Slices of Realizability Topoi

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Chapter 1

Introduction

Towards the end of the 1930's, Turing, Kleene, Church and Post proposed several precise mathematical formulations of the notion 'computability'. Not much later, it was shown that the models they proposed all yield equivalent notions of computability. Partly for this reason, it is widely believed that these models capture exactly the intuitive idea of algorithmic computation. The study of these models is usually referred to as recursion theory. Especially Kleene's approach concentrates on the question which (partial) functions from the set of natural numbers to itself are computable. Such a computable function is given by an algorithm, which, mathematically speaking, is a finite set of data. A key insight in recursion theory is the idea that we may represent this finite amount of data itself as a natural number, which is called the Gödel number of the algorithm. As a result of this move, the algorithms and the objects they are applied to, become of the same nature. One gets a partial function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, which, on input (m, n), outputs the result when we apply the algorithm with Gödel number m to the natural number n. One of the first results in recursion theory is the fact that this function is itself computable. With this idea in mind, one could try to approach computability in a more abstract way, by replacing the set of natural numbers with a general nonempty set A. One then considers partial maps from $A \times A$ to A satisfying certain properties. Such a structure is known as a *partial combinatory algebra*, or PCA for short. The most well-known examples of a PCA is Kleene's model of computation.

In 1945, Kleene used his model of computation to give an interpretation for constructive arithmetic. In the paper *On the Interpretation of Intuitionistic Number Theory* [4], he introduced the notion of *recursive realizability.* This concept is meant to capture the constructive content of statements about arithmetic in a way that can be understood in a classical metatheory. In this way, recursive realizability may be regarded as a classical semantics for constructive arithmetic.

While recursive realizability is based on notions from recursion theory, it can also be studied from the point of view of category and topos theory. In the paper *The Effective Topos* [2], Martin Hyland constructed a specific (elementary) topos, known as the *effective topos*, such that the notions 'true in the effective topos for the natural numbers' and 'realizable in Kleene's sense' coincide. Andrew Pitts generalized Hyland's construction, using the notion of a *tripos*, to build analogous topoi on the basis of arbitrary PCAs. This construction leads to the class of *realizability topoi*, that is, the class of topoi that arise out of a PCA via Pitts' construction. These topoi will be the main object of study in this thesis.

At this point, it is useful to mention another distinguished class of topoi, that has been studied extensively, namely the class of *Grothendieck topoi*. A Grothendieck topos is a category of sheaves on a site. It is known that the class of Grothendieck topoi, like the class of (elementary) topoi itself, is closed under slicing. This is not the case for the class of realizability topoi, as we will in fact show below. Therefore, one may ask: what are these slices of realizability topoi, then? More precisely, what do these slices look like? Which properties do they have? Which of these properties do they share with realizability topoi? And can we find an interesting class of topoi that contains all realizability topoi and that *is* closed under slicing? These questions will occupy us in this thesis.

One of the questions above asks which properties are shared by realizability topoi and their slices. In order to answer this question, it is useful to have an idea which properties are characteristic of realizability topoi themselves in the first place. We observe that both the class of Grothendieck topoi and the class of realizability topoi are specified in an *intensional* way, that is, in terms of how one may construct them. If one wishes to have a Grothendieck topos, one considers a site and takes its category of sheaves. Similarly, if one wants a realizability topos, one takes a PCA and carries out Pitts' construction. Such an intensional description can be contrasted with an *extensional* desciption, which describes the class in terms of the properties its members have. For Grothendieck topoi, such an extensional description has been given by J. Giraud. More recently, J. Frey has given an extensional characterization of the class of realizability topoi as well in his PhD thesis [1]. His characterization is formulated in the language of *fibrations*, that we will introduce below. In order to make Frey's result applicable to our present project, we reformulate Frey's characterization more directly in the language of category theory.

Let us briefly outline the thesis. First of all, in Chapter 2, we define PCAs, present some examples, and take the first steps into the world of category theory. In Chapter 3, we define the realizability topos of a PCA, we show that it is a topos, and we study some special objects that may be encountered in this topos. Next, in Chapter 4, we consider Frey's result and present our reformulation of it. Finally, in Chapter 5, we turn our attention to slices of realizability topoi. First, we give an explicit description of certain of these slices, namely slices over assemblies. Then, we present an extensionally described class of topoi that contains all realizability topoi and is closed under slicing over projective objects, and we give a related, but not extensionally described class that contains all realizability topoi and is closed under slicing over all objects.

Since we will spell out the definition of partial combinatory algebras and their realizability topoi in detail, it is not necessary to have prior knowledge of these concepts in order to read this thesis. We do require a familiarity with basic recursion theory and various notions from category theory, most importantly adjunctions, regular categories, categorical logic, the definition of an elementary topos, and Lawvere-Tierney topologies and their sheaves. Throughout this thesis, the category Set of sets and functions plays a special role. When working in Set, we shall adopt the usual set theoretical conventions and notation. We assume the Axiom of Choice throughout this thesis, but when we use it, we shall also explicitly mention that we do so.

Chapter 2

Partial Combinatory Algebras

In the chapter, we introduce the structures that will be of primary importance to us: partial combinatory algebras, or PCAs for short. Intuitively, such structures are a 'model of computation'. Section 2.1 gives two equivalent definitions of a PCA, and develops some important notation. In Section 2.2, we describe some basic constructions that will solidify the intuition that a PCA is a model of computation. Several examples of PCAs are offered in Section 2.3. In Section 2.4, we introduce the idea of a $\mathcal{P}A$-valued predicate, which will be important in the next chapter. Finally, in Section 2.5, we show how every PCA gives rise to a certain regular and cartesian closed category, called the category of assemblies.

2.1 Definition of a PCA

In this section, we introduce the notion of a PCA. We follow the exposition by J. van Oosten in his book *Realizability* [6].

Definition 2.1.1. A *partial applicative structure*, abbreviated *PAS*, is a nonempty set A equipped with a *partial* map from $A \times A$ to A, called *application*. We denote application by juxtaposition: $(a, b) \mapsto ab$. We say that A is *total* if the application map is total.

Since we do not require the application map to be associative (which it will generally not be), it matters how we bracket our expressions. To avoid an overload of brackets, we adopt the convention that abc is short for $(ab)c$. Before we continue, we first introduce the notion of a term.

Definition 2.1.2. Let A be a PAS. We fix a countably infinite set of *variables*. The set of *terms over A* is defined recursively as follows.

(i) For every element $a \in A$, there is a constant symbol that we also denote by a, which is a term over A.

(ii) Every variable is a term over A.

(iii) If s and t are terms over A, then (st) is a term over A.

A term that does not contain variables is called *closed*.

For terms, we adopt the same convention on brackets as above. Typically, we will use the letters a, b, c to range over elements of A, and letters u, v, w, x, y, z to range over variables. When reasoning with terms, we use the following terminology.

Definition 2.1.3. Let A be a PAS.

(i) For closed terms t and elements $a \in A$, we define the relation $t{\downarrow}a$, read as t *denotes* a, by recursion on t.

  (a) $a{\downarrow}a$ for all $a \in A$;

  (b) $st{\downarrow}a$ if and only if there exist $b, c \in A$ such that $s{\downarrow}b$, $t{\downarrow}c$, bc is defined and $bc = a$.

(ii) We say that a closed term t *denotes*, written $t\downarrow$, if there exists an $a \in A$ such that t denotes a.

(iii) For closed terms s and t, we write $s \simeq t$ if either s and t both do not denote, or there exists an element $a \in A$ such that s and t both denote a. In this case, we say that s and t are *Kleene equal*.

(iv) For closed terms s and t, we write $s = t$ if there is an $a \in A$ such that s and t both denote a.

We notice that a term can only denote if all its subterms denote. Clearly, a closed term t can denote at most element of A. So it t denotes, we can speak of *the* element it denotes. Accordingly, we can view closed terms that denote as elements of A. In such cases, we will not distinguish, in our notation, a closed term that denotes and the element it denotes. We can reformulate the definition of $s \simeq t$ to: either s and t both do not denote, or they both denote the same element. And we have that $s = t$ if and only if s and t denote the same element.

We can also define a notion of *substitution* of terms in terms. We write $t[s/x]$ for the result of substituting the term s for the variable x in t. We adopt similar notation for substituting for more than one variable. If we display the free variables of t by writing $t(x_1, \ldots, x_n)$, then we write $t(a_1, \ldots, a_n)$ for the result of substituting a_i for x_i, where $a_i \in A$.

Now we are ready to define partial combinatory algebras.

Definition 2.1.4. Let A be a PAS. We say that A is *combinatorially complete* if, for all $n \geq 0$ and all terms $t(x_1, \ldots, x_{n+1})$, there exists an element $a \in A$ such that for all $a_1, \ldots, a_{n+1} \in A$, we have

(i) $aa_1 \cdots a_n\downarrow$;

(ii) $aa_1 \cdots a_{n+1} \simeq t(a_1, \ldots, a_{n+1})$.

A *partial combinatory algebra*, abbreviated PCA, is a combinatorially complete PAS.

Intuitively, a PCA is a 'model of computation'. The elements of the PCA can be viewed as algorithms, that can be applied to one another. Combinatorial completeness then says that for every term t, the PCA has an algorithm that computes (the substitution instances of) t.

For any given PAS, it can be rather difficult to check that it is combinatorially complete. Therefore, we give an alternative characterization of combinatorial completeness. The proof of this result also develops some convenient notation.

Theorem 2.1.5 (Feferman)**.** *Let A be a PAS. Then A is a PCA if and only if there exist elements* $\mathsf{k}, \mathsf{s} \in A$ *satisfying:*

(i) $\mathsf{k}ab = a$;

(ii) $\mathsf{s}ab\downarrow$;

(iii) $\mathsf{s}abc \simeq ac(bc)$,

for all $a, b, c \in A$.

Proof. First, suppose A is a PCA. Applying combinatorial completeness to the term $t(x, y) = x$ immediately yields an element $\mathsf{k} \in A$ satisfying (i). Similarly, applying combinatorial completeness to the term $t(x, y, z) = xz(yz)$ yields an element $\mathsf{s} \in A$ satisfying (ii) and (iii). Conversely, suppose we have $\mathsf{k}, \mathsf{s} \in A$ satisfying (i), (ii) and (iii). For variables x and terms t, we define the term $\langle x\rangle t$ by recursion on t:

(i) $\langle x\rangle t$ is the term $\mathsf{k}t$ if t is a constant symbol or a variable different from x;

(ii) $\langle x\rangle x$ is the term skk;

(iii) $\langle x\rangle(st)$ is the term $\mathsf{s}(\langle x\rangle s)(\langle x\rangle t)$.

We notice that $\mathsf{k}a$ denotes for every $a \in A$, since $\mathsf{k}aa = a$ should denote as well. We also notice that $\mathsf{skk}a \simeq \mathsf{k}a(\mathsf{k}a) \simeq a$ for all $a \in A$. Using induction on terms, one easily shows that for any term $t = t(x, x_1, \ldots, x_n)$, the following hold:

(i) the free variables of $\langle x\rangle t$ are those of t, except x;

(ii) $(\langle x\rangle t)[a_1/x_1, \ldots, a_n/x_n]$ denotes for all $a_1, \ldots, a_n \in A$;

(iii) $((\langle x\rangle t)[a_1/x_1, \ldots, a_n/x_n])a \simeq t(a, a_1, \ldots, a_n)$ for all $a, a_1, \ldots, a_n \in A$.

Now let a term $t(x_1, \ldots, x_{n+1})$ be given and define

$$\langle x_1 \cdots x_{n+1}\rangle t = \langle x_1\rangle(\langle x_2\rangle(\cdots\langle x_{n+1}\rangle t\cdots)).$$

For $a_1, \ldots, a_{n+1} \in A$, we see that

$$(\langle x_1 \cdots x_{n+1}\rangle t)a_1 \cdots a_n \simeq (\langle x_{n+1}\rangle t)[a_1/x_1, \ldots, a_n/x_n],$$

which denotes, and

$$\begin{aligned}(\langle x_1 \cdots x_{n+1}\rangle t)a_1 \cdots a_{n+1} &\simeq ((\langle x_{n+1}\rangle t)[a_1/x_1, \ldots, a_n/x_n])a_{n+1}\\ &\simeq t(a_1, \ldots, a_{n+1}).\end{aligned}$$

We conclude that $\langle x_1 \cdots x_{n+1}\rangle t$ is our desired element. □

The elements k and s are called *combinators*. It should be observed that there may be more than one choice for the combinators k and s. In the sequel, we shall assume that for every PCA A we have made an explicit choice for k and s. This means that we can also deploy the notation $\langle x\rangle t$ and $\langle x_1, \ldots x_{n+1}\rangle t$ defined in the proof of Theorem 2.1.5.

Remark 2.1.6. Working with the notation $\langle x\rangle t$ requires some care. We notice a few subtleties.

(i) The operation $\langle x\rangle(\cdot)$ does not respect $\simeq$ or $=$. That is, if we have closed terms s and t such that $s \simeq t$ or $s = t$, then it is not in general true that $\langle x\rangle s = \langle x\rangle t$.

(ii) The operation $\langle x\rangle(\cdot)$ does not commute with substitution. That is, if $t(x, y)$ is a term, we do not in general have that $(\langle x\rangle t)[s/y] = \langle x\rangle t[s/y]$ for every closed term s.

(iii) Suppose $t(x)$ is a term. While it is true that $(\langle x\rangle t)a \simeq t(a)$ for every $a \in A$, we do not in general have that $(\langle x\rangle t)s \simeq t(s)$ for every closed term s. This is so because s may not denote.

2.2 Some Basic Constructions in a PCA

In this section, we describe some basic constructions that can be carried out in any PCA. Using these constructions, we can code basic algorithmic operations inside the PCA.

Let A be a PCA. First of all, we write i for (the element denoted by) the closed term skk. As we saw in the proof of Theorem 2.1.5, we have $\mathsf{i}a = a$ for all $a \in A$. So i functions as an algorithm that does nothing. Next, we write $\overline{\mathsf{k}}$ for ki. Then for all $a, b \in A$, we have

$$\overline{\mathsf{k}}ab \simeq \mathsf{ki}ab \simeq \mathsf{i}b \simeq b,$$

so $\overline{\mathsf{k}}$ works like k but outputs the second entry. We will view k as 'true' and $\overline{\mathsf{k}}$ as 'false'. Now i also functions as a conditional operator: we have $\mathsf{ik}ab = a$ and $\mathsf{i}\overline{\mathsf{k}}ab = b$ for all $a, b \in A$. We can also construct a conditional operator for given closed terms. Let s and t be closed terms, and define the closed term

$$r = \langle x\rangle x(\langle y\rangle s)(\langle y\rangle t)\mathsf{k}.$$

It satisfies

$$r\mathsf{k} \simeq \mathsf{k}(\langle y\rangle s)(\langle y\rangle t)\mathsf{k} \simeq (\langle y\rangle s)\mathsf{k} \simeq s \quad \text{and}$$
$$r\overline{\mathsf{k}} \simeq \overline{\mathsf{k}}(\langle y\rangle s)(\langle y\rangle t)\mathsf{k} \simeq (\langle y\rangle t)\mathsf{k} \simeq t.$$

Remark 2.2.1. The reader may wonder why we included the 'dummy variable' y in the definition of r and did not simply take $r = \langle x\rangle xst$. The reason for this is as follows. Suppose that s denotes but that t does not. Then $(\langle x\rangle xst)\mathsf{k} \simeq \mathsf{k}st$, which does not denote since t does not denote. However, s does denote, so $(\langle x\rangle xst)\mathsf{k} \not\simeq s$. Since an expression of the form $\langle y\rangle t$ always denotes, this scenario is averted. ◊

We also have a pairing operation inside A. Let p be the closed term $\langle xyz\rangle zxy$. Then for all $a, b \in A$, we have that $\mathsf{p}ab \simeq (\langle z\rangle zxy)[a/x, b/y]$, which always denotes. Now let p_0 be the closed term $\langle v\rangle v\mathsf{k}$ and let p_1 be the closed term $\langle v\rangle v\overline{\mathsf{k}}$. Then for all $a, b \in A$, we have

$$\mathsf{p}_0(\mathsf{p}ab) \simeq (\mathsf{p}ab)\mathsf{k} \simeq ((\langle z\rangle zxy)[a/x, b/y])\mathsf{k} \simeq \mathsf{k}ab \simeq a \quad \text{and}$$
$$\mathsf{p}_1(\mathsf{p}ab) \simeq (\mathsf{p}ab)\overline{\mathsf{k}} \simeq ((\langle z\rangle zxy)[a/x, b/y])\overline{\mathsf{k}} \simeq \overline{\mathsf{k}}ab \simeq b.$$

So $\mathsf{p}ab$ codes the pair (a,b), and we can retrieve the elements a and b from $\mathsf{p}ab$ using the terms p_0 and p_1.

Many other constructions from recursion theory can be carried out inside A. For example, for each natural number n, there exists a representative $\overline{n}$ of n in A, called its *Curry numeral*. These Curry numerals are constructed in such a way that every partial recursive function is represented inside A. That is, for every partial recursive function $F\colon \mathbb{N}^k \to \mathbb{N}$, there should exist an $a_F \in A$ satisfying: if $F(n_1,\ldots,n_k)$ is defined, then $a_F\overline{n_1}\cdots\overline{n_k} = \overline{F(n_1,\ldots,n_k)}$. Furthermore, one may use the Curry numerals to code finite sequences of elements of A inside A, in such a way that all the elementary operations on finite sequences, such as concatenation, are represented by an element of A. Developing all this theory is rather tedious, so we will not expand on this here, but instead refer the interested reader to Section 1.3 of the book *Realizability* [6]. The availability of a coding of finite sequences will be used in Section 3.5.

2.3 Examples of PCAs

In this section, we give a few examples of PCAs. In some examples, we shall only define the underlying set of the PCA and the relevant application, but not prove that this actually yields a PCA. The paradigmatic example is the following.

Example 2.3.1. For a natural number m, we write φ_m for the partial recursive function with Gödel number m. This yields a PCA whose underlying set is $\mathbb{N}$, and where application is given by $(m,n) \mapsto \varphi_m(n)$. Combinatorial completeness follows from the fact that the application function is itself partial recursive, and from the *Smn*-theorem. This PCA is called *Kleene's first model*, and we denote it by $\mathcal{K}_1$. $\Diamond$

Kleene's first model fits rather well with the intuition that a PCA consists of algorithms that can be applied to each other. In this case, these algorithms are finitary objects, namely the natural numbers. There are also examples of PCAs whose underlying sets do not consist of finitely respresented objects.

Example 2.3.2. We consider the set $\mathbb{N}^\mathbb{N}$ of all functions from the set of natural numbers to itself. We suppose that we are given a certain bijective coding $\langle\cdot\rangle$ from the set of finite sequences of natural numbers to the set of natural numbers. For an $\alpha \in \mathbb{N}^\mathbb{N}$ and an $n \in \mathbb{N}$, we write $\overline{\alpha}(n)$ for the code $\langle \alpha(0),\ldots,\alpha(n-1)\rangle$. For an element $\alpha \in \mathbb{N}^\mathbb{N}$, we can define a partial function F_α from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$, as follows. Let $\beta \in \mathbb{N}^\mathbb{N}$ and $k \in \mathbb{N}$. We say that $F_\alpha(\beta) = k$ if and only if there exists a natural number n such that $\alpha(\overline{\beta}(n)) = k+1$, while $\alpha(\overline{\beta}(m)) = 0$ for all $m < n$. If such an n does not exist for any $k \in \mathbb{N}$, then we say that $F_\alpha(\beta)$ is undefined.
Now we can define the relevant application map. Let $\alpha, \beta \in \mathbb{N}^\mathbb{N}$ be given. For an $n \in \mathbb{N}$, we write $\langle n\rangle * \beta$ for the element $\beta' \in \mathbb{N}^\mathbb{N}$ given by $\beta'(0) = n$ and $\beta'(k) = \beta(k-1)$ for $k > 0$. If $F_\alpha(\langle n\rangle * \beta)$ is defined for all $n \in \mathbb{N}$, then $\alpha\beta$ is also defined and given by $\alpha\beta(n) = F_\alpha(\langle n\rangle * \beta)$ for all $n \in \mathbb{N}$. Otherwise, $\alpha\beta$ is not defined. This application function yields a PCA, which is called *Kleene's second model* and denoted by $\mathcal{K}_2$.
The underlying set $\mathbb{N}^\mathbb{N}$ of $\mathcal{K}_2$ carries a natural topology, namely the *Baire space topology*. This topology is obtained by giving $\mathbb{N}$ the discrete topology and giving $\mathbb{N}^\mathbb{N}$ the product topology. For an $\alpha \in \mathbb{N}^\mathbb{N}$, let us write G_α for the partial function from $\mathbb{N}^\mathbb{N}$ to itself given by $\beta \mapsto \alpha\beta$.

Then we have that $G_\alpha\colon \operatorname{dom}(G_\alpha) \to \mathbb{N}^\mathbb{N}$ is continuous, where $\operatorname{dom}(G_\alpha)$ is equipped with the subspace topology. Conversely, every partial function from $\mathbb{N}^\mathbb{N}$ to itself that is continuous on its domain, can be extended to a partial function of the form G_α, for a certain $\alpha \in \mathbb{N}^\mathbb{N}$. ♢

Example 2.3.3. Our next example also comes hand in hand with a certain notion of continuity. We consider the powerset $\mathcal{P}\mathbb{N}$ of the natural numbers. On the set $2 = \{0, 1\}$, we have the Sierpinski topology $\{\emptyset, \{1\}, \{0, 1\}\}$. Now we give $2^\mathbb{N}$ the product topology, and by identifying $\mathcal{P}\mathbb{N}$ with $2^\mathbb{N}$ in the canonical way, we also get a topology on $\mathcal{P}\mathbb{N}$. A function $f\colon \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is continuous with respect to this topology if and only if

$$f(A) = \bigcup \{f(p) \mid p \subseteq A \text{ finite}\},$$

for all $A \subseteq \mathbb{N}$. In particular, such a function is completely determined by its values on $\mathcal{P}_{\text{fin}}\mathbb{N}$, the set of finite subsets of $\mathbb{N}$.
We can use the idea of defining a function by prescibing its values on the finite subsets of $\mathbb{N}$ to make $\mathcal{P}\mathbb{N}$ into a PCA. We fix bijections $\langle \cdot, \cdot \rangle\colon \mathbb{N}^2 \to \mathbb{N}$ and $e_{(\cdot)}\colon \mathcal{P}_{\text{fin}}\mathbb{N} \to \mathbb{N}$. Now we can describe a continuous function $F\colon \mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ completely by the set of all pairs $\langle m, n\rangle$ such that $m \in F(e_n)$. Inspired by this observation, we define, for $A, B \in \mathcal{P}\mathbb{N}$, the set

$$AB = \{m \mid \exists n \in \mathbb{N}(e_n \subseteq B \text{ and } \langle m, n\rangle \in A)\}.$$

By construction, a function $\mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$ is continuous if and only if it is of the form $B \mapsto AB$ for a certain $A \subseteq \mathbb{N}$. The application map $(A, B) \mapsto AB$ makes $\mathcal{P}\mathbb{N}$ into a PCA, called *Scott's graph model*. It is worth noticing this PCA is *total*. ♢

Example 2.3.4. Our final example is a bit silly, but nevertheless interesting to mention. It is the *trivial PCA* $\mathbb{I}$ whose underlying set is a singleton $\{*\}$ and whose application is given by $** = *$. It is instructive to contrast this example with other PCAs. If a PCA has more than one element, then all the Curry numerals mentioned in the previous section are distinct, which means that the PCA is automatically infinite. So the trivial PCA is the only finite PCA.
As we shall see below, the constructions based on PCAs typically have a constructive flavour, and are governed by intuitionistic logic. However, if we take the trivial PCA, then these constructive structures usually collapse into classical ones. ♢

2.4 $\mathcal{P}A$-valued Predicates

A *predicate* on a given set X is a subset of X. In classical logic, such a subset of X is given by its characteristic function $X \to \{0, 1\}$. We think of the value 0 as 'false' and of the value 1 as 'true'. The idea of a $\mathcal{P}A$-predicate generalizes this situation. Instead of having only the possible truth values 0 and 1, we allow all subsets of a given PCA A as possible truth values. This leads to the following definition.

Definition 2.4.1. Let A be a PCA and let X be a set.

(i) A *$\mathcal{P}A$-valued predicate* on X is an element of $(\mathcal{P}A)^X$, that is, a function $\phi\colon X \to \mathcal{P}A$. For $a \in A$ and $x \in X$, we also express the fact that $a \in \phi(x)$ by saying that *a realizes* $\phi(x)$, or that a is a *realizer* for $\phi(x)$.

(ii) Given two $\mathcal{P}A$-valued predicates ϕ and ψ on X, we say that $\phi \leq \psi$ if there exists an element $a \in A$ satisfying: if $x \in X$ and $b \in \phi(x)$, then $ab\downarrow$ and $ab \in \psi(x)$. If this holds, we say that a *realizes* $\phi \leq \psi$.

Intuitively, the elements of $\phi(x)$ are 'evidence' that the statement $\phi(x)$ indeed holds. If we recall the intuition of a PCA as a model of computation, then a realizer for $\phi \leq \psi$ can be viewed as an algorithm that computes evidence for $\psi(x)$, given evidence for $\phi(x)$. Notice that this algorithm should not depend on x.

The elementary operations we can perform in a PCA give rise to the following result.

Proposition 2.4.2. *Let A be a PCA and let X be a set. Then $((\mathcal{P}A)^X, \leq)$ is a Heyting prealgebra.*

Proof. In this proof, ϕ, ψ and χ will be variables ranging over $(\mathcal{P}A)^X$. First of all, we have to check that $((\mathcal{P}A)^X, \leq)$ is a preorder. We see that i always realizes $\phi \leq \phi$, so $\leq$ is reflexive. Now suppose a realizes $\phi \leq \psi$ and that b realizes $\psi \leq \chi$. Then it is not difficult to check that $\langle u \rangle b(au)$ realizes $\phi \leq \chi$, so $\leq$ is also transitive.
We define the predicate $\bot \colon X \to \mathcal{P}A$ by $\bot(x) = \emptyset$ for all $x \in X$, and we define the predicate $\top \colon X \to \mathcal{P}A$ by $\top(x) = A$ for all $x \in X$. Then i always realizes the two inequalities $\bot \leq \phi \leq \top$, so $((\mathcal{P}A)^X, \leq)$ has a bottom element and a top element.
Given ϕ and ψ, we define the predicate $\phi \wedge \psi$ by

$$(\phi \wedge \psi)(x) = \{\mathsf{p}ab \mid a \in \phi(x), b \in \psi(x)\} \quad \text{for } x \in X.$$

We claim that $\phi \wedge \psi$ is a meet of ϕ and ψ. First of all, we notice that p_0 and p_1 realize $\phi \wedge \psi \leq \phi$ and $\phi \wedge \psi \leq \psi$, respectively. Furthermore, if a realizes $\chi \leq \phi$ and b realizes $\chi \leq \psi$, then $\langle u \rangle \mathsf{p}(au)(bu)$ realizes $\chi \leq \phi \wedge \psi$.
Next, given ϕ and ψ, we define the predicate $\phi \vee \psi$ by

$$(\phi \vee \psi)(x) = \{\mathsf{pk}a \mid a \in \phi(x)\} \cup \{\mathsf{p}\overline{\mathsf{k}}b \mid b \in \psi(x)\} \quad \text{for } x \in X.$$

We claim that $\phi \vee \psi$ is a join of ϕ and ψ. First of all, we notice that $\langle u \rangle \mathsf{pk}u$ and $\langle u \rangle \mathsf{p}\overline{\mathsf{k}}u$ realize $\phi \leq \phi \vee \psi$ and $\psi \leq \phi \vee \psi$, respectively. Furthermore, if a realizes $\phi \leq \chi$ and b realizes $\psi \leq \chi$, then $\phi \vee \psi \leq \chi$ is realized by the 'case distinction operator'

$$[a, b] = \langle u \rangle \mathsf{p}_0 u(\langle v \rangle a(\mathsf{p}_1 u))(\langle v \rangle b(\mathsf{p}_1 u))\mathsf{k}.$$

Indeed, let $x \in X$ and $c \in \phi(x)$. Then

$$\begin{aligned}
[a, b](\mathsf{pk}c) &\simeq \mathsf{p}_0(\mathsf{pk}c)(((\langle v \rangle a(\mathsf{p}_1 u))[\mathsf{pk}c/u])(((\langle v \rangle b(\mathsf{p}_1 u))[\mathsf{pk}c/u])\mathsf{k} \\
&\simeq \mathsf{k}(((\langle v \rangle a(\mathsf{p}_1 u))[\mathsf{pk}c/u])(((\langle v \rangle b(\mathsf{p}_1 u))[\mathsf{pk}c/u])\mathsf{k} \\
&\simeq (((\langle v \rangle a(\mathsf{p}_1 u))[\mathsf{pk}c/u])\mathsf{k} \\
&\simeq a(\mathsf{p}_1(\mathsf{pk}c)) \\
&\simeq ac,
\end{aligned}$$

which denotes and is an element of $\chi(x)$. Similarly, we can show that if $x \in X$ and $c \in \psi(x)$, then $[a, b](\mathsf{p}\overline{\mathsf{k}}c) \simeq bc$, which denotes and is an element of $\chi(x)$.
Finally, given ϕ and ψ, we define the predicate $\phi \to \psi$ as

$$(\phi \to \psi)(x) = \{a \in A \mid \forall b \in \phi(x)\,(ab\downarrow \text{ and } ab \in \psi(x))\} \quad \text{for } x \in X.$$

We claim that $\phi \to \psi$ is a Heyting implication of ϕ and ψ. Indeed, suppose that a realizes $\chi \wedge \phi \leq \psi$. Then $\langle uv \rangle a(\mathsf{p}uv)$ realizes $\chi \leq \phi \to \psi$. Conversely, if a realizes $\chi \leq \phi \to \psi$, then $\langle u \rangle a(\mathsf{p}_0 u)(\mathsf{p}_1 u)$ realizes $\chi \wedge \phi \leq \psi$. □

The proof above even gives explicit choices for the logical operations $\bot$, $\top$, $\wedge$, $\vee$ and $\to$ on $((\mathcal{P}A)^X, \leq)$, that we will use from now on.

Remark 2.4.3. In the proof that $\phi \vee \psi$ is indeed a join of ϕ and ψ, we used dummy variables v for the same reason as we explained in Remark 2.2.1. As a result, we had to take some care in substituting $\mathsf{pk}c$ for u, because we could not just replace the u inside the scope of $\langle v \rangle$ by $\mathsf{pk}c$, in view of Remark 2.1.6(ii). So we had to wait for the $\langle v \rangle$ to disappear before we could get the expression $a(\mathsf{p}_1(\mathsf{pk}c))$ and finish the calculation. ◊

We now study various interesting maps between the preorders $((\mathcal{P}A)^X, \leq)$.

Definition 2.4.4. Let A be a PCA and let $f\colon X \to Y$ be a function.

(i) The map $f^*\colon ((\mathcal{P}A)^Y, \leq) \to ((\mathcal{P}A)^X, \leq)$ is defined by $f^*(\phi)(x) = \phi(f(x))$ for a predicate $\phi\colon Y \to \mathcal{P}A$ and an element $x \in X$.

