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Higher Gauge Theory

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Abstract

We generalize several aspects of gauge theory to 2-gauge theory. We consider 2-transport on principal 2-bundles with strict 2-group fiber by categorifying a usual definition of principal bundles, following [SW11]. We show that locally 2-transport induces a 2-functor from the 2-groupoid of bigons up to thin homotopy to the 2-group. By generalizing the non-Abelian Stokes' Theorem to the 2-group setting, we are able to prove a direct generalization of the Ambrose-Singer Theorem for 2-bundles. Finally we further develop the theory of surface holonomy. In particular we elucidate how such a theory depends on the choice of marking of a surface. We also show that for covering 2-groups the surface holonomy of any connection taking values in a torus computes an invariant of the bundle. Roughly the first half of the thesis is dedicated to a comprehensive introduction to gauge theory and 2-category theory before proceeding to 2-gauge theory.

Contents

Introduction

From a mathematical point of view, gauge theory studies connections on principal bundles and their holonomy. This has a wide variety of applications in contemporary mathematics, mathematical physics, and physics. For example, the celebrated standard model of particle physics is conveniently stated in terms of gauge theory, and holonomy appears in (mathematical) physics as Wilson loops. A current trend in mathematics is to try to generalize many well-known objects to higher categories. In this thesis we will develop 2-gauge theory; that is, gauge theory generalized to the 2-categorical setting. In this setting we will show one has natural generalizations of principal bundles, connections and parallel transport. Many important facts about these objects naturally translate to the higher categorical setting, if phrased in the right way.

Higher gauge theory has potential applications to physics. Gerbes, which are related to 2-bundles, are already used in physics. Furthermore we will define Wilson surfaces from a purely mathematical point of view, which appear in numerous physics papers. Finally $U(1)$ surface holonomy is related to Dirac monopoles, and for other gauge groups it may also have important physical interpretations.

In the first section we carefully introduce many aspects of 'classical' gauge theory. A good understanding of the classical theory is necessary before attempting to generalize it. In the second section we introduce the language and methods of 2-category theory, in which 2-gauge theory will be stated. In the third section we develop 2-gauge theory, mainly based on the works of Urs Schreiber and Konrad Waldorf. We try to develop the theory from a mostly geometric point of view and avoid unnecessary 'abstract nonsense'. Our main result in this section is a generalization of the Ambrose-Singer Theorem using a non-Abelian Stokes' Theorem. In the last section we make an attempt to generalize holonomy to 'surface holonomy'. It turns out that naive attempts to generalize holonomy to surfaces fails for various reasons, and the language of 2-gauge theory makes it much easier. We then compute surface holonomy explicitly for some simple situations and relate the result to Chern-Weil classes.

1. Gauge Theory

Gauge theory studies parallel transport and connections on principal bundles, often in the context of physics. We will start by quickly recalling the basic notions of principal bundles, connections and parallel transports. The main result of this part is that parallel transport defines a functor $\mathcal{P}_1(M) \to \mathcal{G}$**-tor**. To state and prove this result properly we need to develop a number of important properties of connections and their parallel transport. The main reference for this material is [KN63], although neither the non-Abelian Stokes' Theorem nor the concept of thin homotopy appears there; for this we refer the reader to [SW07] instead.

1.1. Principal bundles

Throughout this section, and indeed throughout the whole text, we let G be a compact connected Lie group, and let M be a smooth manifold.

Definition 1.1

A *principal G-bundle* $\pi : P \to M$ consists of a space P, a (left) Lie group action $G \circlearrowleft P$ and a G-invariant surjective submersion π (i.e. $\pi(g \cdot m) = \pi(m)$), such that the following map is a diffeomorphism:

$$\tau : G \times P \to P \times_M P, \qquad (g, p) \mapsto (p, g \cdot p)$$

There are many definitions equivalent to this one, but this one seems to be the most elegant and the easiest to 'categorify', as we shall later do to obtain the definition of a 2-bundle.

We pick the convention that groups act on the left instead of right. The reason for this is that the notation of left group actions is more compatible with that of function composition. For example

$$(g \cdot h) \cdot m = L_{gh}(m) = L_g \circ L_h(m),$$

whereas for right multiplication we have

$$R_{gh} = R_h \circ R_g.$$

There is no standard convention in the literature, although our main reference, Schreiber & Waldorf [SW07, SW11, SW13a], use right actions. This will cause a large number of slight differences between our results. Some formulas become slightly simpler for left actions and some for right. It is mostly a matter of taste.

The fact that τ is an isomorphism means that the fibers of π are all isomorphic to G. Since π is G-invariant, the action of G descents to one on each fiber. This means that every fiber is in fact a G-torsor (i.e. a space with a free and transitive group action by G). More concretely, given any two points (p, q) in the same fiber there is a unique $q : p \in G$ such that

$$(q : p) \cdot p = q. \tag{1.1}$$

Therefore choosing any $p \in \pi^{-1}(x)$ gives us an explicit isomorphism $G \to \pi^{-1}(x)$, namely $g \mapsto p \cdot g$ with inverse $q \mapsto q : p$. It is worth noting how this map changes when changing basepoint p. Let $g \in G$ then note that

$$(g(q : p)g^{-1}) \cdot g \cdot p = g \cdot q.$$

Thus we conclude that

$$(g \cdot q) : (g \cdot p) = g(q : p)g^{-1}. \tag{1.2}$$

Recall that every surjective submersion admits local sections [Lee03, Thm. 4.26]. Every local section gives a local trivialization; if $\sigma : U \to P$ is a section, then $\phi : G \times U \to P$ given by $\phi(g, x) = g \cdot \sigma(x)$ is a local trivialization precisely because τ is an isomorphism (or because the fibers are G-torsors). Thus P can also be thought of as a locally trivializable G fiber bundle together with a group action on the fibers. Since the group action on P is free we have $M \cong P/G$. In fact all free Lie group actions define a principal bundle over the quotient in this way (so long as the quotient is a manifold, which always happens if the action is proper or in particular if the group is compact).

Seeing a principal G-bundle as a locally trivializable G-bundle, the bundle is determined by its transition functions. That is, if we choose a cover U then a cochain $U^{[2]} \to G$ determines a principal G-bundle. If G is Abelian then (isomorphism classes of) G-bundles are classified by $H^1(M, G)$, but for non-Abelian G one would first need to know what non-Abelian cohomology is. Going further in this direction, one can define G-gerbes as the geometric objects classified by $H^2(M, G)$, and this is one way to lift principal bundles to a higher categorical setting (but not the approach we will take).

A morphism of G-bundles $F : P \to Q$ is an equivariant bundle map. That is, a map such that the following diagrams commute:

$$\begin{tikzcd} P \arrow[rr, "F"] \arrow[dr] & & Q \arrow[dl] \\ & M & \end{tikzcd} \qquad \begin{tikzcd} G \times P \arrow[r, "\mathrm{Id} \times F"] \arrow[d] & G \times Q \arrow[d] \\ P \arrow[r, "F"] & Q \end{tikzcd}$$

More generally if P and Q are not above the same space, the equivariant bundle map F can cover some map $f : M \to N$. By equivariance, any morphism of $F : P \to Q$ of G bundles over the same space is an isomorphism (or if the two bundles are not over the same space, but F covers some diffeomorphism $M \to N$). It is clear that F is a local isomorphism; given a section $\sigma : U \to P$ we get an isomorphism $P|_U \to Q|_U$, $g \cdot \sigma(x) \mapsto F(g \cdot \sigma(x)) = g \cdot F(\sigma(x))$. Now F is a bijective local diffeomorphism, and therefore in particular a diffeomorphism and its inverse is a bundle map.

Just as for fiber bundles one can define pullbacks of principal bundles. Given a map $f : M \to N$ and a bundle $\pi : P \to N$ we can define $f^*P \to M$ as $f^*P = P_\pi \times_f M$ with the obvious induced G-action.

1.2. Lie algebra valued forms

Central to gauge theory is the notion of a connection on a principal bundle. Before giving a definition we will quickly recall the theory of Lie algebra valued differential forms, as its notation will be frequently used. See also section 2.4 in [Mor97]. Let $\mathfrak{g}$ be a Lie-algebra, then define $\Omega^n(M, \mathfrak{g}) = \Omega^n(M) \otimes \mathfrak{g}$ the *n-forms on M with values in* $\mathfrak{g}$. Locally one can write $\xi \in \Omega^n(M, \mathfrak{g})$ as $\xi_I^\alpha dx^I \otimes X_\alpha$, where x^i are local coordinates, I is a multi-index with $|I| = n$, and X_α is a basis of $\mathfrak{g}$. Here we use the familiar convention that repeated indices are summed over. This notion supports ordinary operations on differential forms; suppose $\xi \in \Omega^p(M, \mathfrak{g})$ and $\eta \in \Omega^q(M, \mathfrak{g})$, then $\xi \wedge \eta \in \Omega^{p+q}(M) \otimes \mathfrak{g} \otimes \mathfrak{g}$ is defined locally by $\xi_I^\alpha \eta_J^\beta dx^I \wedge dx^J \otimes X_\alpha \otimes X_\beta$.

One can also use the structure of the Lie-algebra for some extra operations, for example we define $[\xi \wedge \eta] \in \Omega^{p+q}(M, \mathfrak{g})$ by $\xi_I^\alpha \eta_J^\beta dx^I \wedge dx^J \otimes [X_\alpha, X_\beta]$. This has a different skew symmetry than ordinary wedging of forms; the form part contributes a sign $(-1)^{pq}$ whereas the Lie algebra part contributes a -1 sign. For example, for a 1-form $A \in \Omega^1(M, \mathfrak{g})$ one does not a priori have $[A \wedge A] = 0$. In general we thus have $[\xi \wedge \eta] = (-1)^{\deg(\xi)\deg(\eta)+1}[\eta \wedge \xi]$. By using the Jacobi identity one can also show that $[[\xi \wedge \xi] \wedge \xi] = 0$ for any Lie algebra valued form. For operations that 'act only on the differential form part' like the de Rham differential or Lie derivation, the sign rules don't change; $d[\xi \wedge \eta] = [d\xi \wedge \eta] + (-1)^{\deg(\xi)}[\xi \wedge d\eta]$. Heuristically operators like d 'don't see the Lie algebra part'.

One can also do things that 'only see the Lie algebra part'. For example if we have a pairing $\mathrm{Tr} : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ then we can define a map $\Omega^*(M) \otimes \mathfrak{g} \otimes \mathfrak{g} \to \Omega^*(M)$ locally defined as

$$\xi_I^{\alpha\beta} dx^I X_\alpha \wedge X_\beta \mapsto \xi_I^{\alpha\beta} dx^I \,\mathrm{Tr}(X_\alpha, X_\beta).$$

Such a pairing is of particular interest because it is Ad-invariant, i.e. $\mathrm{Tr}(\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)) = \mathrm{Tr}(X, Y)$ for any $g \in G$. In terms of differential forms this means $\mathrm{Tr}(\mathrm{Ad}_g(\xi \wedge \eta)) = \mathrm{Tr}(\xi \wedge \eta)$. In general any map of Lie algebras will induce a map of Lie algebra valued forms in this way. Ordinary operations on differential forms will commute with such maps, e.g. $d\,\mathrm{Tr} = \mathrm{Tr}\, d$.

1.3. Connections

The main objects of study in gauge theory are connections on principal bundles. These objects admit several different points of view, all of which are useful. In fact we will give four equivalent definitions of a connection, and this section is entirely devoted to stating these definitions and proving they are equivalent. A connection is the object that allows us to define what it means for a path in a principal bundle to be horizontal with respect to the group action. Thus a connection is just a horizontal distribution on P compatible with the group action:

Definition 1.2

Let $\pi : P \to M$ be a principal G-bundle, then π induces a map $d\pi : TP \to TM$. Its kernel is a subbundle $V = \ker d\pi \subset TP$. A *horizontal distribution* on P is a rank $\dim M$ subbundle H of TP such that $H \oplus V = TP$. A horizontal distribution is a *connection* if $H_{g\cdot p} = (dL_g)_p H_p$ for all $g \in G$ and $p \in P$.

While fairly conceptual, this definition is hard to work with. Connections are more conveniently defined in the language of differential forms.

Definition 1.3

Let $P \to M$ be a principal G-bundle. Then a connection is a 1-form $A \in \Omega^1(P, \mathfrak{g})$ such that:

1. For all $g \in G$ we have $L_g^* A = \mathrm{Ad}_g A$.
2. For every $\xi \in \mathfrak{g}$ consider the *fundamental vector field* X_ξ, then $A(X_\xi) = \xi$.

Here $X_\xi(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp(t\xi) \cdot p$.

The equivalence between the two definitions is sketched as follows. First, given a connection form A, one can check that $\ker A \subset TP$ is a horizontal distribution. Given a horizontal distribution, the orthogonal decomposition $TP = H \oplus V$ gives us a projection $v : TP \to V \subset TP$. The action $G \circlearrowright P$ allows us to identify $V \cong M \times \mathfrak{g}$, namely the 'vertical part' is given by the fundamental vector fields X_ξ for $\xi \in \mathfrak{g}$. Then for a vector field X define $A(X) = v(X) \in C^\infty(M) \otimes \mathfrak{g}$ by this identification. We then check that this is a connection 1-form.

Every Lie group G defines a principal bundle $G \to *$ over a point with the left action of G on itself. This principal bundle has a unique connection $\theta \in \Omega^1(G, \mathfrak{g})$, known as the *Maurer-Cartan form*. The tangent space $T_g G$ is spanned by fundamental vector fields X_ξ. On a Lie group they are given by

$$(X_\xi)_g = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \exp(t\xi) \cdot g = (R_g)_* \xi. \tag{1.3}$$

Therefore if θ is a connection, it has to satisfy:

$$\theta_g(v) = (R_g^{-1})_* v \in T_e G = \mathfrak{g}, \qquad \forall v \in T_g G. \tag{1.4}$$

We then check the other condition for θ being a connection. Let $v \in T_g G$ and $h \in G$ then

$$\begin{aligned}(L_h^*\theta)_g(v) &= \theta_{hg}((L_h)_* v) = (R_{hg}^{-1})_*(L_h)_*(v) \\ &= (L_h)_*(R_h^{-1})_*(R_g^{-1})_*(v) = (L_h)_*(R_h^{-1})_*\theta_g(v) \\ &= \mathrm{Ad}_h\, \theta(v).\end{aligned}$$

Thus θ is indeed the unique connection on $G \to *$.

If $G = \mathrm{GL}_n$ then $\mathfrak{gl}_n = \mathbb{R}^{n^2}$ and we can see GL_n as a subset of $\mathfrak{gl}_n$. The Lie algebra is generated by E_j^i which are matrices with 0 everywhere except at ji where they have value 1. Around $g \in \mathrm{GL}_n$ we get local coordinates g_j^i by identifying an open neighborhood of g with $\mathfrak{gl}_\mathfrak{n}$, via the identification $T\,\mathrm{GL}_n = \mathrm{GL}_n \times \mathfrak{gl}_n$. Alternatively we can define these coordinates as those such that $\left\langle dg_j^i, E_l^k \right\rangle = \delta_j^i \delta_l^k$. In these coordinates we claim that the Maurer-Cartan form on GL_n is of form

$$\theta = (dg)g^{-1} = (dg\, g^{-1})_j^i E_i^j = dg_k^i (g^{-1})_j^k E_i^j. \tag{1.5}$$

Indeed if $\xi = \xi_j^i E_i^j \in \mathfrak{g}$ then $(X_\xi)_g = \xi \cdot g = \xi_k^i g_j^k E_i^j$ and note that

$$\left\langle dg, (X_\xi)_g \right\rangle = \xi_k^i g_j^k \left\langle dg_i^j, E_i^j \right\rangle = \xi_k^i g_j^k = \xi g. \tag{1.6}$$

Thus we compute

$$\left\langle (dg)g^{-1}, (X_\xi)_g \right\rangle = \xi_k^i g_l^k (g^{-1})_j^l E_j^i = \xi_k^i E_j^i = \xi, \tag{1.7}$$

showing that this is indeed the Maurer-Cartan form. This is powerful, because the same formula also applies to any subgroup of $\mathrm{GL}(n)$:

Proposition 1.4

Suppose $j : G \hookrightarrow H$ is an embedding of Lie groups, and $\theta \in \Omega^1(H, \mathfrak{h})$ the Maurer-Cartan form on H, then

$$j^*\theta \in \Omega^1(G, j_*\mathfrak{g})$$

is the Maurer-Cartan form on G under the identification $j_*\mathfrak{g} \cong \mathfrak{g}$.

Proof: This is a simple computation. Let $\xi \in \mathfrak{g}$ and $g \in G$ then,

$$j^*\theta((X_\xi)_g) = \theta(j_*((R_g)_* X)) = \theta((R_{j(g)})_*(j_*(X)) = j_*(X). \qquad \blacksquare$$

Thus if $G \subset \mathrm{GL}(n)$ (as happens for any compact Lie group, cf. prop. A.3) then with abuse of notation we can write

$$\theta = dgg^{-1}. \tag{1.8}$$

This notation will turn out to simplify some computations considerably, since we in matrix algebras we can treat the group and Lie algebra on equal footing. Now we can use the theory of the Maurer-Cartan form to obtain another equivalent definition of a connection.

Definition 1.5

Let $P \to M$ be a principal G-bundle. Then a connection is a 1-form $A \in \Omega^1(P, \mathfrak{g})$ such that for all $g : M \to G$ we have

$$L_g^* A = \operatorname{Ad}_g A + g^*\theta \quad (= gAg^{-1} + dgg^{-1}), \tag{1.9}$$

with θ the Maurer-Cartan form on G.

Note the similarity to definition 1.3 of a connection. The difference now is that instead of taking constant $g \in G$ we allow g to vary over M. The advantage is that the second condition we had before now becomes superfluous, and a connection can be completely defined in the language of forms. It also makes directly clear what happens to $L_g^* A$ when g is not constant, which is something we will use almost immediately.

Proposition 1.6

Definitions 1.3 and 1.5 of a connection 1-form are equivalent.

Proof: Let A be a connection according to definition 1.3. Let $\phi : G \times P \to P$ be the action and consider $\phi^* A$. For a fixed $g \in G$ and an $X \in T_x P$ there is an $\widehat{X} \in T_{g,x} P$ such that $\phi_* \widehat{X} = X$. We have $\phi^* A(\widehat{X}) = L_g^* A(X) = \operatorname{Ad}_g A(X)$. For a fixed $\xi \in \mathfrak{g}$ we get an associated $\widehat{\xi} \in T_{x,g} P$ from $TG \cong G \times \mathfrak{g}$ and $\phi^* A(\widehat{\xi}) = A(X_\xi) = \xi$. Since such $\widehat{X}$ and $\widehat{\xi}$ together span $T(G \times P)$ we conclude

$$\phi^* A_{(x,g)} = \operatorname{Ad}_g A + \theta. \tag{1.10}$$

Now for any $g : M \to G$ we get a map $\widetilde{g} : P \to G \times P$, $\widetilde{g}(x) = (g \circ \pi(x), x)$. Note that $\widetilde{g}^* \phi^* A = \operatorname{Ad}_g A + g^*\theta$ whereas $\widetilde{g} \circ \phi = L_g$, proving that the definition 1.3 implies definition 1.5. For the inverse implication we note that because $L_g^* A = \operatorname{Ad}_g A + g^*\theta$ for all $g : M \to G$ we must have $\phi^* A = \operatorname{Ad}_g A + \theta$. Then we simply compute $A(X_\xi) = \phi^* A(\widehat{\xi}) = \theta(X_\xi) = \xi$. ■

Suppose we have a trivial bundle $P = M \times G$. This bundle admits a section $s : M \to P$, and $s^* A \in \Omega^1(M, \mathfrak{g})$. Let $X \in T_p P$ be given by $X = \gamma'(0)$ for some path $\gamma : I \to P$, and let $Y = \pi_*(X) = (\pi \circ \gamma)'(0)$. Note that there is then a $g : I \to G$ such that $g(t) \cdot s(t) = \gamma(t)$ (indeed $g = \gamma : s$), which also means that $X = (L_g)_* s_* Y$. From this we conclude:

$$A(X) = A(\gamma'(0)) = A([g \cdot (s \circ \pi \circ \gamma)]'(0)) = (L_g^* \pi^* s^* A)(Y). \tag{1.11}$$

Hence we can fully recover $A \in \Omega^1(P, \mathfrak{g})$ from $s^* A \in \Omega^1(M, \mathfrak{g})$. Now suppose we have a trivializing cover $\{U_i\}$ of M, then the trivializations (i.e. sections $U_i \to P|_{U_i}$) give us $A_i \in \Omega^1(U_i, \mathfrak{g})$ for each i using the procedure above, indeed $A_i = (s_i)^*(A|_{P|_{U_i}})$. If $g_{ij} : U_{ij} \to G$ is a transition function defined by $s_j : s_i$ then we have

$$A_j = s_j^* A = s_i^* L_{g_{ij}}^* A = s_i^*(gAg^{-1} + dgg^{-1}) = gA_i g^{-1} + dgg^{-1}. \tag{1.12}$$

In other words, we can give a fourth definition of a connection in terms of local data satisfying a certain transformation rule. This is the definition most often used in physics, and is very useful for doing concrete computations.

Definition 1.7

Let $P \to M$ be a principal G-bundle with trivializing cover $\{U_i\}$ and transition functions $g_{ij} : U_{ij} \to G$. Then a connection on P is a collection of 1-forms $A_i \in \Omega^1(U_i, \mathfrak{g})$ satisfying the transformation rule

$$A_j = \mathrm{Ad}_g\, A_i + g^*\theta = gA_i g^{-1} + dgg^{-1}. \tag{1.13}$$

1.4. Parallel transport

The most useful property of connections is that it allows use to lift paths in the base space to 'horizontal paths' in the principal bundle.

Definition 1.8

Let $\pi : P \to M$ be a principal G bundle with connection $A \in \Omega^1(P, \mathfrak{g})$ and let $\gamma : I \to M$ be a path. A *horizontal lift* $\widetilde{\gamma} : I \to P$ of γ is any path such that $\pi \circ \widetilde{\gamma} = \gamma$ and $A(\widetilde{\gamma}'(t)) = 0$ for all $t \in I$.

Proposition 1.9

Any path has a horizontal lift, unique up to multiplication by a constant $g \in G$.

Proof: Let $\gamma : I \to M$ be a path. Since I is contractible $\gamma^* P$ is trivial and we can without loss of generality assume P itself is trivial. Then $\widetilde{\gamma}(t) = (g, \gamma(t)) \in G \times M$ is a lift of γ for any $g \in G$. In general this lift is not horizontal, but any horizontal lift α must be of form $\alpha(t) = f(t) \cdot \widetilde{\gamma}(t)$ for some $f : I \to G$, and we can moreover assume $f(0) = 1$. Then α being a horizontal lift is equivalent to:

$$0 = A\left(\frac{\mathrm{d}}{\mathrm{d}t}(f(t) \cdot \widetilde{\gamma}(t))\right) = L_f^* A(\widetilde{\gamma}'(t)) = fA(\widetilde{\gamma}'(t))f^{-1} + dff^{-1}. \tag{1.14}$$

In other words f should satisfy the initial value problem[1]

$$f'(t) = -f(t) \cdot A(\widetilde{\gamma}'(t)), \qquad f(0) = 1. \tag{1.15}$$

[1] For right actions the initial value problem would instead read $f'(t) = -A(\widetilde{\gamma}'(t))f(t)$, $f(0) = 1$.

We will prove below that this initial value problem has a unique solution. The precise form of the solution will be of interest later. Given this fact it is clear that fixing $\widetilde{\gamma}(0)$, horizontal lifts exist and are unique. ■

Proposition 1.10

Let $A : I \to \mathfrak{g}$ then the initial value problem

$$g'(t) = g(t)A(t), \qquad g(0) = 1$$

has a unique solution $g : I \to G$, called the *path ordered exponential of A*, given by

$$g(t) = \mathcal{P}\exp \int_0^t A := \sum_{n=0}^{\infty} \int_0^t \cdots \int_0^{t_n} A(t_n) \cdots A(t_1) dt_n \cdots dt_1 \tag{1.16}$$

$$= 1 + \int_0^t A(t_1)dt_1 + \int_{t>t_1>t_2} A(t_2)A(t_1)dt_1 dt_2 + \cdots \tag{1.17}$$

If G is Abelian then this expression simplifies to

$$g(t) = \exp\left(\int_0^t A(t)\right). \tag{1.18}$$

Proof: The initial value problem can be reformulated into the integral equation

$$g(t) = 1 + \int_0^t g(t_1)A(t_1)dt_1. \tag{1.19}$$

One can then iterate this and obtain

$$\begin{aligned} g(t) &= 1 + \int_0^t \left(1 + \int_0^{t_1} g(t_2)A(t_2)dt_2\right) A(t_1)dt_1 \\ &= 1 + \int_0^t A(t_1)dt + \int_{t>t_1>t_2} g(t_2)A(t_2)A(t_1)dt_1 dt_2. \end{aligned}$$

One can repeat this iteration indefinitely, and the solution will be any fixed point of this iteration. The existence and uniqueness of such a fixed point is guaranteed by the Banach Fixed Point Theorem. This procedure leads exactly to formulas (1.16)-(1.17). In the Abelian case we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp\left(\int_0^t A(t)\right) = \exp\left(\int_0^t A(t)\right) A(t), \tag{1.20}$$

showing that equation (1.18) is indeed valid. Note that in the non-Abelian case there is no such simple formula for the derivative of an exponential. ■

Connections on principal bundles also give us a notion of *parallel transport*. If we take some path $\gamma : x \to y$ then given any point $p \in P_x$, we can take the unique horizontal lift of γ starting at p. Its endpoint will be some point in P_y. This way we actually obtain a map $P_x \to P_y$ know as the *parallel transport along* γ denoted tra_γ. If $\widetilde{\gamma}$ is a horizontal lift of γ then $g \cdot \widetilde{\gamma}$ for $g \in G$ is also a horizontal lift of γ. Thus by uniqueness $\mathrm{tra}_\gamma(g \cdot p) = g \cdot \mathrm{tra}_\gamma(p)$, and parallel transport is actually an equivariant map.

Proposition 1.10 gives us an explicit formula for this parallel transport. Let $\gamma : I \to M$ and choose some trivialization $\tau : I \times G \to \gamma^* P$. Then $A \in \Omega^1(P, \mathfrak{g})$ induces a map $\widehat{A} : I \to \mathfrak{g}$ by first pulling A to $\gamma^* P$ and then applying τ^*. Consider then,

$$\widetilde{\gamma}(t) = \mathcal{P} \exp \int_0^t -\widehat{A}. \tag{1.21}$$

This is a map $I \to G$ and hence a path in $I \times G$. Then one checks that $\tau_* \widetilde{\gamma}$ is a horizontal lift of γ starting at $\tau(0, 1)$. This thus means that if $p = \tau(0, 1)$, then

$$\mathrm{tra}_\gamma(p) : \tau(1, 1) = \widetilde{\gamma}(1) = \mathcal{P} \exp \int_0^1 -\widehat{A}. \tag{1.22}$$

This perspective shows that parallel transport behaves well with respect to concatenation of paths. To state better what we mean we start with a lemma.

