. When $f_3 = 0$ intersects $l_1 = 0$ in Q with multiplicity a_1 and intersects $l_2 = 0$ in Q with multiplicity a_2 then a_1 or a_2 is 1 and $k = 1 + a_1 + a_2$.
If there are other singularities on V, then there is an A_k singularity. When $f_3 = 0$ intersects l_1 or l_2, at some points other than Q, with multiplicity $k + 1$. All different ways this can occur is found in table 4.2. Where the number stand for the intersections of f_3 away from $Q = (l_1 = l_2 = 0)$. For example $1^2.2$ stands for two intersections $f_3 = l_1 = 0$ with multiplicity 1 away from Q, hence an intersection $f_3 = l_1 = 0$ at Q of multiplicity 1 as well. And one intersection $f_3 = l_2 = 0$ of multiplicity 2 away from Q, hence one intersection $f_3 = l_2 = 0$ of multiplicity 1. Thus in total one A_3 at $P = (0 : 0 : 0 : 1)$ and an A_1 at another point.

4.3.3 Possible other singularities when f_2 is of rank 1.

If the rank of f_2 is 1, then the singularity type of P is D_4, D_5 or E_6 as described in subsection 4.2.3. For these types of singularities there can be no more than one isolated singularity when they lie on a cubic surface. This is seen by looking at the normal forms giving in subsection 4.2.3, because for all three cases there is just one normal form which does not have any other singularities there will be no more.

4.4 Complete classes of cubic surfaces in $\mathbb{P}^3$

4.4.1 Lines on a cubic surface

As seen in subsection 4.1 the smooth cubic surface has exactly 27 lines. This will not be the case for the cubic surface V with an isolated singularity on it. Some of the lines on V will be double or even triple on V. Bruce and Wall calculated for different types of cubic surfaces the number of lines on the surface [9]. They found a formula to calculate the number of lines on the cubic surface with isolated singularities V. If the number of singularities is k, and the sum of their Milnor number is c then the number of distinct lines on V is

$$\frac{1}{2}(8-c)(7-c)+k-1 \tag{4.48}$$

Bruce and Wall see the Milnor number as the linear restrictions put on the cubic surface. Here an A_1 type singularity is just one restriction on the surface, in general for every A_k singularity the number of restrictions is k.
In subsection 3.2.4 it is shown that the Milnor number by the simple singularities is the same as the subscript. Knörrer and Miller proved this equation for cubic surfaces [10] by using blow ups and the orbits of the singularities where Bruce and Wall calculated every case separately. The number of lines on a cubic surface will be used to classify cubic surfaces over the real numbers. This is done by looking at which of those lines are real and which are complex.

4.4.2 Total Classes

This will give the classifications of the cubic surfaces with at most isolated singularities over algebraically closed surface and the number of lines lying on them. There are 22 projectively different complex cubic surfaces summed up in table 4.3.

4.4.3 Modern view

There is a more intuitive view to come to the same classification. This has to do with the so called passages of the smooth cubic model [6]. A smooth cubic surface has 7 passages which are linked. This can be seen in the surface left in figure 4.7. Where the seven passages are the three holes in the surface, the

Types:	Nonsing.	A_1	$2A_1$	A_1, A_2	A_1, A_3
Lines:	27	21	16	11	7
Types:	A_1, A_4	A_1, A_5	$3A_1$	$2A_1, A_2$	$2A_1, A_3$
Lines:	4	2	12	8	5
Types:	$A_1, 2A_2$	$4A_1$	A_2	$2A_2$	$3A_2$
Lines:	5	9	15	7	3
Types:	A_3	A_4	A_5	D_4	D_5
Lines:	10	6	3	6	3
Types:	E_6	$\hat{E}_6$			
Lines:	1	∞			

Table 4.3: Possible configuration of singularities on a cubic surface in $\mathbb{P}^3$

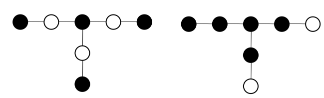

Figure 4.6: Coxeter diagrams collapsed passages.

three entrances to the middle from the left, right and back, the last is the entry to the middle from below. The entry from below is linked to the three holes which are individually links to one of the passages to the middle. This is given schematically to the right of the image of the surface.
The right diagram in figure 4.7 is the non collapsed Coxeter diagram of the $\hat{E}_6$ singularity, which is cone over a non-singular cubic surface. Every time a passage is collapsed, a singularity emerges. A collapsed passage is denoted by colouring the corresponding circle black. In chapter 3 the coxeter diagrams of the simple singularities are given. The structure of the collapsed passages will give a diagram with non connected Coxeter diagrams of simple singularities. An example of this is shown in figure 4.6. The left diagram consists of four disconnected collapsed passages. Every collapsed passage corresponds to an A_1 Coxeter diagram, and the corresponding surface is a surface with four A_1 singularities. The right diagram consists of four coloured nodes, which is the same as the Coxeter diagram of the D_4 singularity. Hence the right diagram corresponds to a cubic surface with a D_4 singularity.
By collapsing the passages and looking at the corresponding diagrams the 21 different configurations of singularities on a cubic surface can be found. The position of the collapsed passage does not matter, the only thing to look at is the structure of the singularities in the diagram. Hence the 7 diagrams with one collapsed passage, or one coloured node, all correspond to a surface with an A_1 singularity and are projectively equivalent. These are counted as one class

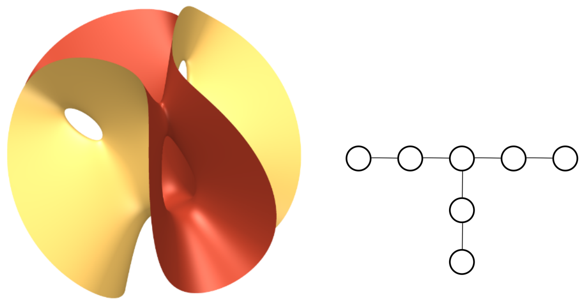

Figure 4.7: Smooth cubic surface with seven passages with its Coxeter diagram.

in our classification.

4.5 Real Models of Cubic Surfaces

The theorems and the classification of the cubic surfaces in the last section is done in an algebraically closed field. The models of the cubic surfaces however are in $\mathbb{R}$. This means for cubic surfaces which projectively fall in the same category over the complex numbers, may have distinct different models over the reals. For the next part the real part of the cubic surface V is denoted as $V_{\mathbb{R}}$. It is shown already that every smooth cubic surface V can be constructed by blowing up six points $P_1, ..., P_6$ in general position in $\mathbb{P}^2$. From now on we will call these six points Σ. Then define $r(V)$ as the number of real pairs of complex points in Σ, or -1 if it is not the blow up of 6 points in general position. This is called the reality index. For a singular cubic surface X take $r(V) = r(\tilde{V})$ where $\tilde{V}$ is the desingularization of V, thus taking the reality index of the desingularised surface V. If a function $F(X : Y : Z : T)$ defines a real cubic surface then all coefficients of the terms $F(X : Y : Z : T)$ are real.

4.5.1 Real Smooth Cubic Surfaces

The classification of the real smooth cubic surfaces is done by looking at which of the lines on the surface are real. In the beginning of this chapter in theorem 4.1 it has been proven that on every smooth cubic surface on a algebraically closed field there lie exactly 27 lines. But nothing has been said about the configuration of these lines. This will be done now.
When the intersection of a plane and the cubic surface contains three lines we speak of a tritangent plane. This naturally happens when we look at the plane on which two known intersecting lines lie, call these two lines a_1 and b_2. This gives rise to a third line c_{12} which intersects both a_1 and b_2. In the notation of Schläfli [18] this tritangent plane is denoted by (12). Similarly the tritangent plane (21) contains the line a_2, b_1, c_{12}.