(ii) The maps $\exists_f, \forall_f\colon ((\mathcal{P}A)^X, \leq) \to ((\mathcal{P}A)^Y, \leq)$ are defined by

$$\exists_f(\phi)(y) = \bigcup_{f(x)=y} \phi(x) \quad \text{and}$$
$$\forall_f(\phi)(y) = \{a \in A \mid \forall b \in A \forall x \in X\, (\text{if } f(x) = y, \text{ then } ab{\downarrow} \text{ and } ab \in \phi(x))\}$$

for $\phi\colon X \to \mathcal{P}A$ and $y \in Y$.

The algebras of $\mathcal{P}A$-valued predicates and the maps f^*, $\exists_f$ and $\forall_f$ between them will play a major role in Section 3.1, where we introduce the realizability tripos. In anticipation of this, we prove some elementary properties of the maps introduced above. First, we need the following definition.

Definition 2.4.5. Suppose we have preorders $(B, \leq)$ and $(C, \leq)$ and functions $B \underset{g}{\overset{f}{\leftrightarrows}} C$. We say that f is *left adjoint* to g, or that g is *right adjoint* of f, written $f \dashv g$, if we have $f(c) \leq b$ if and only if $c \leq g(b)$ for all $b \in B$ and $c \in C$.

It should be observed that, if $f \dashv g$, then the functions f and g are automatically order preserving. Indeed, suppose that $b \leq b'$ for certain $b, b' \in B$. Since $g(b) \leq g(b)$, we also have $f(g(b)) \leq b \leq b'$, which yields $g(b) \leq g(b')$. The argument for f is dual. If we see f and g as functors between the preorder categories $(B, \leq)$ and $(C, \leq)$, then Definition 2.4.5 states exactly what it means for f and g to be adjoint as functors. So our terminology is consistent.

Proposition 2.4.6. *Let A be a PCA and let $f\colon X \to Y$ be a function. Then:*

(i) f^* is a map of Heyting prealgebras;

(ii) we have $\exists_f \dashv f^* \dashv \forall_f$, and in particular, $\exists_f$ and $\forall_f$ are order preserving.

Proof. (i) Using the choices for the logical operations from the proof of Proposition 2.4.2, it is not difficult to show that f^* commutes with these operations. For example, in the case of conjunction, we have that

$$\begin{aligned} f^*(\phi \wedge \psi)(x) &= (\phi \wedge \psi)(f(x)) \\ &= \{\mathsf{p}ab \mid a \in \phi(f(x)), b \in \psi(f(x))\} \\ &= \{\mathsf{p}ab \mid a \in f^*(\phi)(x), b \in f^*(\psi)(x)\} \\ &= (f^*(\phi) \wedge f^*(\psi))(x), \end{aligned}$$

for all $\phi, \psi \colon Y \to \mathcal{P}A$ and $x \in X$.
(ii) Let $\phi \colon X \to \mathcal{P}A$ and $\psi \colon Y \to \mathcal{P}A$. Then we observe that a realizes $\exists_f(\phi) \leq \psi$ if and only if a realizes $\phi \leq f^*(\psi)$, which shows the first adjunction. The second adjunction is a bit more difficult. First, suppose that a realizes $f^*(\psi) \leq \phi$. Then $\langle uv \rangle au$ realizes $\psi \leq \forall_f(\phi)$. Indeed, let $y \in Y$ and $b \in \psi(y)$. Then $(\langle uv \rangle au)b \simeq (\langle v \rangle au)[b/u]$ denotes. Furthermore, if we take $c \in A$ and $x \in X$ such that $f(x) = y$, then $b \in \phi(f(x))$, so $((\langle v \rangle au)[b/u])c \simeq ab$ denotes and is an element of $\psi(x)$. That is, $(\langle uv \rangle au)b \in \forall_f(\phi)(y)$, as desired. Conversely, if a realizes $\psi \leq \forall_f(\phi)$, then $\langle u \rangle au\mathsf{k}$ realizes $f^*(\psi) \leq \phi$. Indeed, let $x \in X$ and $b \in \psi(f(x))$. Then $ab\downarrow$ and $ab \in \forall_f(\phi)(f(x))$. By the definition of $\forall_f$, it follows that $(\langle u \rangle au\mathsf{k})b \simeq ab\mathsf{k}$ denotes and is an element of $\phi(x)$, as desired. □

Remark 2.4.7. The definition of $\forall_f$ is somewhat involved. If f is surjective, then we can replace the definition by the simpler

$$\forall_f(\phi)(y) = \bigcap_{f(x)=y} \phi(x) \quad \text{for } \phi \colon X \to \mathcal{P}A \text{ and } y \in Y.$$

Indeed, one easily shows that, *if f is surjective*, then a realizes $f^*(\psi) \leq \phi$ if and only if a realizes $\psi \leq \forall_f(\phi)$, for all $\phi \colon X \to \mathcal{P}A$ and $\psi \colon Y \to \mathcal{P}A$. Therefore, if f is surjective, one may take this as their definition of $\forall_f$ rather than the expression from Definition 2.4.4. ◊

Example 2.4.8. At the beginning of this section, we claimed that $\mathcal{P}A$-valued predicates are a generalization of predicates from classical logic. And indeed, such predicates can be seen as a special case of $\mathcal{P}A$-valued predicates. Let A be the trivial PCA $\mathbb{I}$ from Example 2.3.4. Then the structure $((\mathcal{P}\mathbb{I})^X, \leq)$ is isomorphic to the Boolean algebra $(\mathcal{P}X, \subseteq)$. The map f^* gives the inverse image under f, while $\exists_f$ gives the direct image under f. ◊

2.5 Assemblies

In this section, we introduce *assemblies* over a PCA A. These are the objects of a certain category, called the category of assemblies. In many ways, this is a nice category; but it is not a topos. In the next chapter, we will construct the *realizability topos* over A, that will contain an isomorphic copy of the category of assemblies.

Definition 2.5.1. Let A be a PCA.

(i) An *assembly* (over A) is a pair (X, E), where X is a set and $E \colon X \to \mathcal{P}^*A$ is a function that assigns to each $x \in X$ a *nonempty* set $E(x) \subseteq A$.

(ii) Let (X, E) and (Y, F) be assemblies. A *morphism* $(X, E) \to (Y, F)$ of assemblies is a function $f\colon X \to Y$ such that there exists an element $a \in A$ satisfying: if $x \in X$ and $b \in E(x)$, then $ab\downarrow$ and $ab \in F(f(x))$. We say that such an element a *tracks* f.

The requirement that $E(x)$ should be nonempty is hard to motivate at present, but it will become clear when we describe the way the category of assemblies sits inside the realizability topos. We will also need this requirement in the proof of Proposition 2.5.4.

Proposition 2.5.2. *Let A be a PCA. The assemblies over A and morphisms between them form a category, called the* category of assemblies*, and denoted by* $\mathsf{Asm}(A)$*.*

Proof. Let (X, E) be an assembly. The identity morphism for this assembly will be the identity function on X. This is indeed a morphism, since it is tracked by i.
Given morphisms $(X, E) \xrightarrow{f} (Y, F) \xrightarrow{g} (Z, G)$, their composition will be usual function composition gf. If a tracks f and b tracks g, then $\langle u\rangle b(au)$ tracks gf, which shows that gf is indeed a morphism. □

The remainder of this section will be devoted to establishing several important properties of $\mathsf{Asm}(A)$.

Proposition 2.5.3. *Let A be a PCA. The category* $\mathsf{Asm}(A)$ *is regular and has all finite colimits.*

Proof. First, we check that $\mathsf{Asm}(A)$ has all finite limits. We observe that $(1, 0 \mapsto A)$ is a terminal object of $\mathsf{Asm}(A)$. Indeed, if (X, E) is an assembly, then i tracks the unique function $X \to 1$. Now let assemblies (X, E) and (Y, F) be given. Their product in $\mathsf{Asm}(A)$ is given by $(X \times Y, P_{E,F})$, where

$$P_{E,F}(x, y) = \{\mathsf{p}ab \mid a \in E(x), b \in F(y)\} \quad \text{for } x \in X, y \in Y.$$

The projection maps $\pi_0\colon X \times Y \to X$ and $\pi_1\colon X \times Y \to Y$ are tracked by p_0 and p_1, respectively, so we can view them as morphisms of assemblies. Now suppose we have morphisms of assemblies $f\colon (Z, G) \to (X, E)$ and $g\colon (Z, G) \to (Y, F)$ tracked by a and b respectively. We should show that the function $\langle f, g\rangle\colon Z \to X \times Y$ is a morphism of assemblies $(Z, G) \to (X \times Y, P_{E,F})$. And indeed, it is tracked by $\langle u\rangle \mathsf{p}(au)(bu)$. So $\mathsf{Asm}(A)$ has binary products. Finally, suppose we have a parallel pair of morphisms $f, g\colon (X, E) \to (Y, F)$. Let $X' = \{x \in X \mid f(x) = g(x)\}$. The inclusion $m\colon X' \to X$ is a morphism of assemblies $(X', E|_{X'}) \to (X, E)$, since it is tracked by i. Now suppose we have a morphism of assemblies $h\colon (Z, G) \to (X, E)$ that satisfies $fh = gh$. Then in Set, we know that there exists a unique $k\colon Z \to X'$ such that $mk = h$. This k is also a morphism $(Z, G) \to (X', E|_{X'})$, since any tracker of h is also a tracker of k. So m is an equalizer of f and g in $\mathsf{Asm}(A)$, and we conclude that $\mathsf{Asm}(A)$ has all finite limits.
Now we construct finite colimits in $\mathsf{Asm}(A)$. We observe that $(\emptyset, \emptyset)$ is an initial object of $\mathsf{Asm}(A)$. Indeed, if (X, E) is any assembly, then any element of A tracks the unique function $\emptyset \to X$. Now let assemblies (X, E) and (Y, F) be given. Their coproduct in $\mathsf{Asm}(A)$ is $(X \sqcup Y, C_{E,F})$, where $X \sqcup Y = (\{0\} \times X) \cup (\{1\} \times Y)$ and

$$C_{E,F}(0, x) = \{\mathsf{pk}a \mid a \in E(x)\} \text{ for } x \in X \quad \text{and} \quad C_{E,F}(1, y) = \{\mathsf{p}\bar{\mathsf{k}}b \mid b \in F(y)\} \text{ for } y \in Y.$$

The coproduct inclusions $\kappa_0 \colon X \to X \sqcup Y$ and $\kappa_1 \colon Y \to X \sqcup Y$ are tracked by $\langle u \rangle \mathsf{pk}u$ and $\langle u \rangle \mathsf{p}\bar{\mathsf{k}}u$, respectively. Suppose we have morphisms of assemblies $f \colon (X, E) \to (Z, G)$ and $g \colon (Y, F) \to (Z, G)$ tracked by a and b respectively. Then $[f, g] \colon X \sqcup Y \to Z$ is a morphism of assemblies $(X \sqcup Y, C_{E,F}) \to (Z, G)$, for it is tracked by

$$\langle u \rangle \mathsf{p}_0 u (\langle v \rangle a(\mathsf{p}_1 u))(\langle v \rangle b(\mathsf{p}_1 u))\mathsf{k}.$$

So $\mathsf{Asm}(A)$ has binary coproducts. Finally, suppose we have a parallel pair of morphisms $f, g \colon (X, E) \to (Y, F)$. Let $q \colon Y \twoheadrightarrow Y'$ be a coequalizer of f and g in Set and define the map $F' \colon Y' \to \mathcal{P}^*A$ by $F'(y') = \bigcup_{q(y)=y'} F(y) \neq \emptyset$ for $y' \in Y$. Then q is a morphism $(Y, F) \to (Y', F')$, for it is tracked by i. Now suppose we have a morphism $h \colon (Y, F) \to (Z, G)$ such that $hf = hg$. Then in Set, we know that there exists a unique $k \colon Y' \to Z$ such that $kq = h$. This k is also a morphism of $(Y', F') \to (Z, G)$, since any tracker of h is also a tracker of k. So q is a coequalizer of f and g in $\mathsf{Asm}(A)$, and we conclude that $\mathsf{Asm}(A)$ has all finite colimits.

Before we check that $\mathsf{Asm}(A)$ is regular, we first find a convenient description of regular epis in $\mathsf{Asm}(A)$. Suppose we have a morphism $e \colon (X, E) \to (Y, F)$. For $y \in Y$, we define $F'(y) = \bigcup_{e(x)=y} E(x)$. We claim that e is regular epi if and only if e is surjective and $\mathrm{id}_Y \colon (Y, F) \to (Y, F')$ is a morphism of assemblies. First, suppose that e is regular epi. From the above description of coequalizers, it follows that e must be surjective. We also observe that $e \colon (X, E) \to (Y, F')$ is a morphism of assemblies, since it is tracked by i, so it must factor through $e \colon (X, E) \to (Y, F)$ by a certain $k \colon (Y, F) \to (Y, F')$. Now in Set, we have $ke = e$, and e is surjective, so $k = \mathrm{id}_Y$, as desired. Conversely, suppose that e is surjective and that $\mathrm{id}_Y \colon (Y, F) \to (Y, F')$ is a morphism of assemblies. First of all, we notice that $\mathrm{id}_Y \colon (Y, F') \to (Y, F)$ is always a morphism, since it is tracked by any tracker of e. So we see that $\mathrm{id}_Y \colon (Y, F) \to (Y, F')$ is an isomorphism in $\mathsf{Asm}(A)$. Construct the kernel pair $p_0, p_1 \colon (Z, G) \to (X, E)$ of e in $\mathsf{Asm}(A)$. From our description of finite limits in $\mathsf{Asm}(A)$ above, it is clear that $p_0, p_1 \colon Z \to X$ is also a kernel pair of e in Set. But e is surjective, so it is the coequalizer of p_0 and p_1 in Set. By the construction of coequalizers we gave above, it follows that $e \colon (X, E) \to (Y, F')$ is a coequalizer of p_0 and p_1 in $\mathsf{Asm}(A)$. Since $\mathrm{id}_Y \colon (Y, F) \to (Y, F')$ is an isomorphism in $\mathsf{Asm}(A)$, it follows that $e \colon (X, E) \to (Y, F)$ is also a coequalizer of p_0 and p_1 in $\mathsf{Asm}(A)$.

We check that $\mathsf{Asm}(A)$ is regular. First, let a morphism $f \colon (X, E) \to (Y, F)$ be given. We factor f in Set as $X \xrightarrow{e} \mathrm{Im}\, f \xrightarrow{m} Y$, where e is surjective and m is injective. For $z \in \mathrm{Im}\, f$, define $G(z) = \bigcup_{e(x)=z} E(x) \neq \emptyset$. Then $e \colon (X, E) \to (\mathrm{Im}\, f, G)$ is a morphism (it is tracked by i), and by the above, it follows that e is regular epi. Also, $m \colon (\mathrm{Im}\, f, G) \to (Y, F)$ is a morphism, since any tracker for f is also a tracker of m. Since m is injective, it follows that m is mono in $\mathsf{Asm}(A)$. We conclude that we have regular epi-mono factorizations inside $\mathsf{Asm}(A)$.

It remains to check that regular epis are stable under pullback. From our description of finite limits above, it follows that, up to isomorphism, pullback diagrams in $\mathsf{Asm}(A)$ are of the form

$$\begin{array}{ccc} (W, H) & \xrightarrow{\pi_1} & (Y, F) \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle g} \\ (X, E) & \xrightarrow{f} & (Z, G) \end{array}$$

where $W = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$ and H is $P_{E,F}$ restricted to W. Suppose that f is a regular epi. Then f is surjective, so π_1 must be surjective as well, since we know that

Set is regular. Now define $G'(z) = \bigcup_{f(x)=z} E(x)$ for $z \in Z$ and let a track $(Z,G) \to (Z,G')$. Furthermore, let b track g. For $y \in Y$, we define

$$F'(y) = \bigcup_{(x,y)\in W} H(x,y) = \bigcup_{f(x)=g(y)} \{\mathsf{p}ab \mid a \in E(x), b \in F(y)\}.$$

Then $\langle v\rangle \mathsf{p}(a(bv))v$ tracks $(Y,F) \to (Y,F')$. So π_1 is also a regular epi, as desired. □

Proposition 2.5.4. *Let A be a PCA. The category $\mathsf{Asm}(A)$ is cartesian closed.*

Proof. Let assemblies (X,E) and (Y,F) be given. Their exponential $(Y,F)^{(X,E)}$ can be constructed as (Z,G), where

$$Z = \{f \in Y^X \mid f\colon (X,E) \to (Y,F) \text{ is a morphism}\} \quad \text{and}$$
$$G(f) = \{a \in A \mid a \text{ tracks } f\colon (X,E) \to (Y,F)\} \neq \emptyset \quad \text{for } f \in Z.$$

The evaluation arrow $\mathrm{ev}\colon (Z,G) \times (X,E) = (Z \times X, P_{G,E}) \to (Y,F)$ is given by $\mathrm{ev}(f,x) = f(x)$. This map is tracked by $\langle v\rangle \mathsf{p}_0 v(\mathsf{p}_1 v)$. Now suppose we have a morphism of assemblies $g\colon (W,H) \times (X,E) = (W \times X, P_{H,E}) \to (Y,F)$. Its exponential transpose $\tilde{g}\colon (W,H) \to (Z,G)$ must be given by $g(w) = (x \mapsto g(w,x))$ for $w \in W$. Let a track g and consider the closed term $\langle uv\rangle a(\mathsf{p}uv)$. Let $w \in W$ and $x \in X$ be arbitrary. Then for all $b \in H(w)$ and $c \in E(x)$, we have that $(\langle uv\rangle a(\mathsf{p}uv))b \simeq (\langle v\rangle a(\mathsf{p}uv))[b/u]$, which denotes. Furthermore, we have $\mathsf{p}bc \in P_{H,E}(w,x)$, so

$$(\langle uv\rangle a(\mathsf{p}uv))bc \simeq (((\langle v\rangle a(\mathsf{p}uv))[b/u])c \simeq a(\mathsf{p}bc)$$

denotes and is an element of $F(g(w,x))$. We conclude that $(\langle uv\rangle a(\mathsf{p}uv))b$ tracks the function $(x \mapsto g(w,x))\colon (X,E) \to (Y,F)$ for all $b \in H(w)$. Since $H(w)$ is nonempty, we see that $x \mapsto g(w,x)$ has at least one tracker, so $\tilde{g}$ is well-defined. And we also see that $\langle uv\rangle a(\mathsf{p}uv)$ tracks $\tilde{g}$ itself, so $\tilde{g}$ is a morphism of assemblies, as desired. □

As a final result, we show that there is an interesting adjunction between Set and $\mathsf{Asm}(A)$.

Proposition 2.5.5. *There exist regular functors $\Gamma\colon \mathsf{Asm}(A) \to \mathsf{Set}$ and $\Delta\colon \mathsf{Set} \to \mathsf{Asm}(A)$ such that $\Gamma \dashv \Delta$ and $\Gamma\Delta$ is the identity on Set.*

Proof. We define Γ as the forgetful functor that sends an assembly (X,E) to its underlying set X and that is the identity on arrows. The functor Δ sends a set X to the assembly (X, cst_X), where cst_X is the constant function assigning A to every $x \in X$. Furthermore, Δ is the identity on arrows. For any function $f\colon X \to Y$, we have that i tracks $f\colon \Delta X \to \Delta Y$, so Δ is well-defined.
Clearly, $\Gamma\Delta X = \Gamma(X, \mathrm{cst}_X) = X$ for all sets X, and $\Gamma\Delta f = \Gamma f = f$ for all functions f. Now suppose that (X,E) is an assembly and that Y is a set. For any function $f\colon X \to Y$, we have that i tracks $f\colon (X,E) \to \Delta Y$. This means that the identity is a bijection

$$\mathsf{Set}(\Gamma(X,E), Y) \to \mathsf{Asm}(A)(X, \Delta Y).$$

Clearly, these bijections are natural in X and Y, so $\Gamma \dashv \Delta$.
The functor Γ, being a left adjoint, preserves all colimits. From the description of finite limits in $\mathsf{Asm}(A)$ in the proof of Proposition 2.5.3, it is also clear that Γ preserves them. The functor Δ, being a right adjoint, preserves all limits. From the description of regular epis in the proof of Proposition 2.5.3, it is also clear that Δe is a regular epi if e is surjective. So Γ and Δ are both regular. □

An assembly of the form ΔX will also be called a *constant object*, and Δ is called the *constant objects functor*. This functor is fully faithful.

Remark 2.5.6. As we already noticed, Γ preserves all (finite) colimits. The functor Δ, however, does not, for it does not preserve coproducts. In fact, if X is a set that has more than one element, then ΔX cannot be written as $(Y, F) \sqcup (Z, G)$, where (Y, F) and (Z, G) are noninitial objects of $\mathsf{Asm}(A)$. In particular, $\Delta 2$ is not isomorphic to $\Delta 1 \sqcup \Delta 1$. $\Diamond$

Chapter 3

Realizability Topoi

In this chapter, we study the realizability topos of a PCA A and various of its properties. As a first step, in Section 3.1, we define the realizability tripos of a PCA and explain how such a tripos interprets typed intuitionistic predicate logic. In Section 3.2, we will use this to define the realizability topos and to show that it is in fact a topos. Next, in Section 3.3, we establish some basic facts about the realizability topos, among which the fact that $\mathsf{Asm}(A)$ is (isomorphic to) a full subcategory of the realizability topos. The remainder of the chapter is devoted to the study of two special kinds of objects in the realizability topos: projective objects in Section 3.4 and discrete objects in Section 3.5.

3.1 The Realizability Tripos of a PCA

In Section 2.4, we introduced the Heyting prealgebras $((\mathcal{P}A)^X, \leq)$ of $\mathcal{P}A$-valued predicates on X, and the maps f^*, $\exists_f$ and $\forall_f$ between them. We can see this construction as a contravariant functor from Set to a certain category, that we will call the realizability tripos.

Definition 3.1.1. (i) The category Hpa of Heyting prealgebras has as objects Heyting prealgebras, and as arrows maps of Heyting prealgebras.

(ii) Let A be a PCA. The functor $\mathsf{P}\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{Hpa}$ is defined by

(a) $\mathsf{P}X = ((\mathcal{P}A)^X, \leq)$ for sets X;

(b) $\mathsf{P}f = (f^*\colon \mathsf{P}Y \to \mathsf{P}X)$ for functions $f\colon X \to Y$.

The functor P is called the *realizability tripos* of A.

It is not difficult to check that P is, in fact, a functor. Usually, we will just write f^* instead of $\mathsf{P}f$. Before we can study the relation of the maps $\exists_f$ and $\forall_f$ to P, we need the following notation.

Definition 3.1.2. Let $(B, \leq)$ be a preorder.

(i) We write $\simeq$ for the equivalence relation on B that gives its poset reflection. That is, $b \simeq b'$ if and only if $b \leq b'$ and $b' \leq b$ for $b, b' \in A$. If $b \simeq b'$, we say that b and b' are isomorphic.

(ii) Suppose we have order preserving maps $f, g \colon (B, \leq) \to (C, \leq)$. We say that $f \leq g$ if $f(b) \leq g(b)$ for all $b \in B$, and that $f \simeq g$ if $f(b) \simeq g(b)$ for all $b \in B$. If $f \simeq g$, we say that f and g are isomorphic.

If we view f and g as functors between the poset categories $(B, \leq)$ and $(C, \leq)$, then $f \leq g$ means precisely that there exists a natural transformation $f \Rightarrow g$, and $f \simeq g$ says that f and g are naturally isomorpic. If $f \dashv g$, then another map f' is left adjoint to g if and only if $f \simeq f'$, and similarly for right adjoints. In particular, left/right adjoints are unique up to $\simeq$.

Since we do not really care whether elements or order preserving maps are really equal or just isomorphic, we will write $\simeq$ even when we know that equality holds. So we will express the functoriality of P by $\mathrm{id}_X^* \simeq \mathrm{id}_{\mathsf{P}X}$ for sets X and $(gf)^* \simeq f^*g^*$ for composable functions f and g.

Proposition 3.1.3. *Let A be a PCA and P be its realizability tripos.*

(*i*) *For any set X, we have $\exists_{\mathrm{id}_X} \simeq \mathrm{id}_{\mathsf{P}X} \simeq \forall_{\mathrm{id}_X}$.*

(*ii*) *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are functions, then $\exists_{gf} \simeq \exists_g \circ \exists_f$ and $\forall_{gf} \simeq \forall_g \circ \forall_f$.*

(*iii*) (The Beck-Chevalley Condition) *Suppose*

$$\begin{array}{ccc} W & \xrightarrow{k} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

is a pullback diagram in Set. Then $\exists_k \circ h^* \simeq g^* \circ \exists_f$ and $\forall_k \circ h^* \simeq g^* \circ \forall_f$.

Proof. (i) We know that $\exists_{\mathrm{id}_X} \dashv \mathrm{id}_X^*$ and $\mathrm{id}_{\mathsf{P}X} \dashv \mathrm{id}_{\mathsf{P}X}$. Since we also have $\mathrm{id}_X^* \simeq \mathrm{id}_{\mathsf{P}X}$, it follows that $\exists_{\mathrm{id}_X} \simeq \mathrm{id}_{\mathsf{P}X}$ by the uniqueness of left adjoints. The argument for $\forall$ is dual.
(ii) We know that adjunctions can be composed. So since $\exists_f \dashv f^*$ and $\exists_g \dashv g^*$, we also have $\exists_g \circ \exists_f \dashv f^*g^*$. Furthermore, we have that $\exists_{gf} \dashv (gf)^*$. So since $(gf)^* \simeq f^*g^*$, it follows that $\exists_{gf} \simeq \exists_g \circ \exists_f$ by the uniqueness of left adjoints. The argument for $\forall$ is dual.
(iii) Without loss of generality, $W = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$ and h and k are the obvious projections. For all $\phi \in \mathsf{P}X$ and $y \in Y$, we have

$$\exists_k(h^*(\phi))(y) = \bigcup_{(x,y) \in W} h^*(\phi)(x, y) = \bigcup_{f(x)=g(y)} \phi(x) = \exists_f(\phi)(g(y)) = g^*(\exists_f(\phi))(y),$$

so $\exists_k \circ h^* \simeq g^* \circ \exists_f$ (they are in fact equal). By twisting the pullback diagram, we also find $\exists_h \circ k^* \simeq f^* \circ \exists_g$. By the compositionality of adjunctions, we find that $\exists_h \circ k^* \dashv \forall_k \circ h^*$ and that $f^* \circ \exists_g \dashv g^* \circ \forall_f$, so the seond statement also follows, by the uniqueness of right adjoints. $\square$

Remark 3.1.4. As one may varify by direct computation, the first isomorphism in (i) is in fact an equality. The second one, however, is not in general an equality. The same holds for (ii). The second statement of (iii) is also in fact an equality, as one may compute directly. $\Diamond$

The element $\mathrm{id}_{\mathcal{P}A}$ of $\mathsf{P}(\mathcal{P}A)$ has the special property that, for any $\phi \in \mathsf{P}X$, there exists a function $f\colon X \to \mathcal{P}A$ such that $\phi \simeq f^*(\mathrm{id}_{\mathcal{P}A})$. Indeed, we may take f to be ϕ itself. We call $\mathrm{id}_{\mathcal{P}A}$ a *generic object* for P.

Remark 3.1.5. The term 'realizability tripos' suggests that it is an instantiation of the more general concept 'tripos'. And this is indeed the case. Let $\mathcal{C}$ be a cartesian closed category. A *pseudofunctor* $\mathsf{P}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Hpa}$ is the same as a functor $\mathcal{C}^{\mathrm{op}} \to \mathsf{Hpa}$, except that the requirements $\mathsf{P}(\mathrm{id}_X) = \mathrm{id}_{\mathsf{P}X}$ and $\mathsf{P}(gf) = \mathsf{P}f \circ \mathsf{P}g$ are replaced by the weaker $\mathsf{P}(\mathrm{id}_X) \simeq \mathrm{id}_{\mathsf{P}X}$ and $\mathsf{P}(gf) \simeq \mathsf{P}f \circ \mathsf{P}g$. We say that P is a *tripos* over $\mathcal{C}$ if $\mathsf{P}f$ always has a left adjoint $\exists_f$ and a right adjoint $\forall_f$, these adjoints satisfy the Beck-Chevalley Condition, and there exists a generic element for P. If $\mathcal{C}$ is not cartesian closed but only has finite limits, we can also define the notion of a tripos over $\mathcal{C}$, but we have to replace the existence of a generic object by a stronger requirement. $\diamondsuit$

At the beginning of this section, we promised that the realizability tripos can be used to interpret typed intuitionistic predicate logic. First, we define the relevant language.

Definition 3.1.6. A Set-*typed language* $\mathcal{L}$ will be a set of *relation symbols* such that to each relation symbol R, a finite sequence of sets $(X_1, \ldots, X_n)$ is assigned, called the *type* of R.

(i) For every finite sequence $(X_1, \ldots, X_n, X)$ of sets and every function $f\colon X_1 \times \cdots \times X_n \to X$, we introduce a function symbol, that we shall also denote by f. We now define the class of *terms* over Set by recursion:
   - (a) for every set X, we have a countably infinite set of variables $x_0^X, x_1^X, \ldots$ of type X, and these are all terms of type X;
   - (b) if $t_1, \ldots, t_n$ are terms of type $X_1, \ldots, X_n$ respectively, and $f\colon X_1 \times \cdots \times X_n \to X$ is a function, then $f(t_1, \ldots, t_n)$ is a term of type X.

(ii) We define the class of *formulae* over $\mathcal{L}$ by recursion:
   - (a) $\top$ and $\bot$ are formulae;
   - (b) if R is a relation symbol of type $(X_1, \ldots, X_n)$ and $t_1, \ldots, t_n$ are terms of type $X_1, \ldots, X_n$ respectively, then $R(t_1, \ldots, t_n)$ is a formula.
   - (c) if φ and ψ are formulae, then $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\varphi \to \psi$ are formulae;
   - (d) if φ is a formula and x is a variable (of any type), then $\exists x\, \varphi$ and $\forall x\, \varphi$ are formulae.

Unless specified otherwise, we will indicate the type of a variable by correspondence of lower and upper case letters. That is, x and x' are variables of type X, while y and y' are variables of type Y, etc. If t is a term with free variables $x_1, \ldots, x_n$, then we write $[\mathrm{fv}(t)] = X_1 \times \cdots \times X_n$ (where X_i is the type of x_i). This presupposes that we have ordered the free variables of t in a certain way. But different choices yields isomorphic results for $[\mathrm{fv}(t)]$, so we do not need to specify such a choice explicitly. We adopt a similar notation for formulae.

A term $t(x_1, \ldots, x_n)$ of type X gives rise to a function $[t]\colon [\mathrm{fv}(t)] \to X$ in the obvious way. We now define the interpretation of formulae.

Definition 3.1.7. Let $\mathcal{L}$ be a Set-typed language, let A be a PCA and let P be its realizability tripos. An *interpretation* of $\mathcal{L}$ assigns to each relation symbol R in $\mathcal{L}$ with type $(X_1, \ldots, X_n)$, a predicate $[R] \in \mathsf{P}(X_1 \times \cdots \times X_n)$. Given an interpretation of $\mathcal{L}$, we define the *interpretation* $[\varphi] \in \mathsf{P}([\mathrm{fv}(\varphi)])$ of a formula over $\mathcal{L}$ using recursion on φ.

(i) $[\top]$ and $[\bot]$ are the top and bottom elements of $\mathsf{P}(1)$, respectively.