Lemma 1.11

Let as in proposition 1.10 $A : I \to \mathfrak{g}$, then we can define

$$\mathcal{P} \exp \int_s^t A \tag{1.23}$$

as the (unique) solution to the initial value problem

$$g'(t) = g(t) A(t), \qquad g(s) = 1 \tag{1.24}$$

Then using this notation we have for any $s, t, u \in I$ that

$$\mathcal{P} \exp \int_s^u A = \left(\mathcal{P} \exp \int_s^t A \right) \left(\mathcal{P} \exp \int_t^u A \right). \tag{1.25}$$

Proof: We set $f(s, t) = \mathcal{P} \exp \int_s^t A$, and $g(u) = f(s, t) f(t, u)$. We compute

$$g'(u) = f(s, t) f(t, u) A(u) = g(u) A(u), \qquad g(t) = f(s, t).$$

This initial value problem is solved by $f(s, u)$, therefore by uniqueness of the solution we conclude that $f(s, u) = g(u) = f(s, t) f(t, u)$, which proves the lemma. ∎

Suppose now that $\gamma = \gamma_2 \circ \gamma_1$, i.e.

$$\gamma(t) = \begin{cases} \gamma_1(2t) & t < \frac{1}{2} \\ \gamma_2(2t - 1) & t \geq \frac{1}{2} \end{cases}.$$

And suppose as before we have some trivialization on $\tau : \gamma^* P \cong I \times G$ such that $\tau(0, 1) = p$ then we can write

$$\begin{aligned}
\mathrm{tra}_\gamma(p) : \tau(1,1) &= \mathcal{P}\exp\int_0^1 -\widehat{A} = \left(\mathcal{P}\exp\int_0^{\frac{1}{2}} -\widehat{A}\right)\left(\mathcal{P}\exp\int_{\frac{1}{2}}^1 -\widehat{A}\right) \\
&= (\mathrm{tra}_{\gamma_1}(\tau(0,1)) : \tau(1/2,1)) \cdot (\mathrm{tra}_{\gamma_2}(\tau(1/2,1)) : \tau(1,1)) \\
&= \mathrm{tra}_{\gamma_2}(\mathrm{tra}_{\gamma_1}(\tau(0,1))) : \tau(1,1) \\
&= \mathrm{tra}_{\gamma_2} \circ \mathrm{tra}_{\gamma_1}(p) : \tau(1,1),
\end{aligned}$$

thus showing that in fact

$$\mathrm{tra}_{\gamma_2\circ\gamma_1} = \mathrm{tra}_{\gamma_2} \circ \mathrm{tra}_{\gamma_1}\,. \tag{1.26}$$

Similarly one can also show that for any path $\gamma : x \to y$

$$\mathrm{tra}_{\gamma^{-1}} = \mathrm{tra}_\gamma^{-1}, \tag{1.27}$$

where $\gamma^{-1}(t) = \gamma(1-t)$. Also note that the constant path has trivial parallel transport. From this it would seem there is some sort of groupoid structure on paths, and one would like to 'upgrade' these computations to some kind of functor. One problem is that by the way we parameterize $\gamma_2 \circ \gamma_1$ the composition of paths is not associative, and we don't get a category of paths. The way to go around this is by taking paths modulo some equivalence relation (thin homotopy) and then proving that parallel transport is independent of this equivalence relation. This is discussed in section 1.8 after first developing more tools.

Remark 1.12

In a trivialization $s : M \to P$ we can consider parallel transport to be group valued. That is, for a path $\gamma : x \to y$ we set $\mathrm{tra}(\gamma) = \mathrm{tra}(\gamma)(s(x)) : s(y) \in G$. Note that in this case we have by lemma 1.11 that $\mathrm{tra}(\gamma \circ \gamma') = \mathrm{tra}(\gamma')\,\mathrm{tra}(\gamma)$. If we would work with right instead of left actions this would be the other way around.

Another property of parallel transport is that it behaves well with respect to homomorphisms. This can be stated in terms of a generalization of the familiar $h \circ \exp = \exp \circ h_*$ one has for Lie groups.

Proposition 1.13

Let $h : G \to H$ be a homomorphism of Lie groups, then the following diagram commutes:

$$\begin{array}{ccc} G^I & \xrightarrow{h} & H^I \\ {\scriptstyle \mathcal{P}\exp}\uparrow & & \uparrow{\scriptstyle \mathcal{P}\exp} \\ \mathfrak{g}^I & \xrightarrow[h_*]{} & \mathfrak{h}^I \end{array} \tag{1.28}$$

Proof: The defining equation for the parallel transport of the map $A(t) : I \to \mathfrak{g}$ is given by

$$g'(t) = g(t)A(t).$$

Now we have to be a bit more precise what this equation means. Note that $g'(t) \in T_{g(t)}G$, so we can make sense of this equation as

$$g'(t) = (L_{g(t)})_* A(t). \tag{1.29}$$

Since h is a homomorphism it satisfies $h = L_{h(g(t))} \circ h \circ L^{-1}_{g(t)}$. Next we apply Dh on both sides of equation (1.29):

$$(h \circ g)'(t) = (L_{h \circ g(t)})_*(h_* A(t)).$$

Thus $\mathcal{P}\exp\int_0^t h_* A = h \circ g(t) = h \circ \mathcal{P}\exp\int_0^t A$, as required. ■

A particular case of parallel transport is when $\gamma : x \to x$ is a loop. In that case $\mathrm{tra}_\gamma(p)$ lies in the same fiber for all $p \in P_x$, and hence differs from p by a unique element $\mathrm{tra}_\gamma(p) : p \in G$. Since (cf. eq. (1.2))

$$\mathrm{tra}_\gamma(p \cdot g) : (p \cdot g) = g(\mathrm{tra}_\gamma(p) : p)g^{-1},$$

this group element does not depend on p *up to conjugation*, and this element is known as the *holonomy* around γ at p. The sets of all holonomies form a group (which is well-defined up to conjugation).

Definition 1.14

Let $p \in P$ then the *holonomy group* $\mathrm{Hol}_p \subset G$ is the subgroup of G generated by the holonomies $\mathrm{tra}_\gamma(p) : p$ around all the loops γ based at $\pi(p)$. The *reduced holonomy group* $\mathrm{Hol}_p^0 \subset \mathrm{Hol}_p$ is the (normal) subgroup consisting of holonomies around contractible loops.

Proposition 1.15

Hol_p^0 is the identity component of Hol_p.

Proof: First we note that Hol_p^0 is connected; if γ is a contractible loop and $\Sigma : \gamma \Rightarrow \mathrm{Id}$ is a homotopy, then $\mathrm{tra}(\Sigma_t)(p) : p$ is a path connecting $\mathrm{tra}(\gamma)(p) : p$ to the identity. Next we will show that Hol_p^0 is of countable index in Hol_p. This implies the Lie algebras of the two groups must coincide. Hence Hol_p^0 must be the identity component of Hol_p, since connected subgroups are classified by sub lie algebras (cf. Thm. A.15). If γ_0, γ_1 are homotopic loops, then $\gamma_0^{-1}\gamma_1$ is contractible, thus $\mathrm{tra}(\gamma_0)(p) : p$ and $\mathrm{tra}(\gamma_1)(p) : p$ land in the same Hol_p^0 coset, and $\gamma \mapsto \mathrm{tra}(\gamma)(p) : p$ descends to a surjective map $\pi_1(M, m) \to \mathrm{Hol}_p \,/\, \mathrm{Hol}_p^0$. Since $\pi_1(M, m)$ is at most countably infinite, so is $\mathrm{Hol}_p \,/\, \mathrm{Hol}_p^0$, which concludes the proof. ■

1.5. Curvature

Connections give distributions. A natural question is to ask whether or not such a distribution is integrable. It turns out this is measured by the curvature of the connection. The curvature is a $\mathfrak{g}$ valued 2-form which exhibits many useful properties. To define it we first note that connections give the following operation on forms.

Definition 1.16

Recall that a connection A gives in particular a decomposition $TP = V \oplus H$ into vertical and horizontal vectors. Let $h : TP \to H$ be the projection to the horizontal part. Then define the *exterior covariant derivative* associated to A by

$$D : \Omega^*(P) \to \Omega^*(P) \qquad D\phi = (d\phi) \circ h. \tag{1.30}$$

Definition 1.17

The *curvature* of a connection $A \in \Omega^1(P, \mathfrak{g})$ is given by $F_A = DA \in \Omega^2(P, \mathfrak{g})$.

It turns out the curvature is a very important object. For one thing, it turns out it transforms well under left multiplication on the fiber; let $g : M \to G$ then

$$L_g^* F_A = \mathrm{Ad}_g F_A. \tag{1.31}$$

To see this, let first $g \in G$ be constant. Let $X, Y \in TP$ and note that we can split them in horizontal and vertical parts X_h, X_v. Note furthermore that $(L_g)_* X_h$ is horizontal and that $(L_g)_*(X_v)$ is vertical. Using this we obtain

$$\begin{aligned}
L_g^*(dA)h(X, Y) &= (dA)h((L_g)_*(X_h + X_v), (L_g)_*(X_h + X_v)) \\
&= (dA)((L_g)_* X_h, (L_g)_* Y_h) \\
&= L_g^*(dA)(X_h, Y_h) = \mathrm{Ad}_g(dA)(X_h, Y_h) \\
&= \mathrm{Ad}_g(dA)h(X, Y).
\end{aligned}$$

Thus the required property holds for constant g. Now let $\phi : G \times P \to P$ be the action and consider $\phi^* F_A$. By similar arguments as in the proof of Proposition 1.6 we note that $\phi^*(F_A)_{x,g} = \mathrm{Ad}_g F_A$. This time there is no Maurer-Cartan form because F_A kills vertical vectors. Again by similar arguments we have that $L_g^* F_A = \mathrm{Ad}_g F_A$ holds for any $g : M \to G$.

The computation is actually more generally true for any basic form:

Definition 1.18

A form $\omega \in \Omega^*(P, \mathfrak{g})$ is *basic* if $L_g^*\omega = \mathrm{Ad}_g \omega$ for all $g \in G$ and $X_\xi \in \ker \omega$ for all $\xi \in \mathfrak{g}$.

Proposition 1.19

Any basic form satisfies the transformation rule

$$L_g^*\omega = \mathrm{Ad}_g \omega$$

for any $g : M \to G$.

Proof: The same argument works as for the curvature form DA ■

Given a connection we can also introduce another derivation operation. Let $\omega \in \Omega^*(P, \mathfrak{g})$ then define[2]

$$d_A\omega := d\omega - \frac{1}{2}[A \wedge \omega]. \tag{1.32}$$

Proposition 1.20

Let $\phi \in \Omega^1(P, \mathfrak{g})$ be basic. Then $D\phi = d_A\phi$.

Proof: We will show $D\phi(X, Y) = d_A\phi(X, Y)$ for any pair X, Y, following the proof in [KN63, p. 79]. First suppose X, Y are both horizontal, then $D\phi(X, Y) = d\phi(X, Y)$ whereas $d_A\phi(X, Y) = d\phi(X, Y) + \frac{1}{2}[A, \phi](X, Y) = d\phi(X, Y)$ since $X, Y \in \ker A$. Now suppose X, Y are both vertical, then both sides are zero since $X, Y \in \ker \phi$. Thus suppose $X = X_\xi$ for $\xi \in \mathfrak{g}$ is vertical and Y is horizontal. Then $D\phi(X_\xi, Y) = d\phi(0, Y) = 0$. On the other hand

$$\begin{aligned} d_A\phi(X_\xi, Y) &= d\phi(X_\xi, Y) - \frac{1}{2}[\xi, \phi(Y)] \\ &= \frac{1}{2}\Big(X_\xi\phi(Y) - Y\phi(X_\xi) - \phi([X_\xi, Y])\Big) - \frac{1}{2}[\xi, \phi(Y)] \\ &= \frac{1}{2}\Big(X_\xi\phi(Y) - \phi([X_\xi, Y])\Big) - \frac{1}{2}[\xi, \phi(Y)]. \end{aligned}$$

First of all since $L_{\exp t\xi}$ is the time t flow of X_ξ we have

$$[X_\xi, Y]_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (L_{\exp -t\xi})_* Y_{\exp t\xi \cdot p},$$

[2]For right actions the sign of the second term would be different.

which is zero since horizontal vector fields are invariant. Thus it remains to show that

$$X_\xi \phi(Y) = [\xi, \phi(Y)].$$

This follows because

$$X_\xi \phi(Y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (L^*_{\exp t\xi}\phi)(Y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{\exp t\xi}\, \phi(Y) = \mathrm{ad}_\xi\, \phi(Y) = [\xi, \phi(Y)].$$

Thus we conclude $D\phi = d_A\phi$. ■

Theorem 1.21 (Structure Equation)

The curvature satisfies

$$F_A = DA = d_A A = dA - \frac{1}{2}[A \wedge A]. \tag{1.33}$$

Proof: If A were basic, then this would follow trivially from the previous proposition. However $A(X_\xi) = \xi \neq 0$, and is therefore not basic, although $A \circ h$ is certainly basic. Furthermore $F_A = DA = D(Ah) = d_A(Ah)$, therefore we just need to prove $d_A A(X,Y) = d_A(Ah)(X,Y)$ for all X, Y. If X, Y are both horizontal this is trivial. If $X = X_\xi$, $Y = X_\eta$ are both vertical then $d_A(Ah)(X,Y) = 0$ and

$$d_A A(X_\xi, X_\eta) = (dA)(X_\xi, X_\eta) - [A(X_\xi), A(X_\eta)] = X_\xi(\eta) - X_\eta(\xi) + A([X_\xi, X_\eta]) - [\xi, \eta] = 0.$$

This is zero because $X_\xi(\eta) = 0$ since η is constant, and because $[X_\xi, X_\eta] = -X_{[\xi,\eta]}$. Now suppose $X = X_\xi$ is vertical and Y horizontal. Then again $d_A(Ah) = 0$, and

$$d_A A(X_\xi, Y) = (dA)(X_\xi, Y) + [A(X_\xi), A(Y)] = YA(X_\xi) - X_\xi A(Y) - A([X_\xi, Y]) = 0,$$

since $Y \in \ker A$ and $YA(X_\xi) = Y\xi = 0$. Thus $d_A(Ah) = d_A A$ and we conclude the structure equation. ■

Recall that according to definition 1.7 a connection locally consists of forms $A_i \in \Omega^1(U_i, \mathfrak{g})$. In terms of this local data the curvature is given by forms $F_i \in \Omega^2(M, \mathfrak{g})$ given by

$$F_i = dA_i - \frac{1}{2}[A_i \wedge A_i]. \tag{1.34}$$

The fact that F is basic then means that

$$F_j = \mathrm{Ad}_{g_{ij}} F_i = g_{ij} F_i g_{ij}^{-1}, \tag{1.35}$$

which is not immediately clear from the structure equation. This can of course be deduced from the structure equation directly, but the computation is fairly difficult. In fact by the same argument as the discussion preceding definition 1.7 we can say something about the local behavior of any basic form:

Proposition 1.22

Let $\omega \in \Omega^n(P, \mathfrak{g})$ be a basic form, and $s_i : U_i \to P|_{U_i}$ a trivialization with transition functions $g_{ij} : U_{ij} \to G$. Then the forms $\omega_i := s_i^* \omega \in \Omega^n(U_i, \mathfrak{g})$ satisfy the transformation rule

$$\omega_j = \mathrm{Ad}_{g_{ij}} \omega_i. \tag{1.36}$$

Furthermore any collection $\omega_i \in \Omega^n(U_i, \mathfrak{g})$ satisfying this rule uniquely determines a basic form.

1.6. Non-Abelian Stokes' Theorem

Suppose $A \in \Omega^1(M)$ is a connection on a trivial $U(1)$ bundle. If Σ is a disk bounding a loop γ then Stokes' Theorem tells us that

$$\int_\gamma A = \int_\Sigma dA = \int_\Sigma F_A. \tag{1.37}$$

And thus the parallel transport around a loop can be expressed using only the curvature of A. This has as a direct consequence that if the curvature vanishes, then the parallel transport around any contractible loop will be zero. A homotopy $\Sigma : \gamma_0 \Rightarrow \gamma_1$ is the same thing as a homotopy $\gamma_0 \gamma_1^{-1} \Rightarrow 1$. Thus this also means that the parallel transport along two homotopic loops is the same if the curvature vanishes. This can be generalized to a similar statement valid for any G-bundle connection, and this is known as the non-Abelian Stokes' Theorem.

Before stating the non-Abelian Stokes' Theorem let us consider a heuristic argument why it should be true. Suppose $\Gamma : \gamma_0 \to \gamma_1$ is a homotopy. Then the loop around the boundary can be decomposed into four smaller loops as shown in figure 1.23.

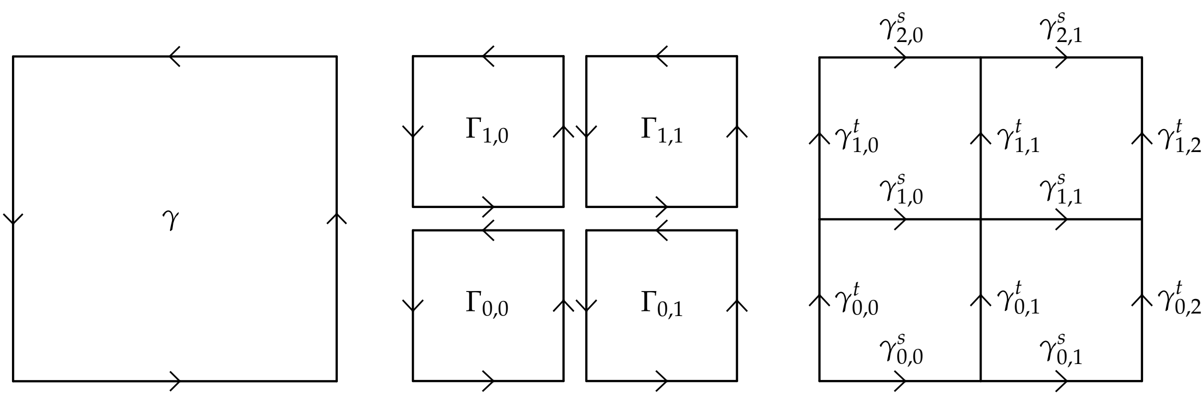


Figure 1.23: Decomposing a loop γ into 4 smaller loops $\Gamma_{i,j}$. In general one can compose γ into arbitrarily small loops, the holonomy around each of which is approximated by the curvature (cf. lemma 1.33).

In terms of the notation of figure 1.23, if we set

$$\Gamma_{0,0} = (\gamma^s_{0,0})^{-1}(\gamma^t_{1,0})^{-1}\gamma^s_{0,1}\gamma^t_{0,0}, \tag{1.38}$$

$$\Gamma_{0,1} = (\gamma^t_{0,0})^{-1}\left[(\gamma^s_{0,1})^{-1}(\gamma^t_{1,1})^{-1}\gamma^s_{0,2}\gamma^t_{0,1}\right]\gamma^t_{0,0}, \tag{1.39}$$

$$\Gamma_{1,0} = (\gamma^s_{0,0})^{-1}\left[(\gamma^s_{1,0})^{-1}(\gamma^t_{2,0})^{-1}\gamma^s_{1,1}\gamma^t_{1,0}\right]\gamma^s_{0,0}, \tag{1.40}$$

$$\Gamma_{1,1} = (\gamma^s_{0,0})^{-1}(\gamma^t_{1,0})^{-1}\left[(\gamma^s_{1,1})^{-1}(\gamma^t_{2,1})^{-1}\gamma^s_{1,2}\gamma^t_{1,1}\right]\gamma^t_{1,0}\gamma^s_{0,0}. \tag{1.41}$$

Then we have that

$$\gamma = (\Gamma_{1,0} \circ \Gamma_{1,1}) \circ (\Gamma_{0,0} \circ \Gamma_{0,1}). \tag{1.42}$$

In principle we can subdivide in this way infinitely. By Lemma 1.33 the holonomy around an infinitesimal loop is given by the curvature, and thus this way we can express the parallel transport around γ as an integral over the curvature. This is made precise by the following theorem.

Theorem 1.24 (non-Abelian Stokes)

Let $A \in \Omega^1(M, \mathfrak{g})$ be a connection on a trivial G bundle. Let $\Gamma : I^2 \to M$ be a homotopy, let Γ_s be the path $\Gamma|_{\{s\}\times I}$, and let $\Gamma_{s,t}$ denote the path $\tau \mapsto \Gamma(s, t\tau)$. Then with respect to any trivialization

$$\mathrm{tra}(\Gamma_0)\,\mathrm{tra}(\Gamma_1)^{-1} = \mathcal{P}\exp\int_0^1 \mathcal{A}, \tag{1.43}$$

where $\mathcal{A} \in \Omega^1(I, \mathfrak{g})$ is defined by

$$\mathcal{A}_s = \left[\int_0^1 \mathrm{d}t\ \mathrm{Ad}_{\mathrm{tra}(\Gamma_{s,t})}\,\Gamma^* F_A(\partial_1, \partial_2)\right]\mathrm{d}s. \tag{1.44}$$

Here ∂_1, ∂_2 denotes the canonical frame on TI^2.

We stress that $\Gamma^* F_A(\partial_1, \partial_2)$ is a $\mathfrak{g}$ valued function, therefore integration produces another $\mathfrak{g}$ valued function and hence taking its path ordered exponent makes sense.

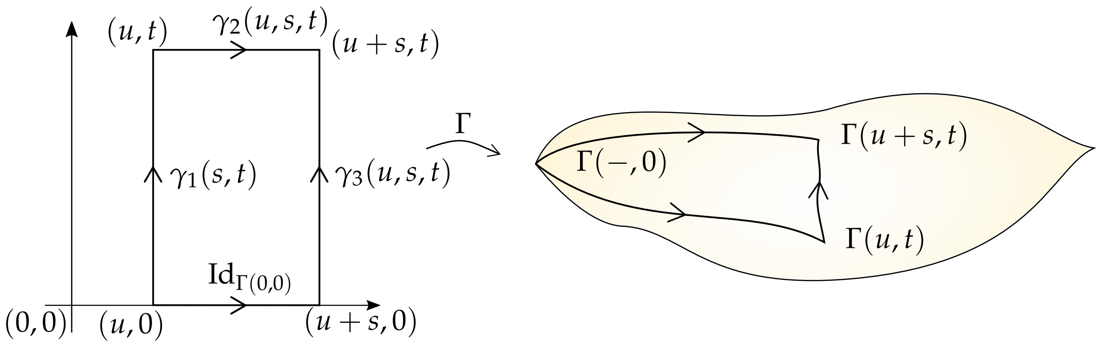


Figure 1.25: Decomposition of $\gamma(u,s,t) = \gamma_3^{-1}(u,s,t) \circ \gamma_2(u,s,t) \circ \gamma_1(s,t)$.

Proof: Let $f(s) = \left(\mathrm{tra}^{-1}_{\Gamma_s} \circ \mathrm{tra}_{\Gamma_0}(p)\right) : p$ for some $p \in P_{\Gamma(0,0)}$. If we have some section $\sigma : \Gamma(I^2) \to P$ with $\sigma(0,0) = p$ we can write this as

$$\mathrm{tra}_{\Gamma_0} \cdot \mathrm{tra}_{\Gamma_s^{-1}} := \left[\mathrm{tra}_{\Gamma_0}(p) : \sigma(0,1)\right] \cdot \left[\mathrm{tra}^{-1}_{\Gamma_s}(\sigma(s,1)) : p\right].$$

It is important to note that there is an inversion of order present here (cf. lemma 1.11). Then consider

$$f^{-1}(u)f(u+s) = \mathrm{tra}_{\Gamma_u}\,\mathrm{tra}_{\Gamma^{-1}_{u+s}}\,.$$

Differentiating with respect to s we obtain the initial value problem:

$$f'(u) = f(u)\,\frac{\partial}{\partial s}\,\mathrm{tra}_{\Gamma_u}\,\mathrm{tra}_{\Gamma^{-1}_{u+s}}\bigg|_{s=0}, \qquad f(0)=0, \tag{1.45}$$

which is solved by a path ordered exponential. Consider the loop

$$\gamma(u,s,t) = \gamma_3^{-1}(u,s,t)\circ\gamma_2(u,s,t)\circ\gamma_1(s,t), \tag{1.46}$$

as in figure 1.25. Note that $\gamma(u,s,1) = \Gamma^{-1}_{u+s}\circ\Gamma_u$, thus to obtain the required formula we have to show that

$$\frac{\partial^2}{\partial s\partial t}\,\mathrm{tra}_{\gamma_1}\,\mathrm{tra}_{\gamma_2}\,\mathrm{tra}^{-1}_{\gamma_3}\bigg|_{(0,t)} = \mathrm{Ad}_{\mathrm{tra}(\Gamma_{u,t})}\,\Gamma^*F_A(\partial_1,\partial_2)\,. \tag{1.47}$$

This is a straightforward computation, and we leave out some of its details.

$$\frac{\partial}{\partial t}\,\mathrm{tra}_{\gamma_1}\,\mathrm{tra}_{\gamma_2}\,\mathrm{tra}^{-1}_{\gamma_3} = \mathrm{tra}_{\gamma_1}\left[-A_{u,t}(\partial_2)\,\mathrm{tra}_{\gamma_2} + \frac{\partial\,\mathrm{tra}_{\gamma_2}}{\partial t} + \mathrm{tra}_{\gamma_2}\,A_{u+s,t}(\partial_2)\right]\mathrm{tra}^{-1}_{\gamma_3}\,.$$

Here $A_{u,t} = (\Gamma^*\sigma^*A)_{u,t}$, which arises through equation 1.15. Now the s derivative:

$$\frac{\partial^2}{\partial s\partial t}\,\mathrm{tra}_{\gamma_1}\,\mathrm{tra}_{\gamma_2}\,\mathrm{tra}^{-1}_{\gamma_3}\bigg|_{(0,t)} = \mathrm{tra}_{\gamma_1}\left[A_{u,t}(\partial_2)A_{u,t}(\partial_1) - \frac{\partial A_{u,t}(\partial_1)}{\partial t} - A_{u,t}(\partial_1)A_{u,t}(\partial_2) + \frac{\partial A_{u,t}(\partial_2)}{\partial s}\right]\mathrm{tra}^{-1}_{\gamma_3}$$

When $s=0$ we have $\gamma_1=\gamma_3=\Gamma_{u,t}$. Identifying the middle term as Γ^*F proves equation (1.47). ■

Note that

$$\mathrm{Ad}_{\mathrm{tra}(\Gamma_{s,t})}\,\Gamma^*F = L_{\mathrm{tra}(\Gamma^*_{s,t})}\Gamma^*F$$

We define $\widetilde{\Gamma}(s,t) = \widetilde{\Gamma}_s(t)$ with $\widetilde{\Gamma}_s$ the horizontal lift of Γ_s starting at $\sigma(\Gamma(0,0))$ (if σ is a section). Then since $\Gamma_*\partial_i$ is a horizontal vector field we obtain

$$\mathrm{tra}(\Gamma_0)\,\mathrm{tra}(\Gamma_1)^{-1} = \mathcal{P}\exp\int_0^1 \mathrm{d}s\int_0^1 \mathrm{d}t\;\widetilde{\Gamma}^*F(\partial_1,\partial_2) \tag{1.48}$$

which makes sense without reference to a section, so long as we fix a basepoint $\widetilde{\Gamma}(s,0)\in P$.

1.7. The Ambrose-Singer Theorem

Note that if X, Y are horizontal then $F_A(X,Y) = dA(X,Y) = -A([X,Y])$ so $F_A = 0$ if and only if the bracket of horizontal vectors is horizontal. That is, if and only if the horizontal distribution defined by A is involutive. In this case we call the connection *flat*. By the non-Abelian Stokes Theorem we note that we can express holonomy around contractible loops in terms of the curvature. In fact it turns out the holonomy group is completely determined by the curvature form. This is made precise by the Ambrose-Singer Theorem [KN63, p. 89], [AS53]. Before stating it, it is convenient to define an equivalence relation on P:

Definition 1.26

Two points $q, p \in P$ are said to be *horizontally equivalent* if there is a path γ such that $\mathrm{tra}_\gamma(p) = q$, i.e. p and q are the endpoints of some horizontal path. This is an equivalence relation, and we write $q \sim p$

Theorem 1.27 (Ambrose-Singer)

Let A be a connection on a G-bundle $\pi : P \to M$. Let $p \in P$ and consider the $\mathrm{Hol}_p \subset G$. Let $\mathfrak{hol}_p \subset \mathfrak{g}$ be its Lie algebra, then

$$\mathfrak{hol}_p = \left\{ F_q(X,Y) \,\middle|\, q \sim p,\ X,Y \in T_qP \right\}. \tag{1.49}$$

Since the reduced holonomy group Hol^0_p is the identity component of Hol_p (cf. prop. 1.15), we deduce the following corollary:

Corollary 1.28

Parallel transport is homotopy independent if and only if the connection is flat (i.e. $F_A = 0$).

We shall provide a proof quite different from the original. The main ingredient in the proof is the non-Abelian Stokes' Theorem; it tells us that every element in Hol^0_p is of form (cf. eq. (1.48))

$$\mathcal{P}\exp \iint \widetilde{\Gamma}^* F,$$

which tells us a lot about the Lie algebra of Hol^0_p:

Lemma 1.29

Every element of $\mathfrak{hol}_p$ is of form

$$\int_0^1 \mathrm{d}t\, \widetilde{\gamma}^* F \tag{1.50}$$

for some loop $\gamma : x \to x$ with $\pi(p) = x$.

Proof: Let $\Omega(M,x)$ be the space of loops at x, then we have a surjection $\psi : \Omega(M,x) \to \mathrm{Hol}_p$ given by $\gamma \mapsto \mathrm{tra}_\gamma(p) : p$. We claim that every path in Hol_p lifts to a path in $\Omega(M, x)$, at least locally. This is equivalent to ψ being a *subduction* of diffeological spaces [IZ13, art. 1.48], and the fact that it is a subduction is clear from [IZ13, art. 8.35]. This means that every $X \in \mathfrak{hol}_p$ can be written as $(\psi_*\Sigma)'(0)$ for Σ a homotopy of loops $x \to x$. By the non-Abelian Stokes' Theorem $(\psi_*\Sigma)(s)$ is of form

$$\mathcal{P}\exp \int_0^s \mathrm{d}s \int_0^1 \mathrm{d}t\, \widetilde{\Sigma}_s^* F$$

differentiating at $s = 0$ gives the required result. ■

To prove the Ambrose-Singer Theorem we only need to prove that $F_q(X, Y) \in \mathfrak{hol}_p$ for any $q \sim p$. First we show that if $q \sim p$, then their holonomy groups coincide.

