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Intuitionistic propositional logic and some special formula classes

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Chapter 1

Introduction

Intuitionistic propositional logic (**IPC**) is based on intuitionism which is a constructive form of mathematics. This means that in order for a mathematical statement to be true we must give an existential proof of it, i.e. we must find a mathematical object in order to prove its existence.

Intuitionism has never replaced classical mathematics as the standard point of view on mathematics seeing as many classical existential theorems do not have a constructive analogue. Nevertheless intuitionism is still widely studied today due to the algorithmic properties of intuitionistic proofs. These proofs are in actuality programs which are remarkably useful in computer science since there are things we can do with programs that we can't do with proofs.

Intuitionistic propositional logic consists of a set of well-formed formulas that make up the language. We are interested in how we can derive such formulas such that we know when a certain mathematical statement is valid/provable in the logic.

In this thesis we intend to show the different ways in which we can construct intuitionistic propositional logic as a formal system. We will discuss the model-theoretic approach which gives meaning to the symbols of our language, and the proof-theoretic approach which tries to find a fixed set of axioms and/or inference rules from which all other formulas in the logic are derivable. We shall provide both a semantic and a syntax of the language. We give two kinds of semantical interpretations the BHK-interpretation and the Kripke semantics and the two syntactic systems namely the Hilbert and natural deduction system. This leads us to both the notion of validity and the notion of provability. Consequently we will show that these two notions correspond to each other via the completeness theorem for **IPC**.

In section 2 we give a general introduction to intuitionism and introduce the different kinds of formalizations, we then show by means of a translation, how we can interpret classical logic in intuitionistic logic. We will end chapter 2 with the completeness theorem for **IPC** and some essential definitions needed for chapter 3.

The final section of the thesis is centered around intermediate logics which are situated between classical and intuitionistic propositional logic. Our main focus lies in the axiomatization of two kinds of intermediate logics, the subframe and the stable logics. In general, to axiomatize a logic means to start with a formal language and a satisfaction relation, that tells us what the strings mean, and from this create a set of axioms and/or inference rules which tells us what the valid rules are for proving new statements in the logic.

Such axiomatizations occur via **NNIL**-formulas, which are formulas with " no nested implications to the left" and **ONNILLI**-formulas which are formulas that are " only **NNIL** to the left of implications". We assume that subframe logics and stable logics are axiomatizable by subframe and stable formulas respectively. For each finite rooted frame $\mathfrak{F}$, we construct its subframe formulas as **NNIL**-formulas with the help of colorful models. We then use this characterization to show that the **NNIL**-class is (up to frame equivalence) the same as the class of subframe formulas. We deduce from this that an intermediate logic is a subframe logic iff it is axiomatizable by **NNIL**-formulas.

A similar process is used for stable logics. We know that the syntactic characterization of stable formulas was an open problem. This problem was resolved in [1], by constructing the stable formulas, for each finite rooted subframe $\mathfrak{F}$, as an **ONNILLI**-formula. We will show how to do this and then use this characterization to show that the **ONNILLI**-class is (up to frame equivalence) the same as the class of stable formulas. As a consequence we prove that an intermediate logic is a stable logic iff it is axiomatizable by **ONNILLI**-formulas.

Our primary source of information lies in [1] and [2], which we discuss in chapter 2 and 3 respectively.

Chapter 2

Intuitionism

Intuitionism is a philosophy of mathematics introduced by L.E.J. Brouwer in the 20$^{\text{th}}$ century. Its origin stems from Brouwer's opposition to non-constructive methods such as Cantor's set theory [3]. It is based on the idea that mathematics is a creation of the mind and not some pre-existing concept to be discovered.

Intuitionists tie mathematical truth and falsity to proof and disproof, this means that for an intuitionist a mathematical statement is true if it can be proven and false if there exists a proof of its absurdity. Therefore intuitionism and classical mathematics deviate in a number of ways, the first of which is that intuitionism does not accept proofs by contradiction, wherein we assume that $\neg p$ is true and derive a contradiction. This stems from the fact that for intuitionists it is necessary to find a mathematical object in order to prove its existence. According to intuitionists $\neg p$ means that it is impossible to prove p and $\neg\neg p$ means that it is impossible to prove that there is no proof of p. This however does not mean that p is true, therefore intuitionism does not accept the principle of double negation implication ($\neg\neg p \to p$), while classical mathematics does. Another classical tautology not accepted by intuitionists is the principle of excluded middle ($p \vee \neg p$) seeing as there are mathematical propositions for which we neither have a proof nor a proof that refutes it. An example of such a proposition is the Riemann hypothesis [4], which is a well known unresolved theorem.

That a certain statement is not currently provable does not imply that in the future it can't become provable. Therefore the dependence of intuitionism on time is essential seeing as an unprovable statement now may become intuitionistically valid in the future.

Besides the rejection of the principle of excluded middle and the double negation implication intuitionism also contradicts classical mathematics in terms of the use of a distinct logic, called the intuitionistic logic. This logic was first introduced by Arend Heyting, who gave a detailed axiomatization of it. The concepts discussed in this chapter come from chapters 2 and 3 of [2].

2.1 Intuitionistic Logic

One of the basic differences between classical and intuitionistic logic is the interpretation of the connectives: $\vee, \wedge, \rightarrow$ and the quantifiers: $\forall, \exists$. In intuitionistic logic a sentence is true if we can indeed find such x such that $\phi(x)$ is true, i.e. if we can provide a constructive method of finding an element n such that $\phi(n)$ is true.
We know that intuitionistic logic omits several classic tautologies, hence we can say that in the most direct sense intuitionistic logic is weaker than classical logic. Consequently the logic has a wider range of semantical interpretations.
Over the years a number of different semantical theories have been proposed to give us a basic understanding of how Brouwer interpreted the different connectives, some of which are:

- The BHK-Interpretation (Brouwer, Heyting and Kolmogorov), which translates the meaning of the logical connectives directly in terms of proof
- Kripke semantics
- Kleene's realizability semantics, where formulas are seen as codes for algorithms
- The topological interpretation, which describes intuitionistic formulas as open sets in a topology

An important way in which these interpretations differ is how simple they make meta-theoretical investigation vs. how much they add to our intuitive understanding of the formulas. The BHK and topological interpretation contrast in the fact that the BHK-interpretation gives us the meaning of the connectives spelled out in terms of truth conditions, while the topological interpretation tells us how we can translate our knowledge about topology in order to understand the formal properties of intuitionistic logic.
The main focus of this chapter is intuitionistic propositional logic **IPC**, which is a subgroup of intuitionistic logic.
Intuitionistic and classical propositional logic share the same language which consists of logical connectives $\wedge, \vee, \rightarrow$, propositional variables: $p, q, ...$ and a constant $\perp$. Formulas in **IPC** are constructed in the following way:

$$\varphi = \perp \mid p \mid \varphi \wedge \ \varphi | \ \varphi \vee \ \varphi | \ \varphi \rightarrow \varphi$$

Assume $\varphi, \psi, \phi, \chi \in$ FORM, with FORM the set of well-formed formulas in **IPC**, p is a random propositional variables and Γ represents a set formulas. We also use the following abbreviations:

- $\neg\varphi$ is defined as $\varphi \rightarrow \perp$
- $\varphi \leftrightarrow \psi$ is defined as $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$

– $\top \equiv \neg\bot$

2.1.1 BHK-Interpretation

The standard description of intuitionistic logic today is the BHK-Interpretation (Brouwer, Heyting and Kolmogorov), which is based on the notion of proof (a convincing natural argument) instead of truth. This interpretation gives a very clear foundation of intuitionistically acceptable principles, one defines a proof of a propositional formula φ in terms of the proof of its constituents which are built up from propositional variables and connectives. The propositions are thus regarded as problems to be solved and their proofs as methods that solve them. We can informally define the connectives as follows, we prove:

- $\varphi \wedge \psi$, by proving both φ and ψ and concluding $\varphi \wedge \psi$
- $\varphi \vee \psi$, by proving φ or ψ and concluding $\varphi \vee \psi$
- $\varphi \rightarrow \psi$, by providing a procedure that convert any proof of φ into a proof of ψ
- $\neg\varphi$ by giving a method of converting any proof of φ into a proof of a contradiction
- There is no proof for $\bot$