(ii) Suppose that R is a relation symbol of type $(X_1, \ldots, X_n)$ and that $t_1, \ldots, t_n$ are terms of type $X_1, \ldots, X_n$ respectively. For $1 \leq i \leq n$, there is a projection $\pi_i \colon [\mathrm{fv}(R(t_1, \ldots, t_n))] \to [\mathrm{fv}(t_i)]$. We define

$$[t] = \langle [t_1] \circ \pi_1, \ldots, [t_n] \circ \pi_n \rangle \colon [\mathrm{fv}(R(t_1, \ldots, t_n))] \to X_1 \times \cdots \times X_n$$

We define $[R(t_1, \ldots, t_n)]$ as $[t]^*([R])$.

(iii) Suppose that φ and ψ are formulae and let $\circ \in \{\wedge, \vee, \to\}$. There are projections $\pi_0 \colon [\mathrm{fv}(\varphi \circ \psi)] \to [\mathrm{fv}(\varphi)]$ and $\pi_1 \colon [\mathrm{fv}(\varphi \circ \psi)] \to [\mathrm{fv}(\psi)]$. Now we define $[\varphi \circ \psi]$ as $\pi_0^*([\varphi]) \circ \pi_1^*([\psi])$, where $\circ$ signifies the relevant operation in the Heyting prealgebra $\mathsf{P}([\mathrm{fv}(\varphi \circ \psi)])$.

(iv) Suppose φ is a formula and x is a variable, and let $\mathsf{Q} \in \{\exists, \forall\}$. There is a projection $\pi \colon [\mathrm{fv}(\varphi)] \to [\mathrm{fv}(\mathsf{Q}x\,\varphi)]$. Now we define $[\mathsf{Q}x\,\varphi] = \mathsf{Q}_\pi([\varphi])$.

We say that a sentence φ (i.e. a formula without free variables) is *valid* (with respect to the interpretation $[\cdot]$) if $[\varphi]$ is the top element of $\mathsf{P}(1)$, that is, if $[\varphi] \subseteq A$ is nonempty. We will sometimes call an element of $[\varphi]$ a *realizer* of φ. We write $\mathsf{P} \models \varphi$.

We shall freely use predicates $\phi \in \mathsf{P}(X_1 \times \cdots \times X_n)$ as relations symbols of our language of type $(X_1, \ldots, X_n)$, understanding that we interpret such a symbol as ϕ itself.

Remark 3.1.8. (i) Let $\varphi(x_1, \ldots, x_n)$ be a formula, and consider $[\varphi] \in \mathsf{P}([\mathrm{fv}(\varphi)])$. If we have $a_i \in X_i$ for $1 \leq i \leq n$, then we will write $[\varphi(a_1, \ldots, a_n)]$ instead of $[\varphi](a_1, \ldots, a_n)$. This may be potentially confusing, since we often use the symbol x to range over elements of X. So when one encounters an expression of the form $[\varphi(x)]$, one should pay attention whether x functions as a variable of type X (which means $[\varphi(x)] \in \mathsf{P}X$) or whether x functions as an element of X (which means $[\varphi(x)] \subseteq A$).

(ii) Recall from Section 2.4 that the definition of $\forall_f$ is somewhat involved. If we are only interested in the validity of a universally quantified sentence $\forall x\,\varphi(x)$, however, then the situation is rather easy. As is not difficult to show, we have that $\mathsf{P} \models \forall x\,\varphi(x)$ if and only if $\bigcap_{x \in X} [\varphi(x)] \neq \emptyset$ (where item (i) of this remark is in force). $\Diamond$

The key results are the following.

Theorem 3.1.9. *Let A be a PCA and let P be its realizability tripos.*

(i) (Substitution Lemma) *Let $\varphi(x, y_1, \ldots, y_n)$ be a formula and let t be a term that is free for x in φ. We have projections $\pi_0 \colon [\mathrm{fv}(\varphi[x/t])] \to [\mathrm{fv}(t)]$ and $\pi_1 \colon [\mathrm{fv}(\varphi[x/t])] \to Y_1 \times \cdots \times Y_n$. Consider the map*

$$\langle \pi_0, [t] \circ \pi_1 \rangle \colon [\mathrm{fv}(\varphi[x/t])] \to X \times (Y_1 \times \cdots \times Y_n) = [\mathrm{fv}(\varphi)].$$

We have $[\varphi[x/t]] = \langle \pi_0, [t] \circ \pi_1 \rangle^*([\varphi])$.

(ii) (Soundness Theorem) *Suppose the sentence φ is provable in typed intuitionistic predicate logic (without equality). Then there exists an $a \in A$ such that $a \in [\varphi]$ for all possible interpretations of our language. In particular, $\mathsf{P} \models \varphi$ with respect to all possible interpretations.*

Proof. These statements can be proved by induction on φ and the proof tree for φ respectively, and make extensive use of the Beck-Chevalley Condition. We will not provide the details here, but the relevant techniques can be found in any textbook on categorical logic. □

3.2 The Realizability Topos of a PCA

Using the theory developed in the previous section, we can start defining the realizability topos.

Definition 3.2.1. Let A be a PCA and let P be its realizability tripos. For a set X, a *partial equivalence relation* on X is a predicate $\sim \in \mathsf{P}(X \times X)$ such that

$$\mathsf{P} \models \forall x x' (x \sim x' \to x' \sim x) \quad \text{and}$$
$$\mathsf{P} \models \forall x x' x'' (x \sim x' \wedge x' \sim x'' \to x \sim x'').$$

In other words, $\sim$ should be symmetric and transitive in the internal logic of P. Explicitly, there should exist and $s \in A$ satisfying: if $x, x' \in X$ and $a \in [x \sim x']$, then $sa\downarrow$ and $sa \in [x' \sim x]$. And there should exist an element $t \in A$ satisfying: if $x, x', x'' \in X$, $a \in [x \sim x']$ and $b \in [x' \sim x'']$, then $t(\mathsf{p}ab)\downarrow$ and $t(\mathsf{p}ab) \in [x \sim x'']$. We formulated Definition 3.2.1 in terms of the logic of P because the Soundness Theorem then makes it easy to derive further properties of $\sim$. For example, it follows immediately that $\mathsf{P} \models \forall x x' (x \sim x' \to x \sim x)$, which we will use frequently. Notice that we do not demand reflexivity, i.e. $\mathsf{P} \models \forall x (x \sim x)$; hence the word ‘partial’.

The objects of the realizability topos over A will be pairs $(X, \sim)$ where $\sim$ is a partial equivalence relation on X. We now turn to defining its arrows.

Definition 3.2.2. Let A be a PCA and let P be its realizability tripos. A *functional relation* between objects $(X, \sim)$ and $(Y, \sim)$ is a predicate $F \in \mathsf{P}(X \times Y)$ such that

$$\mathsf{P} \models \forall xy (F(x,y) \to x \sim x \wedge y \sim y),$$
$$\mathsf{P} \models \forall x x' y y' (F(x,y) \wedge x \sim x' \wedge y \sim y' \to F(x', y')),$$
$$\mathsf{P} \models \forall x y y' (F(x,y) \wedge F(x,y') \to y \sim y') \quad \text{and}$$
$$\mathsf{P} \models \forall x (x \sim x \to \exists y F(x,y)).$$

These requirements express the strictness, the relationality, the single-valuedness and the totality of F, respectively.

Now we can define the realizability topos.

Definition 3.2.3. Let A be a PCA. The *realizability topos* $\mathsf{RT}(A)$ of A is defined as follows. Its objects are pairs $(X, \sim)$, where $\sim$ is a partial equivalence relation on X. An arrow $(X, \sim) \to (Y, \sim)$ is a $\simeq$-isomorphism class of functional relations between $(X, \sim)$ and $(Y, \sim)$. If $f\colon (X, \sim) \to (Y, \sim)$ is an arrow of $\mathsf{RT}(A)$, then we say that a functional relation $F \in f$ *represents* f.

If $F, G \in \mathsf{P}(X \times Y)$ are functional relations between $(X, \sim)$ and $(Y, \sim)$, then $F \simeq G$ can be expressed in the logic of P as $\mathsf{P} \models \forall xy\,(F(x,y) \leftrightarrow G(x,y))$. But in fact, one can intuitionistically derive $\forall xy\,(G(x,y) \to F(x,y))$ from the sentence $\forall xy\,(F(x,y) \to G(x,y))$ and the sentences stating that F and G are functional relations. So by the Soundness Theorem, we have $F \simeq G$ as soon as $\mathsf{P} \models \forall xy\,(F(x,y) \to G(x,y))$, that is, as soon as $F \leq G$. In other words, if we want to show that F and G represent the same arrow in $\mathsf{RT}(A)$, it suffices to check that $F \leq G$.

We now start proving that $\mathsf{RT}(A)$ is, in fact, a topos.

Proposition 3.2.4. *Let A be a PCA. Then $\mathsf{RT}(A)$ is a category.*

Proof. For an object $(X, \sim)$, its identity arrow is represented by $\sim \in \mathsf{P}(X \times X)$. Now suppose F and G represent arrows $(X, \sim) \to (Y, \sim)$ and $(Y, \sim) \to (Z, \sim)$. We define the composition of these arrows as the isomorphism class of $[\exists y\,(F(x,y) \wedge G(y,z))] \in \mathsf{P}(X \times Z)$. It follows from the Soundness Theorem that this is indeed a functional relation and that this isomorphism class does not depend on the choices of F and G. We can also use the Soundness Theorem to show that composing with the identity does nothing and that composition is associative. So $\mathsf{RT}(A)$ is a category. □

Proposition 3.2.5. *Let A be a PCA. Then $\mathsf{RT}(A)$ has finite limits.*

Proof. A terminal object of $\mathsf{RT}(A)$ is given by $(1, \sim)$, where $[0 \sim 0] = A$. If $(X, \sim)$ is an object, then $F \in \mathsf{P}(X \times 1)$ given by $F(x, 0) = [x \sim x]$ is a functional relation between $(X, \sim)$ and $(1, \sim)$. Since any functional relation between these two objects should be strict, it is also clear that this is, up to $\simeq$, the only possible functional relation between these two objects. So $(1, \sim)$ is indeed terminal.
Now let objects $(X \sim)$ and $(Y, \sim)$ be given, and define the projections $\pi_0 \colon X \times Y \to X$ and $\pi_1 \colon X \times Y \to Y$. The product of $(X, \sim)$ and $(Y, \sim)$ is $(X \times Y, \sim)$, where the partial equivalence relation is equal to $[\pi_0(w) \sim \pi_0(w') \wedge \pi_1(w) \sim \pi_1(w')]$ (here w and w' are variables of type $X \times Y$). The function $\mathrm{id}_{X \times Y} \colon X \times Y \to X \times Y$ yields a function symbol $(\cdot, \cdot)$ of our language that takes as input terms of types X and Y respectively, and that outputs a term of type $X \times Y$. Now we can show that

$$\mathsf{P} \models \forall wxy\,(w \sim (x,y) \leftrightarrow \pi_0(w) \sim x \wedge \pi_1(w) \sim y).$$

This shows that may replace all reasoning involving variables of type $X \times Y$ by reasoning about pairs (x, y). The partial equivalence relation on $X \times Y$ we defined above is now completely described by

$$\mathsf{P} \models \forall xx'yy'\,((x,y) \sim (x',y') \leftrightarrow x \sim x' \wedge y \sim y').$$

The first projection $(X \times Y, \sim) \to (X, \sim)$ is represented by $\Pi_0 \in \mathsf{P}(X \times Y \times X)$, where $\mathsf{P} \models \forall xx'y(\Pi_0(x, y, x') \leftrightarrow x \sim x' \wedge y \sim y)$. The second projection is defined analogously. If F and G represent arrows $(Z, \sim) \to (X, \sim)$ and $(Z, \sim) \to (Y, \sim)$, then we have $(Z, \sim) \to (X \times Y, \sim)$ represented by $[F(z,x) \wedge G(z,y)] \in \mathsf{P}(Z \times X \times Y)$. Using the Soundness Theorem, it is easy to check that this is the unique map that has the desired properties.
Now suppose F and G represent a parallel pair of arrows $(X, \sim) \to (Y, \sim)$. We define $(X, \approx)$, where $\approx \in \mathsf{P}(X \times X)$ is defined by $\mathsf{P} \models \forall xx'(x \approx x' \leftrightarrow x \sim x' \wedge \exists y\,(F(x,y) \wedge G(x,y)))$. We have the that $\approx$ is a functional relation between $(X, \approx)$ and $(X, \sim)$, yielding an arrow $(X, \approx) \to (X, \sim)$. If H represents an arrow $(Z, \sim) \to (X, \sim)$ such that

$$\mathsf{P} \models \forall yz\,(\exists x\,(H(z,x) \wedge F(x,y)) \leftrightarrow \exists x\,(H(z,x) \wedge G(x,y))),$$

then H is also a functional relation between $(Z, \sim)$ and $(X, \approx)$, which gives the unique factorization of $(Z, \sim) \to (X, \sim)$ through $(X, \approx) \to (X, \sim)$. So our parallel pair has an equalizer. We conclude that $\mathsf{RT}(A)$ has finite limits. □

Before we show that $\mathsf{RT}(A)$ has power objects, we describe subobjects and their pullbacks. First of all, suppose that the arrow $(X, \sim) \to (Y, \sim)$ represented by $F \in \mathsf{P}(X \times Y)$ has an inverse $(Y, \sim) \to (X, \sim)$ represented by $G \in \mathsf{P}(Y \times X)$. Then we have

$$\mathsf{P} \models \forall x x' (\exists y (F(x, y) \wedge G(y, x')) \leftrightarrow x \sim x') \wedge \forall y y' (\exists x (G(y, x) \wedge F(x, y')) \leftrightarrow y \sim y').$$

By the Soundness Theorem, we get

$$\mathsf{P} \models \forall x x' y (F(x, y) \wedge F(x', y) \to x \sim x') \wedge \forall y (y \sim y \to \exists x F(x, y)). \tag{3.1}$$

Conversely, if F satisfies (3.1), then $(y, x) \mapsto F(x, y)$ in $\mathsf{P}(Y \times X)$ represents an inverse of the arrow represented by F. So F represents an isomorphism in $\mathsf{RT}(A)$ if and only if (3.1) holds.

Now let F represent a monomorphism $(X, \sim) \to (Y, \sim)$. From the description of finite limits in the proof of Proposition 3.2.5, it follows that the kernel pair of this arrow is given by $(X \times X, \sim)$, where $[(x_0, x_1) \sim (x_0' \sim x_1')] = [x_0 \sim x_1 \wedge x_0' \sim x_1' \wedge \exists y (F(x_0, y) \wedge F(x_1, y))]$ for $x_0, x_1, x_0', x_1' \in X$. Since F represents a mono, we see that the arrow $(X, \sim) \to (X \times X, \sim)$ represented by $(x, x_0, x_1) \mapsto [x \sim x_0 \wedge x \sim x_1]$ in $\mathsf{P}(X \times X \times X)$ should be an isomorphism. Now it follows easily from the internal logic of P that we must have

$$\mathsf{P} \models \forall x x' y (F(x, y) \wedge F(x', y) \to x \sim x'). \tag{3.2}$$

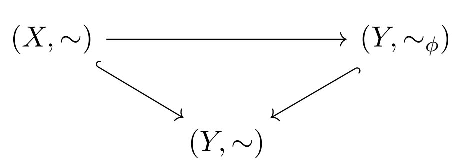
Conversely, if (3.2) holds, then one easily shows that F respresents a mono. So F represents a mono in $\mathsf{RT}(A)$ if and only if (3.2) holds.

Before we can conveniently describe subobjects, we need the following definition.

Definition 3.2.6. Let A be a PCA with realizability tripos P, and let $(X, \sim)$ be an object of $\mathsf{RT}(A)$. An element $\phi \in \mathsf{P}X$ is called a *strict relation* on $(X, \sim)$ if

$$\mathsf{P} \models \forall x (\phi(x) \to x \sim x) \wedge \forall x x' (\phi(x) \wedge x \sim x' \to \phi(x')).$$

If $\phi \in \mathsf{P}Y$ if a strict relation on $(Y, \sim)$, then $(Y, \sim_\phi)$ is also an object of $\mathsf{RT}(A)$, where $\sim_\phi$ is defined by $\mathsf{P} \models \forall y y' (y \sim_\phi y' \leftrightarrow \phi(y) \wedge y \sim y')$. Moreover, $M \in \mathsf{P}(Y \times Y)$ given by $M(y, y') = [\phi(y) \wedge y \sim y']$ represents a monomorphism $(Y, \sim_\phi) \to (Y, \sim)$. If we have two strict relations $\phi, \psi \in \mathsf{P}Y$ on $(Y, \sim)$, then the fact that $(Y, \sim_\phi) \leq (Y, \sim_\psi)$ as subobjects of $(Y, \sim)$ can be expressed in the logic of P as $\mathsf{P} \models \forall x (\phi(x) \to \psi(x))$. This, in turn, just means that the inequality $\phi \leq \psi$ of $\mathcal{P}A$-valued predicates holds. In particular, we have that $(Y, \sim_\phi)$ and $(Y, \sim_\psi)$ represent the same subobject of $(Y, \sim)$ if and only if $\phi \simeq \psi$. Finally, we have that *all* subobjects of $(Y, \sim)$ are given by a strict relation in this way. Indeed, suppose that $(X, \sim) \to (Y, \sim)$ represented by $F \in \mathsf{P}(X \times Y)$ is mono. Define the strict relation ϕ on $(Y, \sim)$ by $\phi(y) = [\exists x F(x, y)]$ for $y \in Y$. Then it easy to show that $F \in \mathsf{P}(X \times Y)$ respresents an *isomorphism* $(X, \sim) \to (Y, \sim_\phi)$ making the triangle



commute. So the lattice of subobjects of $(Y,\sim)$ is isomorphic to the poset reflection of the subpreorder of $((\mathcal{P}A)^Y,\leq)$ on the strict relations on $(Y,\sim)$.

Finally, let us describe the pullbacks of subobjects. Suppose that F represents $(X,\sim)\to(Y,\sim)$ and that ϕ is a strict relation on $(Y,\sim)$. The pullback of $(Y,\sim_\phi)\hookrightarrow(Y,\sim)$ along $(X,\sim)\to(Y,\sim)$ is given by $p\colon(X\times Y,\sim)\hookrightarrow(X,\sim)$, where

$$[(x,y)\sim(x',y')]=[x\sim x'\wedge y\sim y'\wedge\exists y''(F(x,y'')\wedge\phi(y)\wedge y\sim y'')]$$

for $x,x'\in X$ and $y,y'\in Y$. By the Soundness Theorem, we have

$$\mathsf{P}\models\forall xy((x,y)\sim(x,y)\leftrightarrow F(x,y)\wedge\phi(y)).$$

The arrow p can be represented by $(x,y,x')\mapsto[(x,y)\sim(x,y)\wedge x\sim x']$. By the above, this functional relation is isomorphic to $(x,y,x')\mapsto[F(x,y)\wedge\phi(y)\wedge x\sim x']$. By the above, we know that the subobject p of $(X,\sim)$ is given by the strict relation ψ on $(X,\sim)$, where

$$\mathsf{P}\models\forall x'(\psi(x')\leftrightarrow\exists xy(F(x,y)\wedge\phi(y)\wedge x\sim x')\leftrightarrow\exists y(F(x',y)\wedge\phi(y))).$$

We conclude that, in terms of strict relations, the pullback of ϕ along the arrow represented by F is the strict relation $x\mapsto[\exists y(F(x,y)\wedge\phi(y))]$.

Now we can finally prove the following.

Proposition 3.2.7. *Let A be a PCA. Then $\mathsf{RT}(A)$ has power objects.*

Proof. Let $(X,\sim)$ be an object of $\mathsf{RT}(A)$. Recalling that its power object $\mathcal{P}(X,\sim)$ should be the 'object of subobjects' of $(X,\sim)$, we set $\mathcal{P}(X,\sim)=((\mathcal{P}A)^X,\Leftrightarrow)$, where

$$[\phi\Leftrightarrow\psi]=[\forall x(\phi(x)\to x\sim x)\wedge\forall xx'(\phi(x)\wedge x\sim x'\to\phi(x'))\wedge\forall x(\phi(x)\leftrightarrow\psi(x))].$$

for $\phi,\psi\in(\mathcal{P}A)^X$. Notice that is meaningful because ϕ and ψ have corresponding relation symbols in our language. We notice that the sentence

$$(\forall x(R(x)\to x\sim x)\wedge\forall xx'(R(x)\wedge x\sim x'\to R(x'))\wedge\forall x(R(x)\leftrightarrow S(x)))\to$$
$$(\forall x(S(x)\to x\sim x)\wedge\forall xx'(S(x)\wedge x\sim x'\to S(x'))\wedge\forall x(S(x)\leftrightarrow R(x)))$$

is provable in typed intuitionistic logic. If we write χ for this sentence, then there is an $a\in A$ such that $a\in[\chi]$ for all possible interpretations of R and S. In particular, we have that $a\in\bigcap_{\phi,\psi\in(\mathcal{P}A)^X}[\phi\Leftrightarrow\psi\to\psi\Leftrightarrow\phi]$, which shows that $\mathsf{P}\models\forall\phi\psi(\phi\Leftrightarrow\psi\to\psi\Leftrightarrow\phi)$. (Here ϕ and ψ are variables of type $(\mathcal{P}A)^X$.) Similarly, we may show that $\Leftrightarrow$ is transitive in the internal logic of P, so $\Leftrightarrow$ is a partial equivalence relation on $(\mathcal{P}A)^X$.
The element $\in_{(X,\sim)}$ from $\mathsf{P}(X\times(\mathcal{P}A)^X)$ given by

$$\in_{(X,\sim)}(x,\phi)=[\phi\Leftrightarrow\phi\wedge\phi(x)]\quad\text{for }x\in X\text{ and }\phi\in(\mathcal{P}A)^X$$

is a strict relation on $(X,\sim)\times\mathcal{P}(X,\sim)$. We will write $x\in\phi$ instead of $\in_{(X,\sim)}(x,\phi)$ (in particular we drop the subscript for now).
Now let $(Y,\sim)$ be another object of $\mathsf{RT}(A)$ and let $\psi\in\mathsf{P}(X\times Y)$ be a strict relation on $(X\times Y,\sim)$. We need to find a functional relation F between $(Y,\sim)$ and $\mathcal{P}(X,\sim)$ such that

$$\mathsf{P}\models\forall xy(\psi(x,y)\leftrightarrow\exists\phi(F(y,\phi)\wedge x\in\phi)).\qquad(3.3)$$

Moreover, this F should be unique up to isomorphism. It follows from (3.3) and the fact that F should a be functional relation that

$$\mathsf{P} \models \forall y \forall \phi (F(y, \phi) \to y \sim y \wedge \phi \Leftrightarrow \phi \wedge \forall x (x \in \phi \leftrightarrow \psi(x, y))). \tag{3.4}$$

As one can easily verify, the function $(y, \phi) \mapsto [y \sim y \wedge \phi \Leftrightarrow \phi \wedge \forall x (x \in \phi \leftrightarrow \psi(x, y)))]$ is a functional relation between $(Y, \sim)$ and $\mathcal{P}(X, \sim)$ that makes (3.3) hold. Now (3.4) implies that this is, up to isomorphism, the only possible choice for F. □

Putting all these results together, we get:

Theorem 3.2.8 (Pitts)**.** *Let A be a PCA. Then $\mathsf{RT}(A)$ is an elementary topos.*

The most well-known example of a realizability topos is the realizability topos over Kleene's first model $\mathcal{K}_1$. This topos is called the *effective topos*, and is usually denoted by Eff.

3.3 Some Basic Facts about Realizability Topoi

In the previous section, we defined the realizability topos over a PCA A and showed that it is in fact a topos. In this section, we develop some further basic properties of $\mathsf{RT}(A)$ that will be useful to us.

Now that we have a topos, we can do logic in it. If φ is a formula of first order predicate logic with equality, we will write $[\![\varphi]\!]$ for its interpretation in the topos logic. Such interpretations are subobjects, and we know that these are given by strict predicates. In fact, we can describe the logical operators as interpreted in the topos logic completely in terms of the realizability tripos P. First, consider an object $(X, \sim)$ of $\mathsf{RT}(A)$. The largest subobject of $(X, \sim)$, i.e. the identity on $(X, \sim)$, corresponds to the strict relation $[x \sim x] \in \mathsf{P}(X)$ on $(X, \sim)$. The smallest subobject is given by the strict relation $\bot \in \mathsf{P}(X)$. Now suppose that we have subobjects of $(X, \sim)$ given by the strict predicates $\phi, \psi \in \mathsf{P}(X)$. Then their meet and join are given by the strict predicates $[\phi(x) \wedge \psi(x)]$ and $[\phi(x) \vee \psi(x)]$, respectively. At this point, one may be tempted to say that their Heyting implication is given by $[\phi(x) \to \psi(x)]$. However, this is not necessarily a strict predicate on $(X, \sim)$. Instead, the Heyting implication is given by $[x \sim x \wedge (\phi(x) \to \psi(x))] \in \mathsf{P}(X)$. One easily checks that this *is* a strict predicate on $(X, \sim)$ that satisfies the relevant universal property. Now suppose that $\varphi(x, y)$ is a formula, where x and y are variables of type $(X, \sim)$ and $(Y, \sim)$ respectively, and suppose that $[\![\varphi(x, y)]\!]$ is given by the strict predicate ϕ on $(X, \sim) \times (Y, \sim)$. Then $[\![\exists y\, \varphi(x, y)]\!]$ is simply given by the strict predicate $[\exists y\, \phi(x, y)] \in \mathsf{P}(X)$. Universal quantification is a bit more difficult: $[\![\forall y\, \varphi(x, y)]\!]$ is given by the strict predicate $[x \sim x \wedge \forall y (y \sim y \to \phi(x, y))] \in \mathsf{P}(X)$ on $(X, \sim)$. Again, one easily checks that this is indeed a strict predicate on $(X, \sim)$ that satisfies the relevant universal property.

The above is useful for formulating certain categorical concepts in the logic of P. For example, we know that an arrow $f \colon (X, \sim) \to (Y, \sim)$ is a regular epi is and only if $[\![\exists x\, (f(x) = y)]\!]$ is the maximal subobject of $(Y, \sim)$. If f is represented by the functional relation F, then this means that $\mathsf{P} \models \forall y\, (y \sim y \to \exists x\, F(x, y))$.

If $(X, \sim)$ is an object of $\mathsf{RT}(A)$, then it may happen that $[x \sim x] = \emptyset$ for certain $x \in X$. On many occasions, this is annoying, and we may in fact assume without loss of generality that $[x \sim x] \neq \emptyset$ for all $x \in X$. Let us make this precise. For any object $(X, \sim)$ of $\mathsf{RT}(A)$, we have the object $(X_0, \sim_0)$ defined by $X_0 = \{x \in X \mid [x \sim x] \neq \emptyset\}$ and $\sim_0 = \sim|_{X_0 \times X_0}$. Moreover, if F is a functional relation between $(X, \sim)$ and $(Y, \sim)$, then $F_0 = F|_{X_0 \times Y_0}$ is a functional relation between $(X_0, \sim_0)$ and $(Y_0, \sim_0)$. This yields a functor from $\mathsf{RT}(A)$ into its full subcategory $\mathsf{RT}_0(A)$ on objects $(X, \sim)$ such that $[x \sim x] \neq \emptyset$ for all $x \in X$. This functor is a pseudoinverse of the inclusion of $\mathsf{RT}_0(A)$ into $\mathsf{RT}(A)$. Indeed, if $(X, \sim)$ is an object of $\mathsf{RT}(A)$, then $\iota_{(X,\sim)} \colon (X, \sim) \to (X_0, \sim_0)$ represented by $[x \sim x_0] \in \mathsf{P}(X \times X_0)$ is an isomorphism, and these isomorphisms are natural. So $\mathsf{RT}(A)$ is equivalent to $\mathsf{RT}_0(A)$.

At the beginning of this chapter we promised that $\mathsf{Asm}(A)$ is isomorphic to a full subcategory of $\mathsf{RT}(A)$. We start proving this now. Suppose that (X, E) is an assembly. Then we have an object $(X, \sim)$ of $\mathsf{RT}(A)$, where

$$x \sim x' = \begin{cases} E(x) & \text{if } x = x'; \\ \emptyset & \text{if } x \neq x', \end{cases}$$

for all $x, x' \in X$. We will just write (X, E) for this object, therewith viewing every assembly as an object of $\mathsf{RT}(A)$. We notice that the assemblies are precisely those objects $(X, \sim)$ of $\mathsf{RT}(A)$ such that $[x \sim x] \neq \emptyset$ for all $x \in X$ and $[x \sim x'] = \emptyset$ for all *distinct* $x, x' \in X$. In order to relate the morphisms of assemblies to the arrows of $\mathsf{RT}(A)$, we need the following proposition, that also provides us with a rich supply of further arrows of $\mathsf{RT}(A)$.

Proposition 3.3.1. *Let A be a PCA and let P be its realizability tripos. Suppose that $(X, \sim)$ and $(Y, \sim)$ are objects of $\mathsf{RT}(A)$ and that $f \colon X \to Y$ is a function satisfying*

$$\mathsf{P} \models \forall x x' (x \sim x' \to f(x) \sim f(x')). \tag{3.5}$$

Then $F_f := [x \sim x \wedge f(x) \sim y] \in \mathsf{P}(X \times Y)$ is a functional relation between $(X, \sim)$ and $(Y, \sim)$, so f gives rise to an arrow $(X, \sim) \to (Y, \sim)$.

Proof. The fact that F_f is indeed a functional relation follows easily from the internal logic of P. □

A word of warning: it may very well happen that two distinct functions give rise to the *same* arrow in $\mathsf{RT}(A)$. The following lemma describes two occasions on which the situation from Proposition 3.3.1 always applies.

Lemma 3.3.2. *Let A be a PCA and let $(X, \sim)$ and $(Y, \sim)$ be objects of $\mathsf{RT}(A)$. We assume that $[x \sim x] \neq \emptyset$ for all $x \in X$. Then every arrow $(X, \sim) \to (Y, \sim)$ arises from a function $f \colon X \to Y$ as in Proposition 3.3.1 if:*

- *(i) $(Y, \sim)$ is an assembly, or if:*
- *(ii) $[x \sim x]$ is a singleton for all $x \in X$.*

Moreover, in the first case, distinct functions give rise to distinct arrows.

Proof. Let P be the realizability tripos of A and suppose that $F \in \mathsf{P}(X \times Y)$ represents a functional relation between $(X, \sim)$ and $(Y, \sim)$.