In the sketch of the proof of theorem 4.1, the third step was proving that when there is one line l on a smooth cubic surface V, then l lies in five planes $U_1, ..., U_5$ which intersected with V gave a full reduction into three lines of the corresponding curve. Hence $U_i \cap V = l \cup l_1 \cup l_1'$ for l, l_1, l_1' three lines. Which means that every line l on V intersects with exactly 10 other lines. These five planes give 2 intersecting lines per plane. The configuration of all 27 lines can be stated by the notion of a Schläffli double six.

$$\alpha = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix} \tag{4.49}$$

In the proof of theorem 4.1 the different lines which intersect are already studied, the double six will give this some structure.
This notation starts with two skew lines namely a_1 and b_1. Two skew lines will have 5 intersecting lines in common. Then a_1 will intersect with exactly 5 lines which do not intersect b_1, call these lines $b_2, ..., b_6$. In the same way b_1 will intersect with exactly 5 lines which do not intersect with a_1 call these lines $a_2, ..., a_6$. These lines will imply there are the c_{1i} for $i = 2, ..., 6$, which give the five lines intersecting both a_1 and b_1. This gives the 17 first lines. where $a_2, ..., a_6$ are $l_1'', ..., l_5''$ and $b_2, ..., b_6$ are $l_1', ..., l_5'$.
Thus a_i does not intersect a_j for all other j but does intersect with b_j for all $j \neq i$. The c_{ij} completely determine the other 15 lines of the surface. The lines c_{ij} need to intersect six other c_{kl} as well, because they only intersect four of the a_i, b_j. These are exactly the c_{kl} for which $i \neq k$ and $j \neq l$. Every line lies in exactly 5 planes with 3 other lines thus we have $27 \cdot 58 \cdot \frac{1}{3} = 45$ such planes. Now we have the complete configuration of such lines projectively.

Types of real smooth cubic surfaces by Schläfli

The different types of real smooth cubic surfaces Schläfli did by looking at the amount of tritangent planes and lines on the cubic surface which are real. Schläfli found the following different types [18] which all occur:
Type 1: All 27 lines and all 45 planes are real.
Type 2: There lie 15 real lines and 15 real tritangent planes on the surface. The remaining 12 complex lines form a double six where all the complex lines are completely imaginary and the conjugate pairs do not intersect.
Type 3: On the surface there are 7 real lines and 5 real tritangent planes. Trough every real line ther pass 5 real planes but only three contain a real triangles. On the other two, two imaginary lines meet in a real point.
Type 4: There are 3 real lines and 13 real planes. There is 1 real triangle and trough each side ther pass 4 more real planes.
Type 5: There are 3 real lines and 7 real planes. In this case there again is 1 real triangle and trough each side there pass 2 real planes.
A plaster model of a smooth cubic surface with 27 real lines is shown in figure 4.8.

Figure 4.8: Smooth Cubic Surface with 27 real lines, Schilling VII-01

Proof by Segre

In the article Schläfli did not explain how he came to these different types. Later Segre proved the same result in a different way [19]. He does this by degenerating a smooth cubic surface to a surface of three planes. In this deformation the smooth cubic surface will tend to one with a single isolated A_1 singularity.
Start with the following definition:

Definition 4.1. *A complex line l with a real point, hence $l \cap \bar{l} \neq \emptyset$, is called of the 1st kind. A complex line l which is skew to its conjugate, hence $l \cap \bar{l} = \emptyset$ is called of the 2nd kind.*

On a surface which has exactly one isolated A_1 singularity, there will be 6 lines through this singularity. If the singularity is assumed to be at $P = (0:0:0:1)$ then $F(X:Y:Z:T) = Tf_2(X:Y:Z) + f_3(X:Y:Z)$ and the six lines $l_1, ..., l_6$ are the lines for which $f_2(X:Y:Z) = f_3(X:Y:Z) = 0$. Taking two lines, l_i, l_j, they will lie on a plane T and $T \cap V$ will give a third line. All these combinations occur and will give the $6 + \binom{6}{2} = 21$ lines in $\mathbb{P}^3(\mathbb{C})$. These lines can be real or complex. When one of the lines through the singularity is complex its conjugate will be one of the lines through the singularity as well. As they must both be zeroes of $f_2(X:Y:Z)$ which is defined by real coefficients. To further define the reality of the lines on the rest of the surface the different tritangent planes are needed. If a tritangent plane has two real lines on it the third is a real lines as well. If the tritangent plane contains one real and one

	Number of lines trough singularity:		Number of remaining lines:		
Type:	Real	1st kind	Real	1st kind	2nd kind
Ω_1	6	0	15	0	0
Ω_2	4	2	7	0	8
Ω_3	2	4	3	4	8
Ω_4	0	6	3	12	0

Table 4.4: The different types of lines on a singular cubic surface.

complex line through the singularity, then the third line on the tritangent plane is complex. Thus when every pair of 2 lines can be complex or real, then the four types of singular cubic surfaces with a real A_1 singularity in table 4.4 will be the only ones with real lines.

Segre says when a non-singular cubic surface tends to a singular cubic surface then 12 of the lines, constituting of a double six, tend to the 6 lines trough the singularity. This happens in such a way that two corresponding non intersecting lines of the double six tend to coincide in one line trough the singularity. This is because the intersection relations of the 15 lines not through the singularity will still hold. All lines c_{ij} which intersect the a_i in the double six will intersect with b_i as well. Thus naturally these will coincide when the double six collapses. The six lines through the singularity are self conjugate, thus this should be true for the double six as well. The two sextuplets β, γ of the double six are either self conjugate or mutually conjugate. If there is a complex line l_1 in the double six its complex conjugate $\overline{l_1}$ should be in the double six as well. If $l_1, \overline{l_1}$ in the same six γ and another complex pair of lines $l_2, \overline{l_2}$ is spread over the two sixes γ, β. Assume l_2 is in β, then l_2 will intersect l_1 or $\overline{l_1}$, assume $\overline{l_1}$ in a point P. The complex conjugate of P, $\overline{P} = Q$, will lie on l_1 and on $\overline{l_2}$. Thus l_1 and $\overline{l_2}$ will intersect but they lie in the same six γ and by definition do not intersect. Thus this situation cannot happen and the whole six is self conjugate, or both sixes are mutualy conjugate.

Hence the six lines $a_1, ..., a_6$ are either such that they consist of real lines and conjugate pairs, or the six lines $a_1, ..., a_6$ consist of real lines and complex lines $l_i, ..., l_6$ such that their conjugates lie in het lines $b_1, ..., b_6$. Every line through the singularity is the limit of two corresponding and thus skew lines. This gives that a real line can only be the limit of two real lines or a pair of conjugate complex lines. Which gives that the following are the only two cases which occur:

1) A real line through the singularity is the limit of two real lines and a complex line is the limit of two complex lines of the 2nd kind. This happens when the sixes are both self conjugate.

2) A real line through the singularity is the limit of complex conjugate lines of the 2nd kind and a complex line is the limit of two complex lines of the 1st kind. In the case that the sixes are mutually conjugate.

Looking at the cubic surfaces which limit are the types of table 4.4. Every type

	Number of lines:				
Type:	Real:	Complex 1st:	Complex 2nd:	Tritangent planes	Reality Index:
F_1	27	0	0	45	3
F_2	15	0	12	15	2
F_3	7	4	16	5	1
F_4	3	12	12	7	0
F_5	3	24	0	13	-1

Table 4.5: The different types of smooth cubic surfaces.

in table 4.4 can have its six lines made into a double six, where there are two possibilities. The double six are self conjugate or they are mutualy conjugate. Thus for every type in table 4.4 there are two types of smooth cubic surfaces in table 4.5. Some of the types will overlap, and in total there will be 5 types found in this way. These types of real non-singular cubic surfaces are described in table 4.5. Every type Ω_i in table 4.4 will correspond to type F_i or F_{i+1} in table 4.5. From the number of real lines and the construction of the double xis it is possible to calculate the number of real tritangent planes. When the real tritangent planes are counter per type, the types will correspond to the types given by Schläfli.

Finding the reality index by looking at the blow up.

The reality index can be found by looking at the blow up of six points in standard position. Name these points $X_1, ..., X_6$, define a_i the exceptional divisor trough X_i and b_i the line given by the blow up of the conic not going trough X_i. The line c_{ij} is the blow up of the line passing trough X_i and X_j. Certainly a_i and b_j do not intersect if $i = j$ and intersect otherwise, a_i and a_j will not intersect if $i \neq j$ and b_i and b_j will not intersect if $i \neq j$. Thus the following is constructed as a double six.

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix} \tag{4.50}$$

If X_i is a real point then a_i and b_j are real lines, if X_i is complex and $X_j = \overline{X}_i$ then a_i, b_i, a_j, b_j are complex but c_{ij} is real. This gives the reality index given by the different types. From a process of elimination which gives the first four types it is found that F_5 cannot be the blow up of 6 points in standard position and thus has reality index -1. The six points of which F_5 is the blow up then must have a special configuration in $\mathbb{P}^2(\mathbb{C})$.