We can now clearly see why $p \vee \neg p$ does not generally hold since there is no way of guaranteeing a proof of p or a proof of $\neg p$ for all possible propositions. To give a general idea of which statements are valid and not valid we give the following table:

not valid	valid
	$\neg\neg(\varphi \vee \neg\varphi)$
$\varphi \vee \neg\varphi$	$\neg\neg\neg\varphi \leftrightarrow \neg\varphi$
$\neg\neg\varphi \rightarrow \varphi$	$\varphi \rightarrow \neg\neg\varphi$
$\neg(\varphi \wedge \psi) \rightarrow (\neg\varphi \vee \neg\psi)$	$\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$
$(\neg\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \varphi)$	$(\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi)$
$(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \phi)$	$(\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\phi)$
$(\varphi \rightarrow \psi) \rightarrow (\neg\varphi \vee \psi)$	$(\neg\varphi \vee \psi) \rightarrow (\varphi \rightarrow \psi)$
$((\varphi \rightarrow \psi) \rightarrow \varphi) \rightarrow \varphi$	$\neg\neg(\varphi \wedge \psi) \leftrightarrow \neg\neg\varphi \wedge \neg\neg\psi$
$(\neg\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \varphi)$	$\neg\neg(\varphi \rightarrow \psi) \leftrightarrow \neg\neg\varphi \rightarrow \neg\neg\psi$
$(\varphi \rightarrow \psi \vee \chi) \rightarrow (\varphi \rightarrow \psi) \vee (\varphi \rightarrow \chi)$	$(\varphi \rightarrow \psi \wedge \chi) \leftrightarrow (\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)$
$((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow (\varphi \vee \psi)$	$(\varphi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow (\varphi \wedge \psi) \rightarrow \chi)$
	$(\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi) \leftrightarrow ((\varphi \vee \psi) \rightarrow \chi)$

2.1.2 Systems for IPC

Now that we have an interpretation of the connectives we must develop a derivation system which consists of a set of inference rules and/or axioms. Axioms are formulas for which no proof is required, they may be used at the beginning of a proof. Inference rules on the other hand tell us which steps we are allowed to make in a proof.
The derivation system gives us a way to formalize a logic, it determines the validity or invalidity of a conclusion deduced from two or more statements/premises. This means that a derivation system provides us with a way of deriving all provable formulas in the logic. There are many equivalent ways to present the deduction rules of intuitionistic propositional logic, the two most noticeable are:

- The Hilbert style formalization given by A. Heyting
- The Natural Deduction formalization invented by Gentzen in 1935

2.1.2.1 Hilbert type system for IPC

According to the Hilbert type system intuitionistic propositional logic is the smallest set of formulas that contains the following axioms:

- $\varphi \to \varphi \vee \psi$
- $\psi \to \varphi \vee \psi$
- $\varphi \wedge \psi \to \varphi$
- $\varphi \wedge \psi \to \psi$
- $\bot \to \varphi$
- $\varphi \to (\psi \to \varphi)$ (1)
- $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$ (2)
- $(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$

with *modus ponens* as its inference rule:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

Let Γ be the set of well-formed **IPC**-formulas. If φ is derivable/provable from Γ we write $\Gamma \vdash_{\mathbf{IPC}} \varphi$ which is defined as follows:

(a) if $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathbf{IPC}} \varphi$

(b) if φ is a substitution instance of an axiom scheme (i.e., an axiom), then $\Gamma \vdash_{\mathbf{IPC}} \varphi$

(c) if $\Gamma \vdash_{\mathbf{IPC}} \varphi$ and $\Gamma \vdash_{\mathbf{IPC}} \varphi \to \psi$, then $\Gamma \vdash_{\mathbf{IPC}} \psi$, (using *modus ponens*)

The following theorem follows from chapter 3 of [2].

Theorem 2.1 (Deduction Theorem)**.** *If* $\Gamma, \varphi \vdash_{\mathbf{IPC}} \psi$*, then* $\Gamma \vdash_{\mathbf{IPC}} \varphi \to \psi$

Proof. First we show that $\vdash_{\mathbf{IPC}} \varphi \to \varphi$. If we substitute ψ by $(\varphi \to \varphi)$ and χ by φ in (1) and (2) we get $\vdash_{\mathbf{IPC}} (\varphi \to ((\varphi \to \varphi) \to \varphi)) \to ((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi))$ and $\vdash_{\mathbf{IPC}} (\varphi \to ((\varphi \to \varphi) \to \varphi)$. If we substitute ψ by φ in (1) we get $\vdash_{\mathbf{IPC}} \varphi \to (\varphi \to \varphi)$. By applying modus ponens we get our result:

$$\dfrac{\varphi \to (\varphi \to \varphi) \qquad \dfrac{(\varphi \to ((\varphi \to \varphi) \to \varphi)) \to ((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)) \qquad (\varphi \to ((\varphi \to \varphi) \to \varphi)}{(\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)}}{\varphi \to \varphi}$$

In order to prove the theorem we split it into the following three cases:

(i) ψ is an axiom. Then by (b) we have that $\Gamma \vdash_{\mathbf{IPC}} \psi$. We also know that $\Gamma \vdash_{\mathbf{IPC}} \psi \to (\varphi \to \psi)$, since $\psi \to (\varphi \to \psi)$ is an instance of (1). By applying Modus ponens we get that $\Gamma \vdash_{\mathbf{IPC}} \varphi \to \psi$.

(ii) $\psi \in \Gamma \cup \{\varphi\}$. Which consists of the following subcases:

- if $\psi \in \Gamma$, we use rule (a).
- if $\psi = \varphi$, we showed earlier that $\vdash_{\mathbf{IPC}} \varphi \to \varphi$ is true.

(iii) $\Gamma, \varphi \vdash \psi$. The proof follows by induction on the derivation of $\Gamma, \varphi \vdash \psi$. Assuming that $\Gamma, \varphi \vdash \psi$ is derived from $\Gamma, \varphi \vdash_{\mathbf{IPC}} \chi$ and $\Gamma, \varphi \vdash_{\mathbf{IPC}} \chi \to \psi$ via the modus ponens rule. We can use the induction hypothesis for $\Gamma, \varphi \vdash_{\mathbf{IPC}} \chi \to \psi$ and $\Gamma, \varphi \vdash_{\mathbf{IPC}} \chi$ to conclude that $\Gamma \vdash_{\mathbf{IPC}} \varphi \to (\chi \to \psi)$ and $\Gamma \vdash_{\mathbf{IPC}} \varphi \to \chi$. Knowing that $(\varphi \to (\chi \to \psi)) \to ((\varphi \to \chi) \to (\varphi \to \psi))$ is an instance of (2), via modus ponens we derive the following:

$$\dfrac{\dfrac{(\varphi \to (\chi \to \psi)) \to ((\varphi \to \chi) \to (\varphi \to \psi)) \qquad \varphi \to (\chi \to \psi)}{(\varphi \to \chi) \to (\varphi \to \psi)} \qquad \varphi \to \chi}{\varphi \to \psi}$$

Hence $\Gamma \vdash_{\mathbf{IPC}} \varphi \to \psi$, which concludes our proof of the deduction theorem.

□

2.1.2.2 Natural deduction

Another kind of system for **IPC** is the natural deduction system which, contrary to the Hilbert-system, consists only of inference rules. Each of the following rules is valid in the sense that if we find a proof for the premisses above the line, then there must also exist a proof for the conclusion below the line.
We use introduction (I) and elimination (E) rules to construct the following system of natural deduction.

Rules for IPC

$$(A) \qquad \varphi_1, \ldots, \varphi_n \vdash \varphi_i$$

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi}\,(\wedge E1) \qquad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\,(\vee I1) \qquad \frac{\Gamma \vdash \varphi \rightarrow \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi}\,(\rightarrow E) \qquad \frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \rightarrow \psi}\,(\rightarrow I)$$

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}\,(\wedge E2) \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\,(\vee I2) \qquad \frac{\Gamma \vdash \varphi \vee \psi \quad \Gamma, \varphi \vdash \chi \quad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi}\,(\vee E)$$

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}\,(\wedge I) \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\,(\bot E) \qquad \frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \varphi \vdash \neg\chi}{\Gamma \vdash \neg\varphi}\,(\bot I)$$

Where $\Gamma \vdash \varphi$ means that the formula φ is derivable under the assumptions $\Gamma = \{\varphi_1, ..., \varphi_n\}$. We write $\vdash \varphi$ if no assumptions were made, i.e. when $\Gamma = \emptyset$.