(i) Suppose that $(Y, \sim)$ is an assembly, and let $x \in X$. Since we assumed that $[x \sim x] \neq \emptyset$ and since F should be total, there exists a $y \in Y$ such that $F(x, y) \neq \emptyset$. Moreover, if $F(x, y)$ and $F(x, y')$ are both nonempty for certain $y, y' \in Y$, then $[y \sim y']$ must also be nonempty, since F should be functional. Since $(Y, \sim)$ is an assembly, it follows that $y = y'$. Now we can define $f\colon X \to Y$ by sending $x \in X$ to the unique $y \in Y$ such that $F(x, y) \neq \emptyset$. It is easy to check that F is isomorphic to F_f in $\mathsf{P}(X \times Y)$.
If we have two functions $f, g\colon X \to Y$ satisfying (3.5) such that F_f and F_g are isomorphic in $\mathsf{P}(X \times Y)$, then $\mathsf{P} \models \forall x\,(x \sim x \to f(x) \sim g(x))$. Since $[x \sim x] \neq \emptyset$ for all $x \in X$, this implies that $[f(x) \sim g(x)] \neq \emptyset$ for all $x \in X$. As $(Y, \sim)$ is an assembly, this implies that $f(x) = g(x)$ for all $x \in X$, as desired.
(ii) Write $[x \sim x] = \{a_x\}$ for $x \in X$. Since F should be total, there exists a $b \in A$ such that for all $x \in X$, we have that $ba_x\downarrow$ and $ba_x \in \bigcup_{y \in Y} F(x, y)$. Using the Axiom of Choice, we may find a function $f\colon X \to Y$ such that $ba_x \in F(x, f(x))$ for all $x \in X$. Now it is easy to check that F and F_f are isomorphic in $\mathsf{P}(X \times Y)$. □

Suppose that (X, E) and (Y, F) are assemblies. Then by the previous lemma, every arrow $(X, E) \to (Y, F)$ in $\mathsf{RT}(A)$ is given by a function $f\colon X \to Y$ as in Proposition 3.3.1. Since (X, E) is an assembly, the requirement (3.5) is equivalent to:

$$\mathsf{P} \models \forall x\,(E(x) \to F(f(x))).$$

This means exactly that f is a morphism of assemblies. Moreover, distinct morphisms of assemblies give rise to different arrows in $\mathsf{RT}(A)$. So we have a bijection between morphisms of assemblies $(X, E) \to (Y, F)$ and arrows $(X, E) \to (Y, F)$ of $\mathsf{RT}(A)$. Finally, it is easy to check that, modulo this bijection, identities and composition in $\mathsf{Asm}(A)$ and in $\mathsf{RT}(A)$ coincide. So we get a fully faithful embedding of $\mathsf{Asm}(A)$ into $\mathsf{RT}(A)$, as desired. We shall henceforth identify $\mathsf{Asm}(A)$ with the full subcategory of $\mathsf{RT}(A)$ on the assemblies.

For the category of assemblies, we had an adjunction with Set. We can now extend this to the realizability topos. First of all, we can also see the constant objects functor as a functor $\mathsf{Set} \to \mathsf{RT}(A)$. Now we define its left adjoint.

Definition 3.3.3. Let A be a PCA. The functor $\Gamma\colon \mathsf{RT}(A) \to \mathsf{Set}$ is defined as follows.

(i) Let $(X, \sim)$ be an object of $\mathsf{RT}(A)$, and define the equivalence relation $\approx$ on X_0 by $x \approx x'$ if and only if $[x \sim x'] \neq \emptyset$. We set $\Gamma(X, \sim) = X_0/{\approx}$.

(ii) Suppose $f\colon (X, \sim) \to (Y, \sim)$ is an arrow of $\mathsf{RT}(A)$ represented by the functional relation F. We define $\Gamma f\colon \Gamma(X, \sim) \to \Gamma(Y, \sim)$ by $\Gamma f([x]) = [y]$ if and only if $F(x, y) \neq \emptyset$, for all $x \in X$ and $y \in Y$.

It is easy to check that Γ is well-defined. Furthermore, if we restrict Γ to the assemblies, then the result is (isomorphic to) the functor $\Gamma\colon \mathsf{Asm}(A) \to \mathsf{Set}$ we defined in Section 2.5. So no confusion can arise by calling both these functors 'Γ'. We could also have introduced Γ in a different way, namely as the *global sections functor*. A *global section* of an object $(X, \sim)$ is an arrow $(1, \sim) \to (X\ \sim)$ from the terminal object of $\mathsf{RT}(A)$ into $(X, \sim)$, and the global sections functor assigns to $(X, \sim)$ the homset $\mathsf{RT}(A)((1, \sim), (X, \sim))$. If $F \in \mathsf{P}(1 \times X)$ is a functional relation between $(1, \sim)$ and $(X, \sim)$, then the set $\{x \in X \mid F(0, x) \neq \emptyset\}$ is an $\approx$-equivalence class. Moreover, two functional relations represent the same global section if and only if their associated $\approx$-equivalence classes coincide. So we see that the elements of $\Gamma(X, \sim)$ correspond bijectively to the global sections of $(X, \sim)$, and this gives rise to a natural isomorphism between Γ and the global sections functor.

Proposition 3.3.4. *The functors Γ and Δ are both regular, and form a geometric inclusion $(\Gamma \dashv \Delta)\colon \mathsf{Set} \to \mathsf{RT}(A)$.*

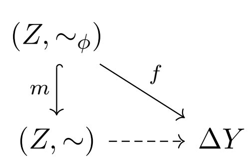
Proof. Let $(X, \sim)$ be an object of $\mathsf{RT}(A)$ and let Y be a set. Then every arrow $(X, \sim) \to \Delta Y$ is given by a function $f\colon X_0 \to Y$ in the sense of Proposition 3.3.1. Moreover, requirement (3.5) now means that $[x \sim x'] \neq \emptyset$ implies that $f(x) = f(x')$. In other words, f should factor through the projection $X_0 \twoheadrightarrow \Gamma(X, \sim)$. By the uniqueness clause of Lemma 3.3.2 and the fact that this projection is epi, it follows that the arrows $(X, \sim) \to \Delta Y$ correspond bijectively to functions $\Gamma(X, \sim) \to Y$, and these bijections are natural. This shows that $\Gamma \dashv \Delta$. It is not hard to show that Γ, being isomorphic to the global sections functor, preserves limits of every type. This shows that $\Gamma \dashv \Delta$ is a geometric morphism and that Γ is regular. Furthermore, Δ is fully faithful, so $\Gamma \dashv \Delta$ is even a geometric inclusion. Finally, suppose that $e\colon X \twoheadrightarrow Y$ is a surjective function. From the description of regular epis given above, it is clear that Δe is a regular epi. So Δ preserves regular epis, which shows that Δ is regular as well. $\square$

Since the adjunction $\Gamma \dashv \Delta$ is a geometric inclusion, its counit is a natural isomorphism. We shall denote its unit by η. For an object $(X, \sim)$ of $\mathsf{RT}(A)$, the arrow $\eta_{(X,\sim)}$ is given by the projection $X_0 \twoheadrightarrow \Gamma(X, \sim)$ in the sense of Proposition 3.3.1. Explicitly, this arrow is represented by the functional relation in $\mathsf{P}(X \times \Gamma(X, \sim))$ that sends $(x, [x'])$ to $[x \sim x]$ if $[x \sim x'] \neq \emptyset$, and that sends $(x, [x'])$ to $\emptyset$ if $[x \sim x'] = \emptyset$. Up to $\simeq$, we can also write this as $(x, [x']) \mapsto [x \sim x \wedge \neg\neg(x \sim x')]$.

We know from topos theory that the geometric inclusion $\mathsf{Set} \to \mathsf{RT}(A)$ must be given by a Lawvere-Tierney topology on $\mathsf{RT}(A)$. This topology is in fact the *double negation topology.* We close this section by showing this. First, we describe the closure operation associated by the double negation topology. Suppose we have an object $(X, \sim)$ of $\mathsf{RT}(A)$ and a subobject given by the strict predicate $\phi \in \mathsf{P}(X)$. From the description of the topos logic we gave at the beginning of this section, it follows that the closure of this subobject is given by the strict predicate $[x \sim x \wedge \neg\neg\phi(x)] \in \mathsf{P}(X)$. In particular, we see that the subobject is closed if and only if $\mathsf{P} \models \forall x(x \sim x \wedge \neg\neg\phi(x) \to \phi(x))$. Furthermore, the subobject is dense if and only if $\mathsf{P} \models \forall x(x \sim x \to \neg\neg\phi(x))$. This, in turn, just means that $\phi(x)$ should be nonempty whenever $[x \sim x]$ is nonempty. The following proposition describes the $\neg\neg$-separated objects and the $\neg\neg$-sheaves.

Proposition 3.3.5. *Let A be a PCA and let $(X, \sim)$ be an object of $\mathsf{RT}(A)$. Then $(X, \sim)$ is $\neg\neg$-separated if and only if $(X, \sim)$ is isomorphic to an assembly, and $(X, \sim)$ is a $\neg\neg$-sheaf if and only if $(X, \sim)$ is isomorphic to a constant object.*

Proof. First, we show that ΔY is a sheaf for every set Y. Suppose we have an object $(Z, \sim)$ and a strict relation $\phi \in \mathsf{P}(Z)$ on $(Z, \sim)$ yielding a dense subobject of $(Z, \sim)$. We should find a unique factorization in every diagram of the form



Transposing this diagram, we see that we need to find a unique factorization in the diagram

$$\begin{array}{ccc} \Gamma(Z, \sim_\phi) & & \\ {\scriptstyle \Gamma m} \big\downarrow & \searrow^{\tilde{f}} & \\ \Gamma(Z, \sim) & \dashrightarrow & Y \end{array}$$

But as is not difficult to show, the fact that our subobject is dense implies that $\Gamma(Z, \sim_\phi) = \Gamma(Z, \sim)$ and that Γm is the identity, so the result is immediate.
The subobjects of a constant object ΔY are given by strict predicates $\phi \in \mathsf{P}(Y)$ on ΔY. Since ΔY is a constant object, the 'strictness' requirement is empty, so every $\phi \in \mathsf{P}(Y)$ gives a subobject of Y. The resulting subobjects are (up to isomorphism) exactly the assemblies (Y', E) with $Y' \subseteq Y$. So the subobjects of the constant objects are, up to isomorphism, exactly the assemblies. In particular, all assemblies are separated.
Conversely, suppose that $(X, \sim)$ is separated. Then the diagonal $(X, \sim) \hookrightarrow (X, \sim) \times (X, \sim)$ should be a closed subobject, which can be expressed in the logic of the realizability tripos P as

$$\mathsf{P} \models \forall x x' (x \sim x \wedge x' \sim x' \wedge \neg\neg(x \sim x') \to x \sim x').$$

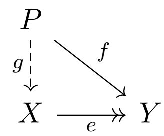
It is not difficult to show that this implies that $\eta_{(X,\sim)} \colon (X, \sim) \to \Delta\Gamma(X, \sim)$ is mono, so $(X, \sim)$ is isomorphic to an assembly.
Finally, suppose that $(X, \sim)$ is a sheaf. Then in particular, $(X, \sim)$ must be separated, so it must be isomorphic to an assembly. Therefore, we assume without loss of generality that we have an assembly (X, E). We notice that $\eta_{(X,E)} \colon (X, E) \hookrightarrow \Delta X$ is dense, so $\mathrm{id}_{(X,E)}$ must factor as $f \circ \eta_{(X,E)}$ for a certain $f \colon \Delta X \to (X, E)$. We notice that $\eta_{(X,E)} = \eta_{(X,E)} \circ f \circ \eta_{(X,E)}$. Since ΔX is a sheaf and $\eta_{(X,E)}$ is dense, we get $f \circ \eta_{(X,E)} = \mathrm{id}_{\Delta X}$ as well, so (X, E) is isomorphic to ΔX, as desired. □

Now we see that Δ is a fully faithful functor into $\mathsf{RT}(A)$ whose image contains, up to isomorphism, exactly the $\neg\neg$-sheaves. Since Γ is left adjoint to Δ, we can conclude that the geometric inclusion $\Gamma \dashv \Delta$ is equivalent to the inclusion of $\neg\neg$-sheaves into $\mathsf{RT}(A)$. Under this identification, $\eta_{(X,\sim)}$ is the universal map from an object $(X, \sim)$ into its sheafification. In particular, $(X, \sim)$ is isomorphic to an assembly if and only if $\eta_{(X,\sim)}$ is mono. Finally, we remark that the full subcategory $\mathsf{Asm}(A)$ of $\mathsf{RT}(A)$ is closed, up to isomorphism, under subobjects, since a subobject of a separated object is always separated. In particular, the inclusion functor $\mathsf{Asm}(A) \to \mathsf{RT}(A)$ is a regular functor. So if we need to calculate a finite limit of assemblies in $\mathsf{RT}(A)$, we can just use the description of finite limits in $\mathsf{Asm}(A)$ given in Section 2.5. Moreover, a morphism of assemblies is a regular epi in $\mathsf{RT}(A)$ if and only if it is a regular epi in $\mathsf{Asm}(A)$.

3.4 Projective Objects

Now that we have developed some basic theory about $\mathsf{RT}(A)$, we study some special objects that can be found there. This section is devoted to the projective objects of $\mathsf{RT}(A)$. We start with the definition of a projective object.

Definition 3.4.1. Let $\mathcal{C}$ be a regular category. An object P of $\mathcal{C}$ is called *projective* if for every regular epi $e\colon X \twoheadrightarrow Y$ and every arrow $f\colon P \to Y$, there exists an arrow $g\colon P \to X$ such that $eg = f$.



We say that $\mathcal{C}$ has *enough projectives* if every object of $\mathcal{C}$ is covered by a projective object. That is, for every object X of $\mathcal{C}$, there should exist a regular epi $P \twoheadrightarrow X$, where P is projective.

This definition also makes sense if we do not work in a regular category. However, by imposing this restriction, we get the following nice result: an object P is projective if and only if every regular epi with codomain P is a split epi.

Before we can describe the projective objects in $\mathsf{RT}(A)$, we need the following terminology.

Definition 3.4.2. Let A be a PCA. An assembly (X, E) over A is called *partitioned* if $E(x)$ is a singleton for all $x \in X$.

Proposition 3.4.3. *Let A be a PCA. Every object of $\mathsf{RT}(A)$ is covered by a partitioned assembly.*

Proof. Let $(X, \sim)$ be an object of $\mathsf{RT}(A)$ and define the partitioned assembly (Y, E) by $Y = \{(x, a) \mid x \in X, a \in [x \sim x]\}$ and $E(x, a) = \{a\}$. We have the obvious projection $\pi\colon Y \to X$, which satisfies (3.5). So we get an arrow $(Y, E) \to (X, \sim)$ represented by F_π. Moreover, we have $\langle u \rangle \mathsf{p}uu \in [x \sim x \to \exists y\, F_\pi(y, x)]$ for all $x \in X$, so $\mathsf{P} \models \forall x(x \sim x \to \exists y\, F_\pi(y, x))$. This means that our arrow $(Y, E) \to (X, \sim)$ is a regular epi, as desired. $\square$

Using this result, we can characterize the projective objects completely.

Proposition 3.4.4. *Let A be a PCA. An object $(X, \sim)$ of $\mathsf{RT}(A)$ is projective if and only if $(X, \sim)$ is isomorphic to a partitioned assembly.*

Proof. First of all, we show that every partitioned assembly (X, E) is projective. For $x \in X$, we write $E(x) = \{a_x\}$. Suppose we have a regular epi $(Y, \sim) \twoheadrightarrow (X, E)$. By Lemma 3.3.2, this arrow must be represented by F_f for a certain function $f\colon Y \to X$. Since F_f represents a regular epi, we must have $\mathsf{P} \models \forall x(E(x) \to \exists y\, F_f(y, x))$. This means that there must be a $b \in A$ such that for all $x \in X$, we have that $ba_x\downarrow$ and $ba_x \in \bigcup_{f(y)=x}[y \sim y \wedge E(x)]$. This means that $\mathsf{p}_0(ba_x)$ also denotes, and is an element of $\bigcup_{f(y)=x}[y \sim y]$ for all $x \in X$. Using the Axiom of Choice, we may find a function $g\colon X \to Y$ such that $f(g(x)) = x$ and $\mathsf{p}_0(ba_x) \in [g(x) \sim g(x)]$ for all $x \in X$. Now we see that $\mathsf{P} \models \forall x(E(x) \to [g(x) \sim g(x)])$, so g gives us an arrow $(X, E) \to (Y, \sim)$ through Proposition 3.3.1. Using the fact that g is a section of f, it is not hard to show that the composition $(X, E) \to (Y, \sim) \twoheadrightarrow (X, E)$ is the identity, so our regular epi splits, as desired.
Conversely, suppose that $(X, \sim)$ is projective. By Proposition 3.4.3, there exist a partitioned assembly (Y, E) and a regular epi $(Y, E) \twoheadrightarrow (X, \sim)$. Since $(X, \sim)$ is projective, this regular epi splits. This means that $(X, \sim)$ is a subobject of (Y, E) and hence given by a strict

predicate $\phi \in \mathsf{P}(Y)$ on (Y, E). Explicitly, $(X, \sim)$ is isomorphic to the assembly (Y_0, E_ϕ), where $Y_0 = \{y \in Y \mid \phi(y) \neq \emptyset\}$ and $E_\phi = (\phi \wedge E)|_{Y_0}$. We also have morphisms of assemblies

$$(Y, E) \underset{m}{\overset{e}{\rightleftarrows}} (Y_0, E_\phi)$$

such that $em = \mathrm{id}_{Y_0}$, and where $m \colon Y_0 \hookrightarrow Y$ is simply the inclusion. From the fact that e should be a morphism, we can derive that $E|_{Y_0} \leq E_\phi$ as inequality in $\mathsf{P}(Y_0)$. But we also clearly have $E_\phi \leq E|_{Y_0}$, so the assembly (Y_0, E_ϕ) is isomorphic to the *partitioned* assembly $(Y_0, E|_{Y_0})$, as desired. □

In particular, scenario (ii) from Lemma 3.3.2 applies when $(X, \sim)$ is projective. We also observe that every constant object is projective, since for any set X, we have that ΔX is isomorphic to (X, E), where $E(x) = \{\mathsf{k}\}$ for all $x \in X$.

Corollary 3.4.5. *Let A be a PCA. The projective objects of $\mathsf{RT}(A)$ are closed under finite limits, and $\mathsf{RT}(A)$ has enough projectives.*

Proof. Using the description of finite limits in $\mathsf{Asm}(A)$ given in Section 2.5, it not difficult to show that a finite limit of partitioned assemblies is again a partitioned assembly. The second statement follows from the previous two propositions. □

To close this section, we introduce a special projective object that will play an important role in the sequel.

Definition 3.4.6. Let A be a PCA. The *object of realizers* of $\mathsf{RT}(A)$ is the partitioned assembly $(A, a \mapsto \{a\})$.

The object of realizers 'generates' all projective objects of $\mathsf{RT}(A)$. In order to make this precise, we need one more definition.

Definition 3.4.7. Let $f \colon (X, E) \to (Y, F)$ be a morphism of assemblies. We say that f is *cartesian* if $E(x) = F(f(x))$ for all $x \in X$.

There is also a more 'categorical' description of cartesian morphisms of assemblies. Let (Y, F) be an assembly and let $f \colon X \to Y$ be any function. The unit $\eta_{(Y,F)} \colon (Y, F) \hookrightarrow \Delta Y$ is given by the identity on Y. Using the description of pullbacks in $\mathsf{Asm}(A)$ given in Section 2.5, we can see that

$$\begin{array}{ccc} (X, E) & \xrightarrow{f} & (Y, F) \\ {\scriptstyle \mathrm{id}_X} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_Y} \\ \Delta X & \xrightarrow[f]{} & \Delta Y \end{array}$$

is a pullback diagram in $\mathsf{Asm}(A)$, where E is defined by $E(x) = F(f(x))$ for all $x \in X$. In particular, we see that $f \colon (X, E) \to (Y, F)$ is cartesian, and conversely, every cartesian arrow with codomain (Y, F) arises in this way. In $\mathsf{RT}(A)$, this means the following: a morphism of assemblies is isomorphic to a cartesian morphism if and only if its naturality square for η is a pullback diagram.

By definition, an assembly is partitioned if and only if it allows a cartesian morphism into the object of realizers. Putting everything together, we get:

Proposition 3.4.8. *Let A be a PCA and let $R = (A, a \mapsto \{a\})$ be the object of realizers of* $\mathsf{RT}(A)$*. The projective objects of* $\mathsf{RT}(A)$ *are exactly the objects that can be obtained by pulling* η_R *back along an arrow between constant objects.*

3.5 Discrete Objects

In this section, we study the discrete objects of $\mathsf{RT}(A)$. Unlike the projective objects, these are not always assemblies. We will follow a different approach in this section. In the previous section, we started with a categorical notion (being projective), and then proceeded to describe the projective objects in $\mathsf{RT}(A)$. In this section, on the other hand, we start by describing a certain type of object in $\mathsf{RT}(A)$, and then we uncover its categorical properties.

Definition 3.5.1. Let A be a PCA. An object of $\mathsf{RT}(A)$ is called *discrete* if it is isomorphic to an object $(D, \sim)$ satisfying $[d \sim d] \cap [d' \sim d'] = \emptyset$ for all *distinct* $d, d' \in D$.

The 'up to isomorphism' part of this definition is not very elegant. We can also give a description of discrete objects that, albeit a bit more involved, is stable under isomorphism.

Proposition 3.5.2. *Let A be a PCA and let $(D, \sim)$ be an object of* $\mathsf{RT}(A)$*. Then $(D, \sim)$ is discrete if and only if there exists an $a \in A$ satisfying: if we have $d, d' \in D$ and an element $b \in [d \sim d] \cap [d' \sim d']$, then $ab\downarrow$ and $ab \in [d \sim d']$.*

Proof. Suppose that $(D, \sim)$ is discrete. Then $(D, \sim)$ is isomorphic to an object $(E, \sim)$ such that $[e \sim e] \cap [e' \sim e'] = \emptyset$ for distinct $e, e' \in E$. Let the functional relation F represent an isomorphism $(D, \sim) \to (E, \sim)$. Since F is a functional relation, there exist elements

$$t \in \bigcap_{d \in D} [d \sim d \to \exists e F(d, e)] \quad \text{and} \quad s \in \bigcap_{d \in D, e \in E} [F(d, e) \to e \sim e].$$

Since F is mono, there also exists an element

$$r \in \bigcap_{d, d' \in D, e \in E} [F(d, e) \wedge F(d', e) \to d \sim d'].$$

Now define $a = \langle x \rangle r(\mathsf{p}(tx)(tx))$ and let $d, d' \in D$ and an element $b \in [d \sim d] \cap [d' \sim d']$ be given. Then there exist $e, e' \in E$ such that $tb \in F(d, e)$ and $tb \in F(d', e')$. Since $s(tb)$ is an element of both $[e \sim e]$ and $[e' \sim e']$, we must have $e = e'$. Now we see that $\mathsf{p}(tb)(tb) \in [F(d, e) \wedge F(d', e)]$ and therefore $ab = r(\mathsf{p}(tb)(tb)) \in [d \sim d']$, as desired.
The converse direction, for the case of the effective topos, is Proposition 3.2.20 from *Realizability* [6]. The proof carries over to the general case, once one has developed the coding of finite sequences in A. $\square$

Corollary 3.5.3. *Let A be a PCA and (D, E) be an assembly over A. Then (D, E) is discrete if and only if we have $E(d) \cap E(d') = \emptyset$ for all distinct $d, d' \in D$.*

Proof. The 'if' direction is clear. For the converse direction, suppose (D, E) is discrete and let $a \in A$ be an element as in Proposition 3.5.2. Suppose we have $d, d' \in D$ such that $E(d)$ and $E(d')$ have an element b in common. Then $ab\downarrow$ and ab realizes the equivalence of d and d'. Since (D, E) is an assembly, we must have $d = d'$, as desired. □

We can use the notion of a cartesian morphism introduced in the previous section to give a categorical description of the discrete objects of $\mathsf{RT}(A)$.

Proposition 3.5.4. *Let A be a PCA and let $(D, \sim)$ be an object of $\mathsf{RT}(A)$. Then $(D, \sim)$ is discrete if and only if for every cartesian regular epi $e\colon (X, E) \twoheadrightarrow (Y, F)$ and every arrow $f\colon (X, E) \to (D, \sim)$, there exists a (necessarily unique, since e is epi) $h\colon (Y, F) \to (D, \sim)$ such that the diagram*

$$\begin{array}{ccc} & & (D, \sim) \\ & \overset{f}{\nearrow} & \Big\uparrow{\scriptstyle h} \\ (X, E) & \xrightarrow[e]{} & (Y, F) \end{array}$$

commutes.

Remark 3.5.5. The fact that e is cartesian, makes one aspect of this situation a bit easier. Recall that a morphism $e\colon (X, E) \to (Y, F)$ of assemblies is regular epi if and only if e is surjective and $\mathrm{id}_Y\colon (Y, F) \to (Y, F')$ is a morphism of assemblies, where $F'(y) := \bigcup_{e(x)=y} E(x)$. Since e is cartesian, we see that, if e is surjective, then $F'(y) = F(y)$ for all $y \in Y$. So we have that e is regular epi as soon as e is surjective. ◊

Proof of Proposition 3.5.4. First, suppose $(D, \sim)$ is discrete. Without loss of generality, we can assume that we have $[d \sim d] \cap [d' \sim d'] = \emptyset$ for distinct $d, d' \in D$. Let G represent f and let K represent e. We define the element $H := [\exists x\, (G(x, d) \wedge K(x, y))] \in \mathsf{P}(Y \times D)$. We claim that this is a functional relation. First of all, the strictness and relationality of H follow immediately from the internal logic of P. Since e is a regular epi, we have $\mathsf{P} \models \forall y\, (F(y) \to \exists x\, K(x, y))$. Now we can also use the Soundness Theorem to derive the totality of H. The single-valuedness of H is complicated. We should show that

$$\mathsf{P} \models \forall y d d' x x'\, (K(x, y) \wedge K(x', y) \wedge G(x, d) \wedge G(x', d') \to d \sim d').$$

We informally describe how to obtain a realizer for this statement. Let $y \in Y$, $d, d' \in D$ and $x, x' \in X$ be given and suppose we have elements from $K(x, y)$, $K(x', y)$, $G(x, d)$ and $G(x', d')$. Since $K(x, y)$ and $K(x', y)$ are both nonempty, we have $e(x) = y = e(x')$, which also means that $E(x) = F(y) = E(x')$. Moreover, from an element of $K(x, y)$, we can compute an element $a \in E(x) = E(x')$. Since G is total, there exists an element $t \in \bigcap_{x \in X} [E(x) \to \exists d\, G(x, d)]$. Since a is an element of both $E(x)$ and $E(x')$, we see that $ta\downarrow$, and there exist $d_0, d_0' \in D$ such that $ta \in G(x, d_0)$ and $ta \in G(x', d_0')$. Since G is relational, there also exists an element $r \in \bigcap_{x \in X, d \in D} [G(x, d) \to d \sim d]$. Now we see that $r(ta)$ is defined and is an element of both $[d_0 \sim d_0]$ and $[d_0' \sim d_0']$. By our assumption, $d_0 = d_0'$. Since G is functional, we can compute, from an element of $G(x, d)$ and the element $ta \in G(x, d_0)$, a realizer of $d \sim d_0$. Similarly, we can compute, from an element of $G(x', d')$ and the element $ta \in G(x', d_0)$, a realizer of $d' \sim d_0$. From realizers of $d \sim d_0$ and $d' \sim d_0$, we can compute a realizer of $d \sim d'$, as desired. So H defines an arrow $h\colon (Y, F) \to (D, \sim)$. Using the relationality of G and the totality of K, it is easy to show that

$$\mathsf{P} \models \forall x d\, (G(x, d) \to \exists y\, (K(x, y) \wedge H(y, d))),$$

so $h \circ e = f$, as desired.
Conversely, suppose that $(D, \sim)$ has the property as stated in the proposition. We define

$$\tilde{D} = \{(d_0, d_1) \in D^2 \mid [d_0 \sim d_0] \cap [d_1 \sim d_1] \neq \emptyset\}.$$

We consider the assembly $(\tilde{D} \sqcup \tilde{D}, E)$, where $E(i, d_0, d_1) = [d_0 \sim d_0] \cap [d_1 \sim d_1]$ for $i \in \{0, 1\}$ and $(d_0, d_1) \in \tilde{D}$. Also consider the assembly $(\tilde{D}, F)$, where $F(d_0, d_1) = [d_0 \sim d_0] \cap [d_1 \sim d_1]$ for $(d_0, d_1) \in \tilde{D}$. Clearly, the projection map $e\colon \tilde{D} \sqcup \tilde{D} \to \tilde{D}\colon (i, d_0, d_1) \mapsto (d_0, d_1)$ is a cartesian regular epi $(\tilde{D} \sqcup \tilde{D}, E) \to (\tilde{D}, F)$.
We also have the function $g\colon \tilde{D} \sqcup \tilde{D} \to D$ that sends $(i, d_0, d_1) \in \tilde{D} \sqcup \tilde{D}$ to $d_i \in D$. By definition,

$$\mathsf{i} \in \bigcap_{(i,d_0,d_1) \in \tilde{D} \sqcup \tilde{D}} [E(i, d_0, d_1) \to d_i \sim d_i],$$

which means that g yields an arrow $f\colon (\tilde{D} \sqcup \tilde{D}, E) \to (D, \sim)$ of $\mathsf{RT}(A)$ through Proposition 3.3.1. It is represented by the functional relation $F_g \in \mathsf{P}((\tilde{D} \sqcup \tilde{D}) \times D)$ given by

$$F_g(i, d_0, d_1, d) = [E(i, d_0, d_1) \wedge d_i \sim d] \quad \text{for } (i, d_0, d_1) \in \tilde{D} \sqcup \tilde{D} \text{ and } d \in D.$$

By assumption, there must exist an arrow $h\colon (\tilde{D}, E) \to (D, \sim)$ such that $h \circ e = f$. Let H represent h. Using the fact that $h \circ e = f$, it is easy to show that there exists an $r \in A$ satisfying: if $b \in F_g(i, d_0, d_1, d)$, then $rb\downarrow$ and $rb \in H(d_0, d_1, d)$. Since H is functional, there also exists an element

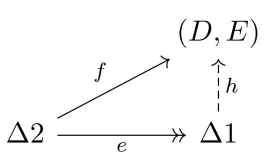
$$s \in \bigcap_{\substack{(d_0,d_1) \in \tilde{D} \\ d,d' \in D}} [H(d_0, d_1, d) \wedge H(d_0, d_1, d') \to d \sim d'].$$

Suppose that $(d_0, d_1) \in \tilde{D}$ and let $b \in [d_0 \sim d_0] \cap [d_1 \sim d_1]$. Then for $i \in \{0, 1\}$, we have $\mathsf{p}bb \in E(i, d_0, d_1) \wedge [d_i \sim d_i] = F_g(i, d_0, d_1, d_i)$ and therefore $r(\mathsf{p}bb) \in H(d_0, d_1, d_i)$. So

$$\mathsf{p}(r(\mathsf{p}bb))(r(\mathsf{p}bb)) \in H(d_0, d_1, d_0) \wedge H(d_0, d_1, d_1),$$

which implies that $s(\mathsf{p}(r(\mathsf{p}bb))(r(\mathsf{p}bb))) \in [d_0 \sim d_1]$. So $a = \langle u \rangle s(\mathsf{p}(r(\mathsf{p}uu))(r(\mathsf{p}uu)))$ has the property stated in Proposition 3.5.2. We conclude that $(D, \sim)$ is discrete. □

Proposition 3.5.6. *Let e be the unique morphism of assemblies $\Delta 2 \twoheadrightarrow \Delta 1$ and let (D, E) be an assembly. Then (D, E) is discrete if and only if, for every morphism of assemblies $f\colon \Delta 2 \to (D, E)$, there exists a (necessarily unique) morphism of assemblies $h\colon \Delta 1 \to (D, E)$ such that the diagram*



commutes.