4.5.2 Real cubic surfaces with isolated singularities

Assume for the next section that V is a cubic surface. Define $V_{\mathbb{R}}$ as the real part of V. For the classification of the real part of the cubic surface we do not care about complex isolated singularities, these will not occur on the real part.

Type:	Equation:	$\mu_{\mathbb{R}}$	ν
A_{2k}^-	$x^{2k+1}+y^2-z^2$	$2k$	0
A_{2k}^+	$x^{2k+1}+y^2+z^2$	0	$k-1$
A_{2k-1}^-	$x^{2k}+y^2-z^2$	$2k-1$	0
A_{2k-1}^+	$x^{2k}-y^2-z^2$	1	$k-1$
$A_{2k-1}^\bullet$	$x^{2k}+y^2+z^2$	1	$k-1$
D_4^-	$x^2y-y^3-z^2$	4	0
D_4^+	$x^2y+y^3+z^2$	2	1
D_5^-	$x^2y+y^4-z^2$	5	0
D_5^+	$x^2y+y^4+z^2$	3	1
E_6^-	$x^3+y^4-z^2$	6	0
E_6^+	$x^3+y^4+z^2$	2	2

Table 4.6: Table of occuring singularities on the real part of a cubic surfaces.

Thus assume that V only has real isolated singularities.
In the classification above over algebraically closed fields it is proven that only $A_1, A_2, A_3, A_4, A_5, D_4, D_5, E_6$ occur as singularities on cubic surfaces. Looking at the normal forms of these singularities these are defined over the reals. The only difference may be the units before some of the terms. These difference in units only occurs in even powered terms. This means that only the types of real singularities with their respective normal forms described in table 4.6 will occur. [10]
Where $\mu_{\mathbb{R}}$ of a certain singularity comes from the resolution of the singularity by blowing up. Every iteration of the resolving of a singularity a point is blown up to an exceptional divisor. In chapter 3 the Coxeter diagrams of such blow up procedure are seen. In the case of the singularities in chapter 3 nothing was said about the complexity of these exceptional divisors. Then $\mu_{\mathbb{R}}$ says something about the number of exceptional divisors, given in the resolution of the singularity, which are real. The types of singularities in table 4.6 will all have their unique resolution of which $\mu_{\mathbb{R}}$ is the number of real exceptional divisors and ν is the number of pairs of complex exceptional divisors. For more information see Durfee [4], [5].
The class of cubic surface with an $A_1^\bullet$ singularity is a special case. There will be no other real singularities on this surface. This is proven in the following lemma.

Lemma 4.7. *If P is a singularity on a cubic surface $V_{\mathbb{R}}$ of the form $A_1^\bullet$ then it is the only singularity on $V_{\mathbb{R}}$.*

Proof Assume there is another singular point $Q \neq P$ on $V_{\mathbb{R}}$. Then the line through P and Q will lie completely on $V_{\mathbb{R}}$ and by extension in $\mathbb{R}^3$. But in a small ball around P it has the normal form $x^2+y^2+z^2=0$ which is an isolated

point of the surface. Thus the line trough P and Q cannot lie in $\mathbb{R}^3$ hence P is the only singularity.
A couple other singularities will not occur as well on the cubic surface but they are harder dealt with.

Lemma 4.8. *The following will not occur on the real part of a cubic surface: (1) $A_3^\bullet$, (2)$A_5^\bullet$ or A_k^+ for $k \geq 4$, (3) D_5^+ and E_6^+.*

Proof: When the cubic surface is real then all coefficients in $F(X:Y:Z:T)$ and f_2 are real. When $P = (0:0:0:1)$ is a singularity of type A_k with $k \geq 2$ or a D_4, D_5 or E_6 singularity, then the homogeneous part of degree 2 in the variables X, Y, Z, $f_2(X, Y, Z)$, factors into two linear factors. Because $f_2(X:Y:Z)$ has only real coefficients these linear factors are both real or are both complex and each others conjugate. Call these linear equations l_1, l_2. In subsection 4.2.2 and 4.2.3 the intersections of $f_2(X:Y:Z)$ and $f_3(X:Y:Z)$, the homogeneous part of degree three in X, Y, Z, are of importance for the classification of the intersections.
(1) For an A_3 singularity f_2 consists of two linear parts, the normal form of $A_3^\bullet$ is $x^2 + y^2 + z^4$ around the point $(0, 0, 0)$. By a linear change of coordinates the linear parts are $x - iy$ and $x + iy$. By another analytic change of coordinates $f_3(X:Y:Z)$ can be written as $2xz^2$. Then the defining polynomital on $U_T = (T = 1)$ is $f(x, y, z) = x^2 + y^2 + 2xz^2$. By the coordinate change $\tilde{x} = x - z^2$ the cubic surface is given by the defining polynomial $F(X:Y:Z:T) = T(X^2 + Y^2) - z^4$. Thus a cubic surface $V_\mathbb{R}$ with an A_3 singularity only has A_3^- or A_3^+ type singularity. Hence an $A_3^\bullet$ type singularity is impossible.
(2) If the singularity at $P = (0:0:0:1)$ is of the form A_k for $k = \{4, 5\}$, then $f_2(x:y:z)$ is of rank 2. This means that $f_2 = 0$ will consist of 2 lines in $\mathbb{P}^2(\mathbb{C})$. The product of these two linear parts l_1, l_2 gives a real polynomial thus these linear parts can both be defined over the reals, or both be complex and each others complex conjugate. By a coordinate change defined over the reals these lines are $(x + y) = 0, (x - y) = 0$ respectively $(x - iy) = 0, (x + iy) = 0$. If $f_3 = 0$ intersects $l_1 = 0$ in a complex point P then $f_3 = 0$ will intersect l_2 in the complex conjugate point $\overline{P}$ which means that $f_3 = 0$ will intersect l_1 and l_2 in the same amount of complex points. In the case of a A_4, A_5 singularity $f_3(0:0:1) = 0$ thus intersect both lines in the point $(0:0:1)$ and will intersect one line with multiplicity greater than 1. If $f_2 = 0$ splits into two complex linear parts then $f_3 = 0$ will intersect one line in 2 complex points and one in at most 1 complex point. This cannot happen which means the linear parts should be real. Thus $f_2(X:Y:Z) = X^2 - Y^2$ which always will give an A_k^- type singularity.
(3) When the cubic surface has a D_5 or E_6 singularity f_2 is the square of a real linear equation. The square of a linear part is only real when the part itself is real. Thus the coordinate change to put $f_2(X:Y:Z) = X^2$ is defined over the real numbers. From now on assume thus that $f_2 = x^2$, then $f_3(x:y:z) = 0$ will have exactly 3 intersections, counted with multplicities, with $x = 0$ in $\mathbb{P}^2(\mathbb{C})$. These three intersections may all be real or are a real point and a complex

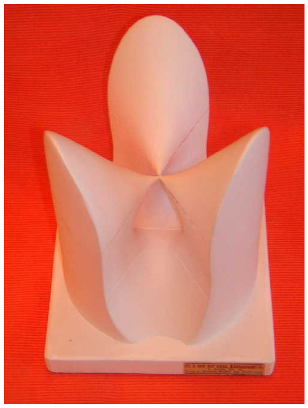

Figure 4.9: Plaster model with D_4^- singularity, Schilling VII-16.

Figure 4.10: Plaster model with D_4^+ singularity, Schilling VII-17.

conjugate pair of points. When there are less than three intersections the latter cannot occur or else counted with multplicities there would be more than 3 intersections. This is not possible by Bezouts theorem and the intersection points are real. This is exactly the case when there is a D_5 or E_6 singularity. Because in both cases f_3 is fully defined over the reals this means putting them in the normal forms $xz^2 + y^2z$ for D_5 and $xz^2 + y^3$ for E_6 can be done by a coordinate change over the reals. Because all other coordinate changes in subsection 4.2.3 are defined over the reals this means that D_5 and E_6 only have local normal forms $x^2 + xy^2 - z^4$ respectively $x^2 + y^3 - z^4$ in $\mathbb{P}^3$. This gives only a D_5^- and E_6^- singularity on the real part of a cubic surface. $\square$

Remark that the surfaces of D_4^- and D_4^+ can occur on a cubic surfaces which is shown in figure 4.10 and 4.9. This reduces the number of singularities we have to look at when making the real cubic models. Some of the singularities which are left can exist on the real cubic surfaces but not combined with any other singularities. Which is given by the following lemma.