Example 2.1. *We are going to show that in **IPC** the following formulas are derivable:*

(1) $\varphi \rightarrow \neg\neg\varphi$

(2) $\neg\neg\neg\varphi \rightarrow \neg\varphi$

using the before described natural deduction system:
(1)

$$\dfrac{\dfrac{\dfrac{\varphi, \neg\varphi \vdash \varphi \qquad \varphi, \neg\varphi \vdash \neg\varphi}{\varphi, \neg\varphi \vdash \bot}\,(\bot E)}{\varphi \vdash \neg\neg\varphi}\,(\bot I)}{\vdash \varphi \rightarrow \neg\neg\varphi}\,(\rightarrow E)$$

(2)

$$
\dfrac{\dfrac{\dfrac{\varphi, \neg\neg\neg\varphi \vdash \neg\neg\neg\varphi \qquad \dfrac{\dfrac{\varphi, \neg\varphi, \neg\neg\neg\varphi \vdash \varphi \qquad \varphi, \neg\varphi, \neg\neg\neg\varphi \vdash \neg\varphi}{\varphi, \neg\varphi, \neg\neg\neg\varphi \vdash \bot}\,(\bot E)}{\varphi, \neg\neg\neg\varphi \vdash \neg\neg\varphi}\,(\bot I)}{\varphi, \neg\neg\neg\varphi \vdash \bot}\,(\bot E)}{\neg\neg\neg\varphi \vdash \neg\varphi}\,(\bot E)}{\vdash \neg\neg\neg\varphi \rightarrow \neg\varphi}\,(\rightarrow E)
$$

2.1.3 Translations

Although intuitionistic logic is often seen as weaker than classical logic, there exists a way of translating classical logic into intuitionistic logic using *Gödel's negative translation*. It works by applying the following rules (from chapter 3 of [2]) in a recursive manner:

- $p^n = \neg\neg p$ (put $\neg\neg$ in front of all atomic formulas)
- $(\varphi \vee \psi)^n = \neg(\neg\varphi^n \wedge \neg\psi^n)$
- $(\varphi \wedge \psi)^n = \varphi^n \wedge \psi^n$
- $(\varphi \rightarrow \psi)^n \rightarrow \varphi^n \rightarrow \psi^n$
- $\bot^n = \bot$

The resulting formula is provable in intuitionistic logic exactly when the original one is provable in classical logic as stated in the following theorem:

Theorem 2.2. $\vdash_{\boldsymbol{CPC}} \varphi \Leftrightarrow \vdash_{\boldsymbol{IPC}} \varphi^n$

It can therefore be said that intuitionistic logic is richer than classical logic seeing as it accepts classical logic in a particular way.
It is also possible to use the Gödel translation to translate **IPC** into modal logic (see [2]).

The following model-theoretic approach to formalizing **IPC** comes from chapter 3 of [2].

2.1.4 Kripke semantics

Kripke semantics for **IPC** is a formal semantics created in the late 1950's and early 1960's by Saul Kripke. The aim of the Kripke semantics is to define the concept of a model and the concept of truth in that model, such that a formula is a tautology if and only if it holds in every world of every Kripke model. This makes the Kripke semantics

a sound and complete semantics for **IPC**.
Assume that PROP is the set of propositional variables: $p_1, ..., p_n$. A Kripke model consists of three items:

(1) A set of worlds/states

(2) An accessibility relation which tells us if one world is accessible from another.

(3) A valuation which determines whether a certain propositional variable is true or false in a given world.

We are going to introduce some basic definitions:

Definition 2.3 (Intuitionistic Kripke frame)**.** An intuitionistic Kripke frame is a pair $\mathfrak{F} = (W, R)$, where W is a non-empty set of possible worlds/states and R is an accessibility relation which is reflexive, transitive and anti-symmetric. When a world v is accessible from another world w we write: wRv.

A Kripke frame $\mathfrak{F} = (W, R)$ is called *rooted* if there exists a world $w \in W$ such that for all other worlds $v \in W$ we have that wRv, i.e. a frame is rooted if it has a least world/state.
A subset $W^{'} \subseteq W$ is called an *upset* of $\mathfrak{F}$ if:

$$\forall w, v \in W \text{ when } w \in W^{'} \text{ and } wRv \text{ then } v \in W^{'}$$

Let $\mathcal{P}$ be the set of all *upsets* of $\mathfrak{F} = (W, R)$ defined as: $\{U \subseteq W \mid \forall w, w^{'}(w \in U \wedge wRw^{'} \Rightarrow w^{'} \in U)\}$, which is closed under $\cap, \cup$ and $\Rightarrow$ with $\emptyset, W \in \mathcal{P}$.

Definition 2.4 (Intuitionistic Kripke model)**.** An intuitionistic Kripke model is a pair $\mathfrak{M} = (\mathfrak{F}, V)$, where:

- $\mathfrak{F}$ is an intuitionistic Kripke frame

- $V : PROP \rightarrow \mathcal{P}$ is an intuitionistic valuation. We say that a proposition p is true in world w if $w \in V(p)$, and false if $w \notin V(p)$

Informally we can think of a Kripke model as an abstract representation of the world, W as a set of states of knowledge and R as a relation where uRv indicates that v is the successor of u, this implies that v is a state of greater knowledge than u.
An intuitionistic valuation can be seen as a function that maps each propositional variable p to a set of worlds in which that propositional variable is true: $V(p) = \{w \mid w \vDash p\}$.
This function is constructed in such a way that each intuitionistic Kripke model is *persistent* which means that if wRv and $w \in V(p)$ then $v \in V(p)$. We can extend the notation $V(p)$ to formulas $V(\varphi) = \{w \mid w \vDash \varphi\}$.
Given an intuitionistic valuation V, truth/validity at a state is defined through the ($\vDash$) relation which satisfies the following rules:

1. $w \vDash p \Leftrightarrow w \in V(p)$
2. $w \vDash p \wedge q \Leftrightarrow w \vDash p$ and $w \vDash q$
3. $w \vDash p \vee q \Leftrightarrow w \vDash p$ or $w \vDash q$
4. $w \vDash p \rightarrow q \Leftrightarrow \forall w'$ s.t. wRw', if $w' \vDash p$, then $w' \vDash q$
5. $w \nvDash \bot$

These rules are also known as the rules for forcing formulas, they tell us how the connectives are interpreted, note also that:

- $V(p \vee q) = V(p) \cup V(q)$
- $V(p \wedge q) = V(p) \cap V(q)$
- $V(p \rightarrow q) = \{x : \text{ for every } y \text{ with } xRy, \text{ if } y \vDash p \text{ then } y \vDash q\}$
- $w \vDash \bot$ is equivalent to $w \in V(\bot) = \emptyset$ which is impossible

Also important to note is that $w \nvDash p$ means that currently there is no proof that p is true, as opposed to $w \vDash \neg p$ which means that there will never exist a proof for p.

Definition 2.5 (Satisfaction relation ($\vDash$))**.** We call $\vDash$ the satisfaction relation. Assume that $\mathfrak{M} = (W, R, V)$ is an intuitionistic Kripke model, with $w \in W$ and ψ a well-formed propositional formula. Then if ψ is true in a world w of the model $\mathfrak{M}$ we write:

$$\mathfrak{M}, w \vDash \psi$$

If ψ is satisfied at every point w in $\mathfrak{M}$ we can say that ψ is true in $\mathfrak{M}$ and write $\mathfrak{M} \vDash \psi$.

Lemma 2.6. *Let $u, v \in W$ and φ a formula in **IPC**, then persistency transfers to formulas i.e if uRv and $u \vDash \varphi$, then $v \vDash \varphi$.*

Proof. If φ is a proposition variable it directly follows from the definition of model in **IPC**.
We also know that $\varphi = \bot$ is impossible via the rules for forcing formulas. Proof of induction using forcing rule 2. and 3. proves $\varphi = \varphi_1 \wedge \varphi_2$ and $\varphi = \varphi_1 \vee \varphi_2$ respectively. Let $\varphi = \varphi_1 \rightarrow \varphi_2$ assume that uRv and $u \vDash \varphi \Leftrightarrow u \vDash \varphi_1 \rightarrow \varphi_2$. We need to prove that $v \vDash \varphi$. Let w be a state s.t. vRw and $w \vDash \varphi_1$. Since R is transitive, uRw. Hence $w \vDash \varphi_2$ follows from $u \vDash \varphi_1 \rightarrow \varphi_2$, and we are done. □

We know that in order for a formula to be a tautology it must be true in every world of every possible model. Thus in order to prove that a certain formula is not a tautology for **IPC** we must provide a Kripke model in which the formula does not satisfy this requirement. Figure 2.1. provides us with such counter-models, they invalidate the following formulas: $p \vee \neg p$, $\neg\neg p \to p$, $(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$. Each node of the tree represents a possible world numbered from 0 to 3 and each letter p, q, r represents a valid propositional variable in that node.