Proposition 1.30

Suppose $\sigma : x \to y$ is such that $\mathrm{tra}_\sigma(p) = q$. Then if $\gamma : x \to x$ is a loop, we have

$$\mathrm{tra}_\sigma \,\mathrm{tra}_\gamma \,\mathrm{tra}_\sigma^{-1}(q) : q = \mathrm{tra}_\gamma(p) : p. \tag{1.51}$$

In other words $\mathrm{tra}_\sigma \mathrm{Hol}_p \mathrm{tra}_\sigma^{-1} = \mathrm{Hol}_p \subset G$. And $\gamma \mapsto \sigma\gamma\sigma^{-1}$ shows $\mathrm{Hol}_p = \mathrm{Hol}_q \subset G$.

Proof: This is a straightforward computation:

$$\begin{aligned}\mathrm{tra}_\sigma \,\mathrm{tra}_\gamma \,\mathrm{tra}_\sigma^{-1}(q) : q &= \mathrm{tra}_\sigma \,\mathrm{tra}_\gamma(p) : q = \mathrm{tra}_\sigma(\mathrm{tra}_\gamma(p) : p \cdot p) : q \\ &= (\mathrm{tra}_\gamma(p) : p)(\mathrm{tra}_\sigma(p) : q) = \mathrm{tra}_\gamma(p) : p.\end{aligned}$$ ■

A less trivial result is that $\mathrm{tra}_\gamma \in \mathrm{Hol}_p$, at least with respect to some appropriate section. This is implied by the Reduction Theorem [KN63, Thm. 2.7.1]:

Theorem 1.31 (Reduction Theorem)

Let $P \to M$ be a principal bundle with connection A. Let $p \in P$ and denote

$$P(p) = \{\, q \in P \mid q \sim p \,\}. \tag{1.52}$$

Then $P(p)$ is a subbundle of P with structure group Hol_p and the connection A is reducible to a connection on $P(p)$, i.e. $A|P(p)$ is a $\mathfrak{hol}_p$-valued connection on $P(p)$.

Thus it remains to show that $F_q(X, Y)$ actually lies in $\mathfrak{hol}_p$. Since for $q \sim p$ the holonomy groups coincide, it is without loss of generality enough to show it for a single point $p \in P$. This is done by showing that the curvature arises as the holonomy around an 'infinitesimal loop'. First we need to make precise what we mean by such an 'infinitesimal loop'. Let $\Gamma : I^2 \to M$ be a map with $\Gamma(0, 0) = m$, then for each $s, t \in I^2$ we can define a new map $\Gamma_{s,t} : I^2 \to M$ by $\Gamma_{s,t}(u, v) = \Gamma(su, tv)$, i.e. be rescaling Γ to a smaller domain. The boundary of $\Gamma_{s,t}$ defines a loop $\gamma_{s,t} : m \to m$, explicitly:

$$\gamma_{s,t}(u) = \begin{cases} \Gamma(4us, 0) & u \in [0, \frac{1}{4}] \\ \Gamma(s, (4u-1)t) & u \in [\frac{1}{4}, \frac{1}{2}] \\ \Gamma((3-4u)s, t) & u \in [\frac{1}{2}, \frac{3}{4}] \\ \Gamma(0, (4-4u)t) & u \in [\frac{3}{4}, 1]. \end{cases} \tag{1.53}$$

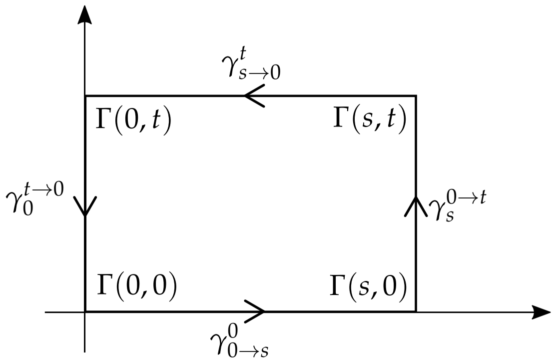


Figure 1.32: Sketch of $\gamma_{s,t}$ from equation (1.53) decomposed into four paths.

Lemma 1.33

Let $p \in P_m$ and suppose A is a connection on P with curvature F. For any $\Gamma : I^2 \to M$ with $\Gamma(0,0) = m$ define $\gamma_{s,t}$ as above, then we have

$$\left.\frac{\partial^2}{\partial s \partial t}\right|_{(0,0)} \operatorname{tra}(\gamma_{s,t})(p) : p = -F_p(X, Y). \tag{1.54}$$

Here X, Y are the horizontal vectors at p respectively corresponding to

$$\left.\frac{\partial \Gamma(s,0)}{\partial s}\right|_0, \qquad \left.\frac{\partial \Gamma(0,t)}{\partial t}\right|_0.$$

Proof: Assume the bundle is trivial over the image of Γ and that $(m, 1)$ corresponds to $p \in P_m$ in the trivialization. Then parallel transport is just group valued, which will make the computation easier. We decompose the path $\gamma_{s,t}$ as sketched in figure 1.32:

$$\gamma_{s,t} = \gamma_0^{t\to 0} \gamma^t_{s\to 0} \gamma_s^{0\to t} \gamma^0_{0\to s}, \tag{1.55}$$

which leaves us to compute

$$\left.\frac{\partial^2}{\partial s \partial t}\right|_{(0,0)} \operatorname{tra}(\gamma^0_{0\to s}) \operatorname{tra}(\gamma_s^{0\to t}) \operatorname{tra}(\gamma^t_{s\to 0}) \operatorname{tra}(\gamma_0^{t\to 0}).$$

We first compute the partial derivative with respect to s:

$$
\begin{aligned}
&-\operatorname{tra}(\gamma^0_{0\to s})A_{\Gamma(s,0)}(X)\operatorname{tra}(\gamma_s^{0\to t})\operatorname{tra}(\gamma^t_{s\to 0})\operatorname{tra}(\gamma_0^{t\to 0})\\
&+\operatorname{tra}(\gamma^0_{0\to s})\left(\frac{\partial}{\partial s}\operatorname{tra}(\gamma_s^{0\to t})\right)\operatorname{tra}(\gamma^t_{s\to 0})\operatorname{tra}(\gamma_0^{t\to 0})\\
&+\operatorname{tra}(\gamma^0_{0\to s})\operatorname{tra}(\gamma_s^{0\to t})A_{\Gamma(s,t)}\operatorname{tra}(\gamma^t_{s\to 0})\operatorname{tra}(\gamma_0^{t\to 0}).
\end{aligned}
$$

Where we use equation (1.15) to compute $\partial_t \operatorname{tra}(\gamma_s^{0\to t})$. To compute $\partial_t \operatorname{tra}(\gamma_0^{t\to 0})$ we also use (1.15) together with the fact that $(f^{-1})'(t) = -(f^{-1})(t)f'(t)(f^{-1})(t)$; this follows from differentiating $1 = f^{-1}f$. Now taking the partial derivative with respect to s and evaluating at 0 we get

$$
\begin{aligned}
&-A(X)\,(-A(Y)+A(Y)) - \frac{\partial}{\partial s}\left(\operatorname{tra}(\gamma_s^{0\to t})A_{\Gamma(s,t)}\right) - A(Y)A(X) + Y(A(X)) + A(X)A(Y)\\
&= -XA(Y) + YA(X) - A(Y)A(X) + A(X)A(Y) = -F(X,Y),
\end{aligned}
$$

as required. Together with the earlier remarks this also proves the Ambrose-Singer Theorem. ∎

1.8. Thin homotopy

One consequence of this is that the parallel transport doesn't change under reparameterization of paths, this is because reparameterizations are given by homotopies that 'don't sweep out any area'. More precisely:

Definition 1.34

Let $h : I^2 \to M$ be a (smooth) homotopy. We say h is a *thin homotopy* if $\operatorname{rank} dh : TI^2 \to TM \leq 1$ everywhere.

This definition is more general than most notions of reparameterization. For example we don't require that $\operatorname{rank} dh = 1$ everywhere; it could well be zero. This is useful because it allows us to parameterize paths in such a way as to have 'sitting instants' near its endpoints, i.e. we can use thin homotopies to make paths constant for some finite time at its endpoints. This will make composition of paths in the smooth setting more natural and simple.

Corollary 1.35

Parallel transport is *thin homotopy invariant*, i.e. *thinly homotopic* paths have the same parallel transport

Proof: Let $h : I^2 \to M$ be a thin homotopy. Then $h^*F = 0$, since F is 2-form and h is at most of rank one. Hence by the non-Abelian Stokes' Theorem $\operatorname{tra}(h_0) = \operatorname{tra}(h_1)$. ∎

Definition 1.36

For any manifold M we define the *path groupoid* $\mathcal{P}_1(M)$ to be the groupoid with objects points in M and morphisms $x \to y$ *thin homotopy classes* of paths $x \to y$. The identity morphisms $x \to x$ are thin homotopy classes of constant paths. To compose two morphisms, let $[\gamma] : x \to y$ and $[\gamma'] : y \to z$ be two thin homotopy classes of paths. Then $[\gamma \circ \gamma'] : x \to z$ is defined by first applying a thin homotopy to γ and γ' so that the paths are constant near their endpoints, and then concatenating the two and taking the thin homotopy class of the result. This ensures that the concatenated path is again smooth, and that the composition is associative. Inversion is $[\gamma] \to [\gamma^{-1}]$ where γ^{-1} is the reversed path.

To see that the inversion map is correct we need to show that $\gamma \circ \gamma^{-1}$ is thinly homotopic to a constant path. We can define a thin homotopy as:

$$h(s,t) = \begin{cases} \gamma(st) & t \leq \frac{1}{2} \\ \gamma(s - s(t-1/2))) & t \geq \frac{1}{2} \end{cases}$$

This is thin because everywhere its differential is proportional to $\gamma'(u)$. If we can compose two paths γ,γ', then $\mathrm{tra}_{\gamma\circ\gamma'} = \mathrm{tra}_{\gamma} \circ \mathrm{tra}_{\gamma'}$ by uniqueness of parallel transport (cf. page 11). This amounts to tra defining a functor $\mathcal{P}_1(M) \to G$-**tor**.

Definition 1.37

Let G by a Lie group, and let G-**tor** be the category of G-torsors, i.e. of smooth manifolds equipped with a free and transitive G-action.[a] The morphisms in this category are G-equivariant maps.

[a] G acting on itself by multiplication is a G-torsor. Any G-torsor is necessarily isomorphic to G, but not canonically so. The isomorphism $X \cong G$ amounts to choosing an $x \in X$ and sending it to $e \in G$ and using the action to uniquely extend it to an equivariant map $X \to G$.

Proposition 1.38

Let $P \to M$ be a G-bundle with connection. Parallel transport defines a functor $\mathcal{P}_1(M) \to G$-**tor**. It sends objects $x \in M$ to the G-torsors P_x, and thin homotopy classes of paths $[\gamma] : x \to y$ to $\mathrm{tra}_\gamma : P_x \to P_y$.

Remark 1.39

If $s : M \to P$ is a section and $\gamma : x \to y$ then we get a map $F : \mathcal{P}_1(M) \to G$ given by $\gamma \mapsto \mathrm{tra}_\gamma(s(x)) : s(y)$. By lemma 1.11 this satisfies $F(\gamma_1 \circ \gamma_2) = F(\gamma_2) \cdot F(\gamma_1)$. Thus seeing G as a category with one object and composition $g \circ h = h \cdot g$ then F is a functor $\mathcal{P}_1(M) \to G$.

This functor actually captures the full information of the bundle P and its connection. Let $\Omega_m(M)$ denote the loop space of M at m. Then note that any connection on a principal P bundle gives a functor $F : \Omega_m(M) \to G$, after choosing a basepoint $p \in P_m$. Call any map $\phi : U \to \Omega_m(M)$ such that the map $U \times I \to M$, $(u,t) \mapsto \phi(u)(t)$ is smooth a *plot* (i.e. we give $\Omega_m(M)$ a diffeological structure, cf. section 3.5). Then F is smooth in the sense that $F \circ \phi : U \to G$ is smooth for any plot. The converse of this also true: [Bar90].

Theorem 1.40

Let $F : \Omega_m(M) \to G$ be a smooth functor such that F is thin-homotopy invariant. Then there is a principal bundle $P \to M$ with connection and a basepoint $p \in P_m$ such that F arises from the parallel transport of P.

Thus the functorial point of view of parallel transport captures all the information of both the connection and bundle. This justifies the fact that when considering 2-transport over 2-bundles we will start with a functorial point of view and derive from this a definition of 2-transport in terms of connection forms.

1.9. Wilson loops

Let us consider again holonomy around loops $\gamma : I \to M$ (cf. page 12). We know that up to conjugation they are determined by the group element $\mathrm{tra}_\gamma(p) : p$ for $p \in T_{\gamma(0)}P$. Loops can be identified with maps $S^1 \to M$, except to get a map $I \to M$ we need to fix a basepoint. We can wonder what happens to the holonomy if we change basepoint. Let $\gamma : S^1 \to M$ and for $x \in S^1$ denote $\gamma_x : I \to M$ the map obtained by setting x as the basepoint. If $y \in S^1$ is another point, then denote $\gamma_{x,y} : I \to M$ to be the path obtained by restricting γ to the interval $[x,y] \subset S^1$ starting at x and going counterclockwise to y. Then,

$$\mathrm{tra}_{\gamma_y} = \mathrm{tra}_{\gamma_{x,y}}\, \mathrm{tra}_{\gamma_x}\, \mathrm{tra}^{-1}_{\gamma_{x,y}} \tag{1.56}$$

since $\gamma_y = \gamma_{x,y}\gamma_x\gamma_{x,y}^{-1}$ up to thin homotopy. Let $p \in T_{\gamma(y)}P$ and let $q = \mathrm{tra}^{-1}_{\gamma_{x,y}}(p)$ then

$$\begin{aligned}
\mathrm{tra}_{\gamma_{x,y}}\, \mathrm{tra}_{\gamma_x}\, \mathrm{tra}^{-1}_{\gamma_{x,y}}(p) &= \mathrm{tra}_{\gamma_{x,y}}\, \mathrm{tra}_{\gamma_x}(q) \\
&= (\mathrm{tra}_{\gamma_x}(q) : q) \cdot \mathrm{tra}_{\gamma_{x,y}}(q) \\
&= (\mathrm{tra}_{\gamma_x}(q) : q) \cdot p.
\end{aligned}$$

And thus we conclude

$$\mathrm{tra}_{\gamma_y}(p) : p = \mathrm{tra}_{\gamma_x}(q) : q, \tag{1.57}$$

and *up to conjugation* one can define the holonomy of the loop $\gamma : S^1 \to M$. In other words, γ defines an element $\mathrm{hol}_\gamma \in G/G$, where G/G denotes the quotient of G by the action of conjugation. Suppose we now have some representation $\phi : G \to \mathrm{Aut}(V)$. Then $\mathrm{Tr} : \mathrm{Aut}(V) \to \mathbb{R}$ is conjugation invariant and thus descends to a map $G/G \to \mathbb{R}$ (also denoted Tr). Thus one can consider the *Wilson loop*:

$$W_\gamma(A) = \mathrm{Tr}(\mathrm{hol}_\gamma) = \mathrm{Tr}(\mathrm{tra}_{\gamma_x}(p) : p) = \mathrm{Tr}\left(\mathcal{P}\exp\int_{S^1} -\rho(A)\right), \tag{1.58}$$

which is then independent on the choice of $x \in S^1$ and $p \in T_{\gamma(x)}P$. The last equation is abuse of notation and denotes the element of G obtained from evaluating equation (1.21) at $t = 1$ some choice of basepoint. Such notation is common in physics literature. One interesting property of the Wilson loop is that it's a *gauge invariant quantity*.

Definition 1.41

Let $\mathcal{A}$ denote the affine space of connections on some principal G bundle P. A function $S : \mathcal{A} \to \mathbb{R}$ is said to be *gauge invariant* if $S(L_g^*A) = S(A)$ for any $g : M \to G$.

The action of G on P is physically interpreted as a symmetry. Any two gauge-equivalent connections A, A' (i.e. $\exists g$ such that $L_g^*A = A'$) represent the same physical state. Thus any physical observable must be gauge invariant.

Proposition 1.42

Wilson loops are gauge invariant

Proof: Without loss of generality suppose $G \subset \mathrm{GL}_n$ and let $\gamma : I \to M$ be a loop, then we will show that

$$\mathcal{P}\exp\int_I -L_g^*A = g(0)\left(\mathcal{P}\exp\int_I -A\right)g^{-1}(0), \tag{1.59}$$

for any connection on $I \times G$ and any $g : I \to G$ such that $A(0) = A(1)$ and $g(0) = g(1)$. Suppose $f : I \to G$ solves the initial value problem

$$f'(t) = -f(t)A(t), \qquad f(0) = 1.$$

Let us define

$$f_g(t) = g(0)f(t)g(t)^{-1}. \tag{1.60}$$

Then we compute

$$\begin{aligned}
f_g'(t) &= g(0)f'(t)g(t)^{-1} - g(0)f(t)g(t)^{-1}g'(t)g(t)^{-1} \\
&= -g(0)f(t)A(t)g(t)^{-1} - f_g(t)(dgg^{-1})(t) \\
&= -f_g(t)\big(g(t)A(t)g(t)^{-1} + (dgg^{-1})(t)\big) \\
&= -f_g(t)L_g^*A(t).
\end{aligned}$$

Here we computed $(g^{-1})'(t)$ using

$$0 = (gg^{-1})' = g'g^{-1} + g(g^{-1})'. \tag{1.61}$$

Note that $f_g(0) = g(0)f(0)g(0)^{-1} = 1$ and thus $f_g(1)$ gives the parallel transport along γ of $L_g^* A$ by proposition 1.10. We thus have

$$\mathcal{P}\exp\int_I -L_g^* A = f_g(1) = g(0)f(1)g(1)^{-1} = g(0)\left(\mathcal{P}\exp\int_I -A\right)g^{-1}(0),$$

since $g(0) = g(1)$. Now using this result we have that if $g : S^1 \to M$ and $x \in S^1$ is some basepoint then

$$W_\gamma(L_g^* A) = \mathrm{Tr}(g(x)(\mathrm{tra}_{\gamma_x}(p) : p)g(x)^{-1}) = \mathrm{Tr}(\mathrm{tra}_{\gamma_x}(p) : p) = W_\gamma(A). \quad \blacksquare$$

1.10. Chern-Weil theory

Principal G bundles admit classifying spaces:

Theorem 1.43

For every Lie group G there exists a space BG together with a G-bundle $EG \to BG$, such that the total space EG is contractible and for each M there is an isomorphism

$$[M, BG] \to G\text{-bund}(M), \qquad [f] \mapsto [f^* EG], \tag{1.62}$$

where $[X, Y]$ denotes the set of homotopy classes of maps $X \to Y$, and G-bund is the set of isomorphism classes of G bundles over M. We call BG the *classifying space* and EG the *universal bundle*.

These spaces are typically infinite dimensional (some exceptions are G finite or contractible). The only requirement on EG is that it's a contractible space with a free G action, and therefore EG and BG are certainly not unique. The fact that EG is contractible means in particular that $\pi_i(EG)$ is trivial for all i. From the long exact sequence in homotopy of the fibration $G \to EG \to BG$ we then obtain $\pi_{i+1}(G) \cong \pi_i(BG)$. For example $\pi_1(G) = \pi_2(BG) = [S^2, BG] = G(S^2)$. Thus G-bundles over S^2 are the same thing as elements of $\pi_1(G)$. This can also been seen from the clutching construction; we can trivialize a bundle over two hemispheres and the transition function is then given by a map $I \times S^1 \to G$. The isomorphism class of this bundle then only depends on the homotopy class of the transition function.

For $U(n)$ the classifying space is fairly explicit. Let $\mathrm{Gr}(n,k)$ be the Grassmannian of n-planes in $\mathbb{C}^k$. Then $BU(n) = \mathrm{Gr}(n, \infty)$, i.e. the direct limit obtained from the inclusions $\mathrm{Gr}(n,k) \to \mathrm{Gr}(n,l)$ for $l > k$. Similarly $EG = V_n(\mathbb{C}^\infty)$, or n-tuples of orthonormal vectors in $\mathbb{C}^\infty$. Recall that any compact

Lie group is isomorphic to a subgroup of $U(n)$ for an appropriately large n (cf. proposition A.4). The free $U(n)$ action on $V_k(\mathbb{C}^\infty)$ induces a free action G action after we embed G in $U(n)$, and therefore $BG = V_k(\mathbb{C}^\infty)/G$.

The classifying space gives us a large class of 'characteristic classes'. Let $c \in H^i(BG, \mathbb{R})$ and let $P \to M$ be a bundle classified by $f : M \to BG$. Then we can consider $f^*c \in H^i(M, \mathbb{R})$, which is an invariant of principal bundles. It is a characteristic class in the sense that if we have a map $g : N \to M$ then to g^*P we associate the class $g^*(f^*c)$, and thus the invariant behaves well with respect to pull backs. Recall that a $U(n)$-principal bundle is just a rank n complex vector bundle. To such vector bundles we can associate Chern classes. It turns out the class f^*c is necessarily a polynomial in Chern classes. For general G it turns out that this class will always be a polynomial in the curvature form. We will make this precise.

Consider the ring of polynomials $\mathrm{Sym}(\mathfrak{g}^*)$ on $\mathfrak{g}$. The adjoint action of G on $\mathfrak{g}$ extends to an action Ad^* on $\mathrm{Sym}(\mathfrak{g}^*)$:

$$\mathrm{Ad}_g^* \alpha(X_1, \ldots, X_n) = \alpha(\mathrm{Ad}_g X_1, \ldots, \mathrm{Ad}_g X_n). \tag{1.63}$$

We call a polynomial α *invariant* if $\mathrm{Ad}_g^* \alpha = \alpha$ for all $g \in G$. Let $I^*(\mathfrak{g})$ denote the space of invariant polynomials on $\mathfrak{g}$. Applying these polynomials to a power of the curvature of a connection we obtain characteristic classes. [FH13, KN69]

Theorem 1.44

Given a connection A on a bundle $\pi : P \to M$ and $\alpha \in I^l(\mathfrak{g})$ we consider $\alpha(F^l) \in \Omega^{2l}(P)$. This satisfies the following properties:

- There is a unique closed form $f \in \Omega^{2l}(M)$ such that $\pi^* f = \alpha(F^l)$
- The induced element $[f] \in H^{2l}(M, \mathbb{R})$ is independent of choice of connection.
- The induced map $w : I^*(\mathfrak{g}) \to H^*(M, \mathbb{R})$ is an algebra homomorphism (the *Weil homomorphism*).
- There is an isomorphism $\widehat{w} : I^l(\mathfrak{g}) \cong H^{2l}(BG, \mathbb{R})$.[a]
- If $\phi : M \to BG$ classifies P, then $\phi^* \widehat{w}(\alpha) = w(\alpha)$. In particular w is natural with respect to pull-backs.

[a] This is essentially the same as the Weil homomorphism, except one should be careful when talking about connections and even differential forms on infinite dimensional spaces. For a precise statement refer to [FH13].

For $U(n)$ the relation to Chern classes is as follows. A vector bundle E of rank n is equivalent to a $U(n)$ bundle over the same space. If F is the curvature of some connection, then consider

$$\det\left(I_n - \frac{F}{2\pi i}\right) = \sum_{k=0}^{n} \alpha_k(F),$$

where each α_k is an invariant polynomial of degree k. Then by Theorem XII.3.1 in [KN69] we have $c_k(E) = w(\alpha_k)$.

2. 2-Category Theory

We will assume basic familiarity with category theory. For reference the reader can refer to [Lan71]. In its essence 2-categories are a simple generalization of ordinary categories. In ordinary categories we have objects and morphisms between them. In 2-categories we also have '2-morphisms' between morphisms. More generally one can go on in such a way to define n-categories, but we shall not go beyond 2-categories in this thesis. Such categories arise naturally in many situations. For example, if one has a category of categories with functors as morphisms, then one can see natural transformations as morphisms between functors. Or one can consider the path groupoid $\mathcal{P}_1(M)$ for some smooth manifold (cf. def. 1.36), then (thin homotopy classes of) homotopies are the 2-morphisms for this category. The main reference for this chapter is the nLab, besides this any reference will only cover some part of what we discuss here. A significant amount of information is also found scattered throughout appendices and introductions of papers on higher gauge theory, see for example [BL03, BS04, SW11].

We begin with an informal definition of a 2-category, a more formal definition follows later. A *(strict)-2-category* is a category $\mathcal{C}$ with objects $\mathcal{C}_0$ ('0-morphisms') and morphisms $\mathcal{C}_1$ ('1-morphisms') together with a collection of '2-morphisms' $\mathcal{C}_2$ and source and target maps $s, t : \mathcal{C}_2 \to \mathcal{C}_1$. We write $\eta : f \Rightarrow g$ if the source and target of η are f and g respectively. We then require that f and g share the same source and target (say respectively x, y) so that we can write η as

$$\begin{tikzcd} y & x \arrow[l, bend right, "f"'] \arrow[l, bend left, "g"] \arrow[l, phantom, "\Downarrow\eta"] \end{tikzcd}$$

One can compose 2-morphisms, just like one can compose 1-morphisms. But now there is both a 'vertical' and 'horizontal' composition. Diagrammatically horizontal composition is written like this:

$$\begin{tikzcd} z & y \arrow[l, bend right, "f'"'] \arrow[l, bend left, "g'"] \arrow[l, phantom, "\Downarrow\eta'"] & x \arrow[l, bend right, "f"'] \arrow[l, bend left, "g"] \arrow[l, phantom, "\Downarrow\eta"] & \mapsto & z & x \arrow[l, bend right, "f'\circ f"'] \arrow[l, bend left, "g'\circ g"] \arrow[l, phantom, "\Downarrow\eta'\circ\eta"] \end{tikzcd} \tag{2.1}$$

and vertical composition is written like this:

$$\begin{tikzcd} y & x \arrow[l, bend right=60, "f"'] \arrow[l, "g" description] \arrow[l, bend left=60, "h"] \arrow[l, phantom, shift right=3, "\Downarrow\eta"] \arrow[l, phantom, shift left=3, "\Downarrow\eta'"] & \mapsto & y & x \arrow[l, bend right, "f"'] \arrow[l, bend left, "h"] \arrow[l, phantom, "\Downarrow\eta'\bullet\eta"] \end{tikzcd} \tag{2.2}$$

Horizontal and vertical composition satisfy an interchange law guaranteeing that in a diagram with vertical and horizontal compositions the order in which we take vertical/horizontal composition doesn't matter. Consider the diagram

$$\begin{tikzcd} z & & y \arrow[ll] \arrow[ll, bend right=50] \arrow[ll, bend left=50] \arrow[ll, phantom, shift right=2ex, "\Downarrow\xi"] \arrow[ll, phantom, shift left=2ex, "\Downarrow\xi'"] & & x \arrow[ll] \arrow[ll, bend right=50] \arrow[ll, bend left=50] \arrow[ll, phantom, shift right=2ex, "\Downarrow\eta"] \arrow[ll, phantom, shift left=2ex, "\Downarrow\eta'"] \end{tikzcd} \tag{2.3}$$

then we require the following *coherence condition*

$$(\xi' \bullet \xi) \circ (\eta' \bullet \eta) = (\xi' \circ \eta') \bullet (\xi \circ \eta). \tag{2.4}$$

Naturally we also require the existence of an identity 2-morphism $\mathrm{Id}_f : f \Rightarrow f$ for every morphism f, which should satisfy some obvious properties.

Remark 2.1

We write these diagrams of 2-morphisms with 1-morphisms going from right to left instead of from left to right. This is to make notation more in line with the convention that function composition is read right to left. For example we write

$$f \circ g = \; z \xleftarrow{f} y \xleftarrow{g} x \;,$$

and we don't have to invert the order to translate between these two notations. The same is true for 2-morphisms, of which composition is also read right to left. Not all authors use this convention.

Remark 2.2

We use $\circ$ for horizontal composition and $\bullet$ for vertical composition. Notation differs throughout the literature. We chose to use $\circ$ for horizontal composition because it is the most like composition of 1-morphisms, whereas the vertical composition really comes from the extra structure and has no parallel in 1-morphism composition.

The informal definition we just gave is that of a *strict* 2-category, which can be thought of as a category with some extra structure. In a more general 2-category one does not assume an underlying category and allows composition of 1-morphisms to be associative only up to some appropriate invertible 2-morphism. We do not need such generality and will therefore consider this easier case.