Lemma 4.9. *Let V be a cubic surface defined over the reals. If there are multiple isolated singularities on $V_{\mathbb{R}}$ then they are all of the type A_k^-.*

Proof. First start with the A_1 singularity. Observe that for the normal forms it holds $A_1^- = -A_1^+$, this means A_1^- and A_1^+ are projectively equivalent. Then every A_1 singuarity, except $A_1^\bullet$, can be seen as an A_1^- singularity.
For A_k for $k \geq 2$ look at the linear parts of $f_2 = l_1 \cdot l_2$. When both l_1 and l_2 are complex and they are each others conjugate, then the only real point $l_1 = 0$ or $l_2 = 0$ is the point where $l_2 = l_1 = 0$. An extra singularity is a multiple intersection of $f_2 = 0$ with $f_3 = 0$ at Q away from $l_2 = l_3 = 0$. Which means that the point Q is a complex point. This will give a complex singularity at Q but $Q \notin X_{\mathbb{R}}$. Hence when multiple singularities lie on $V_{\mathbb{R}}$ the lines l_1, l_2 are real

and the quadratic part of the normal form can be made by a coordinate change to $x^2 - y^2$, and all singularities are of the form A_k^-.
This is the case when one of the singularities is of the form A_k. When a singularity on the cubic surface V is of the form D_4, D_5 or E_6 then there lie no other singularities on the cubic surface V. This is proven in subsection 4.3.3.
Thus if there lie multiple isolated singularities on a cubic surface V then all of them are of the type A_k^-. $\square$

By the classification of the cubic surfaces with isolated singularities over the complex number and the lemmas 4.7, 4.8 and 4.9, all tools for the classification over the reals are there. In this next section all real cubic surface with isolated singularities will be done.

Classification of real cubic surface with isolated singularities

Using the same reasoning as the classification of cubic surfaces over the complex numbers. We can set the singularity p at $(0 : 0 : 0 : 1)$ and get the defining equation of the cubic surface

$$F = T \cdot f_2(X : Y : Z) + f_3(X : Y : Z) \tag{4.51}$$

It is already proven that if f_2 is irreducible then P is of the type A_1. Also it is proven that every intersection $f_2 = f_3 = 0$ gives a line on the surface. In all cases their are six such points counted with multiplicity over the complex numbers. Let Q be a real point such that $Q \neq P$ and $f_2(Q) = f_3(Q) = 0$, then the whole line PQ is real. In the same way if it is complex then the line will be complex. For the classification all singularities are real thus an intersection of multiplicity greater then 1 of $f_2 = f_3 = 0$ is always a real point. This gives a way to classify the surfaces with rational singularities on them. The following cases occur:
Type 1: The curve $C = f_2(X : Y : Z) = 0$ is irreducible and has real points. This means P is a singularity of type A_1 and C is a conic which can be writen as $x^2 + y^2 - z^2$. Every intersection Q of C with $f_3(X : Y : Z) = 0$ can be real or complex. If Q is a complex point then $f_2(\overline{Q}) = f_3(\overline{Q}) = 0$ for the complex conjugate $\overline{Q}$. Furthermore the intersection of $f_3 = 0$ with $f_2 = 0$ at the complex point Q has the same multiplicity as $f_3 = 0$ with $f_2 = 0$ at the point $\overline{Q}$. Thus the 6 intersection points can consist of $0, 1, 2$ or 3 pairs of complex conjugate points and the rest real points. In figure 4.11 this is graphically denote. Every real intersection of $f_2 = 0$ and $f_3 = 0$ is denoted by a point on the circle, and every complex pair of intersections is denoted by a point inside the circle.
In the first row of figure 4.11 non of the intersections of $f_2 = f_3 = 0$ is of multiplicity greater than 1, this means that they all denote an cubic surface with an A_1 singularity. But they all are different over the reals. The first has 6 real lines through the singularity lying on the cubic surface, the second has 4 lines, the third 2 and the last has no lines lying on the real cubic surface through the singularity.
The second row denotes all cubic surfaces for which $f_2 = f_3 = 0$ has exactly 1

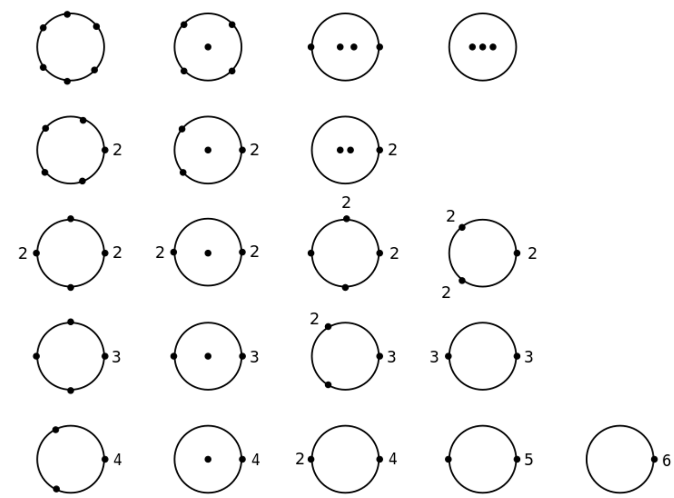

Figure 4.11: List 1

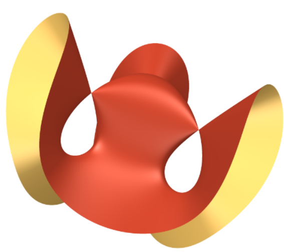

Figure 4.12: Type One

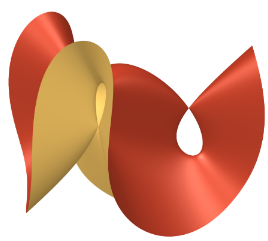

Figure 4.13: Type two

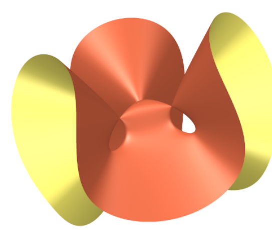

Figure 4.14: Type Three

intersection of mutliplicity 2. This is a cubic surface with 2, A_1 type singularities. Again they are different models which can be seen by the number of lines through the A_1 singularity at $(0:0:0:1)$. The first has 5 lines on the real part of the surface through $(0:0:0:1)$, the second 3 and the last just 1 line. This explains every different surface with an A_1 singularity except for the cubic surfaces with exactly 3 type A_1 singularities. The difference is seen in figures 4.12, 4.13. These models to the first and third diagrams on the third line of figure 4.11. They contain exactly the same amount of line on their real parts and the same number of singularities but, as can bee seen, they are not equivalent.
Type 2: $f_2(X,Y,Z)$ is irreducible and has no real points. Then P is an A_1^+ singularity. By lemma 4.9 this has only one type and by symmetry this is the same as the A_1^- singularity.
Type 3: $f_2(X,Y,Z)$ reduce to two linear forms which both are real. Then P is a singularity of the form A_k^- with $k \geq 2$. In the same way as by type 1 the

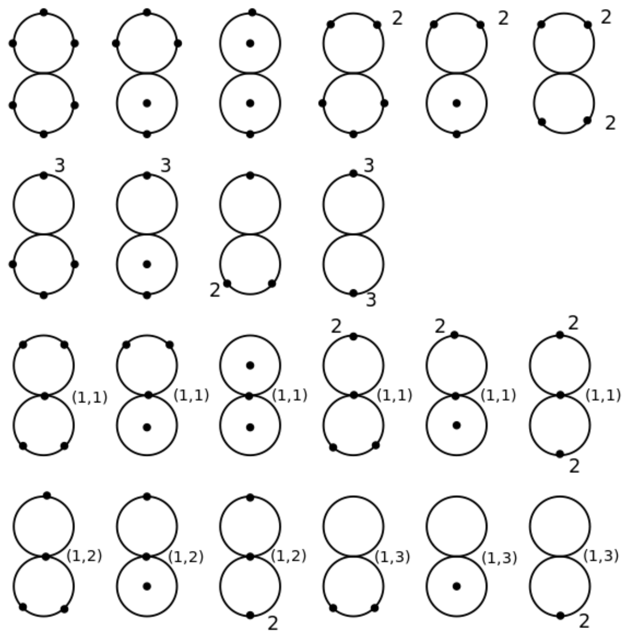

Figure 4.15: List 2

topological different types of cubic surfaces can be found by looking at the six intersections of $f_2 = 0$ and $f_3 = 0$. In figure 4.15 the upper and lower circle stand for the two lines and dots inside the circle correspond to pairs of complex conjugate points intersecting f_2 and f_3.