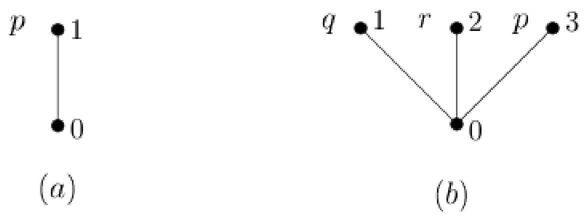

FIGURE 2.1: Counter-models for the propositional formulas

We know that a formula φ is true in the models if and only if $V(\varphi) = W$.
For the model in figure 2.1 (a) we have that $V(p) = \{1\}$, $V(\neg p) = \emptyset$, and $V(\neg\neg p) = \{0, 1\}$, it invalidates the following formulas:

– $p \vee \neg p$ as $0 \nvDash p \vee \neg p$ i.e. $0 \notin V(p \vee \neg p)$, because $V(p \vee \neg p) = V(p) \cup V(\neg p) = \{1\} \cup \emptyset = \{1\}$

– $\neg\neg p \to p$ as $0 \notin V(\neg\neg p \to p)$, because $V(p \to q) = \{x :$ for every y with xRy, if $y \vDash p$ then $y \vDash q\} = \{1\}$

We can thus conclude that both $p \vee \neg p$ and $\neg\neg p \to p$ are not tautologies for **IPC** seeing as neither formulas are true in every world of the model.
For the model in figure 2.1 (b) we have that:

- $V(q) = \{1\}$
- $V(r) = \{2\}$
- $V(p) = \{3\}$
- $V(\neg p \to q) = \{1\}$
- $V(\neg p \to r) = \{2\}$
- $V(q \vee r) = V(q) \cup V(r) = \{2, 3\}$
- $V(\neg p \to q \vee r) = \{1, 2\}$
- $V(\neg p \to q) \vee V(\neg p \to r) = \{1, 2\}$
- $V(\neg p) = V(\neg q) = V(\neg r) = \emptyset$
- $V(\neg\neg p) = V(\neg\neg q) = V(\neg\neg r) = \{0, 1, 2, 3\}$

The model invalidates the formula $(\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$ because: $V((\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)) = \{x :$ for every y with xRy, if $y \vDash (\neg p \to q \vee r)$ then $y \vDash (\neg p \to q) \vee (\neg p \to r)\} = \{1, 2\}$, thus $0 \notin V((\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r))$. Again we can conclude that the formula is not a tautology for **IPC** seeing as it is not valid in every world of our model.

2.1.4.1 Soundness and Completeness

Previously we obtained both a semantic and a deduction system for intuitionistic propositional logic. Consequently we wonder whether they agree on which inferences are correct i.e. whether they are equal. This question consists of two parts:

Soundness If a formula is provable/derivable it must be true/valid:

$$\vdash_{\mathbf{IPC}} \varphi \Rightarrow \vDash_{\mathbf{IPC}} \varphi \tag{2.1}$$

Completeness If a formula is true/valid it must be derivable/provable:

$$\vDash_{\mathbf{IPC}} \varphi \Rightarrow \vdash_{\mathbf{IPC}} \varphi \tag{2.2}$$

The key to showing completeness is the Lindenbaum lemma. There are two different formulations of the Lindenbaum lemma, the first one (shown in [5]) considers maximal consistent sets of formulas used in classical proofs by which intuitively a maximal consistent set is constructed by taking a maximal set Γ such that $\bot$ is not derived from it. We on the other hand consider theories with the disjunction property as illustrated in the lemma below (which is treated in Chapter 3 of [2]). We say that a set of formulas Γ has the disjunction property if $\varphi \vee \psi \in \Gamma$ implies that either $\varphi \in \Gamma$ or $\psi \in \Gamma$. A set of formulas is called a *theory* if it is closed under **IPC**-consequence.

Lemma 2.7 (Lindenbaum Lemma). *If* $\Gamma \nvdash_{\mathbf{IPC}} \psi \to \chi$, *then there exists a theory* Δ *with the following properties:*

(i) $\Gamma \subset \Delta$

(ii) Δ *has the disjunction property*

(iii) $\psi \in \Delta$ *and* $\chi \notin \Delta$

Proof. We begin by enumerating all formulas: $\phi_0, \phi_1, ...$ and defining the following:

- $\Delta_0 = \Gamma \cup \{\psi\}$
- $\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\phi_n\} & \text{, if this does not prove } \chi \\ \Delta_n & \text{, otherwise} \end{cases}$

where Δ is the union of all Δ_n, and neither Δ nor any of the Δ_n prove χ. We assume Δ to be a theory. We are now going to show via contradiction that Δ has the disjunction property.

Assume $\phi \vee \psi \in \Delta$, $\phi \notin \Delta$ and $\psi \notin \Delta$. Because $\psi, \phi \notin \Delta$ and Δ is the union of all Δ_n we know that $\psi, \phi \notin \Delta_{n+1}$. Let $\phi = \phi_m$ and $\psi = \phi_n$ with $m < n$ then $\Delta_{n+1} \neq \Delta_n \cup \{\phi_n\}$ and $\Delta_{n+1} \neq \Delta_n \cup \{\phi_m\}$ we can thus conclude with the definition of Δ_{n+1} that χ must be provable in both $\Delta_n \cup \{\phi\}$ and $\Delta_n \cup \{\psi\}$ and thus also in $\Delta_n \cup \{\phi \vee \psi\}$. This is impossible however since we assumed that Δ does not prove χ and $\Delta_n \cup \{\phi \vee \psi\} \subseteq \Delta$. We can now conclude that Δ satisfies the disjunction property s.t. $\psi \in \Delta$ and $\chi \notin \Delta$. (When we say that φ is provable in Δ we mean that $\varphi \in \Delta$ and thus $\Delta \vdash_{\mathbf{IPC}} \varphi$). □

Interestingly the previously given proof for the Lindenbaum lemma is not constructive since it makes an essential use of the axiom of choice.

Definition 2.8. The canonical model $\mathfrak{M}^C = (W^C, R^C, V^C)$ of **IPC** is a Kripke model, where W^C is the set of all consistent theories with the disjunction property and R^C is the inclusion. The canonical valuation V^C is defined by putting: $\Gamma \vDash p$ if $p \in \Gamma$.

Theorem 2.9 (**Completeness theorem for IPC**). *$\vdash_{\mathbf{IPC}} \varphi$ if φ is valid in all Kripke models for **IPC**.*

Proof. We prove that $\vDash_{\mathbf{IPC}} \varphi$ implies $\vdash_{\mathbf{IPC}} \varphi$ via contraposition (i.e. $\nvdash_{\mathbf{IPC}} \varphi$ implies $\nvDash_{\mathbf{IPC}} \varphi$).
Thus assume for a random formula χ that $\Gamma \nvdash_{\mathbf{IPC}} \chi$, then we also have that $\Gamma \nvdash_{\mathbf{IPC}} \top \to \chi$. We can apply the Lindenbaum lemma to $\Gamma \nvdash_{\mathbf{IPC}} \top \to \chi$ which tells us that there must exist a theory Δ with the disjunction property that includes Γ s.t. $\top \in \Delta$ and $\chi \notin \Delta$. By the definition of canonical model Δ is a world in this model and since $\chi \notin \Delta$ we have that, $\mathfrak{M}, \Delta \nvDash \chi$.
Hence there is a Kripke model $\mathfrak{M}$, and a world in it, namely Δ where $\Delta \nvDash \chi$. This is exactly what we wanted to show. □

We can conclude that Kripke models are both sound and complete with respect to intuitionistic logic, even though the completeness theorem is not constructively valid.

2.1.4.2 Operations on Kripke frames

In this section we define some basic operations on Kripke frames and models. We recall that a subset $W^{'} \subseteq W$ is called an *upset* of $\mathfrak{F} = (W, R)$ if:

$$\forall w, v \in W \text{ if } w \in W^{'} \text{ and } wRv \text{ then } v \in W^{'}$$

A frame $\mathfrak{F}^{'} = (W^{'}, R^{'})$ is called a *subframe* of $\mathfrak{F}$ if $W^{'} \subseteq W$ and $R^{'}$ is the restriction of R to $W^{'}$.