Proof. Since $e\colon \Delta 2 \twoheadrightarrow \Delta 1$ is cartesian and regular epi, the 'only if' direction follows from Proposition 3.5.4. Conversely, suppose that (D, E) has the property as stated in the proposition. Let d and d' be elements from D such that we have an element $a \in E(d) \cap E(d')$. Define the function $f\colon 2 \to D$ by $f(0) = d$ and $f(1) = d'$. This is a morphism of assemblies $\Delta 2 \to (D, E)$, for it is tracked by $\mathsf{k}a$. So there must exist a morphism of assemblies $h\colon \Delta 1 \to (D, E)$ such that $h \circ e = f$ (as morphisms of assemblies, hence as *functions*). Now we get $d = f(0) = h(e(0)) = h(0) = h(e(1)) = f(1) = d'$. We conclude that (D, E) is discrete. □

Example 3.5.7. There exists a *non-discrete* object $(X, \sim)$ of Eff such that every arrow $f\colon \Delta 2 \to (X, \sim)$ factors through $e\colon \Delta 2 \twoheadrightarrow \Delta 1$. Let B be any non-decidable subset of $\mathbb{N}$. We take $X = \mathbb{N}$ and define $\sim\colon \mathbb{N} \times \mathbb{N} \to \mathcal{P}(\mathbb{N})$ as follows:

$$[2n \sim 2n] = [2n+1 \sim 2n+1] = \begin{cases} \{3n, 3n+1\} & \text{if } n \in B; \\ \{3n, 3n+2\} & \text{if } n \notin B, \end{cases}$$

$$[2n \sim 2n+1] = [2n+1 \sim 2n] = \begin{cases} \{3n+1\} & \text{if } n \in B; \\ \{3n+2\} & \text{if } n \notin B, \end{cases}$$

and $[m \sim n] = \emptyset$ if $\lfloor \frac{m}{2} \rfloor \neq \lfloor \frac{n}{2} \rfloor$. The symmetry of $\sim$ is tracked by i. A tracker of transitivity is given by the following algorithm. Suppose we have input $\langle m, n \rangle$. If $3 \nmid m$, then output m. If $3 \mid m$ and $3 \nmid n$, then output n. Finally, if $3 \mid m, n$, then output m.
Let an arrow $f\colon \Delta 2 \to (\mathbb{N}, \sim)$ be given. Since $\Delta 2$ is projective, this arrow is in fact given by a function $g\colon 2 \to \mathbb{N}$. Since $0, 1 \in 2$ have a common realizer in $\Delta 2$, we see that $g(0)$ and $g(1)$ must have a common realizer in $(\mathbb{N}, \sim)$. This means that $g(0), g(1) \in \{2n, 2n+1\}$ for a certain $n \in \mathbb{N}$. Now let $h\colon \Delta 1 \to (\mathbb{N}, \sim)$ be given by the function $k\colon 1 \to \mathbb{N}\colon 0 \mapsto 2n$. We check that $f = h \circ e$. We notice that f is represented by $F_g \in \mathsf{P}(2 \times D)$, which is isomorphic to $(i, m) \mapsto [g(i) \sim m]$. Moreover, the arrow $h \circ e$ is given by the function $ke\colon 2 \to D\colon i \mapsto 2n$ in the sense of Proposition 3.3.1. We have that $F_{ke} \in \mathsf{P}(2 \times D)$ is isomorphic to $(i, m) \mapsto [2n \sim m]$. Suppose that $n \in B$. We notice: if $m \notin \{2n, 2n+1\}$, then $[g(i) \sim m] = \emptyset$, and if $m \in \{2n, 2n+1\}$, then $3n+1 \in [g(i) \sim m]$. The same holds for $[2n \sim m]$. This implies that the functional relations F_g and F_{ke} are isomorphic, so f and $h \circ e$ are indeed equal. If $n \notin B$, then $f = h \circ e$ follows analogously.
But $(\mathbb{N}, \sim)$ is not discrete. Indeed, suppose there were an $a \in \mathbb{N}$ satisfying: if we have $b \in [m \sim m] \cap [n \sim n]$, then $ab\downarrow$ and $ab \in [m \sim n]$. This means that the partial recursive function φ_a satisfies: $\varphi_a(3n) = 3n+1$ if $n \in B$, and $\varphi_a(3n) = 3n+2$ if $n \notin B$. But this means that B is decidable, contradiction. ◊

Similar examples may be constructed over any nontrivial PCA A. A subset B of an arbitrary PCA A is called *decidable* if there exists an $a \in A$ satisfying: $ab\downarrow$ for all $b \in A$, and $ab = \mathsf{k}$ if $b \in B$, while $ab = \overline{\mathsf{k}}$ if $b \notin B$. If A is nontrivial, then $\mathsf{k} \neq \overline{\mathsf{k}}$, so an element $b \in A$ yields at most one decidable subset of A. Since $|A| < |\mathcal{P}A|$, any nontrivial PCA A must have nondecidable subsets, which allows one to construct an example as above.

For the trivial pca $\mathbb{I}$, an object as in Example 3.5.7 cannot be constructed. This is so because every object of the $\mathsf{RT}(\mathbb{I})$ is (isomorphic to) an assembly. In fact, Γ and Δ form an equivalence between $\mathsf{RT}(\mathbb{I})$ and Set. Using this, it is easy to show that the only discrete objects of $\mathsf{RT}(\mathbb{I})$ are the initial and terminal objects.

Chapter 4

Characterization of Realizability Topoi

In Section 3.2, we defined the realizability topos $\mathsf{RT}(A)$ over a given PCA A. We may say that this yields an 'intensional' description of the class of all realizability topoi: this class is delineated as the class of all topoi that may be constructed in a certain way. As we mentioned in the introduction, there is an analogy with the class of Grothendieck topoi here. In his PhD Thesis [1], Jonas Frey used the theory of *fibrations* to give an 'extensional' description of the class of realizability topoi, that is, a description purely in terms of the categorical properties that realizability topoi have. Because Frey relies on the theory of fibrations, his characerization theorem ([1], Corollary 4.11.7(iii)) is formulated in a rather abstract way. In this chapter, our goal is to bring Frey's theorem 'down to earth'. There are two circumstamces that are helpful here. First of all, we shall freely assume that the Axiom of Choice holds in Set, something that Frey explicitly does not do. Secondly, it turns out that the presence of the global sections functor in Frey's theorem allows the theorem to be simplified considerably.

First of all, in Section 4.1, we briefly explain the notion of a fibration and introduce an example of a fibration that will figure prominently in the remainder of the section. In Section 4.2, Section 4.3 and Section 4.4, we consider various notions concerning fibrations that Frey introduces, and we formulate these notions in simpler ways. To this end, we will make progressively stronger assumptions throughout these sections. Finally, in Section 4.5, we formulate our version of the characterization theorem, and show that it follows from Frey's version.

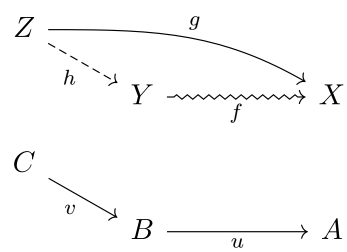
In this chapter, we study the categorical properties of realizability topoi, rather than realizability topoi themselves. In the previous two chapters, the letter 'A' invariably stood for a PCA. In this chapter, on the other hand, the letter 'A' will typically denote an arbitrary object of the base category (to be defined below).

4.1 The Language of Fibrations

Let us start by defining fibrations.

Definition 4.1.1. Let $\mathscr{C} : |\mathscr{C}| \to \mathcal{B}$ be a functor.

(i) Let $f : Y \to X$ in $|\mathscr{C}|$ and $u : B \to A$ in $\mathcal{B}$ such that $\mathscr{C}(f) = u$ be given. We say that f is *cartesian* if, for all $g : Z \to X$ in $|\mathscr{C}|$ and all $v : C \to B$ in $\mathcal{B}$ such that $\mathscr{C}(g) = uv$, there is a unique $h : Z \to Y$ such that $\mathscr{C}(h) = v$ and $fh = g$. We write $f : Y \rightsquigarrow X$ to indicate that f is cartesian.



We use squiggly arrows to denote cartesian arrows.

(ii) We say that $\mathscr{C}$ is a *fibration* if, for any arrow $u : B \to A$ in $\mathcal{B}$ and every object X of $|\mathscr{C}|$ such that $\mathscr{C}(X) = A$, there exists a cartesian arrow $f : Y \rightsquigarrow X$ in $|\mathscr{C}|$ such that $\mathscr{C}(f) = u$. We call such an f a *cartesian lifting of* X *along* u.

The category $|\mathscr{C}|$ is called the *total category* of $\mathscr{C}$, while $\mathcal{B}$ is called the *base category*.

As the diagram above already indicates, the action of $\mathscr{C}$ is typically indicated by vertical alignment. If X is an object of the total category such that $\mathscr{C}(X) = A$, then we say that X *lies above* A. We employ a similar terminology for arrows. If A is an object of the base category, then an arrow f of the total category is called a *vertical arrow over* A if f lies above id_A. The objects of the total category lying above A, together with the vertical arrows over A, form a subcategory $\mathscr{C}_A$ of the total category, called the *fiber* of A.

The cartesian liftings from Definition 4.1.1 are not necessarily unique, but we do have the following.

Proposition 4.1.2. *Let* $\mathscr{C} : |\mathscr{C}| \to \mathcal{B}$ *be a fibration.*

(*i*) *Suppose* $f : Y \rightsquigarrow X$ *is cartesian and* $g, h : Z \to Y$ *are such that* $fg = fh$ *and* $\mathscr{C}(g) = \mathscr{C}(h)$. *Then* $g = h$.

(*ii*) *Let* $u : B \to A$ *in* $\mathcal{B}$ *and* X *in* $|\mathscr{C}|$ *such that* $\mathscr{C}(X) = A$. *Then a cartesian lifting of* X *along* u *is unique up to unique vertical isomorphism. That is, if* $f : Y \rightsquigarrow X$ *and* $f' : Y' \rightsquigarrow X$ *are two cartesian liftings of* X *along* u, *then there exists a unique vertical isomorphism* $i : Y \to Y'$ *such that* $f'i = f$.

Proof. Statement (i) follows directly from the 'uniqueness' part in the universal property of cartesian arrows. Now consider the second statement. By the universal property of f', there exists a unique arrow $i : Y \to Y'$ such that $\mathscr{C}(i) = \mathrm{id}_B$ (so i is vertical) and $f'i = f$. Similarly, there exists a vertical arrow $j : Y' \to Y$ such that $fj = f'$. Now we have $fji = f'i = f =$

$f \circ \mathrm{id}_Y$ and $\mathscr{C}(ji) = \mathrm{id}_B = \mathscr{C}(\mathrm{id}_Y)$, so by part (i) of the proposition, $ji = \mathrm{id}_Y$. Similarly, $ij = \mathrm{id}_{Y'}$, so i is an isomorphism. As we already remarked above, i is the only vertical arrow $Y \to Y'$ such that $f'i = f$, which shows uniqueness. □

We now assume that for every arrow $u\colon B \to A$ in the base category and every object X above A, we have specific *choice* of a cartesian lifting $\eta_{u,X}\colon u^*X \rightsquigarrow X$. If we have an arrow $f\colon X \to Y$ of $\mathscr{C}_A$, then by the universal property of $\eta_{u,Y}$, there exists a unique arrow of $\mathscr{C}_B$ making the diagram

$$\begin{array}{ccc} u^*X & \overset{\eta_{u,X}}{\rightsquigarrow} & X \\ {\scriptstyle u^*f}\downarrow & & \downarrow{\scriptstyle f} \\ u^*Y & \underset{\eta_{u,Y}}{\rightsquigarrow} & Y \end{array}$$

commute. This yields a functor $u^*\colon \mathscr{C}_A \to \mathscr{C}_B$, which is called the *reindexing functor*. Notice that identity arrows are always cartesian. Now using Proposition 4.1.2, we can show that id_A^* is naturally isomorphic to $\mathrm{id}_{\mathscr{C}_A}$. Moreover, if we have $C \xrightarrow{v} B \xrightarrow{u} A$, then $(uv)^*$ is naturally isomorphic to v^*u^*, since the composition of two cartesian arrows is again cartesian. This means that the assignment $A \mapsto \mathscr{C}_A$ for objects A of the base category and $u \mapsto u^*$ for arrows of the base category yields a contravariant pseudofunctor from the base category to the category of categories.

The following will be the most important example of a fibration.

Example 4.1.3. Let $F\colon \mathcal{B} \to \mathcal{C}$ be a functor and suppose that $\mathcal{C}$ has pullbacks. The *comma category* $\mathcal{C}{\downarrow}F$ has as objects triples (X, A, x), where X is an object of $\mathcal{C}$, A is an object of $\mathcal{B}$ and $x\colon X \to FA$. An arrow $(X, A, x) \to (Y, B, y)$ is a pair of arrows $(f\colon X \to Y, u\colon A \to B)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ FA & \xrightarrow{Fu} & FB \end{array}$$

commutes. It is easy to verify that (f, u) is cartesian if and only if this square is a pullback diagram. Given that $\mathcal{C}$ has pullbacks, it is now clear that the projection $\mathcal{C}{\downarrow}F \to \mathcal{B}$ is a fibration, and we denote it by $\mathrm{gl}_F(\mathcal{C})$. For an object A of $\mathcal{B}$, the fiber of A is the slice category $\mathcal{C}/FA$. Furthermore, if $u\colon B \to A$, the reindexing functor $u^*\colon \mathcal{C}/FA \to \mathcal{C}/FB$ is given by pullback along Fu.

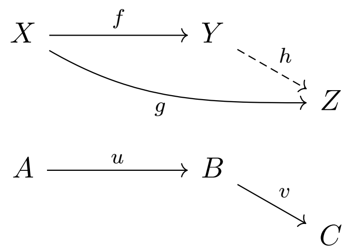
In the situation we shall consider, $\mathcal{B}$ and $\mathcal{C}$ will be regular categories, and F will be a regular functor. In this case, the fibration $\mathrm{gl}_F(\mathcal{C})$ is a so-called *positive pre-stack*. We will not need this concept here, however, so we do not define it. Besides, the concept is not really more general than the situation described in this example. Indeed, using a result called Moens' Theorem, it can be shown that, up to a certain notion of equivalence, every positive pre-stack arises as $\mathrm{gl}_F(\mathcal{C})$ for some regular functor $F\colon \mathcal{B} \to \mathcal{C}$. ◊

Example 4.1.4. In Chapter 3, we defined what it means for a morphism of assemblies to be cartesian. This use of terminology suggests that we have an instantiation of the more general notion of being cartesian we defined here. And so it is. Let A be a PCA. Then the functor $\Gamma\colon \mathsf{Asm}(A) \to \mathsf{Set}$ is a fibration. Given a function $f\colon X \to Y$ and an assembly (Y, F), a cartesian lifting of (Y, F) along f is given $f\colon (X, E) \to (Y, F)$, where $E(x) = F(f(x))$ for all

$x \in X$. We see that, up to isomorphism, the cartesian morphisms of assemblies (in the sense of Definition 3.4.7) are the cartesian arrows of the fibration $\Gamma\colon \mathsf{Asm}(A) \to \mathsf{Set}$. ◊

The follow definition gives the dual notion of 'cartesian'.

Definition 4.1.5. Let $\mathscr{C}\colon |\mathscr{C}| \to \mathcal{B}$ be a fibration. Let $f\colon X \to Y$ be an arrow of $|\mathscr{C}|$ lying above $u\colon A \to B$. We say that f is *cocartesian* if for all $v\colon B \to C$ and $g\colon X \to Z$ lying above vu, there exists a unique $h\colon Y \to Z$ lying above v such that $hf = g$.



Example 4.1.6. It is not difficult to check that in the fibration $\mathrm{gl}_F(\mathcal{C})$ from Example 4.1.3, an arrow (f, u) of $\mathcal{C}{\downarrow}F$ is cocartesian if and only if f is an isomorphism. ◊

Finally, we define what it means for a fibration to have finite limits.

Definition 4.1.7. Let $\mathscr{C}\colon |\mathscr{C}| \to \mathcal{B}$ be a fibration. We say that $\mathscr{C}$ *has finite limits* if both the total category and the base category have finite limits, and $\mathscr{C}$ preserves them.

If $\mathscr{C}\colon |\mathscr{C}| \to \mathcal{B}$ is a fibration, where $\mathcal{B}$ has finite limits, then $\mathscr{C}$ has finite limits if and only if all the fibers have finite limits, and these finite limits are preserved by reindexing. We will not show this here, but instead refer the reader to [5], Theorem 8.5.

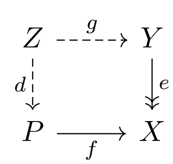
4.2 Projective Objects in a Fibration

Now that we introduced the fibrational framework, we can begin to study some special objects.

Definition 4.2.1. Let $\mathscr{C}\colon |\mathscr{C}| \to \mathcal{B}$ be a fibration.

(i) An arrow of the total category is called *cover-cartesian* if it is cartesian and lies above a regular epi in the base.

(ii) We say that an object $P \in |\mathscr{C}|$ is *f-projective* (with respect to $\mathscr{C}$) if, given e and f as in the diagram



where e is vertical and a regular epi in its fiber, there exist an object Z and arrows d and g such that d is cover-cartesian, and the square commutes.

Remark 4.2.2. These definitions are taken from [1], Definition 2.3.5(i) and Definition 3.1.15(i), but with the following important differences.

(i) In [1], Definition 3.1.15(i) is different. Instead of just requiring that P has the property mentioned in (ii) above, it is required that P' has this property whenever $P' \rightsquigarrow P$ is cartesian. In this way, the f-projective objects are automatically closed under reindexing. We do not require this here, however.

(ii) Frey calls the objects defined in [1], Definition 3.1.15(i) just 'projective'. We call the objects defined in (ii) 'f-projective' to distinguish them from the projective objects defined in Definition 3.4.1.

(iii) Frey defines the concepts above only for (positive) pre-stacks, whereas we defined them for arbitrary fibrations. If one is working in a (positive) pre-stack, then cover-cartesian maps are automatically regular epis in the total category ([1], Lemma 2.3.7). We will not need this here, but one may easily check it for the example $\mathrm{gl}_F(\mathcal{C})$, where $F\colon \mathcal{B} \to \mathcal{C}$ is a regular functor. $\Diamond$

Definition 4.2.1 mentions arrows that are regular epi *in their fibers*. In the example $\mathrm{gl}_F(\mathcal{C})$, the fibers are slices. The following (well-known) result allows us to handle regular epis in a slice.

Proposition 4.2.3. *Let $\mathcal{C}$ be a category that has binary products, and let X be an object of $\mathcal{C}$. Then the domain functor* $\mathrm{dom}\colon \mathcal{C}/X \to \mathcal{C}$ *preserves and creates all colimits. In particular, an arrow $e\colon (Y, y) \to (Z, z)$ of $\mathcal{C}/X$ is a regular epi if and only if e is a regular epi in $\mathcal{C}$.*

Proof. There is a functor $X^*\colon \mathcal{C} \to \mathcal{C}/X$ that sends objects C to $(C \times X, \pi_2)$ and that sends an arrow $f\colon C \to D$ to $f \times \mathrm{id}_X$. Let (C, c) be an object of $\mathcal{C}/X$ and let D be an object of $\mathcal{C}$. There is a natural bijection between arrows $f\colon C \to D$ of $\mathcal{C}$ and arrows $g\colon C \to D \times X$ such that $\pi_2 \circ g = c$. This bijection is given by $f \mapsto \langle f, c\rangle$. In other words, there is a natural bijection $\mathcal{C}(\mathrm{dom}(C, c), D) \to (\mathcal{C}/X)((C, c), X^*D)$, meaning there is an adjunction $\mathrm{dom} \dashv X^*$. So dom, being a left adjoint, preserves all colimits.
The corresponding comonad $L = \mathrm{dom} \circ X^*\colon \mathcal{C} \to \mathcal{C}$ is the product functor $(-) \times X$. Its counit ε is given by $\varepsilon_C = \pi_1 : C \times X \to C$, for objects C of $\mathcal{C}$. The comultiplication δ is given by $\delta_C = \mathrm{id}_C \times d\colon C \times X \to C \times X \times X$, where $d\colon X \to X \times X$ is the diagonal. An L-coalgebra is an arrow $h\colon C \to C \times X$ such that $\pi_1 \circ h = \mathrm{id}_C$. A map of L-coalgebras $(C, h) \to (D, k)$ is an arrow $f\colon C \to D$ such that $(f \times \mathrm{id}_X) \circ h = k \circ f$. The unit η of the adjunction $\mathrm{dom} \dashv X^*$ is given by $\eta_{(C,c)} = \langle \mathrm{id}_C, c\rangle\colon (C, c) \to (C \times X, \pi_2)$. This means that the comparison functor $K\colon \mathcal{C}/X \to L{-}\mathsf{CoAlg}$ is given by

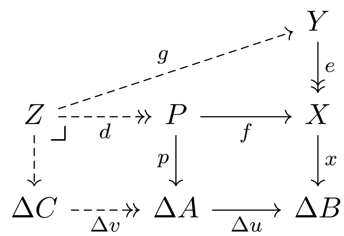
$$(C, c) \mapsto (\langle \mathrm{id}_C, c\rangle\colon C \to C \times X)$$

for objects (C, c) of $\mathcal{C}/X$, and K is the identity on arrows. It is not difficult to see that K is an isomorphism of categories. This means that dom is comonadic, and in particular, it creates all colimits.
Now suppose that $e\colon (Y, y) \to (Z, z)$ is a coequalizer of the parallel pair $a, b\colon (W, w) \rightrightarrows (Y, y)$ in $\mathcal{C}/X$. Because dom preserves colimits, we see that e is a coequalizer of $a, b\colon W \rightrightarrows Y$ in $\mathcal{C}$. Conversely, suppose e is the coequalizer of the parallel pair $a, b\colon W \rightrightarrows Y$ in $\mathcal{C}$. Then $ay = aez = bez = by$, so a and b are both arrows $(W, ay) \to (Y, y)$ in $\mathcal{C}/X$. Since dom creates colimits, e must be a coequalizer of a and b in $\mathcal{C}/X$. $\square$

In the remainder of this chapter, we suppose that we have a regular category $\mathcal{C}$ and a regular functor $\Delta\colon \mathsf{Set} \to \mathcal{C}$. The following proposition relies on the Axiom of Choice.

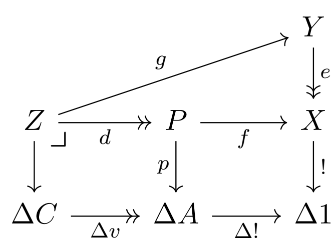
Proposition 4.2.4. *An object $p\colon P \to \Delta A$ of $\mathcal{C}{\downarrow}\Delta$ is f-projective w.r.t. the fibration $\mathrm{gl}_\Delta(\mathcal{C})$ if and only if P is projective in $\mathcal{C}$.*

Proof. First, we use Proposition 4.2.3 to spell out the definition of an f-projective object in the case where $\mathscr{C}$ is the fibration $\mathrm{gl}_\Delta(\mathcal{C})$. For every diagram of the form

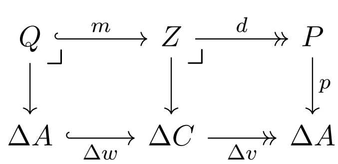


(4.1)

where $\Delta u \circ p = x \circ f$ and e is a regular epi, we can append a pullback diagram on the left, and find an arrow $g\colon Z \to Y$, such that v is surjective and $eg = fd$.
Suppose that $p\colon P \to \Delta A$ is projective, and let a regular epi $e\colon Y \to X$ and a morphism $f\colon P \to X$ be given. Apply the fact that $p\colon P \to \Delta A$ is projective to the diagram (4.1) with $B = 1$. Since $\Delta 1$ is a terminal object of $\mathcal{C}$, we have $\Delta ! \circ p = ! \circ f$. We find a pullback diagram as above, and an arrow $g\colon Z \to Y$ such that $eg = fd$:



Since v is surjective, it has a section $w\colon A \to C$. We append another pullback diagram:



Since $\Delta v \circ \Delta w = \Delta(v \circ w) = \Delta \mathrm{id}_A = \mathrm{id}_{\Delta A}$, we see that dm must be an isomorphism. This means that d has a section $m' := m \circ (dm)^{-1}\colon P \to Z$. Now let $g' = g \circ m'$. We have

$$e \circ g' = e \circ g \circ m' = f \circ d \circ m' = f \circ \mathrm{id}_P = f,$$

as desired.
Conversely, suppose that P is projective. Then we can complete the diagram (4.1) with $Z = P$, $C = A$, and g chosen in such a way that $eg = f$. $\square$

4.3 Indecomposable Objects

In this section, we suppose that our regular functor $\Delta\colon \mathsf{Set} \to \mathcal{C}$ has a left adjoint $\Gamma\colon \mathcal{C} \to \mathsf{Set}$. We will use the following terminology.

Definition 4.3.1. Suppose we have functors $\mathcal{C} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{D}$ such that there exists an adjunction $L \dashv R$. Let η be the unit of this adjunction.

(i) We call an object X of $\mathcal{D}$ an *assembly* if η_X is mono.

(ii) Suppose $f: X \to Y$ is an arrow of $\mathcal{D}$, where X and Y are assemblies. We call f *cartesian* if the naturality square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \eta_X}\big\downarrow & & \big\downarrow{\scriptstyle \eta_Y} \\ RLX & \xrightarrow[RLf]{} & RLY \end{array}$$

is a pullback diagram.

We know from Chapter 3 that this terminology agrees, at least up to isomorphism, with the case where we consider the adjunction $\mathsf{Set} \underset{\Delta}{\overset{\Gamma}{\leftrightarrows}} \mathsf{RT}(A)$ for a certain PCA A.

It should be emphasized that the notions 'assembly' and 'cartesian' only make sense with respect to a given adjunction. In this section, this will be the adjunction $\Gamma \dashv \Delta$ between Set and our regular category $\mathcal{C}$.

Proposition 4.3.2. *Suppose we have functors $\mathcal{C} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{D}$ such that there exists an adjunction $L \dashv R$. Then an object X of $\mathcal{D}$ is an assembly if and only if there exists a monomorphism $X \hookrightarrow RC$ for some object C of $\mathcal{C}$.*

Proof. Suppose $a: X \hookrightarrow RC$ is mono, and let $\tilde{a}: LX \to C$ be its transpose. Then $a = R\tilde{a} \circ \eta_X$ and a is mono, so η_X must be mono as well. The other direction is true by definition. $\square$

The following definition is again inspired by [1].

Definition 4.3.3. Let $\mathscr{C}: |\mathscr{C}| \to \mathcal{B}$ be a fibration. We say that an object X of the total category is *indecomposable* (w.r.t. $\mathscr{C}$), if for every f and s as in the diagram

$$\begin{array}{ccc} X & & \\ {\scriptstyle m}\big\downarrow & \searrow^{f} & \\ Y & \xrightarrow[s]{} & S \end{array}$$

where s is cocartesian, there exists a *unique* m making the triangle commute.

Again, the definition differs from the one given in [1] (Definition 3.1.15(ii)) because we formulate it for arbitrary fibrations and we do not demand the indecomposable objects to be closed under reindexing.

We also formulate what indecomposability means for the case where $\mathscr{C}$ is the fibration $\mathrm{gl}_\Delta(\mathcal{C})$. An object $x: X \to \Delta A$ is indecomposable iff for every diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & \Delta B \\ {\scriptstyle x}\big\downarrow & \overset{\Delta m}{\nearrow} & \big\downarrow{\scriptstyle \Delta s} \\ \Delta A & \xrightarrow[\Delta u]{} & \Delta C \end{array}$$

such that $\Delta u \circ x = \Delta s \circ f$, there exists a unique mediating function $m\colon A \to B$ such that $sm = u$ and $\Delta m \circ x = f$. Now we can prove the following result.

Proposition 4.3.4. *An object $x\colon X \to \Delta A$ of $\mathcal{C}{\downarrow}\Delta$ is indecomposable w.r.t. $\mathrm{gl}_\Delta(\mathcal{C})$ if and only if it is isomorphic to the object $\eta_Y\colon Y \to \Delta\Gamma Y$, for some object Y of $\mathcal{C}$.*

Proof. First, suppose that $x\colon X \to \Delta A$ is indecomposable. Let $\tilde{x}\colon \Gamma X \to A$ be the transpose of $x\colon X \to \Delta A$ under the adjunction. Then we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \Delta\Gamma X \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle \Delta\tilde{x}} \\ \Delta A & \xrightarrow[\Delta\mathrm{id}_A]{} & \Delta A \end{array}$$

This means that there exists a function $m\colon A \to \Gamma X$ such that $\tilde{x} \circ m = \mathrm{id}_A$ and $\Delta m \circ x = \eta_X$. We get

$$\Delta(m \circ \tilde{x}) \circ \eta_X = \Delta m \circ \Delta\tilde{x} \circ \eta_X = \Delta m \circ x = \eta_X = \Delta\mathrm{id}_{\Gamma X} \circ \eta_X.$$

By the uniqueness part of the universal property of an adjunction, we get that $m \circ \tilde{x} = \mathrm{id}_{\Gamma X}$, so m is a bijection. We conclude that $(\mathrm{id}_X, m)\colon x \to \eta_X$ is an isomorphism in $\mathcal{C}{\downarrow}\Delta$.
For the other direction, we show that $\eta_Y\colon Y \to \Delta\Gamma Y$ is indecomposable for all objects Y of $\mathcal{C}$. Suppose we have a commutative diagram of the form

$$\begin{array}{ccc} Y & \xrightarrow{f} & \Delta B \\ {\scriptstyle \eta_Y}\downarrow & & \downarrow{\scriptstyle \Delta s} \\ \Delta\Gamma Y & \xrightarrow[\Delta u]{} & \Delta C \end{array}$$

There exists a unique arrow $m\colon \Gamma Y \to B$ such that $\Delta m \circ \eta_Y = f$, namely the transpose of f under the adjunction. We have

$$\Delta(sm) \circ \eta_Y = \Delta s \circ \Delta m \circ \eta_Y = \Delta s \circ f = \Delta u \circ \eta_Y,$$

so we get $sm = u$, as desired. □

We close this section by proving a result that we will need in Section 4.5.

Lemma 4.3.5. *Let Y be an assembly and let*

$$\begin{array}{ccc} Y & \xrightarrow{e} & X \\ {\scriptstyle \eta_Y}\big\downarrow & & \downarrow{\scriptstyle \eta_X} \\ \Delta\Gamma Y & \xrightarrow[\Delta u]{} & \Delta\Gamma X \end{array}$$

be a commutative diagram. If e splits as $e \circ m = \mathrm{id}_X$, then X is also an assembly, and m is cartesian.