Type 4: $f_2(X, Y, Z)$ reduce to two complex conjugate lines. Then $f_2(X, Y, Z)$ can be writen as $X^2 + Y^2$ and the singularity is of the form A_2^+ or A_3^+. By 4.8 it cannot be A_4^+ or A_5^+. Both singularities give unique real surfaces.

Type 5: $f_2(X, Y, Z04)$ reduce to the quadratic of a linear form. By the complex classification P is a D_4 a D_5 or a E_6 singularity. When P is a D_4 singularity there are two situations. The first is when $f_2 = 0$ and $f_3 = 0$ intersect in 3 real points and the second is when the intersection points consist of a real points and two complex conjugate points. In the first case by a real linear change of coordinates the points are $(1:0:0), (1,1,0), (1,-1,0)$ and the function is localy $x^2y - y^3 + z^2$. In the second case a real coordinate transformation can give the points $(1:0:0), (1:i:0), (1:-i:0)$ and the normal form is $x^2y + y^3 - z^2$. This gives the D_4^+ and D_4^- singularity.

In lemma 4.8 it is proven that D_5^+ and E_6^+ do not occur on the real cubic surfaces. Thus there is just one real model of the D_5 and E_6^+ singularities. This gives 45 types of real cubic surfaces and they are given in figure 4.16 at the end of the chapter. The 45 are divided into the 5 smooth cubic surfaces, 20 with an A_1^- singularity, the $A_1^\bullet$, 13 with an A_k^- but not A_1^-, 2 with an A_k^+ with $k \geq 2$, and the four surfaces with $D_4^-, D_4^+, D_5^-, E_6^-$.

Number of real lines on cubic surfaces.

In the classification over an algebraically closed field the number of lines lying on the cubic surfaces are shown in subsection 4.4.1. Knörrer and Miller proved the case for the real cubic surfaces as well. This is a variation on equation 4.48. Knörrer and Miller proved that for a cubic surface V, with reality index $r(V)$, number of singularities k, sum of the Milnor numbers over the reals $\mu_{\mathbb{R}}$ given in table 4.6 and sum of $\nu(V)$ of the singularities, then the number of real lines lying on the cubic surface is calculated by equation 4.52.

$$\frac{(2+2r(V)-\mu_{\mathbb{R}}(V))(1+r(V)-\mu_{\mathbb{R}}(V)}{2}-(r(V)-2)+k-\nu(V) \qquad (4.52)$$

Schilling Number:	**Singularities:**	**Real lines:**	**Reality Index:**
VII-1	$\emptyset$	27	3
	$\emptyset$	15	2
	$\emptyset$	7	1
	$\emptyset$	3	0
	$\emptyset$	3	-1
	A_1^-	21	3
	A_1^-	11	2
	A_1^-	5	1
	A_1^-	3	0
	$A_1^\bullet$	3	0
	$2A_1^-$	16	3
	$2A_1^-$	8	2
	$2A_1^-$	4	1
VII-8, VII-7	$3A_1^-$	12	3
	$3A_1^-$	12	3
	$3A_1^-$	6	2
VII-2,...,VII-6	$4A_1^-$	9	3
	$A_1^- A_2^-$	11	3
	$A_1^- A_2^-$	5	2
	$2A_1^- A_2^-$	8	3
	$A_1^- 2A_2^-$	5	3
	$A_1^- A_3^-$	7	3
	$A_1^- A_3^-$	3	2
VII-12,	$2A_1^- A_3^-$	5	3
VII-14	$A_1^- A_4^-$	4	3
VII-15	$A_1^- A_5^-$	2	3
VII-10	A_2^-	15	3
	A_2^-	7	2
	A_2^-	3	1
VII-11	A_2^+	3	0
	$2A_2^-$	7	3
	$2A_2^-$	3	2
VII-9	$3A_2^-$	3	3
	A_3^-	10	3
	A_3^-	4	2
VII-13	A_3^-	2	1
	A_3^+	4	1
	A_4^-	6	3
	A_4^-	2	2
	A_5^-	3	3
	A_5^-	1	2
VII-16	D_4^-	6	3
VII-17	D_4^+	2	1
VII-18	D_5^-	3	3
VII-19	E_6^-	1	3

Figure 4.16: Different Topological Real Cubic Surfaces

Chapter 5

Ruled Surfaces

In this chapter the models of ruled cubic and quartic surface are treated. Starting with the ruled cubic surfaces and Schillings models of the different types. Then there will be a short mathematical background of the quartic ruled surfaces taken from [14]. To begin some background of ruled surfaces is needed. This background is based on the work of Edge [22].
First the definition of a ruled surface is needed.

Definition 5.1. *A surface V is called a ruled surface iff for every point $x \in V$ there exist a line l through x such that $l \subset V$.*

A ruled surface thus is a surface consisting of the union of all straight lines on the surface. Any line can be defined by two points lying on it. Thus a ruled surface can be defined by two curves and the lines between two points on the curves. Take two curves in $\mathbb{P}^3$, C_1 and C_2 and take a correspondence between them. Two corresponding points can be joined by a line, giving a ruled surface. The degree of this ruled surface is formalized in the following theorem.

Theorem 5.1. *The order of a ruled surface given by two curves C_1, C_2 in $\mathbb{P}^3$ of degree n_1, n_2, with a correspondence of degree (α_1, α_2). Then the degree of the ruled surface is, when the curves do not intersect, $n_1\alpha_2 + n_2\alpha_1$. Hence a 1-to-1 correspondence gives in a general a ruled surface of degree $n_1 + n_2$. Every ruled surface which is not a cone over a curve can be constructed in this way.*

Theorem 5.1 is not true when the two curves intersect at a point P, and the point P on curve C_1 corresponds to the same point P on C_2. Such a point is called a united point. Then the amount of united points and multiplicity of such a united point i is counted. This will need to be subtracted from the degree of the surface. Thus the degree of the surface then becomes $n_1\alpha_2 + n_2\alpha_1 - i$. For the proof see [22].
When working in $\mathbb{P}^3$ the tangent plane T_PV of a point P of the ruled surface V of degree n contains the line l which passes through P. Looking at $T_PV \cap V$ it consists of the line l and a curve C of order $n-1$. If P is chosen generally the curve C is irreducible and does not intersect l. Then l will meet C in $n-1$

Figure 5.1: The plaster model of Cayleys ruled surface, Schilling VII-22

distinct points. Through each point of C corresponds a line on V, and every line on V other than l will meet C in one point. Thus there is one point on C of which l is the corresponding line going through it. On the other $n-2$ points there will pass other generators. Thus the line l on V is met by $n-2$ other generating lines on a ruled surface of degree n. Concluding on a ruled surface V of order n, every general line l will meet $n-2$ other generating lines of V. This gives rise to a singular or double curve C' on V such that C' will meet every general line of V in $n-2$ points. A ruled surface in $\mathbb{P}^3$ can thus be classified by its double curve.

5.1 Ruled Cubic Surfaces

There are two types of ruled cubic surfaces in $\mathbb{P}^3(\mathbb{R})$, the reducible and irreducible cubic surfaces. The reducible cubic surfaces consist of a plane and a quadric or three planes. The more intereseting type is the irreducible ruled cubic surface. In subsection 4.2.5 it is already stated that most of these ruled surfaces start as irreducible cubic surfaces with a singular line on them. Start again with an irreducible cubic surface V with non isolated singularities. Every generic plane intersecting V is irreducible. Looking at the tangent plane T_PV of a generic point P then the whole line l on which P lies lies on T_PV making the whole line a double line in $T_PV \cap V$. Thus every line on V will intersect exactly one other line. The generic plane intersected with the surface V has only one singular point. Then the subset of all singular points on V is a line. By a linear change of coordinates this is the line $l : (x = y = 0)$. Requiring every point on l to be singular gives the requirement that every term of the polynomial $F(X : Y : Z : T)$ is divisible by X^2, Y^2 or XY. Putting the defining equation $F(X : Y : Z : T)$ of V in the following general form:

$$F(X : Y : Z : T) = f_3(X, Y) + Zf_2(X, Y) + Tg_2(X, Y) \qquad (5.1)$$

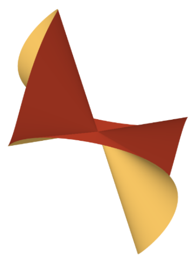

Figure 5.2: Cone over a cuspidal cubic. Case (1)