Definition 2.10 (Generated subframe)**.** A frame $\mathfrak{F}^{'} = (W^{'}, R^{'})$ is called a *generated subframe* of $\mathfrak{F} = (W, R)$ if:

- $W^{'} \subseteq W$
- $R^{'}$ is a restriction of R to $W^{'}$, thus $R^{'} = R \cap W^{'}$
- $W^{'}$ is an *upset* of $\mathfrak{F}$

Definition 2.11 (p-morphism)**.** Let $\mathfrak{F} = (W, R)$ and $\mathfrak{F}^{'} = (W^{'}, R^{'})$ be Kripke frames.

1. A map $f : W \to W^{'}$ is called a *p-morphism between* $\mathfrak{F}$ *and* $\mathfrak{F}^{'}$ if for every $w, v \in W$ and $w^{'} \in W^{'}$ we have that:

   (a) wRv implies $f(w)R^{'}f(v)$

   (b) $f(w)R^{'}w^{'}$ implies that $\exists u \in W$ s.t. wRu and $f(u) = w^{'}$

   We can say that a map f is *monotonic* if it satisfies condition 1(a). $\mathfrak{F}^{'}$ is called a *p-morphic image* of $\mathfrak{F}$ if f is a surjective p-morphism from $\mathfrak{F}$ onto $\mathfrak{F}^{'}$.

2. A map $f : W \to W^{'}$ is called a *p-morphism between* $\mathfrak{M}$ *and* $\mathfrak{M}^{'}$ if f is a p-morphism between $\mathfrak{F}$ and $\mathfrak{F}^{'}$ and for every $w \in W$ and $p \in PROP$ we have that:

$$\mathfrak{M}, w \vDash p \Leftrightarrow \mathfrak{M}^{'}, f(w) \vDash p. \tag{2.3}$$

A Kripke model $\mathfrak{M}^{'}$ is a submodel of another Kripke model $\mathfrak{M}$, if $\mathfrak{M}^{'}$ is obtained by restricting the set of nodes of $\mathfrak{M}$. A formula φ is *preserved under submodels*, if for all models $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}^{'} = (W', R', V')$ where $\mathfrak{M}^{'}$ is a submodel of $\mathfrak{M}$, if $w \in W$ and $\mathfrak{M}, w \vDash \varphi$, then $\mathfrak{M}^{'}, w \vDash \varphi$.
A model $\mathfrak{M}$ is a *p-morphic image* of another model $\mathfrak{M}^{'}$ if the map f between these models is surjective.
We say that the map f between models is *valuation preserving* if it satisfies (2.3). If a map f is a monotonic map between models then it must also be a monotonic map between the underlying frames, which in turn is valuation preserving.

We say that two formulas φ, ψ are *frame equivalent* if for any frame $\mathfrak{F}$ we have that $\mathfrak{F} \vDash \varphi$ iff $\mathfrak{F} \vDash \psi$.

Definition 2.12 (intuitionistic general frame)**.** An *intuitionistic general frame* is a triple $\mathfrak{F} = (W, R, \mathcal{P})$, where (W, R) is an intuitionistic Kripke frame and $\mathcal{P}$ is a set of upsets s.t. $\emptyset, W \in \mathcal{P}$. The set $\mathcal{P}$ must also be closed under $\cap$, $\cup$ and $\Rightarrow$, which is defined as:

$$W \Rightarrow W^{'} := \{w \in W \mid \forall v(wRv \wedge v \in W \to v \in W^{'})\}$$

The elements of $\mathcal{P}$ are called *admissible sets*. A *valuation* on an intuitionistic general frame is a map defined as: $V : PROP \to \mathcal{P}$.
If $\mathfrak{M} = (\mathfrak{F}, V)$, with $\mathfrak{F}$ a intuitionistic general frame, then the model is called a *general intuitionistic model* or *general model* in short.

Definition 2.13 (refined)**.** A general frame $\mathfrak{F} = (W, R, \mathcal{P})$ is called *refined* if

$$\forall w, v \in W : \neg(wRv) \text{ implies that } \exists U \in \mathcal{P} \text{ s.t. } w \in U \text{ and } v \notin U.$$

Definition 2.14 (compact)**.** A general frame $\mathfrak{F} = (W, R, \mathcal{P})$ is called *compact* if for every $\mathcal{X} \subseteq \mathcal{P} \cup \{W \backslash U \mid U \in \mathcal{P}\}$, if every intersection of finitely many elements of $\mathcal{X}$ is non-empty, then $\bigcap \mathcal{X} \neq \emptyset$.

If the general frame $\mathfrak{F} = (W, R, \mathcal{P})$ is *refined* and *compact* it is called *descriptive*. A *descriptive valuation* is a map defined as: $V : PROP \to \mathcal{P}$. If $\mathfrak{F}$ is a descriptive frame and V a descriptive valuation, then $\mathfrak{M} = (\mathfrak{F}, V)$ is called a *descriptive model*. The validity of formulas in such a model (frame) is defined as in the Kripke case with the exception that it ranges over all descriptive valuation.
Every intermediate logic is complete with respect to a class of descriptive frames. We introduce the following definitions:

Definition 2.15 (generated subframe of a descriptive frame)**.** We say that $\mathfrak{F}^{'} = (W^{'}, R^{'}, \mathcal{P}^{'})$ is a *generated subframe of a descriptive frame* $\mathfrak{F} = (W, R, \mathcal{P})$ if:

- $(W^{'}, R^{'})$ is a generated subframe of (W, R)
- $\mathcal{P}^{'} = \{U \cap W^{'} \mid U \in \mathcal{P}\}$

If we also have that for every $U \subseteq W^{'}$ s.t. $W^{'} \backslash U \in \mathcal{P}^{'}$ we have $W \backslash R^{-1}(U) \in \mathcal{P}$ then $\mathfrak{F}^{'}$ is called the *subframe of a descriptive frame* $\mathfrak{F}$.

Definition 2.16 (p-morphism between descriptive frames)**.** We call a map $f : W \to W^{'}$ a *p-morphism* between $\mathfrak{F} = (W, R, \mathcal{P})$ and $\mathfrak{F}^{'} = (W^{'}, R^{'}, \mathcal{P}^{'})$ if:

- f is a p-morphism between (W, R) and $(W^{'}, R^{'})$, thus $\forall w, v \in W$ and $w^{'} \in W^{'}$:
  - wRv implies $f(w)R^{'}f(v)$
  - $f(w)R^{'}w^{'}$ implies that $\exists u \in W$ s.t. $wRu \wedge f(u) = w^{'}$
- for every $U^{'} \in \mathcal{P}^{'}$ we have $f^{-1}(U^{'}) \in \mathcal{P}$

Chapter 3

Intermediate logics

When we add schemes to the Hilbert type system of **IPC** we obtain the so called *Intermediate logics*. An example of this is that when we add $\neg\varphi \vee \varphi$ we obtain classical propositional logic (**CPC**) as a result, i.e. **CPC** = **IPC** + $\neg\varphi \vee \varphi$.
The intermediate logics are logics which are situated between **IPC** and **CPC**. We want to develop an axiomatization of these logics, such that we have a clear description of the ($\vdash$)-relation which tells you the valid rules for proving new statements in the logic. Such an axiomatization was developed using splitting-formulas also known as *Jankov-de Jongh formulas*. Using these formulas Jankov and de Jongh both developed an axiomatization of the intermediate logics via Heyting algebras and Kripke frames respectively, in [2] it is explained in detail. Large classes of intermediate logics are axiomatizable by such formulas but not all. The majority of the formulas not axiomatizable by Jankov-de Jongh formulas are axiomatizable by *subframe-formulas* (first introduced by Zakharyaschev). In this section we will discuss the results from [1]. We show how to give a syntactic characterization of subframe and stable formulas via **NNIL**-formulas and **ONNILLI**-formulas respectively. From this we are going to show that an intermediate logic is called a subframe/stable logic iff it is axiomatizable by subframe/stable formulas, which in turn (up to frame equivalence) coincide with **NNIL**/**ONNILLI** formulas.