Remark 2.3

From now on, whenever we speak of 2-categories we will always mean *strict* 2-categories.

For the sake of precision we give a more complete definition of a 2-category using the notion of an enriched category. The informal definition is good enough for most purposes, and therefore we will not go into detail.

Definition 2.4

Let V be a monoidal category then a *V-enriched category* or a *category enriched over V* is a small category $\mathcal{C}$ together with the following data:

- A *hom-object* $\hom(a,b)$ in V for each pair of objects a,b in $\mathcal{C}$
- For each triple a,b,c a *composition morphism*: $\hom(b,c) \otimes \hom(a,b) \to \hom(a,b)$
- For each a an *identity element* $1_A : \mathbb{1} \to \hom(a,a)$.

Such that the composition is both associative and unital.

Definition 2.5

A *strict 2-category* is a category enriched over **Cat**.

What this definition means is that fixing two objects a,b in a 2-category $\mathcal{C}$, we get a category $\hom(a,b)$. The objects of $\hom(a,b)$ are the morphisms $a \to b$ (1-morphisms in $\mathcal{C}$) and the morphisms of $\hom(a,b)$ are 2-morphisms in $\mathcal{C}$. Composition of morphisms in $\hom(a,b)$ corresponds to *vertical* composition of 2-morphisms. The composition functor $\hom(b,c) \times \hom(a,b) \to \hom(a,c)$ corresponds to composition of 1-morphisms and *horizontal* composition of 2-morphisms. The fact that this is a functor guarantees the coherence condition (2.4).

Definition 2.6

Let $\mathcal{C}$ and $\mathcal{D}$ be categories enriched over V. Then an *enriched functor* $F : \mathcal{C} \to \mathcal{D}$ consists of a function (of sets) $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ between the objects of $\mathcal{C}$ and $\mathcal{D}$ and for each pair a,b a morphism (in V)

$$F_{a,b} : \hom_{\mathcal{C}}(a,b) \to \hom_{\mathcal{D}}(F_0(a), F_0(b))$$

of hom objects. We require that this enriched functor respects composition and units.

Definition 2.7

A *2-functor F* between (strict) 2-categories $\mathcal{C}$ and $\mathcal{D}$ is a **Cat**-enriched functor. More concretely, it associates to each object $a \in \mathcal{C}_0$ an object $F_0 a \in \mathcal{D}_0$. To each morphism $f : a \to b$ it associates a morphism $F_1 f : F_0 a \to F_0 b$ respecting units and composition. To every 2-morphism $\alpha : f \Rightarrow g$ it associates a 2-morphism $\mathrm{tra}^2\,\alpha : F_1 f \Rightarrow F_1 g$ respecting vertical/horizontal composition and units.

2.1. Pasting

We can generalize vertical and horizontal composition of 2-morphisms to composition of labeled planar graphs. This procedure is known as pasting, and is well defined because of our coherence law relating vertical and horizontal composition. Instead of a 2-morphism we can consider diagrams like this:

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\swarrow"] & b \arrow[d, "h"] \\
c \arrow[r, "i"'] & d
\end{tikzcd}$$

The double arrow $\Rightarrow$ means we have some 2-morphism $h \circ f \Rightarrow i \circ g$. As long as there is a clear starting and ending vertex such a diagram always makes sense. For example we can have triangles or heptagons as well:

$$\begin{tikzcd}
a \arrow[r] \arrow[dr, bend right] \arrow[dr, phantom, "\swarrow"] & b \arrow[d] \\
& c
\end{tikzcd}
\qquad
\begin{tikzcd}
& a \arrow[dl] & b \arrow[l] \arrow[dr] & \\
c \arrow[dr] & \arrow[r, phantom, "\Longrightarrow"] & & d \arrow[dl] \\
& e \arrow[r] & f &
\end{tikzcd}
\qquad
\begin{tikzcd}
& a \arrow[r] \arrow[dl] & b \arrow[dr] & \\
c \arrow[dr] & \arrow[r, phantom, "\swarrow"] & & d \arrow[dl] \\
& e \arrow[r] & f &
\end{tikzcd}$$

The important fact is that such diagrams can be composed. For example take two squares, then we can compose them as follows

$$\begin{tikzcd}
a \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & b \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & c \arrow[d] \\
d \arrow[r] & e \arrow[r] & f
\end{tikzcd}
=
\begin{tikzcd}
a \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & b \arrow[d] & \\
d \arrow[r] & e \arrow[r] & f
\end{tikzcd}
\bullet
\begin{tikzcd}
a \arrow[r] & b \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & c \arrow[d] \\
& e \arrow[r] & f
\end{tikzcd}$$

where we recall the bullet $\bullet$ denotes vertical composition ($\circ$ is horizontal composition). The point here is that in order to vertically compose 2-morphisms they need to have a matching source and target morphism. If two diagrams have matching source and target vertices we can compose them horizontally. In order to state in general how we can compose such 2-morphisms, we must make precise what kind of diagrams we allow.

Definition 2.8

A *pasting diagram* is a planar graph G satisfying:

- G has a well-defined *source* vertex, i.e. a unique vertex with only outgoing arrows
- G has a well-defined *sink* vertex, i.e. a unique vertex with only incoming arrows
- For each face/2-cell F there must be a distinct source and target vertices s, t and two distinct directed paths $\sigma(F), \tau(F) : s \to t$ around the boundary of F such that $\sigma(F)\tau(F)^{-1}$ is precisely the boundary of F (respecting orientation).

We can label any pasting diagram by a 2-category $\mathcal{C}$. That is, to each vertex we associate some object in $\mathcal{C}$, to each edge a 1-morphism and to each face a 2-morphism such that source and targets of the morphisms match properly. By using induction on the amount of faces in such a diagram one can prove [Pow90]:

Theorem 2.9 (Pasting Theorem)

Let G be a pasting diagram with source s and sink t. There are unique paths σ and τ such that $\sigma\tau^{-1}$ is the boundary of G respecting orientation. To any labeling of G by a 2-category $\mathcal{C}$ we can associate a unique composite 2-morphism $\alpha : \widetilde{\sigma} \Rightarrow \widetilde{\tau}$, where $\widetilde{\sigma}$ and $\widetilde{\tau}$ are the 1-morphisms in $\mathcal{C}$ associated to σ, τ respectively

The proof is a simple induction on the number of faces.

2.2. Internalization

Often if we have a category we can see the objects and morphisms in this category as objects of some other category. For example a Lie group is a group (a category with one object where all the arrows are invertible) such that the objects and arrows are themselves smooth spaces i.e. objects in the category of smooth manifolds. Formally we say that the category of Lie groups is *internal* to that of smooth manifolds.

Definition 2.10

Let $\mathcal{A}$ be any category then a *category $\mathcal{C}$ internal to $\mathcal{A}$* consists of an object of objects $\mathcal{C}_0 \in \mathcal{A}$ and an object of morphisms $\mathcal{C}_1 \in \mathcal{A}$ together with: source and target morphisms $s, t : \mathcal{C}_1 \to \mathcal{C}_0$, identity assigning morphism $e : \mathcal{C}_0 \to \mathcal{C}_1$ and composition morphism $c : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_1$ satisfying the usual axioms of a category.

Of course one should have an appropriate notion of functors between internalized categories. Sometimes this notion is too restrictive and one needs the notion of anafunctors. For example one is often more interested in Morita equivalences (which are anafunctors) between Lie groupoids than the more restrictive notion of isomorphism. We will however not need this.

Definition 2.11

If $\mathcal{A}$ and $\mathcal{B}$ are categories internal to $\mathcal{C}$ then an *internal functor* $F : \mathcal{A} \to \mathcal{B}$ consists of morphisms $F_0 : \mathcal{A}_0 \to \mathcal{B}_0$ and $F_1 : \mathcal{A}_1 \to \mathcal{B}_1$ which satisfies the usual requirements for a functor (i.e. it should respect source, target, unit and composition).

Of particular interest are categories internal to smooth manifolds. We will elaborate what this entails precisely.

Definition 2.12

A *smooth category* or a *2-space* is a category internal to **Diff**, the category of smooth manifolds

A 2-space $\mathcal{M}$ consists of 2 smooth manifolds M_0 and M_1 together with smooth maps:

- Source and target maps $s, t : M_1 \to M_0$
- An identity map Id : $M_0 \to M_1$
- A composition map $\circ : M_{1s} \times_t M_1$

Satisfying the usual axioms for a category. We can think of M_1 as the 'space of arrows' lying over M_0 and we will denote $f \in M_1$ by $f : s(f) \to t(f)$. In particular any manifold M is a 2-space if we set $M_0 = M_1 = M$ and have $s, t, \mathrm{Id}, \circ$ be equal to the identity morphism. Or we can also see any manifold as a 2-space by setting $M_1 = M$ and $M_0 = \{*\}$. Both turn out in fact to be action groupoids (cf. page 38).

2.3. 2-Groups

Gauge theory relies heavily on Lie groups. In 2-gauge theory we try to replace Lie groups with (Lie) 2-groups and principal bundles with 2-bundles which have 2-groups as fibers. One obtains the notion of a 2-group by internalization. It turns out there are quite a few equivalent definitions of a 2-group. Again we only consider *strict* 2-groups, but we will usually omit the word 'strict' for brevity.

Definition 2.13

A *2-group* is a category internal to **Group**. Equivalently it's a group internal to **Cat**.

This definition is very compact but not very intuitive. Let's decipher what this means in terms of some concrete structure. Let $\mathcal{G}$ be a category internal to **Group**. Its objects form group G_1 and its morphisms a group G_2. On morphisms there are two operations, we can compose them (denote $\bullet$) and we can multiply them (denote $\circ$). Denote the objects by arrows $* \xleftarrow{f} *$ and denote a morphism $\alpha : f \to g$ by

$$
\begin{tikzcd}
* & * \arrow[l, bend right, "f"'] \arrow[l, bend left, "g"] \arrow[l, phantom, "\Downarrow\alpha"]
\end{tikzcd}
$$

Then composition of morphisms corresponds to vertical compositions of such diagrams, and multiplication of 2-morphisms corresponds to horizontal composition of such morphisms. The latter is ensured by the fact that source and target are group homomorphisms. The fact that composition is a group homomorphism ensures the coherence condition (2.4) relating vertical and horizontal composition. Thus we note:

Proposition 2.14

A 2-group is a 2-category with one object where all 1- and 2-morphisms are invertible.

The correspondence between these two points of view is given by the delooping:

Definition 2.15

Let G be a category (or an internalized/enriched/2-category) with one object $*$. Then $\mathbf{B}G$, the *delooping of* G, is the category with as objects the morphisms of G and only identity morphisms. For a 2-category with one object this corresponds to the category of morphisms $* \to *$. [a]

[a] The notation $\mathbf{B}G$ is almost the same as that of the classifying space BG of a group G. This is no accident; in abstract terms the classifying space BG is the geometric realization of the nerve of the delooped category $\mathbf{B}G$ [Seg68]

It turns out 2-groups are also equivalent to a more concrete concept known as a crossed module.

Definition 2.16

A *crossed module* is a tuple (G, H, τ, α) where G, H are groups, $\tau : H \to G$ is a homomorphism and $\alpha : G \to \mathrm{Aut}(H)$ is an action satisfying the following two conditions:

$$\alpha(\tau(h))h' = hh'h^{-1} \tag{2.5}$$
$$g\tau(h)g^{-1} = \tau(\alpha(g)h). \tag{2.6}$$

Before showing the correspondence between 2-groups and crossed modules, let us give a few examples of crossed modules. As two simple examples let G be a group. Then set $H = G$, $\tau = \mathrm{Id}$ and α just the conjugation action of G on itself. This is known as the *inner automorphism* crossed module/2-group. We can also set $H = \{1\}$, which gives the *discrete* crossed module/2-group.

As a less trivial example, let $H \subset G$ be a normal subgroup, and let τ be the inclusion and α the action by conjugation on H. The fact that H is normal means that this is indeed an action, and this defines a crossed module.

Finally, let $Z \subset Z(H)$ be a central subgroup and consider $G = H/G$. Or equivalently suppose G is a central extension of H. Since conjugation by a central element is trivial, G has an action by conjugation on H. The projection map $\tau : H \to G$ together with this action give the structure of a crossed module. We call this *central extension 2-group*. later we will elaborate further on this structure in the case that G and H are Lie groups (cf. p. 39).

Proposition 2.17

Crossed modules correspond to 2-groups

Proof: Let (G, H, τ, α) be a crossed module, then set $\mathcal{G}_0 = \{*\}$, $\mathcal{G}_1 = G$ and $\mathcal{G}_2 = G \ltimes H$, where $\ltimes$ denotes a semi-direct product with respect to α. I.e. as a set $G \ltimes H$ is $G \times H$, and as product we have

$$(g', h') \cdot (g, h) = (g'g, h'\alpha(g')h). \tag{2.7}$$

We can see the pair (g, h) as an arrow $g \Rightarrow \tau(h)g$. We want to give this the structure of a 2-category, thus we need to define vertical/horizontal composition. We propose vertical composition is given by

$$(\tau(h)g, h') \bullet (g, h) = (g, h'h), \qquad \begin{tikzcd} \ast & \ast \arrow[l, bend right=60, "g"'{name=U}] \arrow[l, "\tau(h)g" description, ""{name=M}] \arrow[l, bend left=60, "\tau(h')\tau(h)g"{name=D}] \arrow[from=U, to=M, phantom, "\Downarrow h"] \arrow[from=M, to=D, phantom, "\Downarrow h'"] \end{tikzcd} = \begin{tikzcd} \ast & \ast \arrow[l, bend right=50, "g"'{name=U}] \arrow[l, bend left=50, "\tau(h')\tau(h)g"{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow h'h"] \end{tikzcd} \tag{2.8}$$

and horizontal composition by

$$(g', h') \circ (g, h) = (g'g, h'\alpha(g')h), \qquad \begin{tikzcd} \ast & \ast \arrow[l, bend right=50, "g'"'{name=U1}] \arrow[l, bend left=50, "\tau(h')g'"{name=D1}] \arrow[from=U1, to=D1, phantom, "\Downarrow h'"] & \ast \arrow[l, bend right=50, "g"'{name=U2}] \arrow[l, bend left=50, "\tau(h)g"{name=D2}] \arrow[from=U2, to=D2, phantom, "\Downarrow h"] \end{tikzcd} = \begin{tikzcd} \ast & \ast \arrow[l, bend right=50, "g'g"'{name=U}] \arrow[l, bend left=50, "\tau(h'\alpha(g')h)g'g"{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow h'\alpha(g')h"] \end{tikzcd} \tag{2.9}$$

We have to check that the target on the right hand side is actually correct. We compute:

$$\tau(h')g'\tau(h)g = \tau(h')g'\tau(h)(g')^{-1}g'g = \tau(h')\tau(\alpha(g')h)g'g = \tau(h'\alpha(g')h)g'g, \tag{2.10}$$

which is precisely what we require. Since vertical and horizontal composition are given by group operations, all the 2-morphisms are in particular invertible. To show that this structure gives a 2-group we now only need to show the coherence condition (2.4). Consider,

$$\begin{tikzcd} \ast & \ast \arrow[l, bend right=60, "g'"'{name=U1}] \arrow[l, ""{name=M1}] \arrow[l, bend left=60, ""{name=D1}] \arrow[from=U1, to=M1, phantom, "\Downarrow h'"] \arrow[from=M1, to=D1, phantom, "\Downarrow i'"] & \ast \arrow[l, bend right=60, "g"'{name=U2}] \arrow[l, ""{name=M2}] \arrow[l, bend left=60, ""{name=D2}] \arrow[from=U2, to=M2, phantom, "\Downarrow h"] \arrow[from=M2, to=D2, phantom, "\Downarrow i"] \end{tikzcd} \tag{2.11}$$

Then the left hand side of the coherence equation is

$$((\tau(h')g', i') \bullet (g', h')) \circ ((\tau(h)g, i) \bullet (g, h)) = (g', i'h') \circ (g, ih) = (g'g, i'h'\alpha(g')(ih)).$$

The right hand side of the coherence equation reads

$$
\begin{aligned}
&((\tau(h')g', i') \circ (\tau(h)g, i)) \bullet ((g', h') \circ (g, h)) \\
&\qquad = (\tau(h' \circ h)g'g, i'\alpha(\tau(h')g')(i)) \bullet (g'g, h'\alpha(g')(h)) \\
&\qquad = (\tau(h' \circ h))g'g, i'h'\alpha(g')(i)(h')^{-1}) \bullet (g'g, h'\alpha(g')(h)) \\
&\qquad = (g'g, i'h'\alpha(g')(i)\alpha(g')(h)) \\
&\qquad = (g'g, i'h'\alpha(g')(ih)).
\end{aligned}
$$

which proves the coherence equation and we conclude that from each crossed module we can construct a 2-group. For the converse, suppose $\mathcal{G}$ is a 2-group, then set $G = \mathcal{G}_1$ and

$$H = \{ \mathcal{G}_2 \mid s(h) = 1 \} = \ker s. \tag{2.12}$$

Then let $\tau : H \to G = t|_H$ and define an action $\alpha : G \circlearrowright H$ by

$$\alpha(g)(h) = 1_g \circ h \circ 1_g^{-1} \quad = \quad \begin{tikzcd} \ast & \ast \arrow[l, bend right=50, "g"'] \arrow[l, bend left=50, "g"] \arrow[l, phantom, "\Downarrow \mathrm{Id}_g"] & \ast \arrow[l, bend right=50, "1"'] \arrow[l, bend left=50, "t(h)"] \arrow[l, phantom, "\Downarrow h"] & \ast \arrow[l, bend right=50, "g^{-1}"'] \arrow[l, bend left=50, "g^{-1}"] \arrow[l, phantom, "\Downarrow \mathrm{Id}_{g^{-1}}"] \end{tikzcd} \tag{2.13}$$

We then obtain an isomorphism $\phi : \mathcal{G}_2 \cong G \ltimes H$ by $\phi(h) = (s(h), h \circ 1_{s(h)}^{-1})$. Indeed, let $h = k1_g$ and $h' = k'1_{g'}$ then

$$h \circ h' = k1_g k' 1_{g'} = k1_g k' 1_g^{-1} 1_g 1_{g'} = k\alpha_g(k') 1_g 1_{g'}.$$

Thus

$$\phi(h \circ h') = (gg', k\alpha_g(k')) = (g, k)(g', k'),$$

and ϕ is a homomorphism. Its inverse $(k, g) \mapsto k1_g$ is by the same argument also a homomorphism. To conclude (G, H, τ, α) is a crossed module we need to confirm equations (2.5) and (2.6), the latter of which is obvious. Let $h, h' \in H$ and consider the following diagram

$$\begin{tikzcd} \ast & \ast \arrow[l] \arrow[l, bend right=60, "1"'] \arrow[l, bend left=60] \arrow[l, phantom, shift right=3, "\Downarrow h'"] \arrow[l, phantom, shift left=3, "\Downarrow 1'_{\tau(h')}"] & \ast \arrow[l] \arrow[l, bend right=60, "1"'] \arrow[l, bend left=60] \arrow[l, phantom, shift right=3, "\Downarrow 1_1"] \arrow[l, phantom, shift left=3, "\Downarrow h"] \end{tikzcd} \tag{2.14}$$

We then will compute both sides of the coherence equation (2.4) for this particular diagram. One the one hand we obtain,

$$\text{LHS} = ((\tau(h'), 1) \bullet (1, h')) \circ ((1, h) \bullet (1, 1)) = (1, h') \circ (1, h) = (1, h'h).$$

On the other hand we have

$$\text{RHS} = ((\tau(h'),1) \circ (1,h)) \bullet ((1,h') \circ (1,1)) = (\tau(h'), \alpha(\tau(h'))h) \bullet (1,h') = (1, \alpha(\tau(h'))hh').$$

Equating the two we obtain

$$\alpha(\tau(h'))hh' = h'h,$$

from which we conclude identity (2.5), and consequently the fact that (G, H, τ, α) is a crossed module. Since the central fact of this proof is that $\mathcal{G}_2 \cong G \ltimes H$, we claim that the two constructions are inverse to each other and we in fact obtain an equivalence. ■

One can in principle upgrade this proof to some equivalence of 2-categories [BS76], but we do not wish to go much deeper into this.

Proposition 2.18

$\ker \tau$ is a central subgroup of H

Proof: Let $h \in \ker \tau$ and $h' \in H$. Then

$$hh' = hh'h^{-1}h = (\alpha(\tau(h))h')h = h'h \tag{2.15}$$

where we use property (2.5) of a crossed module. ■

Remark 2.19

From the proof of proposition 2.17 it is clear that a 2-group is also a groupoid, namely under the identification $\mathcal{G}_2 \cong G \ltimes H$ we can set $s(g,h) = g$ and $t(g,h) = \tau(h)g$. One can go on to proof that any 2-group is actually also a group internal **Grpd** rather than just internal to **Cat** [BS76]. Another point of view is that a 2-group is a groupoid which also has a compatible monoidal structure. The groupoid composition then takes the role of vertical composition and the monoidal structure takes the role of horizontal composition.

Definition 2.20

Define *Lie 2-groups* to be a 2-category internal to **Diff** with one object an all morphisms invertible.[a] We define a *smooth crossed module* to be a tuple (G, H, τ, α) where G, H are Lie groups, $\tau : H \to G$ is a homomorphism and α an action $G \circlearrowright H$ satisfying (2.5) and (2.6).

[a] Strictly speaking we only defined what it means for a category to be internal to another category, but one can just as easily ask for a strict 2-category to be internal to another category

By the same proof as that of 2.17 we have:

Proposition 2.21

Smooth 2-groups correspond to smooth crossed modules.

By taking the differentials of τ and α we obtain an infinitesimal version of a crossed module, a strict Lie 2-algebra, so to speak.

Definition 2.22

A *differential crossed module* $(\mathfrak{g}, \mathfrak{g}, \tau_*, \alpha_*)$ is a pair of Lie algebras $\mathfrak{g}, \mathfrak{h}$ together with maps $\tau_* : \mathfrak{h} \to \mathfrak{g}$ and $\alpha_* : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ such that

$$\tau_*(\alpha_*(X)(\xi)) = [X, \tau_*\xi] \tag{2.16}$$

$$\alpha_*(\tau_*(\xi))\eta = [\xi, \eta] \tag{2.17}$$

for all $X \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{h}$. Similarly we can also define $\mathfrak{g} \ltimes \mathfrak{h}$ as the Lie algebra $\mathfrak{g} \times \mathfrak{h}$ with bracket

$$\begin{aligned} [X, Y] &= \mathrm{ad}_X Y, & [X, \xi] &= \alpha_*(X)\xi \\ [\xi, X] &= \mathrm{ad}_{\tau_*(\xi)} X, & [\xi, \eta] &= \mathrm{ad}_\xi \eta \end{aligned} \tag{2.18}$$

for $X, Y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{h}$.

From remark 2.19 we conclude that 2-groups are actually also groupoids. In the case of Lie 2-groups we get the structure of a Lie groupoid. Lie groupoids are interesting objects in their own regard, and deserve mention.

Definition 2.23

A *Lie groupoid* is a groupoid internal to **Diff**. Concretely a Lie groupoid $\mathcal{G}$ consists of two smooth manifolds $\mathcal{G}_0$ and $\mathcal{G}_1$ together with smooth maps $s, t : \mathcal{G}_1 \to \mathcal{G}_0$, an inversion map $i : \mathcal{G}_1 \to \mathcal{G}_1$ a unit map[a] $1 : \mathcal{G}_0 \to \mathcal{G}_1$ and a composition map $\circ : \mathcal{G}_1 {}_s\times_t \mathcal{G}_1 \to \mathcal{G}_1$ such that $\mathcal{G}$ with objects $\mathcal{G}_0$ and morphisms $\mathcal{G}_1$ is a groupoid.

[a] Denote $1(g)$ by either 1_g or Id_g

A broad class of examples of Lie groupoids is that of *action groupoids*. Let G be a Lie group acting on some manifold M. Then define a groupoid $M/\!\!/G$ with $(M/\!\!/G)_0 = M$ and $(M/\!\!/G)_1 = G \times M$. Set $s(g, m) = m$ and $t(g, m) = g \cdot m$. Define the identity map by $1_m = (1, m)$, the inversion by $i(g, m) = (g^{-1}, g \cdot m)$ and the composition by $(g', g \cdot m) \circ (g, m) = (g'g, m)$.

Some of the example 2-groups we gave so far are of this form, if we forget about the extra structure and see a 2-group as a groupoid. The inner automorphism 2-group is obtained as $G/\!\!/G$ by conjugation action. A discrete 2-group is obtained as $G/\!\!/*$, with $*$ the trivial group. More generally if M is a manifold we can define the *discrete groupoid* $M_{\mathrm{dis}} = M/\!\!/*$. Concretely this has $(M_{\mathrm{dis}})_0 = (M_{\mathrm{dis}})_1 = M$ and $s, t = \mathrm{Id}$.

An important example of a smooth 2-group comes from covering groups. Recall from proposition A.14 that coverings $H \to G$ coincide with central extensions of G as well as subgroups $K \subset \pi_1(G)$

by Theorem A.13. This interplay of algebra, topology and geometry gives a rich and useful theory which we will rely on later. For now we will consider the 2-group obtained from such a covering from several angles.

Definition 2.24

Let $t : H \to G$ be a covering of groups. Then since $\ker t$ is central, the conjugation action of H on itself extends to an action $\alpha : G \circlearrowright H$. Namely if $t(h) = g$ and $x \in H$ then define

$$\alpha(g)x = hxh^{-1}. \tag{2.19}$$

We note that (2.5) and (2.6) obviously hold. We call the corresponding crossed module (G, H, t, α) the *covering 2-group*. If H, G are Lie groups then this is a Lie 2-group.
Of particular interest is the case when $H = \widehat{G}$ is the universal cover. In that case we can refer to this crossed module as the *universal covering 2-group* of G.

We will now make the equivalent structure of a 2-group for covering 2-groups explicit. Let $K \subset \pi_1(G)$ be given (for a covering $t : H \to G$ we would have $K = t_*\pi_1(H)$). Then let $\mathcal{G}_1 = G$, and set (cf. Def. 1.36)

$$\mathcal{G}_2 = \mathcal{P}_1(G)\Big/\sim, \tag{2.20}$$

where $\gamma \sim \gamma'$ if $[\gamma^{-1}\gamma'] \in K$. Then we have maps $s, t : \mathcal{G}_2 \to \mathcal{G}_1$ given by

$$t(\gamma) = \gamma(1), \qquad s(\gamma) = \gamma(0), \tag{2.21}$$

in other words the source and target of a path are just its beginning and endpoints. Horizontal composition is pointwise multiplication of paths;

$$(\gamma \circ \gamma')(t) = \gamma(t) \cdot \gamma'(t). \tag{2.22}$$

Vertical composition is concatenation of paths. For $g \in \mathcal{G}_1$ set $\mathrm{Id}_g(t) = g$ to be the constant path. This defines the structure of a 2-group. For this one would have to show the coherence condition (2.4). This can be done by writing an explicit homotopy between the two paths involved. It is however also enough to argue that this 2-group corresponds to the crossed module of Definition 2.24. This follows from the Galois correspondence Theorem A.13. From this point of view $\alpha : \mathcal{G}_1 \circlearrowright \ker s \subset \mathcal{G}_2$ is given by pointwise conjugation;

$$\alpha(g)(\gamma)(t) = g\gamma(t)g^{-1}. \tag{2.23}$$

3. 2-Gauge Theory

Now that we have seen gauge theory and 2-groups, it is time to combine the two. We start with a natural generalization of principal 2-bundles, which have smooth crossed modules as fibers instead of Lie groups. We then wish to define parallel transport for such bundles, however it is not immediately clear what a connection is on 2-bundles. Therefore instead we start with parallel transport as a 2-functor $\mathcal{P}(M) \to \mathcal{G}$**-tor** and from there deduce what a 2-connection on a 2-bundle should be. After that we proceed to describe the parallel transport completely in terms of the 2-connection using a *surface ordered exponentiation* procedure. We will show this satisfies all the properties one would want from it, including thin-homotopy invariance. Like our proof for thin-homotopy invariance in the ordinary setting, we will prove a non-Abelian Stokes' Theorem and an Ambrose-Singer Theorem in the progress. We conclude the chapter by considering 2-bundles from the point of view of paths spaces. The main reference for this material is [SW11].

3.1. Principal 2-bundles

Principal 2-bundles are obtained by categorifying the notion of G-bundles. First we consider 2-group actions on 2-spaces. The definition of a 2-group action is obtained by taking the definition of a usual action and 'replacing morphisms by functors'.