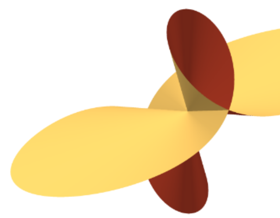

Figure 5.3: Cone over a nodal cubic. Case (2)

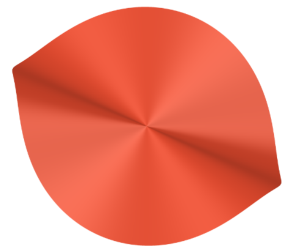

Figure 5.4: Cone over a non singular cubic. Case (3)

Linear transformation can be used to put f_2 and g_2 in one of the following normal forms in $\mathbb{C}^3$:

$$(f_2(X,Y), g_2(X,Y)) = (x^2, y^2), (x^2, xy), (xy, 0), (x^2, 0), (0, 0) \tag{5.2}$$

In the first case the variables Z, T can be adjusted by a linear combination of X and Y to reduce f_3 to zero. Giving the normal form $F(X : Y : Z : T) = X^2Z + Y^2T$. In the second case f_3 can be reduced to a multiple of y^3, or x will be a factor and V reducible. The normal form in this case is $F(X : Y : Z : T) = X^2Z + XYZ + Y^3$. In the last three cases the variable T does not occur in the equation of F. In these cases we have a cone over a cuspidal or nodal cubic curve.
In the first two cases when $(f_2, g_2) = (x^2, y^2), (x^2, xy)$ the ruled cubic surface is not a conic over a curve thus is constructed by a correspondance between two curves. By theorem 5.1 these curves are a conic and a line in a 1-to-1 correspondance. The conic is called the director conic and the line is called the director line or directrix.
The special case where the director conic and the director line meet is called Cayleys ruled surface. Cayleys ruled surface is not given by a $1-1$ correspondence between a conic and a line but by a $(2-1)$ correspondence. Every point on the line will correspond to two points on the conic. This would normally give a ruled surface of degree 4. By theorem 5.1, when the line and the conic meet in a united point, the degree of the surface will drop. The united point on Cayleys ruled surface is counted once, dropping the degree by 1 and giving a cubic ruled surface. This is most easily seen on the model in image 5.1 where the double line is the director line and the directerix conic is the circle etched in the middle.
When the ruled surface is a cone over a cubic curve there are three different cases: (1) the cubic is singular and cuspidal, (2) the cubic is singular and nodal or (3) the cubic is non singular. Where the cone over a non singular cubic curve corresponds with an $\hat{E}_6$ singularity. These different cases are all shown in figures 5.2, 5.3 and 5.4.

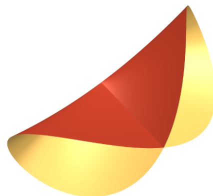

Figure 5.5: $(f_2, g_2) = (x^2, y^2)$

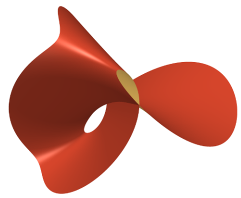

Figure 5.6: $(f_2, g_2) = (x^2, xy)$

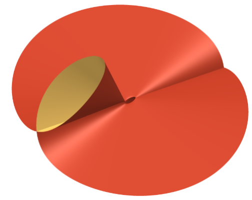

Figure 5.7: $(f_2, g_2) = (x^2, x^2 - y^2)$

5.1.1 Real Ruled Cubic Surfaces

In $\mathbb{R}^3$ the classification is different than in $\mathbb{C}^3$. The base form is the same as in equation 5.2. The following theorem will give the different types.

Theorem 5.2. *If V a cubic ruled surface in $\mathbb{R}^3$ with defining equation in the normal form, as in equation 5.2. Then the pair (f_2, g_2) is projectively equivalent, over $\mathbb{R}$, to one of the following:*

$$(f_2(x,y), g_2(x,y)) = (x^2, y^2), (x^2, xy), (xy, x^2 - y^2), (xy, 0), (x^2, 0), (0, 0) \quad (5.3)$$

Proof: For the first part assume $\frac{f_2}{g_2}$ is not constant.
Start with the case when not both f_2 and g_2 are indefinite. Then one of f_2, g_2 is definite, assume f_2 is definite. Then by a linear change over the reals $f_2 = x^2 + y^2$. There exists an orthogonal transformation M such that f_2 stays fixed over M and g_2 will be $g_2 = ax^2 + by^2$. Assumed was $\frac{f_2}{g_2}$ not a constant thus $a \neq b$. By linear transformations over Z and T it is possible to get $(f_2, g_2) = (x^2, y^2)$.
If both f_2 and g_2 are indifinite then by a tranformation $f_2 = xy$, and g_2 can be written by another orthogonal transformation as $ax^2 - by^2$ with $a, b \geq 0$. $(f_2, g_2) = (xy, x^2 - y^2)$. If a or b is 0 then $(f_2, g_2) = (xy, x^2)$.
If $\frac{f_2}{g_2}$ is a constant then by a linear transformation on z, $g_2 = 0$. The surface is given by a cone over a cubic curve or is a plane. If on the surface cubic lies a line of singularities then the different curves are a nodal or cuspidal. This gives the possibilities $(f_2, g_2) = (x^2, 0), (xy, 0)$. $\square$

Most ruled cubic surfaces form the same category in $\mathbb{C}^3$ as in $\mathbb{R}^3$, the only different category is $(f_2, g_2) = (xy, x^2 - y^2)$. By linear transformation over z and t all terms in f_3 can be eliminated and the normal form is $f(x, y, z) = zxy + x^2 - y^2 +^3$. The different types of ruled cubic surfaces which aren't cones over curves are given in figures 5.5, 5.6 and 5.7.

5.2 Quartic Ruled Surfaces

In this section the quartic ruled surfaces are treated. These are the ruled surface which can be given as the zeroset of a homogeneous polynomial of degree 4 in

$\mathbb{P}^3(\mathbb{R})$. Every line in can be parametrized by two points and the ruled surface can be given by the correspondance between two curves. In the beginning of the chapter the ruled surfaces are categorized by a correspondance between points on a curve and a line. The quartic surfaces have more variety and therefore are harder to classify. This will be done by looking at the double curve of the quartic ruled surface, which was the curve of the surface on which lines intersect and can be found by finding the singular points. Most quartic surfaces can be constructed by a $(2,2)$ correspondence between the double curves lying on it. The following theorem will give all possibilities of double curves on quartic surfaces.

Theorem 5.3. *Let V be a ruled surface, not a cone over a curve, given as the zeroset of a homogeneous polynomial of degree 4 in $\mathbb{P}^3(\mathbb{C})$. Let $C \subset V$ be the singular locus of V then C is one of the following*

- *C consists of two skew double line.*
- *C consists of a double line and a double conic.*
- *C consists of a double space curve of degree three.*
- *C consists of a triple line.*

For the proof see [22]. All of the different types will be treated next. In the proof by Edge he starts in the same way as the cubic surfaces by taking the tangent plane at a general point. This he intersects with the ruled surface V to get a line and an irreducible curve of degree three. If the surface V is a cone over a quartic the intersection would constist of 4 lines,

5.2.1 Two skew double lines.

First a definition is needed.

Definition 5.2. *A bi-homogeneous form $F \subset P^3(\mathbb{C})$ of type (i,j) is a polynomial in variables x_1, x_2, y_1, y_2 such that:*

$$F(\lambda x_1, \lambda x_2, \mu y_1, \mu y_2) = \lambda^i \mu^j F(x_1, x_2, y_1, y_2) \tag{5.4}$$

for all $\lambda, \mu \in \mathbb{C}$.

Now a construction of a quartic ruled surface is possible from the two skew double lines.

Theorem 5.4. *Let Ω be the rational map*

$$\Omega : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \tag{5.5}$$

$$(x_1 : x_2 : y_1 : y_2) \to ((x_1 : x_2), (y_1 : y_2)) \tag{5.6}$$

Then the closure $\Omega^{-1}(C)$ of the curve $C : \mathbb{P}^1 \times \mathbb{P}^1$ is a ruled surface V. If C is given by a bi-homogeneous polynomial F then V is the zeroset of a bi-homogeneous polynomail F too.