Proof. Notice that m must be mono. We have $\Delta\Gamma m \circ \eta_X = \eta_Y \circ m$ and the latter composition is mono, so η_X must be mono as well, i.e. X is an assembly. We also have

$$\Delta(u \circ \Gamma m) \circ \eta_X = \Delta u \circ \Delta\Gamma m \circ \eta_X = \Delta u \circ \eta_Y \circ m = \eta_X \circ e \circ m = \eta_X = \Delta\mathrm{id}_{\Gamma X} \circ \eta_X,$$

and by the universal property of the adjunction, we get $u \circ \Gamma m = \mathrm{id}_{\Gamma X}$. So Γm is split mono, which means that $\Delta\Gamma m$ is split mono as well.
Now we show that m is cartesian. Let $p\colon Z \to Y$ and $q\colon Z \to \Delta\Gamma X$ be given such that $\eta_Y \circ p = \Delta\Gamma m \circ q$. We have

$$\begin{aligned} \eta_Y \circ m \circ e \circ p &= \Delta\Gamma m \circ \eta_X \circ e \circ p = \Delta\Gamma m \circ \Delta u \circ \eta_Y \circ p \\ &= \Delta\Gamma m \circ \Delta u \circ \Delta\Gamma m \circ \eta_Y = \Delta\Gamma m \circ q = \eta_Y \circ p, \end{aligned}$$

and since η_Y is mono, we get $m \circ e \circ p = p$. Now suppose that $k\colon Z \to X$ satisfies $m \circ k = p$ and $\eta_X \circ k = q$. Then in particular, we have $e \circ p = e \circ m \circ k = k$. Conversely, we have $m \circ e \circ p = p$, and also

$$\Delta\Gamma m \circ \eta_X \circ e \circ p = \eta_Y \circ m \circ e \circ p = \eta_Y \circ p = \Delta\Gamma m \circ q,$$

which implies $\eta_X \circ e \circ p = q$, since $\Delta\Gamma m$ is mono. □

Proposition 4.3.6. *Let D be an assembly. Then the following are equivalent.*

(i) For every arrow $x\colon X \to \Delta A$, there exists a commutative diagram

$$\begin{array}{ccccc} D & \xleftarrow{\;f\;} & P & \xrightarrow{\;e\;} & X \\ {\scriptstyle \eta_D}\downarrow & & \downarrow{\scriptstyle \eta_P} & & \downarrow{\scriptstyle x} \\ \Delta\Gamma D & \xleftarrow[\Delta u]{} & \Delta\Gamma P & \xrightarrow[\Delta v]{} & \Delta A \end{array}$$

where P is projective in $\mathcal{C}$.

(ii) $\mathcal{C}$ has enough projectives, and for every projective object P of $\mathcal{C}$, we have that P is an assembly and there exists a cartesian arrow $f\colon P \to D$.

Proof. (i)⇒(ii). The fact that $\mathcal{C}$ has enough projectives follows immediately by applying the property in (i) to $A = \Gamma X$ and $x = \eta_X$, for objects X of $\mathcal{E}$.
Now let Q be projective in $\mathcal{C}$. Apply the property in (i) with $X = Q$, $A = \Gamma Q$ and $x = \eta_Q$, and find e, f, u, v and P as in the diagram. Since Q is projective and e is regular epi, the arrow e splits as $e \circ m = \mathrm{id}_Q$. Now apply Lemma 4.3.5 to see that Q is an assembly and that m is cartesian. Now append the naturality square for m and η, which is a pullback diagram, to the pullback diagram obtained from (i) to get the pullback diagram

$$\begin{array}{ccc} D & \xleftarrow{\;fm\;} & Q \\ {\scriptstyle \eta_D}\downarrow & & \downarrow{\scriptstyle \eta_Q} \\ \Delta\Gamma D & \xleftarrow[\Delta(u \circ \Gamma m)]{} & \Delta\Gamma Q \end{array}$$

Now notice that, because $\Gamma \dashv \Delta$ is an adjunction, we can define $\Gamma(fm)$ as the unique arrow $x\colon \Gamma Q \to \Gamma D$ such that $\eta_D \circ f \circ m = \Delta x \circ \eta_Q$. This means that $u \circ \Gamma m = \Gamma(fm)$, so $fm\colon Q \to D$ is cartesian, as desired.
(ii)⇒(i). Let $a\colon X \to \Delta A$ be given. Apply (ii) to find $e\colon P \twoheadrightarrow X$ with P projective; then P is an assembly. Now we have the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\;e\;} & X \\ \downarrow{\scriptstyle \eta_P} & & \downarrow{\scriptstyle a} \\ \Delta\Gamma P & \xrightarrow{\Delta(\widetilde{ae})} & \Delta A \end{array}$$

and there exists an cartesian arrow $f\colon P \to D$. Now we can satisfy (i) by taking $u = \Gamma f$ and $v = \widetilde{ae}$. □

4.4 Discrete Objects in a Fibration

In this section, we suppose that Δ has a left adjoint Γ that preserves finite limits, and that the counit of the adjunction $\Gamma \dashv \Delta$ is a natural isomorphism. In the case where $\mathcal{C}$ is a topos, this means that $(\Gamma \dashv \Delta)\colon \mathsf{Set} \to \mathcal{C}$ is a geometric inclusion. Instead of demanding that the counit of the adjunction is an isomorphism, we can also require Δ to be fully faithful. One particularly important occasion on which these requirements are fulfilled, is when Γ is the global sections functor $X \mapsto \mathcal{C}(1, X)$. It is easy to show that the global sections functor in fact preserves all limits. Furthermore, we have natural bijections

$$\Gamma\Delta A = \mathcal{C}(1, \Delta A) \cong \mathsf{Set}(\Gamma 1, A) \cong A,$$

where the second bijection arises from the fact that $\Gamma 1$ a singleton, since Γ preserves terminal objects. It is not difficult to see that the resulting bijection $\Gamma\Delta A \to A$ is in fact ε_A, where ε is the counit of $\Gamma \dashv \Delta$.

In Section 4.5, we will need the following result.

Proposition 4.4.1. *Let D be an assembly. Then the following are equivalent.*

(i) If $u\colon A \to \Gamma D$ is a function and

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & D \\ {\scriptstyle x}\big\downarrow & & \big\downarrow{\scriptstyle \eta_D} \\ \Delta A & \xrightarrow[\Delta u]{} & \Delta\Gamma D \end{array}$$

is a pullback diagram, then $x\colon X \hookrightarrow \Delta A$ is f-projective and indecomposable w.r.t. $\mathrm{gl}_\Delta(\mathcal{C})$.

(ii) If X is an assembly and $f\colon X \to D$ is cartesian, then X is projective.

Proof. (i)⇒(ii). Suppose X is an assembly and that we have an cartesian arrow $f\colon X \to D$. Then $\Gamma f\colon \Gamma X \to \Gamma D$ is a function, and the pullback of η_D along $\Delta\Gamma f$ must be f-projective and indecomposable. This means that it must be isomorphic to an object of the form $\eta_P\colon P \hookrightarrow \Delta\Gamma P$ with P projective. Since f is cartesian, the pullback of η_D along $\Delta\Gamma f$ is also $\eta_X\colon X \to \Delta\Gamma X$. This means that η_P and η_X are isomorphic in $\mathcal{C}{\downarrow}\Delta$. In particular, P and X are isomorphic in $\mathcal{C}$, so X is projective.
(ii)⇒(i). Suppose that $g\colon A \to \Gamma D$ is a function, and that

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & D \\ {\scriptstyle x}\big\downarrow & & \big\downarrow{\scriptstyle \eta_D} \\ \Delta A & \xrightarrow[\Delta u]{} & \Delta\Gamma D \end{array}$$

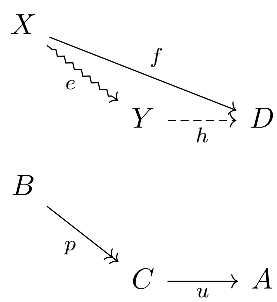
is a pullback diagram. Applying the finite limit preserving functor Γ yields another pullback diagram in Set. But $\Gamma\eta_D$ is an isomorphism (it is the inverse of $\varepsilon_{\Gamma D}$), so Γx must be an isomorphism as well, i.e. a bijection. We conclude that $m := \varepsilon_A \circ \Gamma x\colon \Gamma X \to A$ is also a bijection. We observe that

$$\Delta m \circ \eta_X = \Delta\varepsilon_A \circ \Delta\Gamma x \circ \eta_X = \Delta\varepsilon_A \circ \eta_{\Delta A} \circ x = \mathrm{id}_{\Delta A} \circ x = x,$$

so $(\mathrm{id}_X, m)\colon x \to \eta_X$ is an iso in $\mathcal{C}{\downarrow}\Delta$, which means that $x\colon X \hookrightarrow \Delta A$ is indecomposable w.r.t. $\mathrm{gl}_\Delta(\mathcal{C})$. Furthermore, we now know that η_X must be mono, since x is mono, so X is an assembly. We also have $\Delta(um) \circ \eta_X = \Delta u \circ \Delta m \circ \eta_X = \Delta u \circ x = \eta_D \circ f$, so um must be equal to Γf. We conclude that f is cartesian, so X must be projective in $\mathcal{E}$. We conclude that $x\colon X \hookrightarrow \Delta A$ is a projective indecomposable object of $\mathrm{gl}_\Delta(\mathcal{E})$, as desired. □

We now study the following definition from [1] (Definition 4.11.4).

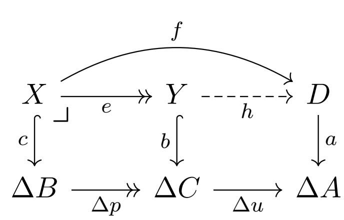
Definition 4.4.2. Let $\mathscr{C}\colon |\mathscr{C}| \to \mathcal{B}$ be a fibration. We call an object D of the total category lying over the object A in the base category *f-discrete* (w.r.t. $\mathscr{C}$) if for every diagram of the form



where f is over up and X is subterminal in the fiber of C, there exists an h over u such that $he = f$. (An object X of a certain category is *subterminal* if for every object of the category, there exists at most one arrow from that object to X.)

In Definition 4.11.4 from [1], these objects are called 'discrete', but we will use this term differently (see Definition 4.4.3 below). Again, Frey defines these objects only for positive pre-stacks. In this case, e as in the definition above is always a regular epi, since it is cover-cartesian, which means that the required h is automatically unique.

Now let us try to describe the f-discrete objects of $\mathcal{C}{\downarrow}\Delta$ w.r.t. $\mathrm{gl}_\Delta(\mathcal{C})$. We notice that an arrow $X \to C$ is a subterminal object of the slice category $\mathcal{C}/C$ if and only if this arrow is monic. So we get: an object $a\colon D \to \Delta A$ of $\mathcal{C}{\downarrow}\Delta$ is discrete if and only if for every diagram of the form



such that p is surjective and $\Delta(up) \circ c = af$, there exists an arrow h making the diagram commute. We notice the following: since p is surjective and Δ is regular, we have that Δp is a regular epi. Since regular epis are stable under pullback in $\mathcal{C}$, we see that e is a regular epi as well. So h as is the diagram above is automatically unique.

In the next section, we shall be interested in the case where A is equal to 1. In this case, the object $!\colon D \to \Delta 1$ is discrete if and only if for every diagram of the form

$$\begin{array}{ccc} & & D \\ & \overset{f}{\nearrow} & \uparrow{\scriptstyle h} \\ X & \overset{e}{\twoheadrightarrow} & Y \\ {\scriptstyle c}\downarrow & \lrcorner & \downarrow{\scriptstyle b} \\ \Delta B & \underset{\Delta p}{\twoheadrightarrow} & \Delta C \end{array}$$

with p and e regular epi, we can find h such that $he = f$. We emphasize again that such an h is necessarily unique. We can simplify the situation a bit further. In accordance with what we found in Section 3.5, we introduce a categorical notion of discreteness.

Definition 4.4.3. An object D of $\mathcal{C}$ is called *discrete* if for every cartesian regular epi $e\colon X \twoheadrightarrow Y$ and every arrow $f\colon X \to D$, there exists a (necessarily unique) $h\colon Y \to D$ such that $he = f$.

We emphasize that this definition only makes sense in the presence of our adjunction $\Gamma \dashv \Delta$, because the notion 'cartesian' occurs in it. The following lemma will help us to link Definition 4.4.3 to the f-discrete objects above.

Proposition 4.4.4. *If an arrow $f\colon X \to Y$ in $\mathcal{C}$ between assemblies fits into a pullback diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle c}\downarrow & \lrcorner & \downarrow{\scriptstyle b} \\ \Delta B & \xrightarrow[\Delta p]{} & \Delta C \end{array}$$

then f is cartesian.

Proof. We can factor the pullback diagram as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \eta_Y}\downarrow & & \downarrow{\scriptstyle \eta_X} \\ \Delta\Gamma X & \xrightarrow[\Delta\Gamma f]{} & \Delta\Gamma Y \\ {\scriptstyle \Delta\tilde{c}}\downarrow & & \downarrow{\scriptstyle \Delta\tilde{b}} \\ \Delta B & \xrightarrow[\Delta p]{} & \Delta C \end{array}$$

Now it suffices to show that the bottom square is a pullback diagram. Since Γ preserves finite limits, we see that

$$\begin{array}{ccc} \Gamma X & \xrightarrow{\Gamma f} & \Gamma Y \\ {\scriptstyle \Gamma c}\downarrow & \lrcorner & \downarrow{\scriptstyle \Gamma b} \\ \Gamma\Delta B & \xrightarrow[\Gamma\Delta p]{} & \Gamma\Delta C \end{array}$$

is a pullback diagram. Because the unit ε of the adjunction is a natural isomorphism, the naturality square

$$\begin{array}{ccc} \Gamma\Delta B & \xrightarrow[\Gamma\Delta p]{} & \Gamma\Delta C \\ {\scriptstyle \varepsilon_C}\downarrow & & \downarrow{\scriptstyle \varepsilon_B} \\ B & \xrightarrow[p]{} & C \end{array}$$

is also a pullback diagram. Now appending these two pullback diagrams and applying the limit preserving functor Δ yields the result. □

Corollary 4.4.5. *Let D be an object of $\mathcal{C}$. The object $!\colon D \to \Delta 1$ of $\mathcal{C}{\downarrow}\Delta$ is f-discrete w.r.t. $\mathrm{gl}_\Delta(\mathcal{C})$ if and only if D is discrete in $\mathcal{C}$.*

Proof. This follows immediately from Proposition 4.4.4, together with the observation that, if e is regular epi, then Γe is surjective. □

4.5 The Characterization Theorem

In this section, we present our version of the characterization theorem and show that it follows from the characterization theorem found by Frey in [1]. The following theorem is Corollary 4.11.7(iii) from [1].

Theorem 4.5.1. *Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is equivalent to a realizability topos if and only if $\mathcal{C}$ is exact and locally cartesian closed, and there exists a regular functor $\Delta\colon \mathsf{Set} \to \mathcal{C}$ that is left adjoint to the global sections functor $\Gamma\colon \mathcal{C} \to \mathsf{Set}$, and there exists a monomorphism $\pi\colon D \hookrightarrow \Delta A$ such that:*

(1)' For every cartesian arrow $X \rightsquigarrow \pi$ in $\mathcal{C}{\downarrow}\Delta$, we have that X is f-projective and indecomposable w.r.t. $\mathrm{gl}_\Delta(\mathcal{C})$.

(2)' $D \to \Delta 1$ is f-discrete w.r.t. $\mathrm{gl}_\Delta(\mathcal{C})$.

(3)' The subfibration $\mathscr{A}$ of $\mathrm{gl}_\Delta(\mathcal{C})$ generated by π is closed under finite limits.

(4)' For every object Y of $\mathcal{C}{\downarrow}\Delta$, there exists an object $X \in |\mathscr{A}|$ and a diagram

$$X \xrightarrow{s} \bullet \xrightarrow{e} Y$$

where s is cocartesian and e is vertical and a regular epi in its fiber.

A few remarks about (3)' and (4)' are in order. By the subfibration $\mathscr{A}$ generated by π, we mean the fibration $\mathscr{A}\colon |\mathscr{A}| \to \mathsf{Set}$, where $|\mathscr{A}|$ is the full subcategory of objects X of $\mathcal{C}{\downarrow}\Delta$ such that there exists a cartesian arrow $X \rightsquigarrow \pi$ and $\mathscr{A}$ is the obvious projection. In the current case, $|\mathscr{C}|$ consists of all objects of $\mathcal{C}{\downarrow}\Delta$ that may be obtained by pulling $\pi\colon D \hookrightarrow \Delta A$ back along an arrow in the image of Δ.

Since Δ preserves finite limits (being a right adjoint), one easily shows that $\mathcal{C}{\downarrow}\Delta$ also has finite limits, and that that these are computed coordinatewise. This implies that $\mathrm{gl}_\Delta(\mathcal{C})$ has finite limits. In (3)', we demand that $\mathscr{A}$ also has finite limits, and that it inherits these from $\mathrm{gl}_\Delta(\mathcal{C})$. That is, the finite limit of a diagram in $|\mathscr{A}|$ should again be in $|\mathscr{A}|$.

Using all our work from this chapter, we can simplify this result to the following theorem.

Theorem 4.5.2. *Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is equivalent to a realizability topos if and only if $\mathcal{C}$ is exact and locally cartesian closed, and there exists a regular functor $\Delta\colon \mathsf{Set} \to \mathcal{C}$ that is right adjoint to the global sections functor $\Gamma\colon \mathcal{C} \to \mathsf{Set}$, and there exists an assembly D such that:*

(*1*) *An object X of $\mathcal{C}$ is projective if and only if it is an assembly and there exists a cartesian arrow $f\colon X \to D$.*

(*2*) *D is discrete in $\mathcal{C}$.*

(*3*) *The projective objects of $\mathcal{C}$ are closed under finite limits.*

(*4*) *$\mathcal{C}$ has enough projectives.*

Proof. We show that the characterization given in Theorem 4.5.1 is equivalent to the characterization given in the current theorem. Throughout this proof, we suppose that $\mathcal{C}$ is exact and locally cartesian closed, and that we have a regular functor $\Delta\colon \mathsf{Set} \to \mathcal{C}$ that is right adjoint to the global sections functor $\Gamma\colon \mathcal{C} \to \mathsf{Set}$.
Suppose that we have $\pi\colon D \hookrightarrow \Delta A$ such that (1)' – (4)' from Theorem 4.5.1 hold. Since $\mathrm{id}_\pi\colon \pi \rightsquigarrow \pi$ is cartesian, we see that π must be f-projective and indecomposable. So without loss of generality, π is $\eta_D\colon D \hookrightarrow \Delta\Gamma D$, where D is a projective assembly. Now we see that (1)' is equivalent to statement (i) of Proposition 4.4.1. Furthermore, given (1)', statement (4)' is equivalent to statement (i) of Proposition 4.3.6. Now it follows from Proposition 4.4.1 and Proposition 4.3.6 that we have (1) and (4). Furthermore, (2) follows from (2)' by Corollary 4.4.5. It remains to show (3). First, we notice that every f-projective indecomposable object of $\mathcal{C}{\downarrow}\Delta$ w.r.t. $\mathrm{gl}_\Delta(\mathcal{C})$ is a pullback of η_D along an arrow in the image of Δ. Indeed, this follows from (1), Proposition 4.2.4 and Proposition 4.3.4. Together with (1)', this means that $|\mathscr{A}|$ contains *exactly* the f-projective indecomposable objects of $\mathcal{C}{\downarrow}\Delta$. Now (3) follows from the fact that every projective object P of $\mathcal{C}$ is the domain of the f-projective indecomposable object $\eta_P\colon P \hookrightarrow \Delta\Gamma P$ of $\mathcal{C}{\downarrow}\Delta$, and the fact that the finite limits in $\mathcal{C}{\downarrow}\Delta$ are computed coordinatewise.
Next, suppose that we have an assembly D such that (1) – (4) hold. We take π to be $\eta_D\colon D \hookrightarrow \Delta\Gamma D$. Since we have (1) and (4), we also have statement (i) of Proposition 4.4.1 and statement (i) of Proposition 4.3.6. As we saw above, these two together are equivalent to (1)' and (4)'. Furthermore, we have (2)' by Corollary 4.4.5. It remains to show (3)'. As above, we see that $|\mathscr{A}|$ contains precisely the f-projective and indecomposable objects of $\mathcal{C}{\downarrow}\Delta$. Suppose $M\colon \mathcal{I} \to \mathcal{C}{\downarrow}\Delta$ is a finite diagram such that Mi is f-projective and indecomposable for all objects i of $\mathcal{I}$. Then without loss of generality, Mi is $\eta_{M'i}\colon M'i \to \Delta\Gamma M'i$ for some projective object $M'i$ of $\mathcal{C}$. If we write $M'j = Mj$ for arrows j of $\mathcal{I}$, then $M'\colon \mathcal{I} \to \mathcal{C}$ is a diagram. By (3), we have a limiting cone $(P, (p_i)_{i\in\mathcal{I}_0})$ for this diagram, where P is projective. Since Δ and Γ preserve finite limits, we see that $(\Delta\Gamma P, (\Delta\Gamma p_i)_{i\in\mathcal{I}_0})$ is a limiting cone for the diagram $\Delta\Gamma M'\colon \mathcal{I} \to \mathcal{C}$. Now the limit of M in $\mathcal{C}{\downarrow}\Delta$ can be computed as $x\colon P \to \Delta\Gamma P$, where x is the unique arrow such that $\Delta\Gamma p_i \circ x = \eta_{Mi} \circ p_i$ for all $i \in \mathcal{I}_0$. But $x = \eta_P$ fulfills this requirement, so the limit of M is equal to $\eta_P\colon P \to \Delta\Gamma P$, which is again f-projective and indecomposable. We conclude that $\mathscr{A}$ is closed under finite limits, as desired. □

Of course, there is an easier way to prove the 'only if' direction, by noticing that for a realizability topos, there exists an assembly D that actually has properties (1) – (4), instead of deriving it from Theorem 4.5.1. And in fact we *have* done this, since we have shown in Chapter 3 that (1)–(4) are satisfied if we take D to be the objects of realizers. It is instructive, nevertheless, to see roughly which properties stated in Theorem 4.5.1 correspond to which properties in Theorem 4.5.2.

Chapter 5

Slices of Realizability Topoi

In this chapter, we study slices of realizability topoi. These are not, in general, realizability topoi again. There are some some special cases in which we can say something useful, however. First, in Section 5.1, we give an explicit description of categories of the form $\mathsf{RT}(A)/(I, E)$, where A is a PCA and (I, E) is an assembly over A. Next, we take a more abstract point of view again and investigate which properties mentioned in Theorem 4.5.2 can be preserved under slicing. As long as we are slicing over projective objects, the answer turns out to be: quite much. In order to obtain our results, we must first develop some theory about limits, projective objects (Section 5.2) and sheaves (Section 5.3) in slice categories. Finally, in Section 5.4, we first present a result about slicing over projective objects (Theorem 5.4.1), and then we derive a corollary about slicing over arbitrary objects (Theorem 5.4.2).

5.1 Slicing over Assemblies

In this section, A will be an arbitrary PCA, and P will be the corresponding realizability tripos. We will use the following abbreviations.

Definition 5.1.1. (i) Let X be a set and let $\sim\in \mathsf{P}(X \times X)$. We write per($\sim$) for the sentence

$$\forall xx'(x \sim x' \to x' \sim x) \wedge \forall xx'x''(x \sim x' \wedge x' \sim x'' \to x \sim x'')$$

stating that $\sim$ is a partial equivalence relation.

Now suppose $(X, \sim_X)$ and $(Y, \sim_Y)$ are objects of $\mathsf{RT}(A)$.

(ii) For $F \in \mathsf{P}(X \times Y)$, we write funcrel($F, \sim_X, \sim_Y$) for the sentence

$$\begin{aligned} &\forall xy\,(F(x, y) \to x \sim_X x \wedge y \sim_Y y) \\ \wedge\ &\forall xx'yy'\,(F(x, y) \wedge x \sim_X x' \wedge y \sim_Y y' \to F(x', y')) \\ \wedge\ &\forall xyy'\,(F(x, y) \wedge F(x, y') \to y \sim_Y y') \\ \wedge\ &\forall x\,(x \sim_X x \to \exists y\, F(x, y)). \end{aligned}$$

stating that F is a functional relation between $(X, \sim_X)$ and $(Y, \sim_Y)$.

(iii) Suppose $F, G \in \mathsf{P}(X \times Y)$. We write $F \leq G$ for the sentence $\forall xy (F(x,y) \to G(x,y))$, stating that $F \leq G$ as $\mathcal{P}(A)$-valued predicates on $X \times Y$.

There exists an element $r \in A$ satisfying, for any $\sim_X \in \mathsf{P}(X \times X)$, $\sim_Y \in \mathsf{P}(Y \times Y)$ and $F, G \in \mathsf{P}(X \times Y)$: if

$$a \in [\text{funcrel}(F, \sim_X, \sim_Y)], b \in [\text{funcrel}(G, \sim_X, \sim_Y)] \text{ and } c \in [F \leq G],$$

then $rabc\downarrow$ and $rabc \in [G \leq F]$. This r can be contructed the Soundness Theorem and the fact that the sentence

$$(\text{funcrel}(T, R, S) \wedge \text{funcrel}(U, R, S) \wedge (T \leq U)) \to (U \leq T)$$

is intuitionistically valid (where R, S, T and U are relation symbols of the relevant type).

Consider an assembly $J = (I, E)$ of $\mathsf{RT}(A)$. Our goal is to give an explicit description of the slice topos $\mathsf{RT}(A)/J$. We will do this by defining a topos $\mathsf{RT}(A)_J$ that is equivalent to $\mathsf{RT}(A)/J$. In dealing with $\mathsf{RT}(A)/J$, the following observation will be useful. Suppose $(X, \sim) \to J$ is an object of $\mathsf{RT}(A)/J$, where $(X, \sim)$ satisfies $[x \sim x] \neq \emptyset$ for all $x \in X$. Then the arrow $(X, \sim) \to J$ must be given by a function $k \colon X \to I$ in the sense of Proposition 3.3.1. In this setting, requirement (3.5) comes down to the following two conditions.

(i) We have $[x \sim x'] = \emptyset$ whenever $k(x) \neq k(x')$.

(ii) There exists an element $a \in A$ satisfying: if $k(x) = k(x') = i$ and $b \in [x \sim x']$, then $ab\downarrow$ and $ab \in E(i)$.

The arrow $(X, \sim) \to J$ is represented by the functional relation $F_k \in \mathsf{P}(X \times I)$, which is given by

$$F_k(x, i) = \begin{cases} [x \sim x \wedge E(i)] & \text{if } k(x) = i; \\ \emptyset & \text{otherwise.} \end{cases}$$

But as we just saw, if $k(x) = i$, then an element of $E(i)$ may be computed from an element of $[x \sim x]$. So F_k is isomorphic to $K \in \mathsf{P}(X \times I)$, where

$$K(x, i) = \begin{cases} [x \sim x] & \text{if } k(x) = i; \\ \emptyset & \text{otherwise.} \end{cases}$$

This is the representation we will use when dealing with objects from $\mathsf{RT}(A)/J$.

We now start the definition of the category $\mathsf{RT}(A)_J$.

Definition 5.1.2. A *J-object* is a family $(X_i, \sim_{X_i})_{i \in I}$, where $\sim_i \in \mathsf{P}(X_i \times X_i)$, such that there exists a $u \in A$ satisfying: if $i \in I$ and $a \in E(i)$, then $ua\downarrow$ and $ua \in [\text{per}(\sim_{X_i})]$.

We notice that, if $(X_i, \sim_{X_i})_{i \in I}$ is a J-object, then $(X_i, \sim_{X_i})$ is an object of $\mathsf{RT}(A)$ for all $i \in I$. However, for $(X_i, \sim_{X_i})_{i \in I}$ to be a J-object, it is not (in general) sufficient that all $(X_i, \sim_i)$ are objects of $\mathsf{RT}(A)$. We must not only know that all $[\text{per}(\sim_{X_i})]$ are nonempty, but we must be able to compute a realizer for $\text{per}(\sim_{X_i})$ from an element of $E(i)$. We will say that the realizers of the statements $\text{per}(\sim_{X_i})$ must be given *uniformly*.

Definition 5.1.3. Let two J-objects $(X_i, \sim_{X_i})_{i\in I}$ and $(Y_i, \sim_{Y_i})_{i\in I}$ be given.

(i) A *J-functional relation* between these J-objects is a family $(F_i)_{i\in I}$ where $F_i \in \mathsf{P}(X_i \times Y_i)$ such that there exists an element $v \in A$ satisfying: if $i \in I$ and $a \in E(i)$, then $va\downarrow$ and $va \in [\text{funcrel}(F_i, \sim_{X_i}, \sim_{Y_i})]$.

(ii) If $(F_i)_{i\in I}$ and $(G_i)_{i\in I}$ are J-functional relations between $(X_i, \sim_{X_i})_{i\in I}$ and $(Y_i \sim_{Y_i})_{i\in I}$, then we write $(F_i)_{i\in I} \leq_J (G_i)_{i\in I}$ if there exists an element $w \in A$ satisfying: if $i \in I$ and $a \in E(i)$, then $wa\downarrow$ and $wa \in [F_i \leq G_i]$.

We notice that, if $(F_i)_{i\in I}$ is a J-functional relation, then all the F_i are functional relations, and, if $(F_i)_{i\in I} \leq_J (G_i)_{i\in I}$, then $F_i \leq G_i$ for all $i \in I$. Again, the converses of these statements are not true in general: the realizers must be given uniformly. Using the element $r \in A$ we introduced above, we can prove the following: if $(F_i)_{i\in I}$ and $(G_i)_{i\in I}$ are J-functional relations, $(G_i)_{i\in I} \leq_J (F_i)_{i\in I}$ as soon as $(F_i)_{i\in I} \leq_J (G_i)_{i\in I}$.

Definition 5.1.4. The category $\mathsf{RT}(A)_J$ has J-objects as objects and $\leq_J$-isomorphism classes of J-functional relations as arrows. The identity arrows and compositions are computed componentwise as one would compute them inside $\mathsf{RT}(A)$.

There are some things to check here. For example, we need to check that, for any J-object $(X_i, \sim_{X_i})_{i\in I}$, the family $(\sim_{X_i})_{i\in I}$ is a J-functional relation. This may be done using the Soundness Theorem and the fact that the sentence $\text{per}(R) \to \text{funcrel}(R, R, R)$ is intuitionistically valid. Similarly, one may check that composition in $\mathsf{RT}(A)_J$ is well-defined, and that $\mathsf{RT}(A)/J$ is actually a category. We may also show that $\mathsf{RT}(A)_J$ is equivalent to its full subcategory $\mathsf{RT}_0(A)_J$ on the J-objects $(X_i, \sim_{X_i})_{i\in I}$ such that $[x \sim_{X_i} x] \neq \emptyset$ for all $i \in I$ and $x \in X_i$.

We claim that $\mathsf{RT}(A)/J$ and $\mathsf{RT}(A)_J$ are equivalent. It turns out that it is more convenient to show that $\mathsf{RT}_0(A)$ and $\mathsf{RT}_0(A)_J$ are equivalent. So from now on, we shall tacitly assume that all objects from $\mathsf{RT}(A)$ we consider are actually from $\mathsf{RT}_0(A)$. Let a J-object $(X_i, \sim_{X_i})_{i\in I}$ be given. We define the object $(X, \sim_X)$ of $\mathsf{RT}(A)$ by $X = \bigsqcup_{i\in I} X_i = \bigcup_{i\in I}(\{i\} \times X_i)$ and

$$[(i, x) \sim_X (i', x'))] = \begin{cases} [x \sim_{X_i} x' \wedge E(i)] & \text{if } i = i'; \\ \emptyset & \text{otherwise,} \end{cases}$$

for $i, i' \in I$, $x \in X_i$ and $x' \in X_{i'}$. We claim that this is indeed an object of $\mathsf{RT}(A)$. We sketch an algorithm realizing the symmetry of $\sim_X$. If we have an element b from $[(i, x) \sim_X (i', x')]$, then $i = i'$ and we can compute an element a of $E(i)$ from b. From a, we can compute an element of $[\text{per}(\sim_{X_i})]$, hence also a realizer of the symmetry of $\sim_{X_i}$. From b, we can also compute an element of $[x \sim_{X_i} x']$. Now we have enough data to compute an element from $[x' \sim_{X_i} x] = [x' \sim_{X_{i'}} x]$. Since we had already computed an $a \in E(i) = E(i')$, we can also get an element from $[x' \sim_{X_{i'}} x \wedge E(i')] = [(i', x') \sim_X (i, x)]$, as desired. Transitivity is similar.