Definition 3.1

Let $\mathcal{G}$ be a Lie 2-group (cf. def. 2.20) and $\mathcal{M}$ a 2-space (cf. def. 2.11). Then a left action $\mathcal{G} \circlearrowright \mathcal{M}$ consists of a smooth functor[a] $R : \mathbf{B}\mathcal{G} \times \mathcal{M} \to \mathcal{M}$ such that $R(1, m) = m$ and $R(\mathrm{Id}_g, f) = f$ for all $g \in \mathcal{G}_1$, $m \in \mathcal{M}_0$ and $f \in \mathcal{M}_1$. And such that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{B}\mathcal{G} \times \mathbf{B}\mathcal{G} \times \mathcal{M} \arrow[r, "\circ \times \mathrm{Id}"] \arrow[d, "R \times \mathrm{Id}"'] & \mathbf{B}\mathcal{G} \times \mathcal{M} \arrow[d, "R"] \\
\mathbf{B}\mathcal{G} \times \mathcal{M} \arrow[r, "R"'] & \mathcal{M}
\end{tikzcd} \tag{3.1}$$

[a]We defined a Lie 2-group $\mathcal{G}$ as a type of 2-category and $\mathcal{M}$ is a category, therefore in order to speak of functors we need to turn $\mathcal{G}$ into a category. This is done by taking the delooping $\mathbf{B}\mathcal{G}$ (cf. Def 2.15).

It should be noted that this means there is an $\mathcal{G}_1$ action on $\mathcal{M}_0$ and a $\mathcal{G}_2$ action on $\mathcal{M}_1$ such that if $h : g \Rightarrow g'$ and $f : p \to p'$ then $h \cdot f$ is an arrow $g \cdot p \to g' \cdot p'$. The fact that R is a functor means that if we have

$$p'' \xleftarrow{f'} p' \xleftarrow{f} p \ , \qquad g'' \xLeftarrow{h'} g' \xLeftarrow{h} g \ ,$$

then we require that

$$(h' \cdot f') \circ (h \cdot f) = (h' \bullet h) \cdot (f \circ f'). \tag{3.2}$$

Using this we can now define a principal 2-bundle by 'categorifying' definition 1.1;

Definition 3.2

Let $\mathcal{G}$ be a 2-group and $\mathcal{M}$ a 2-space. A *principal $\mathcal{G}$-2-bundle* $\pi : \mathcal{P} \to \mathcal{M}$ consists of a 2-space $\mathcal{P}$, a (left) Lie 2-group action $\mathcal{G} \circlearrowright \mathcal{P}$ and a $\mathcal{G}$-invariant surjective submersion functor $\pi : \mathcal{P} \to \mathcal{M}$ such that the following functor is an equivalence of smooth categories:

$$\tau = (\pi_2, R) : \mathbf{B}\mathcal{G} \times \mathcal{P} \to \mathcal{P} \times_M \mathcal{P} \tag{3.3}$$

That is, on objects $g \in \mathcal{G}_1$, $p \in \mathcal{P}_0$:

$$(g, p) \mapsto (p, g \cdot p) = (p, R(g, p)) \tag{3.4}$$

and on arrows $h : g \to g'$, $f : p \to p'$

$$(h, f) \mapsto (f, h \cdot f) = (f, R(h, f)), \tag{3.5}$$

where $h \cdot f$ is a map $g \cdot p \to g' \cdot p'$.

By $\pi : \mathcal{P} \to \mathcal{M}$ being a surjective submersion functor we mean that it is a functor $\mathcal{P} \to \mathcal{M}$ which induces a surjective submersion on the spaces of objects and arrows. This in particular means that $\mathcal{P}$ is a $\mathcal{G}_1$-bundle over $\mathcal{P}_0$ and a $\mathcal{G}_2$-bundle over $\mathcal{P}_1$, since a 2-group action in particular gives group actions on the point and arrow space. Thus restricting to $\mathcal{P}_0$ or $\mathcal{P}_1$ we recover exactly definition 1.1 of a principal bundle. However a 2-bundle is more than just a pair of 2 principal bundles; there are also maps $s, t : \mathcal{P}_1 \to \mathcal{P}_0$, as well as a composition of arrows on $\mathcal{P}_1$. However this point of view does give us local trivializations for 2-bundles.

Proposition 3.3

Principal $\mathcal{G}$-2-bundles admit local trivializations.

Proof: Note that $\mathcal{P}_0 \to \mathcal{M}_0$ and $\mathcal{P}_1 \to \mathcal{M}_1$ both admit local sections, since these maps are submersions. Therefore locally we can find a functor $\sigma : \mathcal{U} \to \mathcal{P}|_{\mathcal{U}}$ such that $\pi\sigma = \mathrm{Id}$. Then we obtain a local trivialization using the map

$$\begin{tikzcd} \mathbf{B}\mathcal{G} \times \mathcal{U} \arrow[r, "\mathrm{Id} \times \sigma"] & \mathbf{B}\mathcal{G} \times \mathcal{P}\big|_{\mathcal{U}} \arrow[r, "\tau"] \arrow[rr, bend right, "R"'] & (\mathcal{P} \times_M \mathcal{P})\big|_{\mathcal{U}} \arrow[r, "\pi_2"] & P\big|_{\mathcal{U}} \end{tikzcd} . \tag{3.6}$$

We claim that the following map is its inverse:

$$\mathbf{B}\mathcal{G} \times \mathcal{U} \xleftarrow{\mathrm{Id} \times \pi} \mathbf{B}\mathcal{G} \times \mathcal{P}\big|_{\mathcal{U}} \xleftarrow{\tau^{-1}} (\mathcal{P} \times_M \mathcal{P})\big|_{\mathcal{U}} \xleftarrow{(\sigma\pi, \mathrm{Id})} P\big|_{\mathcal{U}} . \tag{3.7}$$

Denote $\tau^{-1}(p,q)$ by $(q : p, p)$ (whether $q, p \in \mathcal{M}_0$ or $\mathcal{M}_1$). This notation makes sense because we must have $R\tau^{-1} = \pi_2$. Then let $(g, m) \in \mathbf{B}\mathcal{G} \times \mathcal{U}$ then (3.7) $\circ$ (3.6) gives

$$\begin{aligned}(g,m) &\mapsto (g,\sigma(m)) \mapsto (\sigma(m), g\cdot\sigma(m)) \mapsto g\cdot\sigma(m) \mapsto \\ &\mapsto (\sigma(m), g\cdot\sigma(m)) \mapsto (g,\sigma(m)) \mapsto (g,m),\end{aligned}$$

on the other hand let $p \in \mathcal{P}|_{\mathcal{U}}$ then (3.6) $\circ$ (3.7) gives

$$\begin{aligned}p &\mapsto (\sigma\pi(p), p) \mapsto (p : \sigma\pi(p), \sigma\pi(p)) \mapsto (p : \sigma\pi(p), \pi(p)) \mapsto \\ &\mapsto (p : \sigma\pi(p), \sigma\pi(p)) \mapsto (\sigma\pi(p), p) \mapsto p.\end{aligned}$$

Thus this is indeed a local trivialization. ■

Note that this is actually exactly the same argument as that used to show an ordinary principal G-bundle is locally trivializable given definition 1.1. From the local trivializations of a 2-bundle we can also conclude that the total space $\mathcal{P}$ is in fact a groupoid.

Remark 3.4

We can relax the definition of a 2-bundle by not requiring that τ is an equivalence of categories but instead just a weak equivalence. In that case we don't get a locally trivializable bundle and the discussion becomes more complicated. This is because to construct the local trivialization we have to explicitly use the inverse τ^{-1}, and having a weak inverse is just not good enough. One can for such 'weak'-2-bundles (as opposed to 'strict' ones) also define parallel transport and 2-holonomy as we shall do, but it's considerably more complicated; see [Wal16, Wal17].

One particularly simple 2-space is M_{dis} for M any manifold (cf. page 38). A principal $\mathcal{G}$-2-bundle over M_{dis} consists of a G-bundle $\mathcal{P}_0$ over M and a $G \ltimes H$-bundle $\mathcal{P}_1$ over M together with equivariant maps $s, t : \mathcal{P}_1 \to \mathcal{P}_0$ and a compatible notion of unit and composition on $\mathcal{P}_1$.

Given a Lie group G let $\mathcal{G}$ be the universal covering 2-group of G (cf. 2.24). From this point of view, starting with any principal G-bundle $P \to M$ we can obtain a $\mathcal{G}$-2-bundle over M_{dis} simply by taking a 'fiberwise path groupoid' of P. More precisely $\mathcal{P}_0 = P$ and

$$\mathcal{P}_1 = \{\, [\gamma : p \to q] \mid p, q \in P_x, x \in M \}\,, \tag{3.8}$$

where $[\gamma : p \to q]$ denotes the homotopy class of γ . This space has a $G \ltimes \widehat{G}$ action obtained by pointwise multiplication:

$$((g,h)\cdot\gamma)(t) = h(t)\cdot g\cdot\gamma(t). \tag{3.9}$$

This defines a principal $G \ltimes \widehat{G}$-bundle over M, which is easily seen in a local trivialization. These two bundles together define a principal $\mathcal{G}$-2-bundle. The source and target maps $\mathcal{P}_1 \to \mathcal{P}_0$ are obtained by respectively taking the begin and endpoints of the paths, and the identity map $\mathcal{P}_0 \to \mathcal{P}_1$ is obtained by taking constant paths. The fact that $\pi : \mathcal{P} \to M_{\text{dis}}$ is a surjective submersion functor and that τ is an isomorphism both immediately follow from the fact that $\mathcal{P}_0$ and $\mathcal{P}_1$ are themselves principal bundles.

Definition 3.5

We will call the 2-bundle above the *covering 2-bundle* and denote it by $\mathcal{P}(P)$.

This bundle will be our primary example of 2-bundles, and we will perform some computations specifically for this bundle in section 4.2.

3.2. Parallel transport in 2-bundles: transport 2-functors

Our main aim for now is to define parallel transport along bigons (i.e. 2-transport) as a generalization of parallel transport along paths. We will restrict our attention to principal $\mathcal{G}$-2-bundles over M_{dis}. From the outset it's not clear what parallel transport should be in 2-bundles. In this section we will start with a very natural (but vague) notion of 2-transport and use it to derive the notion of a 2-connection. A 2-connection can then be used to give an unnatural but more practical (i.e. computable) definition of parallel transport. This section mostly serves as motivation why the definition given in the next section is a good definition. We note the similarity with ordinary connections where the most natural definition (i.e. a horizontal distribution) is not the most practical one (i.e. a 1-form on the total space).

Parallel transport in G-bundles behaves well under composition of paths and under changes of paths by thin holonomy. Any good definition of a parallel transport for 2-bundles should satisfy similar properties. To this end we define the path 2-groupoid $\mathcal{P}_2(M)$ as a natural generalization of the path groupoid $\mathcal{P}_1(M)$ of definition 1.36.

Definition 3.6

A *bigon* $\Sigma : I \times I \to M$ is a homotopy of paths $\Sigma_0 \Rightarrow \Sigma_1$, where $\Sigma_t = \Sigma|_{\{t\}\times I}$. A homotopy of bigons $h : I \times I \times I \to M$ is a homotopy such that h_t is a bigon for all t with the same source and target path (up to thin homotopy). A homotopy of bigons is *thin* if $\operatorname{rank} dh : TI^3 \to TM \leq 2$ everywhere. Similar to definition 1.36 we define the *path 2-groupoid* $\mathcal{P}_2(M)$ to be the (smooth) 2-groupoid[a] with

- Objects: points in M
- 1-Morphisms: thin homotopy classes of paths in M
- 2-Morphisms: thin homotopy classes of bigons between paths in M.

[a] A (strict) 2-groupoid is a 2-category where all morphisms are invertible

Here one should note that thin homotopies are themselves bigons, and vertical composition with thin homotopies doesn't change the thin homotopy class of a bigon. Thus a thin homotopy class of bigons can correctly be seen as a morphism between thin homotopy classes of paths.

Mimicking proposition 1.38 we want parallel transport to give us a 2-functor

$$\mathrm{tra} : \mathcal{P}_2(M) \to \mathcal{G}\textbf{-tor}. \tag{3.10}$$

For now we ignore the fact that we haven't defined $\mathcal{G}$-torsors. Note that this by itself is not a definition of parallel transport for 2-bundles, rather we pose this as a requirement any good definition should satisfy. It should also be a generalization of ordinary parallel transport, so that if we restrict such a 2-functor tra to the objects and 1-morphisms we recover the functor $\mathrm{tra}^1 : \mathcal{P}_1(M) \to G\textbf{-tor}$ for some connection. Finally it should be locally trivializable, i.e. if $\mathcal{P}|_U$ is trivializable then we should obtain a functor $\mathcal{P}_2(U) \to \mathcal{G}$ in a compatible way. Stating precisely what local trivializability means is technical and we refer the reader to [SW13a, SW13b]. Using these three requirements we will naturally arrive at a definition of parallel transport for 2-bundles. First we give a definition of the 2-category $\mathcal{G}\textbf{-tor}$.

Definition 3.7

A $\mathcal{G}$-*2-torsor* is a 2-space with a free and transitive $\mathcal{G}$ action. $\mathcal{G}$-2-torsors form the objects of a 2-category $\mathcal{G}\textbf{-tor}$. A morphism of 2-spaces is a smooth functor, and a morphism $F : \mathcal{M} \to \mathcal{N}$ of $\mathcal{G}$-2-torsors is an *equivariant smooth functor*. If $R : \mathbf{B}\mathcal{G} \times \mathcal{M} \to \mathcal{M}$ and $R' : \mathbf{B}\mathcal{G} \times \mathcal{N} \to \mathcal{N}$ are the respective $\mathcal{G}$ actions we require the following diagram to commute for F to be equivariant:

$$\begin{tikzcd} \mathbf{B}\mathcal{G} \times \mathcal{M} \arrow[r, "\mathrm{Id} \times F"] \arrow[d, "R"'] & \mathbf{B}\mathcal{G} \times \mathcal{N} \arrow[d, "R'"] \\ \mathcal{M} \arrow[r, "F"'] & \mathcal{N} \end{tikzcd} \tag{3.11}$$

A 2-morphism $\eta : F \Rightarrow F'$ is a natural transformation of the functors. That is, a smooth map $\eta : \mathcal{M}_0 \to \mathcal{N}_1$ such that the following diagram commutes for every $f : x \to y$ in $\mathcal{M}$

$$\begin{tikzcd} F_0(x) \arrow[r, "\eta(x)"] \arrow[d, "F_1(f)"'] & F_0'(x) \arrow[d, "F_1'(f)"] \\ F_0(y) \arrow[r, "\eta(y)"'] & F_0'(y) \end{tikzcd} \tag{3.12}$$

Lemma 3.8

A $\mathcal{G}$-2-torsor is the same thing as a principal $\mathcal{G}$-2-bundle over a point.

PROOF: Let $\mathcal{P} \to *$ be a $\mathcal{G}$-2-bundle over a point, then the fact that

$$\tau : \mathbf{B}\mathcal{G} \times \mathcal{P} \to \mathcal{P} \times \mathcal{P}$$

is a smooth equivalence of categories means in particular that for each pair (f, g) there is a unique $g : f$ such that $R(g : f, f) = g$ (similarly for objects). Then transitivity is equivalent to existence of such an element, and freedom is equivalent to uniqueness. ■

It turns out that any 1-morphism of $\mathcal{G}$**-tor** is completely determined by what it does on objects. Indeed suppose $F : \mathcal{M} \to \mathcal{N}$ is an equivariant smooth functor between $\mathcal{G}$-2-torsors and let $p \in \mathcal{M}_0$. By transitivity of the $\mathcal{G}^2$ action on morphisms, there is for each $f : p \to q$ an $h \in \mathcal{G}^2$ such that $h \cdot \mathrm{Id}_p = f$. By equivariance we have that

$$F_1(f) = h \cdot F_1(\mathrm{Id}_p) = h \cdot \mathrm{Id}_{F_0(p)} = (f : \mathrm{Id}_p) \cdot \mathrm{Id}_{F_0(p)}\,. \tag{3.13}$$

In fact we note that the entire functor F is determined by its image of a single point. Compare this to the fact that any map of G-torsors is determined by the image of a single point.

Similarly a 2-morphism is also completely determined by the image of a single point. Let $\eta : F \Rightarrow F'$ be a 2-morphism of $\mathcal{G}$-2-torsors and $F, F' : \mathcal{M} \to \mathcal{N}$. Then let $p \in \mathcal{M}_0$, the aim is to write $\eta(g \cdot p)$ in terms of $\eta(p)$ and g for any $g \in G$. To this end let $g \in G$ and let $f : p \to g \cdot p$ be any morphism, and consider the following diagram:

$$\begin{array}{ccc} F_0(p) & \xrightarrow{\eta(p)} & F_0'(p) \\ {\scriptstyle F_1(f)}\big\downarrow & & \big\downarrow{\scriptstyle F_1'(f)} \\ F_0(g \cdot p) & \xrightarrow[\eta(g\cdot p)]{} & F_0'(g \cdot p) \end{array} \tag{3.14}$$

We can always write $f = (1, h) \cdot \mathrm{Id}_p$ for some unique $h \in H$ with $\tau(h) = g$, since the action is free and transitive. Using this we then write

$$\begin{aligned} F_1(f) &= (1, h) \cdot \mathrm{Id}_{F_0(p)} \\ F_1'(f) &= (1, h) \cdot \mathrm{Id}_{F_0'(p)} = (F_0'(p) : F_0(p), h) \cdot \mathrm{Id}_{F_0(p)} = (\tau(\xi), h) \cdot \mathrm{Id}_{F_0(p)} \\ \eta(p) &= (1, \xi) \cdot \mathrm{Id}_{F_0(p)} \\ \eta(g \cdot p) &= (g, \xi^g) \cdot \mathrm{Id}_{F_0(p)}\,. \end{aligned}$$

Here ξ, ξ^g are some elements of H, and we wish to write ξ^g in terms of ξ. From (3.14) we obtain the following identity:

$$(g, \xi^g) \bullet (1, h) = (\tau(\xi), h) \bullet (1, \xi), \tag{3.15}$$

which is equivalent to

$$\xi^g = h \xi h^{-1} = \alpha_g(\xi). \tag{3.16}$$

Thus we have shown that a 2-morphism of $\mathcal{G}$-2-torsors also only depends on its value on a single object. Now we want to apply this to describe better what a parallel transport 2-functor tra : $\mathcal{P}_2(M) \to \mathcal{G}$**-tor** must look like. Let $\mathcal{P} \to M_{\mathrm{dis}}$ be a $\mathcal{G}$-2-bundle, $A \in \Omega^1(\mathcal{P}_0, \mathfrak{g})$ a connection and $\mathrm{tra}^1 : \mathcal{P}_1(M) \to G$**-tor** the associated parallel transport functor. First of all F must send objects $x \in M$ to fibers $\mathcal{P}_x$. By equation (3.13) it should send a 1-morphism $\gamma : x \to y$ to the 1-morphism of

$\mathcal{G}$-2-torsors:

$$\begin{aligned} p \quad &\mapsto \quad \mathrm{tra}^1_\gamma(p) && (3.17)\\ f : p \to q \quad &\mapsto \quad (f : \mathrm{Id}_p) \cdot \mathrm{Id}_{\mathrm{tra}^1_\gamma(p)}, && (3.18)\end{aligned}$$

and this is in fact the only consistent definition. To each bigon $\Sigma : \Sigma_0 \Rightarrow \Sigma_1$ it should associate a natural transformation $\mathrm{tra}^1_{\Sigma_0} \Rightarrow \mathrm{tra}^1_{\Sigma_1}$. For a given $p \in (\mathcal{P}_0)_x$ this is determined by an element $\xi_p \in H$ with $\tau(\xi_p) = \mathrm{tra}^1_{\Sigma_1}(p) : \mathrm{tra}^1_{\Sigma_0}(p)$. If we take a $g \in G$ then $\xi_{p\cdot g} = \alpha_g(p)$ by equation (3.16). As a sanity check we compute that this has the correct target

$$\tau(\alpha_g(\xi_p)) = g\tau(\xi_p)g^{-1} = g\Big(\mathrm{tra}^1_{\Sigma_1}(p) : \mathrm{tra}^1_{\Sigma_0}(p)\Big)g^{-1} = \mathrm{tra}^1_{\Sigma_1}(g\cdot p) : \mathrm{tra}^1_{\Sigma_0}(g \cdot p),$$

where in the last equality we used equation (1.2). To summarize we have the following proposition:

Proposition 3.9

2-Transport associates to each $\Sigma : \gamma_0 \Rightarrow \gamma_1$, $\gamma_i : x \to y$ a map $\mathrm{tra}^2(\Sigma) : (\mathcal{P}_0)_x \to H$ such that

$$\tau(\mathrm{tra}^2(\Sigma)(p)) \cdot \mathrm{tra}^1(\gamma_0)(p) = \mathrm{tra}^1(\gamma_1)(p). \tag{3.19}$$

Moreover it is equivariant in the sense that

$$\mathrm{tra}^2(\Sigma)(g \cdot p) = \alpha_g\left(\mathrm{tra}^2(\Sigma)(p)\right). \tag{3.20}$$

It turns out that if we assume 2-transport is locally trivial then we obtain some 2-form $B \in \Omega^2(\mathcal{P}_0, \mathfrak{h})$. The precise definition of local triviality of a 2-functor is a little complicated. A naive way to think about local triviality is that if we restrict to a chart domain, then F can be transformed into a functor $F : \mathcal{P}_2(M) \to \mathcal{G}$. This can be made precise, but we choose not to do so, since it would require introducing a lot of technical machinery. For the moment suppose we have a 2-functor

$$\mathrm{tra} : \mathcal{P}_2(M) \to \mathcal{G}, \tag{3.21}$$

then we want to obtain a map $\bigwedge^2 TM \to \mathfrak{h}$ (i.e. a form). This should somehow glue to a form $B \in \Omega^2(\mathcal{P}_0, \mathfrak{h})$, but we will just paint the local picture of this form. To this end fix $m \in M$ and $v_1, v_2 \in T_mM$, and suppose we have some map

$$\Gamma : \mathbb{R}^2 \to M \qquad \Gamma(0,0) = m, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\Gamma(s,0) = v_1, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\Gamma(0,t) = v_2. \tag{3.22}$$

One easily obtains such a map explicitly in any chart. For each $s, t \in \mathbb{R}^2$ one canonically obtains a thin homotopy class of bigons $\Sigma_{s,t}$ in $\mathbb{R}^2$, see figure 3.10. This gives us a map $\Sigma : \mathbb{R}^2 \to B^2\mathbb{R}^2$ where $B^2\mathbb{R}^2$ is the space of thin homotopy classes of bigons in $\mathbb{R}^2$.

By taking a pushforward by $\Gamma : \mathbb{R}^2 \to M$ we get a map $\Gamma_* : B^2\mathbb{R}^2 \to B^2M$. Applying $\mathrm{tra}^2 : B^2M \to$

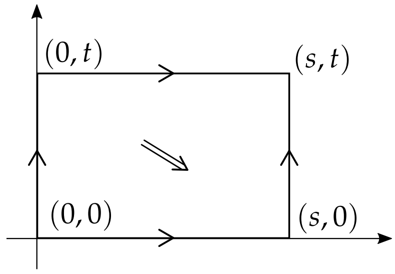


Figure 3.10: Canonical bigon $\Sigma_{s,t}$ in $\mathbb{R}^2$. Any two bigons in $\mathbb{R}^2$ with this source and target are necessarily thinly homotopic since by dimensional reasons any homotopy of bigons is thin.

$G \ltimes H$ and then projecting to H by we obtain a map:

$$F_\Gamma : \mathbb{R}^2 \xrightarrow{\Sigma} B^2\mathbb{R}^2 \xrightarrow{\Gamma_*} B^2M \xrightarrow{\mathrm{tra}^2} G \ltimes H \xrightarrow{p_H} H \ . \tag{3.23}$$

Then we can differentiate this map twice at the origin to get an element of $\mathfrak{h}$. This makes sense since $F_\Gamma(0,t) = 1 = F_\Gamma(s,0)$, since this corresponds to taking 2-transport along a thin bigon. Thus $\partial_s F_\Gamma(s,t)|_{(0,t)} : \mathbb{R} \to TH$ is zero at the origin and hence $\partial_t \partial_s F_\Gamma|_{(0,0)}$ defines an element of $\mathfrak{h}$ (and the same for the other order of derivation). We then have [SW11, Lem. 2.6]:

Proposition 3.11

Let $m \in M$ and $v_1, v_2 \in T_mM$ and suppose $\Gamma : \mathbb{R}^2 \to M$ satisfies equation (3.22). Then

$$\frac{\partial^2}{\partial s \partial t} F_\Gamma \Big|_{(0,0)} = \frac{\partial^2}{\partial t \partial s} F_\Gamma \Big|_{(0,0)} \quad \in \mathfrak{h}$$

is independent of choice of Γ satisfying (3.22).

Proof: Suppose Γ_0, Γ_1 are two maps satisfying equation (3.22). Let Σ_0, Σ_1 be the induced maps $\mathbb{R}^2 \to B^2M$, then we will construct a homotopy between Σ_0 and Σ_1. We work in a chart so that we can assume $M = \mathbb{R}^n$, then we can start with the homotopy

$$h : I \times \mathbb{R}^2 \to B^2M, \qquad h_\tau(s,t)(u,v) = (1-\tau)\Sigma_0(s,t)(u,v) + \tau\Sigma_1(s,t)(u,v),$$

where the addition is defined with respect to the chart. Because Γ_0, Γ_1 are the same up to linear order in some neighborhood $(0,0) \in V \subset \mathbb{R}^2$, the homotopy h factors through a map $I \times V \to I \times V$ of form $(\tau, s, t) \mapsto ((s^2 + t^2)\tau, s, t)$ (for a more precise argument see [SW11, Lem. 2.6]). Thus

$$\frac{\partial^2}{\partial s \partial t} h_\tau \Big|_{(0,0)}$$

is independent of τ. Hence so is

$$\frac{\partial^2}{\partial s \partial t} \mathrm{tra}^2 \circ h_\tau \Big|_{(0,0)},$$

which proves the proposition. ■

Proposition 3.12

Proposition 3.11 gives us a map $B : TM \times_M TM \to \mathfrak{h}$. We claim that this gives a form $B \in \Omega^2(M, \mathfrak{h})$. For this we need to check:

- The map is skew-symmetric: $B_m(v_1, v_2) = -B_m(v_2, v_1)$
- The map is linear: $B_m(v_1 + \lambda v_2, v_3) = B_m(v_1, v_3) + \lambda B_m(v_2, v_3)$
- The map is smooth

Proof: We will follow [SW11, Lem. 2.7] but omit some details. Denote $\overline{\Gamma}(s,t) = \Gamma(t,s)$, and note that if Γ satisfies (3.22) then so does $\overline{\Gamma}$ with v_1, v_2 interchanged. The map $\mathbb{R}^2 \to B^2M$ obtained from $\overline{\Gamma}$ is precisely the vertical inverse to the map $\mathbb{R}^2 \to B^2M$ obtained from Γ. Vertical inversion in $G \ltimes H$ corresponds to inversion in H thus $F_{\overline{\Gamma}} = F_{\Gamma}^{-1}$. Taking derivatives we obtain the required skew symmetry.

For linearity assume we are working in a chart so that $M = \mathbb{R}^n$. This can be done without loss of generality since we are only interested in an arbitrarily small neighborhood around m. Then let Γ^{v_1,v_2} satisfy (3.22), and note that

$$\Gamma^{v_1+\lambda v_2, v_3} = \Gamma^{v_1,v_3} + \lambda \Gamma^{v_2,v_3}, \tag{3.24}$$

since derivation of maps $\Gamma : \mathbb{R}^2 \to \mathbb{R}^n$ is a linear operation. Note that we can add and scale bigons in $\mathbb{R}^n$ pointwise, and one can conclude

$$\frac{\partial^2}{\partial s \partial t} F_{\Gamma^{v_1+\lambda v_2, v_3}}\Big|_{(0,0)} = \frac{\partial^2}{\partial s \partial t} F_{\Gamma^{v_1,v_3}}\Big|_{(0,0)} + \lambda \frac{\partial^2}{\partial s \partial t} F_{\Gamma^{v_2,v_3}}\Big|_{(0,0)},$$

which shows linearity.