Proof The map $\Omega^{-1}(C)$ is not defined on the lines $l_1 = (a_1 : a_2 : 0 : 0)$ and $l_2 = (0 : 0 : b_1 : b_2)$. Let V be the closure of $\Omega^{-1}(C)$. Then for every point $P = (x_1 : x_2 : y_1 : y_2) \in V \setminus (l_1 \cup l_2)$, $\Omega(P) = ((x_1 : x_2), (y_1 : y_2)) \in C$. Thus $F((x_1 : x_2), (y_1 : y_2)) = 0$ and the closure of

$$\Omega^{-1}((x_1 : x_2), (y_1 : y_2)) = \{(\lambda x_1 : \lambda x_2 : \mu y_1 : \mu y_2)|\ (\lambda, \mu) \in \mathbb{P}^1(\mathbb{C})\}$$

Thus the closure connects the points $(x_1 : x_2 : 0 : 0)$ on l_1 with $(0 : 0 : y_1 : y_2)$ on l_2 by a line. Because this is true for every point the surface, V is a ruled surface. $\square$

This theorem only gives a construction of ruled surfaces from two curves. Theorem 5.5 will show that every quartic surface with two different double lines is a ruled surface.

Theorem 5.5. *Let $V \subset \mathbb{P}^3(\mathbb{C})$ be a surface given by a homogeneous polynomial F of degree 4. Suppose the lines $l_1 = (a_1 : a_2 : 0 : 0)$ and $l_2 = (0 : 0 : b_1 : b_2)$ are double lines on V. Then F is bi-homogeneous of degree $(2, 2)$ and V is a ruled surface.*

Proof. The homogeneous polynomial F can be written as

$$F(x_1 : x_2 : y_1 : y_2) = F_{(4,0)} + F_{(3,1)} + F_{(2,2)} + F_{(1,3)} + F_{(0,4)} \tag{5.7}$$

Where $F_{(i,j)}$ is bi-homogeneous of degree (i, j). Since $F = 0$ on the lines l_1 and l_2 this means that $F_{(4,0)} = F_{(0,4)} = 0$. Furthermore the lines are double lines. Thus $F_{(3,1)}(x_1 : x_2 : y_1 : y_2) = 0$ for all $(x_1, x_2) \neq 0$, hence $F_{(3,1)} = 0$. By the same reasoning $F_{(1,3)} = 0$ and can conclude $F = F_{(2,2)}$. Thus every ruled surface with the two double lines l_1 and l_2 are bi-homgeneous of degree $(2, 2)$. Now it will be proven that every point will lie on a line on the surface. Take a point on V then it either lies on one of the lines l_1, l_2 or is a point $(a_1 : a_2 : b_1 : b_2)$ where $(a_1 : a_2) \neq (0, 0)$ and $(b_1, b_2) \neq (0, 0)$. The defining polynomial of V is bi-homogeneous of degree $(2, 2)$ thus $F(\lambda a_1 : \lambda_2 : \mu b_1 : \mu b_2) = \lambda^2\mu^2 F(a_1 : a_2 : b_1 : b_2) = 0$ for every $(\lambda : \mu) \in \mathbb{P}^1(\mathbb{C})$. This is a line on the surface thus the point $(a_1 : a_2 : b_1 : b_2)$ lies on a line on the surface. Thus every point lies on a line on the surface. $\square$

By a linear transformation in $\mathbb{C}$ every such pair of double lines can be put in the normal form of l_1 and l_2 above. Thus theorem 5.5 tells every surface of degree 4 with two skew double lines is a ruled surface and is of this form. The quartic ruled surfaces are modeled by Schillings company with wireframes. The quartic ruled surface with two non intersecting double lines are the models XIII-01 through XIII-05 [17]. Model XIII-04 of the schilling catalogue is shown in image 5.8, and has two conjugated complex double lines.

5.2.2 A double conic and a double line.

When the double curve, of the quartic ruled surface V, consist of a double conic C and a double line l with $C \cap l$ is a point P and the conic and the line

Figure 5.8: Quartic ruled surface with two conjugated imaginary double lines, Schilling XIII-04

lie not in the same plane. In the space $\mathbb{P}^3(\mathbb{C}) = (x_1 : x_2 : y_1 : y_2)$ fix the line $l := (y_1 = y_2 = 0)$ and the conic $C := (x_1 = y_2^2 - x_2 y_1 = 0)$. Then the line can be parametrized as $\{(\lambda : 1 : 0 : 0) | \lambda \in \mathbb{C}\}$ and the conic as $\{(0 : 1 : \mu^2 : \mu) | \mu \in \mathbb{C}\}$. Every line on the ruled surface is the connection of a point on the line and the conic by a straight line these lines can be parametrized as $\{(0 : 1 : \mu^2 : \mu) + \rho(\lambda : 1 : 0 : 0)\}$. This gives points $(\rho\lambda : \rho + 1 : \mu^2 : \mu) = (x_1 : x_2 : y_1 : y_2)$. The variables λ and μ now can be expressed in the following way

$$\lambda = \frac{y_1}{y_2} \quad \mu = \frac{x_1 y_1}{x_2 y_1 - y_2^2} \tag{5.8}$$

Which points are connected is given by a symmetric bi-homogeneous curve $F : \mathbb{P}^1 \times \mathbb{P}^1 \cong l \times C$ of bidegree $(2, 2)$. This can be used to find a general form for the defining curve of the ruled surface. The ruled surface V has the following equation

$$\frac{y_2^2 (x_2 y_1 - y_2^2)^2}{y_1^2} F\left(\frac{y_1}{y_2}, \frac{x_1 y_1}{x_2 y_2 - y_2^2}\right) = 0 \tag{5.9}$$

Here the equation is divided by y_1^2 because the conic and line intersect. The equation has the following form

$$a_{ij} y_1^{i+j-2} x_1^j y_2^{2-i} (x_2 y_2 - y_2^2)^{2-j} = 0 \tag{5.10}$$

In [14] it has been proven that the only occuring pairs in the bi-homogeneous curve F are $(2, 2), (1, 1), (2, 0) + (0, 2), (0, 0)$. The curve is singular at $(0, 0)$ thus $(i, j) = (0, 0)$ does not occur. Thus the ruled surface of degree four with a double conic and a double line have defining equations:

$$a_{22} x_1^2 y_1^2 + a_2 (x_1^2 y_2^2 \pm (x_2 y_1 - y_2^2)^2) + a_{11} x_1 x_2 (x_2 y_1 - y_2^2) = 0 \tag{5.11}$$

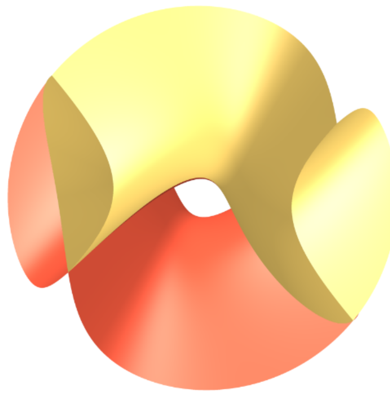

Figure 5.9: Quartic Ruled Surface with Double Line and Double Conic

If there was not an intersection between the conic and the lines theorem 5.1 will tell the constructed surface has order 6. This is seen in the construction, if the conic and line did not intersect the term y^2 could not be divided out of the polynomial and the polynomial would have had order 6.
Schilling made only one model of the quartic ruled surface with a double line and double conic. This is model XIII-08. In figure 5.9 a ruled surface with a double conic and double line is shown. The double conic can be seen but the double line does not lie on the affine part shown. The double line is the projective line $z = t = 0$ which is not part of the affine space.