3.1 NNIL-formulas

The **NNIL**-formulas were introduced by A. Visser in 1985 when he was studying propositional admissible rules for theories, like Heyting algebra's (**HA**). They are preserved under taking submodels of Kripke models, which implies that they are also preserved under taking subframes (Corollary 3.6 of [1]). We are going to show that **NNIL**-formulas axiomatize the intermediate logics. The **NNIL** class can be viewed as a fragment of **IPC**, since it is generated like the language of **IPC**, restricting the formation rule for implications by only allowing propositional variables in the antecedent.
NNIL-formulas in normal form are defined by:

$$\varphi := \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid p \to \varphi$$

where **NNIL** stands for "no nested implication to the left". This means that these formulas are equivalent to conjunctions/disjunctions of implications which have only propositional variables in the antecedents of implications. In other words **NNIL** is the smallest class of formulas satisfying:

- $p \in$**NNIL**, for every $p \in PROP$
- $\varphi \vee \psi \in$**NNIL**, if $\varphi, \psi \in$**NNIL**
- $\varphi \wedge \psi \in$**NNIL**, if $\varphi, \psi \in$**NNIL**
- $\bot \in$**NNIL**
- $\varphi \to \psi \in$**NNIL**, if φ does not contain an implication

The conjunctions and disjunctions in front of implications can be removed using the following rules:

$$\varphi \vee \psi \to \phi \Leftrightarrow (\varphi \to \phi) \wedge (\psi \to \phi)$$

$$\varphi \wedge \psi \to \phi \Leftrightarrow \varphi \to (\psi \to \phi)$$

$$\varphi \to \psi \wedge \phi \Leftrightarrow (\varphi \to \psi) \wedge (\varphi \to \phi)$$

We give some examples

NNIL-formulas	non-**NNIL**-formulas
$p \to (q \to r)$	$(p \to q) \to r \wedge q$
$p \to (q \to (r \to s))$	$(p \to q) \to r$
$p \vee q \to r$	$(p \to \bot) \to \bot$
$(p \to (q \to \bot)) \vee (q \to r)$	$(p \to q) \to (r \to p)$

We have the following theorem, from [1], which shows that the **NNIL**-formulas are exactly the ones preserved under submodels.

Theorem 3.1.

(1) *If* $\varphi \in$ ***NNIL*** *then* φ *is preserved under submodels, i.e.* $\mathfrak{M}, w \vDash \varphi \Rightarrow \mathfrak{N}, w \vDash \varphi$, *where* $\mathfrak{N}$ *is a submodel of* $\mathfrak{M}$ *and* $w \in \mathfrak{N}$.

(2) *If* φ *is preserved under submodels, then there exists* $\psi \in$ ***NNIL*** *s.t.* ***IPC*** $\vdash \psi \leftrightarrow \varphi$.

3.1.1 Subframe formulas

If a logic is axiomatizable by subframe formulas it is called a *subframe logic*. There are continuous many subframe logics, each of which is closed under taking subframes and has the finite model property. This means that any non-theorem of the logic is falsified by some finite model of the logic. Subframe formulas for intuitionistic logic were defined by Zakharyaschev, we call a formula φ a *subframe-formula* if it is preserved under subframes, this means that: for all frames $\mathfrak{F} = (W, R)$ and $\mathfrak{F}^{'} = (W^{'}, R^{'})$ where $\mathfrak{F}^{'}$ is a subframe of $\mathfrak{F}$, if $\mathfrak{F} \vDash \varphi$, then $\mathfrak{F}^{'} \vDash \varphi$.
Our goal in this section is to give a characterization of subframe formulas and show that the class of **NNIL**-formulas is (up to frame equivalence) the same as the class of subframe formulas. In order to do this we construct our subframe formula as a **NNIL**-formula. We also show that an intermediate logic is a subframe logic iff it is axiomatized by **NNIL**-formulas. Such **NNIL**-axiomatizations is constructed for each finite rooted frame $\mathfrak{F}$ by composing a **NNIL**-formula from a colorful model $\mathfrak{M}$ on that frame that fails on a descriptive frame $\mathfrak{B}$ iff $\mathfrak{F}$ is a p-morphic image of a subframe of $\mathfrak{B}$.
We introduce the following definition in order to define *subframe formulas* in the **NNIL**-form.

Definition 3.2 (Color and colorful models)**.** We fix n propositional variables $p_1, ..., p_n$ and associate each world w of a descriptive model $\mathfrak{M}$ with an n-color. Such n-color is denoted by $col(w)$ and consists of a sequence $i_1...i_n$ which is defined as follows:

$$i_k = \begin{cases} 1, & \text{if } w \vDash p_k \\ 0, & \text{if } w \nvDash p_k. \end{cases}$$

with $k = 1, ..., n$.
A finite model $\mathfrak{M} = (\mathfrak{F}, V)$ is said to be *colorful* if the number of propositional variables is $|W|$ and, for each $w \in W$ there is a propositional variable p_w s.t. $v \vDash p_w$ iff wRv.

We order colors according to the $\leq$ relation i.e.:

- $i_1...i_n \leq i_1^{'}...i_n^{'}$, if for every $k = 1, ..., n$ we have that $i_k \leq i_k^{'}$
- $i_1...i_n < i_1^{'}...i_n^{'}$, if $i_1...i_n \leq i_1^{'}...i_n^{'}$ and $i_1...i_n \neq i_1^{'}...i_n^{'}$

With any finite frame $\mathfrak{F}$ containing a least element, we can associate a colorful model $\mathfrak{M} = (\mathfrak{F}, V)$ called the *colorful model corresponding to* $\mathfrak{F}$ by introducing a propositional letter p_w for each $w \in \mathfrak{F}$ and letting V be such that $V(p_w) = R(w) = \{w^{'} \mid wRw^{'}\}$. For each *colorful* model $\mathfrak{M} = (\mathfrak{F}, V)$ if $v, w \in W$ we have:

- $w \neq v$ and wRv iff $col(w) < col(v)$,
- $w = v$ iff $col(w) = col(v)$.

Next we are going to inductively define *subframe formulas* in the **NNIL**-form. Assume $\mathfrak{F} = (W, R)$ to be a finite rooted frame and $\mathfrak{M} = (\mathfrak{F}, V)$ the corresponding colorful model. Let:

$-prop(v) := \{p_k : v \vDash p_k, k \leq n\}$
$-notprop(v) := \{p_k : v \nvDash p_k, k \leq n\}$

If r is the root of $\mathfrak{F}$ then we call $\beta(\mathfrak{F}) := \beta(r)$ the *subframe formula of* $\mathfrak{F}$, which is a **NNIL**-formula that is recursively defined as follows:

- If v is a maximal point of $\mathfrak{M}$ then:

$$\beta(v) := \bigwedge prop(v) \to \bigvee notprop(v)$$

- If $w \in \mathfrak{M}$ and $w_1, ..., w_m$ are immediate successors of w then:

$$\beta(w) := \bigwedge prop(w) \to \bigvee notprop(w) \vee \bigvee_{i=1}^{m} \beta(w_i)$$

  where $\beta(w_i)$ is already defined for each w_i.

Thus we constructed the **NNIL**-subframe formula $\beta(\mathfrak{F})$ using a colorful model on the finite rooted frame $\mathfrak{F}$. This formula satisfies the following condition for each descriptive frame $\mathfrak{B}$:

$$\mathfrak{B} \nvDash \beta(\mathfrak{F}) \Leftrightarrow \mathfrak{F} \text{ is a p-morphic image of a subframe of } \mathfrak{B}. \tag{3.1}$$

The proof of this statement lies in the fact that if,

$$\bigwedge prop(v) \to \bigvee notprop(v)$$

is false somewhere, thus $\bigwedge prop(v)$ is true and $\bigvee notprop(v)$ is false, then some node above it has to have the same color as v, i.e. $\exists r$ s.t. $v \leq r$ with $col(v) = col(r)$. If on the other hand

$$\bigwedge prop(v) \to \bigvee notprop(v) \vee \bigvee_{i=1}^{m} \beta(v_i)$$

is false somewhere then some state above v has to have the same color as v with above it states with the same color as the v_i, i.e. $\exists r$ s.t. $v \leq r$ with $col(v) = col(r)$ and for all successor of r: $r_1, ...r_n$ we have that $col(v_i) = col(r_j)$. Thus when $\beta(\mathfrak{F})$ is false we will have the properly colored states in the right order.
We recall that two formulas φ, ψ are considered *frame equivalent* if for any frame $\mathfrak{F}$ we have that $\mathfrak{F} \vDash \varphi$ iff $\mathfrak{F} \vDash \psi$.

Corollary 3.3.

(1) *An intermediate logic L is a subframe logic $\Leftrightarrow$ L is axiomatizable by **NNIL**-formulas.*

(2) *The class of **NNIL**-formulas coincides with the class of subframe formulas.*

Proof. .

(1) It was shown in [1] that every subframe logic is axiomatizable by **NNIL**-formulas of the form $\beta(\mathfrak{F})$. On the other hand if we assume that L is axiomatizable by **NNIL**-formulas, then because every **NNIL**-formula is preserved under subframes we have that the class of rooted descriptive frames of L, $\mathbb{DF}(L)$, must be closed under subframes. From [1] we know that L is a subframe logic iff $\mathbb{DF}(L)$ is closed under subframes. Thus we can conclude that L is a subframe logic.