For smoothness, let $\phi : \mathbb{R}^n \to M$ be a chart and let

$$D\phi \otimes D\phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n = T\mathbb{R}^n \times_{\mathbb{R}^n} T\mathbb{R}^n \to TM \times_m TM$$

be the induced map. Then we need to show that $B \circ D\phi \otimes D\phi$ is smooth. We obtain a map

$$\Gamma : T\mathbb{R}^n \times_{\mathbb{R}^n} T\mathbb{R}^n \to B^2M, \qquad \Gamma(x,u,v)(s,t) = \phi(x + su + tv).$$

Clearly $\Gamma(u, v_1, v_2)$ satisfies (3.22) for $m = \phi(x), v_1 = D\phi_x(u), v_2 = D\phi_x(v)$. One then sees that

$$B \circ D\phi \otimes D\phi = p_H \circ \mathrm{tra}^2 \circ \Gamma,$$

and the three maps on the right hand side are smooth. Thus we conclude smoothness of B. ■

Proposition 3.13

Let F_A be the curvature of the connection A on $\mathcal{P}_0$. Then B satisfies the *fake-flatness* condition

$$F_A + \tau_*(B) = 0. \tag{3.25}$$

PROOF: Consider $\Sigma : \mathbb{R}^2 \to B^2\mathbb{R}^2$, then let $\gamma_0(s,t)$ and $\gamma_1(s,t)$ denote respectively the source and target paths of $\Sigma(s,t)$. Then the connection A gives us parallel transport maps $F_1(\gamma_i(s,t)) = \mathrm{tra}^1_{\gamma_i(s,t)} : \mathbb{R}^2 \to G$. Note that 2-transport along a bigon $h : h_0 \Rightarrow h_1$ should give us a 2-morphism $\mathrm{tra}^2(h) : F_1(h_0) \Rightarrow F_1(h_1)$ in a local trivialization. Thus we have that

$$\tau(F_\Gamma(s,t)) \cdot F_1(\gamma_0(s,t)) = F_1(\gamma_1(s,t)). \tag{3.26}$$

Note that $\gamma_0^{-1} \circ \gamma_1$ is precisely the path $\gamma_{s,t}$ from lemma 1.33. Thus we have by lemma 1.33 that

$$\tau_* B = \frac{\partial^2}{\partial s \partial t}\bigg|_0 \tau(F_\Gamma(s,t)) = \frac{\partial^2}{\partial s \partial t}\bigg|_0 F_1(\gamma_0^{-1} \circ \gamma_1) = -F. \qquad \blacksquare$$

In summary, a 2-functor $F : \mathcal{P}_2(M) \to \mathcal{G}$ gives us a 2-form $B \in \Omega^2(M, \mathfrak{h})$ with $F_A + \tau_*(B) = 0$. However a 2-functor $\mathrm{tra} : \mathcal{P}_2(M) \to \mathcal{G}$ corresponds to transport on trivial 2-bundles. For non-trivial 2-bundles we need to describe how local triviality gives us such functors, and then we need to show that such a form B glues to a form $B \in \Omega^2(\mathcal{P}_0, \mathfrak{h})$. However the notions of locally trivializable functors (at least those described by Schreiber and Waldorf) are quite complicated, and to continue we would have to spend a lot of time discussing several abstract 2-categorical notions. We want to avoid the use of too much abstract nonsense, and therefore take it on good faith that this can all be formalized. In [SW13a] the relevant notions and statements discussed in full detail. For us the most important thing is that studying 2-functors $F : \mathcal{P}_2(M) \to \mathcal{G}$**-tor** naturally leads us to consider 2-forms $B \in \Omega^2(\mathcal{P}_0, \mathfrak{h})$.

Theorem 3.14 Schreiber&Waldorf

A transport 2-functor $\mathcal{P}_2(M) \to \mathcal{G}$**-tor** (i.e. a smooth 2-functor which is smoothly locally trivializable in some appropriate sense) is equivalent to a 2-bundle $\mathcal{P} \to M_{\mathrm{dis}}$ together with a pair (A, B) with $A \in \Omega^1(\mathcal{P}_0, \mathfrak{g})$ a connection and $B \in \Omega^2(\mathcal{P}_0, \mathfrak{h})$ a basic form such that $F_A + \tau_*(B) = 0$. By basic we mean $L_g^* B = (\alpha_g)_* B$ for any $f : M \to G$.

The property that B is basic is required so that locally it is given by a form on the base space. Thus we can describe 2-transport completely in terms of the pair (A, B). Our goal will now be to describe precisely how such a pair gives us a transport 2-functor $\mathcal{P}_2(M) \to \mathcal{G}$**-tor**.

3.3. Parallel transport in 2-bundles: 2-connections

Using the previous section as a motivation we now introduce the notion of a 2-connection, and we will phrase 2-transport completely in terms of these 2-connections. The aim is to show that it leads to a locally trivializable 2-functor $\mathcal{P}_2(M) \to \mathcal{G}$**-tor** generalizing ordinary parallel transport. All the arguments in this section are essentially based on [SW11].

Definition 3.15

Let $\mathcal{P} \to M_{\text{dis}}$ be a 2-bundle. A 2-connection (A, B) on $\mathcal{P}$ is a pair of forms $A \in \Omega^1(\mathcal{P}_0, \mathfrak{g})$, $B \in \Omega^2(\mathcal{P}_0, \mathfrak{h})$ such that A is a connection on $\mathcal{P}_0$, B is basic (i.e. $L_g^* B = (\alpha_g)_* B$) and such that the pair satisfies the *fake-flatness* condition:

$$\tau_* B + F_A = 0, \tag{3.27}$$

where F_A is the curvature of A.

Recall that 2-transport should give for a bigon $\Sigma : \gamma_0 \Rightarrow \gamma_1$ for $\gamma_i : x \to y$ a morphism $\text{tra}(\gamma_0) \Rightarrow \text{tra}(\gamma_1)$. That is, for $p \in \mathcal{P}_0$ we need to find an $h \in H$ with $\tau(h)\,\text{tra}(\gamma_0)(p) = \text{tra}(\gamma_1)(p)$. Recall that the non-Abelian Stokes' Theorem 1.24 shows us that for a bigon $\Gamma : \Gamma_0 \Rightarrow \Gamma_1$ we have,

$$\text{tra}^1(\Gamma_0)\,\text{tra}^1(\Gamma_1)^{-1}(p) : p = \mathcal{P}\exp \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t\ \widetilde{\Gamma}^* F(\partial_1, \partial_2).$$

Note that the right hand side is in the image of τ by replacing F with $-B$. On the other hand if $\text{tra}^2(\Gamma)$ is the 2-transport along Γ, then its target should satisfy

$$\tau(\text{tra}^2(\Gamma))\,\text{tra}^1(\Gamma_0)(p) = \text{tra}^1(\Gamma_1)(p)$$

Consider therefore the following definition for 2-transport:

$$\text{tra}^2(\Gamma) = \left[\mathcal{P}\exp \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t\ -\widetilde{\Gamma}^* B(\partial_1, \partial_2)\right]^{-1}. \tag{3.28}$$

At least the target of $\text{tra}^2(\Gamma)$ is correct and moreover $\text{tra}^2(\Gamma)$ is trivial for thin bigons, and does not depend on a trivialization. This makes it a good candidate for 2-transport.

Definition 3.16

Let (A, B) be a 2-connection on a $\mathcal{G}$-2-bundle $\mathcal{P}$. If $\Sigma : \gamma_0 \Rightarrow \gamma_1$ is a bigon then we define the 2-transport $\mathrm{tra}^2(\Sigma)$ along Σ to be given by the formula,

$$\mathrm{tra}^2(\Sigma) = \left[\mathcal{P}\exp\int_0^1 \mathrm{d}s\, \mathcal{A}_\Sigma\right]^{-1}, \tag{3.29}$$

where $\mathcal{A}_\Sigma \in \Omega^1(I, \mathfrak{h})$ is defined by

$$\mathcal{A}_\Sigma = \int_0^1 \mathrm{d}t\ -\widetilde{\Sigma}^* B\left(\partial_1, \partial_2\right), \tag{3.30}$$

where $\widetilde{\Sigma}$ is the horizontal lift of Σ with respect to the connection A.

In a local trivialization $\sigma : M \to \mathcal{P}_0$ this instead takes form

$$\begin{aligned}\mathrm{tra}^2(\Sigma) &= \left[\mathcal{P}\exp\int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t\ - L^*_{\mathrm{tra}(\gamma_{s,t})}\Gamma^*\sigma^* B(\partial_1, \partial_2)\right]^{-1} &(3.31)\\ &= \left[\mathcal{P}\exp\int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t\ - \alpha(\mathrm{tra}(\gamma_{s,t}))_*\Gamma^*\sigma^* B(\partial_1, \partial_2)\right]^{-1} &(3.32)\end{aligned}$$

for $\gamma_{s,t}(u) = \Sigma(s, tu)$. For convenience we will often just denote $\mathrm{tra}^2(\Sigma)$ to be the 2-transport in some trivialization without mentioning the trivialization explicitly.

If this is a good definition of 2-transport, it should give us a 2-functor $\mathcal{P}_2(M) \to \mathcal{G}$-**tor**. The fact that it does is something we will check for the remainder of this section. We will need to check the following things:

- 2-transport should be equivariant: $\mathrm{tra}^2(\Sigma)(g \cdot p) = \alpha_g(\mathrm{tra}^2(\Sigma)(p))$.
- 2-transport should respect vertical composition: $\mathrm{tra}^2(\Sigma \bullet \Sigma') = \mathrm{tra}^2(\Sigma) \cdot \mathrm{tra}^2(\Sigma')$.
- 2-transport should respect horizontal composition: if $\Sigma : \gamma_0 \to \gamma_1$, then $\mathrm{tra}^2(\Sigma' \circ \Sigma) = \mathrm{tra}^2(\Sigma) \cdot \alpha_{\mathrm{tra}^1(\gamma_0)}(\mathrm{tra}^2(\Sigma'))$. Compare this to the fact that in a trivialization $\mathrm{tra}^1(\gamma_1\gamma_2) = \mathrm{tra}^1(\gamma_2)\,\mathrm{tra}^1(\gamma_1)$.
- 2-transport should be thin-homotopy invariant

Finally it should be locally trivializable, i.e. in a local trivialization we should obtain a 2-functor $\mathcal{P}_2(M) \to \mathcal{G}$. While stating what this means precisely is difficult, it should not be surprising. We have that $\mathcal{P}_1(M) \to G$-**tor** is locally trivializable, and fixing a basepoint, tra^2 is already H valued. We again refer the reader to [SW13a] to make this precise. Following the same order as [SW11] we will check thin homotopy invariance last. It would be more natural to first prove thin homotopy independence, since vertical/horizontal requires some choice of parameterization. However in the proof of thin homotopy independence we use the way 2-transport behaves under composition as a preliminary result.

Proposition 3.17

2-transport is equivariant, that is if $\Sigma : \gamma_0 \Rightarrow \gamma_1$, $\gamma_i : x \to y$ and $p \in (\mathcal{P}_0)_x$ then

$$\mathrm{tra}^2(\Sigma)(g \cdot p) = \alpha_g \left(\mathrm{tra}^2(\Sigma)(p)\right) \tag{3.33}$$

Proof: Changing the choice of basepoint by g is the same thing as replacing B by $L_g^* B = (\alpha_g)_* B$. Using the functoriality of $\mathcal{P}\exp$ (prop 1.13) we thus have

$$\begin{aligned}
\mathrm{tra}^2(\Sigma)(g \cdot p) &= \left[\mathcal{P}\exp \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t \, - L_g^* \widetilde{\Gamma}^* B(\partial_1, \partial_2)\right]^{-1} \\
&= \left[\mathcal{P}\exp \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t \, - (\alpha_g)_* \widetilde{\Gamma}^* B(\partial_1, \partial_2)\right]^{-1} \\
&= \alpha_g \left[\mathcal{P}\exp \int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t \, - \widetilde{\Gamma}^* B(\partial_1, \partial_2)\right]^{-1}
\end{aligned}$$

■

Proposition 3.18

Let $\Sigma_1 : \gamma_0 \Rightarrow \gamma_1$ and $\Sigma_2 : \gamma_1 \Rightarrow \gamma_2$ be bigons, then

$$\mathrm{tra}^2(\Sigma_2 \bullet \Sigma_1) = \mathrm{tra}^2(\Sigma_2)\,\mathrm{tra}^2(\Sigma_1). \tag{3.34}$$

Additionally $\mathrm{tra}^2(\Sigma) = 1$ for thin bigons and $\mathrm{tra}^2(\overline{\Sigma}) = \mathrm{tra}^2(\Sigma)^{-1}$ (where $\overline{\Sigma}$ is the vertical inverse of Σ).

Proof: The fact that $\mathrm{tra}^2(\Sigma) = 1$ for thin Σ is because $\Sigma^* B$ then vanishes. For vertical inversion note that it is equivalent to inverting the order of the path ordered exponential, which by uniqueness gives the inversion. First we define vertical composition by

$$\Sigma_2 \bullet \Sigma_1(s,t) = \begin{cases} \Sigma_1(2s,t) & s < \frac{1}{2} \\ \Sigma_2(2s-1,t) & s \geq \frac{1}{2} \end{cases} \tag{3.35}$$

One easily convinces oneself that then

$$\mathcal{A}_{\Sigma_2 \bullet \Sigma_1}|_s = \begin{cases} \mathcal{A}_{\Sigma_1}|_{2s} & s < \frac{1}{2} \\ \mathcal{A}_{\Sigma_2}|_{2s-1} & s \geq \frac{1}{2} \end{cases} \tag{3.36}$$

If we then define

$$f_i = \mathcal{P}\exp \int_0^1 \mathrm{d}s \, \mathcal{A}_{\Sigma_i},$$

then by lemma 1.11 we have that

$$\mathrm{tra}^2(\Sigma_2 \bullet \Sigma_1) = (f_1 f_2)^{-1} = f_2^{-1} f_1^{-1} = \mathrm{tra}^2(\Sigma_2)\,\mathrm{tra}^2(\Sigma_1), \tag{3.37}$$

proving the compatibility of 2-transport with vertical composition. ■

We note that this proof used the particular choice of parameterization of $\Sigma_2 \bullet \Sigma_1$. For horizontal composition a natural parameterization would be

$$\Sigma_2 \circ \Sigma_1(s,t) = \begin{cases} \Sigma_1(s,2t) & t < \frac{1}{2} \\ \Sigma_2(s,2t-1) & t \geq \frac{1}{2} \end{cases} \tag{3.38}$$

Unfortunately this is hard to work with in our setting. Somehow it is easier to take the following parameterization, which is thinly homotopic to the former.

$$\Sigma_2 \circ \Sigma_1(s,t) = \begin{cases} \Sigma_1(0,2t) & t \leq \frac{1}{2}, \quad s \leq \frac{1}{2} \\ \Sigma_2(2s,2t-1) & t \geq \frac{1}{2}, \quad s \leq \frac{1}{2} \\ \Sigma_1(2s-1,2t) & t \leq \frac{1}{2}, \quad s \geq \frac{1}{2} \\ \Sigma_2(1,2t-1) & t \geq \frac{1}{2}, \quad s \geq \frac{1}{2} \end{cases} \tag{3.39}$$

Let $\Sigma_i : \gamma_i \Rightarrow \gamma_i'$ then figure 3.19 sketches what this parameterization looks like.

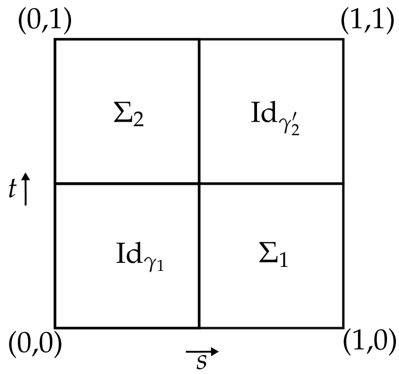


Figure 3.19: Sketch of the parameterization of horizontal composition $\Sigma_2 \circ \Sigma_1(s,t)$ given by equation (3.39) for $\Sigma_i : \gamma_i \Rightarrow \gamma_i'$.

Proposition 3.20

Let $\Sigma_1 : \gamma_1 \Rightarrow \gamma_1'$ and $\Sigma_2 : \gamma_2 \Rightarrow \gamma_2'$ be horizontally composable bigons, i.e. $t(\gamma_1) = s(\gamma_2)$. Then with this parameterization we have

$$\mathrm{tra}^2(\Sigma_2 \circ \Sigma_1) = \mathrm{tra}^2(\Sigma_1)\alpha(\mathrm{tra}^1(\gamma_1), \mathrm{tra}^2(\Sigma_2)). \tag{3.40}$$

Proof: We first consider $\tau(F_{\Sigma_2 \circ \Sigma_1}(s,t))$ in these four quadrants of I^2:

$$\tau(F_{\Sigma_2 \circ \Sigma_1}(s,t)) = \begin{cases} 1 & t \leq \frac{1}{2}, \quad s \leq \frac{1}{2} \\ \mathrm{tra}^1(\gamma_1)\, \tau\big(F_{\Sigma_2}(2s,2t-1)\big)\, \mathrm{tra}^1(\gamma_1)^{-1} & t \geq \frac{1}{2}, \quad s \leq \frac{1}{2} \\ \tau(F_{\Sigma_1})(2s-1,2t) & t \leq \frac{1}{2}, \quad s \geq \frac{1}{2} \\ \tau(F_{\Sigma_1})(2s-1,1) & t \geq \frac{1}{2}, \quad s \geq \frac{1}{2} \end{cases} \tag{3.41}$$

Let us denote

$$f_i = \mathcal{P}\exp\int_0^1 \mathrm{d}s\, \mathcal{A}_{\Sigma_i},$$

then from equation (3.41) and lemma 1.11 we deduce that

$$\mathcal{P}\exp\int_0^1 \mathrm{d}s\, \mathcal{A}_{\Sigma_2\circ\Sigma_1} = (\alpha(\mathrm{tra}^1_{\gamma_1}, f_2)f_1)^{-1} = \mathrm{tra}^2(\Sigma_1)\alpha(\mathrm{tra}^1_{\gamma_1}, \mathrm{tra}^2(\Sigma_2)).$$

■

3.4. 2-Transport and homotopies of bigons

Similarly to how we proved thin-homotopy invariance of parallel transport, we can prove thin-homotopy invariance of 2-transport by first proving a non-Abelian Stokes' Theorem involving a curvature 3-form. The added benefit is that this also gives us a generalization of the Ambrose-Singer Theorem 1.27. As of yet, we don't know what the curvature for a 2-connection is, and we will show that it naturally arises as the 2-transport around an infinitesimal cube, generalizing lemma 1.33.

First we need to show what we mean by the 2-holonomy around the boundary of an infinitesimal cube. Suppose we have a map $h : I^3 \to M$ then for each (r,s,t) we can rescale h to get a map $h_{r,s,t}$ defined by $h_{r,s,t}(x,y,z) = h(rx, sy, tz)$. All of the edges of the cube give paths in M and all faces give bigons in M. All of the edges are of form:

$$_i\gamma^u_{x,y,z}(t) = (x,y,z) + ute_i \tag{3.42}$$

for some $x \in \{0,r\}$, $y \in \{0,s\}$, $z \in \{0,t\}$. Here e_i is the ith coordinate unit vector in I^3. Similarly all the faces keep exactly one coordinate fixed and they are of form

$$_i\Sigma^{u,v}_{x,y,z} = {}_k\gamma \circ {}_j\gamma \Rightarrow {}_j\gamma \circ {}_k\gamma, \tag{3.43}$$

where $j = i+1 \bmod 3$ and $k = i+2 \bmod 3$. If j,k are flipped we denote it $_i\overline{\Sigma}$ instead (for vertical inverse). This choice of orientation is in principal arbitrary, and this choice suits our specific situation. Because these bigons are contained in a plane in $\mathbb{R}^3$, there is only one thin homotopy class of such bigons after fixing orientation. Since we don't yet have thin homotopy invariance, we will fix any consistent parameterization. This notation is useful for facilitating computations. We have sketched the cube with this notation in figure 3.21.

In figure 3.21 we have marked two paths with a different color. We set:

$$\mu = {}_2\gamma \circ {}_3\gamma \circ {}_1\gamma = {}_2\gamma^s_{r,0,t} \circ {}_3\gamma^t_{r,0,0} \circ {}_1\gamma^r_{0,0,0} \tag{3.44}$$

$$\nu = {}_1\gamma \circ {}_3\gamma \circ {}_2\gamma = {}_1\gamma^r_{0,s,t} \circ {}_3\gamma^t_{0,s,0} \circ {}_2\gamma^s_{0,0,0}. \tag{3.45}$$

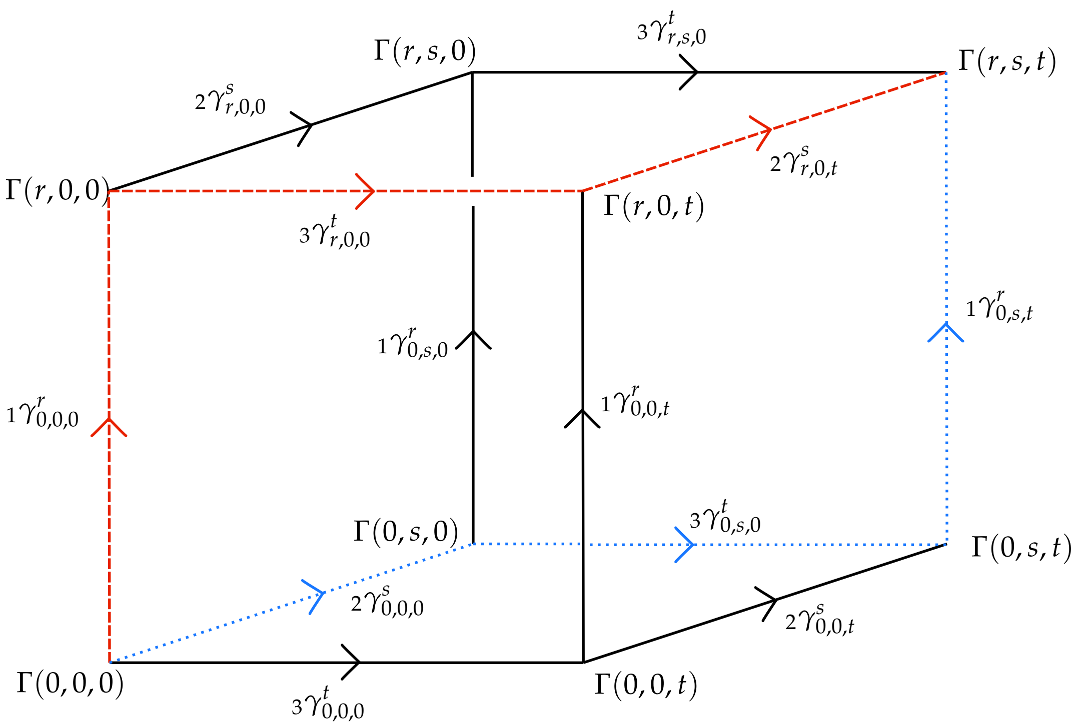


Figure 3.21: A cube in I^3 spanned by $(0,0,0)$ and (r,s,t). The edges are labeled by paths ${}_i\gamma$ as in equation (3.42). The path $\mu = {}_2\gamma \circ {}_3\gamma \circ {}_1\gamma$ is marked red and dashed and the path $\nu = {}_1\gamma \circ {}_3\gamma \circ {}_2\gamma$ is marked blue and dotted.

We can cut open the cube along $\nu^{-1}\mu$ into two pieces as shown in figure 3.22. By pasting, each of the two pieces represents a bigon $\mu \Rightarrow \nu$ or $\nu \Rightarrow \mu$, depending on which orientations of the bigons we choose.

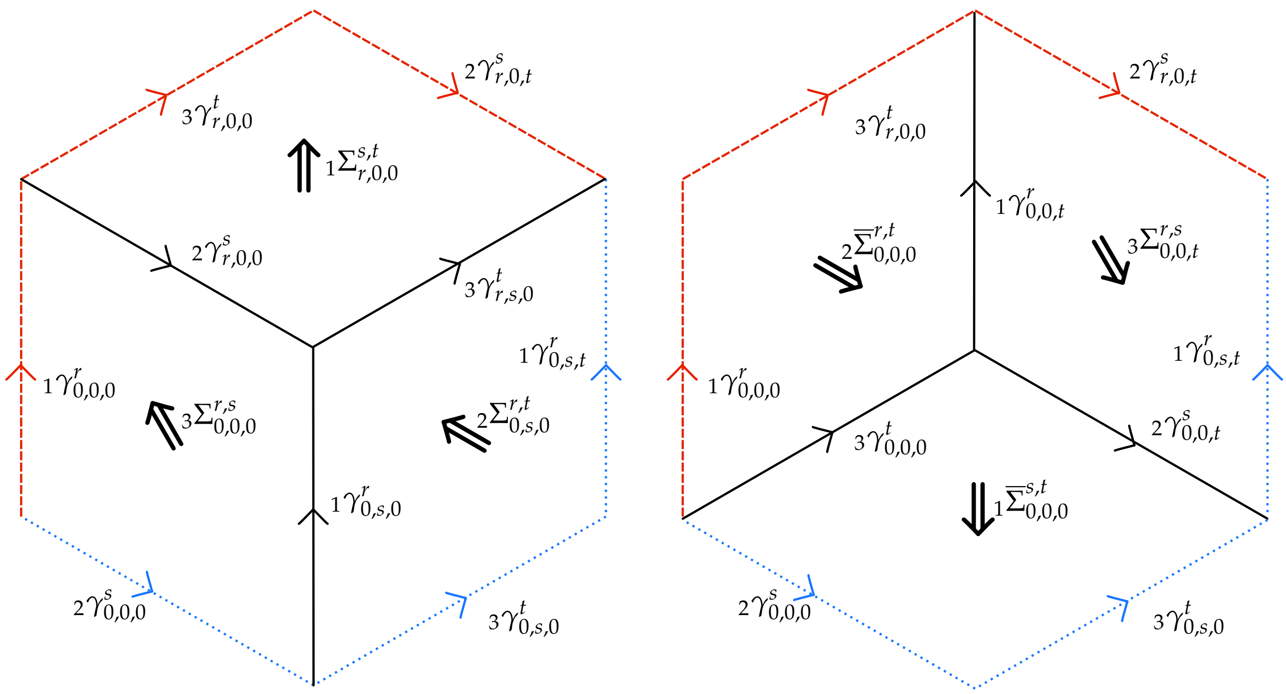


Figure 3.22: Diagram of the cube of figure 3.21 split into two parts with the faces labeled by bigons. Using pasting the left side is a bigon $\nu \Rightarrow \mu$ and the right side a bigon $\mu \Rightarrow \nu$.

Composing the bigons in 3.22 we get on the left side:

$$L(r,s,t) := ({}_1\Sigma^{s,t}_{r,0,0} \circ \mathrm{Id}\,{}_1\gamma^r_{0,0,0}) \bullet (\mathrm{Id}\,{}_3\gamma^t_{r,s,0} \circ {}_3\overline{\Sigma}^{r,s}_{0,0,0}) \bullet ({}_2\Sigma^{r,t}_{0,s,0} \circ \mathrm{Id}\,{}_2\gamma^s_{0,0,0}). \tag{3.46}$$

On the right side we obtain:

$$R(r,s,t) := (\mathrm{Id}\,{}_1\gamma^r_{0,s,t} \circ {}_1\overline{\Sigma}^{s,t}_{0,0,0}) \bullet ({}_3\Sigma^{r,s}_{0,0,t} \circ \mathrm{Id}\,{}_3\gamma^t_{0,0,0}) \bullet (\mathrm{Id}\,{}_2\gamma^s_{r,0,t} \circ {}_2\overline{\Sigma}^{r,t}_{0,0,0}). \tag{3.47}$$

Then we define

$$u(r,s,t) = \mathrm{tra}^2(L \bullet R). \tag{3.48}$$

This is the 2-transport $\mu \Rightarrow \mu$ in M. It turns out that its third order derivative is computed by a 3-form we will call the 2-curvature.