Clearly, the projection $\pi \colon X \to I$ yields an arrow $(X, \sim_X) \to J$ in $\mathsf{RT}(A)$ in the sense of Proposition 3.3.1. So $(X, \sim_X) \to J$ is an object of $\mathsf{RT}(A)/J$.

Now suppose we have another J-object $(Y_i, \sim_{Y_i})_{i\in I}$. We define $(Y, \sim_Y) \to J$ analogously. Suppose $(F_i)_{i\in I}$ is a J-functional relation between $(X_i, \sim_{X_i})_{i\in I}$ and $(Y_i, \sim_{Y_i})_{i\in I}$. We define

$F \in \mathsf{P}(X \times Y)$ by:

$$F(i,x,j,y) = \begin{cases} [F_i(x,y) \wedge E(i)] & \text{if } i = j; \\ \emptyset & \text{otherwise,} \end{cases}$$

for $i, j \in I$, $x \in X_i$ and $y \in X_j$. Then F is a functional relation: the strategy for proving this is similar to the proof that $\sim_X$ is symmetric we gave above. So F represents an arrow $(X, \sim_X) \to (Y, \sim_Y)$ in $\mathsf{RT}(A)$. Using the fact that $F(i,x,j,y) = \emptyset$ if $i \neq j$, one easily checks that the diagram

$$\begin{array}{ccc} (X,\sim_X) & \longrightarrow & (Y,\sim_Y) \\ & \searrow \quad \swarrow & \\ & J & \end{array}$$

commutes.

We can also show (again analogous to proving the symmetry of $\sim_X$) that the above operation is monotone w.r.t. $\leq_J$ and $\leq$. So we get an operation $B \colon \mathsf{RT}_0(A)_J \to \mathsf{RT}_0(A)/J$. It is not difficult to check that B is a functor.

Now we define a pseudoinverse of B. Let an object $(X, \sim_X) \to J$ of $\mathsf{RT}(A)/J$ be given. As we say above, the arrow $(X, \sim_X) \to J$ must be given by a function $k \colon X \to I$, and our arrow is represented by the functional relation $K \in \mathsf{P}(X \times I)$ given by

$$K(x,i) = \begin{cases} [x \sim_X x] & \text{if } k(x) = i; \\ \emptyset & \text{otherwise.} \end{cases}$$

For $i \in I$, we write $X_i = k^{-1}(i)$. Moreover, we write $\sim_{X_i}$ for the restriction of $\sim_X$ to $X_i \times X_i$. Any element of $[\mathrm{per}(\sim_X)]$ is also an element of $[\mathrm{per}(\sim_{X_i})]$, for arbitrary $i \in I$. This means that $(X_i, \sim_{X_i})_{i \in I}$ is a J-object, the element u being $\mathsf{k}b$ for a fixed $b \in [\mathrm{per}(\sim_X)]$.

Now let $(Y, \sim_Y) \to J$ be another arrow of $\mathsf{RT}(A)$, given by the function $l \colon Y \to I$. That is, this arrow is represented by $L \in \mathsf{P}(Y \times I)$ where $L(y,i) = [y \sim_Y y]$ if $l(y) = i$, and $L(y,i) = \emptyset$ otherwise. We define the J-object $(Y_i, \sim_{Y_i})_{i \in I}$ as above. Now suppose we have a functional relation $F \in \mathsf{P}(X \times Y)$ such that the diagram

$$\begin{array}{ccc} (X,\sim_X) & \longrightarrow & (Y,\sim_Y) \\ & \searrow \quad \swarrow & \\ & J & \end{array}$$

commutes. Then K should be ismorphic to the functional relation

$$(x,i) \mapsto \bigcup_{y \in Y} F(x,y) \wedge L(y,i) = \bigcup_{l(y)=i} F(x,y) \wedge [y \sim_Y y].$$

Suppose we have elements $i, j \in I$, and elements $x \in X_i$ and $y \in Y_j$ such that $F(x,y) \neq \emptyset$. Then $l(y) = j$ and $F(x,y) \wedge [y \sim_Y y]$ is nonempty. This means that $K(x,j)$ is nonempty as well. However, $K(x,i)$ is also nonempty, so we must have $i = j$. In other words, if $i \neq j$, then F restricted to $X_i \times Y_j$ always outputs the empty set. If we write F_i for the restriction of F to $X_i \times Y_i$, then F_i is a functional relation between $(X_i, \sim_{X_i})$ and $(Y_i, \sim_{Y_i})$ for all $i \in I$. In fact, every element of $[\text{funcrel}(F, \sim_X, \sim_Y)]$ will also be an element of $[\text{funcrel}(F_i, \sim_{X_i}, \sim_{Y_i})]$, for arbitrary $i \in I$. This allows us to see that $(F_i)_{i\in I}$ is a J-functional relation between $(X_i, \sim_{X_i})_{i\in I}$ and $(Y_i, \sim_{Y_i})_{i\in I}$. Moreover, if $G \in \mathsf{P}(X \times Y)$ is another functional relation, then any element of $[F \leq G]$ will also be an element of $[F_i \leq G_i]$, for arbitrary $i \in I$. So we see that the above construction is monotone w.r.t. $\leq$ and $\leq_J$, meaning that we have an operation $C \colon \mathsf{RT}_0(A)/J \to \mathsf{RT}_0(A)_J$. It is not difficult to see that C is in fact a functor.

It remains to see that the two functors we defined are pseusoinverses. Consider an arrow $(X, \sim_X) \to J$ given by the function $k \colon X \to I$, and let K the functional relation that represents this arrow. Write $C((X, \sim_X) \to J) = (X_i, \sim_{X_i})_{i\in I}$, so $X_i = k^{-1}(i)$ and $\sim_{X_i}$ is $\sim_X$ restricted to $X_i \times X_i$. We have that $BC((X, \sim_X) \to J)$ is the object $(\bigsqcup_{i\in I} X_i, \approx) \to J$ where

$$[(i, x) \approx (i', x')] = \begin{cases} [x \sim_X x' \wedge E(i)] & \text{if } i = i'; \\ \emptyset & \text{otherwise.} \end{cases}$$

Since $[x \sim_X x'] = \emptyset$ if $k(x) \neq k(x')$, we may also write this as

$$[(i, x) \approx (i', x')] = [x \sim_X x' \wedge E(i)].$$

The arrow $(\bigsqcup_{i\in I} X_i, \approx) \to J$ is given by the projection $\pi \colon \bigsqcup_{i\in I} X_i \to I$.

We can now show that $S \in \mathsf{P}(X \times \bigsqcup_{i\in I} X_i)$ given by

$$S(x_0, i, x_1) = [x_0 \sim_X x_1] \quad \text{for} \quad x_0 \in X, x_1 \in X_i$$

is a functional relation, and that the arrow $s \colon (X, \sim) \to (\bigsqcup_{i\in I} X_i, \approx)$ given by S is an isomorphism. Using the fact that $S(x_0, i, x_1) = \emptyset$ if $x_0 \notin X_i$, it is easy to check that s makes the diagram

$$\begin{array}{ccc} (X, \sim) & \xrightarrow{\quad s \quad} & (\bigsqcup_{i\in I} X_i, \approx) \\ & \searrow \quad \swarrow & \\ & J & \end{array}$$

commute. Suppose that $(Y, \sim) \to J$ is another object of $\mathsf{RT}(A)/J$, where the arrow is given by the function $l \colon Y \to I$. Write $BC((Y, \sim) \to J)$ as $(\bigsqcup_{i\in I} Y_i, \approx)$ as we did above. Suppose also that F is a functional relation between X and Y. Then $BC(F)$ is the functional relation $F' \in \mathsf{P}(\bigsqcup_{i\in I} X_i \times \bigsqcup_{i\in I} Y_i)$ given by

$$F'(i, x, j, y) = \begin{cases} [F(x, y) \wedge E(i)] & \text{if } i = j; \\ \emptyset & \text{otherwise,} \end{cases}$$

for $x \in X_i$ and $y \in Y_j$. Recalling that $F(x, y) = \emptyset$ if $k(x) \neq l(y)$, we see that we have $F'(i, x, j, y) = [F(x, y) \wedge E(i)]$ for all $x \in X_i$ and $y \in Y_j$. If $x \in X_i$, then from an element of $F(x, y)$, we can compute an element of $[x \sim_X x]$, from which we can compute an element of $E(i)$. So F' is isomorphic to F'', where $F''(i, x, j, y) = F(x, y)$. Using this description, it is easy to see that the isomorphisms we constructed are natural.

Now let $(X_i, \sim_{X_i})_{i\in I}$ be a J-object. Then $CB(X_i, \sim_{X_i})_{i\in I}$ is the J-object $(\{i\} \times X_i, \approx_{X_i})_{i\in I}$, where $[(i, x) \approx_{X_i} (i, x')] = [x \sim_{X_i} x' \wedge E(i)]$. For all $i \in I$, we have that $S_i \in \mathsf{P}(X_i \times \{i\} \times X_i)$ given by

$$S_i(x_0, i, x_1) = [x_0 \sim_{X_i} x_1] \quad \text{for} \quad x_0, x_1 \in X_i$$

is a functional relation. In fact, the information one needs to construct an element of $[\text{funcrel}(S_i, \sim_{X_i}, \approx_{X_i})]$ is an element of $[\text{per}(\sim_{X_i})]$ and some fixed element of $E(i)$. Since $(X_i, \sim_{X_i})_{i\in I}$ is a J-object, we see that an element of $[\text{funcrel}(S_i, \sim_{X_i}, \approx_{X_i})]$ may be computed from just some fixed element of $E(i)$. That is, $(S_i)_{i\in I}$ is a J-functional relation. In fact, it is an isomorphism in $\mathsf{RT}(A)_J$: its inverse is given by $(T_i)_{i\in I}$, where $T_i \in \mathsf{P}(\{i\} \times X_i \times X_i)$ is given by

$$T_i(i, x_0, x_1) = [x_0 \sim_{X_i} x_1] \quad \text{for} \quad x_0, x_1 \in X_i.$$

The proof that $(T_i)_{i\in I}$ is a J-functional relation is similar to the proof that $(S_i)_{i\in I}$ is.

Suppose $(F_i)_{i\in I}$ is a J-functional relation between $(X_i, \sim_{X_i})_{i\in I}$ and $(Y_i, \sim_{Y_i})_{i\in I}$. Then $CB(F_i)_{i\in I}$ is the J-functional relation $(F'_i)_{i\in I}$, where $F'_i \in \mathsf{P}(\{i\} \times X_i \times \{i\} \times Y_i)$ is given by

$$F'_i(i, x, i, y) = [F_i(x, y) \wedge E(i)].$$

This functional relation is J-isomorphic to $(F''_i)_{i\in I}$, where

$$F''_i(i, x, i, y) = F_i(x, y).$$

Using this description, it is again easy to see that the isomorphisms we constructed are natural. This completes the proof that B and C are pseudoinverses, so $\mathsf{RT}(A)/J$ and $\mathsf{RT}(A)_J$ are indeed equivalent.

Example 5.1.5. Suppose J is a partitioned assembly. Then $E(i)$ is a singleton for all $i \in I$. Let us write $E(i) = \{a_i\}$. A J-object is now a family $(X_i, \sim_{X_i})_{i\in I}$ where $\sim_{X_i} \in \mathsf{P}(X_i \times X_i)$, such that there exists a $u \in A$ satisfying: $ua_i\downarrow$ and $ua_i \in [\text{per}(\sim_{X_i})]$ for all $i \in I$. Similarly, a J-functional relation is a family $(F_i)_{i\in I}$ where $F_i \in \mathsf{P}(X_i \times Y_i)$, such that there exists a $v \in A$ satisfying: $va_i\downarrow$ and $va_i \in [\text{funcrel}(F_i, \sim_{X_i}, \sim_{Y_i})]$ for all $i \in I$. Finally, $(F_i)_{i\in I} \leq_J (G_i)_{i\in I}$ means that there exists a $w \in A$ satisfying: $wa_i\downarrow$ and $wa_i \in [F_i \leq G_i]$ for all $i \in I$. ◊

Example 5.1.6. We consider two projective assemblies with $|I| = 2$.

(i) Let 1 be the terminal object of $\mathsf{RT}(A)$. First, consider the assembly $1 \sqcup 1$. We write it as $(2, E)$, where $E(0) = \{\mathsf{k}\}$ and $E(1) = \{\bar{\mathsf{k}}\}$. If $(X_0, \sim_{X_0})$ and $(X_1, \sim_{X_1})$ are both objects of $\mathsf{RT}(A)$, then $(X_0, \sim_{X_0}, X_1, \sim_{X_1})$ is automatically a $(1 \sqcup 1)$-object. Indeed, the fact that $(X_i, \sim_{X_i})$ is an element of $\mathsf{RT}(A)$ means that there is en element b_i of $[\text{per}(\sim_{X_i})]$, and we can always construct a $u \in A$ such that $u\mathsf{k} = b_0$ and $u\bar{\mathsf{k}} = b_1$. Similarly, if F_0 and F_1 are functional relations, then (F_0, F_1) is automatically a $(1 \sqcup 1)$-functional relation, and if $F_0 \leq G_0$ and $F_1 \leq G_1$, then automatically $(F_0, F_1) \leq_{1\sqcup 1} (G_0, G_1)$.

In other words, the category $\mathsf{RT}(A)_{1\sqcup 1}$ is just $\mathsf{RT}(A)^2$. So we see that $\mathsf{RT}(A)/(1 \sqcup 1)$ is equivalent to $\mathsf{RT}(A)^2$. This should be no surprise, since $\mathcal{E}/(1 \sqcup 1) \cong \mathcal{E}^2$ holds for every topos $\mathcal{E}$.

This example generalizes to arbitrary *finite* coproducts of 1.

(ii) Now consider ths assembly $\Delta 2$. A $\Delta 2$-object is a quadruple $(X_0, \sim_{X_0}, X_1, \sim_{X_1})$ where $\sim_{X_i} \in \mathsf{P}(X_i \times X_i)$ for $i = 0, 1$, and $[\text{per}(\sim_{X_0})] \cap [\text{per}(\sim_{X_1})] \neq \emptyset$. A $\Delta 2$-functional relation between the objects $(X_0, \sim_{X_0}, X_1, \sim_{X_1})$ and $(Y_0, \sim_{Y_0}, Y_1, \sim_{Y_1})$ is a pair (F_0, F_1) where $F_i \in \mathsf{P}(X_i \times Y_i)$ for $i = 0, 1$, and $[\text{funcrel}(F_0, \sim_{X_0}, \sim_{Y_0})] \cap [\text{funcrel}(F_1, \sim_{X_1}, \sim_{Y_1})] \neq \emptyset$. Furthermore, we say that $(F_0, F_1) \leq_{\Delta 2} (G_0, G_1)$ iff $[F_0 \leq G_0] \cap [F_1 \leq G_1] \neq \emptyset$. The arrows of $\mathsf{RT}(A)_{\Delta 2}$ are the $\leq_{\Delta 2}$-isomorphism classes of $\Delta 2$-functional relations. Informally, we may say that $\mathsf{RT}(A)_{\Delta 2}$ differs from $\mathsf{RT}(A)^2$ in the respect that 'all tracking needs to happen simultaneously'.

This example generalizes to ΔB for an arbitrary set B. ♢

Another object we may consider is the object of realizers of $\mathsf{RT}(A)$. Like finite coproducts of 1, this is a discrete projective object. We may therefore guess that the slice of $\mathsf{RT}(A)$ over its object of realizers is equivalent to $\mathsf{RT}(A)^A$. This, however, is not the case. First we prove a lemma, which is a generalization of Proposition 3.2.3 from [6].

Lemma 5.1.7. *Let A be a nontrivial PCA and let $(X_a, \sim_{X_a})_{a\in A}$ be an A-indexed family of objects of $\mathsf{RT}(A)$ such that $(X_a, \sim_{X_a})$ is noninitial for every $a \in A$. Then the coproduct $\coprod_{a\in A}(X_a, \sim_{X_a})$ does not exist in $\mathsf{RT}(A)$.*

Proof. Again, we consider the discrete assembly $1 \sqcup 1$. First, we show that there exist at most $|A|$ arrows in $\mathsf{RT}(A)$ from any given object $(X, \sim)$ of $\mathsf{RT}(A)$ to $1 \sqcup 1$. Assume that $[x \sim_X x] \neq \emptyset$ for all $x \in X$. We know from Lemma 3.3.2(i) that any arrow $f : (X, \sim_X) \to 1 \sqcup 1$ is given by a (unique) function $\tilde{f} : X \to 2$ in the sense of Proposition 3.3.1. Using the Axiom of Choice, we can select, for each f, an $a_f \in A$ such that: for all $x \in X$ and $b \in [x \sim_X x]$, we have that $a_f b\downarrow$, $a_f b = \mathsf{k}$ if $\tilde{f}(x) = 0$, and $a_f b = \overline{\mathsf{k}}$ if $\tilde{f}(x) = 1$. Since A is nontrivial, we have $\mathsf{k} \neq \overline{\mathsf{k}}$, so we must also have $a_f \neq a_g$ for distinct arrows $f, g : (X, \sim_X) \to 1 \sqcup 1$. That is, we have an injective function $\mathsf{RT}(A)((X, \sim_X), 1 \sqcup 1) \to A$, as desired.
Now consider an $a \in A$. We assume without loss of generality that $[x \sim_{X_a} x] \neq \emptyset$ for all $x \in X_a$. Then the fact that $(X_a, \sim_{X_a})$ is noninitial means that $X_a \neq \emptyset$. So the two functions $X_a \to 2$ taking the constant values 0 and 1 are distinct, which also yields two distinct arrows $(X_a, \sim_{X_a}) \to 1 \sqcup 1$. Now, if the coproduct $\coprod_{a\in A}(X_a, \sim_{X_a})$ were to exist, then it would allow at least $2^{|A|}$ arrows $\coprod_{a\in A}(X_a, \sim_{X_a}) \to 1 \sqcup 1$, which contradicts the above. □

Proposition 5.1.8. *Let A be a nontrivial PCA. Then $\mathsf{RT}(A)^A$ is not equivalent to a slice of $\mathsf{RT}(A)$.*

Proof. Suppose $\mathsf{RT}(A)^A$ is equivalent to the slice topos $\mathsf{RT}(A)/(X, \sim)$. We have the adjunction $\mathrm{dom} \dashv (X, \sim)^*$ between $\mathsf{RT}(A)$ and $\mathsf{RT}(A)/(A, \sim)$. We notice that $(X, \sim)^*$ preserves the initial object of $\mathsf{RT}(A)$. This means that we get an adjunction $\mathsf{RT}(A) \underset{R}{\overset{L}{\leftrightarrows}} \mathsf{RT}(A)^A$ such that R preserves the initial object. Write 0 for the initial object of $\mathsf{RT}(A)$. For $a, b \in A$, we define $D_{a,b} = 1$ if $a = b$, and $D_{a,b} = 0$ if $a \neq b$. For $a \in A$, we define the object $D_a = (D_{a,b})_{b\in A}$ of $\mathsf{RT}(A)^A$. Then all the D_a are noninitial, and the coproduct $\coprod_{a\in A} D_a$ exists in $\mathsf{RT}(A)^A$ and is equal to $(1)_{a\in A}$. Suppose that LD_a is initial for a certain $a \in A$. Then we have an arrow $LD_a \to 0$, so we also get an arrow $D_a \to R0$. However, D_a is noninitial, while $R0$ is initial, which is a contradiction. So all the LD_a are noninitial. Since L preserves all colimits, we see that the coproduct $\coprod_{a\in A} LD_a$ exists in $\mathsf{RT}(A)$, which contradicts the previous lemma. □

Example 5.1.9. Let N be the object of realizers of Eff. Then Eff/N is not equivalent to $\mathsf{Eff}^{\mathbb{N}}$. Using the fact that there exist nonrecursive functions $\mathbb{N} \to \mathbb{N}$, we can construct objects $(X_n, \sim_{X_n})$ of Eff for $n \in \mathbb{N}$, such that $(X_n, \sim_{X_n})_{n\in\mathbb{N}}$ is not an N-object. ♢

Example 5.1.10. Let A be Scott's graph model $\mathcal{P}\mathbb{N}$ from Example 2.3.3. The natural numbers object N of $\mathsf{RT}(\mathcal{P}\mathbb{N})$ is the assembly $(\mathbb{N}, E)$, where $E(n) = \{\{n\}\}$ for all $n \in \mathbb{N}$. Now suppose that for each $n \in \mathbb{N}$, we have objects $(X_n, \sim_{X_n})$ of $\mathsf{RT}(\mathcal{P}\mathbb{N})$. Then $(X_n, \sim_n)_{n\in\mathbb{N}}$ is automatically an N-object. Indeed, since $(X_n, \sim_{X_n})$ is an object of $\mathsf{RT}(\mathcal{P}\mathbb{N})$, we have an element $A_n \in \mathcal{P}\mathbb{N}$ of $[\mathrm{per}(\sim_{X_n})]$. The assignment $\{n\} \mapsto A_n$ can always be extended to a continuous function $\mathcal{P}\mathbb{N} \to \mathcal{P}\mathbb{N}$. So there exists a $U \in \mathcal{P}\mathbb{N}$ such that $U\{n\} = A_n \in [\mathrm{per}(\sim_{X_n})]$

for all $n \in \mathbb{N}$, as desired. Using similar arguments for N-functional relations and $\leq_N$, we can show that the topos $\mathsf{RT}(\mathcal{P}\mathbb{N})_N$ is just $\mathsf{RT}(\mathcal{P}\mathbb{N})^{\mathbb{N}}$. So $\mathsf{RT}(\mathcal{P}\mathbb{N})/N$ is equivalent to $\mathsf{RT}(\mathcal{P}\mathbb{N})^{\mathbb{N}}$. $\lozenge$

5.2 Limits and Projectives in a Slice

As promised, we now take a more abstract point of view again. In particular, 'A' will typically stand for an arbitrary set, and not for a PCA.

Projective objects play a crucial role in Theorem 4.5.2. Since we are interested in slices of realizability topoi, we investigate how projective objects behave under slicing. We first need to know how limits behave under slicing. In general, the domain functor $\mathrm{dom}\colon \mathcal{C}/X \to \mathcal{C}$ does not preserve or reflect limits. For example, products in $\mathcal{C}/X$ are constructed as pullbacks in $\mathcal{C}$. There is, however, a special kind of limit that does behave nicely under slicing.

Proposition 5.2.1. *Let $\mathcal{C}$ be a category and let X be an object of $\mathcal{C}$. Suppose $\mathcal{I}$ is a category for which there exists an object t of $\mathcal{I}$ such that $\mathcal{I}(i,t) \neq \emptyset$ for all objects i of $\mathcal{I}$. Then the domain functor $\mathcal{C}/X \to \mathcal{C}$ preserves and reflects all limits of type $\mathcal{I}$.*

Proof. The proof is elementary category theory, and we leave it to the reader. $\square$

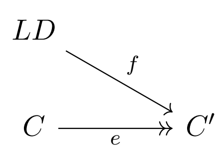
In particular, equalizers and pullbacks are limits of the type referred to above. The following lemma was suggested by J. van Oosten.

Proposition 5.2.2. *Let $L \dashv R$ be an adjunction with unit η, where $\mathcal{C} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{D}$.*

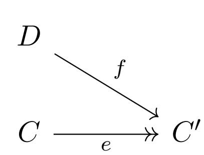
(i) If R preserves regular epis, then L preserves projective objects.

(ii) If the naturality square for η is always a pullback, then L reflects projective objects.

Proof. (i) Let D be a projective object of $\mathcal{D}$, and let arrows



be given. Since e is a regular epi, we have that $Re\colon RC \to RC'$ is also a regular epi. Let $\tilde{f}\colon D \to RC'$ be the transpose of f. Since D is projective, there exists an arrow $g\colon D \to RC$ such that $Re \circ g = \tilde{f}$. Let $\hat{g}\colon LD \to C$ be the transpose of g. Transposing the equality $Re \circ g = \tilde{f}$ yields $e\hat{g} = f$, as desired.
(ii) Let D be an object of $\mathcal{D}$ such that LD is projective, and let arrows



be given. Because L is a left adjoint, it preserves all colimits. This means that Le is a regular epi as well, so since LD is projective, there exists an arrow $g\colon LD \to LC$ such that

$Le \circ g = Lf$. Let $\tilde{g}\colon D \to RLC$ be the transpose of g. If ε is the counit of the adjunction, then we find

$$\begin{aligned}\varepsilon_{LC'} \circ L(RLe \circ \tilde{g}) &= \varepsilon_{LC'} \circ LRLe \circ L\tilde{g} = Le \circ \varepsilon_{LC} \circ L\tilde{g} = Le \circ g = Lf \\ &= \mathrm{id}_{LC'} \circ Lf = \varepsilon_{LC'} \circ L\eta_{C'} \circ Lf = \varepsilon_{LC'} \circ L(\eta_{C'} \circ f).\end{aligned}$$

By the universal property of the adjunction, we get $RLe \circ \tilde{g} = \eta_{C'} \circ f$. Since the naturality square for e and η is a pullback diagram, there exists an arrow $k\colon D \to C$ such that $\eta_C \circ k = \tilde{g}$ and $ek = f$. We conclude that D is projective as well, as desired. □

Let us apply this to the current situation.

Corollary 5.2.3. *Let $\mathcal{C}$ be a category with binary products, and let X be an object of $\mathcal{C}$. Then an object (C,c) of $\mathcal{C}/X$ is projective in $\mathcal{C}/X$ if and only if C is projective in $\mathcal{C}$.*

Proof. Recall from the proof of Proposition 4.2.3 that there exists an adjunction $\mathrm{dom} \dashv X^*$ whose unit η is given by $\eta_{(C,c)} = \langle \mathrm{id}_C, c\rangle\colon (C,c) \to (C \times X, \pi_2)$ for objects (C,c) of $\mathcal{C}/X$. We have to show that dom preserves and reflects projectives. Clearly, X^* preserves regular epis. Indeed, if e is a regular epi, then $X^*e = e \times \mathrm{id}_X$ is a regular epi in $\mathcal{C}$, hence in $\mathcal{C}/X$, by Proposition 4.2.3. By part (i) of the above proposition, dom preserves projectives. To see that dom also reflects projectives, let $f\colon (C,c) \to (D,d)$ be a morphism in $\mathcal{C}/X$. We show that the commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\;f\;} & D \\ {\scriptstyle \langle \mathrm{id}_C, c\rangle}\downarrow & & \downarrow{\scriptstyle \langle \mathrm{id}_D, d\rangle} \\ C \times X & \xrightarrow[f \times \mathrm{id}_X]{} & D \times X \end{array}$$

is a pullback diagram in $\mathcal{C}$. So let E be an object of $\mathcal{C}$ and let $\langle g_0, g_1\rangle\colon E \to C \times X$ and $h\colon E \to D$ be given such that

$$\langle fg_0, g_1\rangle = (f \times \mathrm{id}_X) \circ \langle g_0, g_1\rangle = \langle \mathrm{id}_D, d\rangle \circ h = \langle h, dh\rangle.$$

In other words, we have $fg_0 = h$ and $g_1 = dh$. We want to find a mediating arrow $x\colon E \to C$ such that $fx = h$ and

$$\langle g_0, g_1\rangle = \langle \mathrm{id}_C, c\rangle \circ x = \langle x, cx\rangle.$$

The latter is equivalent to $g_0 = x$ and $g_1 = cx$. Clearly, there is at most one choice for x, namely g_0. We already know that $fg_0 = h$. Furthermore, we have

$$cg_0 = dfg_0 = dh = g_1,$$

so the above diagram is indeed a pullback. Using Proposition 5.2.1, we see that the naturality square for f and η is a pullback diagram in $\mathcal{C}/X$. By part (ii) of the above proposition, dom also reflects projectives. □

Using Corollary 5.2.3, we get the following two results.

Corollary 5.2.4. *Let $\mathcal{C}$ be a category with binary products, and let X be an object of X. If $\mathcal{C}$ has enough projectives, then so does $\mathcal{C}/X$.*

Proof. Let (Y, y) be an object of $\mathcal{C}/X$. Then there must be a projective object P of $\mathcal{C}$ and a regular epi $e \colon P \twoheadrightarrow Y$. By Corollary 5.2.3, (P, ye) is projective in $\mathcal{C}/X$, and by Proposition 4.2.3, the arrow $e \colon (P, ye) \to (Y, y)$ is also a regular epi in $\mathcal{C}/X$. We conclude that $\mathcal{C}/X$ has enough projectives. □

Corollary 5.2.5. *Suppose that $\mathcal{C}$ is a category with finite limits, such that the projective objects of $\mathcal{C}$ are closed under finite limits. Let X be an object of $\mathcal{C}$. Then the projective objects of $\mathcal{C}/X$ (which has finite limits) are closed under finite limits if and only if X is projective.*

Proof. It follows from Proposition 5.2.1 and Corollary 5.2.3 that the projective objects of $\mathcal{C}/X$ are closed under taking pullbacks. Now the result follows by observing that the terminal object (X, id_X) of $\mathcal{C}/X$ is projective in $\mathcal{C}/X$ if and only if X is projective in $\mathcal{C}$. □

5.3 Slicing and sheaves

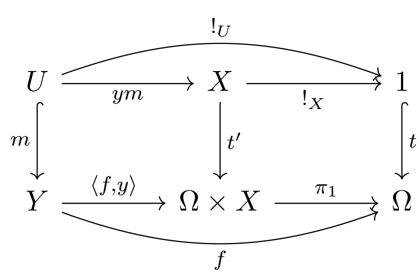
As we have seen in the previous section, item (4) of Theorem 4.5.2 is preserved fully under slicing, while item (3) is preserved partially. We now investigate what other properties from Theorem 4.5.2 we can preserve. In this theorem, the global sections functor Γ plays a major role. For slices, the global sections functor behaves less nicely. Suppose that (Y, y) is an object of $\mathcal{C}/X$. Then a global section of this object is an arrow $f \colon X \to Y$ in $\mathcal{C}$ such that $yf = \mathrm{id}_X$. In particular, the arrow y must be split epi in $\mathcal{C}$. This means that the set of global sections may be empty for many objects of $\mathcal{C}/X$. On the other hand, in a realizability topos, there is (up to isomorphism) only one object without global sections, namely the initial object.

We recall that there is another way in which we can describe the functors Γ and Δ in the case of a realizability topos: they form (up to equivalence) the geometric inclusion of the full subtopos of $\neg\neg$-sheaves of the realizability topos. In this section, we will study the behaviour of sheaves under slicing.

Let $\mathcal{E}$ be a topos with subobject classifier Ω and truth arrow $t \colon 1 \to \Omega$, and let X be an object of $\mathcal{E}$. First, we describe the subobject classifier of the slice topos $\mathcal{E}/X$ explicitly.