(2) We use the fact that the class of **NNIL**-formulas is preserved under subframes to conclude that every **NNIL**-formula is a subframe-formula. Assume φ is preserved under subframes this implies that **IPC** + φ is a subframe logic. Again we know that each subframe logic is axiomatizable by **NNIL**-formulas of the form $\beta(\mathfrak{F})$. Thus the following frames exist $\mathfrak{F}_1, ..., \mathfrak{F}_n$ such that:

$$\mathbf{IPC} + \varphi = \mathbf{IPC} + \bigwedge_{i=1}^{n} \beta(\mathfrak{F}_i) \tag{3.2}$$

i.e. φ is frame-equivalent to $\bigwedge_{i=1}^{n} \beta(\mathfrak{F}_i)$, where $n \in w$. Seeing as every $\beta(\mathfrak{F}_i)$ is a **NNIL**-formula, and the class of **NNIL**-formulas is closed under conjunction, $\bigwedge_{i=1}^{n} \beta(\mathfrak{F}_i)$ must also be a **NNIL**-formula. From this we can conclude that φ is frame-equivalent to a **NNIL**-formula, which in turn are preserved under subframes (corollary 3.6,[1]). Thus the class of **NNIL**-formulas up to frame equivalence coincides with the class of subframe formulas, as φ was chosen randomly.

□

3.2 ONNILLI-formulas

In this section we will discuss the **ONNILLI**-formulas introduced in chapter 3 of [1]. This class is defined using the **NNIL**-formulas. The **ONNILLI**-class is a syntactic class where each **ONNILLI**-formula is preserved in monotonic images of (descriptive and Kripke) frames.

ONNILLI stands for only **NNIL** to the left of implications, i.e. it is the set of formulas $\{\varphi \to \psi \mid \varphi \in \textbf{NNIL} \text{ and } \psi \in \text{BASIC}\}$ closed under conjunctions and disjunctions, where BASIC represents the union of the set of propositional variables with $\top$ and $\bot$ which is also closed under conjunctions and disjunctions.

There are thus no iterations of implications in **ONNILLI**-formulas except of course in the **NNIL** part. We give some examples:

ONNILLI-formulas	non-**ONNILLI**-formulas
$\neg p \vee \neg\neg p \equiv (p \to \bot) \vee (\neg p \to \bot)$	$\neg p \to (q \to r)$
$\neg p \to \bot$	$(p \to \neg q) \to r$
$p \vee q \to r$	$(p \to \bot) \to \bot \to r$

We are going to show that the **ONNILLI**-formulas are up to frame equivalence the stable-formulas and that they are the ones that are preserved under monotonic maps of models. We recall that monotonic maps are maps that preserve orders on frames and also colors on models, i.e. assume that $f : X \to Y$ is a monotonic map on the intuitionistic models $\mathfrak{M} = (X, R, V)$ and $\mathfrak{N} = (Y, R', V')$, then

$$\forall x \in X \text{ and } \forall \varphi \in \textbf{NNIL} \text{ we have } f(x) \vDash \varphi \Rightarrow x \vDash \varphi$$

This is true for every formula seeing as when we assume that the induction hypothesis holds for φ and ψ we get that:

- $f(x) \vDash \varphi \wedge \psi$ then $f(x) \vDash \varphi$ and $f(x) \vDash \psi$ and thus also by the induction hypothesis $x \vDash \varphi$ and $x \vDash \psi$, thus $x \vDash \varphi \wedge \psi$.
- $f(x) \vDash \varphi \vee \psi$ then $f(x) \vDash \varphi$ or $f(x) \vDash \psi$ thus via the induction hypothesis $x \vDash \varphi$ or $x \vDash \psi$, which implies that $x \vDash \varphi \vee \psi$.
- $f(x) \vDash p \to \varphi$ and we know that the induction hypothesis holds for φ thus if xRy and, since f is also valuation preserving, $y \vDash p$ we have that $f(x)Rf(y)$ and $f(y) \vDash p$. Thus, $f(y) \vDash \varphi$ and by the induction hypothesis thus also $y \vDash \varphi$ which implies that $x \vDash p \to \varphi$.

Conversely if we assume that we have a monotonic map $f : X \to Y$ where $x \in X$ and

$$f(x) \vDash \psi \Rightarrow x \vDash \psi \tag{3.3}$$

for any random intuitionistic models $\mathfrak{M} = (X, R, V)$ and $\mathfrak{N} = (Y, R', V')$, then there exists a formula $\varphi \in$ **NNIL** s.t. **IPC** $\vdash \varphi \leftrightarrow \psi$. Since every identity function from a submodel into a larger model is always monotonic, if ψ satisfies (3.3) then it is preserved under submodels. And seeing that **NNIL** is the class of formulas preserved under submodels we can conclude that ψ is equivalent to some **NNIL**-formula (via Theorem 3.1.).

We can do the same thing for **ONNILLI**-formulas.

Proposition 3.4. *Let $f : X \to Y$ be a surjective monotonic map on the rooted intuitionistic models $\mathfrak{M} = (X, R, V)$ and $\mathfrak{N} = (Y, R', V')$. Then for each $\varphi \in$* ***ONNILLI*** *s.t. $\mathfrak{M} \vDash \varphi$ implies that $\mathfrak{N} \vDash \varphi$.*

Proof. We prove this by cases assume the induction hypothesis holds for φ.
Knowing that f is surjective and that $\varphi \in$**ONNILLI**, let $\varphi = \psi \to \phi$ where $\psi \in$**NNIL** and $\phi \in$BASIC. Since f is valuation preserving and $\phi \in$BASIC we obtain that:

$$y \vDash \phi \Leftrightarrow f(y) \vDash \phi \tag{3.4}$$

Since the map f is also surjective, for each element of Y there exists an $x \in X$ s.t. y is of the form $f(x) \in Y$ for some . Let $\mathfrak{M} \vDash \varphi$, thus $\mathfrak{M} \vDash \psi \to \phi$, which implies that $x \vDash \psi \to \phi$ for all $x \in X$. We assume the following:

1. $yR'y'$ and $y' \vDash \psi$, where $y \in Y$ and $y' \in Y'$.
2. $y' = f(x)$

By assumption 1. and 2. we have that $f(x) \vDash \psi$ and $\psi \in$**NNIL** thus, with (3.3), we also have that $x \vDash \psi$.
Since both $x \vDash \psi$ and $x \vDash \psi \to \phi$ (via 1.) we can conclude that $x \vDash \phi$, where $\phi \in$BASIC. With (3.4) we have that $f(x) \vDash \phi$ which is equal to $y' \vDash \phi$ (via assumption 2.), which in turn implies that $y' \vDash \psi \to \phi$ (via 1.)
Again because f is valuation preserving we have that $y \vDash \psi \to \phi$, and since $y \in Y$ was chosen randomly we can conclude that $\mathfrak{N} \vDash \psi \to \phi$, i.e. $\mathfrak{N} \vDash \varphi$.
Let $\varphi = \psi \wedge \phi$ and $\mathfrak{M} \vDash \psi \wedge \phi$, i.e. $x \vDash \psi \wedge \phi$ for all $x \in X$. Again since f is surjective we can assume that $y = f(x)$ with the above proof (for $\varphi = \psi \to \phi$) using induction on $\psi, \phi \in$**ONNILLI**, we have that $y \vDash \psi$ and $y \vDash \phi$ i.e. $y \vDash \psi \wedge \phi$. Hence $\mathfrak{N} \vDash \psi \wedge \phi$.
If on the other hand $\varphi = \psi \vee \phi$ (with $\psi, \phi \in$**ONNILLI**) then since both models $\mathfrak{M}$ and $\mathfrak{N}$ are rooted, assume with roots r and r' respectively, if $\mathfrak{M} \vDash \psi \vee \phi$ then $r \vDash \psi$ or $r \vDash \phi$. Again by the fact that f is surjective we can assume that $f(r) = r'$. We know that $\psi, \phi \in$**ONNILLI** thus by the proofs above (for $\varphi = \psi \to \phi$ and $\varphi = \psi \wedge \phi$) we use induction on ϕ, ψ and we get that: $r' \vDash \psi$ or $r' \vDash \phi$, which implies that $r' \vDash \psi \vee \phi$ and seeing as it was chosen randomly we have that $\mathfrak{N} \vDash \psi \vee \phi$. □

It follows directly from proposition 3.4. that for each $\varphi \in$**ONNILLI**, if $\mathfrak{F} \vDash \varphi$, then $\mathfrak{B} \vDash \varphi$ where both $\mathfrak{F} = (X, R)$ and $\mathfrak{B} = (Y, R^{'})$ are rooted intuitionistic frames connected by a monotonic map $f : X \to Y$ from $\mathfrak{F}$ onto $\mathfrak{B}$.