Lemma 3.23

Define the 2-curvature form of a 2-connection (A, B) by

$$K = dB - \alpha_*(A \wedge B), \tag{3.49}$$

where α is seen as a map $G \times H \to H$. Let $p \in (\mathcal{P}_0)_m$, then for any $\Gamma : I^3 \to M$ with $\Gamma(0,0,0) = m$ we have

$$\left.\frac{\partial^3}{\partial r \partial s \partial t}\right|_0 u(r,s,t) = K_p\left(X, Y, Z\right). \tag{3.50}$$

With X, Y, Z the horizontal vectors (w.r.t. A) at p corresponding to respectively

$$\left.\frac{\partial \Gamma(r,0,0)}{\partial r}\right|_0, \qquad \left.\frac{\partial \Gamma(0,s,0)}{\partial s}\right|_0, \qquad \left.\frac{\partial \Gamma(0,0,t)}{\partial t}\right|_0.$$

Proof. Assume we have a trivialization of $\Gamma^*\mathcal{P}$ and assume A, B are forms on M. Using propositions 3.18 and 3.20 we compute

$$\begin{aligned} u(r,s,t) = &\alpha(\mathrm{tra}^1({}_1\gamma^r_{0,0,0}), \mathrm{tra}^2({}_1\Sigma^{s,t}_{r,0,0})) \cdot \mathrm{tra}^2({}_3\overline{\Sigma}^{r,s}_{0,0,0}) \cdot \\ &\cdot \alpha(\mathrm{tra}^1({}_2\gamma^s_{0,0,0}), \mathrm{tra}^2({}_2\Sigma^{r,t}_{0,s,0})) \cdot \mathrm{tra}^2({}_1\overline{\Sigma}^{s,t}_{0,0,0}) \cdot \\ &\cdot \alpha(\mathrm{tra}^1({}_3\gamma^t_{0,0,0}), \mathrm{tra}^2({}_3\Sigma^{r,s}_{0,0,t})) \cdot \mathrm{tra}^2({}_2\overline{\Sigma}^{r,t}_{0,0,0}). \end{aligned} \tag{3.51}$$

This expression has six terms, and we remark the cyclic symmetry present in the expression. All the terms vanish if we take a single derivative and evaluate at 0. Therefore to compute the third derivative of u at 0, we just need to consider each of the six terms individually. From the definition of tra^2 we have that

$$\left.\frac{\partial^2}{\partial s \partial t}\right|_0 \mathrm{tra}^2({}_1\Sigma^{s,t}_{r,0,0}) = B_{\Gamma(r,0,0)}(Y, Z),$$

where $Y = \partial_s \Gamma(r,s,0)|_0$ and $Z = \partial_t \Gamma(r,0,t)|_0$, and thus

$$\frac{\partial^3}{\partial r \partial s \partial t}\bigg|_0 \operatorname{tra}^2({}_1\Sigma^{s,t}_{r,0,0}) = X(B_m(Y,Z)).$$

From the definition of parallel transport we have that

$$\frac{\partial}{\partial r}\bigg|_0 \operatorname{tra}^1({}_1\gamma^r_{0,0,0}) = -A_{\Gamma(0,0,0)}(X).$$

Hence we conclude,

$$\frac{\partial^3}{\partial r \partial s \partial t}\bigg|_0 \alpha(\operatorname{tra}^1({}_1\gamma^r_{0,0,0}), \operatorname{tra}^2({}_1\Sigma^{s,t}_{r,0,0})) = X(B(Y,Z)) - \alpha(A(X)B(Y,Z))_*.$$

Furthermore we recall that $\overline{\Sigma}(s,t) = \Sigma(1-s,t)$ so that

$$\frac{\partial^2}{\partial r \partial s}\bigg|_0 \operatorname{tra}^2({}_3\overline{\Sigma}^{r,s}_{0,0,0}) = -B_{\Gamma(0,0,0)}\left(X,Y\right),$$

and hence we have that

$$\frac{\partial^3}{\partial r \partial s \partial t}\bigg|_0 \operatorname{tra}^2({}_3\overline{\Sigma}^{r,s}_{0,0,0}) = \frac{\partial}{\partial t}\bigg|_0 - B_{\Gamma(0,0,0)}\left(X,Y\right) = 0.$$

Noting the cyclic symmetry present, the same arguments compute the remaining four terms, and we obtain

$$\begin{aligned}\frac{\partial^3}{\partial r \partial s \partial t}\bigg|_0 u(r,s,t) &= X(B(Y,Z)) + Y(B(Z,X)) + Z(B(X,Y)) - \\ &\quad - \alpha(A(X)B(Y,Z))_* - \alpha(A(Y)B(Z,X))_* - \alpha(A(Z)B(X,Y))_*,\end{aligned}$$

which is precisely $dB - \alpha(A \wedge B)_*$. ■

We can interpret this as the fact that the 2-holonomy around 'the boundary of an infinitesimal cube' is given by the 2-curvature. Interestingly, this 2-curvature can also be obtained as $dB \circ h$, with h a horizontal projection in TP given by the connection A [MP07]. Compare this to the fact that $F = dA \circ h$. Furthermore if we see A and B as $\mathfrak{g} \ltimes \mathfrak{h}$-valued forms, then $K = d_A B$, since the adjoint action of $\mathfrak{g}$ on $\mathfrak{h}$ is by α_*. Therefore it is a very natural generalization to ordinary curvature. Every bigon $\Gamma : \gamma \Rightarrow \gamma$ homotopic to the identity is the boundary of a cube. The parallel transport around such bigons form the elements of a 2-holonomy group based at p.

Definition 3.24

Let (A, B) be a 2-connection on $\mathcal{P}$ and suppose $p \in (\mathcal{P}_0)_x$, then define the *2-holonomy group* $\mathrm{Hol}_p^2 \subset H$ at p by

$$\mathrm{Hol}_p^2 = \left\{ \mathrm{tra}^2(\Gamma)(p) : (\mathrm{Id}_{\mathrm{tra}^1(\gamma)(p)})(p) \,\middle|\, \Gamma : \gamma \Rightarrow \gamma,\ \gamma : x \to y,\ y \in M \right\}. \tag{3.52}$$

We also define the *reduced 2-holonomy group* $(\mathrm{Hol}_p^2)_0$ as the subgroup of Hol_p^2 given by holonomies around bigons $\gamma \Rightarrow \gamma$ homotopic to Id_γ. Both groups are subgroups of $\ker \tau$, and are hence Abelian by proposition 2.18.

Proposition 3.25

$(\mathrm{Hol}_p^2)_0$ is the identity component of Hol_p^2

Proof: First of all if $\Sigma : \gamma \Rightarrow \gamma$ is a contractible bigon, then the homotopy $h : \Sigma : \mathrm{Id}_\gamma$ gives a path $\mathrm{tra}^2(h_t)(p) : (\mathrm{Id}_{\mathrm{tra}^1(\gamma)(p)})(p)$, showing that $(\mathrm{Hol}_p^2)_0$ is connected. Then by the same argument as the proof of proposition 1.15 we only need to show that $(\mathrm{Hol}_p^2)_0$ is of countable index. Note that by proposition 3.20 we have

$$\mathrm{tra}^2(\Sigma \circ \mathrm{Id}_{\gamma^{-1}}) = \alpha(\mathrm{tra}^1_{\gamma^{-1}})\, \mathrm{tra}^2(\Sigma). \tag{3.53}$$

By virtue of the Reduction Theorem 1.31 we have $\mathrm{tra}^1_{\gamma^{-1}} \in \mathrm{Hol}_p$, thus

$$\mathrm{Hol}_p^2 = \left\{ \alpha_g(\mathrm{tra}^2(\Sigma)) \,\middle|\, g \in \mathrm{Hol}_p,\ \Sigma : \mathrm{Id}_x \Rightarrow \mathrm{Id}_x \right\}. \tag{3.54}$$

Suppose $\Sigma, \Sigma' : \mathrm{Id}_x \Rightarrow \mathrm{Id}_x$ are homotopic, then $\overline{\Sigma} \bullet \overline{\Sigma}'$ is contractible, thus $\mathrm{tra}^2(\Sigma)^{-1}\,\mathrm{tra}^2(\Sigma') \in (\mathrm{Hol}_p)^0$. For $g \in \mathrm{Hol}_p^0$ we have $\alpha_g(\mathrm{tra}^2(\Sigma)) = h\,\mathrm{tra}^2(\Sigma)h^{-1} = \mathrm{tra}^2(\Sigma)$ if $t(h) = g$, since $\mathrm{tra}^2(\Sigma)$ is central. Such an h always exists, since any $g \in \mathrm{Hol}_p^0$ is given by tra_γ for a contractible γ; the 2-transport along the homotopy $\gamma \Rightarrow \mathrm{Id}_x$ then gives such an h. Thus we have a surjective homomorphism:

$$\pi_2(M, x) \times \left(\mathrm{Hol}_p \,/\, \mathrm{Hol}_p^0\right) \to \mathrm{Hol}_p^2 \,/\, (\mathrm{Hol}_p^2)_0, \qquad ([\Sigma : \mathrm{Id}_x \Rightarrow \mathrm{Id}_x], [g]) \mapsto [\alpha_g \Sigma].$$

Since $\pi_2(M, x)$ and Hol_p / Hol_p^0 are both countable, this means $(\mathrm{Hol}_p^2)_0$ is of countable index, which proves the proposition. ■

The computation of lemma 3.23 now shows that the Lie algebra elements $K_p(X, Y, Z)$ land inside the Lie algebra Hol_p^2. This suggests a generalization of the Ambrose-Singer Theorem to the setting of 2-bundles. The original proof of the Ambrose-Singer Theorem seems hard to generalize to this setting, although it may be possible to generalize it to the setting of principal bundles over path spaces which we consider in the next section. Instead we will first suggest a generalization to the the non-Abelian Stokes' Theorem and generalize our proof of the Ambrose-Singer Theorem instead. This will also immediately show thin-homotopy invariance.

Theorem 3.26 (Non-Abelian Stokes' Theorem)

Let $h : I^3 \to M$ be a homotopy $h_0 \Rrightarrow h_1$ then

$$\mathrm{tra}^2(h_0)\,\mathrm{tra}^2(h_1^{-1}) = \mathcal{P}\exp \int_0^1 \mathrm{d}r \int_{I^2} \mathrm{d}s\mathrm{d}t\ \mathrm{Ad}_{\mathrm{tra}^2(h_r(s,1))}\, h^*K\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) \tag{3.55}$$

$$= \exp \int_{I^3} h^*K \tag{3.56}$$

Proof: This formula is suggested in [AFG97] using different notation. Also using different notation a very similar formula is proven in [MP07] and [MP08]. We will show it follows as a corollary from lemma A.11 in [SW11]. In all cases the proof is a long and technical computation and we do not see the benefit of repeating the computation in this thesis. Let $h : I^3 \to M$ be a homotopy then instead of considering $u(r,s,t)$ we consider $u_\rho(r,s,t)$ which is obtained by shifting the origin to $(\rho,0,0)$. Then using vertical composition we have that

$$u_\rho(r,1,1) = u(\rho,1,1)^{-1}u(\rho+r,1,1),$$

and thus differentiating with respect to r we obtain the initial value problem

$$\frac{\partial u(r,1,1)}{\partial r} = u(r,1,1)\left.\frac{\partial u_\rho(r,1,1)}{\partial r}\right|_{r=0}, \qquad u(0,1,1) = 0.$$

Thus we need to show that

$$\left.\frac{\partial u_\rho(r,1,1)}{\partial r}\right|_{r=0} = \int_{I^2} \mathrm{d}s\mathrm{d}t\ \mathrm{Ad}_{\mathrm{tra}^2(h_r(s,1))}\, h^*K\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right),$$

or equivalently

$$\left.\frac{\partial}{\partial r}\frac{\partial}{\partial s}\frac{\partial}{\partial t}u_\rho(r,s,t)\right|_{(0,s,t)} = \mathrm{Ad}_{\mathrm{tra}^2(h_r(s,1))}\, h^*K_{(r,s,t)}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right).$$

This formula follows directly from the proof of lemma A.11 in [SW11].

For the final formula, note that $\tau_* K = -dF + [A \wedge F] = -d_A F = 0$, and also recall that $\ker\tau$ is a central subgroup of H (cf. prop. 2.18). Thus in particular $\mathrm{Ad}_{\mathrm{tra}^2(h_r(s,1))} K = K$. Finally since K takes values in an Abelian algebra the path-ordered exponentiation becomes ordinary exponentiation of the integral. ■

Corollary 3.27

Let (A,B) be a 2-connection on a principal $\mathcal{G}$-2-bundle. If $\Sigma_0 : \gamma \Rightarrow \gamma'$ and $\Sigma_1 : \gamma \Rightarrow \gamma'$ are thinly homotopic bigons then $\mathrm{tra}^2(\Sigma_0) = \mathrm{tra}^2(\Sigma_1)$.

Proof: If h is a thin homotopy then $h^*K = 0$. Thus by the non-Abelian Stokes' Theorem we have $\mathrm{tra}^2(\Sigma_0) = \mathrm{tra}^2(\Sigma_1)$. ■

Corollary 3.28

If $\ker \tau$ is discrete (e.g. $\tau : H \to G$ is a covering, or H is semi-simple so that $Z(H)$ is discrete) then (A, B) is always flat; i.e. $K = 0$ and 2-transport is homotopy invariant.

Using this notion and lemma 3.23 we can now state and prove a generalization of the Ambrose-Singer Theorem 1.27:

Theorem 3.29 (Ambrose-Singer)

Let (A, B) be a 2-connection on a principal $\mathcal{G}$-2-bundle $\mathcal{P}$. Then the Lie algebra $\mathfrak{hol}_p^2$ of the 2-holonomy group Hol_p^2 satisfies

$$\mathfrak{hol}_p^2 = \left\{ K_q(X, Y, Z) \,\middle|\, q \in P(p),\ X, Y, Z \in T_q P \right\} \tag{3.57}$$

Proof: Let $\sigma : \pi(p) = x \to y$ be a curve and let $q = \mathrm{tra}^1_\sigma(p)$. Let $\Sigma : \gamma \Rightarrow \gamma$ with $\gamma : y \to z$ be a bigon, then consider the bigon $\Sigma \circ \mathrm{Id}_\sigma$. It has 2-holonomy $\alpha(\mathrm{tra}^1_\gamma, \mathrm{tra}^2(\Sigma))$. By the Reduction Theorem 1.31 $\mathrm{tra}^1_\gamma \in \mathrm{Hol}_p$. In other words, there is also a loop σ at y with $\mathrm{tra}^1_\gamma = \mathrm{tra}^1_\sigma$. Since $\Sigma \circ \sigma$ is a bigon at y, this shows that $\mathrm{Hol}^2_p = \mathrm{Hol}^2_q$. Thus by lemma 3.23, $K_q(X, Y, Z)$ lies inside $\mathfrak{hol}^2_p$ for any q that can be joined to p by a horizontal path. By the non-Abelian Stokes' Theorem 3.26 we note that $\mathfrak{hol}^2_p$ is spanned by elements of form $K_{\Gamma(s,t)}$ for some bigon Γ. By the Reduction Theorem 1.31, we can assume that we are considering K restricted to $P(p)$. This proves the theorem. ∎

3.5. Bundles over path spaces

Now that we have developed the theory of principal 2-bundles it is worthwhile to discuss some other equivalent points of view. One is that of G-gerbes and non-Abelian Čech cocycles, which are proven to be equivalent to our construction of 2-bundles in [NW11]. Instead will consider yet another point of view towards 2-bundles and 2-transport developed in [BS04]. Recall that 2-transport over a bigon has form

$$\mathcal{P} \exp \int_0^1 \mathcal{A},$$

this suggests that $\mathcal{A} \in \Omega^1(I, \mathfrak{h})$ should play the role of some sort of connection. Indeed we will show that if we go to path space $\mathscr{P}(M)$ then $\mathcal{A}$ is locally a connection on a principal bundle over $\mathscr{P}(M)$. Moreover its curvature is

$$(F_{\mathcal{A}})_\gamma = \int_0^1 \alpha(\mathrm{tra}^1_{\gamma_t})K. \tag{3.58}$$

In this section we will first develop the theory of bundles on diffeological spaces and then argue that $\mathcal{A}$ is locally a connection and consider its curvature.

Definition 3.30

A *diffeology* on a set $\mathscr{M}$ is a set of *plots* $\mathcal{U}$, i.e. maps from finite dimensional convex sets $U \to \mathscr{M}$ such that:

- If $\phi : U \to \mathscr{M}$ is a plot, and $\phi' : U' \to U$ is smooth then $\phi \circ \phi'$ is a plot
- Every constant map $U \to \mathscr{M}$ is a plot
- If $\phi : U \to \mathscr{M}$ is a map such that each $\phi|_{U_i}$ is a plot for some cover $\{U_i\}$ of U, then ϕ is also a plot.

We call a set M together with a diffeology $\mathcal{U}$ a *diffeological space*. We call a map $f : \mathscr{M} \to \mathscr{N}$ between two diffeological spaces smooth if for every plot $\phi : U \to \mathscr{M}$ the map $f \circ \phi : U \to \mathscr{N}$ is a plot.

Any smooth manifold has a canonical diffeology; if M is smooth then $\phi : U \to M$ is a plot if and only if its smooth. Moreover the diffeology uniquely determines the smooth structure on M. The concept of diffeological spaces gives a notion of differential geometry on infinite dimensional spaces, such as path spaces.

Definition 3.31

Let M be a smooth manifold (or a diffeological space), then define the *path space*[a]

$$\mathscr{P}(M) = \left\{ \gamma \in M^I \,\middle|\, \gamma \text{ is smooth} \right\}, \tag{3.59}$$

together with the following diffeology: we say $\phi : U \to \mathscr{P}(M)$ is a plot if the associated map $U \times I \to M$, $(u,t) \mapsto \phi(u)(t)$ is smooth.

[a] This is not a groupoid under concatenation of paths with matching source and target since the required reparameterization makes associativity fail.

A lot of concepts of differential geometry can be generalized to diffeological spaces. For example a differential p-form ω on a diffeological space $\mathscr{M}$ assigns to each plot $\phi : U \to M$ a differential form $\omega_\phi \in \Omega^p(U)$ such that for any smooth map $\theta : U' \to U$ we have $\theta^*\omega_\phi = \omega_{\phi\circ\theta}$. Thus we get a set $\Omega^p(\mathscr{M})$ of p-forms. Wedge products and exterior derivation can be defined locally; $(d\omega)_\phi := d(\omega_\phi)$ and $(\omega \wedge \eta)_\phi = \omega_\phi \wedge \theta_\phi$. One also has a natural notion of $\mathfrak{g}$-valued differential forms $\Omega^p(\mathscr{M}, \mathfrak{g})$ which admits similar operations as the finite dimensional version. Moreover definition 1.1 caries over word-by-word to the diffeological setting to give us a notion of principal G bundles over diffeological spaces. In fact G can even be a group in the diffeological category.

Let $A \in \Omega^1(M, \mathfrak{g})$ be a (local) connection on a trivial G-bundle $P \to M$. For any $\omega \in \Omega^p(M, \mathfrak{g})$ and plot $\phi : U \to M$ define

$$\left(\left(\oint_A \omega \right)_\phi \right)_p = \int_0^1 \mathrm{d}t \; \mathrm{Ad}(\mathrm{tra}^1_{\phi(p)_t}) \, (\mathrm{ev}_t \circ \phi)^* \left(i_{\phi(p)'(t)} \omega \right). \tag{3.60}$$

This defines a form $\oint_A \omega \in \Omega^{p-1}(\mathscr{P}(M), \mathfrak{g})$. Naively assuming $\mathscr{P}(M)$ is a smooth space and $\gamma \in \mathscr{P}(M)$ then a 'tangent vector at γ' is just a map $X : I \to TM$, $X(t) \in T_{\gamma(t)}M$. In this naive setting the form reduces to

$$\left(\oint_A \omega\right)_\gamma (X_1, \ldots, X_{p-1}) = \int_0^1 \mathrm{d}t\ \mathrm{Ad}(\mathrm{tra}^1_{\gamma_t})\ \omega(\gamma'(t), X_1(t), \ldots, X_n(t)). \tag{3.61}$$

Of course one must be careful since diffeological spaces don't necessarily admit a 'tangent space' in the ordinary sense. If $\mathcal{P}$ is now a trivial $\mathcal{G}$-2-bundle, then a 2-connection gives forms $A \in \Omega^1(M, \mathfrak{g})$ and $B \in \Omega^2(M, \mathfrak{h})$. We can see $\mathfrak{g}$ and $\mathfrak{h}$ as subspaces of $\mathfrak{g} \ltimes \mathfrak{h}$, the Lie algebra of $G \ltimes H$. Since

$$(g, 1)(1, h)(g^{-1}, 1) = (1, \alpha_g(h)), \tag{3.62}$$

we have that $\mathrm{Ad}_g(X, Y) = (\mathrm{Ad}_g X, (\alpha_g)_* Y)$ for $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$. Thus seeing A, B as $\mathfrak{g} \ltimes \mathfrak{h}$ valued forms we can consider $\oint_A B$. Note that a bigon $\Sigma : I \times I \to M$ induces a plot $\sigma : I \to \mathscr{P}(M)$, $\sigma(s) = \Sigma|_{\{s\} \times I}$. Note furthermore that now

$$\left(\oint_A B\right)_\sigma = \int_0^1 \mathrm{d}t\, \alpha(\mathrm{tra}^1(\gamma_{s,t}))i(\partial_t)\Sigma^* B, \tag{3.63}$$

which is precisely $\mathcal{A}$ from definition 3.16 (the sign comes from the fact that one has to flip ∂_t and ∂_s). The form $\oint_A B$ defines a connection on a trivial bundle over $\mathscr{P}(M)$ and its parallel transport gives us 2-transport. One can then show that the curvature of this form is $\oint_A K$:

$$\mathcal{F} = d \oint_A B + \left[\oint_A B, \oint_A B\right] = \oint_A K$$

The fake-flatness condition plays an essential role here. Computing $d \oint_A B$ is a little complicated and requires a bit of technical machinery, and we refer to [BS04] or [AFG97] for a proof. A homotopy $\widetilde{h} : I^3 \to M$ between bigons induces a bigon $h : I^2 \to \mathscr{P}(M)$. For the same reasons as for parallel transport on ordinary bundles, the parallel transport between thinly homotopic paths $I \to \mathscr{P}(M)$ will be zero. In fact for diffeological bundles we have an Ambrose-Singer Theorem [Mag13]. Thus the 2-transport is thin-homotopy invariant.

One problem with this point of view is that $\oint_A B$ is only locally a connection, and we do not see a way to extend it to a global connection on some bundle over $\mathscr{P}(M)$. Furthermore the fact that $\mathscr{P}(M)$ is infinite dimensional brings in some non-trivial analysis to many of the definitions. While more abstract, seeing 2-bundles from a 2-categorical point of view seems more powerful.

4. Surface Holonomy

4.1. Surface holonomy

Using parallel transport we were able to associate to loops, i.e. maps $S^1 \to M$, some group element (up to conjugation). We want to generalize this to the setting of 2-transport, where we can consider maps $S^2 \to M$ instead. In fact, nothing is stopping us from considering maps $S_g \to M$ where S_g is a surface of genus $g \geq 0$. In order to apply parallel transport to maps $S^1 \to M$ we first need to select a basepoint $x \in S^1$. The essential property of the basepoint is that it allows us to 'cut open' S^1 at x so that we get an interval. For a surface S_g we need to find a loop τ along which we can cut open S_g as to get a square I^2. This approach was developed in [SW13a]. The first statement is that one can indeed always cut open a surface, and the amount of necessary loops depends only on the genus.

Theorem 4.1 (Classification of Surfaces)

Let S_g be a surface of genus g, then one can find a set of $2g$ loops $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ (i.e. maps $S^1 \to S_g$) based at some point $x \in S_g$ such that

$$\pi_1(S_g) = \frac{\langle [\alpha_1], [\beta_1], \ldots, [\alpha_g], [\beta_g] \rangle}{\prod_{i=1}^g [[\alpha_i], [\beta_i]]}, \tag{4.1}$$

where $[\gamma]$ denotes the equivalence class of γ modulo homotopy, and $[\,\bullet\,,\,\bullet\,]$ denotes the commutator. For a sphere the story is a bit easier, since we just have to pick a basepoint. Furthermore there is a surjective map $\Sigma : I^2 \twoheadrightarrow S_g$ such that $\Sigma|_{\mathrm{Int}(I^2)}$ is an embedding and $\Sigma|_{\partial I^2} = \prod_{i=1}^g [\alpha_i, \beta_i]$ up to thin homotopy under a diffeomorphism $\partial I^2 \cong S^1$.

This well-known theorem is proven by considering the universal cover of a surface. For genus 0,1 the theorem is easily shown to be true, and for genus >1 one shows that every surface is the quotient of the hyperbolic half space Γ by a Fuchsian group. One then constructs a fundamental domain for this action in Γ. [Bea83]

This theorem can be interpreted as every surface being presentable as a quotient of a polygon with $2g$ sides as shown in figure 4.2. Such a polygon can be interpreted as a bigon

$$\Sigma : \mathrm{Id}_x \Rightarrow \prod_i [\alpha_i, \beta_i].$$

Given a 2-connection (A, B) on some 2-bundle over S_g we can take the 2-transport of this bigon, giving us a morphism

$$\mathrm{tra}^2(\Sigma) : 1 \Rightarrow \prod_i [\mathrm{tra}^1(\alpha_i), \mathrm{tra}^1(\beta_i)].$$

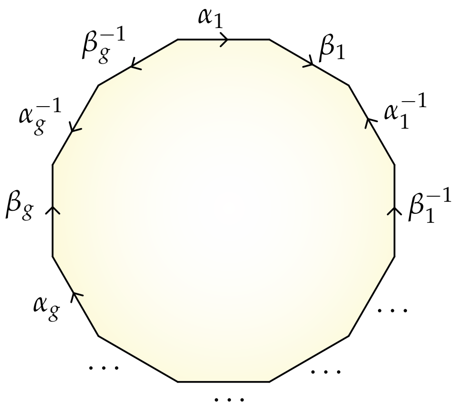


Figure 4.2: Sketch of a fundamental polygon of a surface of genus g. All the edges in this polygon correspond to a loop in S_g, and the all the vertices correspond to the same basepoint in S_g.

Choosing a point $p \in (\mathcal{P}_0)_x$ this gives us an element $h \in H$. Changing p to $g \cdot p$ we see that h changes to $\alpha_g(h)$. Hence such a bigon Σ defines for us an element of

$$H/G = H/\sim, \qquad h \sim h' \text{ if } h = \alpha_g(h') \text{ for some } g \in G. \tag{4.2}$$

This is a good candidate for 'surface holonomy', however a priori this still depends several choices. We list all the choices that we made so far:

- A basepoint $x \in S_g$
- A set of $2g$ loops $\{\alpha_i, \beta_i\}$ giving a presentation of the fundamental group $\pi_1(S_g)$.
- A choice of map $\Sigma : I^2 \to S_g$ and parameterization of Σ into a bigon $\mathrm{Id}_x \Rightarrow \prod_i [\mathrm{tra}^1(\alpha_i), \mathrm{tra}^1(\beta_i)]$.

In full generality we can't prove independence of the second of these choices, but the other choices don't pose much of a problem.

Proposition 4.3

Let $\mathcal{P} \to S_g$ be a $\mathcal{G}$-2-bundle with 2-connection (A, B). Let $\Sigma : I^2 \to S_g$ and $\Sigma' : I^2 \to S_g$ be two maps such that $\Sigma|_{\partial I^2} = \Sigma'|_{\partial I^2}$, then Σ, Σ' induce the same surface holonomy.

Proof: First of all by thin homotopy invariance of 2-transport, we have that it doesn't matter how we parameterize Σ, Σ' into a bigon $\mathrm{Id}_x \Rightarrow \prod_i [\alpha_i, \beta_i]$. The fact that Σ, Σ' agree on the boundary means that they glue to a map $S^2 \to S_g$. If $g > 0$, any such map is contractible ($\pi_2(S_g) = 0$ since the universal cover of S_g is contractible). This contraction then gives a homotopy $\Sigma \Rightarrow \Sigma'$. By dimensional reasons, this homotopy is necessarily thin, and hence Σ and Σ' give the same 2-holonomy. If $g = 0$ the maps Σ, Σ' themselves descent to maps $S^2 \to S^2$. One checks that these maps are of degree 1, and hence Σ, Σ' are homotopic as well. This homotopy is again thin and proves that Σ, Σ' have the same 2-holonomy. ∎

We can also change all the α_i, β_i by thin homotopy. This results in changing the bigon $\mathrm{Id}_x \Rightarrow \prod_i [\alpha_i \beta_i]$

by thin homotopy, which does not affect the result. However in general we can't change α_i, β_i by a homotopy that is not thin; this can change the target of the 2-transport. We can also change basepoint without much of a loss; suppose we have a path $\gamma : y \to x$, then conjugating all the α_i, β_i by γ we get another presentation of the fundamental group. Note that

$$\prod_i [\gamma\alpha_i\gamma^{-1}, \gamma\beta_i\gamma^{-1}] = \gamma \prod_i [\alpha_i, \beta_i]\gamma^{-1},$$

thus such a change of basepoint changes the 2-transport by a factor $\alpha(\mathrm{tra}^1_\gamma)$.

Definition 4.4

A marking $\mathcal{M}$ of a surface S_g of genus g is a choice of basepoint $x \in S_g$ and a cyclically ordered set of g pairs of thin homotopy classes of loops (α_i, β_i) based at x such that $\pi_1(S_g, x) = \langle [\alpha_i], [\beta_i] \rangle \,/\, \prod_i [[\alpha_i], [\beta_i]]$.
One can also consider surfaces with boundary, where one has to fix one additional loop for each boundary component.