Proposition 5.3.1. *Let X be an object of $\mathcal{E}$. Then $t' = \langle t \circ !_X, \mathrm{id}_X \rangle \colon (X, \mathrm{id}_X) \to (\Omega \times X, \pi_2)$ is a subobject classifier in $\mathcal{E}/X$.*

Proof. Let $m \colon (U, ym) \hookrightarrow (Y, y)$ be a subobject in $\mathcal{E}/X$. Then $m \colon U \hookrightarrow Y$ is a subobject in $\mathcal{E}$. We notice that any map $(Y, y) \to (\Omega \times X, \pi_2)$ in $\mathcal{E}/X$ must be of the form $\langle f, y \rangle$, where $f \colon Y \to \Omega$. Now let $f \colon Y \to \Omega$ be any arrow and consider the diagram



It is easy to verify that the right hand square is a pullback diagram. In particular, t' is mono. If the left hand square is a pullback diagram, then the composite square is a pullback diagram as well, by the Pullback Lemma. Conversely, suppose that the composite square is a pullback diagram. Then in particular, it commutes, and this implies that the left hand square commutes as well. Again by the Pullback Lemma, the left hand square must also be a pullback diagram. So we see that the left hand square is a pullback diagram iff the composite square is. By assumption, there exists a unique arrow $f\colon Y \to \Omega$ such that the composite square is a pullback. We conclude that there exists a unique arrow $f\colon Y \to \Omega$ such that the left hand square is a pullback diagram, as desired. □

Remark 5.3.2. From this proof, we find an explicit description of the characteristic arrow of m: it is equal to $\langle \chi_m, y \rangle$, where χ_m is the characteristic arrow of m in $\mathcal{E}$. ◊

It turns out that every Lawvere-Tierney topology on $\mathcal{E}$ yields a topology on the slice $\mathcal{E}/X$.

Proposition 5.3.3. *Suppose $j\colon \Omega \to \Omega$ is a Lawvere-Tierney topology on $\mathcal{E}$. Then*

$$j \times \mathrm{id}_X\colon (\Omega \times X, \pi_2) \to (\Omega \times X, \pi_2)$$

is a Lawvere-Tierney topology on $\mathcal{E}/X$.

Proof. The first two requirements for a Lawvere-Tierney topology follow easily from the corresponding facts for j. We notice that

$$\begin{array}{ccc} \Omega \times \Omega \times X & \xrightarrow{\langle \pi_1, \pi_3 \rangle} & \Omega \times X \\ {\scriptstyle \langle \pi_2, \pi_3 \rangle}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \Omega \times X & \xrightarrow[\pi_2]{} & X \end{array}$$

is a pullback diagram in $\mathcal{E}$. This means that $(\Omega \times \Omega \times X, \pi_3)$ with the displayed projections is a product of $(\Omega \times X, \pi_2)$ with itself in $\mathcal{E}/X$. It is not difficult to check that, with this representation, the subobject $\langle t', t' \rangle\colon (X, \mathrm{id}_X) \hookrightarrow (\Omega \times \Omega \times X, \pi_3)$ is equal to $\langle t \circ !_X, t \circ !_X, \mathrm{id}_X \rangle$. Its characteristic arrow inside $\mathcal{E}$ is the composition $\Omega \times \Omega \times X \to \Omega \times \Omega \xrightarrow{\wedge} \Omega$, where the first arrow is a projection. Using Remark Remark 5.3.2, we see that its classifying arrow inside $\mathcal{E}/X$ is equal to $\wedge \times \mathrm{id}_X\colon (\Omega \times \Omega \times X, \pi_3) \to (\Omega \times X, \pi_2)$. This is the conjunction operation of $\mathcal{E}/X$. Furthermore, the product of $j \times \mathrm{id}_X$ with itself in $\mathcal{E}/X$ is

$$j \times j \times \mathrm{id}_X\colon (\Omega \times \Omega \times X, \pi_3) \to (\Omega \times \Omega \times X, \pi_3).$$

Now the third requirement for a Lawvere-Tierney topology follows immediately from the corresponding fact for j. □

The closure operations associated to these Lawvere-Tierney topologies happen to coincide.

Proposition 5.3.4. *Suppose $j\colon \Omega \to \Omega$ is a Lawvere-Tierney topology on $\mathcal{E}$. Let $m\colon (U, ym) \hookrightarrow (Y, y)$ be a subobject in $\mathcal{E}/X$. Then its closure w.r.t. $j \times \mathrm{id}_X$ is equal to $\bar{m}\colon (\bar{U}, y\bar{m}) \hookrightarrow (Y, y)$, where $\bar{m}\colon \bar{U} \hookrightarrow Y$ is the closure of $m\colon U \hookrightarrow Y$ w.r.t. j.*

Proof. Let χ_m classify m inside $\mathcal{E}$. Then $\langle \chi_m, y \rangle$ classifies m inside $\mathcal{E}/X$. By definition, the closure of m w.r.t. $j \times \mathrm{id}_X$ is classified, in $\mathcal{E}/X$, by $\langle \chi_m, y \rangle \circ (j \times \mathrm{id}_X) = \langle \chi_m j, y \rangle$. So in $\mathcal{E}$, this closure is classified by $\chi_m j$, so it must indeed be $\bar{m}\colon \bar{U} \hookrightarrow Y$. □

Using this proposition, we can give a description of $(j \times \mathrm{id}_X)$-sheaves completely in terms of the topos $\mathcal{E}$ and the topology j. An object (F, f) of $\mathcal{E}/X$ is a sheaf if and only if, for every commutative square of the form

$$\begin{array}{ccc} U & \xrightarrow{a} & F \\ {\scriptstyle m}\big\downarrow & \overset{b}{\dashrightarrow} & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow[y]{} & X \end{array}$$

where m is a (j-)dense subobject, there exists a unique $b\colon Y \to F$ making the two triangles commute.

Example 5.3.5. Let $j = \neg\neg$ be the double negation topology on $\mathcal{E}$. Here $\neg\colon \Omega \to \Omega$ is the characteristic arrow of the falsity arrow $f\colon 1 \hookrightarrow \Omega$. And f, in its turn, is the characteristic arrow of the empty subobject $0 \hookrightarrow 1$.
The empty subobject $0 \hookrightarrow X$ is characterized by $f \circ !_X\colon X \to \Omega$. This means that the falsity arrow of $\mathcal{E}/X$ is equal to $\langle f \circ !_X, \mathrm{id}_X\rangle\colon (X, \mathrm{id}_X) \hookrightarrow (\Omega \times X, \pi_2)$. In $\mathcal{E}$, the subobject $\langle f \circ !_X, \mathrm{id}_X\rangle\colon X \hookrightarrow \Omega \times X$ is characterized by $\neg \circ \pi_1\colon \Omega \times X \to \Omega$. We see that in $\mathcal{E}/X$, the falsity arrow is characterized by $\langle \neg \circ \pi_1, \pi_2\rangle = \neg \times \mathrm{id}_X\colon (\Omega \times X, \pi_2) \to (\Omega \times X, \pi_2)$. So $\neg\neg \times \mathrm{id}_X = (\neg \times \mathrm{id}_X) \circ (\neg \times \mathrm{id}_X)$ is the double negation topology on $\mathcal{E}/X$. $\Diamond$

It is well-known that, given a Lawvere-Tierney topology, there exists a geometric inclusion $(L \dashv i)\colon \mathsf{Sh}_j(\mathcal{E}) \to \mathcal{E}$. The functor i is the inclusion of the full subcategory of j-sheaves into $\mathcal{E}$ itself, while L is known as the *sheafification functor*.

Proposition 5.3.6. *Let $j\colon \Omega \to \Omega$ be a Lawvere-Tierney topology on $\mathcal{E}$ and write L for the sheafification functor. Then there exists a geometric inclusion*

$$\mathsf{Sh}_j(\mathcal{E})/LX \to \mathsf{Sh}_{j \times \mathrm{id}_X}(\mathcal{E}/X).$$

Proof. Let η and ε be the unit resp. counit of the adjunction $L \dashv i$. The adjunction $L \dashv i$ induces an adjunction $\mathsf{Sh}_j(\mathcal{E})/LX \underset{i^*}{\overset{L^*}{\leftrightarrows}} \mathcal{E}/iLX$. The functor $L^*\colon \mathcal{E}/iLX \to \mathsf{Sh}_j(\mathcal{E})/LX$ sends an object $y\colon Y \to iLX$ to its transpose $\tilde{y}\colon LY \to LX$, and sends an arrow f to Lf. The functor $i^*\colon \mathsf{Sh}_j(\mathcal{E})/LX \to \mathcal{E}/iLX$ sends an object $f\colon F \to LX$ to $if\colon iF \to iLX$, and sends an arrow f to if.
We also have an adjunction $\mathcal{E}/iLX \underset{\eta_X^*}{\overset{\Sigma_{\eta_X}}{\leftrightarrows}} \mathcal{E}/X$. Here Σ_{η_X} is just composition with η_X and η_X^* is pullback along η_X. We can compose these adjunctions:

$$\mathsf{Sh}_j(\mathcal{E})/LX \underset{i^*}{\overset{L^*}{\leftrightarrows}} \mathcal{E}/iLX \underset{\eta_X^*}{\overset{\Sigma_{\eta_X}}{\leftrightarrows}} \mathcal{E}/X$$

We write $L' = L^* \circ \Sigma_{\eta_X}$ and $i' = \eta_X^* \circ i^*$. We notice that L' sends an object $y\colon Y \to X$ of $\mathcal{E}/X$ to the transpose of $\eta_X \circ y$, which is just $Ly\colon LY \to LX$. Furthermore, L' sends arrows f of $\mathcal{E}/X$ to Lf. Now it is not difficult to show that L' preserves finite limits, since L does. We show that the counit ε' of this adjunction is an isomorphism. Let $f\colon F \to LX$ be an object of $\mathsf{Sh}_j(\mathcal{E})/LX$. We can construct $\varepsilon'_{(F,f)}$ as follows. Construct a pullback diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & iF \\ {\scriptstyle y}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle if} \\ X & \xrightarrow[\eta_X]{} & iLX \end{array}$$

Then $(Y, y) = i'(F, f)$, and $\varepsilon'_{(F,f)} : (LY, Ly) = L'i'(F, f) \to (F, f)$ is equal to the transpose $\tilde{g} : LY \to F$ of g. If we apply the limit preserving functor L to the above pullback diagram, we see that

$$\begin{array}{ccc} LY & \xrightarrow{Lg} & LiF \\ {\scriptstyle Ly}\downarrow & & \downarrow{\scriptstyle Lif} \\ LX & \xrightarrow[L\eta_X]{} & LiLX \end{array}$$

is also a pullback diagram. But $L\eta_X$ is an isomorphism (it is the inverse of ε_{LX}), so Lg must be an isomorphism as well. Furthermore, we know that ε_F is an isomorphism, so we get that $\tilde{g} = \varepsilon_F \circ Lg$ is also an isomorphism, as desired.
In other words, $(L' \dashv i') : \mathsf{Sh}_j(\mathcal{E})/LX \to \mathcal{E}/X$ is a geometric inclusion. We now show that every object in the image of i' is in fact a sheaf. This implies that the geometric inclusion $L' \dashv i'$ can be restricted in order to obtain the desired geometric inclusion $\mathsf{Sh}_j(\mathcal{E})/LX \to \mathsf{Sh}_{j\times \mathrm{id}_X}(\mathcal{E}/X)$. So let $f : F \to LX$ be an object of $\mathsf{Sh}_j(\mathcal{E})/LX$. We construct $i'(F, f)$ as we did above. Suppose we have the commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{a} & Y & \xrightarrow{g} & iF \\ {\scriptstyle m}\big\downarrow & & {\scriptstyle y}\downarrow & \lrcorner & \downarrow{\scriptstyle if} \\ Z & \xrightarrow[z]{} & X & \xrightarrow[\eta_X]{} & iLX \end{array}$$

where m is a dense subobject. We need to show that there exists a unique $b : Z \to Y$ such that $yb = z$ and $bm = a$. First of all, since iF is a sheaf, there must exists a unique $c : Z \to iF$ such that $cm = ga$. We get

$$\eta_X \circ z \circ m = \eta_X \circ y \circ a = if \circ g \circ a = if \circ c \circ m.$$

Since iLX is a sheaf, this implies $\eta_X \circ z = if \circ c$. By the pullback property, there exists a unique $b : Z \to Y$ such that $yb = z$ and $gb = c$. Now we notice that

$$ybm = zm = ya \quad \text{and} \quad gbm = cm = ga.$$

By the uniqueness part of the pullback property, we get $bm = a$, as desired. Conversely, suppose we have $b' : Z \to Y$ such that $yb' = z$ and $b'm = a$. Then $gb'm = ga$, so gb' must be equal to c. Now we have both $yb' = z$ and $gb' = c$, so $b = b'$, as desired. □

Proposition 5.3.7. *If X is a j-separated object, then the geometric inclusion from Proposition 5.3.6 is an equivalence of categories.*

Proof. Let η' be the counit of the adjunction $L' \dashv i'$. We need to show that for every *sheaf* $f : F \to X$ of $\mathcal{E}/X$, the arrow $\eta'_{(F,f)}$ is an isomorphism.
First, we show that F is separated as well. Suppose $m : U \hookrightarrow Y$ is a dense subobject and that $b, b' : Y \to F$ are two arrows such that $bm = b'm$. Then first of all, since $fbm = fb'm$ and X is separated, we have $fb = fb'$. Now b and b' are both maps $(Y, fb) \to (F, f)$. Since (F, f) is a sheaf, hence separated, and $bm = b'm$, we can conclude that $b = b'$. So F is indeed separated.
We can describe $\eta'_{(F,f)}$ as follows. Construct a pullback diagram

$$\begin{array}{ccc} \tilde{F} & \xrightarrow{g} & iLF \\ {\scriptstyle \tilde{f}}\downarrow & \lrcorner & \downarrow{\scriptstyle iLf} \\ X & \xrightarrow[\eta_X]{} & iLX \end{array}$$

Then $(\tilde{F}, \tilde{f}) = i'L'(F, f)$. Since $\eta_X \circ f = iLf \circ \eta_F$, there exists a unique arrow $x\colon F \to \tilde{F}$ such that $\tilde{f}x = f$ and $gx = \eta_F$. This arrow is $\eta'_{(F,f)}$. In order to show that $\eta'_{(F,f)}$ is an isomorphism, it suffices to show that the naturality square

$$\begin{array}{ccc} F & \xrightarrow{\eta_F} & iLF \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle iLf} \\ X & \xrightarrow[\eta_X]{} & iLX \end{array}$$

is also a pullback. So let $x\colon Y \to iLF$ and $y\colon Y \to X$ be given such that $iLf \circ x = \eta_X \circ y$. Since X and F are separated, η_X and η_F are both mono. Furthermore, η_F is a dense subobject of iLF. Now let $m\colon U \hookrightarrow Z$ be the pullback of η_F along x:

$$\begin{array}{ccc} U & \xrightarrow{a} & F \\ {\scriptstyle m}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \eta_F} \\ Y & \xrightarrow[x]{} & iLF \end{array}$$

Since dense subobjects are stable under pullback, we know that u is also a dense subobject of Y. We have

$$\eta_X \circ y \circ m = iLf \circ x \circ m = iLf \circ \eta_F \circ a = \eta_X \circ f \circ a,$$

and since η_X is mono, it follows that $ym = fa$. Since (F, f) is a sheaf, there exists a unique $b\colon Y \to F$ such that $fb = y$ and $bm = a$. This also implies $\eta_F \circ b \circ m = \eta_F \circ a = x \circ m$, and since iLF is a sheaf, we get $\eta_F \circ b = x$. Conversely, suppose we have $b'\colon Y \to F$ such that $fb' = y$ and $\eta_F \circ b' = x$. Then $\eta_F \circ b' \circ m = x \circ m = \eta_F \circ a$, and since η_F is mono, it follows that $b'm = a$, and therefore $b = b'$, as desired. □

Remark 5.3.8. Suppose X is separated. Under the equivalence from the previous proposition, the inclusion of the sheaf topos

$$\mathsf{Sh}_j(\mathcal{E})/LX \to \mathcal{E}/X$$

is given by i'. The corresponding sheafification functor must be left adjoint to i'. So up to natural isomorphism, the sheafification functor is L'. ◊

Remark 5.3.9. In the above, we showed that if $f\colon F \to X$ is separated, then F is separated as well (under the assumption that X is separated). The converse holds as well: if F is separated, then so is $f\colon F \to X$. ◊

5.4 A Class of Topoi

In this section, we will present a class of topoi that includes all realizability topoi and is closed under slicing over *projective* objects. As we already remarked, the global sections functor does not behave nicely for such slices. Instead, we will describe our class in terms of the adjunction that gives the geometric inclusion of ¬¬-sheaves of our topos. This geometric inclusion will not necessarily be an adjunction with Set, but rather with one of its slices.

In the following, we consider multiple adjunctions simultaneously. We will use terms such as ‘assembly’, ‘cartesian’ and ‘discrete’ (which are relative to a given adjunction) without further qualification. We will always specify to which category the relevant object belongs, so that no confusion can arise with respect to which adjunction these terms must be read.

Theorem 5.4.1. *Let $\mathscr{R}$ be the class of topoi $\mathcal{E}$ such that the following hold:*

(*1*) *$\mathcal{E}$ has enough projectives.*

(*2*) *The projective objects of $\mathcal{E}$ are closed under finite limits.*

(*3*) *The full subtopos of $\neg\neg$-sheaves in $\mathcal{E}$ is equivalent to Set/A for some set A.*

(*4*) *Write $\mathsf{Set}/A \underset{\Delta}{\overset{\Gamma}{\leftrightarrows}} \mathcal{E}$ for the geometric inclusion of $\neg\neg$-sheaves. Then Γ preserves* all *limits, Δ is regular and there exists an assembly D of $\mathcal{E}$ such that:*

   (*a*) *An object X of $\mathcal{E}$ is projective if and only if X is an assembly and there exists an cartesian arrow $f\colon X \to D$.*

   (*b*) *The object D is discrete.*

Then $\mathscr{R}$ contains all realizability topoi and is closed under slicing over projective objects.

Proof. First of all, we notice that for a realizability topos $\mathcal{E}$, the above statements hold for A a singleton.
Now let $\mathcal{E}$ be a topos belonging to the class $\mathscr{R}$ and let X be a projective object in $\mathcal{E}$. We know from Corollary 5.2.4 that $\mathcal{E}/X$ also has enough projectives. Furthermore, since X is projective, we can derive from Corollary 5.2.5 that the projectives in $\mathcal{E}/X$ are closed under finite limits.
Because X is projective, we know in particular that X is an assembly, that is, a $\neg\neg$-separated object. This means that $\mathsf{Set}/\mathrm{dom}\,\Gamma X \cong (\mathsf{Set}/A)/\Gamma X \cong \mathsf{Sh}_{\neg\neg}(\mathcal{E})/\Gamma X$ is equivalent to $\mathsf{Sh}_{\neg\neg}(\mathcal{E}/X)$, and the geometric inclusion of $\neg\neg$-sheaves under the latter equivalence is given by the composite adjunction

$$(\mathsf{Set}/A)/\Gamma X \underset{\Delta^*}{\overset{\Gamma^*}{\leftrightarrows}} \mathcal{E}/\Delta\Gamma X \underset{\eta_X^*}{\overset{\Sigma_{\eta_X}}{\leftrightarrows}} \mathcal{E}/X$$

as we saw in Remark 5.3.8. Again, we write $\Gamma' = \Gamma^* \circ \Sigma_{\eta_X}$ and $\Delta' = \eta_X^* \circ \Delta^*$. We recall that Γ' sends an object $y\colon Y \to X$ of $\mathcal{E}/X$ to $\Gamma y\colon \Gamma Y \to \Gamma X$ and sends an arrow $f\colon (Y,y) \to (Z,z)$ to Γf. Now it is not difficult to show that Γ' preserves all limits, since Γ does. Using Proposition 4.2.3, we can see that Δ^* preserves regular epis, since Δ does. Moreover, since $\mathcal{E}$ is a regular category, we see that η_X^* also preserves regular epis, again using Proposition 4.2.3. So Δ' also preserves regular epis and is therefore a regular functor.
Recall that the unit η' of the adjunction $\Gamma' \dashv \Delta'$ may be constructed as follows. Let $y\colon Y \to X$ be an object of $\mathcal{E}/X$. Construct a pullback diagram

$$\begin{array}{ccc} \tilde{Y} & \xhookrightarrow{m} & \Delta\Gamma Y \\ {\scriptstyle\tilde{y}}\downarrow & & \downarrow{\scriptstyle\Delta\Gamma y} \\ X & \underset{\eta_X}{\hookrightarrow} & \Delta\Gamma X \end{array}$$

Notice that $(\tilde{Y},\tilde{y}) = \Delta'\Gamma'(Y,y)$. Since $\eta_X \circ y = \Delta\Gamma y \circ \eta_Y$, there exists a unique arrow $x\colon Y \to \tilde{Y}$ such that $\tilde{y} \circ x = y$ and $m \circ x = \eta_X$; and this arrow is $\eta'_{(Y,y)}$. Since $m \circ \eta'_{(Y,y)} = \eta_Y$ and m is mono, we have that $\eta'_{(Y,y)}$ is mono if and only if η_Y is mono. That is, (Y,y) is an assembly in

$\mathcal{E}/X$ if and only if Y is an assembly in $\mathcal{E}$ (cf. Remark 5.3.9). Now let $f\colon (Y, y) \to (Z, z)$ be an arrow of $\mathcal{E}/X$, and write $\Delta'\Gamma'(Z, z) = (\tilde{Z}, \tilde{z})$ as above. Consider the squares

$$\begin{array}{ccccc} & & \eta_Y & & \\ Y & \xrightarrow{\eta'_{(Y,y)}} & \tilde{Y} & \hookrightarrow & \Delta\Gamma Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Delta'\Gamma' f} & & \downarrow{\scriptstyle \Delta\Gamma f} \\ Z & \xrightarrow{\eta'_{(Z,z)}} & \tilde{Z} & \hookrightarrow & \Delta\Gamma Z \\ & & \eta_Z & & \end{array}$$

From the definition of η^*_X, it follows that the right hand square is a pullback diagram. Now, if Y and Z are assemblies, we can use the Pullback Lemma and Proposition 5.2.1 to see that f is cartesian in $\mathcal{E}/X$ if and only if f is cartesian in $\mathcal{E}$.

We prove that item (4) holds if we pick $(D \times X, \pi_2)$ to fulfil the role of D in $\mathcal{E}/X$. Since Γ and Δ both preserve limits, we have that $\langle \Delta\Gamma\pi_1, \Delta\Gamma\pi_2\rangle\colon \Delta\Gamma(D \times X) \to \Delta\Gamma D \times \Delta\Gamma X$ is an isomorphism. We claim that $\langle \Delta\Gamma\pi_1, \Delta\Gamma\pi_2\rangle \circ \eta_{D\times X}$ and $\eta_D \times \eta_X$ are the same arrow $D \times X \to \Delta\Gamma D \times \Delta\Gamma X$. Indeed, projecting the left hand side onto the first coordinate yields $\Delta\Gamma\pi_1 \circ \eta_{D\times X}$, while projecting the right hand side onto the first coordinate yields $\eta_D \circ \pi_1$, and these are indeed equal. Of course, the second coordinate is analogous. We can conclude that $\eta_{D\times X}$ is mono, since η_D and η_X are both mono. So $D \times X$ is an assembly in $\mathcal{E}$, which means that $(D \times X, \pi_2)$ is an assembly in $\mathcal{E}/X$.

Since D and X are both projective in $\mathcal{C}$, we have that $D \times X$ is projective in $\mathcal{C}$ as well. So there exists an cartesian arrow $\rho\colon D \times X \to D$. Now suppose that (Y, y) is an object of $\mathcal{E}/X$ such that there exists an cartesian arrow $f\colon (Y, y) \to (D \times X, \pi_2)$ in $\mathcal{E}/X$. Then f is also cartesian in $\mathcal{E}$. Since the composition of two cartesian arrows is again cartesian, by the Pullback Lemma, we have that $\rho f\colon Y \to D$ is cartesian. But this means that Y is projective in $\mathcal{E}$, hence that (Y, y) is projective in $\mathcal{E}/X$.

Conversely, suppose that (P, p) is a projective object of $\mathcal{E}/X$. Then P is projective in $\mathcal{E}$, so there exists an cartesian arrow $f\colon P \to D$. Notice that $\langle f, p\rangle\colon (P, p) \to (D \times X, \pi_2)$ is an arrow of $\mathcal{E}/X$. We claim that $\langle f, p\rangle$ is cartesian in $\mathcal{E}/X$. In order to do this, we show that this arrow is cartesian in $\mathcal{E}$. Let an object Y of $\mathcal{E}$ and arrows $\langle a, y\rangle\colon Y \to D \times X$ and $b\colon Y \to \Delta\Gamma P$ be given such that $\eta_{D\times X} \circ \langle a, y\rangle = \Delta\Gamma\langle f, p\rangle \circ b$. Then

$$\Delta\Gamma f \circ b = \Delta\Gamma\pi_1 \circ \Delta\Gamma\langle f, p\rangle \circ b = \Delta\Gamma\pi_1 \circ \eta_{D\times X} \circ \langle a, y\rangle = \eta_D \circ \pi_1 \circ \langle a, y\rangle = \eta_D \circ a$$

and

$$\Delta\Gamma p \circ b = \Delta\Gamma\pi_2 \circ \Delta\Gamma\langle f, p\rangle \circ b = \Delta\Gamma\pi_2 \circ \eta_{D\times X} \circ \langle a, y\rangle = \eta_X \circ \pi_2 \circ \langle a, y\rangle = \eta_X \circ y.$$

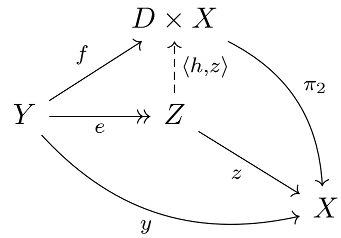
Since $\Delta\Gamma f \circ b = \eta_D \circ a$ and f is cartesian, there exists a unique arrow $k\colon Y \to P$ such that $fk = a$ and $\eta_P \circ k = b$. We have

$$\eta_X \circ p \circ k = \Delta\Gamma p \circ \eta_P \circ k = \Delta\Gamma p \circ b = \eta_X \circ y,$$

and since η_X is mono, we get $pk = y$. So we have $\langle f, p\rangle \circ k = \langle a, y\rangle$. Conversely, if $k'\colon Y \to P$ satisfies $\langle f, p\rangle \circ k' = \langle a, y\rangle$ and $\eta_P \circ k' = b$, then we also have $fk' = a$, so $k = k'$, as desired.

Finally, we show that $(D \times X, \pi_2)$ is discrete in $\mathcal{E}/X$. Let an cartesian regular epi $e\colon (Y, y) \twoheadrightarrow (Z, z)$ in $\mathcal{E}/X$ and an arrow $f\colon (Y, y) \to (D \times X, \pi_2)$ be given. Recall from Proposition 4.2.3 that $e\colon Y \to Z$ is also regular epi in $\mathcal{E}$. By what we have shown above, e is also cartesian in

$\mathcal{E}$.



Notice that $\pi_1 f$ is an arrow $Y \to D$. Since D is discrete in $\mathcal{E}$, there exists an arrow $h\colon Z \to D$ such that $he = \pi_1 f$. Now consider $\langle h, z\rangle\colon Y \to D \times X$. First of all, we have that $\pi_2 \circ \langle h, z\rangle = z$, so $\langle h, y\rangle$ is an arrow $(Z, z) \to (D \times X, \pi_2)$. Furthermore, we have $he = \pi_1 f$ and $ze = y = \pi_2 f$, so $\langle h, z\rangle \circ e = f$, as desired. □

Using the class $\mathscr{R}$, we may also construct a class of topoi that is closed under slicing over *all* objects.

Theorem 5.4.2. *Let $\mathscr{R}_{\twoheadrightarrow}$ be the class of all topoi $\mathcal{E}$ such that there exists a topos $\mathcal{F}$ in $\mathscr{R}$ and a geometric surjection $\mathcal{F} \twoheadrightarrow \mathcal{E}$. Then $\mathscr{R}_{\twoheadrightarrow}$ contains all realizability topoi and is closed under slicing.*

Proof. The first assertion follows from Theorem 5.4.1. Now let $\mathcal{E}$ be a topos belonging to the class $\mathscr{R}_{\twoheadrightarrow}$ and let X be an object of $\mathcal{E}$. Let $\mathcal{F}$ be a topos from $\mathscr{R}$ such that there exists a geometric surjection $\mathcal{F} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{E}$, where L preserves finite limits and L is faithful. Since $\mathcal{F}$ has enough projectives, there exists a projective object P of $\mathcal{F}$ and a regular epi $e\colon P \twoheadrightarrow LX$. By Theorem 5.4.1, we have that $\mathcal{F}/P$ belongs to $\mathscr{R}$. We know from topos theory that $(e^* \dashv \Pi_e)\colon \mathcal{F}/P \twoheadrightarrow \mathcal{F}/LX$ is a geometric surjection. We have the composite adjunction

$$\mathcal{F}/LX \underset{R^*}{\overset{L^*}{\leftrightarrows}} \mathcal{E}/RLX \underset{\eta_X^*}{\overset{\Sigma_{\eta_X}}{\leftrightarrows}} \mathcal{E}/X$$

where η is the unit of $L \dashv R$. Recall that $L' := L^* \circ \Sigma_{\eta_X}$ sends an object $y\colon Y \to X$ of $\mathcal{E}/X$ to $Ly\colon LY \to LX$ and sends an arrow $f\colon (Y, y) \to (Z, z)$ to Lf. This allows us to see that L' preserves finite limits, and that L' is faithful. So th composite adjunction above is a geometric surjection $\mathcal{F}/LX \twoheadrightarrow \mathcal{E}/X$. Composing this geometric surjection with $\mathcal{F}/P \twoheadrightarrow \mathcal{F}/LX$, we get a geometric surjection $\mathcal{F}/P \twoheadrightarrow \mathcal{E}/X$, so $\mathcal{E}/X$ belongs to $\mathscr{R}_{\twoheadrightarrow}$, as desired. □

Chapter 6

Conclusion

In this thesis, we set out to obtain a better understanding of slices of realizability topoi. We summarize the most important results we have obtained.

(i) In Theorem 4.5.2, we formulated Frey's result from [1] in a more direct way that no longer mentions fibrations.

(ii) We (extensionally) defined the class $\mathscr{R}$ that contains all realizability topoi and is closed under slicing over projective objects (Theorem 5.4.1), using many of the properties also mentioned in Theorem 4.5.2. We also presented a related class $\mathscr{R}_{\twoheadrightarrow}$ that contains all realizability topoi and is closed under slicing over all objects (Theorem 5.4.2).

(iii) We gave an explicit description of slices of realizability topoi over assemblies. More precisely, for such a slice, we presented a category equivalent to this slice, defined in a way that is akin to the definition of the realizability topos itself.

Of course, a variety of questions remains open.

(i) 'How large' are the classes $\mathscr{R}$ and $\mathscr{R}_{\twoheadrightarrow}$? That is, what else do they contain besides realizability topoi and their slices (over projective objects)?

(ii) The category $\mathsf{Asm}(A)$ of assemblies over a PCA A has many of the properties mentioned in Theorem 4.5.2, but it is not an exact category. Can we find a similar characterization theorem for categories of assemblies? Is Frey's fibrational framework useful here as well, or does this question require a different approach?

(iii) In the definition of the realizablity topos, the topos Set plays an important role. One can consider realizability topoi over other bases, by replacing Set with another topos. What happens to our results if we consider such other bases? The most pressing problem here seems to be the possible absence of the Axiom of Choice in such a base topos, which will make the treatment of projective objects more involved.

Index of symbols

Index of terms

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