3.2.1 Stable formulas

The set of stable formulas is exactly the one preserved under monotonic images. Our goal is to show that the class of stable formulas (up to frame equivalence) coincides with the class of **ONNILLI**-formulas. In order to show this we construct for each finite rooted frame $\mathfrak{F}$, an **ONNILLI**-formula as its stable formula. As a result we deduce that an intermediate logic is stable if and only if it is axiomatizable by **ONNILLI**-formulas.

Definition 3.5. .
Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a colorful model of a finite rooted frame $\mathfrak{F}$. Then the *stable formula of* $\mathfrak{F}$ is defined as:

$$\gamma(\mathfrak{F}) = \gamma(\mathfrak{M})$$

which is an **ONNILLI**-formula. $\gamma(\mathfrak{M})$ is recursively defined as follows:

$$\gamma(\mathfrak{M}) = \bigvee\{Colors(\mathfrak{M}_w) \to p_{w_1} \vee ... \vee p_{w_m} \,|w \in W, \text{ where } w_1, ..., w_m \text{ all intermediate succesors of } w\}$$

Where:

1. c is an n-color
2. $\psi_c = p_1 \wedge ... \wedge p_k \to q_1 \vee ... \vee q_m$ if:
   - $p_1 ... p_k$ are the propositional variable associated with 1 in c
   - $q_1 ... q_m$ are the propositional variable associated with 0 c
3. $Colors(\mathfrak{M}_w) = prop(w) \wedge \bigwedge\{\psi_c | c \text{ is not the color of any point in } \mathfrak{M}_w\}$,
   where $w \in W$ and $\mathfrak{M}_w$ expresses that only colors in occur.

The ψ_c formula expresses that the color c does not occur i.e. assume $\mathfrak{R} = (W^{'}, R^{'}, V^{'})$ is a model then for $u^{'} \in W^{'}$ we have $u^{'} \vDash \psi_c$ iff, for all $v^{'}$ with $u^{'}R^{'}v^{'}$ we have that if $v^{'} \vDash p_1 \wedge ... \wedge p_k$ then $v^{'} \vDash q_1 \vee ... \vee q_m$ and because the color c is constructed s.t. if a state x has color c (i.e. $col(x) = c$) then $x \vDash p_1, ..., x \vDash p_k$ and $x \nvDash q_1, ..., x \nvDash q_m$ from this we can conclude that $v^{'}$ does not have color c (i.e. $color(v^{'}) \neq color(c)$).

To obtain **ONNILLI**-axiomatizations for each finite rooted frame $\mathfrak{F}$ an **ONNILLI**-formula is constructed from a model $\mathfrak{M}$ on that frame that fails on a descriptive frame $\mathfrak{B}$ iff there is a surjective map from $\mathfrak{B}$ onto $\mathfrak{F}$. Hence we have the following theorem which together with the previously given definition of *stable formulas* solves the open problem on syntactically characterizing formulas that axiomatize *stable logics*. Its proof is given in detail in [1].

Theorem 3.6. *Let $\mathfrak{F} = (W, R)$ be a finite rooted frame and let $\mathfrak{B} = (W', R')$ be a rooted (descriptive or Kripke) frame. Then $\mathfrak{B} \not\models \gamma(\mathfrak{F})$ if and only if there is a surjective monotonic map from $\mathfrak{B}$ onto $\mathfrak{F}$.*

The intermediate logics axiomatizable by stable formulas are called stable logics, there is a continuum of stable logics and all stable logics have the finite model property.
The following theorem is similar to the one given earlier for the subframe formulas.

Theorem 3.7.

(1) *An intermediate logic L is stable $\Leftrightarrow$ L is axiomatizable by* ***ONNILLI****-formulas.*

(2) *The class of* ***ONNILLI****-formulas is up to frame equivalence the class of stable formulas.*

Proof. .

(1) Assume that the intermediate logic L is stable then we know that it is also axiomatizable by stable formulas, and since each stable formula $\gamma(\mathfrak{F})$ is **ONNILLI**, we can conclude that L is axiomatizable by **ONNILLI**-formulas.
Assume on the other hand that L is axiomatized by **ONNILLI**-formulas. Since it followed from proposition 3.4 that every **ONNILLI**-formula is preserved under monotonic images, we can conclude that the class of rooted descriptive frames of L, $\mathbb{DF}(L)$, must be closed under monotonic images. Which in turn implies that L must be stable.

(2) Now assume that the logic L is axiomatizable by **ONNILLI**-formulas. We know from proposition 3.4 that every **ONNILLI**-formula is preserved under monotonic images, and since the set of stable formulas is exactly the one preserved under monotonic images we can conclude that each **ONNILLI**-formula is stable. The least intermediate logic containing a formula φ: **IPC** + φ, where φ is preserved under monotonic images, is a stable logic. From (1) we know that each stable logic is axiomatizable by **ONNILLI**-formulas thus also by $\gamma(\mathfrak{F})$ (since it is an **ONNILLI**-formula). Thus the following frames exist $\mathfrak{F}_1, ..., \mathfrak{F}_n$ s.t.

$$\mathbf{IPC} + \varphi = \mathbf{IPC} + \bigwedge_{i=1}^{n} \gamma(\mathfrak{F}_i) \tag{3.5}$$

i.e. φ is frame-equivalent to $\bigwedge_{i=1}^{n} \gamma(\mathfrak{F}_i)$, where $n \in w$. Seeing as every $\gamma(\mathfrak{F}_i)$ is an **ONNILLI**-formula, and the class of **ONNILLI**-formulas is closed under conjunctions, we get that $\bigwedge_{i=1}^{n} \gamma(\mathfrak{F}_i)$ must also be an **ONNILLI**-formula. From this we can conclude that φ is frame-equivalent to an **ONNILLI**-formula, which in turn are closed under monotonic images, thus the class of **ONNILLI**-formulas up to frame equivalence coincides with the class of all stable formulas, as φ was chosen randomly.

□

In short we have shown that a intermediate logic is called a subframe logic iff it is axiomatized by **NNIL**-formulas. We obtained such axiomatization for each finite rooted frame $\mathfrak{F}$ by constructing a **NNIL**-formula from a colorful model $\mathfrak{M}$ on that frame s.t. it fails on a descriptive frame $\mathfrak{B}$ iff $\mathfrak{F}$ is a p-morphic image of a subframe of $\mathfrak{B}$. We then showed that the class of subframe-formulas (up to frame equivalence) coincides with the class of **NNIL**-formulas.

The same reasoning was used for stable logics. We proved that intermediate logics are called stable logics iff they are axiomatized by **ONNILLI**-formulas. Where the axiomatization for each finite rooted frame $\mathfrak{F}$ was obtained by constructing an stable formula of $\mathfrak{F}$ in the **ONNILLI**-form from a colorful model $\mathfrak{M}$ on that frame s.t. it is refuted in a descriptive frame $\mathfrak{B}$ iff there is a descriptive surjective monotonic map from $\mathfrak{B}$ onto $\mathfrak{F}$. Subsequently we proved that the **ONNILLI**-class (up to frame equivalence) coincides with the class of stable formulas. As a result we deduced that the set of **ONNILLI**-formulas is syntactically defined and axiomatizes all stable logics.

This in turn resolved the open problem on the syntactic characterization of stable formulas.

Bibliography

[1] Dick de Jongh Nick Bezhanshvili. Stable formulas in intuitionistic logic. . URL https://staff.fnwi.uva.nl/n.bezhanishvili/Papers/ONNILLI-final.pdf.

[2] Dick de Jongh Nick Bezhanshvili. Intuitionistic logic. . URL http://www.cs.le.ac.uk/people/nb118/Publications/ESSLLI%2705.pdf.

[3] Cantor's paradox, . URL https://en.wikipedia.org/wiki/Cantor%27s_paradox.

[4] Riemann hypothesis, . URL https://en.wikipedia.org/wiki/Riemann_hypothesis.

[5] D.K. Johnston. Elements of modal logic. URL http://web.uvic.ca/~tiberius/logic/modal.pdf.