Proposition 4.5

Given a marked surface $(S_g, \mathcal{M})$ and a 2-bundle $\mathcal{P} \to M$ with connection (A, B), the procedure above associates to each map $\Sigma : S_g \to M$ a well-defined *surface holonomy* $W(\Sigma) \in H/G$. If we fix a basepoint $p \in \mathcal{P}_0$ and mandate that the basepoint of S_g maps to $\pi(p)$, then we obtain an element $W(\Sigma) \in H$.
If $\mathrm{Tr} : \mathfrak{h} \to \mathbb{R}$ is such that $\mathrm{Tr}(\alpha(g)_* X) = \mathrm{Tr}(X)$ for all g, then in particular $\mathrm{Tr}(W(\Sigma)) \in \mathbb{R}$ is gauge invariant. We call this the *Wilson surface*; if $g = 0$ we call it the *Wilson sphere*.

If we have a punctured surface $S_{g,n}$ with $\chi(S_{g,n}) = 2 - 2g - n < 0$, (e.g. $g \geq 2$) then we can endow $S_{g,n}$ with a hyperbolic structure (i.e. a Riemannian metric with constant curvature -1). We can then require the edges in the marking of the surface to be simple closed geodesics with respect to this structure. This is powerful, because any homotopy class of loops contains a unique geodesic [Thu02, prop. 5.3.1]. Moreover, if we represent a surface as a quotient $\mathbb{H}/\Gamma$ then the boundary of the fundamental domain of this action will give such a marking by geodesics. This still leaves the choice of a fundamental domain, however.

We also have the $\pi_1(S_g)$ admits a presentation with only one relator $\prod_i [\alpha_i, \beta_i]$. Such groups are special, namely if we choose two different presentations then the relators are conjugate [MKS66, Thm. 4.11]. Perhaps together with fixing a hyperbolic metric this allows us to obtain a well-defined surface holonomy as well, up to conjugation. This is however speculation at this point.

4.2. Surface holonomy for covering 2-group 2-bundles

Recall from definition 3.5 we can create from any principal G-bundle P a principal $\mathcal{G}$-2-bundle $\mathcal{P}$ where $\mathcal{G} = (G, \widehat{G}, t, \alpha)$ is the universal covering 2-group of g (cf. def. 2.24). The morphisms of this 2-bundle are just (homotopy classes of) paths in the fibers of P. Suppose A is a connection on P, then since $\tau : \widehat{G} \to G$ induces an isomorphism on the level of Lie algebras, we have that $B = -(\tau_*)^{-1}F$ is a 2-connection on $\mathcal{P}$. In fact it is the only 2-connection with 1-connection A. It will turn out that 2-transport for such 2-bundles is particularly simple. Parallel transport for this type of 2-bundles was studied in great detail in [Par14].

Suppose we have a covering 2-bundle $\mathcal{P}$ of P. Let $\Sigma_t : \Sigma_0 \Rightarrow \Sigma_1$ be some bigon and $p \in P$ then 2-transport gives us a morphism $\mathrm{tra}^1(\Sigma_0)(p) \to \mathrm{tra}^1(\Sigma_1)(p)$, i.e. a path $\mathrm{tra}^1(\Sigma_0)(p) \to \mathrm{tra}^1(\Sigma_1)(p)$. Since Σ_t is a path for each t, one such path is given by $t \mapsto \mathrm{tra}^1(\Sigma_t)(p)$. In fact we can take the 2-transport of $\Sigma_{s,1}$ for any s, which should give a morphism $F(s) : \mathrm{tra}^1(\Sigma_0)(p) \to \mathrm{tra}^1(\Sigma_s)(p)$. This gives us a differential equation:

$$(\tau \circ F)'(s) = (\tau \circ F)(s) A(\partial_s \Sigma(s,1)), \qquad \tau \circ F(s) = 1 \tag{4.3}$$

Since τ_* is an isomorphism and $F(0) = 1$, this uniquely determines $F(s)$ to be the path $F(s)(u) = \mathrm{tra}^1(\Sigma_{su})(p)$. Therefore for the covering 2-bundle $\mathcal{P}$ the only consistent definition of 2-transport is the one given. This means that definition 3.16 with $(A, -\tau_*^{-1}F)$ has to produce the same definition. In fact by Corollary 4.19 in [Par14] any 2-functor $\mathcal{P}_2(M) \to \mathcal{G}$ is of this form, for $\mathcal{G}$ the covering 2-group of G.

We should also note that the 2-curvature of the 2-connection $(A, -\tau_*^{-1}F)$ is zero:

$$K = -d\tau_*^{-1}F + \alpha_*(A \wedge \tau_*^{-1}F) = -\tau_*^{-1}(dF - [A \wedge F]) = -\tau_*^{-1} d_A F = 0. \tag{4.4}$$

The final step, $d_A F = 0$ follows from the fact that F is basic, hence by proposition 1.20: $d_A F = DF = D^2 A = (d^2 A) \circ h = 0$. This can also be seen from another perspective. If $\Sigma, \Sigma' : \gamma \Rightarrow \gamma'$ are homotopic through some homotopy $h : \Sigma \Rrightarrow \Sigma'$ then

$$F_r(s) = \mathrm{tra}^1(h(r,s,1))(p)$$

is a homotopy between the paths $\mathrm{tra}^1(\Sigma_s)(p)$ and $\mathrm{tra}^1(\Sigma'_s)(p)$. Hence the 2-transport along Σ and Σ' produce homotopic paths which thus define the same element of $\widehat{G}$. Thus for this bundle, 2-transport is homotopy independent which also implies that K must be zero.

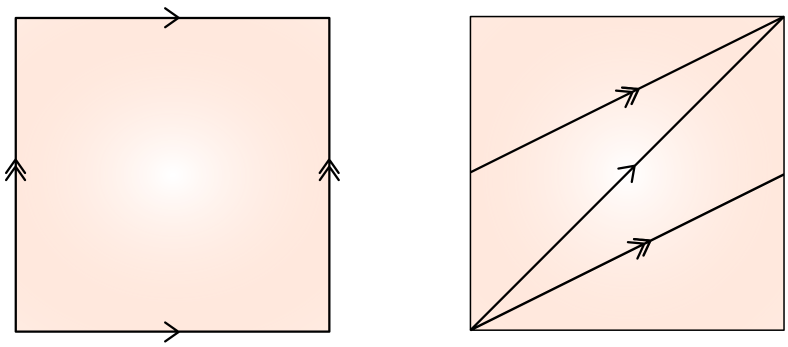
We claimed that the naive definition of surface holonomy we gave in the previous section depends on the choice of marking our surface. Let us show this using an example. Consider the trivial principal G-bundle over a torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates ϕ, θ. We can endow this bundle with the connection $A = Xd\phi + Yd\theta$ for some $X, Y \in \mathfrak{g}$. For a torus we can give an explicit fundamental polygon; take the map $I^2 \to T^2$ given by $(s,t) \mapsto (s,t)$ mod 1. This gives a homotopy $\Sigma : 1 \Rightarrow [\alpha, \beta]$ given by

$$h_s(t) = \begin{cases} (4st, 0) & t \in [0, \frac{1}{4}] \\ (s, (4t-1)s) & t \in [\frac{1}{4}, \frac{1}{2}] \\ ((3-4t)s, s) & t \in [\frac{1}{2}, \frac{3}{4}] \\ (0, (4-4t)s) & t \in [\frac{3}{4}, 1] \end{cases} \tag{4.5}$$

The parallel transport around h_s with respect to A is given by

$$\mathrm{tra}^2(\Sigma)(s) = \exp(-sY)\exp(-sX)\exp(sY)\exp(sX). \tag{4.6}$$

This is because by proposition 1.10 the path ordered exponent in a commutative Lie algebra is just the exponent. In each of the four segments the connection takes values in a torus (spanned by X or Y), and thus the path ordered exponent of A over these segments reduces to an ordinary exponent. We can change the marking by any element of $\mathrm{SL}(2,\mathbb{Z})$ and see how the 2-transport changes. For example we can change the marking by $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ as shown in figure 4.6.



$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

Figure 4.6: Two different markings of the torus with the same basepoint. We get precisely one such marking for each element of $\mathrm{SL}(2,\mathbb{Z})$, and the respective matrices are shown below the two markings.

In this different marking we instead get

$$\mathrm{tra}^2(\Sigma)(s) = \exp(-s(X+2Y))\exp(-s(X+Y))\exp(s(X+2Y))\exp(s(X+Y)). \tag{4.7}$$

To convince oneself that equations 4.6 and 4.7 are really different we can consider $\mathfrak{g} = \mathfrak{so}(3)$ and

$$X = 2\pi \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y = 2\pi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

In that case equation 4.6 gives us $\mathrm{tra}^2(\Sigma)(1) = \mathrm{Id}$ but equation 4.7 gives something much more complicated; we present a numerical approximation:

$$\operatorname{tra}^2(\Sigma)(1) = \begin{pmatrix} 0.791 & -0.417 & -0.449 \\ 0.583 & 0.735 & 0.345 \\ 0.186 & -0.534 & 0.825 \end{pmatrix}$$

In the case of holonomy we could get something well defined by taking Tr. In this case even this is not enough, and we need some careful analysis to produce an invariant out of this.

The main issue that comes into play is that X and Y do not commute, and in any situation where $[X,Y] \neq 0$ the generic situation is that $\operatorname{tra}^2(\Sigma)$ is different for two different markings. On the other hand if $[X,Y] = 0$ then for any choice of marking we would always get the constant path as 2-transport. More generally if the connection takes values in any maximal torus $\mathfrak{t}$ then the 2-transport will only depend on the topology of the principal bundle. To see this, we first classify principal bundles on surfaces.

Proposition 4.7

Principal G-bundles over any surface S are classified by $\pi_1(G) \cong \ker t \subset Z(\widehat{G})$.

Proof: Choose a marking α_i, β_i of S (i.e. representative paths of generators of $\pi_1(S)$). Then let U be a tubular neighborhood of the image of $\prod_i [\alpha_i, \beta_i]$. Per definition U retracts to a wedge of circles. We thus have that any map $U \to \mathbb{B}G$ is contractible, since $\pi_1(\mathbb{B}G) \cong \pi_0(G) = \{1\}$ (we always assume G is connected). Hence any G-bundle over U is trivial. Similarly let $V = S \setminus \prod_i [\alpha_i, \beta_i]$ be the complement of the image of $\prod_i [\alpha_i, \beta_i]$. This subspace is contractible, hence any G-bundle over V is trivial.

Now given a G-bundle $P \to S$, we can trivialize P over U and V. Then the bundle is up to isomorphism completely determined by the homotopy class of the transition function $U \cap V \to G$. Since $U \cap V$ retracts to a circle we can conclude that G-bundles over S are classified by $\pi_1(G)$. The fact that we can see $\pi_1(G)$ as a subgroup of $Z(\widehat{G})$ is by Theorem A.13 and proposition A.14. ■

This classification also gives us a very explicit description of connections on a principal bundle over a surface. On the fundamental polygon of the surface we can choose polar coordinates (r,θ) (with $\theta \in [0, 2\pi)$) by identifying the polygon with a unit disk in $\mathbb{R}^2$. We can then set $U = \{ (r,\theta) \in D^2 \mid r > 0 \}$ and $V = \{ (r,\theta) \in D^2 \mid r < 1 \}$. We can choose the transition function $g : \Sigma^* P|_{U \cap V} \to (U \cap V) \times G$ to be

$$g(r,\theta) = \exp(\theta t_* X), \tag{4.8}$$

with $X \in \widehat{\mathfrak{g}}$ such that $\exp(2\pi X) \in \ker t \subset Z(\widehat{G})$, and thus $\exp(2\pi t_* X) = 1$. Then a connection on P is determined by two one forms $A_{\Sigma(U)} \in \Omega^1(\Sigma(U), \mathfrak{g})$ and $A_{\Sigma(V)} \in \Omega^1(\Sigma(V), \mathfrak{g})$. We then get two one-forms $A_U = \Sigma^* A_{\Sigma(U)}$ and $A_V = \Sigma^* A_{\Sigma(V)}$ on I^2 which together give a connection on $\Sigma^* P$. That is, they satisfy

$$A_V|_{U\cap V} = \mathrm{Ad}_g\, A_U|_{U\cap V} + dgg^{-1} = \mathrm{Ad}_g\, A_U|_{U\cap V} + t_* X\, \mathrm{d}\theta. \tag{4.9}$$

Using this local description we get a particularly nice formula for the 2-transport if $\mathfrak{g}$ is a torus.

Proposition 4.8

Let $P \to S$ be a principal G-bundle corresponding to some $z \in Z(\widehat{G})$ by proposition 4.7. Let $\Sigma : I^2 \to S$ be a marking. If $\mathfrak{t}$ is a maximal torus in G then any connection $A \in \Omega^1(P, \mathfrak{g})$ which takes values strictly in $\mathfrak{t}$ has 2-holonomy $\mathrm{tra}^2(\Sigma) = z$. Put differently, if $Q \subset P$ is a principal $T \subset G$ subbundle embedded in P, then the surface transport of any connection on Q is z.

Proof: Assume that A takes values in a maximal torus $\mathfrak{t} \subset \mathfrak{g}$. We can choose the X as above in such a way that $t_* X \in \mathfrak{t}$. This is because $Z(\widehat{G})$ is the intersection of all maximal tori, hence for each maximal torus $\mathfrak{t}$ in $\widehat{\mathfrak{g}}$ and $z \in Z(\widehat{G})$ there is an $X \in \mathfrak{t}$ such that $\exp(2\pi X) = z$. Equation 4.9 then becomes

$$A_V|_{U\cap V} = A_U|_{U\cap V} + t_* X\, \mathrm{d}\theta. \tag{4.10}$$

This means that there are globally defined functions $f_\theta, f_r : D^2 \to \mathfrak{g}$ such that

$$A_U = f_\theta\, \mathrm{d}\theta + f_r\, \mathrm{d}r, \qquad A_V = (f_\theta + t_* X)\mathrm{d}\theta + f_r\, \mathrm{d}r. \tag{4.11}$$

We can now parameterize $\Sigma : I^2 \to S_g$ as a bigon using the path (in polar coordinates (r, θ))

$$\gamma_s(t) = \begin{cases} \Sigma(3st, 0) & t \in [0, \frac{1}{3}] \\ \Sigma(s, 2\pi(3t-1)) & t \in [\frac{1}{3}, \frac{2}{3}] \\ \Sigma((3-3t)s, 0) & t \in [\frac{2}{3}, 1] \end{cases}$$

as sketched in figure 4.9. Note that for $s = 0$ this path is not the constant path, but rather it is thinly homotopic to the constant path.

Now computing the 2-transport along Σ is a simple integration procedure. For each s we can use A_U or A_V to compute the parallel transport of γ_s. Since A takes values in $\mathfrak{t}$, path ordered exponentiation becomes ordinary exponentiation and we obtain:

$$\begin{aligned} \mathrm{tra}^1(\gamma_s) &= \left(\exp \int_0^1 f_r(sr, 0)\, \mathrm{d}r\right) \left(\exp \int_0^{2\pi} f_\theta(s, \theta)\, \mathrm{d}\theta\right) \left(\exp - \int_0^1 f_r(sr, 0)\, \mathrm{d}r\right) \\ &= \exp \int_0^{2\pi} f_\theta(s, \theta)\, \mathrm{d}\theta, \end{aligned}$$

where that last line follows from commutativity of $\exp \mathfrak{t}$. There is no dependence on X, since $\exp(2\pi t_* X) = 1$. Because A_U and A_V are pullbacks from S we necessarily have that $A_V|_{r=0} = 0$,

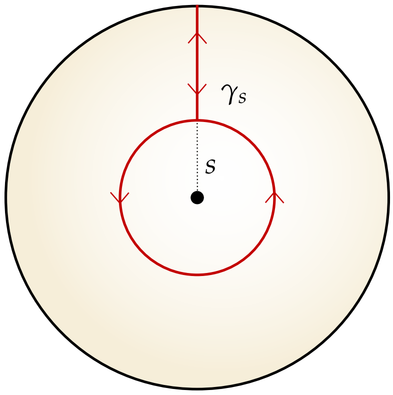


Figure 4.9: Parameterization of the bigon Id $\Rightarrow \prod_i [\alpha_i, \beta_i]$ using polar coordinates.

thus $f_\theta(0, \theta) = -t_* X$. Furthermore since the image of ∂I^2 under Σ is a product of commutators of loops we necessarily have that

$$\int_0^{2\pi} A_U(1, \theta)\, \mathrm{d}\theta = 0.$$

This means that the function $u(r) = \int_0^{2\pi} f_\theta(r, \theta)\, \mathrm{d}\theta$ is a map $I \to \mathfrak{g}$ starting at $-2\pi t_* X$ and ending at 0. Any such map is homotopic to the linear map $s \mapsto -2\pi s X$. Hence the loop $s \mapsto \mathrm{tra}^1(\gamma_s)$ in G is homotopic to the loop $s \mapsto t(\exp(2\pi s X))$, which corresponds to the element $z \in Z(\widehat{G})$ we started with. This computation is a generalization of the examples given in [Par14], where this computation was done in great detail for several specific groups in genus 0. ■

This means that in the case that a connection takes values in a torus, not only is the 2-transport independent of the choice of marking, it doesn't even depend on the connection or choice of torus. The surface holonomy computes a topological invariant of the bundle. From the perspective of definition 3.16 of 2-transport this can also be seen. Let (G, H, α, τ) be any 2-group. If B takes values in some torus $\mathfrak{t} \subset \mathfrak{h}$ and A values in $t_*\mathfrak{t}$, then $\alpha(\mathrm{tra}^1(\gamma_{s,t}))$ is trivial, and thus we obtain for any marking $\Sigma : I^2 \to S$

$$\mathrm{tra}^2(\Sigma) = \exp\left[\int_0^1 \mathrm{d}s \int_0^1 \mathrm{d}t\ \ -\Sigma^* B(\partial_s, \partial_t)\right]^{-1} = \exp \int_S B. \tag{4.12}$$

If we have some representation $\pi : T \to \mathrm{GL}(V)$ then we can consider

$$\det \circ \pi \circ t\ \mathrm{tra}^2(\Sigma) = \exp\left(\int_S \mathrm{Tr}\, \pi_* F\right). \tag{4.13}$$

Notice that the term in the exponential is a Chern-Weil characteristic class. This shows that surface holonomy has in general a non-trivial dependence on the topology of the principal bundle. Perhaps one can therefore think of surface holonomy as a generalization of Chern-Weil numbers.

Wilson surfaces were also studied in [ACM15], and there also torus subbundles were considered and the same Chern-Weil characteristic classes appeared. Their approach is however by considering

$\int_\gamma \operatorname{Tr} \lambda(dgg^{-1} + \operatorname{Ad}_g A)$ where $\lambda \in \mathfrak{t}^*$ is a highest weight, $b : \gamma \to \mathfrak{g}^*$ is any map, and $g : \gamma \to G$ is such that $b(s) = g(s)\lambda g(s)^{-1}$. This appears from physical considerations when studying Wilson Loops. They then noted that form in the integral is exact, with primitive $DP_\lambda(A, g) = \operatorname{Tr} b(F_A - (d_A g g^{-1})^2)$. The integral of this primitive over a surface was then studied, and they found that for a connection taking values in a maximal torus we recover $\int_S \operatorname{Tr} \pi_* F$.

So far our results offer a partial answer to the question posed in [BS04]:

> *"Is the nonabelian surface holonomy in 2-bundles with strict structure 2-group 're-ducible' in some appropriate sense to ordinary abelian surface holonomy?"*

Namely, non-Abelian surface holonomy is not well-defined without choosing a marking, and therefore not reducible to Abelian surface holonomy. However assuming B takes values in a torus (e.g. by restricting to a torus subbundle), the dependence on the marking goes away and we recover the definition of Abelian surface holonomy.

A. Compact Lie Groups

Throughout the thesis we heavily rely on some of the properties of compact Lie groups. Therefore we will review some of these properties in this appendix. We will assume some basic familiarity with the theory of Lie groups in this exposition, and focus specifically on the properties of compact Lie groups. This exposition does not necessarily follow a logical order and is more a collection of important facts about compact Lie groups. All of the material can be found in [Kna02].

Definition A.1

A *compact Lie group* is a Lie group G such that the smooth manifold G is compact. A *compact Lie algebra* is the Lie algebra of a compact group. Throughout this appendix, let G be a compact connected Lie group and $\mathfrak{g}$ be its Lie algebra.

Proposition A.2

G admits a measure μ (the *Haar measure*) which is invariant under both left and right multiplication.

Proposition A.3

Let $\phi : G \to \mathrm{GL}(V)$ be a representation, then V admits an inner product such that Φ is a unitary, i.e. $(\Phi(x)u, v) = (u, \Phi(x)v)$ for all $x \in G$ and $u, v \in V$.

Proof: Take any inner product $\langle \cdot\,,\cdot \rangle$ on V and define

$$(u, v) = \int_G \langle \Phi(x)u,\, \Phi(x)v \rangle \; \mathrm{d}x, \tag{A.1}$$

where we integrate with respect to the Haar measure μ. Unitarity of Φ now follows from the bi-invariance of μ. ■

Proposition A.4

Any compact Lie group admits a faithful finite-dimensional unitary representation. In particular any compact Lie group is isomorphic to a closed subgroup of $U(n)$ for some n.

Proof: The existence of a faithful representation is guaranteed by the Peter-Weyl Theorem, and it can be assumed to be unitary by proposition A.3. ■

Using an embedding $G \hookrightarrow U(n)$ we can assume all compact Lie groups are just matrix groups. This can make notation simpler since this gives an explicit form to the adjoint action, Lie brackets and exponentiation operations.

Proposition A.5

The Lie algebra $\mathfrak{g}$ admits an Ad-invariant inner product, which is also ad-skew symmetric i.e.

$$(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = (X, Y), \qquad (\mathrm{ad}(Z)X, Y) + (X, \mathrm{ad}(Z)Y) = 0.$$

Proposition A.6

$\mathfrak{g}$ is reductive and thus $\mathfrak{g} = Z_{\mathfrak{g}} \oplus [\mathfrak{g}, \mathfrak{g}]$ where $Z_{\mathfrak{g}}$ is the center of $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}]$ is semi-simple In particular a compact Lie group is semi-simple if and only if its center is discrete.

Definition A.7

A *maximal torus* $T \subset G$ is a maximal Abelian subgroup (i.e. it's not contained in a larger one). By compactness $T \cong (S^1)^n$ for some n. Note that $Z(G) \subset T$ for any T and subsequently $Z(G)$ is the intersection of all maximal tori.

Proposition A.8

Let $\mathfrak{t} \subset \mathfrak{g}$ be the Lie algebra of $T \subset G$. Then T is a maximal torus if and only if $\mathfrak{t}$ is a maximal Abelian subalgebra.

Theorem A.9

Let $\mathfrak{t}, \mathfrak{t}'$ be two maximal Abelian subalgebras of $\mathfrak{g}$, then $\mathfrak{t}' = \mathrm{Ad}_g\, \mathfrak{t}$ for some $g \in G$. Alternatively, let $T, T' \subset G$ be maximal tori then $T' = gTg^{-1}$ for some $g \in G$.

Theorem A.10

Let T be a maximal torus, then for each $g \in G$ there is an $h \in G$ such that $g \in hTh^{-1}$.

This can be informally interpreted as saying that every element of a compact Lie group can be diagonalized. That is, in $U(n)$ the set of diagonal matrices $\mathrm{diag}(e^{i\phi_1}, \dots, e^{i\phi_n})$ is a maximal torus, and the previous theorem just means that each element can diagonalized by conjugating it by some other element.

Using Theorem A.10 we show that any element is contained in a maximal torus, and one can also show that on tori the exponential map is surjective. Thus we obtain:

Theorem A.11

The exponential map $\exp : \mathfrak{g} \to G$ is surjective

This is very useful since it allows us to pass easily between the Lie algebra and Lie group.

Finite covers of Lie compact Lie groups are of particular interest. For any semi-simple Lie group G there is a universal covering group $\widehat{G}$, i.e. the unique simply connected finite cover of G. The fact that the cover is finite is because of the following theorem:

Theorem A.12 (Weyl)

If G is compact and semi-simple, then $\pi_1(G)$ is finite. Hence the universal cover $\widehat{G} \to G$ is also a compact Lie group

The semi-simplicity is essential here. If G has a non-discrete center then the fundamental group will contain a cyclic subgroup. The universal covering group has a simple geometric interpretation. Namely it is the set of all homotopy classes of paths rel boundary in G starting at the identity;

$$\widehat{G} = \{ [\gamma] \in \Pi_1(G) \mid s(\gamma) = \gamma(0) = 1 \}, \tag{A.2}$$

where $\Pi_1(G)$ is the fundamental groupoid, i.e. the groupoid of homotopy classes of paths rel boundary. The covering map $t : \widehat{G} \to G$ is then just $t([\gamma]) = \gamma(1)$. The multiplication is given by pointwise multiplication in G. This space is indeed simply connected; take any path $h : I \to \widehat{G}$, then each $h(t)$ is a path in G, so define for each s a new path $h(t)^s(u) = h(t)(su)$. Then $(s,t) \mapsto h(t)^s$ is a homotopy between h and the trivial path in $\widehat{G}$. This is a specific case of a more general phenomenon [Hat01, Thm. 1.38]:

Theorem A.13 (Galois Correspondence)

Let X be a path-connected, locally path-connected, and semilocally simply-connected space (e.g. a connected manifold). Then there is a bijection between the set of isomorphism classes of connected covering spaces $p : \widetilde{X} \to X$ and conjugacy classes of subgroups of $\pi_1(X)$

The correspondence is given on the one hand by associating to $p : \widetilde{X} \to X$ the group $p_*(\pi_1(\widetilde{X}))$. On the other hand given a $K \subset \pi_1(X)$ we set

$$\widetilde{X} = X^K = \{ [\gamma] \in \Pi_1(X) \mid s([\gamma]) = \gamma(0) = 1 \} \Big/ \sim, \tag{A.3}$$

where $[\gamma] \sim [\gamma']$ if $[\overline{\gamma}\gamma'] \in K$. Again $t : X^K \to X$ is just $[\gamma] \mapsto \gamma(1)$. In particular for $K = \{1\}$ we obtain the universal cover. For Lie groups the classification of covers goes even a little further.

Proposition A.14

If $t : G \to H$ is a cover then $\ker t$ is a central subgroup. Thus a cover of a Lie group G is equivalent to a central extension of G.

PROOF: Let $z \in \ker t$ and $g \in G$, then we need to show $zgz^{-1} = g$. At least $t(zgz^{-1}) = t(g)$. Let $K = t_* \pi_1(G)$ and identify z and g with some path in H denoted by ζ, γ respectively. Then we need to show that $\overline{\gamma}(\zeta\gamma\zeta^{-1}) \in K$ where $K = t_*\pi_1(G)$. Actually $\zeta\gamma\zeta^{-1}$ and γ are homotopic; just take the homotopy $s \mapsto \zeta^s\gamma(\zeta^{-1})^s$ (in fact the same argument shows $\pi_1(G)$ is Abelian for any topological group). We conclude that $zgz^{-1} = g$ for all $z \in \ker t$ and $g \in G$ and thus $\ker t$ is central. ∎

Theorem A.15 (Lie Integration)

Let $\mathfrak{g}$ be a Lie algebra, then there is a 1-connected Lie group G with Lie algebra $\mathfrak{g}$. Furthermore if G is 1-connected then for every $\phi : \mathfrak{g} \to \mathfrak{h}$ there is a unique homomorphism $f : G \to H$ such that $\phi = f_*$. In particular, this also means that the 1-connected Lie group integrating any Lie algebra is unique. Finally if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then there is a unique connected Lie subgroup H of G with Lie algebra $\mathfrak{h}$.

Proposition A.16

Let $\mathfrak{g}$ be a semi-simple compact Lie algebra, and let $\widehat{G}$ be the 1-connected Lie group integrating it. Then the other connected Lie groups integrating $\mathfrak{g}$ are classified by central subgroups of $\widehat{G}$.

Proof: If G is any other Lie group with lie algebra $\mathfrak{g}$ then $\widehat{G}$ is necessarily its universal cover. Denote the covering map by $t : \widehat{G} \to G$ then by the Galois correspondence $\ker t \cong \pi_1(G)$. On the other hand if $Z \subset Z(\widehat{G})$ is a central subgroup then $\widehat{G}/Z$ is covered by $\widehat{G}$. Since $\mathfrak{g}$ is semi-simple, $Z(\widehat{G})$ is discrete and hence $\widehat{G}/Z$ has the same Lie algebra as $\widehat{G}$. If $\mathfrak{g}$ is not semi-simple, then $Z(\widehat{G})$ is not discrete and the argument doesn't work. ■

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