5.2.3 A double cubic space curve.

The construction of a quartic ruled surface V with a double cubic space curve is the same as the one with two double lines and the ruled surface with a double conic and double line. The surface is made by connecting two points on the singular part of the surface. This is done by a bi-homogeneous curve on $\mathbb{P}^1 \times \mathbb{P}^1$. This will be done here as well.
First observe that any non singular cubic space curve $N \subset \mathbb{P}^3$ can be parametrized as

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^3 \\ (x_1 : x_2) &\to (x_1^2 : x_1^2 x_2 : x_1 x_2^2 : x_2^3) \end{aligned} \tag{5.12}$$

Every point on the surface V will lie on the connection of two points on the same curve by a straight line and can be expressed as $(1 : \lambda : \lambda^2 : \lambda^3) + \rho(1 : \mu : \mu^2, \mu^3) = (x_1 : x_2 : y_1 : y_2)$. The following relations are found:

$$\begin{aligned} x_2 y_2 - y_1^2 &= \rho\lambda\mu(\mu - \lambda)^2 \\ x_1 y_1 - x_2^2 &= \rho(\lambda - \mu)^2 \\ x_1 y_2 - y_1 x_2 &= \rho(\lambda + \mu)(\mu - \lambda)^2 \end{aligned} \tag{5.13}$$

Figure 5.10: Quartic ruled surface with cubic double curve, Schilling XIII-09

Figure 5.11: Quartic ruled surface with cubic double curve, Schilling XIII-10

This gives the following relations for $\lambda\mu$ and $\lambda + \mu$

$$\lambda\mu = \frac{x_2 y_2 - y_1^2}{x_1 y_1 - x_2^2} \quad (\lambda + \mu) = \frac{x_1 y_2 - y_1 x_2}{x_1 y_1 - x_2^2} \tag{5.14}$$

Find a function $F \subset \mathbb{P} \times \mathbb{P} \cong N \times N$, non-singular and bi-homogeneous of type $(2,2)$. The quartic ruled surface of degree 4 will have defining function

$$(x_1 y_1 - x_2^2)^4 F(\lambda, \mu) = (x_1 y_1 - x_2^2)^4 \cdot (a((\lambda\mu)^2 \pm 1) + b(\lambda + \mu)^2 + c\lambda\mu) \tag{5.15}$$

This will give the ruled surface with defining equation

$$a(X^2 \pm Z^2) + bY^2 + cXZ = 0 \tag{5.16}$$

Where $X = (x_1 y_1 - x_2^2)$, $Y = (x_1 y_2 - x_2 y_1)$ and $Z = (x_2 y_2 - y_1^2)$.
The wireframe models of the quartic surfaces with a double cubic space curve are models 09 and 10 of series $XIII$ by Schilling. Utrecht University has both such models which are shown in figures 5.10 and 5.11

5.2.4 A triple line.

Theorem 5.6. *Let $V \subset \mathbb{P}^3(\mathbb{C})$ be an irreducible surface given by $F = 0$ with F a homogeneous polynomial of degree 4. Suppose the line $l = (x_1 : x_2 : 0 : 0)$ is a triple line, then F has the form:*

$$x_1 F_3(y_1 : y_2) + x_2 G_3(y_1 : y_2) + F_4(y_1 : y_2) \tag{5.17}$$

with F_3, G_3 homogeneous of degree 3 in y_1, y_2 and F_4 homogeneous of degree 4 in y_1, y_2 and V is a ruled surface.

Proof Wirte F in the following form

$$F(x_1 : x_2 : y_1 : y_2) = F_{(4,0)} + F_{(3,1)} + F_{(2,2)} + F_{(1,3)} + F_{(0,4)} \tag{5.18}$$

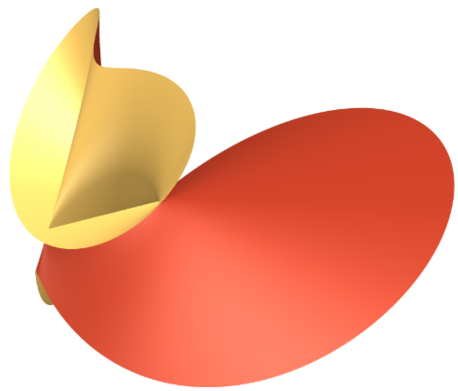

Figure 5.12: Quartic Ruled Surface with a Triple Line

Where $F_{(i,j)}$ is bi-homogeneous of degree (i,j). Since the line l lies on the surface V then $F_{(4,0)} = 0$. Looking on the affine card where $x_2 = 1$ then every point is writen as $(u : 1 : v : w)$ are the coordinates with $u = \frac{x_1}{x_2}, v = \frac{y_1}{x_2}, w = \frac{y_2}{x_2}$. Then F can be rewritten as

$$F = A_4(v,w) + (uA_3(v,w) + B_3(v,w)) + (u^2A_2(v,w) + uB_2(v,w) + C_2(v,w)) + \\ (u^3A_1(v,w) + u^2B_1(v,w) + uC_1(v,w) + D_1(v,w)) \tag{5.19}$$

Here A_i, B_i, C_i, D_i are homogeneous of degree i. The intersection of the surface V with a plane $y_1 = \alpha y_2$ is a curve which contains l as a triple line. This means that the function $F(u : 1 : v : \alpha v)$ has a factor v^3. Thus

$$F(u : 1 : v : \alpha v) = A_4(v : \alpha v) + uA_3(v : \alpha v) + B_3(v : \alpha v) \tag{5.20}$$

Which gives the form in equation 5.17.
The surface V is irreducible thus $gcd(F_3, G_3, F_2) = 1$. Take a point $P = (a_1 : a_2 : b_1 : b_2)$ on V, if P not on the line l then $(b_1, b_2) \neq (0,0)$. For this point if $F_3(b_1 : b_2) = G_3(b_1 : b_2) = 0$ then $F_2(b_1 : b_2) = 0$ and $gcd(F_3, G_3, F_2) \neq 1$. Thus there exist $(c_1, c_2) \neq (0,0)$ such that $c_1F_3(b_1 : b_2) + c_2G_3(b_1 : b_2) = 0$ and the line through $(a_1 : a_2 : b_1 : b_2)$ and $(c_1 : c_2 : 0 : 0)$ lies on V. □

A ruled surface of degree 4 with a triple line is given in figure 5.12. Unfortunatly not the whole of the line is seen as singular because part of the surface is imaginary. The corresponding models of this type in the catalogue of Schilling are XIII-6 and XIII-7

Conclusion

The mathematical background of Schillings models is an extensive one with this thesis just containing the irreducible cubic surfaces and the ruled quartic surfaces. These surfaces correspond to series VII and $XIII$ of Schilling catalogue. All cubic surfaces are classified in chapter 4 with the complex cubic surfaces only differing in the type and number of isolated singularities. The real cubic surfaces could further be catagorized by the number of lines lying on the real part of the cubic.
The cubic ruled surfaces are classified through the use of projective transformations to find all possible normal forms for real cubic ruled surfaces. All these projective transformations do not transform the singular line which always lies on a ruled cubic surface, limiting the options.
Quartic ruled surfaces need to be catagorized differently because there are multiple types of singular curve lying on a quartic ruled surface. Thus these are classified by the curve which is singular on the surface.
This thesis just gives the mathematical background to two of the series of models Schilling gave and does not deal with the real quartic ruled surfaces. This thesis has to small a scope to deal with the mathematical background for all the 39 different sets of models given in Schillings catalogue. This would be a direction a next project could go towards.

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Appendix A

Figures

Almost all photographs of the models are credited to the catalogue of the University of Utrecht. There is one exception namely figure 4.9 which is from the online catalogue of the Rijksuniversiteit of Groningen.
All constructed figures in this thesis are constructed by the use of Surfer. The equations in the table below are the equation used. The figures with an asterix are surfaces for which the affine equations are created by Stephan Holzer and Oliver Labs in [8].

Figure	**Polynomial**
Figure 4.2	$x^2+y^2+z^2+2xyz-1=0$
Figure 4.7*	$4(x^3+3x^2-3xy^2+3y^2+1/2)+3(x^2+y^2)(z-6)+$ $3/2(x^2+y^2-z^3)=0$
Figure 4.12*	$x^3+3x^2-3xy^2+3y^2+z^3+z^2(x+1/2)-4=0$
Figure 4.13*	$x^3+3x^2-3xy^2+3y^2+z^3+2z^2-4=0$
Figure 4.14*	$4(x^3+3x^2-3xy^2+3y^2+1/2)+3(x^2+y^2)(z-6)-$ $z(3+4z+7z^2)-z^2=0$
Figure 5.2	$zx^2-y^3=0$
Figure 5.3	$zx^2+x^3-y^3=0$
Figure 5.4	$xyz+3y^2z+z^3+x^2=0$
Figure 5.5	$zx^2+y^2=0$
Figure 5.6	$zx^2+xy-y^3=0$
Figure 5.7	$zxy+x^2+y^2+x^3-y^3+x^2y=0$
Figure 5.9	$100x^2z^2+50(x^2-(yz-1)^2)+36xy(yz-1)=0$
Figure 5.12	$(y^3-y^2z+z^2y-z^3)+x(y^3-z^3)+(y^2z^2+y^3z+yz^3)=0$